Pattern space: a general framework for tilings, Delone sets, functions and measures to discuss their interrelations, repetitivity and corresponding dynamical systems

March 2017

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A Thesis for the Degree of Ph.D. in Science

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March 2017

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Chapter 1

Introduction

In this thesis we investigate objects such as tilings, Delone sets, functions and measures. In particular, we discuss the following two topics: (1) we study the distribution of configurations inside these objects; (2) we study the almost periodicity of these objects. The following two sections explain each of these two topics in detail.

1.1 Distribution of configurations inside objects such as tilings

To discuss this topic let us start with symbolic dynamics. Let $\{a,b\}$ be a two-point set. The space $\{a,b\}^{\mathbb{Z}}$ is considered as a space of words (sequences). The elements of this space are represented as $(w_i)_{i\in\mathbb{Z}}$, where $w_i \in \{a,b\}$ is the ith coordinate. We define $\sigma\colon \{a,b\}^{\mathbb{Z}} \to \{a,b\}^{\mathbb{Z}}$ by $\sigma((w_i))_i = w_{i+1}$. In other words, the map σ shifts a word to the left. Closed subspaces X of $\{a,b\}^{\mathbb{Z}}$ which is invariant under σ are called subshifts.

Now consider the following three subshifts. First, $\Omega_B = \{a,b\}^{\mathbb{Z}}$ itself is a subshift. If we take $w \in \Omega_B$, $n,m \in \mathbb{Z}_{>0}$ arbitrarily, we can say nothing about $w_{n+1}w_{n+2}\cdots w_{n+m}$ from information of $w_1w_2\cdots w_m$ if n is large.

Second, if the subshift is Ω_{w_0}, the situation is opposite. Here, $w_0 \in \{a,b\}^{\mathbb{Z}}$ is a periodic one, that is, there is $m > 0$ such that $\sigma^m(w_0) = w_0$. Ω_{w_0} is the space of all the shifts of w_0, namely $\Omega_{w_0} = \{\sigma^k(w_0) \mid k \in \mathbb{Z}\}$. In this case, if we take $w \in \Omega_{w_0}$ and $n \in \mathbb{Z}$ arbitrarily, we can predict perfectly what is $w_{n+1}w_{n+2}\cdots w_{n+m}$ from $w_1w_2\cdots w_m$.

Third, if the subshift is the one Ω_{MT} from Morse-Thue substituiton, the situation is in between the above two extreme cases. The words in this subshift are non-periodic and we cannot perfectly predict what happens in one part of such a word from information on another part. However, we can "sometimes predict the behavior in another part a little". In fact, if $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in \{a,b\}$, the finite word $x_1x_2\cdots x_n$ never repeats three times consecutively. Namely, if $w \in \Omega_{MT}, m \in \mathbb{Z}$ and $w_mw_{m+1}\cdots w_{m+n-1} = w_{m+n}w_{m+n+1}\cdots w_{m+2n-1} = x_1x_2\cdots x_n$, then $w_{m+2n}w_{m+2n+1}\cdots w_{m+3n-1} \neq x_1x_2\cdots x_n$. The two-times consecutive appearance of a finite word gives us the information of the non-existence for the third appearance.

Now, tilings are geometric analogues for words. In this thesis we first ask if, for non-periodic tilings, we can predict partially what happens in such a tiling in the distance from what happens in one part of the tiling.

Let us informally define tilings. A collection of subsets of $\mathbb{R}^d$ that are called tiles, such as polygons, that intersect only on their boundaries is called a patch. If a patch covers the whole space $\mathbb{R}^d$, then it is called a tiling. For example, set $d = 2$ and take a square I of side-length 1. The collection $\mathcal{T}_S = \{I + x \mid x \in \mathbb{Z}^2\}$ is an example of tiling. This is crystallographic [1]: in general, a tiling $\mathcal{T}$ of $\mathbb{R}^d$ is said to be crystallographic if there is a basis $\mathcal{B}$ of $\mathbb{R}^d$ such that $\mathcal{T} + x = \mathcal{T}$ for any $x \in \mathcal{B}$ ($\mathcal{T} + x$ is the shift of the tiling $\mathcal{T}$ by the vector x).

As in the case of words, we call a closed space consisting of tilings that is invariant under the $\mathbb{R}^d$-action by translation a subshift. Again let us consider three subshifts.

First, set $d = 2$ and consider two squares I_B, I_W of side-length 1, one black and one white. These I_B and I_W have the same shape but are distinguished. Just as we constructed the periodic tiling $\mathcal{T}_S$ above, we juxtapose these I_B and I_W in a grid, so that we obtain tilings, but in this case with arbitrary arrangement of colors. In other words, we consider all the tilings that are obtained by painting tiles in $\mathcal{T}_S$ in a random way. Collecting all the shifts of all such tilings, we obtain a subshift X_B, which is similar to Ω_B given above. If we take $\mathcal{T} \in X_B$ and $x \in \mathbb{R}^d$ arbitrarily, we can say little about what happens in $\mathcal{T}$ around the point x from the knowledge of what happens around the origin 0: we know where the vertices of the squares are, but can tell nothing about colors.

Second, consider the tiling $\mathcal{T}_S$ given above and take the subshift $X_{\mathcal{T}_S} = \{\mathcal{T}_S + x \mid x \in \mathbb{R}^d\}$. If we pick $\mathcal{S} \in X_{\mathcal{T}_S}$ and $x \in \mathbb{R}^d$ arbitrarily, we can perfectly predict what happens around the point x in $\mathcal{S}$ from information of what happens around the origin 0.

Third, consider the subshift of all Penrose tilings, which lies in between these two extreme cases. Penrose tilings are discovered by Penrose in 1970s. These are constructed by juxtaposing two rhombi with local matching rules (see Figure 1.1). These are not periodic: if $\mathcal{T}_P$ is a Penrose tiling and $x \neq 0$, then $\mathcal{T}_P + x \neq \mathcal{T}_P$. However, the arrangement of tiles in $\mathcal{T}_P$ is not completely random: for example, the corresponding dynamical system for Penrose tilings is not mixing (Theorem 2.2.41). Here, the corresponding dynamical system for a general tiling $\mathcal{T}$ is obtained by taking the closure of the orbit $\{\mathcal{T} + x \mid x \in \mathbb{R}^d\}$ with respect to a "local" topology. The group $\mathbb{R}^d$ acts on this closure and for many examples, including Penrose tilings, this topological dynamical system is uniquely ergodic, so that we can discuss their mixing property.

In this thesis we first discuss non-existence, just as for Morse-Thue words, for non-periodic tilings such as Penrose tilings. In Theorem 2.3.6, we show for a certain (FLC, repetitive and FTT) tiling $\mathcal{T}$, a condition on the corresponding dynamical system is equivalent to a condition on non-existence of patches. In other words, we show the following

[1] some authors call crystallographic tilings periodic tilings or completely periodic tilings.

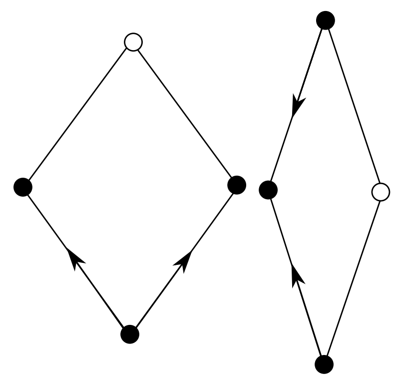

Figure 1.1: Two rhombi to construct Penrose tilings. We juxtapose them in such a way that (1)vertices meet vertices with the same color and (2)edges with arrow meet edges with arrow of the same direction.

two conditions are equivalent:

1. $0 \in \mathbb{R}^d$ is a limit point of the set of topological eigenvalues for the corresponding dynamical system.
2. for any $R_1, R_2 > 0$ and $\varepsilon > 0$, there are $L_1, L_2 > 0$ such that
   (a) $|L_j - R_j| < \varepsilon$ for each $j = 1, 2$, and
   (b) $\mathcal{T}$ has (L_1, L_2)-stripe structure (Definition 2.3.4).

The first condition is on the dynamical system and the second is on the non-existence of patches in the tiling. In plain language, a tiling $\mathcal{T}$ has (L_1, L_2)-stripe structure if, whenever we take $\mathcal{S} \in X_{\mathcal{T}}$, we know translates of a large patch of $\mathcal{S}$ around the origin never appear in $\mathcal{S}$ in a periodic "forbidden area", which is obtained by juxtaposing "bands" of width $2L_2$ with interval L_1 (see Figure 2.2 in page 41).

1.2 A general framework for almost periodic objects such as tilings, Delone sets, functions and measures

The appearance of periodic region in the above result suggests that such tilings are "close to" periodic in a sense. The second topic of this thesis is almost periodicity of objects such as tilings, Delone sets, functions and measures.

To discuss this second topic, let us begin with a historical remark. Crystallographic tilings have been analyzed since a long time ago. First let the dimension be 2 and consider the crystallographic tiling $\mathcal{T}_S$ constructed above. In addition to translations, the tiling $\mathcal{T}_S$

has symmetry by rotations and flips. The symmetry group of $\mathcal{T}_S$ consists of translations, $n\pi/2$-rotations, where $n = 1, 2, 3$, flips, and their compositions. It has been known since a long time ago that there are only 17 isomorphic classes of symmetry groups of crystallographic tilings. Hilbert's 18th problem asked if there are only finitely many such groups in general dimensions. Bieberbach answered this problem affirmatively (see for example [24], Theorem 7.5.3).

The obvious next step of Bieberbach's work is to replace the group of isometries of $\mathbb{R}^d$ with another Lie group and study its lattices. Here, we instead replace the crystallographic tilings with non-periodic tilings that are almost periodic in several senses.

For example, take Penrose tilings given above. As pointed out above, any Penrose tiling $\mathcal{T}_P$ is not periodic. However, $\mathcal{T}_P$ is almost periodic, in the sense that any finite patches that appear in $\mathcal{T}_P$ appear infinitely often in $\mathcal{T}_P$ with bounded gaps. We call tilings with this property weakly repetitive tilings.

In general, if a topological group Γ acts on a metric space (Ω, ρ), a point $\omega \in \Omega$ is Bohr almost periodic if for any $\varepsilon > 0$ the set

$$S_\varepsilon = \{\gamma \in \Gamma \mid \rho(\omega, \gamma^{-1}\omega) < \varepsilon\}$$

admits a compact set $K \subset \Gamma$ such that $S_\varepsilon K = \Gamma$. Weak repetitivity of a tiling is a type of Bohr almost periodicity where Ω is a space of tilings with a "local" metric on which $\Gamma = \mathbb{R}^d$ acts by translation.

Besides tilings, there are several objects that exhibit almost periodicity. Many functions $f\colon \mathbb{R} \to \mathbb{C}$ are Bohr almost periodic: for example, $f(x) = \sin(x) + \sin(\sqrt{2}x)$ is an example of Bohr almost periodic functions. Here, Ω above is in this case the space of all uniformly continuous bounded complex-valued functions on $\mathbb{R}$, on which $\mathbb{R}$ acts by translation. We consider the metric from sup-norm on this space. Note that this f is non-periodic, that is, $f = {}_tf$ only if $t = 0$ (${}_tf(s) = f(s - t)$).

Likewise, certain discrete and closed subsets of $\mathbb{R}^d$ exhibit almost periodicity. For example, model sets are almost periodic with respect to the autocorrelation topology ([16]).

Such almost periodicity plays an important role in aperiodic order, a branch of mathematics which studies objects that are not periodic but are "close to" periodic in some senses, especially in connection with quasicrystals. First, the repetitivity of certain (FLC and FTT) tilings is equivalent to the minimality of the corresponding tiling dynamical system (Proposition 2.1.63). Second, Gouéré [8] proved that, for certain Delone sets, having a type of almost periodicity is equivalent to being pure point diffractive. Here, being pure point diffractive is important in connection with the study of quasicrystals. As to the relations between almost periodicity and pure point diffraction, [29] and [3] are also important. Third, by Baake and Moody [3] and Moody and Strungaru [17], we see that, for certain weighted Dirac combs, having another type of almost periodicity is equivalent to having higher-dimensional periodic structure behind them, that is, being constructed from a cut and project scheme. (See also [32].)

In the context above it is natural to try to understand almost periodicity. A classification of almost periodic structures is an ultimate goal. For example, as a next step from Bieberbach's work, one can ask whether there are only finitely many almost periodic structures, if we restrict the subject of study to a class of almost periodic objects. In order to achieve this goal one have to *define* almost periodic structure and study in a systematic way the almost periodic structures, which, as examples, include tilings, Delone sets, functions and measures. A framework for these objects is obtained by extracting the essence of the theory of tilings. In the theory of tilings, the operation of "cutting off" of a tiling $\mathcal{T}$ by a set $C \subset \mathbb{R}^d$ is important; we "cut off" $\mathcal{T}$ by C by forgetting the tiles in $\mathcal{T}$ which do not lie inside C. We axiomatize the properties that the operation of "cutting off" should have and several objects such as tilings, Delone sets, functions and measures are captured by this axiom. These objects are called abstract patterns. The spaces of abstract patterns are called pattern spaces.

The axiom is sufficient to define local matching uniform structures. In the literature, "local" metrics for the space of tilings or the space of Delone sets are defined and used. This is the topology by which we take the closure when we construct tiling dynamical systems above. With respect to this metric, two tilings are "close" if they coincide inside a large region after a small translation. All the structure we need to define this metric is the $\mathbb{R}^d$ action on the space of patches by translation and the operation of "cutting off"; thus we can define similar "local" metric (or uniform structure) for any pattern spaces. Weak repetitivity can be defined as the Bohr almost periodicity with respect to this metric. Thus one type of almost periodicity is captured in the framework of pattern space.

This axiom is also sufficient to define "locally drivable (LD)" and "mutually locally derivable (MLD)" between two abstract patterns. LD and MLD are originally defined by Baake, Schlottmann and Jarvis ([4]) for tilings (or more generally patterns). Two tilings $\mathcal{P}, \mathcal{Q}$ that are MLD are "similar" and the distribution of finite patches in $\mathcal{P}$ is the same as that of $\mathcal{Q}$. We generalize this concept and make it applicable to any two abstract patterns. Moreover, we give an affirmative answer to the following problem under a mild assumption:

Problem 1. There are several canonical maps, such as

1. the map that sends a Delone set D in a metric space X to a positive measure $\sum_{x \in D} \delta_x$, where δ_x is the Dirac measure at a point x,
2. the map that sends a continuous bounded function f on a locally compact abelian group G to a measure $f d\mu$, where μ is a Haar measure,

and so on. Do these map send an object $\mathcal{P}$ to a one which is MLD with $\mathcal{P}$?

See Proposition 3.2.23, Proposition 3.2.24, Proposition 3.2.25, and Proposition 3.2.31. These show our generalized MLD is a natural concept. This LD and MLD are relevant in the study of almost periodicity because we can show weak repetitivity is propagated by LD (Proposition 4.2.11).

Next, as to LD and MLD, we also answer the following question:

Problem 2. For an abstract pattern $\mathcal{P}$ and an interesting class Σ of abstract patterns, can we describe a condition on $\mathcal{P}$ and Σ that assures that there is $\mathcal{Q} \in \Sigma$ which is MLD with $\mathcal{P}$?

See Theorem 3.3.1. There we describe a condition on $\mathcal{P}$ and a one on Σ (not on the relations between $\mathcal{P}$ and Σ) that assures that there is a $\mathcal{Q} \in \Sigma$ which is MLD with $\mathcal{P}$. The conditions are mild enough so that many interesting examples satisfy them. This Theorem 3.3.1 enables us to "translate" an object $\mathcal{P}$ to an object $\mathcal{Q}$ in another class Σ of objects so that we can use tools that can only be applicable to objects in Σ. Moreover, by this theorem we see that, in order to study abstract patterns up to MLD, in many interesting cases, it suffices to study Delone sets or translation bounded measures. For example, we can show results on non-existence for Delone sets (Lemma 4.3.5 and Lemma 4.3.12). The theorem on non-existence for tilings given above is easily deduced from these results on Delone sets by translating tilings to Delone sets by using Theorem 3.3.1.

As an application of this Theorem 3.3.1, we study pattern equivariant functions, which were defined by Kellendonk [10] and generalized by Rand [23]. We first show that pattern equivariant functions for an object $\mathcal{P}$ are the functions that are LD from $\mathcal{P}$. In other words, we can capture pattern equivariant functions in terms of LD and this simplifies the study of pattern equivariant functions. Next, we show that two objects $\mathcal{P}$ and $\mathcal{Q}$ are MLD if and only if the spaces of the pattern-equivariant functions are the same, under a mild condition. The space of pattern equivariant functions has all the information of the original object up to MLD; in order to analyze certain abstract patterns up to MLD, it suffices to investigate its space of pattern equivariant functions.

Before finishing this introduction let us give a remark. In order to capture other types of almost periodicity (strong and weak almost periodicity for functions, strong, weak, sup and norm almost periodicity for translation bounded measures, and so on), we need an additional structure on pattern spaces. We need information on the "local structures": for example, the local structures of a function are given by the value of the function on each point; the local structure of a Dolone set at a point x is described by the position of the point in D near x relative to x; the local structures of certain tilings are described by elements of Anderson-Putnam complex ([1]). If we can gauge the distances between two local structures, we may define other types of almost periodicity. For example, the distance of two local structures for a function is gauged by the standard metric on $\mathbb{C}$. By this distance we can say two parts of a function are "close" or not, and thus we can define usual strong almost periodicity. However we do not deal with such local structures in this thesis and study only weak repetitivity. We leave the study of other almost periodicities for further research. (See Chapter 5.)

This thesis is organized as follows. In Chapter 2 we follow [20] and give an introductory exposition on the theory of tilings, their continuous hulls and tiling dynamical systems. The argument is based on works by several authors such as Solomyak([27], [28], [31]), Lee-Solomyak([14], [13]) and Robinson ([25]). In Section 2.1 we start from the definition of tiling and introduce their continuous hulls and tiling dynamical systems, followed by an explanation of important concepts such as FLC and repetitivity. In Section 2.2 we introduce substitution rules. Properties such as the non-periodicity and the repetitivity of tilings such as Penrose tilings are proved by their self-similar structure. Such structures are induced by (tiling) substitution rules, which are geometric versions of word substitutions in symbolic dynamics (for word substitution, see a book [21]). We explain important properties of tilings from substitutions.

In Chapter 3 we give a general framework for tilings, Delone sets, functions and measures to discuss local derivability among them, their weak repetitivity and corresponding dynamical systems. In Section 3.1 we define pattern spaces, by which we can capture several space of objects such as tilings, Delone sets, functions and measures. In Section 3.2 we incorporate group actions in the theory of pattern spaces. This enables us to generalize local derivability (LD) and mutual local derivability (MLD). In Section 3.3 we show Theorem 3.3.1, which answer Problem 2 given above. In Section 3.4 we discuss an application of these theory to the theory of pattern equivariant functions.

In Chapter 4 we define local matching topology and the dynamical system which corresponds to a general abstract pattern. This is a generalization of the local matching topology and the dynamical systems for tilings given in Chapter 2. We show that weak repetitivity is captured in terms of almost periodicity (Lemma 4.2.8) and is propagated by local derivability (Proposition 4.2.11). In Section 4.3 we discuss a theorem on non-existence of abstract patterns in certain abstract patterns.

We will finish the thesis with appendices on dynamical systems and uniform spaces.

Notation 1.2.1. For a topological space X and its subset A, the closure of A is denoted by $\overline{A}$ and the open kernel of A is denoted by A°.

For a metric space X, the *closed* ball with its center $x \in X$ and its radius $r > 0$ is denoted by $B(x, r)$.

For a positive integer d, let ρ be the Euclidean metric for the Euclidean space $\mathbb{R}^d$. Let $\mathrm{E}(d)$ be the group of all isometries on the Euclidean space $\mathbb{R}^d$ and $\mathrm{O}(d)$ be the orthogonal group. There is a group isomorphism $\mathbb{R}^d \rtimes \mathrm{O}(d) \to \mathrm{E}(d)$, by which we can identify these two groups. Thus elements of $\mathrm{E}(d)$ are recognized as pairs (a, A) of $a \in \mathbb{R}^d$ and $A \in \mathrm{O}(d)$. For $\mathrm{E}(d)$, define a metric $\rho_{\mathrm{E}(d)}$ by $\rho_{\mathrm{E}(d)}((a, A), (b, B)) = \rho(a, b) + \|A - B\|$, where $\|\cdot\|$ is the operator norm for the operators on the Banach space $\mathbb{R}^d$ with the Euclidean norm. For any closed subgroup Γ of $\mathrm{E}(d)$, the restriction ρ_Γ of $\rho_{\mathrm{E}(d)}$is a left-invariant metric for Γ. Moreover, for any $\gamma, \eta \in \Gamma$, we have

$$\rho(\gamma 0, \eta 0) \leqq \rho_\Gamma(\gamma, \eta) \leqq \rho(\gamma 0, \eta 0) + 2. \tag{1.1}$$

For each $j = 1, 2, \ldots, d$, let $e_j \in \mathbb{R}^d$ be the vector of which ith component is 0 for $i \neq j$ and jth component is 1.

The standard inner product in $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$. That is, for $x = (x_1, x_2, \cdots, x_d)$

and $y = (y_1, y_2, \cdots, y_d)$ in $\mathbb{R}^d$,

$$\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i.$$

For $S \subset \mathbb{R}^d$ set $-S = \{-x \mid x \in S\}$ and for $S_1, S_2 \subset \mathbb{R}^d$ set $S_1 + S_2 = \{x + y \mid x \in S_1, y \in S_2\}$.

We set $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

For any group Γ which acts on a set X, its isotropy group for a point $x \in X$ is denoted by Γ_x. That is, $\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$. For an abstract pattern $\mathcal{P}$ such as patches, uniformly discrete sets, functions, its group of symmetry is denoted by $\operatorname{Sym}_\Gamma \mathcal{P} = \{\gamma \in \Gamma \mid \gamma\mathcal{P} = \mathcal{P}\}$. The orbit $\{\gamma x \mid \gamma \in \Gamma\}$ of x is denoted by $\mathcal{O}_x$.

The identity element of any group is denoted by e.

Chapter 2

General theory of tilings, continuous hulls and tiling dynamical systems

Here, we follow [20] and give an introductory exposition for the theory of tilings.

We will stress the following two points. First, we introduce two topologies on the space of all patches on $\mathbb{R}^d$: the cylinder topology (Definition 2.1.8) and the local matching topology (Definition 2.1.19). Tilings are examples of patches (Definition 2.1.2). Thus these two topologies define two topologies on a space of tilings. We investigate properties of these two topologies and relations between them. Often on the continuous hull of a tiling the relative topologies of these two coincide.

Second, relations between properties of tilings and those of continuous hulls and tiling dynamical systems are stressed. For example, relations between FLC of tilings and compactness of continuous hulls (Corollary 2.1.49), and repetitivity of tilings and minimality of tiling dynamical systems (Proposition 2.1.63) are fundamental. We can prove that for tilings from certain substitutions the corresponding tiling dynamical systems are not mixing (Theorem 2.2.41), and this is derived from a property of distribution of patches in tilings (Remark 2.2.42).

Many results in the literature is on implications of the properties of tilings on the continuous hulls and the corresponding dynamical systems. In Section 2.3 we conversely deduce a property on the distribution of patches in certain tilings from a property of the corresponding dynamical system (the converse of this is also proved).

2.1 Definition of tilings and their properties

Here we introduce patches, tilings and topological spaces consisting of patches. Such topological spaces often admit an $\mathbb{R}^d$ action.

Definition 2.1.1. For any $\mathcal{P} \subset 2^{\mathbb{R}^d}$, the set $\operatorname{supp} \mathcal{P}$ defined by

$$\operatorname{supp} \mathcal{P} = \overline{\bigcup_{T \in \mathcal{P}} T}$$

is called the support of the set $\mathcal{P}$.

The support is the closure of the area that elements $T \in \mathcal{P}$ cover.

Definition 2.1.2. We fix $d \in \mathbb{Z}_{>0}$.

- An open, bounded and nonempty subset of $\mathbb{R}^d$ is called a tile.
- A set $\mathcal{P}$ of tiles such that $S, T \in \mathcal{P}$ and $S \neq T$ imply $S \cap T = \emptyset$ is called a patch. A patch $\mathcal{P}$ is said to be bounded if $\operatorname{supp} \mathcal{P}$ is bounded.
- A patch $\mathcal{T}$ such that $\operatorname{supp} \mathcal{T} = \mathbb{R}^d$ is called a tiling.
- For a tiling $\mathcal{T}$ and a vector $x \in \mathbb{R}^d$, suppose there exists $T \in \mathcal{T}$ such that $T + x \in \mathcal{T}$. Then we call x a return vector for $\mathcal{T}$.

Remark 2.1.3. In the literature, the word "tile" is defined in various ways. Often tiles are defined as compact sets which are "simple". What the word simple means depends on the authors.

For example, it is defined as (1) a subset of $\mathbb{R}^d$ which is homeomorphic to a closed unit ball of $\mathbb{R}^d$ ([1]), (2) a closed polygonal subset of $\mathbb{R}^d$ ([33]), or (3) a subset of $\mathbb{R}^d$ which is compact and equal to the closure of its interior ([5]).

Here we put the simplicity assumption by defining tiles as open sets. This change is not essential and the theory we develop becomes almost the same.

Often we consider labels on tiles in order to distinguish two tiles that are as sets the same. For example, one can prove the unique ergodicity of certain tiling dynamical systems from substitutions by considering labels. On the other hand, considering labels gives an additional complexity in notation. Here we avoid considering labels, and when they are necessary we find a way round by giving a "puncture" to each tile (i.e. remove one point from each tile). Two tiles that are originally the same become after this procedure different if they have different punctures (see Example 2.2.6).

Definition 2.1.4. A tiling $\mathcal{T}$ is said to be periodic if there is $x \in \mathbb{R}^d \setminus \{0\}$ such that its translate by x coincide with itself, that is, $\mathcal{T} + x = \mathcal{T}$. Otherwise a tiling is said to be non-periodic. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is said to be crystallographic if there is a basis $\{b_1, b_2, \ldots, b_d\}$ of $\mathbb{R}^d$ such that $\mathcal{T} + b_i = \mathcal{T}$ for all i.

Example 2.1.5 (Square tiling)**.** For any dimension $d \in \mathbb{Z}_{>0}$, a tiling $\mathcal{T}_s = \{(0,1)^d + v \mid v \in \mathbb{Z}^d\}$ is called Square tiling. This is an example of crystallographic tiling.

Many interesting examples of non-periodic tilings can be constructed from substitution rules, which we will introduce later.

Remark 2.1.6. If $\mathcal{P}$ is a patch, then the set $\mathcal{P}$ is at most countable.

Definition 2.1.7. $\mathrm{Patch}(\mathbb{R}^d)$ denotes the set of all patches in $\mathbb{R}^d$. $\mathrm{Tiling}(\mathbb{R}^d)$ denotes the set of all tilings in $\mathbb{R}^d$.

Next we introduce two topologies on $\mathrm{Patch}(\mathbb{R}^d)$.

Definition 2.1.8. For $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ and a neighborhood U of 0 in $\mathbb{R}^d$, set

$$C(U, \mathcal{P}) = \{\mathcal{Q} \in \mathrm{Patch}(\mathbb{R}^d) \mid \text{there exists } x \in U \text{ such that } \mathcal{P} + x \subset \mathcal{Q}\}.$$

Such sets are called cylinder sets. The topology generated by

$$\{C(U, \mathcal{P}) \mid U\text{: open neighborhood of 0 in } \mathbb{R}^d,\ \mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)\text{: bounded}\} \tag{2.1}$$

is called the cylinder topology.

Remark 2.1.9. The subbasis (2.1) is in fact a basis. For if $n \in \mathbb{Z}_{>0}$, $U_1, U_2, \dots, U_n$ are open neighborhoods of 0, $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n \in \mathrm{Patch}(\mathbb{R}^d)$ are bounded and

$$\mathcal{Q} \in \bigcap_i C(U_i, \mathcal{P}_i),$$

then for each i there is $x_i \in U_i$ such that $\mathcal{P}_i + x_i \subset \mathcal{Q}$. Set $\mathcal{P} = \bigcup_i (P_i + x_i)$. Then $\mathcal{P}$ is a bounded patch and if we take an open neighborhood U of 0 in $\mathbb{R}^d$ small enough, then

$$\mathcal{Q} \in C(U, \mathcal{P}) \subset \bigcap_i C(U_i, \mathcal{P}_i).$$

Lemma 2.1.10. *If $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, the set*

$$\{C(U, \mathcal{Q}) \mid U\text{: neighborhood of } 0 \text{ in } \mathbb{R}^d \text{ and } \mathcal{Q} \subset \mathcal{P}\text{: bounded}\}$$

forms a neighborhood basis for $\mathcal{P}$ with respect to the cylinder topology.

Proof. Suppose $\mathcal{P} \in C(U, \mathcal{P}')$ for some open neighborhood U of 0 and a bounded $\mathcal{P}' \in \mathrm{Patch}(\mathbb{R}^d)$. Then there is $x \in U$ such that $\mathcal{P}' + x \subset \mathcal{P}$. If a neighborhood V of 0 is small enough,

$$\mathcal{P} \in C(V^\circ, \mathcal{P}' + x) \subset C(V, \mathcal{P}' + x) \subset C(U, \mathcal{P}').$$

□

Lemma 2.1.11. *The group* $\mathbb{R}^d$ *acts on* $\mathrm{Patch}(\mathbb{R}^d)$ *by translation:*

$$\mathrm{Patch}(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mathcal{P}, x) \mapsto \mathcal{P} + x \in \mathrm{Patch}(\mathbb{R}^d). \tag{2.2}$$

Furthermore this map is continuous with respect to the cylinder topology.

Proof. Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Take also a neighborhood O with respect to the cylinder topology of $P + x$. To prove the continuity of the map at $(\mathcal{P}, x)$, we may assume O is of the form $O = C(U, \mathcal{P}_0)$ where U is an open neighborhood of 0 in $\mathbb{R}^d$, $\mathcal{P}_0$ is bounded and $\mathcal{P}_0 \subset \mathcal{P} + x$ (cf. Lemma 2.1.10). Take a neighborhood V of x and a neighborhood V' of 0 such that if $y \in V$ and $z \in V'$, then $y - x + z \in U$. If $y \in V$ and $\mathcal{Q} \in C(V', \mathcal{P}_0 - x)$ (cf. Lemma 2.1.10), then there is $z \in V'$ such that $\mathcal{P}_0 - x + z \subset \mathcal{Q}$. We obtain $\mathcal{P}_0 + y - x + z \subset \mathcal{Q} + y$ and $\mathcal{Q} + y \in C(U, \mathcal{P}_0)$. □

Remark 2.1.12. If $\mathcal{T} \in \mathrm{Tiling}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, then $\mathcal{T} + x \in \mathrm{Tiling}(\mathbb{R}^d)$.

Next we define a uniform structure on $\mathrm{Patch}(\mathbb{R}^d)$ and the second topology on it. For a generality of uniform spaces, see Appendix and [6].

Definition 2.1.13. For any subset $\mathcal{P} \subset 2^{\mathbb{R}^d}$ and any subset $S \subset \mathbb{R}^d$ set

$$\mathcal{P} \cap S = \{T \in \mathcal{P} \mid T \subset S\}.$$

The next lemma is easy to prove.

Lemma 2.1.14. *If* $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$*,* $x \in \mathbb{R}^d$ *and* $S \subset \mathbb{R}^d$*, then* $(\mathcal{P} \cap S) + x = (\mathcal{P} + x) \cap (S + x)$*. If moreover* $S_1 \subset S_2 \subset \mathbb{R}^d$*, then* $(\mathcal{P} \cap S_2) \cap S_1 = \mathcal{P} \cap S_1$*.*

Definition 2.1.15. For a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of 0 in $\mathbb{R}^d$, set

$$\mathcal{U}_{K,V} = \{(\mathcal{P}_1, \mathcal{P}_2) \in \mathrm{Patch}(\mathbb{R}^d) \times \mathrm{Patch}(\mathbb{R}^d) \mid \text{there exists } x \in V \text{ such that } \mathcal{P}_1 \cap K = (\mathcal{P}_2 + x) \cap K\}.$$

Remark 2.1.16. If $K_1 \subset K_2$ and $V_1 \supset V_2$, then by Lemma 2.1.14, $\mathcal{U}_{K_1,V_1} \supset \mathcal{U}_{K_2,V_2}$.

Lemma 2.1.17. *The set*

$$\{\mathcal{U}_{K,V} \mid K \subset \mathbb{R}^d\text{: compact and } V\text{: a compact neighborhood of } 0 \text{ in } \mathbb{R}^d\} \tag{2.3}$$

forms a fundamental system of entourages for $\mathrm{Patch}(\mathbb{R}^d)$*.*

Proof. This is a special case of Lemma 4.1.3 and so we omit the proof. □

Definition 2.1.18. Let $\mathfrak{U}$ denote the set of all entourages generated by (2.3) and the uniform space constructed in this way is represented by $(\mathrm{Patch}(\mathbb{R}^d), \mathfrak{U})$.

Recall that a uniform structure on a set defines a topology on that set. In this context $\{\mathcal{U}_{K,V}(\mathcal{P}) \mid K\text{: a compact subset of } \mathbb{R}^d,\ V\text{: a compact neighborhood of } 0\}$ form a neighborhood basis for $\mathcal{P}$. Here $\mathcal{U}(\mathcal{P}) = \{\mathcal{Q} \in \mathrm{Patch}(\mathbb{R}^d) \mid (\mathcal{P}, \mathcal{Q}) \in \mathcal{U}\}$ for each $\mathcal{U} \in \mathfrak{U}$ and $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$.

