Quotients of inverse semigroups, étale groupoids and C*-algebras

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Contents

Chapter 0.

Introduction

The theory of operator algebras is a branch of functional analysis. This theory was initiated to formulate a mathematical framework of quantum mechanics. The theory of operator algebras itself is deeply evolving and interacting with other fields like representation theory, dynamical systems, number theory and so on. Operator algebras are divided into von Neumann algebras and C*-algebras, depending on topologies. In this thesis, we study C*-algebras. Because C*-algebras are highly abstract objects, it used to be difficult to construct a C*-algebra with desired properties. Now there are many ways to construct C*-algebras from mathematical objects like groups, dynamical systems, directed graphs and so on. Many researchers have studied the relation between associated C*-algebras and their ingredients.

In this thesis, we treat C*-algebras associated to étale groupoids. A groupoid is a small category whose morphisms are invertible. An étale groupoid is a groupoid equipped with topology which has discreteness in some sense. Discrete groups and topological spaces are typical examples of étale groupoids. Étale groupoids are associated to many objects like discrete group actions, directed graphs, tilings and so on. Using étale groupoids, we can treat many objects in a unified way.

For an étale groupoid G, one can associate C*-algebras $C^*(G)$ and $C^*_\lambda(G)$, which are called the full groupoid C*-algebra and the reduced groupoid C*-algebra respectively. The study of C*-algebras associated to groupoids was initiated by Renault's lecture note [18]. The class of groupoid C*-algebras is an important class of C*-algebras because it contains a broad class of C*-algebras and groupoid C*-algebras are somewhat treatable. Actually, many researchers have studied the relationship between étale groupoids G and groupoid C*-algebras $C^*(G)$, $C^*_\lambda(G)$. For example, the simplicity of groupoid C*-algebras is studied in [3] while the intermediate subalgebras of groupoid C*-algebras are studied in [4].

As mentioned above, we can construct étale groupoids from many objects. In this thesis, we mainly treat étale groupoids associated to actions of inverse semigroups. An inverse semigroup is a special class of semigroups. Inverse semigroup actions are used to describe the local symmetry of the spaces, while group actions describe the global symmetry of the spaces. When an inverse semigroup acts on a topological space, one can associate an étale groupoid. An inverse semigroup acts on a certain topological space called a spectrum in a natural way. Hence, we can associate an étale groupoid, which is called a universal groupoid, to this action on the spectrum. The study of the universal groupoids is initiated by Paterson [14]. It is a natural task to study the relation between inverse semigroups and the universal groupoids. Because the universal groupoids are constructed only from the algebraic structure of inverse semigroups, it is expected that properties of the universal groupoids should be described in purely algebraic language.

The author of this thesis studies the relation among inverse semigroups, étale groupoids and C*-algebras. This research aims to give algebraic and intuitive description for infinite dimensional phenomena of C*-algebras by using inverse semigroup and étale groupoids. In addition, this research also aims to apply techniques in the theory of C*-algebras to the theory of inverse semigroups and étale groupoids. In short, the purpose of this research is to construct a framework to mutually develop the theory of inverse semigroups, étale groupoids and C*-algebras.

In this thesis, we study the relation among inverse semigroups, étale groupoids and C*-algebras from the view point of quotients. We will prove that quotients of inverse semigroups induce the quotients of étale groupoids. Similarly, we prove that quotients of étale groupoids induce the quotients of C*-algebras. Then we investigate certain quotients such as the abelianization of inverse semigroups, étale groupoids and C*-algebras. The main theorems in this thesis are Theorem A, B and C as described below.

This thesis is organized as follows. Chapter 1 is devoted to preliminaries. We introduce here notions which we use in this thesis.

In Chapter 2, we describe the results in [11]. Quotients of inverse semigroups and C*-algebras are fundamental notions and well-established. On the other hand, quotients of étale groupoids seem to be fundamental notions, but the author could not find them in literatures. Therefore, we establish the notion of quotient étale groupoids. One may imagine that the notion of a quotient étale groupoid is defined as a surjective groupoid homomorphism to another étale groupoid. However, such formal quotients do not induce the quotients of groupoid C*-algebras in a natural way. Therefore, in this thesis, we define the notion of quotient étale groupoids so that the quotients of groupoid C*-algebras are naturally induced. After we define the notion of quotient étale groupoids, we observe that quotients of étale groupoids actually induce the quotients of C*-algebras. Using these facts, we obtain the main theorem (Theorem 2.2.2.4) in this chapter. For an étale groupoid G, we define the abelianization G^{ab}, which is also an étale groupoid. This étale groupoid G^{ab} describes the abelianization of $C^*(G)$ as follows.

Theorem A (Theorem 2.2.2.4)**.** Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Then the abelianization $C^*(G)^{\mathrm{ab}}$ of $C^*(G)$ is isomorphic to $C^*(G^{\mathrm{ab}})$.

The key step in the proof of Theorem 2.2.2.4 is the calculation of one dimensional representations of $C^*(G)$ (Theorem 2.2.1.8). At the end of this chapter, we explain the relation between these theorems and the dual of étale abelian group bundles.

In Chapter 3, we describe the results in [10]. We study the relation between quotients of inverse semigroups and quotients of the universal groupoids. Given an inverse semigroup S, one can associate the universal groupoid $G_u(S)$. We observe that a quotient $S \twoheadrightarrow S/\nu$ of an inverse semigroup S by a congruence ν induces the invariant set F_ν of $G_u(S)$ and the normal subgroupoid $G_u(\ker\nu)_{F_\nu} \subset G_u(S)_{F_\nu}$, where $G_u(S)_{F_\nu}$ denotes the restriction of $G_u(S)$ to F_ν. Now we may consider the quotient étale groupoid $G_u(S)_{F_\nu}/G_u(\ker\nu)_{F_\nu}$ and obtain one of the main theorems (Theorem 3.2.1.3).

Theorem B (Theorem 3.2.1.3)**.** Let S be an inverse semigroup and ν be a congruence on S. Then $G_u(S/\nu)$ is isomorphic to $G_u(S)_{F_\nu}/G_u(\ker\nu)_{F_\nu}$.

This theorem gives us a way to compute the universal groupoids associated to quotient inverse semigroups. Indeed, we compute the universal groupoid associated to the certain quotient inverse semigroups including the abelianizations. We show that the abelianizations of étale groupoids introduced in Chapter 2 corresponds to the abelianizations of inverse semigroups.

Theorem C (Theorem 3.2.2.3)**.** Let S be an inverse semigroup. Then $G_u(S^{\mathrm{ab}})$ is isomorphic to $G_u(S)^{\mathrm{ab}}$.

In the last of this chapter, we give applications and examples. We analyse Clifford inverse semigroups and compute the universal groupoids associated to the free Clifford inverse semigroups (Theorem 3.3.2.7). We also evaluate the number of fixed points in transformation groupoids associated to Boolean actions (Corollary 3.3.3.2).

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Chapter 1.

Preliminaries

1.1 Preliminaries

In this section, we recall fundamental notions which we use in this thesis. First, we recall the definitions and properties of C*-algebras, inverse semigroups and étale groupoids. Then we explain how to associate C*-algebras to étale groupoids. Finally we define the notion of inverse semigroup actions and explain how to construct étale groupoids from given inverse semigroup actions.

1.1.1 C*-algebras

In this subsection, we recall fundamental notions about C*-algebras. See [22] or [5] for details.

In this thesis, we assume that a coefficient of a vector space is the field of complex numbers $\mathbb{C}$. A $\mathbb{C}$-algebra is a $\mathbb{C}$-vector space with a multiplication which is compatible with the structure of $\mathbb{C}$-vector space. A Banach space is a complete normed space.

Definition 1.1.1.1. A Banach space A is called a Banach algebra if A is equipped with a multiplication such that $\|ab\| \leq \|a\|\|b\|$ holds for all $a, b \in A$. A C*-algebra is a Banach algebra A equipped with an involution $A \ni a \mapsto a^* \in A$ such that

1. $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$ holds for all $\alpha, \beta \in \mathbb{C}$ and $a, b \in A$;
2. $(a^*)^* = a$ holds for all $a \in A$;
3. $(ab)^* = b^* a^*$ holds for all $a, b \in A$; and
4. $\|a^* a\| = \|a\|^2$ holds for all $a \in A$.

Definition 1.1.1.2. Let A and B be C*-algebras. A map $\pi\colon A \to B$ is called a *-homomorphism if π is a linear map which satisfies $\pi(a_1 a_2) = \pi(a_1)\pi(a_2)$ and $\pi(a_1^*) = \pi(a_1)^*$ for all $a_1, a_2 \in A$. C*-algebras A and B are said to be isomorphic if there exist *-homomorphisms $\pi\colon A \to B$ and $\sigma\colon B \to A$ such that $\sigma \circ \pi = \mathrm{id}_A$ and $\pi \circ \sigma = \mathrm{id}_B$

hold. Equivalently, A and B are isomorphic if there exists a bijective *-homomorphism $\pi\colon A \to B$.

The continuity of a *-homomorphism is automatically deduced as follows.

Proposition 1.1.1.3 ([22, Proposition 5.2 in Chapter I])**.** Let A and B be C*-algebras. Then every *-homomorphism $\pi\colon A \to B$ is norm decreasing (i.e. $\|\pi(a)\| \leq \|a\|$ holds for all $a \in A$). Moreover, π is injective if and only π is an isometry (i.e. $\|\pi(a)\| = \|a\|$ holds for all $a \in A$).

Example 1.1.1.4. Let X be a locally compact Hausdorff space. A continuous function $f\colon X \to \mathbb{C}$ is said to vanish at infinity if for all $\varepsilon > 0$, the set

$$\{x \in X \mid |f(x)| \geq \varepsilon\}$$

is a compact subset of X. The set of all continuous functions on X is denoted by $C(X)$. The set of all elements in $C(X)$ vanishing at infinity is denoted by $C_0(X)$. Then $C(X)$ and $C_0(X)$ are $\mathbb{C}$-algebras with respect to the pointwise operations. For $f \in C_0(X)$, define $\|f\| := \sup_{x\in X}|f(x)|$. Then this defines a norm on $C_0(X)$ and $C_0(X)$ is a Banach algebra. Since $C_0(X)$ has an involution defined by the pointwise complex conjugation, $C_0(X)$ is a C*-algebra. Note that $C_0(X) = C(X)$ if and only if X is compact.

We recall the Gelfand-Naimark duality. Let A be a commutative C*-algebra. We denote the set of characters of A by $\Delta(A)$. Recall that $\Delta(A)$ is the set of all nonzero *-homomorphisms from A to $\mathbb{C}$. It is known that $\Delta(A)$ is a subset of A^*, the dual space of A, and a locally compact Hausdorff space with respect to the weak* topology.

Theorem 1.1.1.5 (Gelfand-Naimark)**.** Let A be a commutative C*-algebra. Then A is isomorphic to $C_0(\Delta(A))$. Moreover, if a locally compact Hausdorff space Y satisfies $A \simeq C_0(Y)$, then Y is homeomorphic to $\Delta(A)$.

Another typical example of C*-algebras is a C*-algebras of bounded linear operators on a Hilbert space.

Example 1.1.1.6. Let H be a Hilbert space. We denote the set of all bounded linear operators on H by $B(H)$. Then $B(H)$ is a C*-algebra. Recall that the norm of $x \in B(H)$ is defined by

$$\|x\| := \sup_{\|\xi\|\leq 1} \|x\xi\|$$

and the involution of $x \in B(H)$ is defined to be the operator $x^* \in B(H)$ which satisfies

$$\langle \xi | x^* \eta \rangle = \langle x\xi | \eta \rangle$$

for all $\xi, \eta \in H$. The existence and uniqueness of x^* follow from Riesz representation theorem.

For a C*-algebra A, a *-homomorphism from A to $B(H)$ is called a *-representation of A on H. The next theorem states that every C*-algebra is isomorphic to a subalgebra of $B(H)$.

Theorem 1.1.1.7 (Gelfand-Naimark)**.** Let A be a C*-algebra. Then there exist a Hilbert space H and an injective *-representation $\pi\colon A \to B(H)$.

1.1.2 Inverse semigroups

In this subsection, we recall fundamental notions about inverse semigroups. See [13] or [14] for more details.

Let S be a semigroup, which is a set equipped with a multiplication with the associative law. For $s \in S$, an element $t \in S$ is called a generalized inverse of s if t satisfies $sts = s$ and $tst = t$. A semigroup S is called an inverse semigroup if each $s \in S$ admits a unique generalized inverse, which is denoted by $s^* \in S$.

Example 1.1.2.1. A group Γ is an inverse semigroup. Note that $s^* = s^{-1}$ holds for all $s \in \Gamma$.

Example 1.1.2.2. Let $n \in \mathbb{N}$ be a natural number. For $1 \leq i, j \leq n$, define $e_{i,j} \in M_n(\mathbb{C})$ to be the $n \times n$ matrix whose (i, j)th entry is 1 and the other entries are zero. Then

$$S := \{e_{i,j} \in M_n(\mathbb{C}) \mid 1 \leq i, j \leq n\} \cup \{0\}$$

is an inverse semigroup with respect to the usual product of matrices. Note that $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$ and $e_{i,j}^* = e_{j,i}$ hold for all $i, j, k, l \in \{1, 2, \ldots, n\}$, where $\delta_{k,l}$ denotes the Kronecker delta.

Example 1.1.2.3. Let X be a topological space. A partial homeomorphism on X is a homeomorphism between open subsets of X. For a partial homeomorphism f on X, the domain and range of f are denoted by $\operatorname{dom} f$ and $\operatorname{ran} f$ respectively. The set of all partial homeomorphisms on X is denoted by I_X. Then I_X is an inverse semigroup with respect to the multiplication defined by the composition of maps. We remark that the composition $f \circ g \in I_X$ of $f \in I_X$ and $g \in I_X$ is defined on $\operatorname{dom} g \cap g^{-1}(\operatorname{dom} f \cap \operatorname{ran} g)$. Also remark that the inverse of $f \in I_X$ is the inverse map f^{-1}.

Let S be an inverse semigroup. By a subsemigroup of an inverse semigroup S, we mean a subset $T \subset S$ closed under the multiplication and generalized inverse $s \mapsto s^*$. We denote the set of all idempotents in S by $E(S) := \{e \in S \mid e^2 = e\}$. It is known that $E(S)$

is a commutative subsemigroup of S. A zero element is a unique element $0 \in S$ such that $0s = s0 = 0$ holds for all $s \in S$. A unit is a unique element $1 \in S$ such that $1s = s1 = s$ holds for all $s \in S$. An inverse semigroup does not necessarily have a zero element nor a unit. A subsemigroup N of S is said to be normal if $E(S) \subset N$ and $sns^* \in N$ holds for all $s \in S$ and $n \in N$. An order on S is defined by declaring that $s \leq t$ if $st^*t = s$ for $s, t \in S$. Remark that $e \leq f$ is equivalent to $ef = e$ for $e, f \in E(S)$. Note that $E(S)$ is a meet semilattice. Indeed, one can see that the infimum of $e, f \in E(S)$ is ef.

We recall that the notion of congruences. A congruence is an equivalence relation which is compatible with the multiplication of an inverse semigroup.

Definition 1.1.2.4. Let S be an inverse semigroup. An equivalence relation ν on S is called a congruence if $(s, t) \in \nu$ implies $(as, at) \in \nu$ and $(sa, ta) \in \nu$ for all $s, t, a \in S$.

One can see that S/ν is an inverse semigroup such that the quotient map $S \to S/\nu$ is a semigroup homomorphism.

Example 1.1.2.5. Let Γ be a group. There is a one-to-one correspondence between congruences on Γ and normal subgroups of Γ. For a congruence ν on Γ, define

$$N_\nu := \{n \in \Gamma \mid (e, n) \in \nu\},$$

where $e \in \Gamma$ is the unit element. Then N_ν is a normal subgroup of Γ. Conversely, for a normal subgroup $N \subset \Gamma$, define an equivalence relation ν_N on Γ by declaring that $(s, t) \in \nu_N$ if $s^{-1}t \in N$ for $s, t \in \Gamma$. Then ν_N is a congruence on Γ. One can see that $\nu = \nu_{N_\nu}$ and $N_{\nu_N} = N$ hold for all congruences ν and normal subgroups $N \subset \Gamma$.

Remark 1.1.2.6. As in Example 1.1.2.5, the notion of congruences corresponds to the notion of normal subgroups in case of groups. For a general inverse semigroup S, it is known that a congruence on S corresponds to a congruence pair, which is a pair of normal inverse subsemigroups of S and a normal congruence on $E(S)$ with some compatibilities. We do not explain a congruence pair any more because we do not use facts about a congruence pair. See [15] for details.

A congruence ρ on $E(S)$ is said to be normal if $(e, f) \in \rho$ implies $(ses^*, sfs^*) \in \rho$ for all $e, f \in E(S)$ and $s \in S$. One of the typical examples of normal congruences is $E(S) \times E(S)$. Assume that ρ is a normal congruence on $E(S)$. Define an equivalence relation $\nu_{\rho,\min}$ on S by declaring that $(s, t) \in \nu_{\rho,\min}$ if $(s^*s, t^*t) \in \rho$ and $se = te$ holds for some $e \in E(S)$ with $(e, s^*s) \in \rho$. Then $\nu_{\rho,\min}$ is the minimum congruence on S such that its restriction to $E(S)$ coincides with ρ. One can see that $\nu_{E(S)\times E(S),\min}$ is the least congruence such that the quotient inverse semigroup becomes a group. We call $S/\nu_{E(S)\times E(S),\min}$ the maximal group image of S.

Remark. We check that $\nu_{\rho,\min}$ is the minimum congruence on S such that its restriction to $E(S)$ coincides with ρ for the reader's convenience. It is easy to see that $\nu_{\rho,\min}$ is an equivalence relation on S. We show that $\nu_{\rho,\min}$ is a congruence on S. Assume that $(s,t) \in \nu_{\rho,\min}$ and $a \in S$. It suffices to show $(as, at), (sa, ta) \in \nu_{\rho,\min}$. We have $(s^*s, t^*t) \in \rho$ and there exists $e \in E(S)$ such that $se = te$ and $(s^*s, e) \in \rho$. Since we have $(s^*s, e) \in \rho$ and ρ is a congruence on $E(S)$, it follows $(s^*a^*as, s^*a^*ase) = (s^*a^*ass^*s, s^*a^*ase) \in \rho$. One can see that $(t^*a^*at, t^*a^*ate) \in \rho$ in the same way. Since we have

$$s^*a^*ase = (se)^*a^*a(se) = (te)^*a^*a(te) = t^*a^*ate,$$

it follows $(s^*a^*as, t^*a^*at) \in \rho$ from $(s^*a^*as, s^*a^*ase) \in \rho$ and $(t^*a^*at, t^*a^*ate) \in \rho$. From $se = te$, it follows that

$$ass^*a^*ase = ase = ate = att^*a^*ate = ats^*a^*ase.$$

Combining with $(s^*a^*as, s^*a^*ase) \in \rho$, we obtain $(as, at) \in \nu_{\rho,\min}$. Next we show $(sa, ta) \in \nu_{\rho,\min}$. Since ρ is normal and $(s^*s, t^*t) \in \rho$, we have $(a^*s^*sa, a^*t^*ta) \in \rho$. In addition, we have $(a^*s^*sa, a^*ea) \in \rho$ and

$$saa^*ea = sea = tea = taa^*ea.$$

Hence, we obtain $(sa, ta) \in \nu_{\rho,\min}$.