Definition 2.1.19. The topology on $\mathrm{Patch}(\mathbb{R}^d)$ defined by the uniform structure $\mathfrak{U}$ is called the local matching topology.

The uniform structure is metrizable; see Lemma B.0.16. We can describe a metric explicitly as follows.

For two patches $\mathcal{P}_1, \mathcal{P}_2$ of $\mathbb{R}^d$, set

$$\Delta(\mathcal{P}_1, \mathcal{P}_2) = \Big\{0 < r < \frac{1}{\sqrt{2}} \mid \text{there exist } x, y \in B(0, r) \text{ such that } (\mathcal{P}_1 + x) \cap B(0, 1/r) = (\mathcal{P}_2 + y) \cap B(0, 1/r)\Big\}.$$

Then define

$$\rho(\mathcal{P}_1, \mathcal{P}_2) = \inf\left(\Delta(\mathcal{P}_1, \mathcal{P}_2) \cup \left\{\frac{1}{\sqrt{2}}\right\}\right). \tag{2.4}$$

Remark 2.1.20. It is tempting in the definition of the tiling metric to replace $\Delta(\mathcal{P}_1, \mathcal{P}_2)$ above with

$$\left\{ 0 < r < \frac{1}{\sqrt{2}} \;\middle|\; \text{there exists } y \in B(0, r) \text{ such that } \mathcal{P}_1 \cap B(0, 1/r) = (\mathcal{P}_2 + y) \cap B(0, 1/r) \right\},$$

because this definition seems to simplify the following proofs. However if we define the function ρ in this way ρ does not become a metric; it is not necessarily true that $\rho(\mathcal{T}_1, \mathcal{T}_2) = \rho(\mathcal{T}_2, \mathcal{T}_1)$ for two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$. Here is an easy counterexample: take small $r > 0$, and consider three copies of a tile $(-1, 1)^d$. Give each of them a puncture in three different ways so that we obtain three different tiles (or equivalently, put three different labels to each of the copies so that we can distinguish them). Let S, T, U denote the three tiles. Set $\mathcal{T}_1 = \{S\} \cup (\{T\} + 2\mathbb{Z}^d \setminus \{0\})$ and $\mathcal{T}_2 = (\{S\} \cup (\{U\} + 2\mathbb{Z}^d \setminus \{0\})) + (r, 0, 0, \dots 0)$. Then $\rho(\mathcal{T}_1, \mathcal{T}_2) = \frac{1}{\sqrt{8+d}}$ and $\rho(\mathcal{T}_2, \mathcal{T}_1) = \frac{1}{\sqrt{(3-r)^2+d-1}}$.

It is easy to prove that ρ in (2.4) is a metric on $\mathrm{Patch}(\mathbb{R}^d)$. To prove $\rho(\mathcal{T}_1, \mathcal{T}_2) = 0$ implies $\mathcal{T}_1 = \mathcal{T}_2$, we use the following lemma.

Lemma 2.1.21. *Let T be a tile and $\mathcal{P}$ be a patch. Suppose $x_1, x_2, \dots$ are elements of $\mathbb{R}^d$ such that $x_n \to 0$ as $n \to \infty$ and $T + x_n \in \mathcal{P}$ for all n. Then $T \in \mathcal{P}$.*

To prove the triangle inequality, one has to use the fact that $\frac{1}{\varepsilon} - \eta > \frac{1}{\varepsilon+\eta}$ whenever $0 < \varepsilon < \frac{1}{\sqrt{2}}$ and $0 < \eta < \frac{1}{\sqrt{2}}$.

Lemma 2.1.22. *The local matching uniform structure and the uniform structure given by the metric ρ are the same.*

Next we collect several properties of local matching uniform structure.

Lemma 2.1.23. *The local matching topology is Hausdorff.*

Proof. Since it is metrizable, the statement is clear. We prove a generalization of this in Corollary 4.1.11. □

Lemma 2.1.24. *With respect to the local matching topology, the action* (2.2) *is jointly continuous.*

Proof. We prove this in Lemma 4.1.14 in a more general context. □

Proposition 2.1.25. *The uniform space* $(\mathrm{Patch}(\mathbb{R}^d), \mathfrak{U})$ *is complete.*

Proof. We prove this in Proposition 4.1.16 in a more general setting. □

Proposition 2.1.26. *The local matching topology is stronger than the cylinder topology.*

Proof. Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. For any bounded patch $\mathcal{P}_0 \subset \mathcal{P}$ and an open neighborhood U_0 of 0 in $\mathbb{R}^d$ (cf. Lemma 2.1.10), take a compact neighborhood U of 0 such that $U \subset U_0$ and set $K = U + \operatorname{supp} \mathcal{P}_0$. If $\mathcal{Q} \in \mathcal{U}_{K,U}^{-1}(\mathcal{P})$, then there is $x \in U$ such that

$$\mathcal{Q} \cap K = (\mathcal{P} + x) \cap K.$$

Since $\operatorname{supp}(\mathcal{P}_0 + x) \subset K$, $\mathcal{P}_0 + x \subset (\mathcal{P} + x) \cap K \subset \mathcal{Q}$. Then $\mathcal{Q} \in C(U, \mathcal{P}_0) \subset C(U_0, \mathcal{P}_0)$. This argument shows that $\mathcal{U}_{K,U}^{-1}(\mathcal{P}) \subset C(U_0, \mathcal{P}_0)$. □

Lemma 2.1.27. *Suppose $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Patch}(\mathbb{R}^d)$ and $\mathcal{P}_1 \subset \mathcal{P}_2$. Take $S \subset \mathbb{R}^d$ such that $\operatorname{supp} \mathcal{P}_1 \supset S$. Then $\mathcal{P}_1 \cap S = \mathcal{P}_2 \cap S$.*

Definition 2.1.28. For each $R > 0$, set

$$\mathrm{Tiling}_R(\mathbb{R}^d) := \{\mathcal{T} \in \mathrm{Tiling}(\mathbb{R}^d) \mid \sup_{T \in \mathcal{T}} \operatorname{diam} T < R\}.$$

Proposition 2.1.29. *For any $R > 0$, on $\mathrm{Tiling}_R(\mathbb{R}^d)$, the relative topologies of the local matching topology and the cylinder topology coincide.*

Proof. Take $\mathcal{T} \in \mathrm{Tiling}_R(\mathbb{R}^d)$. Take also a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$. Set $K' = K + \overline{B(0,R)}$ and $\mathcal{P}_0 = \mathcal{T} \cap K'$. Note that $\operatorname{supp} \mathcal{P}_0 \supset K$. If $\mathcal{S} \in C(-V, \mathcal{P}_0) \cap \mathrm{Tiling}_R(\mathbb{R}^d)$, there is $x \in V$ such that $\mathcal{P}_0 - x \subset \mathcal{S}$. By Lemma 2.1.14 and Lemma 2.1.27,

$$(\mathcal{S} + x) \cap K = \mathcal{P}_0 \cap K = (\mathcal{T} \cap K') \cap K = \mathcal{T} \cap K,$$

and $\mathcal{S} \in \mathcal{U}_{K,V}(\mathcal{T})$. Hence

$$\mathcal{T} \in C(-V, \mathcal{P}_0) \cap \mathrm{Tiling}_R(\mathbb{R}^d) \subset \mathcal{U}_{K,V}(\mathcal{T}).$$

We see on $\mathrm{Tiling}_R(\mathbb{R}^d)$ the cylinder topology is stronger than the local matching topology and together with Proposition 2.1.26 we see they are equal on $\mathrm{Tiling}_R(\mathbb{R}^d)$. □

Remark 2.1.30. With respect to the local matching topology, $\mathrm{Tiling}_R(\mathbb{R}^d)$ is a closed subset of $\mathrm{Patch}(\mathbb{R}^d)$. However $\mathrm{Tiling}(\mathbb{R}^d)$ is not closed in $\mathrm{Patch}(\mathbb{R}^d)$ as the following example shows.

Example 2.1.31. Consider a tiling $\mathcal{T}_s$ of $\mathbb{R}^d$ defined by $\mathcal{T}_s = \{(0,1)^d + x \mid x \in \mathbb{Z}^d\}$. We start from this tiling $\mathcal{T}_s$ and replace tiles with larger ones. For any $n \in \mathbb{Z}_{>0}$, choose $x_n \in \mathbb{Z}^d$ such that for any two distinct n and m, $((0,n)^d + x_n) \cap ((0,m)^d + x_m) = \emptyset$. To $\mathcal{T}_s$, we add tiles $(0,n)^d + x_n, n = 2, 3, \ldots$ and remove tiles with side-length 1 that intersect these tiles with side-length $2, 3, \ldots$. The resulting tiling is represented by $\mathcal{T}_s'$ and this consists of translates of $(0,n)^d$, $n = 1, 2, 3, \ldots$. This tiling is not in $\mathrm{Tiling}_R(\mathbb{R}^d)$ for any $R > 0$ and a sequence $(\mathcal{T}_s' - x_n - y_n)_n$, where $y_n = (\frac{1}{2}n, \frac{1}{2}n, \ldots, \frac{1}{2}n)$ for each n, converges to $\emptyset$ with respect to the local matching topology.

2.1.1 Finite local complexity and finite tile type

Definition 2.1.32. On $2^{\mathbb{R}^d}$ (the set of all subsets of $\mathbb{R}^d$), define an equivalence relation $\approx$ by

$$A \approx B \iff \text{there exists } x \in \mathbb{R}^d \text{ such that } A = B + x.$$

On the set $2^{2^{\mathbb{R}^d}}$ of all subsets of $2^{\mathbb{R}^d}$, we define an equivalence relation $\sim$ by

$$\mathcal{P}_1 \sim \mathcal{P}_2 \iff \text{there exists } x \in \mathbb{R}^d \text{ such that } \mathcal{P}_1 = \mathcal{P}_2 + x.$$

Definition 2.1.33. An element $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ has finite local complexity (FLC) if the quotient set

$$\{(\mathcal{P} + x) \cap K \mid x \in \mathbb{R}^d\}/\!\sim$$

is finite for any compact $K \subset \mathbb{R}^d$.

Definition 2.1.34. An element $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ has finite tile type (FTT) if $\mathcal{P}/\approx$ is finite. In this case there exists a finite set $\mathcal{A}$ of tiles such that

- For any $P \in \mathcal{A}$, we have $0 \in P$, and
- For any $T \in \mathcal{P}$, there is a unique $P \in \mathcal{A}$ and a (necessarily unique) $x \in \mathbb{R}^d$ such that $T = P + x$.

Such a set $\mathcal{A}$ is called an alphabet for the FTT patch $\mathcal{P}$.

Given a finite non-empty set $\mathcal{A}$ of tiles that are not pairwise translationally equivalent, for any $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$ set $c_{\mathcal{A}}(P+x) = x$. For $\mathcal{P} \subset \mathcal{A}+\mathbb{R}^d$, set $c_{\mathcal{A}}(\mathcal{P}) = \{c_{\mathcal{A}}(T) \mid T \in \mathcal{P}\}$.

In Proposition 2.1.37 we give a characterization of FLC and FTT.

Definition 2.1.35. For a patch $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ and $S \subset \mathbb{R}^d$, set

$$\mathcal{P} \sqcap S = \{T \in \mathcal{P} \mid \overline{T} \cap S \neq \emptyset\}.$$

Lemma 2.1.36. *For any subsets $\Pi_1, \Pi_2 \subset 2^{2^{\mathbb{R}^d}}$, suppose the following conditions;*

- *for any $\mathcal{P}_1 \in \Pi_1$ there are $\mathcal{P}_2 \in \Pi_2$ and $x \in \mathbb{R}^d$ such that $\mathcal{P}_1 + x \subset \mathcal{P}_2$,*
- *each $\mathcal{P}_2 \in \Pi_2$ is finite, and*
- *$\Pi_2/\sim$ is finite.*

Then $\Pi_1/\sim$ is finite.

Proposition 2.1.37. *For $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, the following conditions are equivalent;*

1. *$\mathcal{P}$ has FTT and FLC.*
2. *$\mathcal{P}$ has FTT and $\{\mathcal{P}' \subset \mathcal{P} \mid \operatorname{diam}\operatorname{supp}\mathcal{P}' < R\}/\sim$ is finite for all $R > 0$.*
3. *$\{\mathcal{P} \cap B(x,R) \mid x \in \mathbb{R}^d\}/\sim$ is finite for any $R > 0$ and $\mathcal{P}$ has FTT.*
4. *$\{\mathcal{P} \sqcap (K+x) \mid x \in \mathbb{R}^d\}/\sim$ is finite for any compact $K \subset \mathbb{R}^d$.*
5. *$\mathcal{P}$ has FTT and $c_{\mathcal{A}}(\mathcal{P}) - c_{\mathcal{A}}(\mathcal{P})$ is discrete and closed in $\mathbb{R}^d$, for any alphabet $\mathcal{A}$.*

Proof. 1⇒2. For any $R > 0$, if $\mathcal{P}' \subset \mathcal{P}$ and $\operatorname{diam}\overline{\operatorname{supp}\mathcal{P}'} < R$, either $\mathcal{P}' = \emptyset$ or we can take $x \in \operatorname{supp}\mathcal{P}'$. In the latter case $\mathcal{P}' \subset \mathcal{P} \cap (x + \overline{B(0,R)})$ and Lemma 2.1.36 applies.

2⇒3. For any $x \in \mathbb{R}^d$, we have $\operatorname{diam}\operatorname{supp}(\mathcal{P} \cap B(x,R)) < 2R$. Lemma 2.1.36 applies.

3⇒4. Set $r = \max_{T \in \mathcal{P}} \operatorname{diam} T$. For any compact $K \subset \mathbb{R}^d$, take $R > 0$ such that $K \subset B(0, R-r)$. For any $x \in \mathbb{R}^d$ we have $\mathcal{P} \sqcap (K+x) \subset \mathcal{P} \cap B(x,R)$ and Lemma 2.1.36 implies (4).

4⇒5. First by taking $K=\{0\}$ we see $\mathcal{P}/\approx$ is finite and so $\mathcal{P}$ has FTT. Take $R>0$ arbitrarily. We shall show that $(c_{\mathcal{A}}(\mathcal{P})-c_{\mathcal{A}}(\mathcal{P}))\cap B(0,R)$ is finite. Set $K=\overline{B(0,R)}$. There is a finite $F\subset\mathbb{R}^d$ such that if $x\in\mathbb{R}^d$ there are $y\in F$ and $z\in\mathbb{R}^d$ for which

$$\mathcal{P}\sqcap(K+x)=(\mathcal{P}\sqcap(K+y))+z.$$

Take $a\in(c_{\mathcal{A}}(\mathcal{P})-c_{\mathcal{A}}(\mathcal{P}))\cap B(0,R)$. Then there are $P_1,P_2\in\mathcal{A}$ and $a_1,a_2\in\mathbb{R}^d$ such that $P_i+a_i\in\mathcal{P}$ $(i=1,2)$ and $a=a_1-a_2$. By $a_2\in P_2+a_2$ and $\|a_1-a_2\|<R$, we have $P_2+a_2\in\mathcal{P}\sqcap(K+a_1)$, and

$$\begin{aligned}a\in&(c_{\mathcal{A}}(\mathcal{P}\sqcap(K+a_1))-c_{\mathcal{A}}(\mathcal{P}\sqcap(K+a_1))\\=&(c_{\mathcal{A}}(\mathcal{P}\sqcap(K+b))-c_{\mathcal{A}}(\mathcal{P}\sqcap(K+b)))\end{aligned}$$

for some $b\in F$. Hence

$$(c_{\mathcal{A}}(\mathcal{P})-c_{\mathcal{A}}(\mathcal{P}))\cap B(0,R)\subset\bigcup_{b\in F}(c_{\mathcal{A}}(\mathcal{P}\sqcap(K+b))-c_{\mathcal{A}}(\mathcal{P}\sqcap(K+b))).\qquad(2.5)$$

Since F is finite and $\mathcal{P}\sqcap(K+b)$ is finite by FTT, the right-hand side of (2.5) is finite.

5⇒1. Take a compact $K\subset\mathbb{R}^d$ arbitrarily. Set $C=(c_{\mathcal{A}}(\mathcal{P})-c_{\mathcal{A}}(\mathcal{P}))\cap(K-K)$ and $\mathcal{P}'=\mathcal{A}+C$. If $x\in\mathbb{R}^d$ and $\mathcal{P}\cap(K+x)\neq\emptyset$, then take $P_0\in\mathcal{A}$ and $x_0\in\mathbb{R}^d$ such that $P_0+x_0\in\mathcal{P}\cap(K+x)$. If we arbitrarily take $P_1\in\mathcal{A}$ and $x_1\in\mathbb{R}^d$ such that $P_1+x_1\in\mathcal{P}\cap(K+x)$, then $x_1-x_0\in C$. This implies that $\mathcal{P}\cap(K+x)-x_0\subset\mathcal{P}'$. Since $\mathcal{P}'$ is finite, by Lemma 2.1.36 $\{\mathcal{P}\cap(K+x)\mid x\in\mathbb{R}^d\}/\sim$ is finite. □

Remark 2.1.38. Example 2.1.31 is an example of tiling which has FLC but does not have FTT.

We then introduce another characterization of FLC.

Definition 2.1.39. Let $\mathcal{P}$ be a patch and take $T\in\mathcal{P}$. We inductively define coronas $C^n(T,\mathcal{P}),n=0,1,\dots$ by

$$\begin{aligned}C^0(T,\mathcal{P})&=\{T\}\\C^{n+1}(T,\mathcal{P})&=\mathcal{P}\sqcap\operatorname{supp}C^n(T,\mathcal{P}).\end{aligned}$$

Also set $C^\infty(T,\mathcal{P})=\bigcup_{n\in\mathbb{Z}_{>0}}C^n(T,\mathcal{P})$.

Note that if $\mathcal{T}$ has finite tile type, any of its coronas are finite sets. Now we prove another characterization of FLC (Proposition 2.1.41), which will be useful when we try to prove that an example of substitution rule has FLC.

Lemma 2.1.40. *Let $\mathcal{P}$ be a non-empty finite patch. If there is a connected set $C\subset\operatorname{supp}\mathcal{P}$ such that $C\cap\overline{T}\neq\emptyset$ for any $T\in\mathcal{P}$, then for any $T\in\mathcal{P}$ we have $\mathcal{P}=C^\infty(T,\mathcal{P})$.*

Proof. For $S, T \in \mathcal{P}$ we have either

$$C^{\infty}(T, \mathcal{P}) = C^{\infty}(S, \mathcal{P}) \tag{2.6}$$

or

$$\operatorname{supp} C^{\infty}(T, \mathcal{P}) \cap \operatorname{supp} C^{\infty}(S, \mathcal{P}) = \emptyset.$$

If the equation (2.6) holds, we set $S \sim T$. $\mathcal{P}/\!\sim$ is a finite set and the equivalence class including $S \in \mathcal{P}$ is $C^{\infty}(S, \mathcal{P})$. There are $k \in \mathbb{Z}_{>0}$ and $T_1, T_2, \ldots, T_k \in \mathcal{P}$ such that $\mathcal{P}/\!\sim = \{C^{\infty}(T_1, \mathcal{P}), C^{\infty}(T_2, \mathcal{P}), \ldots, C^{\infty}(T_k, \mathcal{P})\}$. We have

$$C \subset \operatorname{supp} \mathcal{P} = \bigcup_{i=1}^{k} \operatorname{supp} C^{\infty}(T_i, \mathcal{P})$$

and for each i, $\operatorname{supp} C^{\infty}(T_i, \mathcal{P}) \cap C \neq \emptyset$. By the connectivity of C, we necessarily have $k = 1$. For any $T \in \mathcal{P}$, $\mathcal{P} = C^{\infty}(T_1, \mathcal{P}) = C^{\infty}(T, \mathcal{P})$. □

Proposition 2.1.41. *Let $\mathcal{T}$ be a tiling of finite tile type. Then the following conditions are equivalent:*

1. *$\mathcal{T}$ has FLC.*
2. *$\{C^1(T, \mathcal{T}) \mid T \in \mathcal{T}\}/\!\sim$ is finite.*
3. *$\{C^n(T, \mathcal{T}) \mid T \in \mathcal{T}\}/\!\sim$ is finite for any $n \in \mathbb{Z}_{>0}$.*

Proof. We use Lemma 2.1.36 for several times.

(1)⇒(2): Set $\Pi_0 = \{C^1(T, \mathcal{T}) \mid T \in \mathcal{T}\}$ and $\Pi_1 = \{\mathcal{T} \cap B(x, 2r) \mid x \in \mathbb{R}^d\}$, where $r > \max_{T \in \mathcal{T}} \operatorname{diam} T$.

(2)⇒(3): We prove by induction on n. Suppose we have proved $\{C^n(T, \mathcal{T}) \mid T \in \mathcal{T}\}/\!\sim$ is finite. There is a finite set $\mathcal{F}_n \subset \mathcal{T}$ such that, if $T \in \mathcal{T}$, there are $F \in \mathcal{F}_n$ and $v \in \mathbb{R}^d$ for which $C^n(T, \mathcal{T}) = C^n(F, \mathcal{T}) + v$. By (2) it can be shown that there is a finite set $\mathcal{F}_1 \subset \mathcal{T}$ such that, for any $T \in \mathcal{T}$, there are $E \in \mathcal{F}_1$ and $v \in \mathbb{R}^d$ for which $T = E + v$ and $C^1(T, \mathcal{T}) = C^1(E, \mathcal{T}) + v$. For $E \in \mathcal{F}_1$ and $F \in \mathcal{F}_n$, set

$$V(E, F) = \{v \in \mathbb{R}^d \mid E + v \in C^n(F, \mathcal{T})\}.$$

Set

$$\mathcal{P} = \bigcup_{F \in \mathcal{F}_n} \bigcup_{E \in \mathcal{F}_1} (C^1(E, \mathcal{T}) + V(E, F)).$$

Then $\mathcal{P}$ is a finite set, which is not necessarily a patch. We can show that $\mathcal{P} \supset C^n(F, \mathcal{T})$ for each $F \in \mathcal{F}_n$. Using this we can show that for any $T \in \mathcal{T}$ there is $v \in \mathbb{R}^d$ such that $C^{n+1}(T, \mathcal{T}) + v \subset \mathcal{P}$. It follows that $\{C^{n+1}(T, \mathcal{T}) \mid T \in \mathcal{T}\}/\!\sim$ is finite.

(3)⇒(1): Take $R > 0$ and let N be an integer which is large enough. Take $x \in \mathbb{R}^d$. By Lemma 2.1.40, for any $S \in \mathcal{T} \sqcap B(x, R)$,

$$\mathcal{T} \sqcap B(x, R) = C^{\infty}(S, \mathcal{T} \sqcap B(x, R)) = C^N(S, \mathcal{T} \sqcap B(x, R)) \subset C^N(S, \mathcal{T}).$$

Since $\{C^N(T, \mathcal{T}) \mid T \in \mathcal{T}\}/\sim$ is finite, $\mathcal{T}$ has FLC. □

Definition 2.1.42. For $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, set

$$X_{\mathcal{P}} = \overline{\{\mathcal{P} + x \mid x \in \mathbb{R}^d\}}$$

with respect to the local matching topology.

Lemma 2.1.43. *If $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, $\mathcal{Q} \in X_{\mathcal{P}}$ and $x \in \mathbb{R}^d$, then $\mathcal{Q} + x \in X_{\mathcal{P}}$.*

Proof. There is a sequence (x_n) of $\mathbb{R}^d$ such that $\mathcal{Q} = \lim_n (P + x_n)$. By Lemma 2.1.24 $\mathcal{Q} + x = \lim(\mathcal{P} + x_n + x) \in X_{\mathcal{P}}$. □

Definition 2.1.44. A subset $X \subset \mathrm{Patch}(\mathbb{R}^d)$ has FLC if

$$\{\mathcal{P} \cap (K + x) \mid x \in \mathbb{R}^d, \mathcal{P} \in X\}/\sim$$

is finite for any compact $K \subset \mathbb{R}^d$.

Remark 2.1.45. If X is invariant under translation, X has FLC if and only if

$$\{\mathcal{P} \cap K \mid \mathcal{P} \in X\}/\sim$$

is finite for any compact $K \subset \mathbb{R}^d$. If there are only finitely many tile types in X, that is, $(\bigcup_{\mathcal{P} \in X} \mathcal{P})/\approx$ is finite, then by Lemma 2.1.36 X has FLC if and only if

$$\{\mathcal{P} \cap B(x, R) \mid x \in \mathbb{R}^d, \mathcal{P} \in X\}/\sim$$

is finite for any $R > 0$.

Lemma 2.1.46. *Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. Then the following two conditions are equivalent;*

1. *$\mathcal{P}$ has FLC.*

2. *$X_{\mathcal{P}}$ has FLC.*

Proof. 1⇒2. Take any compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$. If $\mathcal{Q} \in X_{\mathcal{P}}$, then there is $x \in \mathbb{R}^d$ such that $\mathcal{P} + x \in \mathcal{U}_{K,V}(\mathcal{Q})$. This implies that there is $y \in V$ such that $(\mathcal{P} + x + y) \cap K = \mathcal{Q} \cap K$ and so

$$\{\mathcal{Q} \cap K \mid \mathcal{Q} \in X_{\mathcal{P}}\} = \{(\mathcal{P} + x) \cap K \mid x \in \mathbb{R}^d\}.$$

Since $X_{\mathcal{P}}$ is translation invariant (Lemma 2.1.43), we see $X_{\mathcal{P}}$ has FLC.

2⇒1. This direction is clear because $\{\mathcal{P} + x \mid x \in \mathbb{R}^d\} \subset X_{\mathcal{P}}$. □

Lemma 2.1.47. *Let X be an FLC subspace of* $\mathrm{Patch}(\mathbb{R}^d)$*. For any sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ of X, any compact $K \subset \mathbb{R}^d$ and any compact neighborhood V of $0 \in \mathbb{R}^d$, we can take a subsequence $\mathcal{P}_{n_1}, \mathcal{P}_{n_2}, \dots$ of $(\mathcal{P}_n)_n$ such that $(\mathcal{P}_{n_j}, \mathcal{P}_{n_k}) \in \mathcal{U}_{K,V}$ for any $j, k > 0$.*

Proof. Set $K' = K - V$. By FLC there is a subsequence $\mathcal{P}_{n_1}, \mathcal{P}_{n_2}, \dots$ and $x_1, x_2, \dots \in \mathbb{R}^d$ such that for any $j > 0$ we have

$$\mathcal{P}_{n_1} \cap K' = (\mathcal{P}_{n_j} \cap K') + x_j.$$

If $\mathcal{P}_{n_1} \cap K' = \emptyset$, we have nothing to prove and we may assume that we can take $T \in \mathcal{P}_{n_1} \cap K'$. Take $x \in T$, then $x - x_j \in K'$ for each j and we see $(x_j)_j$ is a bounded sequence. By taking subsequence again we may assume that $x_j - x_k \in V$ for any j, k. For any $j, k > 0$,

$$\mathcal{P}_{n_k} \cap K' = (\mathcal{P}_{n_j} \cap K') + x_j - x_k = (\mathcal{P}_{n_j} + x_j - x_k) \cap (K' + x_j - x_k)$$

and by Lemma 2.1.14,

$$\mathcal{P}_{n_k} \cap K = (\mathcal{P}_{n_j} + x_j - x_k) \cap K,$$

which implies $(\mathcal{P}_{n_k}, \mathcal{P}_{n_j}) \in \mathcal{U}_{K,V}$. □

Note that since the uniform space $(\mathrm{Patch}(\mathbb{R}^d), \mathfrak{U})$ is metrizable, for any $X \subset \mathrm{Patch}(\mathbb{R}^d)$ the following two conditions are equivalent:

- X is totally bounded, that is, for any $\mathcal{U} \in \mathfrak{U}$ there is a finite $F \subset X$ such that $X \subset \bigcup_{\mathcal{P} \in F} \mathcal{U}(\mathcal{P})$.
- For any sequence in X, there is a Cauchy subsequence of it.

Note also that any $X \subset \mathrm{Patch}(\mathbb{R}^d)$ is compact if and only if it is closed and totally bounded.

Lemma 2.1.48. *For any $X \subset \mathrm{Patch}(\mathbb{R}^d)$, consider the following conditions;*

1. *X has FLC.*
2. *X is totally bounded with respect to $\mathfrak{U}$.*

Then condition 1 always implies condition 2 and the converse holds if X is invariant under translation and the set $(\bigcup_{\mathcal{P} \in X} \mathcal{P})/\approx$ is finite (that is, there are only finitely many tile types up to translation).

Proof. 1⇒2. Take countably many open sets $O_1, O_2, \dots$ and a countable neighborhood basis $\{V_n \mid n > 0\}$ of 0 consisting of compact sets such that

- $K_n := \overline{O_n}$ is compact for each n, and
- $\bigcup_n O_n = \mathbb{R}^d$.

Take a sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ of X. By Lemma 2.1.47, we can take a subsequence $(\mathcal{P}_n^{(1)})$ of $(\mathcal{P}_n)$ such that $(\mathcal{P}_n^{(1)}, \mathcal{P}_m^{(1)}) \in \mathcal{U}_{K_1,V_1}$ for any $n, m > 0$. We further take a subsequence $(\mathcal{P}_n^{(2)})$ of $(\mathcal{P}_n^{(1)})$ such that $(\mathcal{P}_n^{(2)}, \mathcal{P}_m^{(2)}) \in \mathcal{U}_{K_2,V_2}$ for any $n, m > 0$. Proceeding in this way we can take subsequences $(\mathcal{P}_n^{(k)})_n$ for $k = 1, 2, \dots$. Set $Q_n = \mathcal{P}_n^{(n)}$ for each n, then $(\mathcal{Q}_n)_n$ is a Cauchy subsequence of $(\mathcal{P}_n)_n$.

2⇒1. Assume X is invariant under translation and $\bigcup_{\mathcal{P}\in X} \mathcal{P}/\approx$ is finite. Take a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of 0. By condition 2 there is a finite set $F \subset X$ such that

$$X \subset \bigcup_{\mathcal{P}\in F} \mathcal{U}_{V+K,V}(\mathcal{P}).$$

For any $\mathcal{Q} \in X$ there are $\mathcal{P} \in F$ and $x \in V$ such that $(\mathcal{Q} + x) \cap (K + V) = \mathcal{P} \cap (K + V)$, and

$$(\mathcal{Q} \cap K) + x = (\mathcal{Q} + x) \cap (K + x) \subset (\mathcal{Q} + x) \cap (K + V) = \mathcal{P} \cap (K + V).$$

Since $\mathcal{P} \cap (K + V)$ is finite for each $\mathcal{P} \in F$, by Lemma 2.1.36 $\{\mathcal{Q} \cap K \mid \mathcal{Q} \in X\}/\sim$ is finite. □

Corollary 2.1.49. *Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. Consider the following two conditions;*

1. *$\mathcal{P}$ has FLC.*
2. *$X_{\mathcal{P}}$ is compact with respect to the local matching topology.*

Then 1 always implies 2 and if $\mathcal{P}$ has FTT 2 implies 1.

Proof. Clear by Lemma 2.1.46, Lemma 2.1.48 and the fact that if $\mathcal{P}$ has FTT then $(\bigcup_{\mathcal{Q}\in X_{\mathcal{P}}} \mathcal{Q})/\approx$ is finite. □

Remark 2.1.50. If a tiling $\mathcal{T}$ has FTT, then on $X_{\mathcal{T}}$ the cylinder topology and the local matching topology coincide (Proposition 2.1.29). Thus if a tiling $\mathcal{T}$ has FLC and FTT the space $X_{\mathcal{T}}$ is compact with respect to the both topologies.

For some tiling $\mathcal{T}$ there is a patch $\mathcal{P}$ in $X_{\mathcal{T}}$ which is not a tiling ($\operatorname{supp} \mathcal{P} \neq \mathbb{R}^d$). For example, for the tiling $\mathcal{T}_s'$ in Example 2.1.31, we have $\emptyset \in X_{\mathcal{T}_s'}$. For tilings with finite tile type, we have the following lemma.

Definition 2.1.51. If a tiling $\mathcal{T}$ satisfies the condition

$$\sup_{T\in\mathcal{T}} \operatorname{diam} T < \infty \tag{2.7}$$

then we say $\mathcal{T}$ has bounded tile type.

Note that if $\mathcal{T}$ has FTT, it has bounded tile type.

Lemma 2.1.52. *If $\mathcal{T}$ has bounded tile type, then any $\mathcal{S} \in X_{\mathcal{T}}$ is a tiling.*

Proof. Take $\mathcal{S} \in X_{\mathcal{T}}$ and $R > 0$ arbitrarily. Take $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ such that $\frac{1}{\varepsilon} - \varepsilon > R + \sup_{T \in \mathcal{T}} \operatorname{diam} T$. There is $x \in \mathbb{R}^d$ such that $\rho(\mathcal{S}, \mathcal{T} + x) < \varepsilon$, and there is $y \in \mathbb{R}^d$ such that $\mathcal{S} \cap B(0, \frac{1}{\varepsilon} - \varepsilon) = (\mathcal{T} + x + y) \cap B(0, \frac{1}{\varepsilon} - \varepsilon)$. It follows that $\operatorname{supp} \mathcal{S} \sqcap B(0,R) = \operatorname{supp}(\mathcal{T} + x + y) \sqcap B(0,R) \supset B(0,R)$. ☐

The following another characterization of FLC will be useful.

Definition 2.1.53. For $R > 0$ and a tiling $\mathcal{T}$ of finite tile type with a set of proto-tiles $\mathcal{A}$, set

$$\Pi_{\mathcal{T},R,\mathcal{A}} = \{(\mathcal{T} - x) \cap B(0,R) \mid x \in \mathbb{R}^d, (\mathcal{T} - x) \cap \mathcal{A} \neq \emptyset\}.$$

Remark 2.1.54. This is a tiling-version of language for sequences.

Proposition 2.1.55. *Let $\mathcal{T}$ be a tiling of finite tile type with a set of proto-tiles $\mathcal{A}$. Then the following conditions are equivalent:*

1. *$\mathcal{T}$ has FLC.*
2. *$\Pi_{\mathcal{T},R,\mathcal{A}}$ is finite for all $R > 0$.*

Proof. (1)⇒ (2): It is enough to show that $\Pi_{\mathcal{T},R,\mathcal{A}}$ is finite for any $R > \max_{P \in \mathcal{A}} \operatorname{diam} P$ since there is a surjection $\Pi_{\mathcal{T},R,\mathcal{A}} \to \Pi_{\mathcal{T},R',\mathcal{A}}$ for $R' < R$.

Suppose $R > \max_{P \in \mathcal{A}} \operatorname{diam} P$. It suffices to show for each $\mathcal{P} \in \Pi_{\mathcal{T},R,\mathcal{A}}$, the set $Z_{\mathcal{P}} = \{z \in \mathbb{R}^d \mid \mathcal{P} + z \in \Pi_{\mathcal{T},R,\mathcal{A}}\}$ is finite since $\Pi_{\mathcal{T},R,\mathcal{A}}/\sim$ is finite by (1). Take $\mathcal{P} \in \Pi_{\mathcal{T},R,\mathcal{A}}$. Define a map $\varphi \colon Z_{\mathcal{P}} \to \mathcal{P}$ as follows. For $z \in Z_{\mathcal{P}}$, there is a unique $P \in (\mathcal{P} + z) \cap \mathcal{A}$. In fact, there are $y \in \mathbb{R}^d$ and $P \in \mathcal{A}$ such that $\mathcal{P} + z = (\mathcal{T} - y) \cap B(0,R)$ and $P \in (\mathcal{T} - y) \cap \mathcal{A}$. Since we assumed R was large enough, $P \in \mathcal{P} + z$. Set $\varphi(z) = P - z$. This map φ is injective. In fact, if $\varphi(z_1) = \varphi(z_2)$, take $P_i \in (\mathcal{P} + z_i) \cap \mathcal{A}$. Then $P_1 - z_1 = P_2 - z_2$ and so $P_1 = P_2$ and $z_1 = z_2$. Since $\mathcal{P}$ is finite, we see $Z_{\mathcal{P}}$ is finite.

(2)⇒(1): Take $R > 0$. Set $\Pi_1 = \{(\mathcal{T} - x) \cap B(0,R) \mid x \in \mathbb{R}^d\}$ and we show $\Pi_1/\sim$ is finite. Take $L > \max\{R, \max_{P \in \mathcal{A}} \operatorname{diam} P\}$ and set $\Pi_2 = \{(\mathcal{T} - x) \cap B(0,L) \mid x \in \mathbb{R}^d\}$. By Lemma 2.1.36, it suffices to show $\Pi_2/\sim$ is finite. Take $(\mathcal{T} - x) \cap B(0,L) \in \Pi_2$. Choose $P \in \mathcal{A}$ and $y \in \mathbb{R}^d$ such that $P + y \in (\mathcal{T} - x) \cap B(0,L)$. Then $\|y\| < L$ and $P \in \mathcal{T} - x - y$. We have

$$(\mathcal{T} - x) \cap B(0,L) - y = (\mathcal{T} - x - y) \cap B(-y, L) \subset (\mathcal{T} - x - y) \cap B(0, 2L).$$

We have proved that, for any $\mathcal{P} \in \Pi_2$ there are $y \in \mathbb{R}^d$ and $\mathcal{P}' \in \Pi_{\mathcal{T},2L,\mathcal{A}}$ such that $\mathcal{P} - y \subset \mathcal{P}'$. $\Pi_{\mathcal{T},2L,\mathcal{A}}/\sim$ is finite since $\Pi_{\mathcal{T},2L,\mathcal{A}}$ is finite by assumption. Therefore by Lemma 2.1.36, $\Pi_2/\sim$ is finite. ☐

2.1.2 Repetitivity

Definition 2.1.56. A subset $S \subset \mathbb{R}^d$ is said to be relatively dense if there is a compact $K \subset \mathbb{R}^d$ such that $S + K = \mathbb{R}^d$.

Definition 2.1.57. Take $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$. $\mathcal{P}$ is said to be repetitive if for any bounded patch $\mathcal{Q} \subset \mathcal{P}$, the set

$$\{x \in \mathbb{R}^d \mid \mathcal{Q} + x \subset \mathcal{P}\}$$

is relatively dense.

Lemma 2.1.58. *For any $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$, the following two conditions are equivalent;*

1. *$\mathcal{P}$ is repetitive.*
2. *For any bounded $\mathcal{Q} \subset \mathcal{P}$, there is an $R > 0$ such that the following condition holds:*

$$\text{For any } a \in \mathbb{R}^d \text{ there is } x \in \mathbb{R}^d \text{ such that } \mathcal{P} \cap B(a, R) \supset \mathcal{Q} + x.$$

Proof. 1⇒2. Take a bounded $\mathcal{Q} \subset \mathcal{P}$. We may assume $\mathcal{Q} \neq \emptyset$. Take a translate $\mathcal{Q}'$ of $\mathcal{Q}$ such that $0 \in \operatorname{supp} \mathcal{Q}'$. Since $S = \{x \in \mathbb{R}^d \mid \mathcal{Q}' + x \subset \mathcal{P}\}$ is relatively dense, there is $R_0 > 0$ such that $S + B(0, R_0) = \mathbb{R}^d$. For any $a \in \mathbb{R}^d$ there is $x \in S \cap B(a, R_0)$. Then $\mathcal{Q}' + x \subset \mathcal{P} \cap B(a, R_0 + \operatorname{diam} \operatorname{supp} \mathcal{Q})$. Thus 2 is satisfied for $R = R_0 + \operatorname{diam} \operatorname{supp} \mathcal{Q}$.

2⇒1. For any bounded $\mathcal{Q} \subset \mathcal{P}$, either $\mathcal{Q} = \emptyset$ or there is a translate $\mathcal{Q}'$ of $\mathcal{Q}$ such that $0 \in \operatorname{supp} \mathcal{Q}'$. Consider the latter case. Let $R > 0$ be a constant for $\mathcal{Q}$ in condition 2. For any $a \in \mathbb{R}^d$, there is $x \in \mathbb{R}^d$ such that $\mathcal{P} \cap B(a, R) \supset \mathcal{Q}' + x$. Then $x \in \overline{B(a, R)}$ and we see $S_{\mathcal{Q}'} = \{x \in \mathbb{R}^d \mid \mathcal{Q}' + x \subset \mathcal{P}\}$ is relatively dense. Since $S_{\mathcal{Q}} = \{x \in \mathbb{R}^d \mid \mathcal{Q} + x \subset \mathcal{P}\}$ is a translate of $S_{\mathcal{Q}'}$, the set $S_{\mathcal{Q}}$ is relatively dense. □

Definition 2.1.59. Let $\mathcal{P}$ be a patch. A patch $\mathcal{Q}$ is $\mathcal{P}$-legal if there is $x \in \mathbb{R}^d$ such that $\mathcal{Q} + x \subset \mathcal{P}$.

Definition 2.1.60. Define an equivalence relation $\sim_{\mathrm{LI}}$ on $\mathrm{Patch}(\mathbb{R}^d)$ as follows. For any two patches $\mathcal{P}_1, \mathcal{P}_2$, we have $\mathcal{P}_1 \sim_{\mathrm{LI}} \mathcal{P}_2$ if and only if $\mathcal{P}_1$-legality and $\mathcal{P}_2$-legality are equivalent for bounded patches, that is,

- for any bounded $\mathcal{Q} \subset \mathcal{P}_1$ there is $x \in \mathbb{R}^d$ such that $\mathcal{Q} + x \subset \mathcal{P}_2$, and
- for any bounded $\mathcal{Q} \subset \mathcal{P}_2$ there is $x \in \mathbb{R}^d$ such that $\mathcal{Q} + x \subset \mathcal{P}_1$.

Two patches $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P}_1 \sim_{\mathrm{LI}} \mathcal{P}_2$ are said to be locally indistinguishable. The equivalence class including $\mathcal{P}$ is represented by $[\mathcal{P}]_{\mathrm{LI}}$.

Lemma 2.1.61. *Take $R > 0$ and $\mathcal{T} \in \mathrm{Tiling}_R(\mathbb{R}^d)$ arbitrarily. Then $[\mathcal{T}]_{\mathrm{LI}} \cap \mathrm{Tiling}_R(\mathbb{R}^d) \subset X_{\mathcal{T}}$.*

Proof. Take $\mathcal{T}' \in [\mathcal{T}]_{\mathrm{LI}} \cap \mathrm{Tiling}_R(\mathbb{R}^d)$. For any compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$, there is $x \in \mathbb{R}^d$ such that $\mathcal{T}' \sqcap K + x \subset \mathcal{T}$. Since $\mathcal{T}' \sqcap K$ covers K, by Lemma 2.1.27 we have $\mathcal{T}' \cap K = (\mathcal{T} - x) \cap K$. This implies that $\mathcal{T} - x \in \mathcal{U}_{K,V}(\mathcal{T}')$. Since K and V were arbitrary, $\mathcal{T}' \in X_{\mathcal{T}}$. □

Lemma 2.1.62. *For any tiling $\mathcal{T}$ of $\mathbb{R}^d$, $\mathcal{S} \in X_{\mathcal{T}}$ and bounded $\mathcal{P} \subset \mathcal{S}$, there is $x \in \mathbb{R}^d$ such that $\mathcal{P} + x \subset \mathcal{T}$.*

Proof. Set $K = \operatorname{supp} \mathcal{P}$ and take an arbitrary compact neighborhood V of $0 \in \mathbb{R}^d$. There is $x \in \mathbb{R}^d$ such that $\mathcal{T} + x \in \mathcal{U}_{K,V}(\mathcal{S})$. Then there is $y \in V$ such that $\mathcal{P} = \mathcal{S} \cap K = (\mathcal{T} + x + y) \cap K$, and $\mathcal{P} - x - y \subset \mathcal{T}$. □

Proposition 2.1.63. *Take $R > 0$ and $\mathcal{T} \in \mathrm{Tiling}_R(\mathbb{R}^d)$ arbitrarily. Consider the following three conditions;*

1. *$\mathcal{T}$ is repetitive.*
2. *$[\mathcal{T}]_{\mathrm{LI}} \cap \mathrm{Tiling}_R(\mathbb{R}^d) = X_{\mathcal{T}}$.*
3. *The action $\mathbb{R}^d \curvearrowright X_{\mathcal{T}}$ is minimal.*

Then always condition 1 implies condition 2 and condition 2 and condition 3 are equivalent. If $\mathcal{T}$ has FLC, then condition 2 implies condition 1.

Proof. 1⇒2. Take $\mathcal{T}' \in X_{\mathcal{T}}$. If $\mathcal{P} \subset \mathcal{T}$ is a bounded patch, there is $R_0 > 0$ such that for any $a \in \mathbb{R}^d$ there is $x \in \mathbb{R}^d$ with $\mathcal{T} \cap B(a, R_0) \supset \mathcal{P} + x$. Set $K = \overline{B(0, R_0)}$ and take an arbitrary compact neighborhood V of $0 \in \mathbb{R}^d$. There exists $x \in \mathbb{R}^d$ such that $\mathcal{T} + x \in \mathcal{U}_{K,V}(\mathcal{T}')$. This means that there is $y \in V$ such that $(\mathcal{T} + x + y) \cap K = \mathcal{T}' \cap K$. By the property of R_0 there is $z \in \mathbb{R}^d$ such that $\mathcal{T} \cap B(-x - y, R_0) \supset \mathcal{P} + z$. Then

$$\mathcal{P} + x + y + z \subset (\mathcal{T} + x + y) \cap K = \mathcal{T}' \cap K,$$

and so $\mathcal{P} + x + y + z \subset \mathcal{T}'$. By Lemma 2.1.62 we have $\mathcal{T}' \in [\mathcal{T}]_{\mathrm{LI}}$. Hence $X_{\mathcal{T}} \subset [\mathcal{T}]_{\mathrm{LI}}$. Since $\mathrm{Tiling}_R(\mathbb{R}^d)$ is closed with respect to the local matching topology in $\mathrm{Patch}(\mathbb{R}^d)$, $X_{\mathcal{T}} \subset \mathrm{Tiling}_R(\mathbb{R}^d)$ and together with Lemma 2.1.61 we obtain condition 2.

2.⇒3. Take $\mathcal{T}', \mathcal{T}'' \in X_{\mathcal{T}}$. Take a compact $K \subset \mathbb{R}^d$ and a compact neighborhood V of $0 \in \mathbb{R}^d$. By condition 2 there is $x \in \mathbb{R}^d$ such that $\mathcal{T}' \sqcap (K + B(0, R)) + x \subset \mathcal{T}''$, and $\mathcal{T}' \cap K = (\mathcal{T}'' - x) \cap K$ by Lemma 2.1.27. This means that $\mathcal{T}'' - x \in \mathcal{U}_{K,V}(\mathcal{T}')$.

3⇒2. Take $\mathcal{T}' \in X_{\mathcal{T}}$. Take an arbitrary bounded non-empty patch $\mathcal{P} \subset \mathcal{T}$. Set $K = \operatorname{supp} \mathcal{P}$ and take a compact neighborhood V of $0 \in \mathbb{R}^d$. By minimality there is $x \in \mathbb{R}^d$ such that $\mathcal{T}' + x \in \mathcal{U}_{K,V}(\mathcal{T})$. There is $y \in V$ such that $(\mathcal{T}' + x + y) \cap K = \mathcal{T} \cap K \supset \mathcal{P}$ and $\mathcal{P} - x - y \subset \mathcal{T}'$. By Lemma 2.1.61 and Lemma 2.1.62 we obtain condition 2.

Finally we assume that $\mathcal{T}$ has FLC and satisfies condition 2 and we will prove condition 1. Suppose conversely that $\mathcal{T}$ is not repetitive. Then there are bounded $\mathcal{P} \subset \mathcal{T}$, $a_1, a_2, \ldots \in \mathbb{R}^d$ and $R_1, R_2, \ldots > 0$ such that

- The sequence (R_n) is monotone increasing and $\lim R_n = \infty$, and
- For each n the patch $\mathcal{T} \cap B(a_n, R_n)$ does not contain any translates of $\mathcal{P}$.

By Corollary 2.1.49 we can take a subsequence $(\mathcal{T} - a_{n_j})_j$ of the sequence $(\mathcal{T} - a_n)_n$ that converges to a tiling $\mathcal{T}_0 \in X_{\mathcal{T}}$. For any $R > 0$ and any compact neighborhood V of $0 \in \mathbb{R}^d$ there is $j_0 \in \mathbb{Z}_{>0}$ such that

$$j \geq j_0 \Rightarrow \mathcal{T} - a_{n_j} \in \mathcal{U}_{\overline{B(0,R)},V}(\mathcal{T}_0).$$

For large j, there is $x_j \in V$ such that

$$\begin{aligned} \mathcal{T}_0 \cap \overline{B(0,R)} &= (\mathcal{T} - a_{n_j} + x_j) \cap \overline{B(0,R)} \\ &\subset ((\mathcal{T} - a_{n_j}) \cap B(0, R_{n_j})) + x_j. \end{aligned}$$

This means there are no translates of $\mathcal{P}$ inside $\mathcal{T}_0 \cap B(0,R)$. Since R was arbitrary, there are no translates of $\mathcal{P}$ inside $\mathcal{T}_0$ and so $\mathcal{T}_0 \notin [\mathcal{T}]_{\mathrm{LI}}$. This contradicts condition 2. $\square$

2.2 Substitution rules

As was mentioned there are several ways to construct tilings of $\mathbb{R}^d$. In this section we introduce one of the ways, namely the way from substitution rules. After definitions we introduce some of important results.

Definition 2.2.1. Let $\mathcal{A}$ be a finite set of tiles in $\mathbb{R}^d$. Set

$$\mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d) = \{\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d) \mid \text{any tile } T \in \mathcal{P} \text{ is a translate of a tile in } \mathcal{A}\}.$$

Lemma 2.2.2. *The set* $\mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ *is a closed subset of* $\mathrm{Patch}(\mathbb{R}^d)$ *with respect to the local matching topology.*

Proof. Take $\mathcal{P} \in \overline{\mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)}$ and $T \in \mathcal{P}$. This patch $\mathcal{P}$ and an element $\mathcal{Q} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ coincide, after a small translation, inside a large ball around the origin. Thus for some $\mathcal{Q} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ a translate of T appears in $\mathcal{Q}$ and T is a translate of an element of $\mathcal{A}$. $\square$

Definition 2.2.3. A linear map $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ is said to be expanding if $|\lambda| > 1$ for any eigenvalue λ for φ.

Definition 2.2.4. A substitution rule (of $\mathbb{R}^d$) is a triple $(\mathcal{A}, \varphi, \omega)$ where

- $\mathcal{A}$ is a finite nonempty set of tiles in $\mathbb{R}^d$,
- φ is an expanding linear map of $\mathbb{R}^d$, and

- ω is a map $\omega\colon \mathcal{A} \to \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ such that

$$\operatorname{supp} \omega(P) = \overline{\varphi(P)}.$$

Tiles in $\mathcal{A}$ are called proto-tiles for the substitution rule.

Remark 2.2.5. In plain language, a substitution rule is an operation to expand each proto-tile, subdivide it and obtain a patch consisting of translates of proto-tiles. The following example will illuminate this point.

We can also consider substitution rules with rotations or flips. Radin's pinwheel tiling [22] is an example. We do not deal with such substitution rules in this article.

Example 2.2.6 (Figure2.1)**.** Set $\tau = \frac{1+\sqrt{5}}{2}$. Take the interior of the triangle which has side-length 1,1, and τ, and remove one point anywhere from the left side or the right side. Moreover take the interior of the triangle of the side-length τ, τ and 1, and remove one point from the left side or the right side. The proto-tiles of this substitution are the copies of these two punctured triangles by $2n\pi/10$-rotations and flip, where $n = 0, 1, \dots, 9$. There are 40 proto-tiles.

The expansion map is τI, where I is the identity map. The map ω is depicted in Figure 2.1. The image of the other proto-tiles by ω is defined accordingly, so that ω and rotations, ω and flips will commute.

Tilings for this substitution are called Robinson triangle tilings. Such tilings are known to be related (MLD) to Penrose tilings by kites and darts.

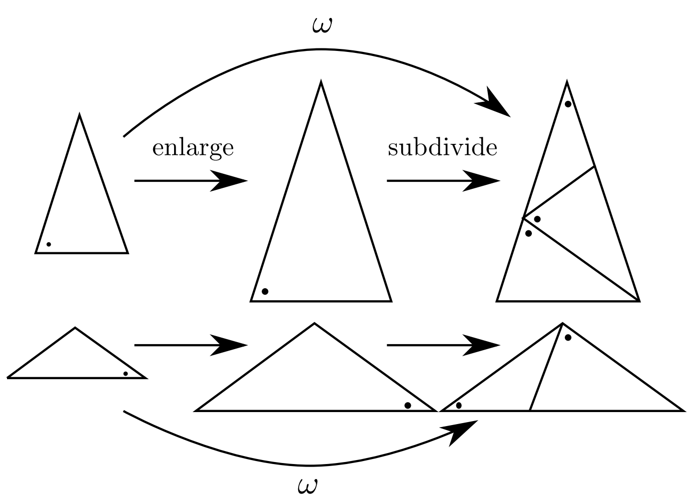

Figure 2.1: Example of substitution

Definition 2.2.7. For a substitution rule $(\mathcal{A}, \varphi, \omega)$, $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$, we set a patch $\omega(P + x) \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ by

$$\omega(P + x) = \omega(P) + \varphi(x).$$

An easy computation shows the next lemma:

Lemma 2.2.8. *Let $(\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then* $\operatorname{supp} \omega(P + x) = \varphi(\overline{P}) + \varphi(x) = \varphi(\overline{P + x})$.

Definition 2.2.9. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Define a map $\omega_\sigma \colon \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d) \to \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ by

$$\omega_\sigma(\mathcal{P}) = \bigcup_{T \in \mathcal{P}} \omega(T).$$

Lemma 2.2.10. *For any substitution rule σ, the map ω_σ is well defined, that is, for any $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ we have $\omega_\sigma(\mathcal{P}) \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$. Moreover the following conditions hold:*

- *For any $\mathcal{P}_1, \mathcal{P}_2, \ldots \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$, if $\bigcup_n \mathcal{P}_n$ is a patch, then we have $\omega_\sigma(\bigcup \mathcal{P}_n) = \bigcup \omega_\sigma(\mathcal{P}_n)$.*
- *For any $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$, $\operatorname{supp} \omega_\sigma(\mathcal{P}) = \varphi(\operatorname{supp} \mathcal{P})$.*
- *For any $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $m \in \mathbb{Z}_{>0}$, $\omega_\sigma^m(\mathcal{P} + x) = \omega_\sigma^m(\mathcal{P}) + \varphi^m(x)$.*

The following lemma also holds for the local matching topology, but we omit the proof.

Lemma 2.2.11. *For any substitution rule $\sigma = (\mathcal{A}, \varphi, \omega)$, the map ω_σ is continuous with respect to the cylinder topology.*

Proof. Take $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$. Take any finite $\mathcal{Q} \subset \omega_\sigma(\mathcal{P})$ and a neighborhood U of 0 in $\mathbb{R}^d$ (cf. Lemma 2.1.10). For any $T \in \mathcal{Q}$ there is $S_T \in \mathcal{P}$ such that $T \in \omega(S_T)$. Set $\mathcal{P}' = \{S_T \mid T \in \mathcal{Q}\}$ and $U' = \varphi^{-1}(U)$. Then $\omega_\sigma(C(U', \mathcal{P}') \cap \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)) \subset C(U, \mathcal{Q})$. □

Remark 2.2.12. Often in the literature the letter σ is suppressed and ω_σ is simply written as ω. Clearly, $\omega_\sigma(\{P + x\}) = \omega(P + x)$ for $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$.

Definition 2.2.13. A substitution rule $(\mathcal{A}, \varphi, \omega)$ is said to be primitive if there is $K \in \mathbb{Z}_{>0}$ such that for any $P, P' \in \mathcal{A}$ there is $x \in \mathbb{R}^d$ with $P + x \in \omega_\sigma^K(\{P'\})$.

Definition 2.2.14. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. A patch $\mathcal{P} \in \mathrm{Patch}(\mathbb{R}^d)$ is said to be σ-legal if there are $P \in \mathcal{A}, n \in \mathbb{Z}_{>0}$ and $x \in \mathbb{R}^d$ such that

$$\mathcal{P} \subset \omega_\sigma^n(\{P + x\}).$$

Definition 2.2.15. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule of $\mathbb{R}^d$. Define

$$X_\sigma = \{\mathcal{T} \in \mathrm{Tiling}(\mathbb{R}^d) \mid \text{if } \mathcal{P} \subset \mathcal{T} \text{ is a finite patch, then } \mathcal{P} \text{ is } \sigma\text{-legal}\}.$$

In the following arguments we show X_σ is not empty.

Lemma 2.2.16. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. There are $P \in \mathcal{A}$, $m > 0$, and $x \in \mathbb{R}^d$ such that*

- $P + x \in \omega_\sigma^m(\{P\})$*, and*
- $\overline{P + x} \subset \varphi^m(P)$.

Proof. Since φ is expanding, for any $P \in \mathcal{A}$, there are $m > 0, x \in \mathbb{R}^d$ and $P' \in \mathcal{A}$ such that

$$P' + x \in \omega_\sigma^m(\{P\}), \text{ and} \tag{2.8}$$

$$\overline{P' + x} \subset \varphi^m(P). \tag{2.9}$$

If for some m, x the conditions (2.8) and (2.9) hold, we write $P \rightsquigarrow P'$.

We have a sequence $P_1, P_2, \dots$ of $\mathcal{A}$ such that for each n we have $P_n \rightsquigarrow P_{n+1}$. Since $\mathcal{A}$ is finite, for some k, l with $k < l$ we obtain $P_k = P_l$. Thus it suffices to show that if P, P', P'' are in $\mathcal{A}$ and $P \rightsquigarrow P'$ and $P' \rightsquigarrow P''$ hold, then $P \rightsquigarrow P''$. But this is clear by a simple computation. □

Lemma 2.2.17. *Take a finite nonempty set $\mathcal{A}$ of tiles. Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map. Let $\omega\colon \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d) \to \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ be a map such that $\operatorname{supp}\omega(\mathcal{P}) = \varphi(\operatorname{supp}\mathcal{P})$ for each $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$. Suppose there is $\mathcal{P}_0 \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ such that*

- $\mathcal{P}_0 \subset \omega(\mathcal{P}_0)$*, and*
- $\operatorname{supp}\mathcal{P}_0 \subset \varphi(\operatorname{supp}\mathcal{P}_0)^\circ$.

Then there is $r > 0$ such that $\operatorname{supp}\omega^n(\mathcal{P}_0) \supset \varphi^n(B(0, r))$ for any $n \in \mathbb{Z}_{>0}$.

Proof. For each n we have $(\operatorname{supp}\mathcal{P}_0)^\circ \supset \varphi^{-n}(\operatorname{supp}\mathcal{P}_0) \supset \varphi^{-n-1}(\operatorname{supp}\mathcal{P}_0)$. Take $x \in \varphi^{-1}(\operatorname{supp}\mathcal{P}_0)$. Then $0 = \lim_n \varphi^{-n}(x) \in \varphi^{-1}(\operatorname{supp}\mathcal{P}_0) \subset (\operatorname{supp}\mathcal{P}_0)^\circ$. There exists $r > 0$ such that $B(0, r) \subset \operatorname{supp}\mathcal{P}_0$. For each n

$$\begin{aligned}
\operatorname{supp}\omega^n(\mathcal{P}_0) &= \varphi(\operatorname{supp}\omega^{n-1}\mathcal{P}_0) \\
&= \varphi^2(\operatorname{supp}\omega^{n-2}\mathcal{P}_0) \\
&= \cdots \\
&= \varphi^n(\operatorname{supp}\mathcal{P}_0) \\
&\supset \varphi^n(B(0, r)).
\end{aligned}$$

□

Proposition 2.2.18. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then there are $P \in \mathcal{A}$, $b \in \mathbb{R}^d$ and $m \in \mathbb{Z}_{>0}$ such that $P + b \in \omega_\sigma^m(\{P + b\})$ and*

$$\bigcup_{n>0} \omega_\sigma^{nm}(\{P + b\})$$

is a tiling in X_σ.

Proof. By Lemma 2.2.16, there are $P \in \mathcal{A}, a \in \mathbb{R}^d$ and $m \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned} P + a &\in \omega_\sigma^m(\{P\}), \text{ and} \\ \overline{P + a} &\subset \varphi^m(P). \end{aligned}$$

Since φ is expansive, the linear map $I - \varphi^m$ is invertible. Set $b = (I - \varphi^m)^{-1}(a)$. Then we have

$$P + b \in \omega_\sigma^m(\{P + b\}), \text{ and} \tag{2.10}$$

$$\overline{P + b} \subset \varphi^m(P + b). \tag{2.11}$$

Set

$$\mathcal{T} = \bigcup_{n>0} \omega_\sigma^{nm}(\{P + b\}).$$

By (2.10), $\mathcal{T}$ is a patch. Moreover $\operatorname{supp} \omega_\sigma^m(\mathcal{P}) = \varphi^m(\operatorname{supp} \mathcal{P})$ for any $\mathcal{P} \in \operatorname{Patch}_{\mathcal{A}}(\mathbb{R}^d)$ and $\operatorname{supp}\{P + b\} \subset \varphi^m((\operatorname{supp}\{P + b\})^\circ)$. Applying Lemma 2.2.17 for $\mathcal{P}_0 = \{P + b\}$ and $\omega = \omega_\sigma^m$, we see $\operatorname{supp} \mathcal{T} = \mathbb{R}^d$ and so $\mathcal{T}$ is a tiling.

Finally if $\mathcal{P}$ is a finite subset of $\mathcal{T}$, then for some n, the patch $\mathcal{P}$ is included in $\omega_\sigma^{mn}(\{P + b\})$ and so $\mathcal{P}$ is σ-legal. Thus $\mathcal{T}$ is in X_σ. □

Lemma 2.2.19. *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ such that $\sup_{T \in \mathcal{T}} \operatorname{diam} T < r$ for some $r > 0$. Then for any subset $S \subset \mathbb{R}^d$, $\operatorname{supp} \mathcal{T} \cap (S + \overline{B(0, r)}) \supset S$.*

Lemma 2.2.20. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then X_σ is closed in $\operatorname{Patch}(\mathbb{R}^d)$ (and in $\operatorname{Patch}_{\mathcal{A}}(\mathbb{R}^d)$) with respect to the local matching topology.*

Proof. Take $\mathcal{T} \in \overline{X_\sigma}$. For any compact $K \subset \mathbb{R}^d$ and any compact neighborhood V of $0 \in \mathbb{R}^d$, there is $\mathcal{T}' \in \mathcal{U}_{K,V}(\mathcal{T}) \cap X_\sigma$. We can take $x \in V$ such that $\mathcal{T} \cap K = (\mathcal{T}' + x) \cap K$. Thus if $\mathcal{P}$ is a finite subset of $\mathcal{T}$, by taking K large enough, we see that there are $\mathcal{T}' \in X_\sigma$ and $x \in \mathbb{R}^d$ such that $\mathcal{P} - x \subset \mathcal{T}'$. Since $\mathcal{P} - x$ is σ-legal, $\mathcal{P}$ is also σ-legal. Next, for any compact L set $K = L + \overline{B(0, r)}$ where $r > \max_{P \in \mathcal{A}} \operatorname{diam} P$. By the above argument and Lemma 2.2.19,

$$\begin{aligned} \operatorname{supp} \mathcal{T} &\supset \operatorname{supp} \mathcal{T} \cap K \\ &= \operatorname{supp}(\mathcal{T}' + x) \cap K \\ &\supset L \end{aligned}$$

for some $\mathcal{T}' \in X_\sigma$ and $x \in \mathbb{R}^d$. It follows that $\operatorname{supp} \mathcal{T} = \mathbb{R}^d$ and so $\mathcal{T} \in X_\sigma$. □

Remark 2.2.21. By Proposition 2.1.29, X_σ is closed in $\mathrm{Tiling}_R(\mathbb{R}^d)$ with respect to the cylinder topology for any $R > \max_{P\in\mathcal{A}} \operatorname{diam} P$.

Remark 2.2.22. Since a translate of σ-legal patch is again σ-legal, it is clear that X_σ is invariant under translation.

Lemma 2.2.23. *If $\mathcal{T} \in X_\sigma$, then $\omega_\sigma(\mathcal{T}) \in X_\sigma$.*

Proof. Take a finite $\mathcal{P} \subset \omega_\sigma(\mathcal{T})$. For any $T \in \mathcal{P}$ there is $S_T \in \mathcal{T}$ such that $T \in \omega(S_T)$. Set $\mathcal{P}' = \{S_T \mid T \in \mathcal{P}\}$, then $\mathcal{P} \subset \omega_\sigma(\mathcal{P}')$. Since $\mathcal{P}'$ is σ-legal, there are P, x, n such that $\mathcal{P}' \subset \omega_\sigma^n(\{P + x\})$, and $\mathcal{P} \subset \omega_\sigma^{n+1}(\{P + x\})$. This means that $\mathcal{P}$ is σ-legal. Moreover $\operatorname{supp} \omega_\sigma(\mathcal{T}) = \varphi(\operatorname{supp} \mathcal{T}) = \mathbb{R}^d$ by Lemma 2.2.10. □

Proposition 2.2.24 ([1], Proposition 2.2)**.** *Let $(\mathcal{A}, \varphi, \omega)$ be a substitution rule. Then $\omega_\sigma \colon X_\sigma \to X_\sigma$ is surjective.*

The following easy lemmas will be useful later.

Lemma 2.2.25. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule and take $n \in \mathbb{Z}_{>0}$. Then*

$$\omega_\sigma^n(\mathcal{P}) = \bigcup_{T\in\mathcal{P}} \omega_\sigma^n(\{T\})$$

for any $\mathcal{P} \in \mathrm{Patch}_{\mathcal{A}}(\mathbb{R}^d)$.

Definition 2.2.26. For a substitution rule $\sigma = (\mathcal{A}, \varphi, \omega)$ and $n \in \mathbb{Z}_{>0}$, define a substitution rule σ^n by $\sigma^n = (\mathcal{A}, \varphi^n, \omega^n)$ where $\omega^n(P) = \omega_\sigma^n(\{P\})$ for each $P \in \mathcal{A}$.

Remark 2.2.27. If σ is primitive, then so is σ^n for any n.

Lemma 2.2.28. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule and take $n \in \mathbb{Z}_{>0}$. Then $(\omega_\sigma)^n = (\omega^n)_{\sigma^n}$ (the iterate of ω_σ coincides with the map associated to σ^n in regard to Definition 2.2.9).*

Lemma 2.2.29. *Let σ be a primitive substitution. Then for any $n \in \mathbb{Z}_{>0}$ we have $X_\sigma = X_{\sigma^n}$.*

Proof. Take $\mathcal{T} \in X_{\sigma^n}$ and a finite subset $\mathcal{P} \subset \mathcal{T}$. There are $P \in \mathcal{A}, m > 0$ and $x \in \mathbb{R}^d$ such that $\mathcal{P} \subset \omega_\sigma^{nm}(\{P + x\})$ (cf. Lemma 2.2.28). This shows that $\mathcal{P}$ is σ-legal and $\mathcal{T} \in X_\sigma$.