It is easy to show that the restriction of $\nu_{\rho,\min}$ to $E(S)$ coincides with ρ. Let ν be a congruence on S whose restriction to $E(S)$ is ρ. We show $\nu_{\min,\rho} \subset \nu$. Take $(s,t) \in \nu_{\min,\rho}$. Then there exists $e \in E(S)$ such that $se = te$ and $(s^*s, e), (t^*t, e) \in \rho \subset \nu$. By $(s^*s, e), (t^*t, e) \in \nu$, we have $(s, se), (t, te) \in \nu$. Combining with $se = te$, we obtain $(s,t) \in \nu$. Therefore, $\nu_{\rho,\min}$ is the minimum congruence on S whose restriction to $E(S)$ is ρ.

An inverse semigroup S is said to be Clifford if $s^*s = ss^*$ holds for all $s \in S$. One can verify that an inverse semigroup S is Clifford if and only if $se = es$ holds for all $s \in S$ and $e \in E(S)$. A congruence ν on S is said to be Clifford if S/ν is Clifford. Similarly, a congruence ν is said to be commutative if S/ν is commutative.

1.1.3 Étale groupoid

In this subsection, we recall fundamental notions about étale groupoids.

Definition 1.1.3.1. A groupoid is a set G together with a unit space $G^{(0)} \subset G$, domain and range maps $d, r\colon G \to G^{(0)}$ and a multiplication

$$G^{(2)} := \{(\alpha, \beta) \in G \times G \mid d(\alpha) = r(\beta)\} \ni (\alpha, \beta) \mapsto \alpha\beta \in G$$

such that

1. for all $x \in G^{(0)}$, $d(x) = x$ and $r(x) = x$ hold,
2. for all $\alpha \in G$, $\alpha d(\alpha) = r(\alpha)\alpha = \alpha$ holds,
3. for all $(\alpha, \beta) \in G^{(2)}$, $d(\alpha\beta) = d(\beta)$ and $r(\alpha\beta) = r(\alpha)$ hold,
4. if $(\alpha, \beta), (\beta, \gamma) \in G^{(2)}$, we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$,
5. every $\gamma \in G$, there exists $\gamma' \in G$ which satisfies $(\gamma', \gamma), (\gamma, \gamma') \in G^{(2)}$ and $d(\gamma) = \gamma'\gamma$ and $r(\gamma) = \gamma\gamma'$.

Since the element γ' in (5) is uniquely determined by γ, γ' is called the inverse of γ and denoted by γ^{-1}. A subgroupoid of G is a subset of G which is closed under the inverse and multiplication. For $U \subset G^{(0)}$, we define $G_U := d^{-1}(U)$ and $G^U := r^{-1}(U)$. We define also $G_x := G_{\{x\}}$ and $G^x := G^{\{x\}}$ for $x \in G^{(0)}$. The isotropy bundle of G is denoted by $\mathrm{Iso}(G) := \{\gamma \in G \mid d(\gamma) = r(\gamma)\}$. Note that $\mathrm{Iso}(G)$ is a subgroupoid of G. If G satisfies $G = \mathrm{Iso}(G)$, then G is called a group bundle over $G^{(0)}$. A group bundle G is said to be abelian if G_x is an abelian group for all $x \in G^{(0)}$.

Definition 1.1.3.2. A topological groupoid G is a groupoid equipped with a topology where the multiplication and inverse of G are continuous. A topological groupoid G is said to be étale if the domain map $d\colon G \to G^{(0)}$ is a local homeomorphism (namely, for all $\alpha \in G$, there exists an open neighborhood U of α such that $d(U) \subset G^{(0)}$ is an open set and the restriction map $d|_U$ is a homeomorphism onto $d(U)$).

An étale topological groupoid is called an étale groupoid for short. Remark that the range map of an étale groupoid G is also a local homeomorphism, since $r(\alpha) = d(\alpha^{-1})$ holds for all $\alpha \in G$ and the inverse $\alpha \mapsto \alpha^{-1}$ is a homeomorphism.

We give typical examples of étale groupoids.

Example 1.1.3.3. A topological space is an étale groupoid such that $G = G^{(0)}$ holds. A discrete group is an étale groupoid such that $G^{(0)}$ is a singleton.

Example 1.1.3.4. Let X be a topological space, Γ be a discrete group and $\alpha\colon \Gamma \curvearrowright X$ be an action. The transformation groupoid $\Gamma \ltimes_\alpha X$ is defined as follows. Define $\Gamma \ltimes_\alpha X := \Gamma \times X$ as a topological space. The unit space of $\Gamma \ltimes_\alpha X$ is X, which is identified with the subset of $\Gamma \ltimes_\alpha X$ via an inclusion $X \ni x \mapsto (e, x) \in \Gamma \ltimes_\alpha X$. The source map and range map are defined by $s((t, x)) = x$ and $r((t, x)) = \alpha_t(x)$ respectively for $(t, x) \in \Gamma \ltimes_\alpha X$. For a pair $(t_1, y), (t_2, x) \in \Gamma \ltimes_\alpha X$ with $y = \alpha_{t_2}(x)$, their multiplication is defined by $(t_1, y) \cdot (t_2, x) := (t_1 t_2, x)$. An inverse is given by $(t, x)^{-1} = (t^{-1}, \alpha_t(x))$. Then, $\Gamma \ltimes_\alpha X$ is an étale groupoid.

In some literatures, the condition that the domain map $d\colon G \to G^{(0)}$ is a local homeomorphism in Definition 1.1.3.2 is replaced by the condition that the domain map $d\colon G \to$

G is a local homeomorphism. As in Proposition 1.1.3.5, these definitions are equivalent.

Proposition 1.1.3.5 ([8, Proposition 3.2])**.** Let G be an étale groupoid. Then the unit space $G^{(0)}$ is an open subset of G. In particular, the domain and range maps d, r are local homeomorphisms as maps from G to G.

Definition 1.1.3.6. Let G be an étale groupoid. A subset $U \subset G$ is called a bisection if the restrictions $d|_U$ and $r|_U$ are injective.

Remark 1.1.3.7. By the definition of an étale groupoid, the set of open bisections is a basis of G. For an open bisection $U \subset G$, $d|_U$ and $r|_U$ are homeomorphisms onto their images since they are open maps.

We often use the fact that the multiplication map of an étale groupoid is an open map.

Proposition 1.1.3.8 ([14, Proposition 2.2.4])**.** Let G be an étale groupoid and $U, V \subset G$ be open sets. Then a set $UV := \{\alpha\beta \in G \mid \alpha \in U, \beta \in V, d(\alpha) = r(\beta)\} \subset G$ is an open set. Furthermore, if $U, V \subset G$ are open bisections, UV is also an open bisection.

Definition 1.1.3.9. Let G_1 and G_2 be groupoids. A map $\Phi\colon G_1 \to G_2$ is called a groupoid homomorphism if $(\Phi(\alpha), \Phi(\beta)) \in G_2^{(2)}$ and $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$ hold for all $(\alpha, \beta) \in G_1^{(2)}$.

As a morphism between étale groupoids, we often consider a groupoid homomorphism which is a local homeomorphism. Whether an étale groupoid homomorphism is a local homeomorphism or not is determined by its behaviour on the unit spaces as the following proposition shows. This proposition follows from the definition of étale groupoids.

Proposition 1.1.3.10. Let G and H be étale groupoids. A groupoid homomorphism $\Phi\colon G \to H$ is a local homeomorphism if and only if $\Phi|_{G^{(0)}}\colon G^{(0)} \to H^{(0)}$ is a local homeomorphism.

As in the case of group actions, the notion of invariant sets is defined for étale groupoids.

Definition 1.1.3.11. Let G be a groupoid. A subset $F \subset G^{(0)}$ is said to be invariant if $d(\gamma) \in F$ implies $r(\gamma) \in F$ for all γ. A point $x \in G^{(0)}$ is called a fixed point if $\{x\} \subset G^{(0)}$ is invariant.

Note that a set $F \subset G^{(0)}$ is invariant if and only if $G^{(0)} \setminus F$ is invariant. If $F \subset G^{(0)}$ is invariant, then $G_F = G_F \cap G^F$ holds and $G_F \subset G$ is a subgroupoid whose unit space is F. In particular, $G_x \subset G$ is a discrete group if $x \in G^{(0)}$ is a fixed point.

Proposition 1.1.3.12. Let G be an étale groupoid with the Hausdorff unit space $G^{(0)}$. Then the set of all fixed points $F \subset G^{(0)}$ is a closed subset.

Proof. We show that $G^{(0)} \setminus F \subset G^{(0)}$ is an open set. Take $x \in G^{(0)} \setminus F$. Then there exists $\gamma \in G$ such that $x = d(\gamma)$ and $x \neq r(\gamma)$. Take an open bisection U which contains γ. Let $S_U \colon d(U) \to r(U)$ denote a homeomorphism defined by $S_U(d(\alpha)) = r(\alpha)$ for each $\alpha \in U$. Since $G^{(0)}$ is Hausdorff, there exist open sets $U_1, V_1 \subset G^{(0)}$ such that $d(\gamma) \in U_1$, $r(\gamma) \in V_1$ and $U_1 \cap V_1 = \emptyset$. By the continuity of S_U, there exists an open set $U_2 \subset U$ such that $\gamma \in U_2$ and $r(U_2) \subset V_1$. Now one can see $x \in d(U_2) \subset G^{(0)} \setminus F$. Therefore, $G^{(0)} \setminus F \subset G^{(0)}$ is an open set. □

We will use the next proposition for the set of all fixed points.

Proposition 1.1.3.13. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $U, F \subset G^{(0)}$ be an invariant open and closed subset respectively. Then $G_U \subset G$ is an open subgroupoid of G and an étale groupoid with the locally compact Hausdorff unit space U in the relative topology. Similarly, $G_F \subset G$ is a closed subgroupoid of G and an étale groupoid with the locally compact Hausdorff unit space F in the relative topology.

Proof. Observe that U and F are locally compact Hausdorff spaces in the relative topology of $G^{(0)}$. Now it is clear that G_U and G_F are étale groupoids. □

1.1.4 Étale groupoid C*-algebras

Following Connes's idea in [7], we associate a C*-algebra to an étale groupoid which is not necessarily Hausdorff. If G is not Hausdorff, the set of all continuous functions on G is not enough to capture the structure of G. The key idea is to consider functions which are continuous on some open set but not necessarily continuous on the whole space. See [8, Section 3] for more details.

We assume that the unit space $G^{(0)}$ is a locally compact Hausdorff space with respect to the relative topology when we consider groupoid C*-algebras. Remark that the whole space G is not necessarily a Hausdorff space. Since $d \colon G \to G^{(0)}$ is a local homeomorphism, G has a basis which consists of locally compact Hausdorff subsets.

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. For an open Hausdorff subset $U \subset G$, we denote the set of all continuous functions on U with compact support by $C_c(U)$. We regard an element in $C_c(U)$ as an element in $\mathrm{Funct}(G)$, the vector space of all complex valued functions on G, by defining it to be 0 outside of U. We define $\mathcal{C}(G) := \operatorname{span} \bigcup_U C_c(U) \subset \mathrm{Funct}(G)$, where the union is taken over all open Hausdorff subsets $U \subset G$.

If G is Hausdorff, then $\mathcal{C}(G)$ coincides with $C_c(G)$. If G is not Hausdorff, an element in $\mathcal{C}(G)$ may not be continuous.

Proposition 1.1.4.1 ([8, Proposition 3.10])**.** Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Take an open basis $\{U_i\}_{i\in I}$ of G consisting of open Hausdorff subsets. Then $\mathcal{C}(G)$ is the linear span of $\bigcup_{i\in I} C_c(U_i)$. In particular, $\mathcal{C}(G)$ is the linear span of $\bigcup_U C_c(U)$, where the union is taken over all open bisections of G.

Proof. This follows from the partition of unity argument. □

Definition 1.1.4.2. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Recall that $\mathcal{C}(G)$ is equipped with a structure of $\mathbb{C}$-vector space by pointwise addition and scalar multiplication. The multiplication $f * g \in \mathcal{C}(G)$ and involution $f^* \in \mathcal{C}(G)$ of $f, g \in \mathcal{C}(G)$ are defined by

$$f * g(\gamma) = \sum_{\beta \in G_{d(\gamma)}} f(\gamma\beta^{-1})g(\beta),\ f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Then $\mathcal{C}(G)$ is a *-algebra under these operations.

One can see that $C_c(G^{(0)}) \subset \mathcal{C}(G)$ is a *-subalgebra.

Lemma 1.1.4.3 ([8, Proposition 3.14])**.** Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $f \in \mathcal{C}(G)$. Then there exists a constant $C_f \geq 0$ such that $\|\rho(f)\| \leq C_f$ for all Hilbert spaces H and *-homomorphisms $\rho\colon \mathcal{C}(G) \to B(H)$.

Proof. We may assume that $f \in C_c(U)$ for some open bisection $U \subset G$. One can see that $f^* * f \in C_c(G^{(0)})$. Since $C_c(G^{(0)})$ is a union of commutative C*-algebras, we have $\|\rho(h)\| \leq \sup_{x\in G^{(0)}}|h(x)|$ for all $h \in C_c(G^{(0)})$. Then we obtain $\|\rho(f)\|^2 = \|\rho(f^* * f)\| \leq \sup_{x\in G^{(0)}}|f^* * f(x)| < \infty$. □

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. We denote the left regular representation by $\lambda_x\colon \mathcal{C}(G) \to \ell^2(G_x)$ at $x \in G^{(0)}$, which is defined by

$$\lambda_x(f)\delta_\beta = \sum_{d(\alpha)=r(\beta)} f(\alpha)\delta_{\alpha\beta}$$

for $f \in \mathcal{C}(G)$ and $\beta \in G_x$. One can see that $\bigoplus_{x\in G^{(0)}} \lambda_x$ is a faithful *-representation of $\mathcal{C}(G)$. The reduced norm of $f \in \mathcal{C}(G)$ is defined by

$$\|f\| := \sup_{x\in G^{(0)}} \|\lambda_x(f)\|.$$

We denote the reduced groupoid C*-algebra of G by $C^*_\lambda(G)$, which is the completion of $\mathcal{C}(G)$ by the reduced norm.

The universal norm of $f \in \mathcal{C}(G)$ is defined by

$$\|f\| := \sup\{\|\rho(f)\| \mid \rho\colon \mathcal{C}(G) \to B(H) \text{ is a *-representation}\}.$$

By Lemma 1.1.4.3, the universal norm takes values in $[0, \infty)$. Since the left regular representations of $\mathcal{C}(G)$ induces a faithful *-representation of $\mathcal{C}(G)$, the universal norm becomes a C*-norm (see [6, Section 4]). The completion of $\mathcal{C}(G)$ by universal norm is denoted by $C^*(G)$. We shall remark that every *-representation of $\mathcal{C}(G)$ induces the *-representation of $C^*(G)$. Note that the inclusion $C_c(G^{(0)}) \subset \mathcal{C}(G)$ extends to $C_0(G^{(0)}) \subset C^*(G)$.

Proposition 1.1.4.4. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $F \subset G^{(0)}$ be a closed invariant set. Then the restriction $\mathcal{C}(G) \ni f \mapsto f|_{G_F} \in \mathcal{C}(G_F)$ extends to the surjective *-homomorphism $C^*(G) \to C^*(G_F)$.

Proof. First, we check that $f|_{G_F} \in \mathcal{C}(G_F)$ for all $f \in \mathcal{C}(G)$. We may assume that $f \in C_c(U)$ for some open Hausdorff subset $U \subset G$, since $\mathcal{C}(G)$ is spanned by $\bigcup_U C_c(U)$, where the union is taken over all open Hausdorff subsets $U \subset G$. Defining $V := G_F \cap U$, V is a Hausdorff open subset of G_F. Then $f|_{G_F}$ is contained in $C_c(V) \subset \mathcal{C}(G_F)$.

Direct calculations imply that the restriction $\mathcal{C}(G) \ni f \mapsto f|_{G_F} \in \mathcal{C}(G_F)$ is a *-homomorphism.

Next, we show that the restriction $\mathcal{C}(G) \ni f \mapsto f|_{G_F} \in \mathcal{C}(G_F)$ is surjective. Note that $\{G_F \cap U \mid U \subset G \text{ is an open Hausdorff subset}\}$ is an open basis of G_F. Take an open Hausdorff subset $U \subset G$ and $f \in C_c(G_F \cap U)$ arbitrarily. Put $V := G_F \cap U$. Since $V \subset U$ is a closed subset of U and $f \in C_c(V)$, there exists $\tilde{f} \in C_c(U)$ such that $\tilde{f}|_V = f$ by the Tietze extension theorem. Now we obtain $\tilde{f} \in \mathcal{C}(G)$ such that $\tilde{f}|_{G_F} = f$. By Proposition 1.1.4.1, $\mathcal{C}(G_F)$ is the linear span of $\bigcup_U C_c(G_F \cap U)$, where the union is taken over all open Hausdorff subsets $U \subset G$. Therefore, the restriction $\mathcal{C}(G) \ni f \mapsto f|_{G_F} \in \mathcal{C}(G_F)$ is surjective.

By the universality of $C^*(G)$, the restriction $\mathcal{C}(G) \ni f \mapsto f|_{G_F} \in \mathcal{C}(G_F)$ extends to the *-homomorphism $C^*(G) \to C^*(G_F)$. Since the image of $C^*(G)$ is dense in $C^*(G_F)$, $C^*(G) \to C^*(G_F)$ is surjective.[1)] □

1.1.5 Étale groupoids associated to inverse semigroup actions

In this subsection, we recall how to construct an étale groupoid from inverse semigroup actions.

Let X be a topological space. We denote by I_X the inverse semigroup of homeomorphisms between open sets in X. An action $\alpha\colon S \curvearrowright X$ of an inverse semigroup S on X is a semigroup homomorphism $S \ni s \mapsto \alpha_s \in I_X$. For $e \in E(S)$, we denote the domain of α_e by D_e^α. Then α_s is a homeomorphism from $D_{s^*s}^\alpha$ to $D_{ss^*}^\alpha$. In this thesis, we always

1) Now we use the fact that the image of a *-homomorphism becomes a closed subset. See [22, Chapter I].

assume that $\bigcup_{e\in E(S)} D^\alpha_e = X$ holds.

For an action $\alpha\colon S \curvearrowright X$, we associate an étale groupoid $S \ltimes_\alpha X$ as the following. First we put the set $S * X := \{(s,x) \in S \times X \mid x \in D^\alpha_{s^*s}\}$. Then we define an equivalence relation $\sim$ on $S * X$ by $(s,x) \sim (t,y)$ if

$$x = y \text{ and there exists } e \in E(S) \text{ such that } x \in D^\alpha_e \text{ and } se = te.$$

Set $S \ltimes_\alpha X := S * X/{\sim}$ and denote the equivalence class of $(s,x) \in S * X$ by $[s,x]$. The unit space $S \ltimes_\alpha X$ is X, where X is identified with the subset of $S \ltimes_\alpha X$ via the injection

$$X \ni x \mapsto [e,x] \in S \ltimes_\alpha X, x \in D^\alpha_e.$$

The source and range maps are defined by

$$d([s,x]) = x,\, r([s,x]) = \alpha_s(x)$$

for $[s,x] \in S \ltimes_\alpha X$. The product of $[s,\alpha_t(x)], [t,x] \in S \ltimes_\alpha X$ is $[st,x]$. The inverse is $[s,x]^{-1} = [s^*, \alpha_s(x)]$. Then $S \ltimes_\alpha X$ is a groupoid with these operations. For $s \in S$ and an open set $U \subset D^\alpha_{s^*s}$, define

$$[s,U] := \{[s,x] \in S \ltimes_\alpha X \mid x \in U\}.$$

These sets form an open basis of $S \ltimes_\alpha X$. With this structure, $S \ltimes_\alpha X$ is an étale groupoid.

Let S be an inverse semigroup. Now we define the spectral action $\beta\colon S \curvearrowright \widehat{E(S)}$. A character on $E(S)$ is a nonzero semigroup homomorphism from $E(S)$ to $\{0,1\}$, where $\{0,1\}$ is an inverse semigroup with the usual product. The set of all characters on $E(S)$ is denoted by $\widehat{E(S)}$. We view $\widehat{E(S)}$ as a locally compact Hausdorff space with respect to the topology of pointwise convergence. Define

$$N^e_P := \{\xi \in \widehat{E(S)} \mid \xi(e) = 1, \xi(p) = 0 \text{ for all } p \in P\}$$

for $e \in E(S)$ and a finite subset $P \subset E(S)$. Then these sets form a basis for the topology on $\widehat{E(S)}$. For $e \in E(S)$, we define $D^\beta_e := \{\xi \in \widehat{E(S)} \mid \xi(e) = 1\}$. For each $s \in S$ and $\xi \in D^\beta_{s^*s}$, define $\beta_s(\xi) \in D^\beta_{ss^*}$ by $\beta_s(\xi)(e) = \xi(s^*es)$, where $e \in E(S)$. Then β is an action $\beta\colon S \curvearrowright \widehat{E(S)}$, which we call the spectral action of S. Now the universal groupoid of S is defined to be $G_u(S) := S \ltimes_\beta \widehat{E(S)}$.

Chapter 2.

Quotients of étale groupoids and the abelianizations of groupoid C*-algebras

In this chapter, we review main results obtained in [11]. First, we introduce the notion of quotient étale groupoids in Section 2.1. Then we observe that quotients of étale groupoids induce quotients of groupoid C*-algebras. In section 2.2, we investigate the abelianizations of groupoid C*-algebras. For a given étale groupoid, we construct an étale group bundle which describes the abelianization of the original associated groupoid C*-algebra.

2.1 Quotients of étale groupoids

In this section we introduce the notion of quotient étale groupoids. Then we will observe that a quotient of an étale groupoid induces a *-homomorphism of a groupoid C*-algebra.

2.1.1 Quotients of étale groupoids

In this subsection we introduce the notion of quotient étale groupoids. First, we define normal subgroupoids and quotient groupoids. Then we show that quotient groupoids of étale groupoids by open normal subgroupoids again become étale.

Definition 2.1.1.1. Let G be a groupoid. A subgroupoid $H \subset G$ is said to be normal if

1. $G^{(0)} \subset H \subset \mathrm{Iso}(G)$ holds and
2. $\alpha H \alpha^{-1} \subset H$ holds for all $\alpha \in G$.

Definition 2.1.1.2. Let G be a groupoid and $H \subset G$ be a normal subgroupoid. Then we define an equivalence relation $\sim$ on G by declaring that $\alpha \sim \beta$ if $d(\alpha) = d(\beta)$ and $\alpha\beta^{-1} \in H$. We denote the quotient set $G/{\sim}$ by G/H.

We prove some lemmas needed to define the groupoid structure of a quotient groupoid.

Lemma 2.1.1.3. Let G be a groupoid and $H \subset G$ be a normal subgroupoid. Suppose that $\alpha, \alpha' \in G$ satisfy $\alpha \sim \alpha'$. Then we have $d(\alpha) = d(\alpha')$ and $r(\alpha) = r(\alpha')$.

Proof. It follows that $d(\alpha) = d(\alpha')$ from the definition of $\alpha \sim \alpha'$. Since $\alpha\alpha'^{-1} \in H \subset \mathrm{Iso}(G)$, we have $r(\alpha) = r(\alpha\alpha'^{-1}) = d(\alpha\alpha'^{-1}) = r(\alpha')$. □

Lemma 2.1.1.4. Let G be a groupoid and $H \subset G$ be a normal subgroupoid. Suppose that $\alpha, \alpha', \beta, \beta' \in G$ satisfy $\alpha \sim \alpha'$, $\beta \sim \beta'$, $d(\alpha) = r(\beta)$. Then we have $d(\alpha') = r(\beta')$ and $\alpha\beta \sim \alpha'\beta'$.

Proof. By Lemma 2.1.1.3, we have $d(\alpha) = d(\alpha')$ and $r(\beta) = r(\beta')$. Using $d(\alpha) = r(\beta)$, we obtain $d(\alpha') = r(\beta')$.

The last assertion follows from a simple calculation. Indeed, we have $d(\alpha\beta) = d(\beta) = d(\beta') = d(\alpha'\beta')$ and

$$\alpha\beta(\alpha'\beta')^{-1} = \alpha\beta\beta'^{-1}\alpha'^{-1} = (\alpha\beta\beta'^{-1}\alpha^{-1})(\alpha\alpha'^{-1}) \in H.$$

Note that $\alpha\beta\beta'^{-1}\alpha^{-1} \in H$, since H is normal. □

Definition 2.1.1.5. Let G be a groupoid, $H \subset G$ be a normal subgroupoid and $q\colon G \to G/H$ be the quotient map. A groupoid structure of G/H is defined as follows:

- a unit space $(G/H)^{(0)}$ is $q(G^{(0)})$, which can be identified with $G^{(0)}$ via an injection $q|_{G^{(0)}}$;
- domain and range maps $d, r\colon G/H \to G^{(0)}$ are defined by $d(q(\gamma)) := q(d(\gamma))$; $r(q(\gamma)) := q(r(\gamma))$ for $\gamma \in G$;
- a multiplication of G/H is defined by $q(\alpha)q(\beta) := q(\alpha\beta)$ for $\alpha, \beta \in G$ with $d(\alpha) = r(\beta)$.

One can see that the inverse map of G/H satisfies $q(\gamma)^{-1} = q(\gamma^{-1})$ for $\gamma \in G$. Then G/H is a groupoid under these operations.

Remark 2.1.1.6. The operations of G/H are well-defined by Lemma 2.1.1.3 and Lemma 2.1.1.4.

If G is a topological groupoid, then we consider the quotient topology as a topology of G/H.

Lemma 2.1.1.7. Let G be an étale groupoid and $H \subset G$ be an open normal subgroupoid. Then the quotient map $q\colon G \to G/H$ is an open map. Furthermore, q is a local homeomorphism.

Proof. Let $U \subset G$ be an open subset. Then $q^{-1}(q(U)) = UH$ is an open subset of G by Proposition 1.1.3.8. Hence, $q(U) \subset G/H$ is an open subset by the definition of the quotient topology.

Next, we show that the quotient map $q\colon G \to G/H$ is a local homeomorphism. Fix a $\gamma \in G$. Then take an open bisection $U \subset G$ with $\gamma \in U$. One can see that $q|_U$ is injective. Since q is an open map, $q|_U$ is a homeomorphism onto an open subset $q(U) \subset G$. Hence, q is a local homeomorphism. □

Observe that $q|_{G^{(0)}}\colon G^{(0)} \to (G/H)^{(0)}$ is homeomorphic.

Proposition 2.1.1.8. Let G be an étale groupoid and $H \subset G$ be an open normal subgroupoid. Then G/H is an étale groupoid.

Proof. First, we show the continuity of the inverse $G/H \ni \delta \mapsto \delta^{-1} \in G/H$. One can see that the map $G \ni \gamma \mapsto q(\gamma)^{-1} \in G/H$ is continuous, since the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\text{inverse}} & G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ G/H & \xrightarrow{\text{inverse}} & G/H. \end{array}$$

By the definition of the quotient topology, the inverse of G/H is continuous.

Next, we show that the multiplication of G/H is continuous. Take $(q(\alpha), q(\beta)) \in (G/H)^{(2)}$ and an open set $U \subset G/H$ such that $q(\alpha)q(\beta) \in U$. Since $\alpha\beta \in q^{-1}(U)$ and $q^{-1}(U) \subset G$ is open, there exist open sets $V_1, V_2 \subset G$ such that $\alpha \in V_1, \beta \in V_2$ and $V_1V_2 \subset q^{-1}(U)$. Since the subsets $V_1, V_2 \subset G$ are open, $q(V_1), q(V_2) \subset G/H$ are also open. One can see that $q(\alpha) \in q(V_1)$, $q(\beta) \in q(V_2)$ and $q(V_1)q(V_2) = q(V_1V_2) \subset U$. Therefore, the multiplication of G/H is continuous.

Finally, we show that G/H is étale. Since the restriction $q|_{G^{(0)}}$ gives a homeomorphism from $G^{(0)}$ to $(G/H)^{(0)}$, $(G/H)^{(0)}$ is a locally compact Hausdorff space. One can see that the domain map $d\colon G/H \to (G/H)^{(0)}$ is a local homeomorphism, since we have Lemma 2.1.1.7 and the following diagram is commutative for every open bisection $U \subset G$:

$$\begin{array}{ccc} U & \xrightarrow{q} & q(U) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ d(U) & \xrightarrow{q} & d(q(U)). \end{array}$$

Therefore, G/H is an étale groupoid. □

Now we obtain the next theorem by Lemma 2.1.1.7 and Proposition 2.1.1.8,

Theorem 2.1.1.9 ([11, Theorem 3.10]). Let G be an étale groupoid and $H \subset G$ be an open normal subgroupoid. Then the sequence of the étale groupoids

$$H \xhookrightarrow{\text{inclusion}} G \xrightarrow{q} G/H$$

is exact, that is, $q^{-1}((G/H)^{(0)}) = H$.

We have the fundamental homomorphism theorem. The proof is straightforward.

Proposition 2.1.1.10. Let G and H be étale groupoids and $\Phi\colon G \to H$ be a continuous groupoid homomorphism which is a local homeomorphism. Assume that Φ is injective on $G^{(0)}$. Then $\ker \Phi := \Phi^{-1}(H^{(0)})$ is an open normal subgroupoid of G. Moreover there exist an isomorphism $\widetilde{\Phi}\colon G/\ker\Phi \to \Phi(G)$ which makes the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & H \\ {\scriptstyle Q}\downarrow & \nearrow_{\widetilde{\Phi}} & \\ G/\ker\Phi & & \end{array},$$

where $Q\colon G \to G/\ker\Phi$ denotes the quotient map.

As in the case of topological groups, Hausdorffness of a quotient groupoid can be characterized as follows.

Proposition 2.1.1.11. Let G be an étale groupoid and $H \subset G$ be an open normal subgroupoid. Then G/H is Hausdorff if and only if $H \subset G$ is closed.

Proof. Recall that an étale groupoid G is Hausdorff if and only if its unit space $G^{(0)}$ is a closed subset of G (see, for example, [20, Lemma 2.3.2]). If G/H is Hausdorff, $(G/H)^{(0)} \subset G/H$ is closed. Hence, $H = q^{-1}((G/H)^{(0)})$ is a closed subset of G.

Suppose that $H \subset G$ is closed. Since q is an open map, $(G/H) \setminus (G/H)^{(0)} = q(G \setminus H) \subset G/H$ is open. Hence, $(G/H)^{(0)} \subset G/H$ is closed, which implies that G/H is Hausdorff. □

Proposition 2.1.1.12. Let G be an étale groupoid. Then the interior of isotropy $\mathrm{Iso}(G)^\circ \subset \mathrm{Iso}(G)$ is a normal subgroupoid.

Proof. We show that $\mathrm{Iso}(G)^\circ$ is normal. By Proposition 1.1.3.5, $G^{(0)}$ is contained in $\mathrm{Iso}(G)^\circ$. Take $\alpha \in G$ and $\gamma \in \mathrm{Iso}(G)^\circ$ with $d(\alpha) = r(\gamma)$. There exist open bisections $U, V \subset G$ with $\alpha \in U$ and $\gamma \in V \subset \mathrm{Iso}(G)$. Then, by Proposition 1.1.3.8, $UVU^{-1} \subset G$ is an open subset which contains $\alpha\gamma\alpha^{-1}$. Since U is bisection and $V \subset \mathrm{Iso}(G)$, we have $UVU^{-1} \subset \mathrm{Iso}(G)$. Therefore, $\alpha\gamma\alpha^{-1} \in \mathrm{Iso}(G)^\circ$ and $\mathrm{Iso}(G)^\circ$ is an open normal subgroupoid. $\square$

An étale groupoid $G/\mathrm{Iso}(G)^\circ$, which is a special case of quotient groupoids, coincides with a groupoid of germs of the canonical action (see [19, Section 3]). One can see that $G/\mathrm{Iso}(G)^\circ$ is effective[1)].

2.1.2 *-homomorphisms induced by quotients of étale groupoids

For an étale groupoid G and an open normal subgroupoid $H \subset G$, we have obtained the quotient étale groupoid G/H. Next, we see that the quotient map $q\colon G \to G/H$ induces a *-homomorphism $C^*(G) \to C^*(G/H)$.

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. For $f \in \mathcal{C}(G)$, we define $\tilde{f}\colon G/H \to \mathbb{C}$ by

$$\tilde{f}(\gamma) := \sum_{q(\alpha)=\gamma} f(\alpha)$$

for $\gamma \in G/H$. Then the following proposition holds.

Proposition 2.1.2.1. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $H \subset G$ be an open normal subgroupoid. Then $\mathcal{C}(G) \ni f \mapsto \tilde{f} \in \mathcal{C}(G/H)$ is a surjective *-homomorphism.

Proof. First, we show $\tilde{f} \in \mathcal{C}(G/H)$. We may assume that there exists an open bisection $U \subset G$ such that $f|_U \in C_c(U)$ and $f|_{G\setminus U} = 0$. Then $q(U) \subset G/H$ is an open bisection and $\tilde{f}|_{q(U)} = f \circ (q|_U)^{-1} \in C_c(q(U))$, since $q|_U$ is a homeomorphism onto the image. Moreover, one can see that $\tilde{f}_{(G/H)\setminus q(U)} = 0$. Hence, $\tilde{f} \in C_c(q(U)) \subset \mathcal{C}(G/H)$.

We show that $\mathcal{C}(G) \ni f \mapsto \tilde{f} \in \mathcal{C}(G/H)$ is a *-homomorphism. We only check that $\mathcal{C}(G) \ni f \mapsto \tilde{f} \in \mathcal{C}(G/H)$ preserves the multiplications, since it is easy to check that this map is linear and preserves the involution. For all $f, g \in \mathcal{C}(G)$ and $\gamma' \in G/H$, we have

$$\begin{aligned}
\widetilde{f*g}(\gamma') &= \sum_{q(\gamma)=\gamma'} f*g(\gamma) = \sum_{q(\gamma)=\gamma'}\sum_{\alpha\beta=\gamma} f(\alpha)g(\beta) = \sum_{q(\alpha\beta)=\gamma'} f(\alpha)g(\beta),\\
\tilde{f}*\tilde{g}(\gamma') &= \sum_{\alpha'\beta'=\gamma'} \tilde{f}(\alpha')\tilde{g}(\beta') = \sum_{\alpha'\beta'=\gamma'}\sum_{q(\alpha)=\alpha'}\sum_{q(\beta)=\beta'} f(\alpha)g(\beta)\\
&= \sum_{q(\alpha\beta)=\gamma'} f(\alpha)g(\beta).
\end{aligned}$$

Finally, we show that $\mathcal{C}(G) \ni f \mapsto \tilde{f} \in \mathcal{C}(G/H)$ is surjective. Note that

$$\{q(U) \subset G/H \mid U \subset G \text{ is an open bisection}\}$$

1) Recall that an étale groupoid G is said to be effective if $G^{(0)} = \mathrm{Iso}(G)^\circ$ holds

is an open basis of G. Let $U \subset G$ be an open bisection and $f \in C_c(q(U))$. One can see that $q|_U$ is a homeomorphism onto its image. Define $g := f \circ q|_U \in C_c(U)$. Then we have $\widetilde{g} = f$. By Proposition 1.1.4.4, $\mathcal{C}(G/H)$ is the linear span of $\bigcup_U C_c(q(U))$, where the union is taken over all open bisections $U \subset G$. Hence, $\mathcal{C}(G) \ni f \mapsto \tilde{f} \in \mathcal{C}(G/H)$ is a surjective *-homomorphism. □

By Proposition 2.1.2.1, the map $\mathcal{C}(G) \ni f \mapsto \tilde{f} \in \mathcal{C}(G/H) \subset C^*(G/H)$ is a *-homomorphism. By the definition of the universal norm of $\mathcal{C}(G)$, we have $\|\tilde{f}\| \leq \|f\|$ for all $f \in \mathcal{C}(G)$. Therefore, the *-homomorphism in Proposition 2.1.2.1 extends to the *-homomorphism $Q\colon C^*(G) \to C^*(G/H)$. Since the image of Q is dense in $C^*(G/H)$, Q is surjective (see, for example, [2, Corollary II.5.1.2]).

We make some observations on a Cuntz-Krieger uniqueness theorem in the remainder of this section.

Lemma 2.1.2.2. Let $Q\colon C^*(G) \to C^*(G/H)$ be the *-homomorphism as above. Then $\ker Q \cap C_0(G^{(0)}) = \{0\}$ holds.

Proof. Since the universal norm of a function in $C_c(G^{(0)})$ coincides with the supremum norm, $Q|_{C_c(G^{(0)})}$ is isometric. Therefore, $Q|_{C_0(G^{(0)})}$ is isometric and $\ker Q \cap C_0(G^{(0)}) = \{0\}$. □

Lemma 2.1.2.3. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $H \subset G$ be an open normal subgroupoid. Then the *-homomorphism $Q\colon C^*(G) \to C^*(G/H)$ induced by Proposition 2.1.2.1 is injective if and only if $H = G^{(0)}$.