Next, take $\mathcal{T} \in X_\sigma$ and finite $\mathcal{P} \subset \mathcal{T}$. There are $P \in \mathcal{A}, m > 0$ and $x \in \mathbb{R}^d$ such that $\mathcal{P} \subset \omega_\sigma^m(\{P + x\})$. There is $K \in \mathbb{Z}_{>0}$ as in Definition 2.2.13. Take $l \in \mathbb{Z}_{>0}$ such that $nl \geq K + m$. We can take $y \in \mathbb{R}^d$ such that $P + y \in \omega_\sigma^{nl-m}(\{P\})$. Then

$$\mathcal{P} \subset (\omega_{\sigma^n}^n)^l(\{P + \varphi^{m-nl}(x - y)\}),$$

and so $\mathcal{P}$ is σ^n-legal. □

Definition 2.2.30. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. If the set

$$\{\omega_\sigma^n(\{P\}) \cap B(x, R) \mid P \in \mathcal{A}, n > 0, x \in \mathbb{R}^d\}/\sim$$

is finite for each $R > 0$, then σ is said to have FLC.

Note that by Proposition 2.1.29, on X_σ the relative topologies of the local matching topology and the cylinder topology coincide. We endow X_σ this relative topology.

Lemma 2.2.31. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a primitive substitution rule. Then the following conditions are equivalent:*

1. *σ has FLC.*
2. *X_σ has FLC.*
3. *X_σ is compact.*

Proof. 1⇒2. Suppose σ has FLC. Take a positive number $R > 0$. Take $\mathcal{T} \in X_\sigma$ and $x \in \mathbb{R}^d$, and set $\mathcal{P} = \mathcal{T} \cap B(x, R)$. By definition of X_σ there are $P \in \mathcal{A}, n > 0$ and $y \in \mathbb{R}^d$ such that $\mathcal{P} \subset \omega_\sigma^n(\{P + y\})$. For some $z \in \mathbb{R}^d$ a translate of $\mathcal{P}$ appears inside $\omega_\sigma^n(\{P\}) \cap B(z, R)$. Thus by Lemma 2.1.36,

$$\{\mathcal{T} \cap B(x, R) \mid x \in \mathbb{R}^d, \mathcal{T} \in X_\sigma\}/\sim$$

is finite.

2⇒1. Take $R > 0$ arbitrarily. If $P \in \mathcal{A}$, then by primitivity and Lemma 2.2.23, there is $\mathcal{T} \in X_\sigma$ such that $P \in \mathcal{T}$. Take $n \in \mathbb{Z}_{>0}$ and $x \in \mathbb{R}^d$. Then $\omega_\sigma^n(\{P\}) \cap B(x, R) \subset \omega_\sigma^n(\mathcal{T}) \cap B(x, R)$. Since $\omega_\sigma^n(\mathcal{T}) \in X_\sigma$ (Lemma 2.2.23), by Lemma 2.1.36 and condition 2,

$$\{\omega_\sigma^n(\{P\}) \cap B(x, R) \mid P \in \mathcal{A}, n > 0, x \in \mathbb{R}^d\}/\sim$$

is finite.

The equivalence of 2 and 3 follows from Lemma 2.2.20 and Lemma 2.1.48. □

Remark 2.2.32. It is known that given a substitution rule $\sigma = (\mathcal{A}, \varphi, \omega)$, it is often possible to prove FLC of σ by observing coronas (Definition 2.1.39) in iterates $\omega_\sigma^n(\{P\})$ for any $P \in \mathcal{A}$ and small $n \in \mathbb{Z}_{>0}$.

Remark 2.2.33. By Lemma 2.2.29 and Lemma 2.2.31, we see that for any primitive σ and $n > 0$, σ has FLC if and only if σ^n has FLC.

If σ has FLC we obtain a topological dynamical system $(X_\sigma, \mathbb{R}^d)$ by the action of translations.

Proposition 2.2.34. *If $\sigma = (\mathcal{A}, \varphi, \omega)$ is primitive, then $(X_\sigma, \mathbb{R}^d)$ is minimal and any $\mathcal{T} \in X_\sigma$ is repetitive.*

Proof. Let K be a positive integer appearing in Definition 2.2.13. Take $r > \max_{P\in\mathcal{A}} \operatorname{diam} P$. Take $\mathcal{T}, \mathcal{S} \in X_\sigma$ and a finite $\mathcal{P} \subset \mathcal{T}$ arbitrarily. By the definition of X_σ, there are $P \in \mathcal{A}, y \in \mathbb{R}^d$ and $n > 0$ such that $\mathcal{P} \subset \omega_\sigma^n(\{P + y\})$. Take $R > 0$ such that $\varphi^{K+n}B(0, r) \subset B(0, R)$. We claim

$$\text{for any } x \in \mathbb{R}^d, \text{ there is a translate of } \mathcal{P} \text{ in } \mathcal{S} \cap B(x, R). \tag{2.12}$$

Take $x \in \mathbb{R}^d$. By Proposition 2.2.24, there is $\mathcal{S}' \in X_\sigma$ such that $\omega_\sigma^{n+K}(\mathcal{S}') = \mathcal{S}$. We can take $T \in \mathcal{S}'$ such that $\varphi^{-n-K}(x) \in \overline{T}$. Then there is a translate of P in $\omega_\sigma^K(\{T\})$, and there is a translate of $\mathcal{P}$ in $\omega_\sigma^{n+K}(\{T\})$. Since $\mathcal{S} \supset \omega_\sigma^{n+K}(\{T\})$ and $\operatorname{supp} \omega_\sigma^{n+K}(\{T\}) \subset B(x, R)$, there is a translate of $\mathcal{P}$ in $\mathcal{S} \cap B(x, R)$. Thus the claim (2.12) is proved. This firstly means that a translate of $\mathcal{S}$ contains $\mathcal{P}$. By Lemma 2.1.10, this implies that for any neighborhood of $\mathcal{T}$, a translate of $\mathcal{S}$ is a member of that neighborhood. This means that $(X_\sigma, \mathbb{R}^d)$ is minimal. Secondly the claim (2.12) shows that (by considering the case where $\mathcal{S} = \mathcal{T}$) $\mathcal{T}$ is repetitive. □

Remark 2.2.35. This proposition shows that, if σ is primitive then $X_\sigma = X_\mathcal{S}$ for any $\mathcal{S} \in X_\sigma$.

Definition 2.2.36. Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. A tiling $\mathcal{T} \in \operatorname{Patch}_\mathcal{A}(\mathbb{R}^d)$ is called a fixed point if $\omega_\sigma(\mathcal{T}) = \mathcal{T}$. A repetitive tiling of FLC which is a fixed point of some substitution rule is called a self-affine tiling.

Lemma 2.2.37. *Let σ be a substitution rule and $\mathcal{T}$ be its fixed point. If $\mathcal{T}$ is repetitive, then $\mathcal{T} \in X_\sigma$. If σ is primitive and $\mathcal{T} \in X_\sigma$, then $\mathcal{T}$ is repetitive.*

Proof. Suppose $\mathcal{T}$ is repetitive. Take a finite $\mathcal{P} \subset \mathcal{T}$. There exists $R > 0$ such that for any $x \in \mathbb{R}^d$ there is $y \in \mathbb{R}^d$ with $\mathcal{T} \cap B(x, R) \supset \mathcal{P} + y$. For arbitrary $T \in \mathcal{T}$, if n is large enough the support of the patch $\omega_\sigma^n(\{T\})$ contains a ball of radius R. Hence a translate of $\mathcal{P}$ appears in $\omega_\sigma^n(\{T\})$, and so $\mathcal{P}$ is σ-legal. Hence $\mathcal{T} \in X_\sigma$. The converse under the assumption of primitivity is proved in Proposition 2.2.34. □

Lemma 2.2.38. *For any primitive substitution rule σ there is $n > 0$ such that σ^n admits a repetitive fixed point.*

Proof. This is clear by Proposition 2.2.18, Lemma 2.2.29 and Lemma 2.2.37. □

Remark 2.2.39. Often we assume a primitive substitution admits a repetitive fixed point because we may replace the original substitution σ with σ^n for some n.

Theorem 2.2.40 ([27], [13])**.** *If a substitution rule σ is primitive and FLC, then the corresponding topological dynamical system $(X_\sigma, \mathbb{R}^d)$ is uniquely ergodic, that is, it admits a unique invariant probability measure.*

We recall mixing property of dynamical systems in Definition A.0.9.

Theorem 2.2.41 ([27], Theorem 4.1)**.** *Let σ be a primitive substitution of FLC. Then the dynamical system $(X_\sigma, \mathbb{R}^d, \mu)$ is not mixing, where μ is the unique invariant probability measure.*

Remark 2.2.42. The proof of the previous theorem is decomposed into two parts. Let $\mathcal{T}$ be a repetitive fixed point. First, we can prove the following: take any $T \in \mathcal{T}$ and any vector x such that $T + x \in \mathcal{T}$. Then there is $c > 0$ such that for any finite patch $\mathcal{P}$ and $n \in \mathbb{Z}_{>0}$, we have

$$\lim_N \frac{L(\mathcal{P} \cup (\mathcal{P} + \varphi^n(x)), \mathcal{T} \cap A_N)}{m(A_N)} \geqq c\frac{L(\mathcal{P}, \omega^n(T))}{m(\varphi^n(T))}. \tag{2.13}$$

Here,

- $L(\mathcal{P}, \mathcal{Q}) = \mathrm{card}\{x \in \mathbb{R}^d \mid \mathcal{P} + x \subset \mathcal{Q}\}$ (the number of translates of $\mathcal{P}$ inside $\mathcal{Q}$) for any patch $\mathcal{P}, \mathcal{Q}$.
- A_N is the ball of radius N with its center 0, or more generally (A_N) is a van Hove sequence.

The left-hand side of inequality (2.13) is called the frequency of the patch $\mathcal{P} \cup (\mathcal{P} + \varphi^n(x))$. In plain language, this inequality means that there is positive probability of finding another translate of $\mathcal{P}$ after finding a translate of $\mathcal{P}$ in the tiling $\mathcal{T}$ and moving our attention by a vector $\varphi^n(x)$ from that position.

Next, from this fact about the distribution of patches we can prove the property of the dynamical system, i.e. , that the dynamical system is not mixing. This is an example of a relation between distribution of patches in tilings and corresponding tiling dynamical systems.

Solomyak [28] proved the recognizability of certain substitution rules, which is a tiling analogue of [18].

Theorem 2.2.43 ([28])**.** *Let σ be a primitive substitution rule of FLC. Then $\omega_\sigma \colon X_\sigma \to X_\sigma$ is injective if and only if each $\mathcal{T} \in X_\sigma$ is non-periodic.*

In this theorem the "if" part is hard to show. For the "only if" part see for example [1], Proposition 2.3.

For examples of substitution rule the following lemma is useful to prove that ω_σ is injective.

Lemma 2.2.44. *Let $\sigma = (\mathcal{A}, \varphi, \omega)$ be a substitution rule. Suppose that the following three conditions*

- *$P \in \mathcal{A}$,*
- *$\mathcal{P}$ is a σ-legal finite patch, and*

- $\omega(P) \subset \omega_\sigma(\mathcal{P})$,

imply $P \in \mathcal{P}$. Then $\omega_\sigma \colon X_\sigma \to X_\sigma$ is injective.

Proof. Take $\mathcal{T}, \mathcal{S} \in X_\sigma$ and assume $\omega_\sigma(\mathcal{T}) = \omega_\sigma(\mathcal{S})$. Take $T \in \mathcal{T}$ arbitrarily. Set $\mathcal{P} = \mathcal{S} \sqcap \overline{T}$. Then $\operatorname{supp} \omega(T) \subset \operatorname{supp} \omega_\sigma(\mathcal{P})$. Since $\omega(T) \subset \omega_\sigma(\mathcal{S})$, we have $\omega(T) \subset \omega_\sigma(\mathcal{P})$. There are $P \in \mathcal{A}$ and $x \in \mathbb{R}^d$ such that $T = P + x$. We have $\omega(P) \subset \omega_\sigma(\mathcal{P} - x)$ and by the assumption of this lemma we obtain $P \in \mathcal{P} - x$, and so $T \in \mathcal{P} \subset \mathcal{S}$. Hence $\mathcal{T} \subset \mathcal{S}$ and so $\mathcal{T} = \mathcal{S}$. $\square$

By the following theorem we see for certain dynamical systems from substitution, topological and measurable eigenvalues coincide and any measurable eigenfunction can be taken continuous. (These notions are explained in Appendix.)

Theorem 2.2.45 ([31], Theorem 3.13)**.** *Let $(\mathcal{A}, \varphi, \omega)$ be a primitive tiling substitution of FLC. Assume there is a repetitive fixed point $\mathcal{T}$ for this substitution. Then for $\xi \in \mathbb{R}^d$, the following conditions are equivalent:*

1. *ξ is a topological eigenvalue for the topological dynamical system $(X_\sigma, \mathbb{R}^d)$;*
2. *ξ is a measurable eigenvalue for the measure-preserving system $(X_\sigma, \mathbb{R}^d, \mu)$, where μ is the unique invariant probability measure;*
3. *ξ satisfies the following two conditions:*
   (a) For any return vector z (cf. Definition 2.1.2) for $\mathcal{T}$, we have
   $$\lim_{n \to \infty} e^{2\pi i \langle \varphi^n(z), \xi \rangle} = 1, \tag{2.14}$$
   and
   (b) if $z \in \mathbb{R}^d$ and $\mathcal{T} + z = \mathcal{T}$, then
   $$e^{2\pi i \langle z, \xi \rangle} = 1.$$

Definition 2.2.46.

- An algebraic integer $\lambda > 1$ is called a Pisot number if any Galois conjugates μ except λ itself satisfy $|\mu| < 1$.
- Let Λ be a finite non-empty set of algebraic integers. We say Λ is a Pisot family if the following condition holds:

  if $\lambda \in \Lambda$, $\mu \notin \Lambda$ and λ and μ are Galois conjugate, then $|\mu| < 1$.

For example, $\tau = \frac{1+\sqrt{5}}{2}$ is a Pisot number because τ and $\frac{1-\sqrt{5}}{2}$ are all of its Galois conjugates. A one-point set $\{\tau\}$ forms a Pisot family.

For a linear map $\varphi \colon \mathbb{R}^d \to \mathbb{R}^d$, its spectrum $\operatorname{sp}(\varphi)$ is by definition the set of all eigenvalues.

For a linear map $\phi \colon \mathbb{R}^d \to \mathbb{R}^d$, its adjoint is denoted by ϕ^* and its spectrum is denoted by $\operatorname{sp}(\phi)$.

Theorem 2.2.47 ([14], Theorem 2.8)**.** *Let $(\mathcal{A}, \phi, \omega)$ be a primitive substitution rule of FLC. Assume ϕ is diagonalizable over $\mathbb{C}$ and all the eigenvalues are algebraic conjugates of the same multiplicity. Then the following conditions are equivalent:*

1. *The set* $\mathrm{sp}(\phi)$ *is a Pisot family.*
2. *The set of (topological and measurable) eigenvalues for $(X_\omega, \mathbb{R}^d)$ is relatively dense.*

Finally we briefly mention pseudo-self-affine tilings. For the definition of $\overset{\mathbb{R}^d}{\to}$ and MLD, see Definition 3.2.16.

Definition 2.2.48 ([30])**.** A repetitive FLC tiling $\mathcal{T}$ is called a pseudo-self-affine tiling if $\phi(\mathcal{T}) \overset{\mathbb{R}^d}{\to} \mathcal{T}$ for some expanding linear map $\phi\colon \mathbb{R}^d \to \mathbb{R}^d$.

Remark 2.2.49. Self-affine tilings are pseudo-self-affine tilings.

Theorem 2.2.50 ([30])**.** *Let $\mathcal{T}$ be a pseudo-self-affine tilings with an expansion map ϕ. Then for any $k \in \mathbb{Z}_{>0}$ sufficiently large, there exists a tiling $\mathcal{T}'$ which is self-affine with expansion ϕ^k such that $\mathcal{T}$ is MLD with $\mathcal{T}'$.*

2.3 Results on relation between properties of tilings and properties of the corresponding dynamical systems

Proposition 2.1.63 describes a relation between the distribution of patches in a tiling and a property of the corresponding dynamical system. Here we mention another relation.

Definition 2.3.1. We endow a metric $\rho_{\mathbb{T}}$ on $\mathbb{T}$ by identifying $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$. In other words we set

$$\rho_{\mathbb{T}}(e^{2\pi i\theta}, e^{2\pi i\theta'}) = \min_{n\in\mathbb{Z}} |\theta - \theta' + n|$$

for any $\theta, \theta' \in \mathbb{R}$. This gives a well-defined metric on $\mathbb{T}$ that generates the standard topology of $\mathbb{T}$.

Definition 2.3.2. Take $a, b \in \mathbb{R}^d$ such that $\|a\| = 1$. Take also positive real numbers $R_1, R_2 > 0$. Set

$$S(a, b, R_1, R_2) = \{x \in \mathbb{R}^d \mid \langle x - b, a\rangle \in R_1\mathbb{Z} + [-R_2, R_2]\}.$$

Remark 2.3.3. $S(a, b, R_1, R_2)$ is the union of "bands" with width $2R_2$ and intervals R_1.

Definition 2.3.4. Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ and $L_1, L_2 > 0$. We say $\mathcal{T}$ has (L_1, L_2)-stripe structure if there are $a \in \mathbb{R}^d$ with $\|a\| = 1$ and $R_0 > 0$ such that

$$\{y \in \mathbb{R}^d \mid (\mathcal{T} - x) \cap B(0, R) = (\mathcal{T} - y) \cap B(0, R)\} \subset S(a, x, L_1, L_2) \tag{2.15}$$

for each $x \in \mathbb{R}^d$.

Remark 2.3.5. In plain language, the inclusion (2.15) means that, if we take a patch $\mathcal{P} \subset \mathcal{T}$ around the point x which is large enough, there is a "forbidden area" of the appearance of the translate of $\mathcal{P}$. The forbidden area is a periodic one which is obtained by juxtaposing bands of width $2L_2$. (See Figure 2.2 in page 41.) This is a statement on non-existence which we discussed in Introduction.

We will show the following theorem. That the first condition implies the the second is essentially [19], Theorem 3.5.

Theorem 2.3.6. *Take a tiling which is of FTT and has FLC. Consider the following two conditions:*

1. $0 \in \mathbb{R}^d$ *is a limit point of the group of eigenvalues of the corresponding dynamical system* $(X_{\mathcal{T}}, \mathbb{R}^d)$.
2. *For any* $R_1, R_2, \varepsilon > 0$*, there are* $L_1, L_2 > 0$ *such that*
   (a) $|R_j - L_j| < \varepsilon$ *for each* $j = 1, 2$*, and*
   (b) $\mathcal{T}$ *has* (L_1, L_2)*-stripe structure.*

Then the first condition always implies the second and if $\mathcal{T}$ *is repetitive the second one implies the first.*

Proof. We show a generalization of this theorem in Theorem 4.3.7 and Theorem 4.3.13. □

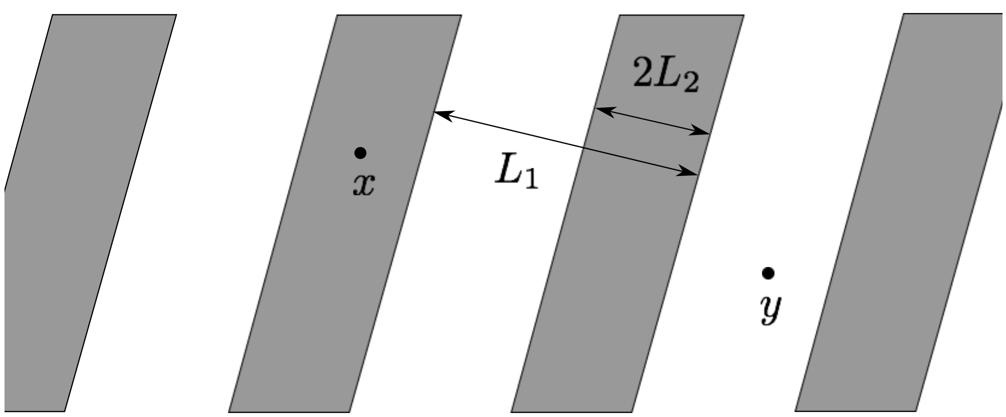

Figure 2.2: (L_1, L_2)-stripe structure. The situation of the tiling $\mathcal{T}$ around the point x is different from the one around the points, such as y, oudside the shaded region.

Remark 2.3.7. Consider a primitive FLC substitution with injective substitution map and such that the expansion map is diagonalizable and all the eigenvalues are algebraic conjugates of the same multiplicity. If the spectrum of the expansion map is a Pisot family, the self-affine tilings $\mathcal{T}$ for this substitution satisfies the first condition in Theorem 2.3.6. Indeed, $\mathcal{T}$ is non-periodic and so by Theorem 2.2.45, $a \in \mathbb{R}^d$ is an eigenvalue if $\lim_n e^{2\pi i \langle a, \varphi^n(x) \rangle} = 1$ for any return vector x. There is a non-zero eigenvalue a by Theorem 2.2.47; by the above remark $(\varphi^*)^{-k}(a)$, where $*$ denotes the adjoint, is an eigenvalue for all $k > 0$; since $\lim_k (\varphi^*)^{-k}(a) = 0$, we have arbitrary small non-zero eigenvalues.

Remark 2.3.8. Even if we know $\mathcal{T}$ has stripe structure, we do not know how large the $R > 0$ in Definition 2.3.4 is. If $\mathcal{T}$ is a self-affine tiling with the same condition as in Remark 2.3.7, a Delone set D (consisting of what is called control points) is locally derivable from $\mathcal{T}$. The set D in this case is a Meyer set([14], Corollary 2.13); by the characterization of Meyer sets [15], we see

$$\{\chi \in (\mathbb{R}^d)\hat{} \mid \text{for any } x \in D, \text{ we have } |\chi(x) - 1| < \varepsilon\}$$

is relatively dense for any $\varepsilon > 0$. Thus we have a forbidden area of the appearance of any small but nonempty patches.

Chapter 3

A general framework for objects such as tilings, Delone sets, functions and measures

3.1 General theory of pattern spaces

In this section X represents a nonempty topological space unless otherwise stated. First in Subsection 3.1.1, we define "pattern space". Several spaces such as the space of patches and space of point sets have an operation of "cutting off": for example, for a discrete set $D \subset \mathbb{R}^d$ and a subset C of $\mathbb{R}^d$, we can "cut off" D by the window C by taking intersection $D \cap C$. We axiomatize the properties that such cutting-off operation should have and obtain the notion of pattern space. Several space of objects such as patches, point sets, functions and measures are captured in this framework. In Subsection 3.1.2 we introduce an order relation on pattern spaces, which is the inclusion between two patches when the pattern space is the set of all patches. In Subsection 3.1.3 we study the operation of "gluing" objects to obtain a new objects. This is an abstract framework to capture the usual operation of taking union. Finally, in Subsection 3.1.4 we define zero elements, which is the empty-set in the pattern space of all patches and is zero function in the pattern space of all functions.

3.1.1 Definition and examples of pattern space

Definition 3.1.1. The set of all closed subsets of X is denoted by $\mathcal{C}(X)$.

Definition 3.1.2. A non-empty set Π equipped with a map

$$\Pi \times \mathcal{C}(X) \ni (\mathcal{P}, C) \mapsto \mathcal{P} \cap C \in \Pi \tag{3.1}$$

such that

1. $(\mathcal{P} \cap C_1) \cap C_2 = \mathcal{P} \cap (C_1 \cap C_2)$ for any $\mathcal{P} \in \Pi$ and any $C_1, C_2 \in \mathcal{C}(X)$, and
2. for any $\mathcal{P} \in \Pi$ there exists $C_{\mathcal{P}} \in \mathcal{C}(X)$ such that

$$\mathcal{P} \cap C = \mathcal{P} \iff C \supset C_{\mathcal{P}},$$

   for any $C \in \mathcal{C}(X)$,

is called a pattern space over X. The map (3.1) is called the scissors operation of the pattern space Π. The closed set $C_{\mathcal{P}}$ that appears in 2. is unique. It is called the support of $\mathcal{P}$ and is represented by $\operatorname{supp} \mathcal{P}$. Elements in Π are called abstract patterns in Π.

Remark 3.1.3. Note that the symbol $\cap$ is used for two different meanings: sometimes it refers to the scissors operation given to a pattern space; sometimes it refers to the intersection of two subsets of X.

Lemma 3.1.4. *Let Π be a pattern space over X. For any $\mathcal{P} \in \Pi$ and $C \in \mathcal{C}(X)$, we have $\operatorname{supp}(\mathcal{P} \cap C) \subset (\operatorname{supp} \mathcal{P}) \cap C$.*

Proof.

$$(\mathcal{P} \cap C) \cap ((\operatorname{supp} \mathcal{P}) \cap C) = (\mathcal{P} \cap \operatorname{supp} \mathcal{P}) \cap C = \mathcal{P} \cap C.$$

□

Example 3.1.5 (The space of patches in a metric space)**.** Let X be a metric space. An open, nonempty and bounded subset of X is called a tile (in X). A set $\mathcal{P}$ of tiles such that if $S, T \in \mathcal{P}$, then either $S = T$ or $S \cap T = \emptyset$ is called a patch (in X). The set of all patches in X is denoted by $\operatorname{Patch}(X)$. For $\mathcal{P} \in \operatorname{Patch}(X)$ and $C \in \mathcal{C}(X)$, set

$$\mathcal{P} \cap C = \{T \in \mathcal{P} \mid T \subset C\}.$$

With this scissors operation $\operatorname{Patch}(X)$ becomes a pattern space over X. For $\mathcal{P} \in \operatorname{Patch}(X)$, its support is

$$\operatorname{supp} \mathcal{P} = \overline{\bigcup_{T \in \mathcal{P}} T}.$$

Patches $\mathcal{P}$ with $\operatorname{supp} \mathcal{P} = X$ are called tilings.

Example 3.1.6 (The space of all locally finite subsets of a metric space)**.** Let X be a metric space. Let $\operatorname{LF}(X)$ be the set of all locally finite subsets of X; that is,

$$\operatorname{LF}(X) = \{D \subset X \mid \text{for all } x \in X \text{ and } r > 0,\ D \cap B(x, r) \text{ is finite}\}.$$

With the usual intersection $\operatorname{LF}(X) \times \mathcal{C}(X) \ni (D, C) \mapsto D \cap C \in \operatorname{LF}(X)$ of two subsets of X, $\operatorname{LF}(X)$ is a pattern space over X. For any $D \in \operatorname{LF}(X)$, its support is D itself.

Example 3.1.7 (The space of all uniformly discrete subsets)**.** We say, for $r > 0$, a subset D of a metric space (X, ρ) is r-uniformly discrete if $\rho(x, y) > r$ for any $x, y \in D$ with $x \neq y$. The set $\mathrm{UD}_r(X)$ of all r-uniformly discrete subsets of X is a pattern space over X by the usual intersection as a scissors operation. If D is r-uniformly discrete for some $r > 0$, we say D is uniformly discrete. The set $\mathrm{UD}(X) = \bigcup_{r>0} \mathrm{UD}_r(X)$ of all uniformly discrete subsets of X is also a pattern space over X.

Example 3.1.8. With the usual intersection of two subsets of X as a scissors operation, the set 2^X of all subsets of X and $\mathcal{C}(X)$ are pattern spaces over X.

Example 3.1.9 (The space of maps)**.** Let Y be a nonempty set. Take one element $y_0 \in Y$ and fix it. The pattern space $\mathrm{Map}(X, Y, y_0)$ is defined as follows: as a set the space is equal to $\mathrm{Map}(X, Y)$ of all mappings from X to Y; for $f \in \mathrm{Map}(X, Y, y_0)$ and $C \in \mathcal{C}(X)$, the scissors operation is defined by

$$(f \cap C)(x) = \begin{cases} f(x) & \text{if } x \in C \\ y_0 & \text{if } x \notin C. \end{cases}$$

With this operation $\mathrm{Map}(X, Y, y_0)$ is a pattern space over X and for $f \in \mathrm{Map}(X, Y, y_0)$ its support is $\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq y_0\}}$.

Example 3.1.10 (The space of measures)**.** Let X be a locally compact σ-compact metric space. Let $C_c(X)$ be the space of all continuous and complex-valued functions on X which have compact supports. Its dual space $C_c(X)^*$ with respect to a suitable topology consists of Radon charges, that is, the maps $\Phi \colon C_c(X) \to \mathbb{C}$ such that there is a unique positive Borel measure m and a Borel measurable map $u \colon X \to \mathbb{T}$ such that

$$\Phi(\varphi) = \int_X \varphi u dm$$

for all $\varphi \in C_c(X)$. For such Φ and $C \in \mathcal{C}(X)$ set

$$(\Phi \cap C)(\varphi) = \int_C \varphi u dm$$

for each $\varphi \in C_c(X)$. Then the new functional $\Phi \cap C$ is a Radon charge. With this operation $C_c(X)^* \times \mathcal{C}(X) \ni (\Phi, C) \mapsto \Phi \cap C \in C_c(X)^*$, the space $C_c(X)^*$ becomes a pattern space over X.

Note that if m is a positive measure on X and $u \colon X \to \mathbb{C}$ is a bounded Borel map (not necessarily $\mathbb{T}$-valued), then $\Phi \colon C_c(X) \ni \varphi \mapsto \int \varphi u dm$ is a Radon charge. If $C \in \mathcal{C}(X)$, then

$$(\Phi \cap C)(\varphi) = \int_C \varphi u dm,$$

for each $\varphi \in C_c(X)$.

Next we investigate pattern subspaces. The relation between a pattern space and its pattern subspace is similar to the one between a dynamical system and its invariant subspace.

Definition 3.1.11. Let Π be a pattern space over X. Suppose a non-empty subset Π' of Π satisfies the condition

$$\mathcal{P} \in \Pi' \text{ and } C \in \mathcal{C}(X) \Rightarrow \mathcal{P} \cap C \in \Pi'.$$

Then Π' is called a pattern subspace of $\mathcal{P}$.

Remark 3.1.12. If Π' is a pattern subspace of a pattern space Π, then Π' is a pattern space by restricting the scissors operation.

Example 3.1.13. Let X be a topological space. Then $\mathcal{C}(X)$ is a pattern subspace of 2^X. If X is a metric space, then $\mathrm{LF}(X)$ is a pattern subspace of $\mathcal{C}(X)$ and $\mathrm{UD}_r(X)$ is a pattern subspace of $\mathrm{UD}(X)$ for each $r > 0$. If moreover the closed balls are compact, $\mathrm{UD}(X)$ is a pattern subspace of $\mathrm{LF}(X)$.

Next we investigate two ways to construct new pattern spaces from old ones; taking product and taking power set.

Lemma 3.1.14. *Let Λ be an index set and $\Pi_\lambda, \lambda \in \Lambda$, is a family of pattern spaces over X. The direct product $\prod_\lambda \Pi_\lambda$ becomes a pattern space over X by a scissors operation*

$$(\mathcal{P}_\lambda)_{\lambda\in\Lambda} \cap C = (\mathcal{P}_\lambda \cap C)_{\lambda\in\Lambda}.$$

for $(\mathcal{P}_\lambda)_\lambda \in \prod_\lambda \Pi_\lambda$ and $C \in \mathcal{C}(X)$. The support is given by $\mathrm{supp}(\mathcal{P}_\lambda)_\lambda = \overline{\bigcup_\lambda \mathrm{supp}\,\mathcal{P}_\lambda}$.

Definition 3.1.15. Under the same condition as in Lemma 3.1.14, we call $\prod \Pi_\lambda$ the product pattern space of $(\Pi_\lambda)_\lambda$.

Lemma 3.1.16. *Let Π be a pattern space over X. The set 2^Π of all subsets of Π is a pattern space over X by a scissors operation*

$$\Xi \cap C = \{\mathcal{P} \cap C \mid \mathcal{P} \in \Xi\}, \tag{3.2}$$

for any $\Xi \in 2^\Pi$ and $C \in \mathcal{C}(X)$. The support is given by $\mathrm{supp}\,\Xi = \overline{\bigcup_{\mathcal{P}\in\Xi} \mathrm{supp}\,\mathcal{P}}$.

Definition 3.1.17. The power set 2^Π of a pattern space Π, endowed with the scissors operation in equation (3.2), is called the power pattern space of Π.

Next, we define a notion which will be useful later. Maps and elements of 2^X (and so uniformly discrete subsets of X) always satisfy this condition; a patch (and so a tiling) satisfies this condition if and only if the diameters of tiles in that patch are bounded from above.

Definition 3.1.18. Let Π be a pattern space over a metric space X. For any element $\mathcal{P} \in \Pi$, we say $\mathcal{P}$ consists of bounded components if there is $R_\mathcal{P} > 0$ such that for any $x \in \mathrm{supp}\,\mathcal{P}$, we have $x \in \mathrm{supp}(\mathcal{P} \cap B(x, R_\mathcal{P}))$.