Proof. It is clear that the *-homomorphism $Q\colon C^*(G) \to C^*(G/H)$ is injective if $H = G^{(0)}$. Suppose that $G^{(0)} \subsetneq H$ and take $\gamma_0 \in H \setminus G^{(0)}$. There exists an open bisection $U \subset G$ with $\gamma_0 \in U \subset H$. By the Urysohn lemma, there exists $f_1 \in C_c(U)$ with $f_1(\gamma_0) = 1$. Define $f_2 \in C_c(G^{(0)})$ by

$$f_2(\gamma) = \begin{cases} f_1 \circ (d|_U)^{-1}(\gamma) & (\gamma \in d(U)) \\ 0 & (\gamma \in G^{(0)} \setminus d(U)). \end{cases}$$

We have $f := f_1 - f_2 \neq 0$, since $f(\gamma_0) = 1$. One can see that $Q(f) = 0$, which implies that Q is not injective. □

Recall that an étale groupoid G is said to be effective if $G^{(0)} = \mathrm{Iso}(G)^\circ$.

Corollary 2.1.2.4 (cf. [3, Proposition 5.5])**.** Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Assume that every nonzero ideal $I \subset C^*(G)$ satisfies $I \cap C_0(G^{(0)}) \neq \{0\}$. Then G is effective.

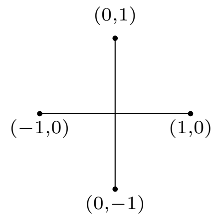


Figure 2.1   Picture of X in Example 2.1.2.6

Proof. By Proposition 2.1.1.12, $\mathrm{Iso}(G)^\circ$ is a normal subgroupoid of G. Letting $Q\colon C^*(G) \to C^*(G/\mathrm{Iso}(G)^\circ)$ be the *-homomorphism induced by Proposition 2.1.2.1, we have $\ker Q \cap C_0(G^{(0)}) = \{0\}$ by Lemma 2.1.2.2. The assumption implies that $Q\colon C^*(G) \to C^*(G/\mathrm{Iso}(G)^\circ)$ is injective. Therefore, we obtain $\mathrm{Iso}(G)^\circ = G^{(0)}$ by Lemma 2.1.2.3. □

Remark 2.1.2.5. It was proved in [3, Proposition 5.5] that Corollary 2.1.2.4 holds for Hausdorff étale groupoids. In [3, Proposition 5.5], the authors use the augmentation representation, which seems to work for non-Hausdorff étale groupoids.

As shown in Proposition 2.1.2.1, the quotient map $G \to G/\mathrm{Iso}(G)^\circ$ of étale groupoids induces the *-homomorphism $C^*(G) \to C^*(G/\mathrm{Iso}(G)^\circ)$. Using this *-homomorphism, we obtain the proof of Corollary 2.1.2.4, which seems to be more direct than that in [3, Proposition 5.5].

The converse of Corollary 2.1.2.4 does not hold for non-Hausdorff étale groupoids. Indeed, Exel showed that there exists an effective étale groupoid G such that there exists a nonzero ideal $I \subset C^*(G)$ with $I \cap C_0(G^{(0)}) \neq \{0\}$ in [9] (cf. Example 2.1.2.6).

Example 2.1.2.6 ([9, Section 2])**.** Let $X := ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1]) \subset \mathbb{R}^2$ (see figure 2.1) and $K := \{e, s, t, st\}$ be the Klein group, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We define an action σ of K on X by

$$\sigma_s((x,y)) = (-x,y), \quad \sigma_t((x,y)) = (x,-y), \quad \sigma_{st}((x,y)) = (-x,-y)$$

for $(x,y) \in X$.

Consider the transformation groupoid $G := K \ltimes_\sigma X$ (see Example 1.1.3.4). One can see that

$$\begin{aligned}\mathrm{Iso}(G) = G^{(0)} &\cup \{(s,(0,y)) \in G \mid y \in [-1,1]\} \\ &\cup \{(t,(x,0)) \in G \mid x \in [-1,1]\} \cup \{(st,(0,0))\}.\end{aligned}$$

Moreover, we have $\mathrm{Iso}(G)^\circ = \mathrm{Iso}(G) \setminus \{(s,(0,0)), (t,(0,0)), (st,(0,0))\}$. Since $\mathrm{Iso}(G)^\circ$ is not closed in G (for example, $(s,(0,0)) \in \overline{\mathrm{Iso}(G)^\circ} \setminus \mathrm{Iso}(G)^\circ$), the quotient étale groupoid $G/\mathrm{Iso}(G)^\circ$ is not Hausdorff by Proposition 2.1.1.11. In [9], Exel shows that there exists a nonzero ideal $I \subset C^*(G/\operatorname{Iso}(G)^\circ)$ with $I \cap C_0((G/\operatorname{Iso}(G)^\circ)^{(0)}) = \{0\}$, although it is effective.

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. In [6], the authors defined the notion of singularity for an element of $C^*_\lambda(G)$. An element $a \in C^*_\lambda(G)$ is said to be singular if the interior of $\{\gamma \in G \mid \langle \delta_\gamma | \lambda_{d(\gamma)}(a)\delta_{d(\gamma)}\rangle \neq 0\}$ is empty, where $\delta_\gamma \in \ell^2(G_{d(\gamma)})$ denotes the delta function at $\gamma \in G_{d(\gamma)}$. In [6], the authors proved the following theorem.

Theorem 2.1.2.7 ([6, Theorem 4.4])**.** Let G be a second countable étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Assume that G is effective and $C^*_\lambda(G)$ has no nonzero singular element. Then every nonzero ideal $I \subset C^*_\lambda(G)$ satisfies $I \cap C_0(G^{(0)}) \neq \{0\}$.

By the universality of $C^*(G)$, the left representation extends to the *-representation $\lambda_x \colon C^*(G) \to B(\ell^2(G_x))$. Following [6], we say that an element $a \in C^*(G)$ is singular if the interior of $\{\gamma \in G \mid \langle \delta_\gamma | \lambda_{d(\gamma)}(a)\delta_{d(\gamma)}\rangle \neq 0\}$ is empty. A uniqueness theorem for $C^*(G)$ implies that $C^*(G)$ has no nonzero singular elements.

Proposition 2.1.2.8. Let G be a second countable étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Assume that every nonzero ideal $I \subset C^*(G)$ satisfies $I \cap C_0(G^{(0)}) \neq \{0\}$. Then $C^*(G)$ has no nonzero singular elements.

Proof. Observe that the canonical surjective *-homomorphism $C^*(G) \to C^*_\lambda(G)$ is isomorphic by the assumption. Note that G is effective by Proposition 2.1.2.4. We define $S := \{x \in G^{(0)} \mid G_x \cap G^x = \{x\}\}$. One can see that S is an invariant set. Moreover, S is a dense subset of $G^{(0)}$ by [19, Proposition 3.6]. Therefore, letting $\pi := \bigoplus_{x \in S} \lambda_x$, π is injective on $C_0(G^{(0)})$. Then π is injective by the assumption.

Let $a \in C^*(G)$ be a singular element. By [6, Lemma 4.2], we have

$$d(\{\gamma \in G \mid \langle \delta_\gamma | \lambda_{d(\gamma)}(a)\delta_{d(\gamma)}\rangle \neq 0\}) \subset G^{(0)} \setminus S.$$

Using this fact, we show $\pi(a) = 0$. Take $x \in S$. Assume that there exist $\alpha, \beta \in G_x$ such that

$$\langle \delta_\alpha | \lambda_x(a)\delta_\beta\rangle \neq 0.$$

Then we have

$$\langle \delta_{\alpha\beta^{-1}} | \lambda_{d(\beta^{-1})}(a)\delta_{d(\beta^{-1})}\rangle = \langle \delta_\alpha | \lambda_x(a)\delta_\beta\rangle \neq 0.$$

It follows that $r(\beta) = d(\beta^{-1}) \notin S$. This contradicts the fact that $x = d(\beta) \in S$ and S is invariant. Now we have $\langle \delta_\alpha | \lambda_x(a)\delta_\beta\rangle = 0$ for all $\alpha, \beta \in G_x$ and therefore $\lambda_x(a) = 0$ holds for all $x \in S$. Now we have $\pi(a) = 0$, which implies $a = 0$. Hence, $C^*(G)$ has no nonzero singular element. □

2.2 The abelianizations of étale groupoid C*-algebras

In this section we calculate the abelianizations of étale groupoid C*-algebras. First, we recall the abelianizations of C*-algebras, following [1, Definition 2.8]. For a C*-algebra A, its abelianization is defined by $A^{\mathrm{ab}} = A/I$, where $I \subset A$ is the closed two-sided ideal generated by $\{xy - yx \in A \mid x, y \in A\}$. The abelianization A^{ab} is a commutative C*-algebra with the following universality: for all commutative C*-algebra B and *-homomorphism $\pi\colon A \to B$, there exists the unique *-homomorphism $\tilde{\pi}\colon A^{\mathrm{ab}} \to B$ such that $\tilde{\pi} \circ q = \pi$, where $q\colon A \to A^{\mathrm{ab}}$ denotes the quotient map.

2.2.1 One dimensional representations of a groupoid C*-algebra

For a C*-algebra A, we denote the set of all one-dimensional nondegenerate representations of A by $\Delta(A)$. Namely, $\Delta(A)$ is the set of all nonzero *-homomorphisms from A to $\mathbb{C}$. We suppose that $\Delta(A)$ is equipped with the pointwise convergence topology. If A is commutative, $\Delta(A)$ is known as the Gelfand spectrum of A. First, we calculate $\Delta(C^*(G))$.

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $x \in G^{(0)}$ be a fixed point of G. Note that G_x is a discrete group. We temporarily denote the surjection in Proposition 1.1.4.4 by $Q_x\colon C^*(G) \to C^*(G_x)$. Also, we denote the circle group by $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. For a group homomorphism $\chi\colon G_x \to \mathbb{T}$, the map $C_c(G_x) \ni f \mapsto \sum_{\gamma \in G_x} \chi(\gamma) f(\gamma) \in \mathbb{C}$ is a *-homomorphism. This *-homomorphism extends to the *-homomorphism $C^*(G_x) \to \mathbb{C}$, which we also denote by $\chi\colon C^*(G_x) \to \mathbb{C}$.

Definition 2.2.1.1. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$, $x \in G^{(0)}$ be a fixed point and $\chi\colon G_x \to \mathbb{T}$ be a group homomorphism. Then we define a *-homomorphism $\varphi_{x,\chi}\colon C^*(G) \to \mathbb{C}$ by $\varphi_{x,\chi} := \chi \circ Q_x$.

We will show that all elements of $\Delta(C^*(G))$ have this form (Theorem 2.2.1.8).

Proposition 2.2.1.2. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $\varphi \in \Delta(C^*(G))$. Then there exists unique $x_\varphi \in G^{(0)}$ which satisfies $\varphi(f) = f(x_\varphi)$ for all $f \in C_0(G^{(0)})$.

Proof. We have $\varphi|_{C_0(G^0)} \neq 0$ since $C_0(G^{(0)})$ has an approximate identity of $C^*(G)$. Therefore, $\varphi|_{C_0(G^{(0)})}$ belongs to $\Delta(C_0(G^{(0)}))$. Now the existence and uniqueness of $x_\varphi \in G^{(0)}$ follow from the Gelfand-Naimark theorem. □

Proposition 2.2.1.3. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $\varphi \in \Delta(C^*(G))$. Then $x_\varphi \in G^{(0)}$ as defined in Proposition 2.2.1.2 is a fixed point.

PROOF. Assume that $\gamma \in G$ satisfies $d(\gamma) = x_\varphi$. We will show $r(\gamma) = x_\varphi$. There exists an open bisection $U \subset G$ with $\gamma \in U$. Take $n_\gamma \in C_c(U)$ which satisfies $n_\gamma(\gamma) = 1$. Note that we have $n_\gamma^* * n_\gamma \in C_c(G^{(0)})$ and $n_\gamma^* * n_\gamma(x_\varphi) = |n_\gamma(\gamma)|^2 = 1$. Fix $f \in C_c(G^{(0)})$ arbitrarily.

Direct calculations show that $n_\gamma^* * f * n_\gamma(x_\varphi) = \overline{n_\gamma(\gamma)} f(r(\gamma)) n_\gamma(\gamma) = f(r(\gamma))$. On the other hand, one can see that $n_\gamma^* * f * n_\gamma \in C_c(G^{(0)})$. Then we have

$$n_\gamma^* * f * n_\gamma(x_\varphi) = \varphi(n_\gamma^* * f * n_\gamma) = \varphi(n_\gamma^*)\varphi(f)\varphi(n_\gamma) = \varphi(n_\gamma^* * n_\gamma)\varphi(f) = f(x_\varphi).$$

Therefore, $f(r(\gamma)) = f(x_\varphi)$ holds for all $f \in C_c(G^{(0)})$, which implies $r(\gamma) = x_\varphi$. Hence, $x_\varphi \in G^{(0)}$ is a fixed point of G. □

Proposition 2.2.1.4. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$, $\varphi \in \Delta(C^*(G))$ and $\gamma \in G_{x_\varphi}$. Take an open bisection $U_\gamma \subset G$ with $\gamma \in U_\gamma$ and $f_\gamma \in C_c(U_\gamma)$ with $f_\gamma(\gamma) = 1$. Then $\varphi(f_\gamma)$ is independent of the choice of U_γ and f_γ. Moreover, we have $\varphi(f_\gamma) \in \mathbb{T}$.

PROOF. First, we show $\varphi(f_\gamma) \in \mathbb{T}$. Since $f_\gamma^* * f_\gamma \in C_0(G^{(0)})$, we have

$$|\varphi(f_\gamma)|^2 = \varphi(f_\gamma^* * f_\gamma) = f_\gamma^* * f_\gamma(x_\varphi) = |f_\gamma(\gamma)|^2 = 1.$$

Therefore, $\varphi(f_\gamma) \in \mathbb{T}$.

Second, we show that $\varphi(f_\gamma)$ is independent of the choice of U_γ and f_γ. Assume that $f_\gamma \in C_c(U_\gamma)$ and $g_\gamma \in C_c(V_\gamma)$ satisfies $f_\gamma(\gamma) = g_\gamma(\gamma) = 1$, where U_γ and $V_\gamma \subset G$ are open bisections. Find a function $h \in C_c(d(U_\gamma \cap V_\gamma)) \subset C_c(G^{(0)})$ such that $h(d(\gamma)) = 1$. Recall that $d(\gamma) = r(\gamma) = x_\varphi$ since x_φ is a fixed point. Also, note that $\varphi(h) = h(x_\varphi) = 1$. Putting $\widetilde{f_\gamma} := f_\gamma * h$ and $\widetilde{g_\gamma} = g_\gamma * h$, we have that $\widetilde{f_\gamma}$ and $\widetilde{g_\gamma}$ are contained in $C_c(U_\gamma \cap V_\gamma)$. Then it follows that $\widetilde{f_\gamma}^* * \widetilde{g_\gamma} \in C_0(G^{(0)})$ and

$$\begin{aligned}\overline{\varphi(f_\gamma)}\varphi(g_\gamma) &= \overline{\varphi(h)\varphi(f_\gamma)}\varphi(g_\gamma)\varphi(h) = \varphi(\widetilde{f_\gamma}^* * \widetilde{g_\gamma}) \\ &= \widetilde{f_\gamma}^* * \widetilde{g_\gamma}(x_\varphi) = \overline{h(r(\gamma)) f_\gamma(\gamma)} g_\gamma(\gamma) h(d(\gamma)) = 1.\end{aligned}$$

Now we have $\varphi(f_\gamma) = \varphi(g_\gamma)$ since $\varphi(f_\gamma) \in \mathbb{T}$. □

Proposition 2.2.1.5. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$ and $\varphi \in \Delta(C^*(G))$. We define $\chi_\varphi \colon G_{x_\varphi} \to \mathbb{T}$ by $\chi_\varphi(\gamma) := \varphi(f_\gamma)$, where $\gamma \in G_{x_\varphi}$ and $f_\gamma \in \mathcal{C}(G)$ is a function as in Proposition 2.2.1.4. Then $\chi_\varphi \colon G_{x_\varphi} \to \mathbb{T}$ is a group homomorphism.

Proof. Take $\alpha, \beta \in G_{x_\varphi}$. We show $\chi_\varphi(\alpha)\chi_\varphi(\beta) = \chi_\varphi(\alpha\beta)$. Take $f_\alpha, f_\beta \in \mathcal{C}(G)$ as in Proposition 2.2.1.4. It follows that $f_\alpha * f_\beta \in C_c(U)$ for some open bisection $U \subset G$ and $f_\alpha * f_\beta(\alpha\beta) = 1$. Hence, we have

$$\chi_\varphi(\alpha\beta) = \varphi(f_\alpha * f_\beta) = \varphi(f_\alpha)\varphi(f_\beta) = \chi_\varphi(\alpha)\chi_\varphi(\beta)$$

by the definition of χ_φ. □

Proposition 2.2.1.6. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Then we have $\varphi = \varphi_{x_\varphi,\chi_\varphi}$ for all $\varphi \in \Delta(C^*(G))$.

Proof. Take $f \in C_c(U)$, where $U \subset G$ is an open bisection. It suffices to show that $\varphi(f) = \varphi_{x_\varphi,\chi_\varphi}(f)$, since $C^*(G)$ is generated by such functions. Note that $f^* * f \in C_c(G^{(0)})$. If $G_{x_\varphi} \cap f^{-1}(\mathbb{C} \setminus \{0\}) = \emptyset$, then we have $0 = f^* * f(x_\varphi) = |\varphi(f)|^2$. Since the restriction of $f|_{G_{x_\varphi}}$ is zero, it follows that $\varphi_{x_\varphi,\chi_\varphi}(f) = 0 = \varphi(f)$. If $G_{x_\varphi} \cap f^{-1}(\mathbb{C} \setminus \{0\}) \neq \emptyset$, $G_{x_\varphi} \cap f^{-1}(\mathbb{C} \setminus \{0\})$ is a singleton because f is supported on an open bisection. Let $\gamma \in G_{x_\varphi} \cap f^{-1}(\mathbb{C} \setminus \{0\})$ be the unique element of $G_{x_\varphi} \cap f^{-1}(\mathbb{C} \setminus \{0\})$. Observe that $F := f/f(\gamma) \in C_c(U)$ satisfies $F(\gamma) = 1$. Now we have

$$\varphi_{x_\varphi,\chi_\varphi}(f) = f(\gamma)\chi_\varphi(\gamma) = f(\gamma)\varphi(F) = \varphi(f).$$

Hence, we have $\varphi_{x_\varphi,\chi_\varphi} = \varphi$. □

Proposition 2.2.1.7. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$, $x \in G^{(0)}$ be a fixed point and $\chi\colon G_x \to \mathbb{T}$ be a group homomorphism. Then $x = x_{\varphi_{x,\chi}}$ and $\chi = \chi_{\varphi_{x,\chi}}$.