3.1.2 An order on pattern spaces

Definition 3.1.19. Let Π be a pattern space over X. We define a relation $\geqq$ on Π as follows: for each $\mathcal{P}, \mathcal{Q} \in \Pi$, we set $\mathcal{P} \geqq \mathcal{Q}$ if

$$\mathcal{P} \cap \operatorname{supp} \mathcal{Q} = \mathcal{Q}.$$

Lemma 3.1.20. *1. If $\mathcal{P} \geqq \mathcal{Q}$, then $\operatorname{supp} \mathcal{P} \supset \operatorname{supp} \mathcal{Q}$.*

2. The relation $\geqq$ is an order on Π.

Proof. If $\mathcal{P} \geqq \mathcal{Q}$, then

$$\mathcal{Q} \cap \operatorname{supp} \mathcal{P} = \mathcal{P} \cap \operatorname{supp} \mathcal{P} \cap \operatorname{supp} \mathcal{Q} = \mathcal{P} \cap \operatorname{supp} \mathcal{Q} = \mathcal{Q}.$$

Thus $\operatorname{supp} \mathcal{P} \supset \operatorname{supp} \mathcal{Q}$. Next we prove that $\geqq$ is an order. $\mathcal{P} \geqq \mathcal{P}$ is clear. If $\mathcal{P} \geqq \mathcal{Q}$ and $\mathcal{Q} \geqq \mathcal{P}$, then $\operatorname{supp} \mathcal{P} = \operatorname{supp} \mathcal{Q}$ and $\mathcal{P} = \mathcal{P} \cap \operatorname{supp} \mathcal{P} = \mathcal{P} \cap \operatorname{supp} \mathcal{Q} = \mathcal{Q}$. Finally, if $\mathcal{P} \geqq \mathcal{Q} \geqq \mathcal{R}$, then $\operatorname{supp} \mathcal{P} \supset \operatorname{supp} \mathcal{Q} \supset \operatorname{supp} \mathcal{R}$ and $\mathcal{P} \cap \operatorname{supp} \mathcal{R} = \mathcal{P} \cap \operatorname{supp} \mathcal{Q} \cap \operatorname{supp} \mathcal{R} = \mathcal{Q} \cap \operatorname{supp} \mathcal{R} = \mathcal{R}$, and so $\mathcal{P} \geqq \mathcal{R}$. □

Lemma 3.1.21. *1. If $\mathcal{P} \in \Pi$ and $C \in \mathcal{C}(X)$, then $\mathcal{P} \geqq \mathcal{P} \cap C$.*

2. If $\mathcal{P}, \mathcal{Q} \in \Pi$, $C \in \mathcal{C}(X)$ and $\mathcal{P} \geqq \mathcal{Q}$, then $\mathcal{P} \cap C \geqq \mathcal{Q} \cap C$.

Proof. The statements follow from Lemma 3.1.4.
1. $\mathcal{P} \cap \operatorname{supp}(\mathcal{P} \cap C) = \mathcal{P} \cap \operatorname{supp} \mathcal{P} \cap C \cap \operatorname{supp}(\mathcal{P} \cap C) = \mathcal{P} \cap C \cap \operatorname{supp}(\mathcal{P} \cap C) = \mathcal{P} \cap C$.
2. $\mathcal{P} \cap C \cap \operatorname{supp}(\mathcal{Q} \cap C) = \mathcal{P} \cap \operatorname{supp} \mathcal{Q} \cap C \cap \operatorname{supp}(\mathcal{Q} \cap C) = \mathcal{Q} \cap C$. □

Definition 3.1.22. Let Ξ be a subset of Π. If the supremum of Ξ with respect to the order $\geqq$ defined in Definition 3.1.19 exists in Π, it is denoted by $\bigvee \Xi$.

Lemma 3.1.23. *If a subset $\Xi \subset \Pi$ admits the supremum $\bigvee \Xi$, then $\operatorname{supp} \bigvee \Xi = \overline{\bigcup_{\mathcal{P} \in \Xi} \operatorname{supp} \mathcal{P}}$.*

Proof. Set $C = \overline{\bigcup_{\mathcal{P} \in \Xi} \operatorname{supp} \mathcal{P}}$. Since $\bigvee \Xi \geqq \mathcal{P}$ for any $\mathcal{P} \in \Xi$, by Lemma 3.1.21 $\operatorname{supp} \bigvee \Xi \supset \operatorname{supp} \mathcal{P}$ for each $\mathcal{P} \in \Xi$. Since the support is closed, we have $\operatorname{supp} \bigvee \Xi \supset C$. If $\operatorname{supp} \bigvee \Xi$ is strictly larger than C we have a following contradiction. Since $(\operatorname{supp} \bigvee \Xi) \cap C \subset C \neq \operatorname{supp} \bigvee \Xi$, the two abstract patterns $\bigvee \Xi$ and $(\bigvee \Xi) \cap C$ are different and $\bigvee \Xi \geqq (\bigvee \Xi) \cap C$ by Lemma 3.1.21. On the other hand, $(\bigvee \Xi) \cap C$ majorizes Ξ. These contradict the fact that $\bigvee \Xi$ is the supremum. □

Remark 3.1.24. It is not necessarily true that any element $\mathcal{P}_0$ in Π that majorizes Ξ and $\operatorname{supp} \mathcal{P}_0 = \overline{\bigcup_{\mathcal{P} \in \Xi} \operatorname{supp} \mathcal{P}}$ is the supremum of Ξ.

The following lemma will be useful later.

Lemma 3.1.25. *Let F_j be a finite subset of X for $j = 1, 2$. Take a positive real number r such that for each $j = 1, 2$, any two distinct elements $x, y \in F_j$ satisfy $\rho(x, y) > 4r$. Suppose for each j and $x \in F_j$, there corresponds $\mathcal{P}^j_x \in \Pi$ such that $\emptyset \neq \operatorname{supp} \mathcal{P}^j_x \subset B(x, r)$. Suppose also there is $Q^j = \bigvee\{\mathcal{P}^j_x \mid x \in F_j\}$ for $j = 1, 2$. Then the following statements hold:*

1. *If $\operatorname{supp} \mathcal{Q}^1 \subset \operatorname{supp} \mathcal{Q}^2$, then for each $x \in F_1$ there is a unique $y \in F_2$ such that $\operatorname{supp} \mathcal{P}^1_x \cap \operatorname{supp} \mathcal{P}^2_y \neq \emptyset$. In this case $\operatorname{supp} \mathcal{P}^1_x \subset \operatorname{supp} \mathcal{P}^2_y$ holds.*
2. *If $\operatorname{supp} \mathcal{Q}^1 = \operatorname{supp} \mathcal{Q}^2$, then for each $x \in F_1$ there is a unique $y \in F_2$ such that $\operatorname{supp} \mathcal{P}^1_x = \operatorname{supp} \mathcal{P}^2_y$.*
3. *If $\mathcal{Q}^1 = \mathcal{Q}^2$, then for each $x \in F_1$ there is a unique $y \in F_2$ such that $\mathcal{P}^1_x = \mathcal{P}^2_y$.*

Proof. 1. By Lemma 3.1.23, $\operatorname{supp} \mathcal{Q}^j = \bigcup_{x \in F_j} \operatorname{supp} \mathcal{P}^j_x$ for each $j = 1, 2$. For each $x \in F_1$, there is $y \in F_2$ such that $\operatorname{supp} \mathcal{P}^1_x \cap \operatorname{supp} \mathcal{P}^2_y \neq \emptyset$. If there is another $y' \in F_2$ such that $\operatorname{supp} \mathcal{P}^1_x \cap \operatorname{supp} \mathcal{P}^2_{y'} \neq \emptyset$, then $B(x, r) \cap B(y, r) \neq \emptyset$ and $B(x, r) \cap B(y', r) \neq \emptyset$ and so $\rho(y, y') \leqq 4r$. By definition of r, we have $y = y'$. This shows the uniqueness of y. The uniqueness implies the last statement.

2. By 1., for each $x \in F_1$ there is $y \in F_2$ such that $\operatorname{supp} \mathcal{P}^1_x \subset \operatorname{supp} \mathcal{P}^2_y$. Applying 1. again, there is $x' \in F_1$ such that $\operatorname{supp} \mathcal{P}^2_y \subset \operatorname{supp} \mathcal{P}^1_{x'}$. We have $\operatorname{supp} \mathcal{P}^1_x \subset \operatorname{supp} \mathcal{P}^1_{x'}$ and by applying the uniqueness in 1., we see $x = x'$.

3. By 2., for each $x \in F_1$ there is $y \in F_2$ such that $\operatorname{supp} \mathcal{P}^1_x = \operatorname{supp} \mathcal{P}^2_y$. Then

$$\mathcal{P}^1_x = \mathcal{Q}^1 \cap \operatorname{supp} \mathcal{P}^x_1 = \mathcal{Q}^2 \cap \operatorname{supp} \mathcal{P}^y_2 = \mathcal{P}^2_y.$$

The uniqueness follows from the uniqueness in 1. □

3.1.3 Glueable pattern spaces

In this subsection X is a metric space with a metric ρ and Π is a pattern space over X.

Often we want to "glue" abstract patterns to obtain a larger abstract pattern. For example, suppose Ξ is a collection of patches such that if $\mathcal{P}, \mathcal{Q} \in \Xi$, $S \in \mathcal{P}$ and $T \in \mathcal{Q}$, then we have either $S = T$ or $S \cap T = \emptyset$. Then we can "glue" patches in Ξ, that is, we can take a union $\bigcup_{\mathcal{P} \in \Xi} \mathcal{P}$, which is also a patch. Pattern spaces in which we can "glue" abstract patterns are called glueable pattern spaces (Definition 3.1.28).

Definition 3.1.26.

1. Two abstract patterns $\mathcal{P}, \mathcal{Q} \in \Pi$ are said to be compatible if there is $\mathcal{R} \in \Pi$ such that $\mathcal{R} \geqq \mathcal{P}$ and $\mathcal{R} \geqq \mathcal{Q}$.
2. A subset $\Xi \subset \Pi$ is said to be pairwise compatible if any two elements $\mathcal{P}, \mathcal{Q} \in \Pi$ are compatible.

3. A subset $\Xi \subset \Pi$ is said to be locally finite if for any $x \in X$ and $r > 0$, the set $\Xi \cap B(x,r)$, which was defined in (3.2), is finite.

Lemma 3.1.27. *Let Ξ be a subset of Π and take $C \in \mathcal{C}(X)$. Then the following hold.*

1. *If Ξ is locally finite, then so is $\Xi \cap C$.*
2. *If Ξ is pairwise compatible, then so is $\Xi \cap C$.*

Proof. 1. Suppose there are $x \in X$, $r > 0$ such that $\Xi \cap C \cap B(x,r)$ is infinite. There are $\mathcal{P}_1, \mathcal{P}_2, \dots$ in Ξ such that $\mathcal{P}_n \cap C \cap B(x,r)$ are all distinct. However by local finiteness of Ξ, there are distinct n and m such that $\mathcal{P}_n \cap B(x,r) = \mathcal{P}_m \cap B(x,r)$; this implies that $\mathcal{P}_n \cap C \cap B(x,r) = \mathcal{P}_m \cap C \cap B(x,r)$ and leads to a contradiction.

2. Take $\mathcal{P}, \mathcal{Q} \in \Xi$ arbitrarily. By Definition 3.1.26, there is $\mathcal{R} \in \Xi$ such that $\mathcal{R} \geqq \mathcal{P}$ and $\mathcal{R} \geqq \mathcal{Q}$. By Lemma 3.1.21, we have $\mathcal{R} \cap C \geqq \mathcal{P} \cap C$ and $\mathcal{R} \cap C \geqq \mathcal{Q} \cap C$ and so $\mathcal{P} \cap C$ and $\mathcal{Q} \cap C$ are compatible. □

Definition 3.1.28. A pattern space Π over a metric space X is said to be glueable if the following two conditions hold:

1. If $\Xi \subset \Pi$ is both locally finite and pairwise compatible, then there is the supremum $\bigvee \Xi$ for Ξ.
2. If $\Xi \subset \Pi$ is both locally finite and pairwise compatible, then for any $C \in \mathcal{C}(X)$,
$$\bigvee(\Xi \cap C) = (\bigvee \Xi) \cap C. \tag{3.3}$$

Remark 3.1.29. By Lemma 3.1.27, for $\Xi \subset \Pi$ which is locally finite and pairwise compatible and $C \in \mathcal{C}(X)$ the left-hand side of the equation (3.3) makes sense.

Lemma 3.1.30. *Let Π be glueable and Λ be a set. For each $\lambda \in \Lambda$, let $\Xi_\lambda \subset \Pi$ be a subset and suppose $\bigcup_\lambda \Xi_\lambda$ is locally finite and pairwise compatible. Then for each λ, the set Ξ_λ is locally finite and pairwise-compatible and if we set $\mathcal{Q}_\lambda = \bigvee \Xi_\lambda$, the set $\{\mathcal{Q}_\lambda \mid \lambda \in \Lambda\}$ is locally finite and pairwise-compatible and*
$$\bigvee \bigcup_\lambda \Xi_\lambda = \bigvee \{\mathcal{Q}_\lambda \mid \lambda \in \Lambda\}.$$

Proof. Set $\mathcal{P} = \bigvee \bigcup_\lambda \Xi_\lambda$. For each $\lambda \in \Lambda$ and $\mathcal{Q} \in \Xi_\lambda$, we have $\mathcal{P} \geqq \mathcal{Q}$ and so $\mathcal{P} \geqq \mathcal{Q}_\lambda$. This in particular shows that $\{\mathcal{Q}_\lambda \mid \lambda\}$ is pairwise compatible. Moreover, since for each $x \in X$ and $r > 0$,
$$\{\mathcal{Q}_\lambda \cap B(x,r) \mid \lambda \in \Lambda\} = \{\bigvee(\Xi_\lambda \cap B(x,r)) \mid \lambda \in \Lambda\}, \tag{3.4}$$
$\Xi_\lambda \cap B(x,r) \subset (\bigcup_\lambda \Xi_\lambda) \cap B(x,r)$ and $(\bigcup \Xi_\lambda) \cap B(x,r)$ is finite by assumption, the set (3.4) is finite: the set $\{\mathcal{Q}_\lambda \mid \lambda\}$ is locally finite.

If $\mathcal{P}'$ is a majorant for $\{\mathcal{Q}_\lambda \mid \lambda\}$, then $\mathcal{P}' \geqq \mathcal{Q}$ for each $\lambda \in \Lambda$ and $\mathcal{Q} \in \Xi_\lambda$, and so $\mathcal{P}' \geqq \mathcal{P}$. As was mentioned above, $\mathcal{P}$ is a majorant for $\{\mathcal{Q}_\lambda \mid \lambda\}$, and so it is its supremum. □

We finish this subsection with examples.

Example 3.1.31. Consider $\Pi = \mathrm{Patch}(X)$ (Example 3.1.5). In this pattern space, for two elements $\mathcal{P}, \mathcal{Q} \in \mathrm{Patch}(X)$, the following statements hold:

1. $\mathcal{P} \geqq \mathcal{Q} \iff \mathcal{P} \supset \mathcal{Q}$.
2. $\mathcal{P}$ and $\mathcal{Q}$ are compatible if and only if for any $T \in \mathcal{P}$ and $S \in \mathcal{Q}$, either $S = T$ or $S \cap T = \emptyset$ holds.

If $\Xi \subset \mathrm{Patch}(X)$ is pairwise compatible, then $\mathcal{P}_\Xi = \bigcup_{\mathcal{P}\in\Xi} \mathcal{P}$ is a patch, which is the supremum of Ξ. If $C \in \mathcal{C}(X)$, then

$$(\bigvee \Xi) \cap C = (\bigcup_{\mathcal{P}\in\Xi} \mathcal{P}) \cap C = \bigcup (\mathcal{P} \cap C) = \bigvee (\Xi \cap C).$$

$\mathrm{Patch}(X)$ is glueable.

Example 3.1.32. For the pattern space 2^X in Example 3.1.8, two elements $A, B \in 2^X$ are compatible if and only if

$$\overline{A} \cap B \subset A \text{ and } A \cap \overline{B} \subset B. \tag{3.5}$$

In fact, if A and B are compatible, then there is a majorant C. By $C \supset A \cup B$,

$$A \cup (\overline{A} \cap B) = (A \cup B) \cap \overline{A} = C \cap \overline{A} \cap (A \cup B) = A \cap (A \cup B) = A,$$

and so $\overline{A} \cap B \subset A$. A similar argument shows that $\overline{B} \cap A \subset B$. Conversely, if the condition (3.5) holds, then $(A \cup B) \cap \overline{A} = A \cup (B \cap \overline{A}) = A$ and similarly $(A \cup B) \cap \overline{B} = B$, and so $A \cup B$ is a majorant for A and B.

Suppose $\Xi \subset 2^X$ is locally finite and pairwise compatible. Note that $\bigcup_{A\in\Xi} \overline{A} = \overline{\bigcup_{A\in\Xi} A}$. Set $A_\Xi = \bigcup_{A\in\Xi} A$. For each $A \in \Xi$, $A_\Xi \cap \overline{A} = \bigcup_{B\in\Xi} (B \cap \overline{A}) = A$; A_Ξ is a majorant of Ξ. If B is also a majorant for Ξ, then

$$B \cap \overline{A_\Xi} = B \cap (\bigcup_{A\in\Xi} \overline{A}) = \bigcup_{A\in\Xi} (B \cap \overline{A}) = \bigcup_{A\in\Xi} A = A_\Xi,$$

and so $B \geqq A_\Xi$. It turns out that A_Ξ is the supremum for Ξ. Moreover, if $C \in \mathcal{C}(X)$, then $A_\Xi \cap C = \bigcup_{A\in\Xi} (A \cap C) = \bigvee (\Xi \cap C)$. Thus 2^X is a glueable space.

Remark 3.1.33. Let Π_0 be a glueable pattern space and $\Pi_1 \subset \Pi_0$ a pattern subspace. For any subset $\Xi \subset \Pi_1$, if it is pairwise compatible in Π_1, then it is pairwise compatible in Π_0. Moreover, whether a set is locally finite or not is independent of the ambient pattern space in which the set is included. For a subset $\Xi \subset \Pi_1$ which is locally finite and pairwise compatible in Π_1, since Π_0 is glueable, there is the supremum $\bigvee \Xi$ in Π_0. If this supremum in Π_0 is always included in Π_1, then Π_1 is glueable.

By this remark it is easy to see the pattern spaces $\mathcal{C}(X)$ (Example 3.1.8), $\mathrm{LF}(X)$ (Example 3.1.6), and $\mathrm{UD}_r(X)$ (Example 3.1.7, r is an arbitrary positive number) are glueable.

However, $\mathrm{UD}(X)$ (Example 3.1.7) is not necessarily glueable. For example, set $X = \mathbb{R}$. Set $\mathcal{P}_n = \{n, n + \frac{1}{n}\}$ for each integer $n \neq 0$. Each $\mathcal{P}_n$ is in $\mathrm{UD}(\mathbb{R})$, $\Xi = \{\mathcal{P}_n \mid n \neq 0\}$ is locally finite and pairwise compatible, but it does not admit the supremum.

For the rest of this subsection we show that $\mathrm{Map}(X, Y, y_0)$ (Example 3.1.9) is glueable, where X is a metric space, Y a set and $y_0 \in Y$.

Lemma 3.1.34. *Two maps $f, g \in \mathrm{Map}(X, Y, y_0)$ are compatible if and only if $f|_{\operatorname{supp} f \cap \operatorname{supp} g} = g|_{\operatorname{supp} f \cap \operatorname{supp} g}$.*

Proof. Suppose f and g are compatible. Take a majorant $h \in \mathrm{Map}(X, Y, y_0)$. For each $x \in \operatorname{supp} f \cap \operatorname{supp} g$,

$$f(x) = (h \cap \operatorname{supp} f)(x) = h(x) = (h \cap \operatorname{supp} g)(x) = g(x).$$

Conversely suppose $f|_{\operatorname{supp} f \cap \operatorname{supp} g} = g|_{\operatorname{supp} f \cap \operatorname{supp} g}$. Define a map $h \in \mathrm{Map}(X, Y, y_0)$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{supp} f \\ g(x) & \text{if } x \in \operatorname{supp} g \\ y_0 & \text{otherwise.} \end{cases}$$

This is well-defined. Next, $h \geqq f$ because

$$\begin{aligned} (h \cap \operatorname{supp} f)(x) &= \begin{cases} h(x) & \text{if } x \in \operatorname{supp} f \\ y_0 & \text{if } x \notin \operatorname{supp} f \end{cases} \\ &= \begin{cases} f(x) & \text{if } x \in \operatorname{supp} f \\ y_0 & \text{if } x \notin \operatorname{supp} f \end{cases} \\ &= f(x) \end{aligned}$$

for any $x \in X$. Similarly $h \geqq g$ and so f and g are compatible. □

Lemma 3.1.35. *For $f \in \mathrm{Map}(X, Y, y_0)$, $x \in X$ and two positive numbers $r > s > 0$, we have $\operatorname{supp}(f \cap B(x, r)) \supset (\operatorname{supp} f) \cap B(x, s)$. Consequently, $(\operatorname{supp}(f \cap B(x, r))) \cap B(x, s) = (\operatorname{supp} f) \cap B(x, s)$.*

Proof. Take $x' \in (\operatorname{supp} f) \cap B(x, s)$. For any $\varepsilon > 0$, there is $x'' \in B(x', \varepsilon)$ such that $f(x'') \neq y_0$. If ε is small enough, this x'' is in $B(x, r)$ and so $(f \cap B(x, r))(x'') \neq y_0$. Since ε was arbitrary, $x' \in \operatorname{supp}(f \cap B(x, r))$. □

Lemma 3.1.36. *Let Ξ be a subset of* $\mathrm{Map}(X, Y, y_0)$*. Take $x \in X$ and two numbers r, s such that $r > s > 0$. Then if $\Xi \cap B(x, r)$ is finite, then $\{(\operatorname{supp} f) \cap B(x, s) \mid f \in \Xi\}$ is finite.*

Proof. Clear by Lemma 3.1.35. $\square$

Lemma 3.1.37. *If $\Xi \subset \mathrm{Map}(X, Y, y_0)$ is locally finite, then $\bigcup_{f \in \Xi} \operatorname{supp} f$ is closed.*

Proof. Take $x \in X \setminus (\bigcup_{f \in \Xi} \operatorname{supp} f)$. Since $\Xi \cap B(x, 1)$ is finite, by Lemma 3.1.36, $\{(\operatorname{supp} f) \cap B(x, \frac{1}{2}) \mid f \in \Xi\}$ is finite. There is $r > 0$ such that $B(x, r) \cap \operatorname{supp} f = \emptyset$ for any $f \in \Xi$. $\square$

Proposition 3.1.38. $\mathrm{Map}(X, Y, y_0)$ *is glueable.*

Proof. Suppose $\Xi \subset \mathrm{Map}(X, Y, y_0)$ is locally finite and pairwise compatible. Set

$$
\begin{aligned}
f_\Xi(x) &= \begin{cases} f(x) & \text{if there is } f \in \Xi \text{ such that } x \in \operatorname{supp} f \\ y_0 & \text{otherwise} \end{cases} \\
&= \begin{cases} f(x) & \text{if there is } f \in \Xi \text{ such that } f(x) \neq y_0 \\ y_0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

This is well-defined by Lemma 3.1.34. For each $f \in \Xi$ and $x \in X$,

$$
\begin{aligned}
(f_\Xi \cap \operatorname{supp} f)(x) &= \begin{cases} f_\Xi(x) & \text{if } x \in \operatorname{supp} f \\ y_0 & \text{if } x \notin \operatorname{supp} f \end{cases} \\
&= \begin{cases} f(x) & \text{if } x \in \operatorname{supp} f \\ y_0 & \text{if } x \notin \operatorname{supp} f \end{cases} \\
&= f(x),
\end{aligned}
$$

and so $f_\Xi \geqq f$. In other words, f_Ξ is a majorant for Ξ.

Next, we prove that f_Ξ is the supremum for Ξ. To this end, we first claim $\operatorname{supp} f_\Xi = \bigcup_{f \in \Xi} \operatorname{supp} f$. It is clear that $\{x \in X \mid f_\Xi(x) \neq y_0\} \subset \bigcup_{f \in \Xi} \operatorname{supp} f$ because if $f_\Xi(x) \neq y_0$, then there is $f \in \Xi$ such that $f(x) \neq y_0$. Together with Lemma 3.1.37, we see $\operatorname{supp} f_\Xi \subset \bigcup_{f \in \Xi} \operatorname{supp} f$. Since f_Ξ is a majorant for Ξ, the reverse inclusion is clear, and so $\operatorname{supp} f_\Xi = \bigcup_{f \in \Xi} \operatorname{supp} f$.

To prove that f_Ξ is the supremum, we next take a majorant g for Ξ arbitrarily. Since for $f \in \Xi$ and $x \in \operatorname{supp} f$, we have $g(x) = (g \cap \operatorname{supp} f)(x) = f(x)$, and we obtain

$$
\begin{aligned}
g \cap (\bigcup_{f \in \Xi} \operatorname{supp} f)(x) &= \begin{cases} g(x) & \text{if there is } f \in \Xi \text{ such that } x \in \operatorname{supp} f \\ y_0 & \text{otherwise} \end{cases} \\
&= \begin{cases} f(x) & \text{if there is } f \in \Xi \text{ such that } x \in \operatorname{supp} f \\ y_0 & \text{otherwise} \end{cases} \\
&= f_\Xi(x)
\end{aligned}
$$

for each $x \in X$, and so $g \cap (\operatorname{supp} f_\Xi) = f_\Xi$, namely $g \geqq f_\Xi$. We have shown f_Ξ is the supremum for Ξ; $f_\Xi = \bigvee \Xi$.

It remains to show that $f_\Xi \cap C$ is equal to $\bigvee(\Xi \cap C)$ for each $C \in \mathcal{C}(X)$. This is the case because

$$
\begin{aligned}
(f_\Xi \cap C) &= \begin{cases} f_\Xi(x) & \text{if } x \in C \\ y_0 & \text{otherwise} \end{cases} \\
&= \begin{cases} f(x) & \text{if } x \in C \text{ and } f(x) \neq y_0 \text{ for some } f \in \Xi \\ y_0 & \text{otherwise} \end{cases} \\
&= \begin{cases} (f \cap C)(x) & \text{if there is } f \in \Xi \text{ such that } (f \cap C)(x) \neq y_0 \\ y_0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

□

3.1.4 Zero Element and Its Uniqueness

Definition 3.1.39. Let Π be a pattern space over a topological space X. An element $\mathcal{P} \in \Pi$ such that $\operatorname{supp} \mathcal{P} = \emptyset$ is called a zero element of Π. If there is only one zero element in Π, it is denoted by 0.

Remark 3.1.40. Zero elements always exist. In fact, take an arbitrary element $\mathcal{P} \in \Pi$. Then by Lemma 3.1.4, $\operatorname{supp}(\mathcal{P} \cap \emptyset) = \emptyset$ and so $\mathcal{P} \cap \emptyset$ is a zero element.

Lemma 3.1.41. *If Π is a glueable pattern space over a topological space X, there is only one zero element in Π.*

Proof. The subset $\emptyset$ of Π is locally finite and pairwise compatible. Set $\mathcal{P} = \bigvee \emptyset$. By Lemma 3.1.23, $\mathcal{P}$ is a zero element. If $\mathcal{Q}$ is a zero element, then since $\mathcal{Q}$ is a majorant for $\emptyset$, we see $\mathcal{Q} \geqq \mathcal{P}$. We have $\mathcal{Q} = \mathcal{Q} \cap \emptyset = \mathcal{P}$. □

Lemma 3.1.42. *Let Π be a glueable pattern space over a topological space X. Take a locally finite and pairwise compatible subset Ξ of Π. Then $\bigvee \Xi \cup \{0\}$ exists and $\bigvee \Xi \cup \{0\} = \bigvee \Xi$.*

Proof. For any $\mathcal{P} \in \Pi$, the abstract pattern $\mathcal{P} \cap \emptyset$ is a zero element and by the uniqueness of zero element (Lemma 3.1.41), $\mathcal{P} \cap \emptyset = 0$ and $\mathcal{P} \geqq 0$. Thus $\bigvee \Xi$ is a majorant for $\Xi \cup \{0\}$. If $\mathcal{Q}$ is a majorant for $\Xi \cup \{0\}$, then it is a majorant for Ξ and so $\mathcal{Q} \geqq \bigvee \Xi$. This shows that $\bigvee \Xi$ is the supremum for $\Xi \cup \{0\}$. □

3.2 Γ-pattern spaces over X, or pattern spaces over (X, Γ)

Here we incorporate group actions to the theory of pattern spaces. First we define pattern spaces over (X, Γ), or Γ-pattern spaces over X, where X is a topological space and a group

Γ acts on X by homeomorphisms. We require there is an action of the group Γ on such a pattern space and the scissors operation is equivariant. In Subsection 3.2.2 we define local derivation by using the structure of Γ-pattern spaces. There we show several maps in aperiodic order send an abstract pattern $\mathcal{P}$ to a one which is mutually locally derivable (MLD) with $\mathcal{P}$; we solve the first question in Introduction affirmatively. The final two subsections (Subsection 3.2.3 and Subsection 3.2.4) prepare tools to prove Theorem 3.3.1. In Subsection 3.2.3 we "decompose" abstract patterns via Delone sets. In Subsection 3.2.4 we construct abstract patterns from "building blocks".

3.2.1 Definition and Examples

Setting 1. In this subsection, unless otherwise stated, X is a topological space, Γ is a group that acts on X as homeomorphisms, and Π is a pattern space over X.

Definition 3.2.1. Suppose there is a group action $\Gamma \curvearrowright \Pi$ such that for each $\mathcal{P} \in \Pi, C \in \mathcal{C}(X)$ and $\gamma \in \Gamma$, we have $(\gamma\mathcal{P}) \sqcap (\gamma C) = \gamma(\mathcal{P} \sqcap \mathcal{C})$, that is, the scissors operation is equivariant. Then we say Π is a Γ-pattern space or a pattern space over (X, Γ). For a pattern space Π over (X, Γ), its nonempty subset Σ such that $\mathcal{P} \in \Sigma$ and $\gamma \in \Gamma$ imply $\gamma\mathcal{P} \in \Sigma$ is called a subshift of Π.

Examples are given after lemmas.

Lemma 3.2.2. *Let Π be a pattern space over (X, Γ). For $\mathcal{P}, \mathcal{Q} \in \Pi$ and $\gamma \in \Gamma$, the following statements hold:*

1. $\gamma \operatorname{supp} \mathcal{P} = \operatorname{supp}(\gamma\mathcal{P})$.
2. *If $\mathcal{P} \geqq \mathcal{Q}$, then $\gamma\mathcal{P} \geqq \gamma\mathcal{Q}$.*

Lemma 3.2.3. *Let Π be a pattern space over (X, Γ). Suppose Π' is a pattern subspace of Π. If Π' is closed under the Γ-action, then Π' is a pattern space over (X, Γ).*

Lemma 3.2.4. *Let Λ be a set and $(\Pi_\lambda)_{\lambda\in\Lambda}$ be a family of pattern spaces over (X, Γ). Then Γ acts on the product space $\prod_\lambda \Pi_\lambda$ by $\gamma(\mathcal{P}_\lambda)_\lambda = (\gamma\mathcal{P}_\lambda)_\lambda$ and by this action $\prod_\lambda \Pi_\lambda$ is a pattern space over (X, Γ).*

Proof. That $\prod \Pi_\lambda$ is a pattern space is proved in Lemma 3.1.14. For $\gamma \in \Gamma, (\mathcal{P}_\lambda) \in \prod \Pi_\lambda$ and $C \in \mathcal{C}(X)$, $\gamma((\mathcal{P}_\lambda)_\lambda \sqcap C) = (\gamma(\mathcal{P}_\lambda)_\lambda) \sqcap \gamma C$ by a straightforward computation. □

Definition 3.2.5. The pattern space $\prod \Pi_\lambda$ is called the product Γ-pattern space.

Lemma 3.2.6. *Let Π be a pattern space over (X, Γ). Then the power pattern space 2^Π (Definition 3.1.17) is a pattern space over (X, Γ) by an action $\gamma\Xi = \{\gamma\mathcal{P} \mid \mathcal{P} \in \Xi\}$.*

Example 3.2.7. For $\mathcal{P} \in \operatorname{Patch}(X)$ and $\gamma \in \Gamma$, set $\gamma\mathcal{P} = \{\gamma T \mid T \in \mathcal{P}\}$. This defines an action of Γ on $\operatorname{Patch}(X)$ and makes $\operatorname{Patch}(X)$ a pattern space over (X, Γ).

Example 3.2.8. Let X be a metric space and a group Γ act on X as isometries. 2^X (Example 3.1.8) is a pattern space over (X, Γ). By Lemma 3.2.3, the spaces $\mathrm{LF}(X)$(Example 3.1.6), $\mathcal{C}(X)$ (Example 3.1.8), $\mathrm{UD}(X)$ and $\mathrm{UD}_r(X)$ (Example 3.1.7, $r > 0$) are all pattern spaces over (X, Γ).

Example 3.2.9. Take a non-empty set Y, an element $y_0 \in Y$ and and an action $\phi \colon \Gamma \curvearrowright Y$ that fixes y_0. As was mentioned before (Example 3.1.9), $\mathrm{Map}(X, Y, y_0)$ is a pattern space over X. Define an action of Γ on $\mathrm{Map}(X, Y, y_0)$ by

$$(\gamma f)(x) = \phi(\gamma)(f(\gamma^{-1}x)).$$

For each $f \in \mathrm{Map}(X, Y, y_0)$, $\gamma \in \Gamma$ and $C \in \mathcal{C}(X)$,

$$\begin{aligned}
(\gamma f) \cap (\gamma C)(x) &= \begin{cases} (\gamma f)(x) & \text{if } x \in \gamma C \\ y_0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \phi(\gamma)(f(\gamma^{-1}x)) & \text{if } \gamma^{-1}x \in C \\ \phi(\gamma) y_0 & \text{otherwise} \end{cases} \\
&= \phi(\gamma)(f \cap C)(\gamma^{-1}x) \\
&= \gamma(f \cap C)(x),
\end{aligned}$$

for each $x \in X$ and so $\mathrm{Map}(X, Y, y_0)$ is a pattern space over (X, Γ). This Γ-pattern space is denoted by $\mathrm{Map}_\phi(X, Y, y_0)$. If ϕ sends every group element to the identity, we denote the corresponding space by $\mathrm{Map}(X, Y, y_0)$.