Proof. First, we show $x = x_{\varphi_{x,\chi}}$. Take $f \in C_c(G^{(0)})$ arbitrarily. Then we have

$$f(x_{\varphi_{x,\chi}}) = \varphi_{x,\chi}(f) = f(x)\chi(x) = f(x).$$

Hence, it follows $x = x_{\varphi_x}$.

Next, we show $\chi = \chi_{\varphi_{x,\chi}}$. Take $\gamma \in G_x$ arbitrarily. There exist an open bisection $U \subset G$ with $\gamma \in U$ and $f \in C_c(U)$ with $f(\gamma) = 1$. Then we have

$$\chi_{\varphi_{x,\chi}}(\gamma) = \varphi_{x,\chi}(f) = f(\gamma)\chi(\gamma) = \chi(\gamma).$$

Hence, we have shown $x = x_{\varphi_{x,\chi}}$ and $\chi = \chi_{\varphi_{x,\chi}}$. □

Combining Propositions 2.2.1.6 and 2.2.1.7, we obtain the next theorem.

Theorem 2.2.1.8 ([11, Theorem 4.8])**.** Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Define a set

$$\mathcal{D} := \{(x, \chi) \mid x \in G^{(0)} \text{ is a fixed point and } \chi\colon G_x \to \mathbb{T} \text{ is a group homomorphism}\}.$$

Then the map

$$\mathcal{D} \ni (x, \chi) \longrightarrow \varphi_{x,\chi} \in \Delta(C^*(G))$$

is bijective.

2.2.2 Construction of an étale abelian group bundle G^{ab}

For an étale groupoid G with the locally compact Hausdorff unit space $G^{(0)}$, we construct an étale abelian group bundle G^{ab} so that $C^*(G)^{\mathrm{ab}} \simeq C^*(G^{\mathrm{ab}})$ holds.

Proposition 2.2.2.1. Let G be an étale group bundle with the locally compact Hausdorff unit space $G^{(0)}$. We define the commutator subgroupoid of G by $[G, G] := \bigcup_{x \in G^{(0)}} [G_x, G_x]$, where $[G_x, G_x]$ is the commutator subgroup of G_x. Then $[G, G]$ is an open normal subgroupoid of G.

Proof. It is obvious that $[G, G] \subset G$ is a normal subgroupoid. We show that $[G, G] \subset G$ is open. Take $\gamma \in [G, G]$. By the definition of the commutator subgroup, there exists $\{\alpha_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k \subset G_{d(\gamma)}$ such that

$$\gamma = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \cdots \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1}.$$

Take open bisections $U_j, V_j \subset G$ such that $\alpha_j \in U_j$ and $\beta_j \in V_j$ for all $j = 1, 2, \ldots, k$. We show that $U_1 V_1 U_1^{-1} V_1^{-1} \subset [G, G]$, where we define $U^{-1} := \{\gamma^{-1} \mid \gamma \in U\}$ for $U \subset G$. Fix $\gamma' \in U_1 V_1 U_1^{-1} V_1^{-1}$. Then there exist $\alpha, \alpha' \in U_1$ and $\beta, \beta' \in V_1$ which satisfy $\gamma = \alpha\beta\alpha'^{-1}\beta'^{-1}$. Since G is a group bundle, we have $d(\alpha) = d(\alpha') = d(\beta) = d(\beta')$. We obtain $\alpha = \alpha'$ and $\beta = \beta'$ because U_1 and V_1 are bijections. Therefore, $\gamma' = \alpha\beta\alpha^{-1}\beta^{-1} \in [G, G]$. Similarly, one can show that $U_1 V_1 U_1^{-1} V_1^{-1} U_2 V_2 U_2^{-1} V_2^{-1} \cdots U_k V_k U_k^{-1} V_k^{-1} \subset [G, G]$.

By Proposition 1.1.3.8, $U_1 V_1 U_1^{-1} V_1^{-1} U_2 V_2 U_2^{-1} V_2^{-1} \cdots U_k V_k U_k^{-1} V_k^{-1}$ is an open set and contains γ. Hence, $[G, G] \subset G$ is an open normal subgroupoid. □

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Recall that the set of all fixed points $F \subset G^{(0)}$ is a closed subset of $G^{(0)}$ (Proposition 1.1.3.12). We define $G_{\mathrm{fix}} := G_F$, which is an étale groupoid from Proposition 1.1.3.13. Since we have $G_{\mathrm{fix}} = \mathrm{Iso}(G_{\mathrm{fix}})$, G_{fix} is an étale group bundle.

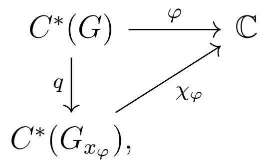
Definition 2.2.2.2. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. We define the abelianization of G by $G^{\mathrm{ab}} := G_{\mathrm{fix}}/[G_{\mathrm{fix}}, G_{\mathrm{fix}}]$.

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Then we have a *-homomorphism $C^*(G) \to C^*(G_{\mathrm{fix}})$ induced by the restriction (Proposition 1.1.4.4). Composing with the *-homomorphism $C^*(G_{\mathrm{fix}}) \to C^*(G^{\mathrm{ab}})$ in Proposition 2.1.2.1, we obtain a *-homomorphism $\pi \colon C^*(G) \to C^*(G^{\mathrm{ab}})$.

Note that $C^*(G)$ is commutative if and only if G is an étale abelian group bundle. In particular, $C^*(G^{\mathrm{ab}})$ is commutative.

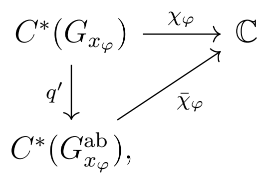
Lemma 2.2.2.3. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Then the map $\Phi\colon\ \Delta(C^*(G^{\mathrm{ab}})) \ni \chi \mapsto \chi \circ \pi \in \Delta(C^*(G))$ is bijective.

Proof. Surjectivity of π implies that Φ is injective. We show that Φ is surjective. Take $\varphi \in \Delta(C^*(G))$. Then we have the fixed point $x_\varphi \in G^{(0)}$ and the group homomorphism χ_φ which makes the following diagram commutative:



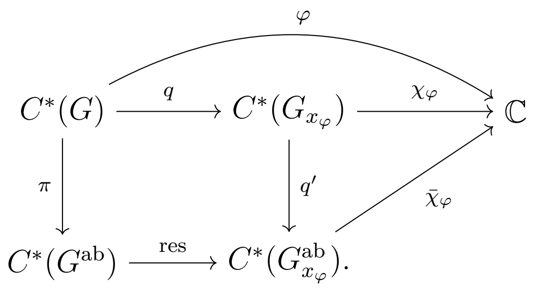
where $q\colon C^*(G) \to C^*(G_{x_\varphi})$ is the *-homomorphism obtained in Proposition 1.1.4.4.

By the universality of $G^{\mathrm{ab}}_{x_\varphi} := (G_{x_\varphi})^{\mathrm{ab}} = (G^{\mathrm{ab}})_{x_\varphi}$, we obtain the group homomorphism $\bar{\chi}_\varphi\colon G^{\mathrm{ab}}_{x_\varphi} \to \mathbb{T}$ which makes the following diagram commutative:



where $q'\colon C^*(G_{x_\varphi}) \to C^*(G^{\mathrm{ab}}_{x_\varphi})$ denotes the *-homomorphism induced by the quotient map $G_{x_\varphi} \to G^{\mathrm{ab}}_{x_\varphi}$.

Let $\mathrm{res}\colon C^*(G^{\mathrm{ab}}) \to C^*(G^{\mathrm{ab}}_{x_\varphi})$ denote the *-homomorphism obtained by the restriction $\mathcal{C}(G^{\mathrm{ab}}) \to \mathcal{C}(G^{\mathrm{ab}}_{x_\varphi})$ (see Proposition 1.1.4.4). Now we have the following commutative diagram:



In particular, we have $\varphi = (\bar{\chi}_\varphi \circ \mathrm{res}) \circ \pi$ and $\bar{\chi}_\varphi \circ \mathrm{res} \in \Delta(C^*(G^{\mathrm{ab}}))$. Hence, Φ is surjective. □

We are now ready to calculate the abelianization of $C^*(G)$.

Theorem 2.2.2.4 ([11, Theorem 4.12])**.** Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Then $C^*(G)^{\mathrm{ab}}$ is isomorphic to $C^*(G^{\mathrm{ab}})$ via the unique isomorphism $\bar{\pi}$ which makes the following diagram commutative:

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\ \pi\ } & C^*(G^{\mathrm{ab}}) \\ {\scriptstyle Q}\big\downarrow & \nearrow_{\bar{\pi}} & \\ C^*(G)^{\mathrm{ab}} & & \end{array}$$

where $Q\colon C^*(G) \to C^*(G)^{\mathrm{ab}}$ denotes the quotient map.

Proof. By the universality of $C^*(G)^{\mathrm{ab}}$, we obtain a *-homomorphism which makes the following diagram commutative:

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\ \pi\ } & C^*(G^{\mathrm{ab}}) \\ {\scriptstyle Q}\big\downarrow & \nearrow_{\bar{\pi}} & \\ C^*(G)^{\mathrm{ab}}. & & \end{array}$$

It is clear that $\bar{\pi}$ is surjective. We show that $\bar{\pi}$ is injective. Suppose that $a \in C^*(G)$ satisfies $\pi(a) = 0$. It suffices to show $Q(a) = 0$, which is equivalent to $\bar{\varphi}(Q(a)) = 0$ for all $\bar{\varphi} \in \Delta(C^*(G)^{\mathrm{ab}})$ since $C^*(G)^{\mathrm{ab}}$ is commutative. Take $\bar{\varphi} \in \Delta(C^*(G)^{\mathrm{ab}})$ and define $\varphi := \bar{\varphi} \circ Q$. Then, by Lemma 2.2.2.3, there exists $\tilde{\varphi} \in \Delta(C^*(G^{\mathrm{ab}}))$ which makes the following diagram commutative:

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\ \pi\ } & C^*(G^{\mathrm{ab}}) \\ & \searrow^{\varphi} & \big\downarrow{\scriptstyle \tilde{\varphi}} \\ & & \mathbb{C}. \end{array}$$

Now we have the following commutative diagram:

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\ \pi\ } & C^*(G^{\mathrm{ab}}) \\ {\scriptstyle Q}\big\downarrow & \searrow^{\varphi} & \big\downarrow{\scriptstyle \tilde{\varphi}} \\ C^*(G)^{\mathrm{ab}} & \xrightarrow{\ \bar{\varphi}\ } & \mathbb{C}. \end{array}$$

Hence, we have $\bar{\varphi}(Q(a)) = \tilde{\varphi}(\pi(a)) = 0$. □

2.2.3 Duals of étale abelian group bundles

Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Since the abelianization of $C^*(G)$ is a commutative C*-algebra, $C^*(G)^{\mathrm{ab}}$ is isomorphic to $C_0(\Delta(C^*(G)^{\mathrm{ab}}))$ via the Gelfand transformation (see, for example, [2, Theorem II.2.2.4]). In this subsection we calculate the Gelfand spectrum $\Delta(C^*(G)^{\mathrm{ab}})$.

For a discrete abelian group Γ, its Pontryagin dual group is defined as the set of all group homomorphisms from Γ to $\mathbb{T}$, which is denoted by $\widehat{\Gamma}$. Then $\widehat{\Gamma}$ is an abelian group with respect to the pointwise multiplication. It is known that $\widehat{\Gamma}$ is a compact abelian topological group with respect to the topology of pointwise convergence.

Proposition 2.2.3.1. Let Γ be a discrete group and $Q\colon C^*(\Gamma) \to C^*(\Gamma^{\mathrm{ab}})$ be the *-homomorphism induced by the quotient map $\Gamma \to \Gamma^{\mathrm{ab}}$. Then the map

$$\Phi\colon \widehat{\Gamma^{\mathrm{ab}}} \ni \chi \mapsto \chi \circ Q \in \Delta(C^*(\Gamma))$$

is a homeomorphism. Hence, $C^*(\Gamma)^{\mathrm{ab}}$ is isomorphic to $C(\widehat{\Gamma^{\mathrm{ab}}})$.

Proof. This follows from the universality of Γ^{ab} and $C^*(\Gamma)$. □

As seen in the previous proposition, the key to calculate $\Delta(C^*(G))$ is the Pontryagin dual.

Definition 2.2.3.2. Let G be an étale abelian group bundle with the locally compact Hausdorff unit space $G^{(0)}$. We define a group bundle $\widehat{G} := \{(\chi, x) \mid x \in G^{(0)}, \chi \in \widehat{G_x}\}$ over $G^{(0)}$.

Note that $\widehat{G}$ is a group bundle such that $\widehat{G}_x = \widehat{G_x} \times \{x\} (\simeq \widehat{G_x})$ for every $x \in G^{(0)}$.

Let G be an étale abelian group bundle with the locally compact Hausdorff unit space $G^{(0)}$ and $(\chi, x) \in \widehat{G}$. Recall that we obtain the *-homomorphism $\varphi_{x,\chi} \in \Delta(C^*(G))$ as in Definition 2.2.1.1.

Definition 2.2.3.3. Let G be an étale abelian group bundle with the locally compact Hausdorff unit space $G^{(0)}$. For each $f \in \mathcal{C}(G)$, we define $\mathrm{ev}_f\colon \widehat{G} \to \mathbb{C}$ by $\mathrm{ev}_f((\chi, x)) = \varphi_{x,\chi}(f)$, where $(\chi, x) \in \widehat{G}$. We define a topology of $\widehat{G}$ as the weakest topology in which ev_f is continuous for all $f \in \mathcal{C}(G)$.

Proposition 2.2.3.4. Let G be an étale abelian group bundle with the locally compact Hausdorff unit space $G^{(0)}$. Then the map

$$\Psi\colon\ \Delta(C^*(G)) \ni \varphi \mapsto (\chi_\varphi, x_\varphi) \in \widehat{G}$$

is a homeomorphism (see Propositions 2.2.1.2 and 2.2.1.5 for the definition of x_φ and χ_φ). Hence, $C^*(G)$ is isomorphic to $C_0(\widehat{G})$

Proof. Proposition 2.2.1.8 states that Ψ is a bijection and Ψ^{-1} is given by $\Psi^{-1}((\chi, x)) = \varphi_{x,\chi}$ for each $(\chi, x) \in \widehat{G}$. For each $f \in \mathcal{C}(G)$, the map $\Delta(C^*(G)) \ni \varphi \mapsto \mathrm{ev}_f((\chi_\varphi, x_\varphi)) = \varphi(f) \in \mathbb{C}$ is continuous. This means that Ψ is continuous. The continuity of Ψ^{-1} follows from approximation arguments. Therefore, Ψ is a homeomorphism. □

Let G be an étale groupoid. Recall that G^{ab} is an étale abelian group bundle.

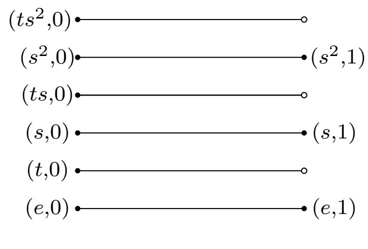
Corollary 2.2.3.5. Let G be an étale groupoid with the locally compact Hausdorff unit space $G^{(0)}$. Then $C^*(G)^{\mathrm{ab}}$ is isomorphic to $C_0(\widehat{G^{\mathrm{ab}}})$.

Proof. Recall that $C^*(G)^{\mathrm{ab}}$ is isomorphic to $C^*(G^{\mathrm{ab}})$ by Theorem 2.2.2.4. Since G^{ab} is an étale abelian group bundle, Proposition 2.2.3.4 implies that $C^*(G^{\mathrm{ab}})$ is isomorphic to $C_0(\widehat{G^{\mathrm{ab}}})$. $\square$

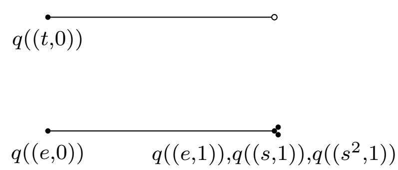
Proposition 2.2.3.6. Let G be an étale abelian group bundle with the locally compact Hausdorff unit space $G^{(0)}$. Then $\widehat{G}$ is a locally compact Hausdorff topological group bundle. Furthermore, $\widehat{G}$ is compact if and only if $G^{(0)}$ is compact.

Proof. It is clear that $\widehat{G}$ is locally compact Hausdorff, since $\widehat{G}$ is homeomorphic to $\Delta(C^*(G))$. In order to show the continuity of the operations, take $f \in \mathcal{C}(G)$ arbitrarily. Then the map $\widehat{G}^{(2)} \ni (\chi_1, \chi_2) \mapsto \mathrm{ev}_f(\chi_1\chi_2) = \mathrm{ev}_f(\chi_1)\,\mathrm{ev}_f(\chi_2) \in \mathbb{C}$ is continuous. Therefore, the multiplication of $\widehat{G}^{(2)}$ is continuous. Similarly, one can show that the inverse is continuous. Hence, $\widehat{G}$ is a locally compact Hausdorff topological group bundle. The last assertion follows from the fact that $G^{(0)}$ is compact if and only if $C^*(G) \simeq C_0(\widehat{G})$ is unital. $\square$

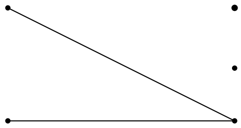
Example 2.2.3.7. We give an example of an étale groupoid G such that G^{ab} is not Hausdorff although G is Hausdorff. Let $\mathfrak{S}_3 = \langle s, t \mid s^3 = t^2 = e, st = ts^2 \rangle = \{e, s, s^2, t, ts, ts^2\}$ be the symmetric group of degree 3 and $A_3 := \{e, s, s^2\} \subset \mathfrak{S}_3$ be the subgroup of even permutations. Let $G := \mathfrak{S}_3 \times [0,1] \setminus \{(t,1) \mid t \notin A_3\}$ be an étale group bundle over $[0,1]$. Then G can be drawn as follows:



One can see that $[G,G] \subset G$ is not closed. By Proposition 2.1.1.11, $G^{\mathrm{ab}} = G/[G,G]$ is not Hausdorff. Indeed, letting $q\colon G \to G^{\mathrm{ab}}$ denote the quotient map, G^{ab} looks as follows:



The dual $\widehat{G^{\mathrm{ab}}}$ of G^{ab} can be drawn as follows:



Note that $\widehat{G^{\mathrm{ab}}}$ is not étale.

Chapter 3.

Invariant sets and normal subgroupoids of universal étale groupoids induced by congruences of inverse semigroups

In this chapter, we review results obtained in [10]. First, we investigate congruences on inverse semigroups from the view point of the spectrum in Section 3.1. In Section 3.2, we show that the universal étale groupoids associated to quotient inverse semigroups can be described by restrictions and quotients of the original universal étale groupoids. In Section 3.3, we give applications and examples of the previous sections.

3.1 Certain least congruences

Recall that an inverse semigroup S is said to be Clifford if $s^*s = ss^*$ holds for all $s \in S$. In addition, a congruence ν on S is said to be Clifford if S/ν is Clifford. The notion of a commutative congruence is defined in the same way. It is known that every inverse semigroup admits the least Clifford congruence and the least commutative congruence. For example, see [16, Proposition III. 6. 7] for the least Clifford congruence and [17] for the least commutative congruence. In this section, we reprove that every inverse semigroup admits the least Clifford congruence and the least commutative congruence by a new method using the spectrum.

3.1.1 Invariant subset of $\widehat{E(S)}$

Let S be an inverse semigroup. Recall that we have the spectral action $\beta\colon S \curvearrowright \widehat{E(S)}$ (see the last of Chapter 1). A subset $F \subset \widehat{E(S)}$ is said to be invariant if $\beta_s(F \cap D_{s^*s}) \subset F$ holds for all $s \in S$. Note that F is invariant if and only if F is invariant as a subset of the universal groupoid $G_u(S)$. First, we observe that an invariant subset induces a normal congruence on $E(S)$ in the next proposition.

Proposition 3.1.1.1. Let S be an inverse semigroup and $F \subset \widehat{E(S)}$ be an invariant subset. We define the set $\rho_F \subset E(S) \times E(S)$ of all pairs $(e, f) \in E(S) \times E(S)$ such that $\xi(e) = \xi(f)$ holds for all $\xi \in F$. Then ρ_F is a normal congruence on $E(S)$.

PROOF. It is obvious that ρ_F is a congruence on $E(S)$. We show that ρ_F is normal. Take $s \in S$ and $(e, f) \in \rho_F$. It suffices to show that $\xi(ses^*) = \xi(sfs^*)$ for all $\xi \in F$. If $\xi(ss^*) = 0$, we have $\xi(ses^*) = \xi(sfs^*) = 0$. Assume that $\xi(ss^*) = 1$. Since F is invariant, we have $\beta_{s^*}(\xi) \in F$. From $(e, f) \in \rho_F$, it follows

$$\xi(ses^*) = \beta_{s^*}(\xi)(e) = \beta_{s^*}(\xi)(f) = \xi(sfs^*).$$

Thus ρ_F is a normal congruence on $E(S)$. □

Let S be an inverse semigroup and ρ be a normal congruence on $E(S)$. Moreover, let $q\colon E(S) \to E(S)/\rho$ denote the quotient map. For $\xi \in \widehat{E(S)/\rho}$, we define $\widehat{q}(\xi) \in \widehat{E(S)}$ by $\widehat{q}(\xi)(e) = \xi(q(e))$, where $e \in E(S)$. Note that $\widehat{q}(\xi)$ is not zero since q is surjective. Then $\widehat{q}\colon \widehat{E(S)/\rho} \to \widehat{E(S)}$ is a continuous map by the definition of the topology of pointwise convergence. One can see that

$$\widehat{q}(\widehat{E(S)/\rho}) = \{\xi \in \widehat{E(S)} \mid \xi(e) = \xi(f) \text{ for all } (e, f) \in \rho\}$$

holds. In particular, $F_\rho := \widehat{q}(\widehat{E(S)/\rho})$ is a closed subset of $\widehat{E(S)}$.

We say that $F \subset \widehat{E(S)}$ is multiplicative if the multiplication of two elements in F also belongs to F whenever it is not zero.

Proposition 3.1.1.2. Let S be an inverse semigroup and ρ be a normal congruence on $E(S)$. Then $F_\rho \subset \widehat{E(S)}$ is a closed multiplicative invariant set.

PROOF. It is easy to show that $F_\rho \subset \widehat{E(S)}$ is a closed multiplicative set. We show that $F_\rho \subset \widehat{E(S)}$ is invariant. Take $\xi \in F_\rho$ and $s \in S$ with $\xi(s^*s) = 1$. To see $\beta_s(\xi) \in F_\rho$, it suffices to show that $\beta_s(\xi)(e) = \beta_s(\xi)(f)$ holds for all $(e, f) \in \rho$. Since ρ is normal, we have $(s^*es, s^*fs) \in \rho$. Hence, we have

$$\beta_s(\xi)(e) = \xi(s^*es) = \xi(s^*fs) = \beta_s(\xi)(f),$$

where the middle equality follows from $\xi \in F_\rho$. □

Proposition 3.1.1.3. Let S be an inverse semigroup. Then $\rho = \rho_{F_\rho}$ holds for every normal congruence ρ on $E(S)$.

PROOF. Assume that $(e, f) \in \rho$. For all $\eta \in \widehat{E(S)/\rho}$, it follows that

$$\widehat{q}(\eta)(e) = \eta(q(e)) = \eta(q(f)) = \widehat{q}(\eta)(f).$$

Therefore, $(e, f) \in \rho_{F_\rho}$.

To show the reverse inclusion, assume that $(e, f) \in \rho_{F_\rho}$. Define $\eta_{q(e)} \in \widehat{E(S)/\rho}$ by

$$\eta_{q(e)}(p) = \begin{cases} 1 & (p \geq q(e)), \\ 0 & (\text{otherwise}), \end{cases}$$

where $p \in E(S)/\rho$. By $(e, f) \in \rho_{F_\rho}$, we have $\eta_{q(e)}(q(f)) = \eta_{q(e)}(q(e)) = 1$. Therefore, $q(f) \geq q(e)$. Similarly we obtain $q(f) \leq q(e)$, so $q(e) = q(f)$ holds. Thus, it follows that $(e, f) \in \rho$. □

We say that $F \subset \widehat{E(S)}$ is unital if F contains the constant function 1.

Lemma 3.1.1.4. Let S be an inverse semigroup and $F \subset \widehat{E(S)}$ be a unital multiplicative set. Assume that F separates $E(S)$ (that is, for $e, f \in E(S)$, $e = f$ is equivalent to the condition that $\xi(e) = \xi(f)$ holds for all $\xi \in F$). Then F is dense in $\widehat{E(S)}$.

Proof. For $e \in E(S)$ and a finite subset $P \subset E(S)$, we define

$$N_P^e := \{\xi \in \widehat{E(S)} \mid \xi(e) = 1, \xi(p) = 0 \text{ for all } p \in P\}.$$

Recall that these sets form an open basis of $\widehat{E(S)}$. Observe that $N_P^e = N_{eP}^e$ holds, where $eP := \{ep \in E(S) \mid p \in P\}$. Now it suffices to show that $F \cap N_P^e \neq \emptyset$ holds for nonempty N_P^e such that $p \leq e$ holds for all $p \in P$.

In case that $P = \emptyset$, the constant function 1 belongs to $F \cap N_P^e$. We may assume that $p \leq e$ holds for all $p \in P$. Since N_P^e is nonempty, we have $e \neq p$ for all $p \in P$. Since F separates $E(S)$, there exists $\xi_p \in F$ such that $\xi_p(e) = 1$ and $\xi_p(p) = 0$ for each $p \in P$. Define $\xi := \prod_{p \in P} \xi_p$; then $\xi \in N_P^e \cap F$. □

Proposition 3.1.1.5. Let S be an inverse semigroup. Then $F = F_{\rho_F}$ holds for every unital multiplicative invariant closed set $F \subset \widehat{E(S)}$.

Proof. It is easy to show that $F \subset F_{\rho_F}$. Let $q\colon E(S) \to E(S)/\rho_F$ denote the quotient map. Then the set $\widehat{q}^{-1}(F)$ is a unital multiplicative closed set which separates $E(S)/\rho_F$. By Lemma 3.1.1.4, $\widehat{q}^{-1}(F) = \widehat{E(S)/\rho_F}$ holds. Therefore, we have $F \supset \widehat{q}(\widehat{q}^{-1}(F)) = \widehat{q}(\widehat{E(S)/\rho_F}) = F_{\rho_F}$. □

Corollary 3.1.1.6. Let S be an inverse semigroup. There is a one-to-one correspondence between normal congruences on $E(S)$ and unital multiplicative invariant closed sets in $\widehat{E(S)}$.

Proof. Just combine Propositions 3.1.1.3 and 3.1.1.5. □

3.1.2 The least Clifford congruences

Let S be an inverse semigroup. Recall that a congruence ρ on S is said to be Clifford if S/ρ is Clifford. For example, $S \times S$ is a Clifford congruence on S. In this subsection, we prove that every inverse semigroup admits the least Clifford congruence (Theorem 3.1.2.3). Our construction of the congruence is based on the fixed points of $\widehat{E(S)}$.

Definition 3.1.2.1. Let S be an inverse semigroup. A character $\xi \in \widehat{E(S)}$ is said to be fixed if $\xi(s^*es) = \xi(e)$ holds for all $e \in E(S)$ and $s \in S$ such that $\xi(s^*s) = 1$. We denote the set of all fixed characters by $\widehat{E(S)}_{\mathrm{fix}}$.

One can see that $\widehat{E(S)}_{\mathrm{fix}}$ is a closed subset of $\widehat{E(S)}$. Moreover, $\widehat{E(S)}_{\mathrm{fix}}$ is a multiplicative set. The fixed characters are characterized in the next proposition.

Proposition 3.1.2.2. Let S be an inverse semigroup and $\xi \in \widehat{E(S)}$. The following conditions are equivalent.

(1) ξ is a fixed character;
(2) ξ can be extended to a semigroup homomorphism $\widetilde{\xi}\colon S \to \{0,1\}$; and
(3) $\xi(s^*s) = \xi(ss^*)$ holds for all $s \in S$.

In this case, $\widetilde{\xi}\colon S \to \{0,1\}$ is the unique extension of ξ.

Proof. If $\xi \in \widehat{E(S)}$ has an extension $\widetilde{\xi}\colon S \to \{0,1\}$, we have

$$\widetilde{\xi}(s) = \widetilde{\xi}(s)^2 = \widetilde{\xi}(s^*s) = \xi(s^*s)$$

for all $s \in S$. Therefore, a semigroup homomorphism extension of ξ is unique if it exists.

We show that (1) implies (2). Assume that $\xi \in \widehat{E(S)}$ is fixed. Then define $\widetilde{\xi}(s)\colon S \to \{0,1\}$ by $\widetilde{\xi}(s) := \xi(s^*s)$ for $s \in S$. For $s, t \in S$, if $\xi(t^*t) = 1$, we have $\widetilde{\xi}(st) = \xi(t^*s^*st) = \xi(s^*s) = \widetilde{\xi}(s)\widetilde{\xi}(t)$. If $\xi(t^*t) = 0$, we have $\widetilde{\xi}(st) = \widetilde{\xi}(s)\widetilde{\xi}(t) = 0$. Thus, $\widetilde{\xi}$ is a semigroup homomorphism.

It is obvious that (2) implies (3). We show that (3) implies (1). Take $s \in S$ with $\xi(s^*s) = 1$. It suffices to show that $\beta_s(\xi)(e) = \xi(e)$ holds for all $e \in E(S)$. This follows from the following direct calculation:

$$\begin{aligned}\beta_s(\xi)(e) &= \xi(s^*es) = \xi((es)^*(es)) = \xi((es)(es)^*)\\ &= \xi(ess^*e) = \xi(ess^*) = \xi(e)\xi(ss^*) = \xi(e)\xi(s^*s) = \xi(e).\end{aligned}$$

Now we have shown that the conditions (1), (2) and (3) are equivalent. □

Definition 3.1.2.3. Let S be an inverse semigroup. We define the normal congruence

$\rho_{\text{Clif}} := \rho_{\widehat{E(S)}_{\text{fix}}}$ on $E(S)$. Furthermore, we define the congruence $\nu_{\text{Clif}} := \nu_{\rho_{\text{Clif}},\min}$ on S and $S^{\text{Clif}} := S/\nu_{\text{Clif}}$.

Lemma 3.1.2.4. Let S be an inverse semigroup, ν be a Clifford congruence on S and $q\colon S \to S/\nu$ be the quotient map. Then a set

$$F_\nu = \{\xi \circ q|_{E(S)} \in \widehat{E(S)} \mid \xi \in \widehat{E(S/\nu)}\}$$

is contained in $\widehat{E(S)}_{\text{fix}}$. Moreover, $\widehat{E(S)}_{\text{fix}} = \widehat{E(S)}$ holds if and only if S is Clifford.

Remark 3.1.2.5. Before proceeding to the proof of Lemma 3.1.2.4, we verify that F_ν in Lemma 3.1.2.4 is well-defined.

Let S be an inverse semigroup, ν be a congruence and $q\colon S \to S/\nu$. Then $q(E(S)) = E(S/\nu)$ holds. Indeed, $q(E(S)) \subset E(S/\nu)$ is obvious. Take $p \in E(S/\nu)$ and $s \in S$ such that $q(s) = p$. Then $p = q(s^*s) \in q(E(S))$. Therefore, we have $q(E(S)) = E(S/\nu)$. Using this fact, one can verify that $\xi \circ q|_{E(S)}$ is a character on $E(S)$ for $\xi \in \widehat{E(S)}$. Indeed, it is easy to see that $\xi \circ q|_{E(S)}$ is a semigroup homomorphism. Since $q(E(S)) = E(S/\nu)$ and q is a surjection, $\xi \circ q|_{E(S)}$ is nonzero. Hence, $\xi \circ q|_{E(S)}$ is a character on $E(S)$.

From the above argument, it follows that F_ν in Lemma 3.1.2.4 is well-defined.

Proof of the Lemma 3.1.2.4. Take $\xi \in \widehat{E(S/\nu)}$ and $s \in S$. Since S/ν is Clifford, we have

$$\xi \circ q(s^*s) = \xi(q(s^*s)) = \xi(q(ss^*)) = \xi \circ q(ss^*).$$

Therefore, $\xi \circ q|_{E(S)}$ is a fixed character by Proposition 3.1.2.2.

Applying what we have shown for the trivial congruence $\nu = \{(s,s) \in S \times S \mid s \in S\}$, it follows that $\widehat{E(S)}_{\text{fix}} = \widehat{E(S)}$ holds if S is Clifford. Assume that $\widehat{E(S)}_{\text{fix}} = \widehat{E(S)}$ holds and take $s \in S$. Define a character $\xi_{s^*s} \in \widehat{E(S)}$ by

$$\xi_{s^*s}(e) = \begin{cases} 1 & (e \geq s^*s), \\ 0 & (\text{otherwise}), \end{cases}$$

where $e \in E(S)$. Since we assume that $\widehat{E(S)}_{\text{fix}} = \widehat{E(S)}$, we have

$$\xi_{s^*s}(ss^*) = \xi_{s^*s}(s^*s) = 1$$

by Proposition 3.1.2.2. Then we have $s^*s \leq ss^*$. It follows that $s^*s \geq ss^*$ from the same argument. Now we have $s^*s = ss^*$ and S is Clifford. □

Now we show that every inverse semigroup admits the Cliffordization. See Definition 3.1.2.3 for the definition of ν_{Clif}.

Theorem 3.1.2.6 ([10, Theorem 3.11])**.** Let S be an inverse semigroup. Then ν_{Clif} is the least Clifford congruence on S.

Proof. First, we show that the congruence ν_{Clif} is Clifford. Take $s \in S$ and $\xi \in \widehat{E(S)}_{\mathrm{fix}}$. Then we have $\xi(s^*s) = \xi(ss^*)$ by Proposition 3.1.2.4. Therefore, $(s^*s, ss^*) \in \nu_{\mathrm{Clif}}$ and ν_{Clif} is a Clifford congruence.

Let ν be a Clifford congruence and $q\colon S \to S/\nu$ be the quotient map. To show that $\nu_{\mathrm{Clif}} \subset \nu$, take $(s,t) \in \nu_{\mathrm{Clif}}$. First, we show that $(s^*s, t^*t) \in \nu$. We define $\eta \in E\widehat{(S/\nu)}$ by

$$\eta(e) = \begin{cases} 1 & (e \geq q(s^*s)), \\ 0 & (\text{otherwise}). \end{cases}$$

By Lemma 3.1.2.4, it follows that $\eta \circ q \in \widehat{E(S)}_{\mathrm{fix}}$. Since $(s,t) \in \nu_{\mathrm{Clif}}$, we have $1 = \eta \circ q(s^*s) = \eta \circ q(t^*t)$, which implies $q(t^*t) \geq q(s^*s)$. The reverse inequality is obtained symmetrically and therefore $q(t^*t) = q(s^*s)$ holds.

Let $\eta \in E\widehat{(S/\nu_{\mathrm{Clif}})}$ be the above character. Since $\eta \circ q$ is a fixed character and $(s,t) \in \nu_{\mathrm{Clif}}$, there exists $e \in E(S)$ such that $\eta \circ q(e) = 1$ and $se = te$ hold. Since $\eta \circ q(e) = 1$, we have $q(e) \geq q(s^*s) = q(t^*t)$ by the definition of η. Now we have $q(s) = q(s)q(e) = q(t)q(e) = q(t)$. Therefore, $(s,t) \in \nu$. □

Corollary 3.1.2.7. Let S be an inverse semigroup, T be a Clifford inverse semigroup and $\varphi\colon S \to T$ be a semigroup homomorphism. Then there exists a unique semigroup homomorphism $\widetilde{\varphi}\colon S^{\mathrm{Clif}} \to T$ which makes the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{\ \varphi\ } & T \\ {\scriptstyle q}\big\downarrow & \nearrow_{\widetilde{\varphi}} & \\ S^{\mathrm{Clif}} & & \end{array},$$

where $q\colon S \to S^{\mathrm{Clif}}$ denotes the quotient map.

3.1.3 The least commutative congruences

We say that a congruence on inverse semigroup is commutative if the quotient semigroup is commutative. In this subsection, we show that every inverse semigroups admits the least commutative congruence.

Recall that we denote the circle group by $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. We view $\mathbb{T} \cup \{0\}$ as an inverse semigroup with the usual product. By $\widehat{S}$, we denote the set of all semigroup homomorphisms from S to $\mathbb{T} \cup \{0\}$.

Definition 3.1.3.1. Let S be an inverse semigroup. We define the commutative congruence ν_{ab} on S as the set of all pairs $(s,t) \in S \times S$ such that $\varphi(s) = \varphi(t)$ holds for all $\varphi \in \widehat{S}$. We define $S^{\mathrm{ab}} := S/\nu_{\mathrm{ab}}$.

One can see that S^{ab} is actually commutative.

Let S be a Clifford inverse semigroup and $e \in E(S)$. We define $H_e := \{s \in S \mid s^*s = e\}$. One can see that H_e is a group with the operation inherited from S. Note that the unit of H_e is e.

In order to show that ν_{ab} is the least commutative congruence, we need the next lemma.

Lemma 3.1.3.2. Let S be a Clifford inverse semigroup and $e \in E(S)$. Then a group homomorphism $\varphi\colon H_e \to \mathbb{T}$ can be extended to a semigroup homomorphism $\widetilde{\varphi}\colon S \to \mathbb{T} \cup \{0\}$.