Example 3.2.10. Let X be a locally compact σ-compact space and a group Γ act on X as homeomorphisms. The dual space $C_c(X)^*$ is a pattern space over X (Example 3.1.10). For $\varphi \in C_c(X)$ and $\gamma \in \Gamma$, set $(\gamma\varphi)(x) = \varphi(\gamma^{-1}x)$. For $\Phi \in C_c(X)^*$ and $\gamma \in \Gamma$, set $\gamma\Phi(\varphi) = \Phi(\gamma^{-1}\varphi)$. Then $C_c(X)^*$ is a pattern space over (X, Γ).

We mention two examples of subshifts.

Example 3.2.11. For a metric space X, its uniformly discrete and relatively dense subsets are called Delone sets. Definition of "uniformly discrete" was given in Example 3.1.7; a subset $D \subset X$ is relatively dense if there is $R > 0$ such that $D \cap B(x.R)^\circ \neq \emptyset$ for each $x \in X$. If D is relatively dense with respect to $R > 0$ and uniformly discrete with respect to $r > 0$ we say D is an (R, r)-Delone set. The set $\mathrm{Del}(X)$ of all Delone sets in X is a subshift of $\mathrm{UD}(X)$.

Example 3.2.12. For a topological space X, a patch $\mathcal{T} \in \mathrm{Patch}(X)$ is called a tiling if $\operatorname{supp} \mathcal{T} = X$. The space of all tilings is a subshift of $\mathrm{Patch}(X)$.

Definition 3.2.13. Let X be a metric space and Γ a group which acts on X as isometries. Let Π be a pattern space over (X, Γ). We say Π is a glueable pattern space over (X, Γ) if it is a glueable pattern space over X. For a glueable pattern space Π, its subshift Σ is said to be glueable if for any pairwise compatible and locally finite $\Xi \subset \Sigma$, we have $\bigvee \Xi \in \Sigma$.

Lemma 3.2.14. *Let Π be a glueable pattern space over (X,Γ), where X is a metric space on which a group Γ acts as isometries. If $\gamma \in \Gamma$ and $\Xi \subset \Pi$ is a subset which is both locally finite and pairwise compatible, then $\gamma\Xi$ (Lemma 3.2.6) is both locally finite and pairwise compatible. In this case we have*

$$\gamma \bigvee \Xi = \bigvee(\gamma\Xi).$$

Proof. If $\mathcal{P} \in \Xi$ and $\mathcal{Q} \in \Xi$, then there is $\mathcal{R} \in \Pi$ such that $\mathcal{R} \geqq \mathcal{P}$ and $\mathcal{R} \geqq \mathcal{Q}$. By Lemma 3.2.2, we see $\gamma\mathcal{R} \geqq \gamma\mathcal{P}$ and $\gamma\mathcal{R} \geqq \gamma\mathcal{Q}$ and so $\gamma\mathcal{P}$ and $\gamma\mathcal{Q}$ are compatible. If $x \in X$ and $r > 0$ then since γ is an isometry, $\gamma^{-1}B(x,r) = B(\gamma^{-1}x, r)$. By

$$\{\gamma\mathcal{P} \cap B(x,r) \mid \mathcal{P} \in \Xi\} = \gamma\{\mathcal{P} \cap B(\gamma^{-1}x, r) \mid \mathcal{P} \in \Xi\},$$

we see this set is finite. We have proved $\gamma\Xi$ is both pairwise compatible and locally finite.

Next we show the latter statement. We use Lemma 3.2.2 several times. For any $\mathcal{P} \in \Xi$, $\gamma \bigvee \Xi \geqq \gamma\mathcal{P}$. This means that $\gamma \bigvee \Xi$ is a majorant for $\gamma\Xi$. To show this is the supremum, take a majorant $\mathcal{R}$ for $\gamma\Xi$. Then $\gamma^{-1}\mathcal{R}$ is a majorant for Ξ and so $\gamma^{-1}\mathcal{R} \geqq \bigvee \Xi$. We have $\mathcal{R} \geqq \gamma \bigvee \Xi$, and so $\gamma \bigvee \Xi$ is the supremum for $\gamma\Xi$. □

3.2.2 Local derivability

Setting 2. In this subsection, X, Y and Z are non-empty metric spaces and Γ is a group which acts on X, Y and Z as isometries.

Local derivability was defined in [4] for tilings or more generally patterns in $\mathbb{R}^d$. Here we generalize it and define local derivability for two abstract patterns $\mathcal{P}_1$ and $\mathcal{P}_2$. Note that these $\mathcal{P}_1$ and $\mathcal{P}_2$ may be in different pattern spaces Π_1 and Π_2, and these Π_1 and Π_2 may be over different metric spaces X and Y. However we assume Π_1 and Π_2 are Γ-pattern spaces for the same group Γ.

Lemma 3.2.15. *Let Π_1 be a pattern space over (X,Γ) and Π_2 a pattern space over (Y,Γ). For two abstract patterns $\mathcal{P}_1 \in \Pi_1$ and $\mathcal{P}_2 \in \Pi_2$, the following two conditions are equivalent:*

1. *There exist $x_0 \in X$, $y_0 \in Y$ and $R_0 \geqq 0$ such that if $\gamma, \eta \in \Gamma$, $R \geqq 0$ and*

$$(\gamma\mathcal{P}_1) \cap B(x_0, R + R_0) = (\eta\mathcal{P}_1) \cap B(x_0, R + R_0),$$

   then

$$(\gamma\mathcal{P}_2) \cap B(y_0, R) = (\eta\mathcal{P}_2) \cap B(y_0, R).$$

2. *For any $x_1 \in X$ and $y_1 \in Y$ there exists $R_1 \geqq 0$ such that if $\gamma, \eta \in \Gamma$, $R \geqq 0$ and*

$$(\gamma\mathcal{P}_1) \cap B(x_1, R + R_1) = (\eta\mathcal{P}_1) \cap B(x_1, R + R_1),$$

   then

$$(\gamma\mathcal{P}_2) \cap B(y_1, R) = (\eta\mathcal{P}_2) \cap B(y_1, R).$$

Proof. It suffices to show only the implication 1.⇒2. By 1., there are x_0, y_0 and R_0 that satisfy the condition in 1. Take $x_1 \in X$ and $y_1 \in Y$ arbitrarily. Set $R_1 = R_0 + \rho_X(x_0, x_1) + \rho_Y(y_0, y_1)$, where ρ_X, ρ_Y are the metrics for X and Y, respectively. Take $\gamma, \eta \in \Gamma$ and $R > 0$ arbitrarily and suppose

$$(\gamma\mathcal{P}_1) \cap B(x_1, R_1 + R) = (\eta\mathcal{P}_1) \cap B(x_1, R_1 + R). \tag{3.6}$$

Since $B(x_0, R + \rho_Y(y_0, y_1) + R_0) \subset B(x_1, R_1 + R)$, by taking scissors operation for both sides of (3.6), we obtain

$$(\gamma\mathcal{P}_1) \cap B(x_0, R + \rho_Y(y_0, y_1) + R_0) = (\eta\mathcal{P}_1) \cap B(x_0, R + \rho_Y(y_0, y_1) + R_0),$$

and so

$$(\gamma\mathcal{P}_2) \cap B(y_0, R + \rho_Y(y_0, y_1)) = (\eta\mathcal{P}_2) \cap B(y_0, R + \rho_Y(y_0, y_1)).$$

By $B(y_1, R) \subset B(y_0, R + \rho_Y(y_0, y_1))$,

$$(\gamma\mathcal{P}_2) \cap B(y_1, R) = (\eta\mathcal{P}_2) \cap B(y_1, R).$$

□

Definition 3.2.16. Let Π_1 be a pattern space over (X, Γ) and Π_2 be a pattern space over (Y, Γ). If $\mathcal{P}_1 \in \Pi_1$ and $\mathcal{P}_2 \in \Pi_2$ satisfy the two equivalent conditions in Lemma 3.2.15, then we say $\mathcal{P}_2$ is locally derivable from $\mathcal{P}_1$ and write $\mathcal{P}_1 \xrightarrow{\Gamma} \mathcal{P}_2$. If both $\mathcal{P}_1 \xrightarrow{\Gamma} \mathcal{P}_2$ and $\mathcal{P}_2 \xrightarrow{\Gamma} \mathcal{P}_1$ hold, we say $\mathcal{P}_1$ and $\mathcal{P}_2$ are mutually locally derivable (MLD) and write $\mathcal{P}_1 \overset{\Gamma}{\leftrightarrow} \mathcal{P}_2$.

The following two lemmas are easy to prove.

Lemma 3.2.17.

1. *Let $\mathcal{P}$ be an abstract pattern in a pattern space over (X, Γ). Then $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{P}$.*
2. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be abstract patterns in pattern spaces over $(X, \Gamma), (Y, \Gamma)$, and (Z, Γ), respectively. If $\mathcal{P} \xrightarrow{\Gamma} \mathcal{Q}$ and $\mathcal{Q} \xrightarrow{\Gamma} \mathcal{R}$, then $\mathcal{P} \xrightarrow{\Gamma} \mathcal{R}$. Consequently, if $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{Q}$ and $\mathcal{Q} \overset{\Gamma}{\leftrightarrow} \mathcal{R}$, then $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{R}$.*

Lemma 3.2.18. *Let Π_1 be a pattern space over (X, Γ) and Π_2 be a pattern space over (Y, Γ). Take two abstract patterns $\mathcal{P}_1 \in \Pi_1$ and $\mathcal{P}_2 \in \Pi_2$ and suppose $\mathcal{P}_1 \xrightarrow{\Gamma} \mathcal{P}_2$. Then for any $\gamma \in \Gamma$, we have $\gamma\mathcal{P}_1 \xrightarrow{\Gamma} \gamma\mathcal{P}_2$.*

We use the following notion in Section 5.

Definition 3.2.19. Let Π be a pattern space over (X, Γ). $\mathcal{P} \in \Pi$ is said to be Delone-deriving if there is a Delone set D in X such that $\mathcal{P} \xrightarrow{\Gamma} D$.

Remark 3.2.20. Delone sets are Delone-deriving. If a tiling consists of finitely many types of tiles up to Γ and each tile T admits a fixed point of its symmetry group $\operatorname{Sym}_\Gamma T$, then the tiling is Delone-deriving.

We next show that local derivability propagates symmetries.

Lemma 3.2.21. *Let X_1, X_2 be metric spaces on which a group Γ acts as isometries. Let Π_j be a glueable pattern space over (X_j, Γ) and $\mathcal{P}_j$ an element of Π_j, for each j. Suppose $\mathcal{P}_1 \xrightarrow{\Gamma} \mathcal{P}_2$ and $\mathcal{P}_2$ consists of bounded components. Then $\operatorname{Sym}_\Gamma \mathcal{P}_1 \subset \operatorname{Sym}_\Gamma \mathcal{P}_2$.*

This is an easy consequence of the following lemma.

Lemma 3.2.22. *Suppose Π is a glueable pattern space over (X, Γ). Take $\mathcal{P} \in \Pi$ which consists of bounded components. For $x_0 \in X$ and positive real numbers $R_1 < R_2 < \dots$ such that $\lim R_n = \infty$, the set*

$$\Xi = \{\mathcal{P} \cap B(x_0, R_n) \mid n = 1, 2, \dots\}$$

is locally finite and pairwise compatible, and $\mathcal{P} = \bigvee \Xi$.

Proof. It follows directly from the definition that Ξ is pairwise compatible. For any closed ball B, the set $\{n \mid B(x_0, R_n) \not\supset B\}$ is finite. All but finitely many elements in $\{\mathcal{P} \cap B(x_0, R_n) \cap B\}$ is equal to $\mathcal{P} \cap B$, and so Ξ is locally finite.

Set $\mathcal{Q} = \bigvee \mathcal{P}$. Since $\mathcal{P}$ is a majorant for Ξ, we have $\mathcal{P} \geqq \mathcal{Q}$. Since $\mathcal{P}$ consists of bounded components, there is $R_{\mathcal{P}} > 0$ such that if $x \in \operatorname{supp} \mathcal{P}$,

$$x \in \operatorname{supp}(\mathcal{P} \cap B(x, R_{\mathcal{P}})).$$

If n is large enough,

$$\operatorname{supp}(\mathcal{P} \cap B(x, R_{\mathcal{P}})) \subset \operatorname{supp}(\mathcal{P} \cap B(x_0, R_n)) \subset \operatorname{supp} \mathcal{Q},$$

and so we have $x \in \operatorname{supp} \mathcal{Q}$. Thus $\operatorname{supp} \mathcal{P} = \operatorname{supp} \mathcal{Q}$ and so $\mathcal{P} = \mathcal{Q}$. ☐

We finish this subsection by showing several canonical maps in aperiodic order send an abstract pattern $\mathcal{P}$ to a one which is MLD with $\mathcal{P}$.

Proposition 3.2.23. *Let (X, ρ) be a metric space and (Γ, ρ_Γ) be a group with a left invariant metric. Assume Γ acts on X as isometries and there is $x_0 \in X$ and $C_0 > 0$ such that*

$$\rho(\gamma x_0, \eta x_0) \leqq \rho_\Gamma(\gamma, \eta) \leqq \rho(\gamma x_0, \eta x_0) + C_0$$

for any $\gamma, \eta \in \Gamma$. Let $\mathcal{T}$ be a tiling of X which has finite tile type and is of discrete symmetry with respect to Γ; in other words, there is a finite set $\mathcal{A}$ of tiles in X such that

- *for each $T \in \mathcal{A}$, the group $\mathrm{Sym}_\Gamma T$ is discrete,*
- *there is $r > 0$ such that $B(x_0, r) \subset T$ for each $T \in \mathcal{A}$, and*
- *for any $S \in \mathcal{T}$ there is a unique $T \in \mathcal{A}$ and $\gamma \in \Gamma$ such that $S = \gamma T$.*

Set $D_T = \{\gamma \in \Gamma \mid \gamma T \in \mathcal{T}\}$. Then the following hold:

1. *there is $s > 0$ such that, for any $S, T \in \mathcal{A}$, $\gamma \in D_T$ and $\eta \in D_S$, if $\rho_\Gamma(\gamma, \eta) < s$, then $S = T$ and $\gamma = \eta$.*
2. *$\bigcup_T D_T$ is relatively dense.*
3. *if we regard $(D_T)_{T\in\mathcal{A}}$ as an abstract pattern in the product $\prod_{T\in\mathcal{A}} \mathrm{UD}_s(\Gamma)$, then $\mathcal{T} \overset{\Gamma}{\leftrightarrow} (D_T)_T$.*

Proof. 1. Take $s > 0$ small enough so that $s < r$ and if $\xi \in \mathrm{Sym}_\Gamma T \setminus \{e\}$ for some $T \in \mathcal{A}$, then $\rho_\Gamma(\xi, e) > s$. Assume $S, T \in \mathcal{A}$, $\gamma \in D_T$, $\eta \in D_S$ and $\rho_\Gamma(\gamma, \eta) < s$. Then $\rho(\gamma x_0, \eta x_0) < r$ and $\gamma T \cap \eta S \supset B(\gamma x_0, r) \cap B(\eta x_0, r) \neq \emptyset$, and so $\gamma T = \eta S$. By definition of finite tile type, we have $S = T$ and $\eta^{-1}\gamma \in \mathrm{Sym}_\Gamma T$. By definition of s, we have $\eta^{-1}\gamma = e$ and $\eta = \gamma$.

2. Take $R > \max_{T\in\mathcal{A}} \operatorname{diam} T$. Take $\eta \in \Gamma$. There is $T \in \mathcal{T} \cap B(\eta x_0, R)$ and there are $S \in \mathcal{A}$ and $\gamma \in \Gamma$ such that $T = \gamma S$. This γ is in D_S. Moreover, since $\gamma x_0 \in B(\eta x_0, R)$, $\rho_\Gamma(\eta, \gamma) \leqq \rho(\gamma x_0, \eta x_0) \leqq R + C_0$. This means $\bigcup D_T$ is relatively dense in Γ with respect to a constant $R + C_0$.

3. We first show $\mathcal{T} \overset{\Gamma}{\to} (D_T)_T$. Take $R_0 > \max_{T\in\mathcal{A}} \operatorname{diam} T$. Take $\gamma, \eta \in \Gamma$ and $R > 0$ and assume

$$(\gamma\mathcal{T}) \cap B(x_0, R + R_0) = (\eta\mathcal{T}) \cap B(x_0, R + R_0). \tag{3.7}$$

To prove $(\gamma D_T) \cap B(e, R) = (\eta D_T) \cap B(e, R)$ for each $T \in \mathcal{A}$, take $T \in \mathcal{A}$ and $\zeta \in D_T$ such that $\gamma\zeta \in B(e, R)$. We have $\gamma\zeta T \in \gamma\mathcal{T}$. Moreover, since $\rho(\gamma\zeta x_0, x_0) \leqq \rho_\Gamma(\gamma\zeta, e) \leqq R$, we see $\gamma\zeta T \subset B(x_0, R + R_0)$ and so $\gamma\zeta T$ is in the set (3.7), and consequently $\eta^{-1}\gamma\zeta T \in \mathcal{T}$. By the definition of D_T this implies that $\eta^{-1}\gamma\zeta \in D_T$ and $\gamma\zeta \in \eta D_T$. We have shown $(\gamma D_T) \cap B(e, R) \subset (\eta D_T) \cap B(e, R)$ and by symmetry the reverse inclusion is obvious. Hence $\mathcal{T} \overset{\Gamma}{\to} (D_T)_T$.

Next we show $(D_T)_{T\in\mathcal{A}} \overset{\Gamma}{\to} \mathcal{T}$. Take $\gamma, \eta \in \Gamma$ and $R > 0$ and suppose

$$(\gamma D_T) \cap B(e, R + C_0) = (\eta D_T) \cap B(e, R + C_0) \tag{3.8}$$

holds for each $T \in \mathcal{A}$. To probe $(\gamma\mathcal{T}) \cap B(x_0, R) = (\eta\mathcal{T}) \cap B(x_0, R)$, we take $S \in \mathcal{T}$ and assume $\gamma S \subset B(x_0.R)$. There are $T \in \mathcal{A}$ and $\xi \in D_T$ such that $S = \xi T$. Then $\gamma\xi x_0 \in \gamma\xi T = \gamma S \subset B(x_0, R)$ and so $\rho(e, \gamma\xi) \leqq R + C_0$. Thus $\gamma\xi$ is in the set (3.8). We have $\eta^{-1}\gamma\xi \in D_T$ and so $\eta^{-1}\gamma\xi T \in \mathcal{T}$, in other words, $\gamma S = \gamma\xi T \in (\eta\mathcal{T}) \cap B(x_0, R)$. We have shown $(\gamma\mathcal{T}) \cap B(x_0, R) \subset (\eta\mathcal{T}) \cap B(x_0, R)$ and the reverse inclusion is clear. □

Proposition 3.2.24. *Let X be a proper metric space on which a group Γ acts as isometries. Let D be a uniformly discrete subset of X and set $\mu = \sum_{x \in D} \delta_x$, the sum of Dirac measures with respect to the vague topology. If we regard D as an abstract pattern of* $\mathrm{UD}(X)$ *(Example 3.2.8) and μ an abstract pattern of $C_c(X)^*$ (Example 3.2.10), we have the following:*

1. *$\mu \cap C = \sum_{x \in D \cap C} \delta_x$ for each $C \in \mathcal{C}(X)$,*
2. *$\gamma\mu = \sum_{x \in \gamma D} \delta_x$, and*
3. *$\mu \overset{\Gamma}{\leftrightarrow} D$.*

Proof. The first two is clear by definition and the third condition follows from the first two conditions. ☐

Proposition 3.2.25. *Let Γ be a locally compact abelian group with a proper invariant metric and μ its Haar measure. Let f be a complex valued continuous bounded function on Γ. If we regard f as an abstract pattern in* $\mathrm{Map}(\Gamma, \mathbb{C}, 0)$ *(Example 3.2.9) and $f d\mu$ as an element of $C_c(\Gamma)^*$ (Example 3.2.10) that sends $\varphi \in C_c(\Gamma)$ to $\int \varphi f d\mu$, we have $f \overset{\Gamma}{\leftrightarrow} f d\mu$.*

Proof. Take $R > 0$ and $s, t \in \Gamma$ and assume

$$(f - s) \cap B(e, R) = (f - t) \cap B(e, R). \tag{3.9}$$

Here, $f - t$ and $f - s$ denote the image of f by the group action. For each $\varphi \in C_c(\Gamma)$, the image by $(f d\mu - s) \cap B(e, R)$ is $\int_{B(e,R)} \varphi(x) f(x + s) d\mu$ and the image by $(f d\mu - t) \cap B(e, R)$ is $\int_{B(e,R)} \varphi(x) f(x + t) d\mu$. By (3.9), for each $x \in B(e, R)$,

$$f(x + t) = f \cap B(t, R)(x + t) = (f - t) \cap B(0, R)(t) = (f - s) \cap B(e, R)(x) = f(x + s),$$

and so the images of φ by $(f d\mu - s) \cap B(e, R)$ and $(f d\mu - t) \cap B(e, R)$ are the same, and so these two maps are the same.

Conversely, suppose $R > 0$, $s, t \in \Gamma$ and

$$(f d\mu - s) \cap B(e, R + 1) = (f d\mu - t) \cap B(e, R + 1).$$

For any $\varphi \in C_c(\Gamma)$ with $\operatorname{supp} \varphi \subset B(e, R + 1)$, we have

$$\int \varphi(x) f(x + s) d\mu(x) = \int \varphi(x) f(x + t) d\mu(x),$$

and so for any $x \in B(e, R)$, we have $f(x + s) = f(x + t)$ and

$$(f - s) \cap B(e, R) = (f - t) \cap B(e, R).$$

☐

For the rest of this subsection $(\mathbb{R}^d, \rho)$ is the Euclidean space with the Euclidean metric and D is a Delone subset (Example 3.2.11) of $\mathbb{R}^d$ which is relatively dense with respect to $R > 0$ and uniformly discrete with respect to $r > 0$.

Definition 3.2.26. For each $x \in D$, set

$$V_x = \{y \in \mathbb{R}^d \mid \rho(x, y) < \rho(x', y) \text{ for any } x' \in D \setminus \{x\}.\}$$

Lemma 3.2.27. *For each $x \in D$, V_x is nonempty and $V_x \subset B(x, R)^\circ$. Moreover,*

$$V_x = \{y \in B(x, R)^\circ \mid \ \rho(x, y) < \rho(x', y) \text{ for each } x' \in D'\} \tag{3.10}$$

for each D' with $D \setminus \{x\} \cap B(x, 2R) \subset D' \subset D \setminus \{x\}$. In particular V_x is open for each $x \in D$.

Proof. If $y \in \mathbb{R}^d$ and $\rho(x, y) < r/2$, then $y \in V_x$. Thus $V_x \neq \emptyset$. If $y \in \mathbb{R}^d \setminus B(x, R)^\circ$, then since there is $x' \in D \cap B(y, R)^\circ$, we have $\rho(x', y) < R \leqq \rho(x, y)$ and so $y \notin V_x$.

Assume $y \in B(x, R)^\circ$ and $\rho(x, y) < \rho(x', y)$ for each $x' \in (D \setminus \{x\}) \cap B(x, 2R)$. If $x' \in D \setminus \{x\}$ and $\rho(x, x') > 2R$, then $\rho(x', y) \geqq \rho(x, x') - \rho(x, y) > R > \rho(x, y)$ and so $y \in V_x$. This observation shows the equality (3.10). □

Definition 3.2.28. For each $x \in D$, set $U_x = V_x \setminus \{x\}$. Set $\mathcal{T} = \{U_x \mid x \in D\}$.

Lemma 3.2.29. *$\mathcal{T}$ is a tiling of $\mathbb{R}^d$.*

Proof. By Lemma 3.2.27, U_x is open, bounded and nonempty. By definition of V_x, if $x \neq x'$ we have $U_x \cap U_{x'} = \emptyset$. Next we take $y \in \mathbb{R}^d$ and show that there is $x \in D$ such that $y \in \overline{U_x}$. To this purpose we may assume that $y \neq x$ for any $x \in D$. Since $\{x \in D \mid \rho(x, y) < R\}$ is finite and nonempty, $F = \{x \in D \mid \rho(x, y) \leqq \rho(x', y) \text{ for any } x' \in D\}$ is nonempty and finite. Take $x \in F$. For each $t \in (0, 1)$, set $y_t = tx + (1-t)y$. Then $\rho(x, y_t) = \|(1-t)(y-x)\|$. If $x' \in D$ and $\{y - x, y - x'\}$ is linearly independent, we have

$$\rho(x', y_t) = \|(1-t)y + tx - x'\| > \|y - x'\| - t\|y - x\| \geqq (1-t)\|y - x\| = \rho(x, y_t).$$

If $x' \in D \setminus \{x\}$ and $\{y - x, y - x'\}$ is linearly dependent, then there is $\lambda \in \mathbb{R}$ such that $x' - y = \lambda(x - y)$. Since $\lambda > 1$ or $\lambda \leqq -1$, we see $\rho(y_t, x) < \rho(y_t, x')$. By these observations we see $y_t \in V_x$, and so $y \in \overline{V_x} = \overline{U_x}$. □

Remark 3.2.30. There is $r > 0$ such that $B(x, r) \subset U_x \cup \{x\}$. Conversely, if $y \in \mathbb{R}^d \setminus U_x$ and there is $r > 0$ such that $B(y, r) \subset U_x \cup \{y\}$, then $x = y$. Thus if $x, y \in D$, $\gamma, \eta \in \Gamma$ and $\gamma U_x = \eta U_y$, then $\gamma x = \eta y$.

Proposition 3.2.31. *Let Γ be a closed subgroup of $E(d)$. If we regard D as an element of $\mathrm{UD}(\mathbb{R}^d)$, which is a pattern space over $(\mathbb{R}^d, \Gamma)$, and $\mathcal{T}$ as an element of $\mathrm{Patch}(\mathbb{R}^d)$, which is also a pattern space over $(\mathbb{R}^d, \Gamma)$, we have $D \overset{\Gamma}{\leftrightarrow} \mathcal{T}$.*

Proof. Take $L > 0$ and $\gamma, \eta \in \Gamma$ and assume

$$(\gamma D) \cap B(0, L + 2R) = (\eta D) \cap B(0, L + 2R). \tag{3.11}$$

Suppose $x \in D$ and $\gamma U_x \subset B(0, L)$. Since $\gamma x \in B(0, L)$, by (3.11), we see $\gamma x \in \eta D$ and $y = \eta^{-1}\gamma x \in D$. By setting $D' = (D \setminus \{x\}) \cap B(\gamma^{-1}0, L + 2R)$ in Lemma 3.2.27, we have

$$\begin{aligned}\gamma U_x &= \gamma\{z \in B(x, R)^\circ \mid \rho(x, z) < \rho(x', z) \text{ for any } x' \in (D \setminus \{x\}) \cap B(\gamma^{-1}0, L + 2R)\} \\ &= \{z \in B(\gamma x, R)^\circ \mid \rho(\gamma x, z) < \rho(x', z) \text{ for any } x' \in' (\gamma D) \cap B(0, L + 2R) \setminus \{\gamma x\}\ \} \\ &= \{z \in B(\eta y, R)^\circ \mid \rho(\eta y, z) < \rho(x', z) \text{ for any } x' \in (\eta D) \cap B(0, L + 2R) \setminus \{\eta y\})\} \\ &= \eta U_y,\end{aligned}$$

and so $\gamma U_x \in \eta\mathcal{T}$. We have shown $(\gamma\mathcal{T}) \cap B(0, L) \subset \eta\mathcal{T}$ and by symmetry this implies that $(\gamma\mathcal{T}) \cap B(0, L) = (\eta\mathcal{T}) \cap B(0, L)$.

Conversely, assume $L > 0$, $\eta, \gamma \in \Gamma$ and

$$(\gamma\mathcal{T}) \cap B(0, L + R) = (\eta\mathcal{T}) \cap B(0, L + R). \tag{3.12}$$

If $x \in D$ and $\gamma x \in B(0, L)$, then $\gamma U_x \subset B(0, L+R)$ and so by (3.12) we have $\gamma U_x \in (\eta\mathcal{T}) \cap B(0, L + R)$. There is $y \in D$ such that $\gamma U_x = \eta U_y$, and so $\gamma x = \eta y \in \eta D$. We have shown $(\gamma D) \cap B(0, L) \subset \eta D$ and by symmetry we obtain $(\gamma D) \cap B(0, L) = (\eta D) \cap B(0, L)$. □

3.2.3 Decomposition of Abstract Patterns by Delone Sets

Setting 3. Here is the setting of this subsection. (X, ρ) is a metric space and Γ is a group which acts on X transitively as isometries. We take $x_0 \in X$ and fix it. Assume there are a left invariant metric ρ_Γ for Γ and $C_0 > 0$ such that

$$\rho(\gamma x_0, \eta x_0) \leqq \rho_\Gamma(\gamma, \eta) \leqq \rho(\gamma x_0, \eta x_0) + C_0$$

holds for any $\gamma, \eta \in \Gamma$. (We use this inequality only for Proposition 3.2.39.) Π is a glueable pattern space over (X, Γ).

Definition 3.2.32. Take an abstract pattern $\mathcal{P} \in \Pi$. We say a pair (D, R) of a Delone set in X and a positive number $R > 0$ decomposes $\mathcal{P}$ if the following three conditions are satisfied:

1. $\mathcal{P} \xrightarrow{\Gamma} D$,
2. $\mathcal{P} = \bigvee\{\mathcal{P} \cap B(x, R) \mid x \in D\}$, and
3. $\max_{x \in D} \operatorname{card} \operatorname{Sym}_{\Gamma_x} \mathcal{P} \cap B(x, R)$ is finite.

Lemma 3.2.33. *If (D, R_0) decomposes $\mathcal{P}$ and $\gamma \in \Gamma$, then $(\gamma D, R_0)$ decomposes $\gamma\mathcal{P}$.*

For the rest of this subsection $\mathcal{P}$ is an element of Π, D a Delone set in X and R_0 a positive real number and assume that (D, R_0) decomposes $\mathcal{P}$.

Lemma 3.2.34. *There are a set Λ and $\mathcal{P}_\lambda \in \Pi$ for each $\lambda \in \Lambda$ such that*

1. *for each $\lambda \in \Lambda$, we have $\operatorname{supp} \mathcal{P}_\lambda \subset B(x_0, R_0)$, and*
2. *for each $x \in D$ there are a unique $\lambda_x \in \Lambda$ and $\gamma \in \Gamma$ such that $\mathcal{P} \cap B(x, R_0) = \gamma \mathcal{P}_{\lambda_x}$ and $x = \gamma x_0$.*

Proof. Define an equivalence relation $\sim$ on D as follows: we have $x \sim y$ if there is $\gamma \in \Gamma$ such that (1) $\gamma x = y$, and (2) $\gamma(\mathcal{P} \cap B(x, R_0)) = \mathcal{P} \cap B(y, R_0)$. Then by taking one point from each equivalence class for $\sim$, we obtain a set Λ.

For each $x \in \Lambda$, take an element $\gamma_x \in \Gamma$ such that $\gamma_x x_0 = x$. Set $\mathcal{P}_x = \gamma_x^{-1}(\mathcal{P} \cap B(x, R_0))$; then Λ and $\mathcal{P}_x, x \in \Lambda$, satisfy the conditions. □

Remark 3.2.35. By the second condition of Lemma 3.2.34, we see $\operatorname{Sym}_{\Gamma_{x_0}} \mathcal{P}_{\lambda_x}$ is conjugate to $\operatorname{Sym}_{\Gamma_x} \mathcal{P} \cap B(x, R_0)$. In particular, $\operatorname{card} \operatorname{Sym}_{\Gamma_{x_0}} \mathcal{P}_\lambda$, where $\lambda \in \Lambda$, is bounded from above.

Definition 3.2.36. The tuple of abstract patterns $(\mathcal{P}_\lambda)_{\lambda \in \Lambda}$ which satisfies the conditions in Lemma 3.2.34 is called the tuple of ingredients for $\mathcal{P}$ with respect to (D, R_0). For each $\lambda \in \Lambda$, set

$$\Gamma_\lambda = \Gamma_\lambda(\mathcal{P}, D, R_0, (\mathcal{P}_\lambda)_\lambda) = \{\gamma \in \Gamma \mid \gamma x_0 \in D \text{ and } \mathcal{P} \cap B(\gamma x_0, R_0) = \gamma \mathcal{P}_\lambda\}$$

and call the tuple $(\Gamma_\lambda)_\lambda$ the recipe for $\mathcal{P}$ with respect to $(D, R_0, (\mathcal{P}_\lambda))$.

The tuple of ingredients are "components" for $\mathcal{P}$, and the recipe describes how we construct $\mathcal{P}$ from the ingredients.