Proof. Define

$$\widetilde{\varphi}(s) = \begin{cases} \varphi(se) & (s^*s \geq e), \\ 0 & (\text{otherwise}). \end{cases}$$

Then one can check that $\widetilde{\varphi}$ is a semigroup homomorphism extension of φ. □

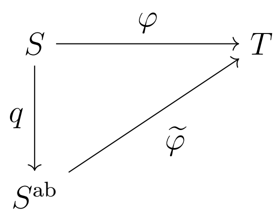
Theorem 3.1.3.3 ([10, Theorem 3.15])**.** Let S be an inverse semigroup. Then ν_{ab} in Definition 3.1.3.1 is the least commutative congruence on S.

Proof. Assume that ν is a commutative congruence. Let $q\colon S \to S/\nu$ denote the quotient map. In order to show $\nu_{\mathrm{ab}} \subset \nu$, take $(s,t) \in \nu_{\mathrm{ab}}$.

First, we show that $q(s^*s) = q(t^*t)$. It suffices to show that $\xi(q(s^*s)) = \xi(q(t^*t))$ holds for all $\xi \in \widehat{E(S/\nu)}$. Note that $\xi \circ q \in \widehat{E(S)}$ is a fixed point by Lemma 3.1.2.4. Since $\xi \circ q$ is a restriction of an element in $\widehat{S}$ by Proposition 3.1.2.2, $\xi(q(s^*s)) = \xi(q(t^*t))$ follows from $(s^*s, t^*t) \in \nu_{\mathrm{ab}}$.

In order to show that $q(s) = q(t)$, it suffices to show that $\psi(q(s)) = \psi(q(t))$ for all group homomorphisms $\psi\colon H_{q(s^*s)} \to \mathbb{T}$, since $H_{q(s^*s)} = \{a \in S/\nu \mid a^*a = q(s^*s)\}$ is an abelian group. By Lemma 3.1.3.2, there exists a semigroup homomorphism extension $\widetilde{\psi} \in \widehat{S/\nu}$ of ψ. Since $\widetilde{\psi} \circ q \in \widehat{S}$ and $(s,t) \in \nu_{\mathrm{ab}}$, we have $\psi(q(s)) = \psi(q(t))$. Therefore, $q(s) = q(t)$ holds. □

Corollary 3.1.3.4. Let S be an inverse semigroup, T be a commutative inverse semigroup and $\varphi\colon S \to T$ be a semigroup homomorphism. Then there exists a unique semigroup homomorphism $\widetilde{\varphi}\colon S^{\mathrm{ab}} \to T$ which makes the following diagram commutative:



,

where $q\colon S \to S^{\mathrm{ab}}$ denotes the quotient map.

3.2 Universal étale groupoids associated to quotient inverse semigroups

3.2.1 General case

Let S be an inverse semigroup and ν be a congruence on S. Let $q\colon S \to S/\nu$ denote the quotient map. Note that

$$F_\nu = \{\xi \circ q|_{E(S)} \in \widehat{E(S)} \mid \xi \in \widehat{E(S/\nu)}\}$$

is a closed invariant subset of $G_u(S)$ as shown in Proposition 3.1.1.2.

We omit the proof of the next proposition, since it is not difficult.

Proposition 3.2.1.1. Let S be an inverse semigroup and $H \subset S$ be a subsemigroup such that $E(S) \subset H$. Then the map

$$G_u(H) \ni [s, \xi] \mapsto [s, \xi] \in G_u(S)$$

is a groupoid homomorphism which is a homeomorphism onto its image. Moreover, the image is an open subgroupoid of $G_u(S)$.

Via the map in the above proposition, we identify $G_u(H)$ with an open subgroupoid of $G_u(S)$. Note that $G_u(S)^{(0)} \subset G_u(H)$ holds.

Let S be an inverse semigroup, ν be a congruence on S and $q\colon S \to S/\nu$ be the quotient map. Define $\ker \nu := q^{-1}(E(S/\nu)) \subset S$. Then $\ker \nu$ is a normal subsemigroup of S. Although $G_u(\ker \nu)$ is not necessarily a normal subgroupoid of $G_u(S)$, the following holds.

Proposition 3.2.1.2. Let S be an inverse semigroup and ν be a congruence on S. Then $G_u(\ker \nu)_{F_\nu}$ is an open normal subgroupoid of $G_u(S)_{F_\nu}$.

Proof. We know that $G_u(\ker \nu)_{F_\nu}$ is an open subgroupoid of $G_u(S)_{F_\nu}$. We show that $G_u(\ker \nu)_{F_\nu}$ is normal in $G_u(S)_{F_\nu}$. Let $q\colon S \to S/\nu$ denote the quotient map.

First, we show $G_u(\ker \nu)_{F_\nu} \subset \mathrm{Iso}(G_u(S)_{F_\nu})$. Take $[n, \xi] \in G_u(\ker \nu)_{F_\nu}$, where $n \in \ker \nu$. Since $\xi \in F_\nu$ holds, there exists $\eta \in \widehat{E(S/\nu)}$ such that $\xi = \eta \circ q$. Since $q(n) \in E(S/\nu)$

holds, we have $q(n^*) \in E(S/\nu)$ and

$$\begin{aligned}\beta_n(\xi)(e) &= \xi(n^*en) = \eta(q(n^*)q(e)q(n)) \\ &= \eta(q(n^*))\eta(q(e))\eta(q(n)) = \eta(q(n^*n))\eta(q(e)) = \xi(e)\end{aligned}$$

for all $e \in E(S)$, where we use $\eta(q(n^*n)) = \xi(n^*n) = 1$ in the last equality. Therefore, $\beta_n(\xi) = \xi$ holds and it follows that $[n, \xi] \in \mathrm{Iso}(G_u(\ker\nu)_{F_\nu})$.

Next we show that $[s, \eta][n, \xi][s, \eta]^{-1} \in G_u(\ker\nu)_{F_\nu}$ holds for all $[n, \xi] \in G_u(\ker\nu)_{F_\nu}$ and $[s, \eta] \in G_u(S)_{F_\nu}$ such that $\eta = \beta_n(\xi)(= \xi)$. One can see that

$$[s, \eta][n, \xi][s, \eta]^{-1} = [sns^*, \beta_s(\eta)].$$

Now it follows that $[s, \eta][n, \xi][s, \eta]^{-1} \in G_u(\ker\nu)_{F_\nu}$ from $sns^* \in \ker\nu$. □

Theorem 3.2.1.3 ([10, Theorem 4.3])**.** Let S be an inverse semigroup and ν be a congruence on S. Then $G_u(S/\nu)$ is isomorphic to $G_u(S)_{F_\nu}/G_u(\ker\nu)_{F_\nu}$.

Proof. Let $q\colon S \to S/\nu$ denote the quotient map. Note that a map

$$\widehat{q}\colon \widehat{E(S/\nu)} \ni \xi \mapsto \xi \circ q \in F_\nu$$

is well-defined by Remark 3.1.2.5. One can verify that $\widehat{q}$ is a homeomorphism.

Define a map

$$\Phi\colon G_u(S)_{F_\nu} \ni [s, \widehat{q}(\xi)] \mapsto [q(s), \xi] \in G_u(S/\nu).$$

Using Proposition 1.1.3.10, one can see that Φ is a groupoid homomorphism which is a local homeomorphism and injective on $G_u(S)^{(0)}_{F_\nu}$. Observe that Φ is surjective.

We show that $\ker\Phi = G_u(\ker\nu)_{F_\nu}$ holds. The inclusion $\ker\Phi \supset G_u(\ker\nu)_{F_\nu}$ is obvious. In order to show that $\ker\Phi \subset G_u(\ker\nu)_{F_\nu}$, take $[s, \widehat{q}(\xi)] \in \ker\Phi$. Since we have $[q(s), \xi] \in G_u(S/\nu)^{(0)}$ and $q(E(S)) = E(S/\nu)$, there exists $e \in E(S)$ such that $[q(s), \xi] = [q(e), \xi]$. There exists $f \in E(S)$ such that $\xi(q(f)) = 1$ and $q(s)q(f) = q(e)q(f)$. Now we have $sf \in \ker\nu$, so it follows that

$$[s, \widehat{q}(\xi)] = [sf, \widehat{q}(\xi)] \in G_u(\ker\nu)_{F_\nu}.$$

This shows that $\ker\Phi = G_u(\ker\nu)_{F_\nu}$.

By Proposition 2.1.1.10, Φ induces an isomorphism $\widetilde{\Phi}$ which makes the following diagram commutative:

$$\begin{array}{ccc} G_u(S)_{F_\nu} & \xrightarrow{\ \Phi\ } & G_u(S/\nu) \\ {\scriptstyle Q}\downarrow & \nearrow{\scriptstyle \widetilde{\Phi}} & \\ G_u(S)_{F_\nu}/G_u(\ker\nu)_{F_\nu} & & \end{array},$$

where Q denotes the quotient map. $\square$

3.2.2 Universal groupoids associated to special quotient inverse semigroups

Minimum congruences associated to normal congruences on semilattices of idempotents

Let S be an inverse semigroup. Recall that a congruence ρ on $E(S)$ is normal if $(e, f) \in \rho$ implies $(ses^*, sfs^*) \in \rho$ for all $s \in S$ and $e, f \in E(S)$. Note that one can construct the least congruence $\nu_{\rho,\min}$ whose restriction to $E(S)$ coincides with ρ. Recall that we can associate the closed invariant subset F_ρ of $G_u(S)$ as shown in Proposition 3.1.1.2.

Proposition 3.2.2.1. Let S be an inverse semigroup and ρ be a normal congruence on $E(S)$. Then $G_u(S/\nu_{\rho,\min})$ is isomorphic to $G_u(S)_{F_\rho}$.

Proof. By Theorem 3.2.1.3, it suffices to show that $G_u(\ker \nu_{\rho,\min})_{F_\rho} = G_u(S)^{(0)}_{F_\rho}$ holds. Let $q\colon S \to S/\nu_{\rho,\min}$ denote the quotient map. Take $[n, \widehat{q}(\xi)] \in G_u(\ker \nu_{\rho,\min})_{F_\rho}$, where $n \in \ker \nu_{\rho,\min}$ and $\xi \in \widehat{E(S/\rho)}$. Since $n \in \ker \nu_{\rho,\min}$, there exists $e \in E(S)$ such that $q(n) = q(e)$. By the definition of $\nu_{\rho,\min}$, there exists $f \in E(S)$ such that $nf = ef$ and $(n^*n, f) \in \rho$ hold. Observe that $\widehat{q}(\xi)(n^*n) = \xi(q(n^*n)) = \xi(q(f)) = \xi(q(e)) = 1$. We have

$$[n, \widehat{q}(\xi)] = [nf, \widehat{q}(\xi)] = [ef, \widehat{q}(\xi)] \in G_u(S)^{(0)}_{F_\rho}.$$

Now we have shown that $G_u(\ker \nu_{\rho,\min})_{F_\rho} = G_u(S)^{(0)}_{F_\rho}$. $\square$

Theorem 3.2.2.2 ([10, Theorem 4.5])**.** Let S be an inverse semigroup. Then $G_u(S^{\mathrm{Clif}})$ is isomorphic to $G_u(S)_{\mathrm{fix}}$.

Proof. Recall the definition of $\nu_{\mathrm{Clif}} = \nu_{\rho_{\mathrm{Clif}},\min}$ (see Definition 3.1.2.3). Since we have Proposition 3.2.2.1, it suffices to show $F_{\rho_{\mathrm{Clif}}} = \widehat{E(S)}_{\mathrm{fix}}$. By Lemma 3.1.2.4, we have $F_{\rho_{\mathrm{Clif}}} \subset \widehat{E(S)}_{\mathrm{fix}}$. To show the reverse inclusion, take $\xi \in \widehat{E(S)}_{\mathrm{fix}}$. By Proposition 3.1.2.2, there exists a semigroup homomorphism extension $\widetilde{\xi}\colon S \to \{0, 1\}$. Since $\{0, 1\}$ is Clifford, there exists a semigroup homomorphism $\eta\colon S^{\mathrm{Clif}} \to \{0, 1\}$ such that $\eta \circ q = \widetilde{\xi}$, where $q\colon S \to S^{\mathrm{Clif}}$ denotes the quotient map. Therefore, we have $\xi = \eta \circ q|_{E(S)} \in F_{\rho_{\mathrm{Clif}}}$. Now we have shown $F_{\rho_{\mathrm{Clif}}} = \widehat{E(S)}_{\mathrm{fix}}$. $\square$

The least commutative congruences

Let S be an inverse semigroup and ν_{ab} be the least commutative congruence (see Proposition 3.1.3.1 and Theorem 3.1.3.3). Recall that the abelianization of S is defined to be $S^{\mathrm{ab}} := S/\nu_{\mathrm{ab}}$.

Theorem 3.2.2.3 ([10, Theorem 4.6])**.** Let S be an inverse semigroup. Then $G_u(S^{\mathrm{ab}})$ is isomorphic to $G_u(S)^{\mathrm{ab}}$.

Proof. By Theorem 3.2.1.3, it suffices to show that $F_{\nu_{\mathrm{ab}}} = \widehat{E(S)}_{\mathrm{fix}}$ and $G_u(\ker \nu_{\mathrm{ab}})_{\mathrm{fix}} = [G_u(S)_{\mathrm{fix}}, G_u(S)_{\mathrm{fix}}]$ hold.

Observe that ν_{ab} is equal to ν_{Clif} on $E(S)$. Indeed, this follows from the fact that $\varphi|_{E(S)} \in \widehat{E(S)}_{\mathrm{fix}}$ holds for all $\varphi \in \widehat{S}$. Therefore, we have $F_{\nu_{\mathrm{ab}}} = \widehat{E(S)}_{\mathrm{fix}}$.

Next we show that $G_u(\ker \nu_{\mathrm{ab}})_{\mathrm{fix}} = [G_u(S)_{\mathrm{fix}}, G_u(S)_{\mathrm{fix}}]$. The inclusion

$$G_u(\ker \nu_{\mathrm{ab}})_{\mathrm{fix}} \supset [G_u(S)_{\mathrm{fix}}, G_u(S)_{\mathrm{fix}}]$$

is easy to show.

Let $q\colon S \to S^{\mathrm{ab}}$ and $q'\colon S \to S^{\mathrm{Clif}}$ denote the quotient maps. Since a commutative inverse semigroup is Clifford, there exists a semigroup homomorphism $\sigma\colon S^{\mathrm{Clif}} \to S^{\mathrm{ab}}$ such that $q = \sigma \circ q'$. To show the reverse inclusion

$$G_u(\ker \nu_{\mathrm{ab}})_{\mathrm{fix}} \subset [G_u(S)_{\mathrm{fix}}, G_u(S)_{\mathrm{fix}}],$$

take $[n, \widehat{q}(\xi)] \in G_u(\ker \nu_{\mathrm{ab}})_{\mathrm{fix}}$, where $n \in \ker \nu_{\mathrm{ab}}$ and $\xi \in \widehat{E(S^{\mathrm{ab}})}$. Since $n \in \ker \nu_{\mathrm{ab}}$, there exists $e \in E(S)$ such that $q(n) = q(e)$. Then we have $q(n^*n) = q(e)$. Since ν_{ab} coincides with ν_{Clif} on $E(S)$, it follows that $q'(n^*n) = q'(e)$. Define

$$H_{q'(e)} = \{s \in S^{\mathrm{Clif}} \mid s^*s = q'(e)\};$$

then $H_{q'(e)}$ is a group in the operation inherited from S^{Clif}. Observe that a unit of $H_{q'(e)}$ is $q'(e)$ and we have $q'(n) \in H_{q'(e)}$. Fix a group homomorphism $\chi\colon H_{q'(e)} \to \mathbb{T}$ arbitrarily. By Proposition 3.1.3.2, χ is extended to the semigroup homomorphism $\widetilde{\chi}\colon S^{\mathrm{Clif}} \to \mathbb{T} \cup \{0\}$. Since $\mathbb{T} \cup \{0\}$ is commutative, there exists a semigroup homomorphism $\overline{\chi}\colon S^{\mathrm{ab}} \to \mathbb{T} \cup \{0\}$ which makes the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{\;q'\;} & S^{\mathrm{Clif}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \widetilde{\chi}} \\ S^{\mathrm{ab}} & \xrightarrow[\;\overline{\chi}\;]{} & \mathbb{T} \cup \{0\} \end{array}.$$

Now we have

$$\chi(q'(n)) = \overline{\chi}(q(n)) = \overline{\chi}(q(e)) = \chi(q'(e)).$$

Since we take a group homomorphism $\chi\colon H_{q'(e)} \to \mathbb{T}$ arbitrarily, it follows that $q'(n) \in [H_{q'(e)}, H_{q'(e)}]$, where $[H_{q'(e)}, H_{q'(e)}]$ denotes the commutator subgroup of $H_{q'(e)}$. Therefore, there exists $s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_m \in S$ such that

$$\begin{aligned} q'(n) &= q'(s_1)q'(t_1)q'(s_1)^*q'(t_1)^* \cdots q'(s_m)q'(t_m)q'(s_m)^*q'(t_m)^* \\ &= q'(s_1t_1s_1^*t_1^* \cdots s_mt_ms_m^*t_m^*). \end{aligned}$$

By the definition of ν_{Clif}, there exists $f \in E(S)$ such that

$$nf = s_1t_1s_1^*t_1^* \cdots s_mt_ms_m^*t_m^*f$$

and $q'(n^*n) = q'(f)$ hold. Then we have

$$\begin{aligned} [n, \widehat{q}(\xi)] &= [nf, \widehat{q}(\xi)] \\ &= [s_1t_1s_1^*t_1^* \cdots s_mt_ms_m^*t_m^*f, \widehat{q}(\xi)] \\ &= [s_1t_1s_1^*t_1^* \cdots s_mt_ms_m^*t_m^*, \widehat{q}(\xi)] \in [G_u(S)_{\mathrm{fix}}, G_u(S)_{\mathrm{fix}}] \end{aligned}$$

Thus, it is shown that $G_u(\ker \nu_{\mathrm{ab}})_{\mathrm{fix}} = [G_u(S)_{\mathrm{fix}}, G_u(S)_{\mathrm{fix}}]$. □

3.3 Applications and examples

3.3.1 Clifford inverse semigroups from the view point of fixed points

A 0-group is an inverse semigroup isomorphic to $\Gamma \cup \{0\}$ for some group Γ. For a group Γ, we denote the 0-group associated to Γ by $\Gamma^0 := \Gamma \amalg \{0\}$. It is clear that every 0-group is a Clifford inverse semigroup. Conversely, we see that every Clifford inverse semigroup is embedded into a direct product of 0-groups. Remark that this fact is already known (see [16, Theorem 2.6]). Using fixed characters, we obtain a new proof.