Lemma 3.2.37. *Let $(\mathcal{P}_\lambda)_{\lambda \in \Lambda}$ be a tuple of ingredients for $\mathcal{P}$ with respect to (D, R_0). Let $(\Gamma_\lambda)_{\lambda \in \Lambda}$ be the recipe for $\mathcal{P}$ with respect to $(D, R_0, (\mathcal{P}_\lambda))_\lambda$. For any $\gamma \in \Gamma$, $(\mathcal{P}_\lambda)_\lambda$ is a tuple of ingredients for $\gamma\mathcal{P}$ with respect to (D, R_0) and $(\gamma\Gamma_\lambda)_\lambda$ is the recipe for $\gamma\mathcal{P}$ with respect to $(D, R_0, (\mathcal{P}_\lambda))_\lambda$.*

Proof. Clear from the definition. □

Remark 3.2.38. Let $(\mathcal{P}_\lambda)_{\lambda \in \Lambda}$ be a tuple of ingredients for $\mathcal{P}$ with respect to (D, R_0). Let (Γ_λ) be the recipe for $\mathcal{P}$ with respect to $(D, R_0, (\mathcal{P}_\lambda))$. Then

$$\{\mathcal{P} \cap B(x, R_0) \mid x \in D\} = \{\gamma \mathcal{P}_\lambda \mid \lambda \in \Lambda, \gamma \in \Gamma_\lambda\}.$$

This implies that $\mathcal{P} = \bigvee \{\gamma \mathcal{P}_\lambda \mid \lambda \in \Lambda, \gamma \in \Gamma_\lambda\}$.

Proposition 3.2.39. *Let $(\mathcal{P}_\lambda)_{\lambda\in\Lambda}$ be a tuple of ingredients for $\mathcal{P}$ with respect to (D, R_0) and (Γ_λ) be the recipe for $\mathcal{P}$ with respect to $(D, R_0, (\mathcal{P}_\lambda))$. If we regard (Γ_λ) as an abstract pattern of $\prod_{\lambda\in\Lambda} 2^\Gamma$, which is a pattern space over (Γ, Γ), (Lemma 3.2.4, Definition 3.2.5, Example 3.2.8) we have*

$$\mathcal{P} \overset{\Gamma}{\leftrightarrow} (\Gamma_\lambda)_\lambda.$$

Proof. A proof of $\mathcal{P} \overset{\Gamma}{\to} (\Gamma_\lambda)_\lambda$. Let $R_1 > 0$ be a constant for the local derivation $\mathcal{P} \overset{\Gamma}{\to} D$ for points x_0 and $e \in \Gamma$ which appears in the definition of local derivability (Definition 3.2.16). Let L_0 be an arbitrary positive real number. Set $L_1 = L_0 + R_0 + R_1$. We assume $\gamma, \eta \in \Gamma$ and

$$(\gamma\mathcal{P}) \cap B(x_0, L_1) = (\eta\mathcal{P}) \cap B(x_0, L_1) \tag{3.13}$$

and show

$$(\gamma\Gamma_\lambda) \cap B(e, L_0) = (\eta\Gamma_\lambda) \cap B(e, L_0) \tag{3.14}$$

for each $\lambda \in \Lambda$.

Take $\lambda \in \Lambda$ and fix it. By (3.13), we see

$$(\gamma D) \cap B(x_0, L_0 + R_0) = (\eta D) \cap B(x_0, L_0 + R_0).$$

Let ζ be an element of Γ_λ such that $\gamma\zeta \in B(e, L_0)$. We claim that $\gamma\zeta \in \eta\Gamma_\lambda$. By the definition of the recipe, $\zeta x_0 \in D$ and $\zeta\mathcal{P}_\lambda = \mathcal{P} \cap B(\zeta x_0, R_0)$. Since $\rho(\gamma\zeta x_0, x_0) \leqq \rho_\Gamma(\gamma\zeta, e) \leqq L_0$, $\gamma\zeta x_0 \in (\gamma D) \cap B(x, L_0) = (\eta D) \cap B(x_0, L_0)$, and so there is $y \in D$ such that $\eta y = \gamma\zeta x_0$. Now

$$\begin{aligned}\gamma\zeta\mathcal{P}_\lambda &= (\gamma\mathcal{P}) \cap B(\gamma\zeta x_0, R_0)\\ &= (\gamma\mathcal{P}) \cap B(x_0, L_1) \cap B(\gamma\zeta x_0, R_0)\\ &= (\eta\mathcal{P}) \cap B(x_0, L_1) \cap B(\eta y, R_0)\\ &= \eta(\mathcal{P} \cap B(y, R_0)).\end{aligned}$$

We have proved $\eta^{-1}\gamma\zeta x_0 \in D$ and $\eta^{-1}\gamma\zeta\mathcal{P}_\lambda = \mathcal{P} \cap B(\eta^{-1}\gamma\zeta x_0, R_0)$, and so $\eta^{-1}\gamma\zeta \in \Gamma_\lambda$, by which we proved the claim. Thus $(\gamma\Gamma_\lambda) \cap B(e, L_0) \subset (\eta\Gamma_\lambda) \cap B(e, L_0)$ and by symmetry we have shown (3.14).

A proof of $(\Gamma_\lambda)_\lambda \overset{\Gamma}{\to} \mathcal{P}$. Let $L_0 > 0$ be an arbitrary positive number and set $L_1 = L_0 + R_0 + C_0$. Assume $\gamma, \eta \in \Gamma$ and

$$(\gamma\Gamma_\lambda) \cap B(e, L_1) = (\eta\Gamma_\lambda) \cap B(e, L_1) \tag{3.15}$$

holds for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ and $\xi \in \Gamma_\lambda$, if we have $(\gamma\xi\mathcal{P}_\lambda) \cap B(x_0, L_0) \neq 0$, then $B(\gamma\xi x_0, R_0) \cap B(x_0, L_0) \neq \emptyset$. This implies $\rho(\gamma\xi x_0, x_0) \leqq L_0 + R_0$ and $\rho_\Gamma(\gamma\xi, e) \leqq L_0 + R_0 + C_0 = L_1$. We have the same observation if we replace γ with η. Thus

$$\begin{aligned}
&\{(\gamma\xi\mathcal{P}_\lambda) \cap B(x_0, L_0) \mid \lambda \in \Lambda, \xi \in \Gamma_\lambda\} \cup \{0\} \\
&\qquad = \{(\gamma\xi\mathcal{P}_\lambda) \cap B(x_0, L_0)) \mid \lambda \in \Lambda,\ \xi \in \Gamma_\lambda \text{ and } \gamma\xi \in B(e, L_1)\} \cup \{0\} \\
&\qquad = \{(\eta\zeta\mathcal{P}_\lambda) \cap B(x_0, L_0) \mid \lambda \in \Lambda,\ \zeta \in \Gamma_\lambda \text{ and } \eta\zeta \in B(e, L_1)\} \cup \{0\} \\
&\qquad = \{(\eta\zeta\mathcal{P}_\lambda) B(x_0, R_0) \mid \lambda \in \Lambda, \zeta \in \Gamma_\lambda\} \cup \{0\}.
\end{aligned}$$

We obtain the desired result by Lemma 3.1.42 and Lemma 3.2.14:

$$\begin{aligned}
(\gamma\mathcal{P}) \cap B(x_0, L_0) &= \bigvee \{(\gamma\xi\mathcal{P}_\lambda) \cap B(x_0, L_0) \mid \lambda \in \Lambda, \xi \in \Gamma_\lambda\} \cup \{0\} \\
&= \bigvee \{(\eta\zeta\mathcal{P}_\lambda) \cap B(x_0, L_0) \mid \lambda \in \Lambda, \zeta \in \Gamma_\lambda\} \cup \{0\} \\
&= (\eta\mathcal{P}) \cap B(x_0, L_0).
\end{aligned}$$

□

Remark 3.2.40. For tilings and Delone sets we have a concept of finite local complexity (FLC). We can generalize this concept to arbitrary pattern spaces. If $\mathcal{P}$ has FLC, then the index set Λ is finite.

3.2.4 Families of building blocks and admissible digits

In the last subsection we studied decomposition of abstract patterns. Here we study construction of abstract patterns from "building blocks".

Setting 4. In this subsection X is a proper metric space and Γ is a group which acts on X as isometries. Let Σ be a glueable subshift inside a glueable pattern space Π over (X, Γ).

Here we define and study "building blocks" and "admissible digits". A family of building blocks is a family of abstract patterns located around a fixed point $x_0 \in X$ such that we can easily construct abstract patterns by "juxtaposing them". For example, take two numbers r, s such that $\frac{1}{2} > r > s > 0$. Set $\mathcal{P} = \{0\}$ and $\mathcal{Q} = \{0, r\}$. These are abstract patterns in $\mathrm{UD}_s(\mathbb{R})$ (Example 3.1.7), which is regarded as a pattern space over $(\mathbb{R}, \mathbb{R})$ by a natural action. We can easily construct s-uniformly discrete set by juxtaposing these two abstract patterns $\mathcal{P}$ and $\mathcal{Q}$. For example, set $\Gamma_{\mathcal{P}} = 4\mathbb{Z}$ and $\Gamma_{\mathcal{Q}} = 4\mathbb{Z} + 2$, then the set $(\mathcal{P} + \Gamma_{\mathcal{P}}) \cup (\mathcal{Q} + \Gamma_{\mathcal{Q}})$ is an s-uniformly discrete set obtained by juxtaposing the two abstract patterns according to $\Gamma_{\mathcal{P}}$ and $\Gamma_{\mathcal{Q}}$. Such a tuple $(\Gamma_{\mathcal{P}}, \Gamma_{\mathcal{Q}})$ is called an admissible digit. The family $\{\mathcal{P}, \mathcal{Q}\}$ becomes a family of building block for $(0, r)$ in the following Definition 3.2.41.

Definition 3.2.41. Take a point $x_0 \in X$ and a positive number $r > 0$ arbitrarily. A subset $\mathfrak{F} \subset \Sigma$ is called a family of building block of Σ for (x_0, r) if the following three conditions are satisfied:

1. $\mathfrak{F} \neq \emptyset$ and $\emptyset \neq \operatorname{supp} \mathcal{P} \subset B(x_0, r)$ for each $\mathcal{P} \in \mathfrak{F}$.
2. If $\gamma, \eta \in \Gamma$, $\mathcal{P}, \mathcal{Q} \in \mathfrak{F}$ and $\rho(\gamma x_0, \eta x_0) > 4r$, then $\gamma\mathcal{P}$ and $\eta\mathcal{Q}$ are compatible.
3. If $\mathcal{P}, \mathcal{Q} \in \mathfrak{F}$, $\gamma \in \Gamma$ and $\gamma\mathcal{P} = \mathcal{Q}$, then $\mathcal{P} = \mathcal{Q}$ and $\gamma x_0 = x_0$.

The elements of $\mathfrak{F}$ are called building blocks for (x_0, r). If a building block $\mathcal{P}$ for (x_0, r) additionally satisfies a condition

$$\operatorname{Sym}_\Gamma \mathcal{P} = \Gamma_{x_0},$$

then $\mathcal{P}$ is called a symmetric building block for (x_0, r).

Let $\mathfrak{F}$ be a family of building block of Σ for (x_0, r). Then a tuple $(\Gamma_\mathcal{P})_{\mathcal{P} \in \mathfrak{F}}$ of subsets $\Gamma_\mathcal{P} \subset \Gamma$ is called an admissible digit if it satisfies the condition

If $\mathcal{P}, \mathcal{Q} \in \mathfrak{F}$, $\gamma \in \Gamma_\mathcal{P}$, $\eta \in \Gamma_\mathcal{Q}$ and $\rho(\gamma x_0, \eta x_0) \leqq 4r$, then $\mathcal{P} = \mathcal{Q}$ and $\gamma\mathcal{P} = \eta\mathcal{Q}$.

Remark 3.2.42. A non-empty subset of a family of building block is again a family of building block.

Lemma 3.2.43. *Let $\mathfrak{F}$ be a family of building block for (x_0, r) and $(\Gamma_\mathcal{P})_{\mathcal{P} \in \mathfrak{F}}$ be an admissible digit. Then $\{\gamma\mathcal{P} \mid \mathcal{P} \in \mathfrak{F}, \gamma \in \Gamma_\mathcal{P}\}$ is locally finite an pairwise compatible.*

Proof. Clear by definition. □

Since a family of building block is inside a glueable subshift, we can freely take a supremum $\bigvee\{\gamma\mathcal{P} \mid \mathcal{P} \in \mathfrak{F}, \gamma \in \Gamma_\mathcal{P}\}$ under the same condition as in Lemma 3.2.43.

We finish this subsection by proving two lemmas which will be useful in Section 5.

Lemma 3.2.44. *Let $\mathfrak{F}$ be a family of building block for (x_0, r). Take a real number $r' > 2r$ arbitrarily. Let $(\Gamma^\lambda_\mathcal{P})_{\mathcal{P} \in \mathfrak{F}}$ be an admissible digit for each λ, where λ belongs to an index set Λ, such that*

1. *for each $\lambda \in \Lambda$, we have $\bigcup_{\mathcal{P} \in \mathfrak{F}} \Gamma^\lambda_\mathcal{P} \neq \emptyset$, and*
2. *for each λ and $\mathcal{P}$, any element $\gamma \in \Gamma^\lambda_\mathcal{P}$ satisfies a condition*

$$\rho(x_0, \gamma x_0) < r' - 2r. \tag{3.16}$$

Set $\mathcal{Q}_\lambda = \bigvee\{\gamma\mathcal{P} \mid \mathcal{P} \in \mathfrak{F}, \gamma \in \Gamma^\lambda_\mathcal{P}\}$ for each $\lambda \in \Lambda$. Then the family $\{\mathcal{Q}_\lambda \mid \lambda \in \Lambda\}$ satisfies the first two conditions of the definition of family of building block (Definition 3.2.41).

Proof. The first condition. Take $\lambda \in \Lambda$ and fix it. Since $\operatorname{supp} \mathcal{Q}_\lambda = \overline{\bigcup_{\mathcal{P}\in\mathfrak{F},\gamma\in\Gamma^\lambda_\mathcal{P}} \operatorname{supp} \gamma\mathcal{P}}$, it is nonempty. We have moreover $\operatorname{supp} \gamma\mathcal{P} \subset B(\gamma x_0, r) \subset B(x_0, r')$ by (3.16), for each $\mathcal{P} \in \mathfrak{F}$ and $\gamma \in \Gamma^\lambda_\mathcal{P}$, and so $\operatorname{supp} \mathcal{Q}_\lambda \subset B(x_0, r')$.

The second condition. Take $\lambda, \mu \in \Lambda$ and $\gamma, \eta \in \Gamma$ such that $\rho(\gamma x_0, \eta x_0) > 4r'$. We show that $\gamma\mathcal{Q}_\lambda$ and $\eta\mathcal{Q}_\mu$ are compatible. For each $\mathcal{P}, \mathcal{Q} \in \mathfrak{F}$, $\xi \in \Gamma^\lambda_\mathcal{P}$ and $\zeta \in \Gamma^\mu_\mathcal{Q}$, by (3.16), we have $\rho(\gamma\xi x_0, \eta\zeta x_0) > 4r$. Thus $\gamma\xi\mathcal{P}$ and $\eta\zeta\mathcal{Q}$ are compatible and so together with Lemma 3.2.43, the set $\Xi_1 \cup \Xi_2$ is locally finite and pairwise compatible. Here,

$$\Xi_1 = \{\gamma\xi\mathcal{P} \mid \mathcal{P} \in \mathfrak{F}, \xi \in \Gamma^\lambda_\mathcal{P}\},$$

and

$$\Xi_2 = \{\eta\zeta\mathcal{Q} \mid \mathcal{Q} \in \mathfrak{F}, \zeta \in \Gamma^\mu_\mathcal{Q}\}.$$

By Lemma 3.1.30 and the fact that $\gamma\mathcal{Q}_\lambda = \bigvee \Xi_1$ and $\eta\mathcal{Q}_\mu = \bigvee \Xi_2$, we see $\gamma\mathcal{Q}_\lambda$ and $\eta\mathcal{Q}_\mu$ are compatible. □

Remark 3.2.45. In Lemma 3.2.44, the third condition is not always satisfied. When we use this lemma in Section 3.3, we prove the third condition in an ad hoc way.

Lemma 3.2.46. *Take $x_0 \in X$ and $r > 0$ arbitrarily. Let $\mathfrak{F}$ be a family of building block for (x_0, r). Take two admissible digits $(\Gamma^1_\mathcal{P})_{\mathcal{P}\in\mathfrak{F}}$ and $(\Gamma^2_\mathcal{P})_{\mathcal{P}\in\mathfrak{F}}$. Suppose both $\bigcup_\mathcal{P} \Gamma^1_\mathcal{P}$ and $\bigcup_\mathcal{P} \Gamma^2_\mathcal{P}$ are finite. Suppose also that*

$$\bigvee\{\gamma\mathcal{P} \mid \mathcal{P} \in \mathfrak{F}, \gamma \in \Gamma^1_\mathcal{P}\} = \bigvee\{\gamma\mathcal{P} \mid \mathcal{P} \in \mathfrak{F}, \gamma \in \Gamma^2_\mathcal{P}\}.$$

Then for any $\mathcal{P} \in \mathfrak{F}$ and $\gamma \in \Gamma^1_\mathcal{P}$ there is $\eta \in \Gamma^2_\mathcal{P}$ such that $\gamma\mathcal{P} = \eta\mathcal{P}$.

Proof. Consider two finite sets

$$F_1 = \{\gamma x_0 \mid \gamma \in \bigcup_\mathcal{P} \Gamma^1_\mathcal{P}\}$$

and

$$F_2 = \{\gamma x_0 \mid \gamma \in \bigcup_\mathcal{P} \Gamma^2_\mathcal{P}\}.$$

For each $x \in F_1$, there are $\mathcal{P} \in \mathfrak{F}$ and $\gamma \in \Gamma^1_\mathcal{P}$ such that $x = \gamma x_0$. Set $\mathcal{P}^1_x = \gamma\mathcal{P}$. This is independent of the choice of $\mathcal{P}$ and γ. Define $\mathcal{P}^2_x$ for each $x \in F_2$ in a similar way. We can apply Lemma 3.1.25. □

3.3 Translation theorem for certain abstract patterns

Here we prove Theorem 3.3.1, which answers the second question given in Introduction. In Subsection 3.3.1 we prepare necessary lemmas to prove this theorem. In Subsection 3.3.2 we give a proof of Theorem 3.3.1.

Setting 5. In this section $X = \mathbb{R}^d$ and Γ is a closed subgroup of $\mathrm{E}(d)$ that contains $\mathbb{R}^d$. Π_1 and Π_2 are glueable pattern spaces over $(\mathbb{R}^d, \Gamma)$. Let Σ be a glueable subshift inside Π_2. We assume Σ contains sufficiently many symmetric building blocks, which means that for each $r > 0$, there is a symmetric building block $\mathcal{P}_r$ for $(0, r)$ (Definition 3.2.41).

In this setting we prove

Theorem 3.3.1. *Let $\mathcal{P}$ be an abstract pattern in Π_1 which consists of bounded components (Definition 3.1.18) and is Delone-deriving (Definition 3.2.19). Then there is an abstract pattern $\mathcal{S}$ in Σ such that $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{S}$. Moreover,* $\operatorname{supp} \mathcal{S}$ *is relatively dense in $\mathbb{R}^d$.*

Remark 3.3.2. This theorem holds if replace $(\mathbb{R}^d, \Gamma)$ with a pair (X, Γ) of a proper metric space X and a group Γ that acts on X transitively as isometries and admits left-invariant proper metric such that inequality (1.1), Lemma 3.3.4, Lemma 3.3.8 and Lemma 3.3.9 hold if we replace 2 on the right-hand side of (1.1) with some positive number and $0 \in \mathbb{R}^d$ in these assertions with some point in X.

Remark 3.3.3. If $\Sigma = \mathrm{UD}_r(\mathbb{R}^d)$, the one-point set $\mathcal{P} = \{0\}$ is a symmetric building block and so this Σ satisfies the condition in Theorem 3.3.1. Thus for any Π_1 and $\mathcal{P} \in \Pi_1$ which satisfy the condition in Theorem 3.3.1, we obtain a uniformly discrete set $\mathcal{S}$ with relatively dense support, that is, a Delone set, which is MLD with $\mathcal{P}$.

In Section 3.4 we give several sufficient conditions for a subshift of functions to have sufficiently many symmetric building blocks. We will be able to apply Theorem 3.3.1 when Σ is a space of certain functions under a mild condition.

3.3.1 Preliminary Lemmas

Lemma 3.3.4. *Let $\mathfrak{G}$ be a subset of Γ_0 which is at most countable. Suppose* $\max_{G \in \mathfrak{G}} \operatorname{card} G < \infty$. *Then for each two numbers r, s such that $r > s > 0$, there are $\varepsilon > 0$ and a point $y_G \in B(0, r)^\circ \setminus B(0, s)$ for each $G \in \mathfrak{G}$ such that*

1. *if $G \in \mathfrak{G}$ and $\gamma \in G \setminus \{e\}$, then $\rho(y_G, \gamma y_G) > \varepsilon$, and*
2. *if $G \neq H$, then $\rho(0, y_G) \neq \rho(0, y_H)$.*

To prove Lemma 3.3.4, we prepare the following notation.

Definition 3.3.5. For any $A \in \mathrm{O}(d)$, $r > 0$ and $\varepsilon \geqq 0$, set

$$S_{A,\varepsilon,r} = \{x \in B(0, r) \mid \rho(Ax, x) \leqq \varepsilon\}.$$

Lemma 3.3.6. *If the order of an element $A \in O(d)$ is less than an integer m, then $S_{A,\varepsilon,r} \subset S_{A,\varepsilon,0} + B(0, \frac{m}{2}\varepsilon)$.*

Proof. Take an element $x \in S_{A,\varepsilon,r}$. Let k be the order of A. Set $y = \frac{1}{k}\sum_{j=0}^{k-1} A^j x$. By convexity of $B(0,r)$, y is in $B(0,r)$, and so $y \in S_{A,0,r}$. Moreover,

$$\begin{aligned}
\rho(x,y) &= \|\frac{1}{k}\sum(A^j x - x)\| \\
&\leqq \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=0}^{j-1}\|A^i x - A^{i+1}x\| \\
&\leqq \frac{1}{k}\sum_{j=0}^{k-1} j\varepsilon \\
&\leqq \frac{m}{2}\varepsilon.
\end{aligned}$$

□

Lemma 3.3.7. *Let m be a positive integer and r be a positive real number. We have $\lim_{\varepsilon\to 0}\mu(S_{A,\varepsilon,r}) = 0$ uniformly for all $A \in O(d)\setminus\{e\}$ such that the order of A is less than m.*

Proof. For each such A there is a $d-1$ dimensional vector subspace V_A of $\mathbb{R}^d$ such that $S_{A,0,r} \subset V_A \cap B(0,r)$, and so $S_{A,\varepsilon,r} \subset (V_A \cap B(0,r)) + B(0, \frac{m}{2}\varepsilon)$. For any $d-1$ dimensional vector subspace V of $\mathbb{R}^d$, the limit $\lim_{\varepsilon\to 0}\mu((V\cap B(0,r)) + B(0,\frac{m}{2}\varepsilon))$ converges uniformly to 0. □

Proof of Lemma 3.3.4. If ε is small enough, for any $A \in \mathrm{O}(d)\setminus\{e\}$ of which order is less than m, $m\mu(S_{A,\varepsilon,r}) < \mu(B(0,r)^\circ \setminus B(0,s))$. This implies that $B(0,r)^\circ\setminus B(0,s)$ is not included in $\bigcup_{A\in G, A\neq e} S_{A,\varepsilon,r}$ for any $G \in \mathfrak{G}$. To take each y_G, we enumerate $\mathfrak{G}$ as $\mathfrak{G} = \{G_1, G_2, \ldots\}$. First take $y_{G_1} \in B(0,r)^\circ \setminus (B(0,s)\cup\bigcup_{A\in G_1, A\neq e} S_{A,\varepsilon,r})$. If we have taken $y_{G_1}, y_{G_2}, \ldots, y_{G_{n-1}}$, we can take $y_{G_n} \in B(0,r)^\circ\setminus(B(0,s)\cup\bigcup_{A\in G_n, A\neq e} S_{A,\varepsilon,r})$ such that $\|y_{G_n}\| \neq \|y_{G_j}\|$ for each $j = 1, 2, \ldots, n-1$. In this way we can take $y_{G_1}, y_{G_2}, \ldots$ with the desired condition. □

We defined e_i in Notation 1.2.1.

Lemma 3.3.8. *For any $r > 0$ there is a subset $F \subset B(0,r)$ such that*

- $1 < \operatorname{card} F < \infty$, *and*
- $\mathrm{Sym}_\Gamma F = \{e\}$.

Proof. Take for each $j = 1, 2, \dots, d$ a positive number $r_j > 0$. Set $F = \{0, r_1e_1, r_2e_2, \dots, r_de_d\}$. If any two r_j's are different but all close to 1, then 0 is the only vector in F such that the distances with any other vectors are close to 1. Thus if $\gamma \in \Gamma$ and $\gamma F = F$, then $\gamma 0 = 0$. Since r_j's are all different,$\gamma r_je_j = r_je_j$ for each j, and since $\{r_1e_1, \dots, r_de_d\}$ is a basis for $\mathbb{R}^d$, γ must be e. □

Lemma 3.3.9. *For any $r > 0$ and $R > 0$ there are $R' > 0$ and $C_1 > 0$ such that, if $x \in \mathbb{R}^d$ and D is a Delone set of $\mathbb{R}^d$ which is relatively dense with respect to R and uniformly discrete with respect to r, then*

$$\operatorname{card}(\operatorname{Sym}_{\Gamma_x} D \cap B(x, R')) < C_1. \tag{3.17}$$

Proof. Take $R' > 0$ large enough so that if $e'_1, e'_2, \dots, e'_d \in \mathbb{R}^d$ and $\|e_j - e'_j\| < \frac{R}{R'-R}$ for each j, then $\{e'_1, e'_2, \dots, e'_d\}$ is linear independent. Set $C_1 > k!$, where k is an integer such that $k > \frac{\mu(B(0,R'+r))}{\mu(B(0,\frac{r}{2}))}$.

Take (R, r)-Delone set D and $x \in \mathbb{R}^d$ arbitrarily. For each $j = 1, 2, \dots, d$, there is $x_j \in D \cap B(x + (R' - R)e_j, R)$. Then for each j we have $\|\frac{1}{R'-R}(x_j - x) - e_j\| < \frac{R}{R'-R}$ and so the set of vectors $\{x_j - x \mid j = 1, 2, \dots, d\}$ is a basis for $\mathbb{R}^d$.

If $\gamma \in \Gamma_x$ and $g(y) = y$ for each $y \in D \cap B(x, R')$, then since γ fixes $x, x_1, x_2, \dots, x_d$, $\gamma = e$. Thus we have an embedding of $\operatorname{Sym}_{\Gamma_x} D \cap B(x, R')$ into the permutation group of the set $D \cap B(x, R')$. Since for any two distinct $y, z \in D \cap B(x, R')$ we have $B(y, r/2) \cap B(z, r/2) = \emptyset$, we see $\mu(B(0, r/2)) \operatorname{card} D \cap B(x, R') \leqq \mu(B(0, R' + r))$. The order of the permutation group is less than C_1 which we take above. We thus see the inequality (3.17). □

3.3.2 Proof of Theorem 3.3.1

Let $\mathcal{P} \in \Pi_1$ be an abstract pattern that consists of bounded components (Definition 3.1.18). Suppose $\mathcal{P}$ is Delone-deriving, that is, there is a Delone set D in $\mathbb{R}^d$ such that $\mathcal{P} \xrightarrow{\Gamma} D$.

Lemma 3.3.10. *There exists $R_0 > 0$ such that (D, R_0) decomposes $\mathcal{P}$ (Definition 3.2.32).*

Proof. The set D is Delone, so that it is relatively dense for a positive $R_D > 0$ and uniformly discrete for $r_D > 0$. For these R_D and r_D, there are R' and C_1 as in Lemma 3.3.9. The abstract pattern $\mathcal{P}$ consists of bounded components so that there is $R_{\mathcal{P}}$ as in Definition 3.1.18. Since D is locally derivable from $\mathcal{P}$, there is a constant $R_{LD} > 0$ for a point $x_0 = y_0 = 0$ as in 1. of Lemme 3.2.15. Take $R_0 > R_D + R_{\mathcal{P}} + R_{LD} + R'$.

The first condition of Definition 3.2.32 is satisfied by the assumption.

The Second Condition of Definition 3.2.32. First we show $\{\mathcal{P} \cap B(x, R_0) \mid x \in D\}$ is locally finite and pairwise compatible. For each $x \in \mathbb{R}^d$ and $r > 0$, we have an inclusion

$$\{y \in D \mid B(y, R_0) \cap B(x, r) \neq \emptyset\} \subset D \cap B(x, R_0 + r)$$

and the latter is finite. Hence $\{\mathcal{P} \cap B(y, R_0) \cap B(x, r) \mid y \in D\}$ is finite, since it is a zero element except for finitely many $y's$ and by Lemma 3.1.41, zero element is unique. On the other hand, pairwise-compatibility is clear since for each $\mathcal{P} \cap B(x, R_0)$, $\mathcal{P}$ is a majorant.

Since Π_1 is glueable, there is the supremum $\mathcal{Q} = \bigvee\{\mathcal{P} \cap B(x, R_0) \mid x \in D\}$. On one hand, we see by Lemma 3.1.23 $\operatorname{supp} \mathcal{Q} = \overline{\bigcup_{x \in D} \operatorname{supp}(\mathcal{P} \cap B(x, R_0))} \subset \operatorname{supp} \mathcal{P}$; on the other hand, if $y \in \operatorname{supp} \mathcal{P}$, then

$$y \in \operatorname{supp}(\mathcal{P} \cap B(y, R_{\mathcal{P}})) \subset \operatorname{supp}(\mathcal{P} \cap B(x, R_0)) \subset \operatorname{supp} \mathcal{Q}$$

for some $x \in D$, and so $\operatorname{supp} \mathcal{P} \subset \operatorname{supp} \mathcal{Q}$; we see $\operatorname{supp} \mathcal{P} = \operatorname{supp} \mathcal{Q}$. Since $\mathcal{P} \geqq \mathcal{P} \cap B(x, R_0)$ for each $x \in D$ and $\mathcal{Q}$ is the supremum of such abstract patterns, we have $\mathcal{P} \geqq \mathcal{Q}$. Thus $\mathcal{P} = \mathcal{P} \cap \operatorname{supp} \mathcal{P} = \mathcal{P} \cap \operatorname{supp} \mathcal{Q} = \mathcal{Q}$ by the definition of order $\geqq$ (Definition 3.1.19).

The Third Condition of Definition 3.2.32. For each $x \in D$, take $\gamma \in \operatorname{Sym}_{\Gamma_x} \mathcal{P} \cap B(x, R_0)$. Then

$$(\gamma \mathcal{P}) \cap B(x, R_0) = \gamma(\mathcal{P} \cap B(x, R_0)) = \mathcal{P} \cap B(x, R_0),$$

and since $\mathcal{P} \xrightarrow{\Gamma} D$ with respect to the constant R_{LD}, we have

$$\gamma(D \cap B(x, R')) = (\gamma D) \cap B(x, R') = D \cap B(x, R').$$

This means that $\gamma \in \operatorname{Sym}_{\Gamma_x} D \cap B(x, R')$. By definition of C_1, $\operatorname{card} \operatorname{Sym}_{\Gamma_x} \mathcal{P} \cap B(x, R_0) \leqq \operatorname{card} \operatorname{Sym}_{\Gamma_x} D \cap B(x, R') < C_1$. □

By Lemma 3.3.10, there is $R_0 > 0$ such that (D, R_0) decomposes $\mathcal{P}$.

By Lemma 3.2.34, there is a set Λ and a tuple of ingredients $(\mathcal{P}_\lambda)_{\lambda \in \Lambda}$. Let $(C_\lambda)_{\lambda \in \Lambda}$ be the recipe for $\mathcal{P}$ with respect to $(D, R_0, (\mathcal{P}_\lambda)_\lambda)$. Then we have the following:

- Λ is a set which is at most countable.
- Since each $\mathcal{P}_\lambda$ is a copy of an abstract pattern of the form $\mathcal{P} \cap B(x, R_0)$ $(x \in D)$ by an element $\gamma \in \Gamma$ such that $\gamma x = 0$, by Definition 3.2.32 we have the following: $G_\lambda = \operatorname{Sym}_{\Gamma_0} \mathcal{P}_\lambda$ is a finite group, for each $\lambda \in \Lambda$, and $\max_\lambda \operatorname{card} G_\lambda < \infty$.
- For each $\lambda \in \Lambda$, C_λ is a subset of Γ such that
$$C_\lambda G_\lambda = C_\lambda. \tag{3.18}$$
- D is a Delone set such that
$$D = \{\gamma 0 \mid \lambda \in \Lambda, \gamma \in C_\lambda\}. \tag{3.19}$$
- There is $r_0 > 0$ such that,
$$\text{if } \lambda, \mu \in \Lambda, \gamma \in C_\lambda, \eta \in C_\mu \text{ and } \rho(\gamma 0, \eta 0) \leqq 4r_0, \text{ then } \gamma 0 = \eta 0, \\ \text{and so } \lambda = \mu \text{ and } \gamma^{-1}\eta \in G_\lambda. \tag{3.20}$$

By Proposition 3.2.39, we have $\mathcal{P} \overset{\Gamma}{\leftrightarrow} (C_\lambda)_\lambda$. To prove $\mathcal{S} \overset{\Gamma}{\leftrightarrow} \mathcal{P}$ for some $\mathcal{S} \in \Sigma$, we construct an abstract pattern $\mathcal{S}$ in Σ such that $\mathcal{S} \overset{\Gamma}{\leftrightarrow} (C_\lambda)_\lambda$. It consists of three steps.

Step 1: construction of $\mathcal{E}$.

By Lemma 3.3.4, there are $y_\lambda \in B(0, \frac{3}{4}r_0) \setminus B(0, \frac{1}{2}r_0)$ for each $\lambda \in \Lambda$ and $r_1 \in (0, \frac{1}{8}r_0)$ such that

- $\inf\{\rho(\gamma y_\lambda, y_\lambda) \mid \lambda \in \Lambda, \gamma \in G_\lambda \setminus \{e\}\} > 4r_1 > 0$, and
- if λ, μ are two distinct elements of Λ, then we have $\rho(0, y_\lambda) \neq \rho(0, y_\mu)$.