Let S be a Clifford inverse semigroup and $\xi \in \widehat{E(S)}$. Since $\{\xi\} \subset \widehat{E(S)}$ is invariant by Lemma 3.1.2.4, we may consider a normal congruence $\rho_\xi := \rho_{\{\xi\}}$ on $E(S)$ and a congruence $\nu_\xi := \nu_{\rho_{\{\xi\}},\min}$ on S. If $\xi = 1$, ρ_ξ coincides with $E(S) \times E(S)$ and S/ν_ξ is the maximal group image of S. We define $S(\xi) := \{q_\xi(s) \in S/\nu_\xi \mid \xi(s^*s) = 1\}$, where $q_\xi \colon S \to S/\nu_\xi$ is the quotient map. Then $S(\xi)$ is a group.

Define the map $\varphi_\xi \colon S \to S(\xi)^0$ by

$$\varphi_\xi(s) := \begin{cases} q_\xi(s) & (\xi(s^*s) = 1), \\ 0 & (\xi(s^*s) = 0). \end{cases} \tag{3.1}$$

Then φ_ξ is a semigroup homomorphism.

Proposition 3.3.1.1. Let S be a Clifford inverse semigroup. Then the semigroup homomorphism

$$\Phi \colon S \ni s \mapsto (\varphi_\xi(s))_{\xi \in \widehat{E(S)}} \in \prod_{\xi \in \widehat{E(S)}} S(\xi)^0$$

is injective. In particular, every Clifford inverse semigroup is embedded into a direct product of 0-groups.

Proof. Assume that $s, t \in S$ satisfy $\Phi(s) = \Phi(t)$. Since we have $\varphi_\xi(s^*s) = \varphi_\xi(t^*t)$ for all $\xi \in \widehat{E(S)}$, it follows that $\xi(s^*s) = \xi(t^*t)$ for all $\xi \in \widehat{E(S)}$. Therefore, we obtain $s^*s = t^*t$.

Define $\xi_{s^*s} \in \widehat{E(S)}$ by

$$\xi_{s^*s}(e) := \begin{cases} 1 & (e \geq s^*s), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $\xi_{s^*s}(s^*s) = \xi_{s^*s}(t^*t) = 1$. Combining with $\varphi_{\xi_{s^*s}}(s) = \varphi_{\xi_{s^*s}}(t)$, we obtain $q_{\xi_{s^*s}}(s) = q_{\xi_{s^*s}}(t)$. Therefore, there exists $e \in E(S)$ such that $\xi_{s^*s}(e) = \xi_{s^*s}(s^*s) = 1$ and $se = te$. It follows that $e \geq s^*s(= t^*t)$ from $\xi_{s^*s}(e) = 1$. Thus, we have shown that $s = t$ and Φ is injective. □

Proposition 3.3.1.2. Let S be a finitely generated Clifford inverse semigroup. Then $\widehat{E(S)}$ is a finite set. More precisely, if S is generated by a finite set $F \subset S$, then $|\widehat{E(S)}|$ is less than or equal to $2^{|F|}$, where $|A|$ denotes the number of elements in a finite set A.

Proof. Take a finite set $F \subset S$ which generates S. Let X denote the set of all nonzero semigroup homomorphisms from S to $\{0, 1\}$. Then the map

$$X \ni \xi \mapsto (\xi(f))_{f \in F} \in \{0, 1\}^F$$

is injective since F generates S. By Proposition 3.1.2.2 and Lemma 3.1.2.4, the map $X \ni \xi \mapsto \xi|_{E(S)} \in \widehat{E(S)}$ is bijective. Since $\widehat{E(S)}$ is embedded into $\{0, 1\}^F$, $\widehat{E(S)}$ is a finite set. □

Corollary 3.3.1.3. Let S be a finitely generated Clifford inverse semigroup. Then S is embedded into a direct sum of finitely many 0-groups.

Let S be a Clifford inverse semigroup and $\xi \in \widehat{E(S)}$. Recall that $G_u(S)_\xi$ is a discrete group. Then $G_u(S)_\xi$ can be computed as the following.

Proposition 3.3.1.4. Let S be a Clifford inverse semigroup and $\xi \in \widehat{E(S)}$. Then $G_u(S)_\xi$ is isomorphic to $S(\xi)$.

Proof. Let $\varphi_\xi \colon S \to S(\xi)^0$ denote the map in Proposition 3.3.1.1. We show that a map defined by

$$\sigma \colon S(\xi) \ni \varphi_\xi(s) \mapsto [s, \xi] \in G_u(S)_\xi$$

is actually well-defined and an isomorphism. First we check that $\varphi_\xi(s) = \varphi_\xi(t)$ is equivalent to $[s, \xi] = [t, \xi]$ for all $s, t \in S$ with $\xi(s^*s) = \xi(t^*t) = 1$. Assume that $s, t \in S$ with $\xi(s^*s) = \xi(t^*t) = 1$ satisfies $\varphi_\xi(s) = \varphi_\xi(t)$. Then there exists $e \in E(S)$ such that $se = te$

and $\xi(e) = \xi(s^*s)(= 1)$ hold by the definition of ν_ξ. Hence, it follows that $[s, \xi] = [t, \xi]$. One can see that $[s, \xi] = [t, \xi]$ implies $\varphi_\xi(s) = \varphi_\xi(t) \in S(\xi)$ by a similar argument. Thus, the map σ is actually well-defined and injective. It is easy to show that σ is a group homomorphism and surjective. Therefore, σ is an isomorphism. □

Fix a character $\xi \in \widehat{E(S)}$. We compute $S(\xi)$ here. Note that $\xi^{-1}(\{1\})$ is a directed set with respect to the order inherited from $E(S)$. For $e \in E(S)$, define $S(e) := \{s \in S \mid s^*s = e\}$. Then $S(e)$ is a group. For $e, f \in E(S)$ with $e \leq f$, define a map $\varphi_e^f \colon S(f) \to S(e)$ by $\varphi_e^f(s) = se$ for $s \in S(f)$. Then φ_e^f is a group homomorphism. One can see that $(S(e), \varphi_e^f)$ is an inductive system of groups.

Proposition 3.3.1.5. Let S be a Clifford inverse semigroup and $\xi \in \widehat{E(S)}$. Then we have the following isomorphism:

$$\varinjlim_{\xi(e)=1} S(e) \simeq S(\xi).$$

PROOF. Let $\varphi_\xi \colon S \to S(\xi)^0$ denote the map in Proposition 3.3.1.1 and put $\Gamma := \varinjlim S(e)$. For $e \in E(S)$ with $\xi(e) = 1$, we define $\sigma_e \colon S(e) \to S(\xi)$ by $\sigma_e(s) := \varphi_\xi(s)$. We obtain a group homomorphism $\widetilde{\sigma} \colon \Gamma \to S(\xi)$. One can see that $\widetilde{\sigma}$ is an isomorphism. □

Combining Propositions 3.3.1.4 and 3.3.1.5, we obtain a proof of the next corollary, which was already proved in [12].

Corollary 3.3.1.6 ([12, Theorem 3.1])**.** Let S be a Clifford inverse semigroup and $\xi \in \widehat{E(S)}$. Then there exists an isomorphism

$$G_u(S)_\xi \simeq \varinjlim_{\xi(e)=1} S(e).$$

Let I be a discrete set and $\{\Gamma_i\}_{i \in I}$ be a family of discrete groups. Then the disjoint union $\coprod_{i \in I} \Gamma_i$ is a discrete group bundle over I in the natural way. Using Propositions 3.3.1.2 and 3.3.1.4, we obtain the next corollary.

Corollary 3.3.1.7. Let S be a finitely generated Clifford inverse semigroup. Then there exists an isomorphism

$$G_u(S) \simeq \coprod_{\xi \in \widehat{E(S)}} S(\xi).$$

For an étale groupoid G with the locally compact Hausdorff unit space $G^{(0)}$, we write $C^*(G)$ (resp. $C^*_\lambda(G)$) to represent the universal (resp. reduced) groupoid C*-algebra of G (see Chapter 1 or [14] for the definitions). Corollary 3.3.1.7 immediately implies the next corollary.

Corollary 3.3.1.8. Let S be a finitely generated Clifford inverse semigroup. Then we have isomorphisms

$$C^*(G_u(S)) \simeq \bigoplus_{\xi \in \widehat{E(S)}} C^*(S(\xi)),\ C^*_\lambda(G_u(S)) \simeq \bigoplus_{\xi \in \widehat{E(S)}} C^*_\lambda(S(\xi)).$$

3.3.2 Free Clifford inverse semigroups

We investigate universal groupoids and C*-algebras associated to free Clifford inverse semigroups on finite sets.

First, we recall the definition of the free groups.

Definition 3.3.2.1. Let X be a set. A free group on X is a pair $(\mathbb{F}(X), \kappa)$ consisting of a group $\mathbb{F}(X)$ and a map $\kappa\colon X \to \mathbb{F}(X)$ such that:

1. $\kappa(X)$ generates $\mathbb{F}(X)$ as a group; and
2. for every group Γ and a map $\varphi\colon X \to \Gamma$, there exists a group homomorphism $\widetilde{\varphi}\colon \mathbb{F}(X) \to \Gamma$ such that $\varphi(x) = \widetilde{\varphi}(\kappa(x))$ holds for all $x \in X$.

We define free inverse semigroups in a similar way.

Definition 3.3.2.2. Let X be a set. A free inverse semigroup on X is a pair $(\mathrm{FIS}(X), \iota)$ consisting of an inverse semigroup $\mathrm{FIS}(X)$ and a map $\iota\colon X \to \mathrm{FIS}(X)$ such that:

1. $\iota(X)$ generates $\mathrm{FIS}(X)$ as an inverse semigroup; and
2. for every inverse semigroup T and map $\varphi\colon X \to T$, there exists a semigroup homomorphism $\widetilde{\varphi}\colon \mathrm{FIS}(X) \to T$ such that $\varphi(x) = \widetilde{\varphi}(\iota(x))$ holds for all $x \in X$.

It is known that free inverse semigroups exist and are unique up to isomorphism. See [13, Section 6.1] for the existence of free inverse semigroups. The uniqueness is obvious.

Definition 3.3.2.3. A free Clifford inverse semigroup on X is a pair $(\mathrm{FCIS}(X), \iota)$ consisting of a Clifford inverse semigroup $\mathrm{FCIS}(X)$ and a map $\iota\colon X \to \mathrm{FCIS}(X)$ such that:

1. $\iota(X)$ generates $\mathrm{FCIS}(X)$ as an inverse semigroup; and
2. for every Clifford inverse semigroup T and map $\varphi\colon X \to T$, there exists a semigroup homomorphism $\widetilde{\varphi}\colon \mathrm{FCIS}(X) \to T$ such that $\varphi(x) = \widetilde{\varphi}(\iota(x))$ holds for all $x \in X$.

Free Clifford inverse semigroups exist and are unique up to isomorphism. Indeed, for a free inverse semigroup $(\mathrm{FIS}(X), \iota)$ and the quotient map $q\colon \mathrm{FIS}(X) \to \mathrm{FIS}(X)^{\mathrm{Clif}}$, one can see that $(\mathrm{FIS}(X)^{\mathrm{Clif}}, q \circ \iota)$ is a free Clifford inverse semigroup on X. The uniqueness is obvious.

Let X be a set. For $A \subset X$, define a map $\chi_A \colon X \to \{0,1\}$ by

$$\chi_A(x) = \begin{cases} 1 & (x \in A), \\ 0 & (x \notin A). \end{cases}$$

Since $\{0,1\}$ is Clifford, χ_A can be extended to the semigroup homomorphism from $\mathrm{FCIS}(X)$ to $\{0,1\}$, which we also denote by χ_A. Every semigroup homomorphism from $\mathrm{FCIS}(X)$ to $\{0,1\}$ is of the form χ_A for a unique $A \subset X$.

By Proposition 3.1.2.2, $\chi_A|_{E(\mathrm{FCIS}(X))}$ is a fixed character if A is not empty. By Lemma 3.1.2.4, all characters on $E(\mathrm{FCIS}(X))$ are fixed characters. Therefore we obtain the next proposition.

Proposition 3.3.2.4. Let X be a finite set. Put $S = \mathrm{FCIS}(X)$. Then the map

$$P(X) \setminus \{\emptyset\} \ni A \mapsto \chi_A|_{E(S)} \in \widehat{E(S)}$$

is bijective, where $P(X)$ denotes the power set of X.

We identify $\chi_A|_{E(\mathrm{FCIS}(X))}$ with χ_A since we can recover χ_A from the restriction $\chi_A|_{E(\mathrm{FCIS}(X))}$.

For a nonempty set $A \subset X$, define $e_A := \prod_{x \in A} \iota(x)^*\iota(x) \in E(\mathrm{FCIS}(X))$. For $e \in E(\mathrm{FCIS}(X))$, the condition that $\chi_A(e) = 1$ is equivalent to the condition that $e \geq e_A$. Using this fact, one can prove the next proposition.

Proposition 3.3.2.5. The map

$$P(X) \setminus \{\emptyset\} \ni A \mapsto e_A \in E(\mathrm{FCIS}(X))$$

is bijective.

In order to apply Proposition 3.3.1.4 for free Clifford inverse semigroups, we prepare the next proposition.

Proposition 3.3.2.6. Let X be a set and $A \subset X$ be a nonempty set. Put $S = \mathrm{FCIS}(X)$. Then $S(\chi_A)$ is isomorphic to the free group $\mathbb{F}(A)$ generated by A.

Proof. If $X = A$, $S(\chi_A)$ is the maximal group image of S. Therefore, $S(\chi_A)$ is isomorphic to $\mathbb{F}(A)$.

We assume $A \subsetneq X$. Let $\varphi_A \colon S \to S(\chi_A)^0$ denote the map defined by

$$\varphi_A(s) = \begin{cases} Q(s) & (\chi_A(s^*s) = 1), \\ 0 & (\chi_A(s^*s) = 0), \end{cases}$$

where $Q\colon S \to S/\nu_{\chi_A}$ denotes the quotient map. By the universality of $\mathbb{F}(A)$, define a group homomorphism $\tau\colon \mathbb{F}(A) \to S(\chi_A)$ such that $\tau(\kappa(a)) = \varphi_A(\iota(a))$ for all $a \in A$. We construct the inverse map of τ. Using the universality of $S = \mathrm{FCIS}(X)$, define a semigroup homomorphism $\sigma\colon S \to \mathbb{F}(A)^0$ which satisfies

$$\sigma(\iota(x)) = \begin{cases} \kappa(x) & (x \in A) \\ 0 & (x \notin A) \end{cases}$$

for $x \in X$. We claim that $(s,t) \in \nu_{\chi_A}$ implies $\sigma(s) = \sigma(t)$ for $s, t \in S$. If $\chi_A(s^*s) = 0$, we have $\sigma(s) = \sigma(t) = 0$. We may assume $\chi_A(s^*s) = 1$. By $(s,t) \in \nu_{\chi_A}$, we have $se_A = te_A$. Since $\sigma(e_A)$ is the unit of $\mathbb{F}(A)$, we have $\sigma(s) = \sigma(t)$. Therefore, we obtain a semigroup homomorphism $\widetilde{\sigma}\colon S(\chi_A)^0 \to \mathbb{F}(A)^0$ which makes the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{\ \sigma\ } & \mathbb{F}(A)^0 \\ {\scriptstyle \varphi_A}\downarrow & \nearrow {\scriptstyle \widetilde{\sigma}} & \\ S(\chi_A)^0 & & \end{array}.$$

Now one can verify that $\widetilde{\sigma}|_{S(\chi_A)}$ is the inverse map of τ. □

Now we have the following Theorem.

Theorem 3.3.2.7 ([10, Theorem 5.15])**.** Let X be a finite set. Then there exists an isomorphism

$$G_u(\mathrm{FCIS}(X)) \simeq \coprod_{A \in P(X) \setminus \{\emptyset\}} \mathbb{F}(A).$$

Proof. Put $S = \mathrm{FCIS}(X)$. By Proposition 3.3.2.4, it follows that

$$\widehat{E(S)} = \{\chi_A \in \widehat{E(S)} \mid A \in P(X) \setminus \{\emptyset\}\}$$

is a finite set. Therefore, we have an isomorphism

$$G_u(S) \simeq \coprod_{A \in P(X) \setminus \{\emptyset\}} G_u(S)_{\chi_A}.$$

By Proposition 3.3.2.6, we obtain the isomorphism in the statement. □

3.3.3 Fixed points of Boolean actions

From [21, Section 5], we recall the notion of Boolean actions. By a locally compact Boolean space, we mean a locally compact Hausdorff space which admits a basis of compact open sets. Let S be an inverse semigroup and X be a locally compact Boolean space. An action $\alpha\colon S \curvearrowright X$ is said to be Boolean if

1. for all $e \in E(S)$, $D_e^\alpha \subset X$ is a compact open set of X; and
2. the family

$$\left\{ D_e^\alpha \cap \bigcap_{f \in P} (X \setminus D_f^\alpha) \mid e \in E(S), P \subset E(S) \text{ is a finite set.} \right\}$$

forms a basis of X.

It is known that $G_u(S)$ has the following universal property for Boolean actions.

Theorem 3.3.3.1 ([21, Proposition 5.5])**.** Let S be an inverse semigroup, X be a Boolean space and $\alpha\colon S \curvearrowright X$ be a Boolean action. Then $S \ltimes_\alpha X$ is isomorphic to $G_u(S)_F$ for some closed invariant subset $F \subset \widehat{E(S)}$.

Corollary 3.3.3.2. Let S be a finitely generated inverse semigroup and $\alpha\colon S \curvearrowright X$ be a Boolean action. Then α has finitely many fixed points. More precisely, the number of fixed points of α is less than or equal to the number of nonzero semigroup homomorphisms from S to $\{0, 1\}$.

Proof. Since we assume that S is finitely generated, the set of all nonzero semigroup homomorphisms from S to $\{0, 1\}$ is a finite set. By Proposition 3.1.2.2, there exists a bijection between the set of all nonzero semigroup homomorphisms from S to $\{0, 1\}$ and $\widehat{E(S)}_{\mathrm{fix}}$. Now Theorem 3.3.3.1 completes the proof. □

Example 3.3.3.3 (cf. [14, Example 3 in Section 4.2])**.** For a natural number $n \in \mathbb{N}$ with $n \geq 2$, the polycyclic inverse monoid P_n is an inverse semigroup which is generated by $\{0, 1, s_1, \dots, s_n\}$ with the relation

$$s_i^* s_j = \delta_{i,j} 1$$

for all $i, j \in \{1, 2, \dots, n\}$. Define $\xi\colon P_n \to \{0, 1\}$ by $\xi(x) = 1$ for all $x \in P_n$. Then ξ is the unique nonzero semigroup homomorphism from P_n to $\{0, 1\}$. Since $0 \in P_n$, ξ is an isolated point of $\widehat{E(P_n)}$. Therefore, every Boolean action of P_n has at most one fixed point, which becomes an isolated point.

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