By Lemma 3.3.8, there are $F \subset B(0, \frac{1}{2}r_1)$ and $r_2 \in (0, \frac{1}{4}r_1)$ such that

- If $x, y \in F$ and $x \neq y$, then $\rho(x, y) > 4r_2$,
- $\operatorname{Sym}_\Gamma F = \{e\}$, and
- $\infty > \operatorname{card} F > 1$.

Take $\gamma_x \in \Gamma$, for each $x \in X$, such that $\gamma_x 0 = x$.

Definition 3.3.11. Let $\mathcal{P}$ be a symmetric building block of Σ for $(0, r_2)$. (Its existence is assumed in Setting 5.) Set $\mathcal{E} = \bigvee\{\gamma_x\mathcal{P} \mid x \in F\}$.

Remark 3.3.12. Since points of F are separated by the distance $4r_2$, by the definition of building block the set $\{\gamma_x\mathcal{P} \mid x \in F\}$ is pairwise compatible. Since it is a finite set, it is locally finite. Its supremum exists.

Lemma 3.3.13. $\operatorname{Sym}_\Gamma \mathcal{E} = \{e\}$.

Proof. Take $\gamma \in \Gamma$ such that $\gamma\mathcal{E} = \mathcal{E}$. Since $\gamma\mathcal{E} = \bigvee\{\gamma\gamma_x\mathcal{P} \mid x \in F\}$, by Lemma 3.2.46, for each $x \in F$ there is $y \in F$ such that $\gamma\gamma_x\mathcal{P} = \gamma_y\mathcal{P}$. By the definition of building block (Definition 3.2.41), we have $\gamma\gamma_x 0 = \gamma_y 0$ and $\gamma x = y$. This implies that $\gamma F \subset F$ and $\gamma F = F$, which implies that $\gamma = e$. □

Lemma 3.3.14. *The set $\{\mathcal{P}, \mathcal{E}\}$ is a family of building block of Σ for $(0, r_1)$.*

Proof. We apply Lemma 3.2.44. The sets $\{e\}$ and $\{\gamma_x \mid x \in F\}$ play the role of admissible digits. If $x \in F$, then

$$\rho(\gamma_x 0, 0) = \rho(x, 0) \leqq \frac{1}{2}r_1 < r_1 - 2r_2,$$

and so by Lemma 3.2.44 the first two axioms for family of building block are satisfied.

Since $\mathcal{P}$ is a building block, we have $\operatorname{Sym}_\Gamma \mathcal{P} \subset \Gamma_0$. Moreover, $\operatorname{Sym}_\Gamma \mathcal{E} = \{e\} \subset \Gamma_0$. Finally we never have $\gamma\mathcal{P} = \mathcal{E}$ for any $\gamma \in \Gamma$. If this holds we have, by Lemma 3.2.46, $\gamma_x\mathcal{P} = \gamma\mathcal{P}$ for any $x \in F$, and this implies $x = \gamma_x 0 = \gamma 0$ for each $x \in F$. This contradicts the fact that $\operatorname{card} F > 1$. □

Step2: construction of $\mathcal{R}_\lambda$.

For each $\lambda \in \Lambda$, set

$$\mathcal{R}_\lambda = \bigvee \{\mathcal{P}\} \cup \{\gamma\gamma_{y_\lambda}\mathcal{E} \mid \gamma \in G_\lambda\}.$$

Lemma 3.3.15. *The set* $\{\mathcal{R}_\lambda \mid \lambda \in \Lambda\}$ *is a family of building block for* $(0, r_0)$*.*

Proof. Since $\gamma 0 = 0$, we have for each $\gamma \in G_\lambda$,

$$\rho(\gamma\gamma_{y_\lambda}0, 0) = \rho(0, y_\lambda) > \frac{1}{2}r_0 > 4r_1,$$

and by definition of y_λ's, for each distinct $\gamma, \eta \in G_\lambda$,

$$\rho(\gamma\gamma_{y_\lambda}0, \eta\gamma_{y_\lambda}0) = \rho(\eta^{-1}\gamma y_\lambda, y_\lambda) > 4r_1$$

we see the pair of $\{e\}$ and $\{\gamma\gamma_{y_\lambda} \mid \gamma \in G_\lambda\}$ forms an admissible digit, for each $\lambda \in \Lambda$. Moreover,

$$\rho(\gamma\gamma_{y_\lambda}0, 0) = \rho(0, y_\lambda) \leqq \frac{3}{4}r_0 < r_0 - 2r_1$$

we see, by Lemma 3.2.44, the first two axioms for the building block are satisfied.

Suppose $\lambda, \mu \in \Lambda$, $\gamma_0 \in \Gamma$ and $\gamma_0\mathcal{R}_\lambda = \mathcal{R}_\mu$. By Lemma 3.2.46, we have $\gamma_0\mathcal{P} = \mathcal{P}$ and so $\gamma_0 0 = 0$. Again by Lemma 3.2.46, there is $\gamma \in G_\mu$ such that $\gamma_0\gamma_{y_\lambda}\mathcal{E} = \gamma\gamma_{y_\mu}\mathcal{E}$, and so by Lemma 3.3.13, $\gamma_0\gamma_{y_\lambda} = \gamma\gamma_{y_\mu}$. This implies that (since γ_0 and γ fix 0)

$$\rho(0, y_\lambda) = \rho(0, \gamma_0\gamma_{y_\lambda}0) = \rho(0, \gamma\gamma_{y_\mu}0) = \rho(0, y_\mu)$$

and so $\lambda = \mu$. □

Lemma 3.3.16. $\mathrm{Sym}_\Gamma \mathcal{R}_\lambda = G_\lambda$ *for each* λ*.*

Proof. Take $\gamma_0 \in \mathrm{Sym}_\Gamma \mathcal{R}_\lambda$. By Lemma 3.2.46, there is $\gamma \in G_\lambda$ such that $\gamma_0\gamma_{y_\lambda}\mathcal{E} = \gamma\gamma_{y_\lambda}\mathcal{E}$ and so by Lemma 3.3.13 we have $\gamma_0 = \gamma \in G_\lambda$.

On the other hand, if $\gamma_0 \in G_\lambda$, then

$$\begin{aligned}\gamma_0\mathcal{R}_\lambda &= \bigvee \{\gamma_0\mathcal{P}\} \cup \{\gamma_0\gamma\gamma_{y_\lambda}\mathcal{E} \mid \gamma \in G_\lambda\} \\ &= \bigvee \{\mathcal{P}\} \cup \{\gamma\gamma_{y_\lambda}\mathcal{E} \mid \gamma \in G_\lambda\} \\ &= \mathcal{R}_\lambda,\end{aligned}$$

since $\mathcal{P}$ is a symmetric building block. □

Step3: Construction of $\mathcal{S}$ and its property.
Define

$$\mathcal{S} = \bigvee\{\gamma\mathcal{R}_\lambda \mid \lambda \in \Lambda, \gamma \in C_\lambda\}.$$

by (3.20), $(C_\lambda)_\lambda$ is an admissible digit for $(\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ and so $\mathcal{S}$ is well-defined.

Lemma 3.3.17. $\mathcal{S} \overset{\Gamma}{\to} D$.

Proof. Let R be an arbitrary positive real number. Set $L = R + 3r_0$. Assume $\gamma, \eta \in \Gamma$ and

$$(\gamma\mathcal{S}) \cap B(0, L) = (\eta\mathcal{S}) \cap B(0, L). \tag{3.21}$$

Set $\Xi = \{\xi\mathcal{R}_\lambda \mid \lambda \in \Lambda, \xi \in C_\lambda\}$, then by (3.21) and Lemma 3.2.14, we see

$$\bigvee(\gamma\Xi \cap B(0, L)) = \bigvee(\eta\Xi \cap B(0, L)).$$

Consider the following two finite sets:

$$F_1 = \{\gamma\xi 0 \mid \lambda \in \Lambda, \xi \in C_\lambda, \gamma\xi\mathcal{R}_\lambda \cap B(0, L) \neq 0\}$$

and

$$F_2 = \{\eta\zeta 0 \mid \lambda \in \Lambda, \zeta \in C_\lambda, \eta\zeta\mathcal{R}_\lambda \cap B(0, L) \neq 0\}.$$

For each $x = \gamma\xi 0 \in F_1$, we consider an abstract pattern $\mathcal{P}^1_x = \gamma\xi\mathcal{R}_\lambda \cap B(0, L)$. This is included in $B(\gamma\xi 0, r_0)$. For F_2 we define $\mathcal{P}^2_x$'s in a similar way. We can apply Lemma 3.1.25 and obtain the following: if $\lambda \in \Lambda$, $\xi \in C_\lambda$ and $\gamma\xi\mathcal{R}_\lambda \cap B(0, L) \neq 0$, there is $\mu \in \Lambda$ and $\zeta \in C_\mu$ such that

$$(\gamma\xi\mathcal{R}_\lambda) \cap B(0, L) = (\eta\zeta\mathcal{R}_\mu) \cap B(0, L).$$

Now we prove $(\gamma D) \cap B(0, R) \subset (\eta D) \cap B(0, R)$. Take an element $\gamma\xi 0$ from the left-hand side set, where $\xi \in C_\lambda$ for some λ and $\gamma\xi 0 \in B(0, R)$. Then $\operatorname{supp} \gamma\xi R_\lambda \subset B(0, R + r_0)$. As in the previous paragraph, there are $\mu \in \Lambda$ and $\zeta \in C_\mu$ such that $\gamma\xi\mathcal{R}_\lambda = (\eta\zeta\mathcal{R}_\mu) \cap B(0, L)$. The support of this abstract pattern is included in $B(0, R + r_0)$ and the support of $\eta\zeta\mathcal{R}_\mu$ has diameter less than $2r_0$; we have $\operatorname{supp}(\eta\zeta\mathcal{R}_\mu) \subset B(0, L)$ and so $\gamma\xi\mathcal{R}_\lambda = \eta\zeta\mathcal{R}_\mu$. Since $(\mathcal{R}_\lambda)_\lambda$ is a family of building block, we see $\lambda = \mu$ and $\gamma\xi 0 = \eta\zeta 0 \in \eta D$. We have proved $(\gamma D) \cap B(0, R) \subset (\eta D) \cap B(0, R)$ and by symmetry the reverse inclusion is true. □

Lemma 3.3.18. *For each $\lambda \in \Lambda$ and $\gamma \in C_\lambda$, we have*

$$\mathcal{S} \cap B(\gamma 0, r_0) = \gamma\mathcal{R}_\lambda.$$

Proof. If $\mu \in \Lambda$, $\eta \in C_\mu$ and $\eta\mathcal{R}_\mu \cap B(\gamma 0, r_0) \neq 0$, then $\rho(\gamma 0, \eta 0) \leqq 2r_0$ and so by (3.20), $\lambda = \mu$ and $\gamma\mathcal{R}_\lambda = \eta\mathcal{R}_\mu$. Hence

$$\begin{aligned}\mathcal{S} \cap B(\gamma 0, r_0) &= \bigvee \{\eta\mathcal{R}_\mu \cap B(\gamma 0, r_0) \mid \mu \in \Lambda, \eta \in C_\mu\} \\ &= \bigvee \{\gamma\mathcal{R}_\lambda \cap B(\gamma 0, r_0)\} \\ &= \gamma\mathcal{R}_\lambda.\end{aligned}$$

□

Lemma 3.3.19. *The pair* (D, r_0) *decomposes* $\mathcal{S}$*.*

Proof. Clear by the definition of $\mathcal{S}$, Lemma 3.3.18 and Lemma 3.3.17 and Lemma 3.3.16. □

Lemma 3.3.20. $(\mathcal{R}_\lambda)_\lambda$ *is a tuple of ingredients for* $\mathcal{S}$ *with respect to* (D, r_0) *and* (C_λ) *is the recipe for* $\mathcal{S}$ *with respect to* $(D, r_0, (\mathcal{R}_\lambda)_{\lambda\in\Lambda})$*.*

Proof. Take $x \in D$ arbitrarily. By (3.19), there are $\lambda \in \Lambda$ and $\gamma \in C_\lambda$ such that $x = \gamma 0$ and by Lemma 3.3.18, $\mathcal{S} \cap B(x, r_0) = \mathcal{S} \cap B(\gamma 0, r_0) = \gamma\mathcal{R}_\lambda$. Uniqueness of such λ is clear since $(\mathcal{R}_\mu)_{\mu\in\Lambda}$ is a family of building block. We have shown that $(\mathcal{R}_\mu)$ is a tuple of ingredients for $\mathcal{P}$ with respect to (D, r_0).

Next we show that $(C_\lambda)_\lambda$ is the recipe. If $\mu \in \Lambda$ and $\gamma \in C_\mu$, then $\gamma 0 \in D$ by (3.19) and $\mathcal{S} \cap B(\gamma 0, r_0) = \gamma\mathcal{R}_\mu$ by Lemma 3.3.18, and so $\gamma \in \Gamma_\mu(\mathcal{S}, D, r_0, (\mathcal{R}_\lambda))$. Conversely, if $\gamma \in \Gamma_\mu(\mathcal{S}, D, r_0, (\mathcal{R}_\lambda))$, then $\gamma 0 \in D$ and $\gamma\mathcal{R}_\mu = \mathcal{S} \cap B(\gamma 0, r_0)$. By (3.19), there is $\nu \in \Lambda$ and $\eta \in C_\nu$ such that $\gamma 0 = \eta 0$, and so by Lemma 3.3.18, $\eta\mathcal{R}_\nu = \mathcal{S} \cap B(\eta 0, r_0)$. This implies that $\gamma\mathcal{R}_\mu = \eta\mathcal{R}_\nu$, and so $\mu = \nu$ and $\eta^{-1}\gamma \in \operatorname{Sym}_\Gamma \mathcal{R}_\mu = G_\mu$. By (3.18), we see $\gamma = \eta\eta^{-1}\gamma \in C_\mu G_\mu = C_\mu$. We have proved $C_\mu = \Gamma_\mu(\mathcal{S}, D, r_0, (\mathcal{R}_\lambda))$ for any $\mu \in \Lambda$. □

Theorem 3.3.21. $\mathcal{S} \overset{\Gamma}{\leftrightarrow} (C_\lambda)_\lambda$*.*

Proof. Clear from Lemma 3.3.20 and Proposition 3.2.39. □

Corollary 3.3.22. $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{S}$*.*

Proof. By Proposition 3.2.39, $\mathcal{P} \overset{\Gamma}{\leftrightarrow} (C_\lambda)$ because (C_λ) is a recipe for $\mathcal{P}$. Combined with Theorem 3.3.21 we have $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{S}$. □

Lemma 3.3.23. $\operatorname{supp} \mathcal{S}$ *is relatively dense.*

Proof. For any $x \in \mathbb{R}^d$ there is $y \in D$ near x. By (3.19), there are $\lambda \in \Lambda$ and $\gamma \in C_\lambda$ such that $y = \gamma 0$. Since $\operatorname{supp} \gamma\mathcal{R}_\lambda \subset B(y, r_0)$, any point in $\operatorname{supp} \gamma\mathcal{R}_\lambda$, which is a point in $\operatorname{supp} \mathcal{S}$, is near x. □

This lemma completes the proof of Theorem 3.3.1.

3.4 An application of Theorem 3.3.1

Here we apply Theorem 3.3.1 to the theory of pattern-equivariant functions.

We start with a definition in an abstract setting:

Definition 3.4.1. Let Π be a pattern space over (X, Γ) and Π' be a pattern space over (Y, Γ), where Γ is a group which acts on metric spaces X and Y respectively as isometries. Let Σ be a subshift of Π'. For each $\mathcal{P} \in \Pi$, we set

$$\Sigma_{\mathcal{P}} = \{\mathcal{Q} \in \Sigma \mid \mathcal{P} \xrightarrow{\Gamma} \mathcal{Q}\}.$$

In order to study the relations between $\mathcal{P}$ and $\Sigma_{\mathcal{P}}$, its maximal elements, that is, elements $\mathcal{Q} \in \Sigma$ such that $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{Q}$, are useful. It may be that there is no maximal elements, but Theorem 3.3.1 gives us a sufficient condition for $\mathcal{P}$ and Σ to admit maximal elements. In the following theorem, using maximal elements, we show $\Sigma_{\mathcal{P}}$ has all of the information on $\mathcal{P}$ up to MLD:

Theorem 3.4.2. *Let Γ be a closed subgroup of $E(d)$ that contains $\mathbb{R}^d$. Let Π, Π', Π'' be glueable pattern spaces over $(\mathbb{R}^d, \Gamma)$ and Σ a glueable subshift of Π'' which has sufficiently many symmetric building blocks. Take $\mathcal{P} \in \Pi$ and $\mathcal{P}' \in \Pi'$ and assume that they consist of bounded components and they are Delone-deriving (Definition 3.2.19). Then we have $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{P}'$ if and only if $\Sigma_{\mathcal{P}} = \Sigma_{\mathcal{P}'}$.*

Proof. If $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{P}'$, then for any $\mathcal{Q} \in \Sigma_{\mathcal{P}}$ we have $\mathcal{P}' \xrightarrow{\Gamma} \mathcal{P} \xrightarrow{\Gamma} \mathcal{Q}$, and so $\mathcal{Q} \in \Sigma_{\mathcal{P}'}$. The converse also holds and so $\Sigma_{\mathcal{P}'} = \Sigma_{\mathcal{P}'}$.

Suppose $\Sigma_{\mathcal{P}} = \Sigma_{\mathcal{P}'}$. By Theorem 3.3.1 there is $\mathcal{Q} \in \Sigma_{\mathcal{P}}$ such that $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{Q}$ (that is, $\mathcal{Q}$ is a maximal element). Since $\mathcal{Q} \in \Sigma_{\mathcal{P}'}$, we have $\mathcal{P}' \xrightarrow{\Gamma} \mathcal{Q}$ and so $\mathcal{P}' \xrightarrow{\Gamma} \mathcal{P}$. Similarly $\mathcal{P} \xrightarrow{\Gamma} \mathcal{P}'$. □

Thus under the assumption of Theorem 3.4.2, in order to analyze $\mathcal{P}$ up to MLD it suffices to investigate $\Sigma_{\mathcal{P}}$.

Next we move on to the theory of pattern equivariant functions. We will show for certain Σ consisting of functions, $\Sigma_{\mathcal{P}}$ is the space of pattern equivariant functions. First we recall the definition of pattern equivariant functions. Kellendonk [10] defined pattern-equivariant functions for tilings or Delone sets in order to study cohomology of the tiling spaces. Rand [23] generalized the definition to incorporate rotations and flips in the 2-dimensional cases. We recall the definitions here.

Definition 3.4.3. Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ and C be a subset of $\mathbb{R}^d$. Set

$$\mathcal{T} \sqcap C = \{T \in \mathcal{T} \mid \overline{T} \cap S \neq \emptyset\}.$$

Kellendonk gave a definition for subsets of $\mathbb{R}^d$, but here we define pattern-equivariant functions for tilings ([9]).

Definition 3.4.4 ([10],[9])**.** Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ and X be a set. A function $f\colon \mathbb{R}^d \to X$ is said to be $\mathcal{T}$-equivariant if there is $R > 0$ such that $x, y \in \mathbb{R}^d$ and $(\mathcal{T} - x) \sqcap B(0, R) = (\mathcal{T} - y) \sqcap B(0, R)$ imply $f(x) = f(y)$.

Definition 3.4.5 ([23])**.** Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$, Γ a closed subgroup of $\mathrm{E}(d)$ that contains $\mathbb{R}^d$, G an abelian group and $\phi\colon \Gamma_0 \to \mathrm{Aut}(G)$ a group homomorphism. Here, $\mathrm{Aut}(G)$ is the group of automorphisms of G. We say a function $f\colon \mathbb{R}^d \to G$ is $\mathcal{T}$-equivariant with representation ϕ, or is ϕ-invariant, if there is $R > 0$ such that $x, x' \in \mathbb{R}^d$, $\gamma \in \Gamma_0$ and

$$(\mathcal{T} \sqcap B(x', R)) - x' = \gamma(\mathcal{T} \sqcap B(x, R) - x)$$

imply $f(x') = \phi(\gamma)(f(x))$.

We show these pattern equivariant functions are captured in terms of local derivability in the following two lemmas (Lemma 3.4.6 and Lemma 3.4.7).

Lemma 3.4.6. *Let $\mathcal{T}$ be a tiling which consists of bounded components. In other words, the diameter of tiles in $\mathcal{T}$ is bounded from above. Then for any $f \in \mathrm{Map}(\mathbb{R}^d, \mathbb{C})$, f is $\mathcal{T}$-equivariant if and only if $\mathcal{T} \overset{\mathbb{R}^d}{\to} f$. Here we regard $\mathcal{T}$ as an element of* $\mathrm{Patch}(\mathbb{R}^d)$ *(Example 3.2.7), which is a pattern space over $(\mathbb{R}^d, \mathbb{R}^d)$, and f as an element of* $\mathrm{Map}(\mathbb{R}^d, \mathbb{C}, 0)$ *(Example 3.2.9), which is a pattern space over $(\mathbb{R}^d, \mathbb{R}^d)$.*

For what follows let $\pi\colon \Gamma \ni (a, A) \mapsto A \in \Gamma_0$ be the projection.

Lemma 3.4.7. *Let $\mathcal{T}$ be a tiling which consists of bounded components. Let Γ be a closed subgroup of $E(d)$ that contains $\mathbb{R}^d$, G an abelian group and $\phi\colon \Gamma_0 \to \mathrm{Aut}(G)$ a group homomorphism. Then for any $f \in \mathrm{Map}(\mathbb{R}^d, G)$, f is $\mathcal{T}$-equivariant with representation ϕ if and only if $\mathcal{T} \overset{\Gamma}{\to} f$. Here $\mathcal{T}$ is regarded as an element of* $\mathrm{Patch}(\mathbb{R}^d)$ *(Example 3.2.7), which is a pattern space over $(\mathbb{R}^d, \Gamma)$, and f is regarded as an element of* $\mathrm{Map}_{\phi\circ\pi}(\mathbb{R}^d, G, e)$ *(Example 3.2.9), which is a pattern space over $(\mathbb{R}^d, \Gamma)$.*

These two lemmas show that pattern equivariant functions are just functions which are locally derivable from the tiling. Thus in the case where $\mathcal{T}$ is a tiling of $\mathbb{R}^d$ that consists of bounded components and Σ is a certain subshift consisting of functions, $\Sigma_{\mathcal{T}}$ (Definition 3.4.1) is just the space of all $\mathcal{T}$-equivariant functions (either in the sense of Definition 3.4.4 or Definition 3.4.5).

We apply Theorem 3.4.2 to this situation where Σ is a space of functions and obtain an insight on pattern equivariant functions. We will show that the space of smooth pattern equivariant functions, with their ranges in $\mathbb{C}^m$, remembers the original abstract pattern up to MLD (Theorem 3.4.10).

Here is the setting: let Γ be a closed subgroup of $\mathrm{E}(d)$ that contains $\mathbb{R}^d$. Take a group homomorphism $\phi\colon \Gamma_0 \to GL_m(\mathbb{C})$. Let $C^\infty_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$ be the subshift of $\mathrm{Map}_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$ consisting of all smooth elements of $\mathrm{Map}_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$. (We say a map $f\colon \mathbb{R}^d \to \mathbb{C}^m$ is smooth if $\langle f(\cdot), v\rangle$ is smooth for any $v \in \mathbb{C}^m$, where $\langle\cdot,\cdot\rangle$ is the standard inner product.) In order to use Theorem 3.4.2 to $\Sigma = C^\infty_{\phi\circ\pi}(\mathbb{R}^d, G, 0)$, we need to show Σ admits sufficiently many symmetric building blocks. We show in two cases there are sufficiently many building blocks (Lemma 3.4.8 and Lemma 3.4.9.)

Lemma 3.4.8. *Suppose there is $v \in \mathbb{C}^m \setminus \{0\}$ such that $\phi(\gamma)v = v$ for each $\gamma \in \Gamma_0$. Then $C^\infty_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$ has sufficiently many symmetric building blocks: in other words, for any $r > 0$ there is a symmetric building block g_r for $(0, r)$.*

Proof. For each $r > 0$, set

$$f_r(x) = \begin{cases} \exp(-\frac{1}{r^2 - \|x\|^2}) & \text{if } \|x\| < r \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}^d$. Then f_r is a smooth real-valued function on $\mathbb{R}^d$. Set $g_r(x) = f_r(x)v$.Then $\emptyset \neq \operatorname{supp} g_r \subset B(0, r)$. Moreover if $\gamma, \eta \in \Gamma$ and $\rho(\gamma 0, \eta 0) > 4r$, then γg_r and ηg_r are compatible since

$$g(x) = \begin{cases} \gamma g_r(x) & \text{if } x \in B(\gamma 0, r) \\ \eta g_r(x) & \text{if } x \in B(\eta 0, r) \\ 0 & \text{otherwise} \end{cases}$$

is a majorant. Finally $\mathrm{Sym}_\Gamma\, g_r = \Gamma_0$. □

Lemma 3.4.9. *Suppose Γ_0 is finite. Then $C^\infty_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$ has sufficiently many building blocks.*

Proof. For any $r > 0$, take $x \in \mathbb{R}^d$ and $r_1 \in (0, r/4)$ such that $\|x\| < r/2$ and if $A \in \Gamma_0$ and $A \neq I$, then $\|Ax - x\| > 4r_1$. Take $v \in \mathbb{C}^m$ and set $f(x) = f_{r_1}(x)v$ (we defined f_{r_1} in the proof of Lemma 3.4.8.) Set $h = \bigvee\{(A, Ax)f \mid A \in \Gamma_0\}$. Then h is a symmetric building block. □

By Lemma 3.4.8, Lemma 3.4.9 and Theorem 3.4.2, we have the following:

Theorem 3.4.10. *Assume the same assumption as in Lemma 3.4.8 or in Lemma 3.4.9. Let Π and Π' be glueable pattern spaces over $(\mathbb{R}^d, \Gamma)$ and take $\mathcal{P}$ and $\mathcal{P}'$ from Π and Π' respectively. Assume $\mathcal{P}$ and $\mathcal{P}'$ are both Delone-deriving and consist of bounded components. Set $\Sigma = C^\infty_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$. Then $\mathcal{P} \overset{\Gamma}{\leftrightarrow} \mathcal{Q}$ if and only if $\Sigma_{\mathcal{P}} = \Sigma_{\mathcal{Q}}$.*

Thus in order to study Delone deriving abstract patterns which consists of bounded components up to MLD, it suffices to study the space $\Sigma_{\mathcal{P}}$ (where $\Sigma = C^{\infty}_{\phi\circ\pi}(\mathbb{R}^d, \mathbb{C}^m, 0)$) of smooth pattern-equivariant functions.

We may regard the space $\Sigma_{\mathcal{P}}$ as the space of functions that reflect the structure of $\mathcal{P}$. Sometimes in mathematics the set of functions that reflect the structure of an object remembers the original object. For example, consider a locally compact abelian group and its dual, or a smooth manifold M and its space $C^{\infty}(M)$ of smooth functions. Theorem 3.4.10 is similar to such phenomena.

Chapter 4

Local matching topology, repetitivity and stripe structure

4.1 The definition and properties of local matching topology

Setting 6. In this section (X, ρ) is a non-empty proper metric space and Γ is a locally compact topological group. Assume Γ acts on X as isometries and the action is jointly continuous. Assume also that there is a left-invariant metric ρ_Γ on Γ which is compatible with the original topology on Γ and the metric ρ_Γ is proper. Let $\mathcal{C}_0(X)$ be the set of all compact subsets of X and $\mathscr{V}$ the set of all compact neighborhoods of $e \in \Gamma$. Let Π be a pattern space over (X, Γ).

In this section we define and investigate local matching topologies on pattern spaces. We use the theory of uniform structure to define them. The uniform structure will be metrizable, but the description of a metric is not simple when Γ is non-commutative, and this is why we prefer uniform structure. With respect to this uniform structure, two abstract patterns $\mathcal{P}$ and $\mathcal{Q}$ in Π are "close" when they match in a "large region" after sliding $\mathcal{Q}$ by "small" $\gamma \in \Gamma$. This is analogous to the product topology of the space $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ is a finite set; in fact we can show on this space the relative topology of the local matching topology on a space of maps coincides with the product topology.

Definition 4.1.1. For $K \in \mathcal{C}_0(X)$ and $V \in \mathscr{V}$, set

$$\mathcal{U}_{K,V} = \{(\mathcal{P}, \mathcal{Q}) \in \Pi \times \Pi \mid \text{there is } \gamma \in V \text{ such that } \mathcal{P} \cap K = (\gamma\mathcal{Q}) \cap K\}.$$

Lemma 4.1.2. *If $K_1 \subset K_2$ and $V_2 \subset V_1$, then $\mathcal{U}_{K_2,V_2} \subset \mathcal{U}_{K_1,V_2}$.*

Lemma 4.1.3. *The set*

$$\{\mathcal{U}_{K,V} \mid K \in \mathcal{C}_0(X), V \in \mathscr{V}\} \tag{4.1}$$

satisfies the axiom of fundamental system of entourages.

Proof. (1) For any $K \in \mathcal{C}_0(X)$, $V \in \mathscr{V}$ and $\mathcal{P} \in \Pi$, we have $(\mathcal{P}, \mathcal{P}) \in \mathcal{U}_{K,V}$ since $\mathcal{P} \cap K = \mathcal{P} \cap K$.

(2) For any K and V, take $(\mathcal{P}, \mathcal{Q}) \in \mathcal{U}_{V^{-1}K,V^{-1}}$. There is $\gamma \in V$ such that

$$\mathcal{P} \cap V^{-1}K = (\gamma^{-1}\mathcal{Q}) \cap V^{-1}K.$$

Multiplying by γ both sides we have

$$(\gamma\mathcal{P}) \cap \gamma V^{-1}K = \mathcal{Q} \cap \gamma V^{-1}K,$$

and so

$$\begin{aligned}(\gamma\mathcal{P}) \cap K &= (\gamma\mathcal{P}) \cap \gamma V^{-1}K \cap K \\ &= \mathcal{Q} \cap \gamma V^{-1}K \cap K \\ &= \mathcal{Q} \cap K.\end{aligned}$$

We have $(\mathcal{Q}, \mathcal{P}) \in \mathcal{U}_{K,V}$ and so $\mathcal{U}^{-1}_{V^{-1}K,V^{-1}} \subset \mathcal{U}_{K,V}$.

(3) By Lemma 4.1.2, for $K_1, K_2 \in \mathcal{C}_0(X)$ and $V_1, V_2 \in \mathscr{V}$, we have

$$\mathcal{U}_{K_1 \cup K_2, V_1 \cap V_2} \subset \mathcal{U}_{K_1,V_1} \cap \mathcal{U}_{K_2,V_2}.$$

(4)Take $K \in \mathcal{C}_0(X)$ and $V \in \mathscr{V}$ arbitrarily. Set $K_1 = (V^{-1}K) \cup K$ and take $V_1 \in \mathscr{V}$ such that $V_1V_1 \subset V$. Note that $V_1 \subset V$. If $(\mathcal{P}_1, \mathcal{P}_2), (\mathcal{P}_2, \mathcal{P}_3) \in \mathcal{U}_{K_1,V_1}$, then there are γ_1 and γ_2 in V_1 such that $\mathcal{P}_1 \cap K_1 = (\gamma_1\mathcal{P}_2) \cap K_1$ and $\mathcal{P}_2 \cap K_1 = (\gamma_2\mathcal{P}_3) \cap K_1$. We have

$$\begin{aligned}(\gamma_1\gamma_2\mathcal{P}_3) \cap K &= \gamma_1((\gamma_2\mathcal{P}_3) \cap K_1) \cap K \\ &= \gamma_1(\mathcal{P}_2 \cap K_1) \cap K \\ &= (\gamma_1\mathcal{P}_2) \cap K \\ &= ((\gamma_1\mathcal{P}_2) \cap K_1) \cap K \\ &= (\mathcal{P}_1 \cap K_1) \cap K \\ &= \mathcal{P}_1 \cap K.\end{aligned}$$

Thus $(\mathcal{P}_1, \mathcal{P}_3) \in \mathcal{U}_{K,V}$. We have proved $\mathcal{U}^2_{K_1,V_1} \subset \mathcal{U}_{K,V}$. □

Definition 4.1.4. Let $\mathfrak{U}$ be the set of all entourages generated by (4.1). The uniform structure defined by $\mathfrak{U}$ is called the local matching uniform structure and the topology defined by it is called the local matching topology.

Next we give a sufficient condition for the local matching topology to be Hausdorff.

Definition 4.1.5. Suppose Π admits a unique zero element 0. An abstract pattern $\mathcal{P} \in \Pi$ is called an atom if $\operatorname{supp}\mathcal{P}$ is compact and

$$\mathcal{Q} \in \Pi \text{ and } \mathcal{Q} \leqq \mathcal{P} \Rightarrow \mathcal{P} = \mathcal{Q} \text{ or } \mathcal{Q} = 0.$$

For $\mathcal{P} \in \Pi$ set

$$A(\mathcal{P}) = \{\mathcal{Q}\colon \text{ atom} \mid \mathcal{Q} \leqq \mathcal{P}\}.$$

A subset $\Sigma \subset \Pi$ is said to be atomistic if for any $\mathcal{P} \in \Pi$ we have $\mathcal{P} = \bigvee A(\mathcal{P})$.

A subset $\Sigma \subset \Pi$ is said to have limit inclusion property if the following condition is satisfied:

$$\begin{gathered}\text{for any } \mathcal{P} \in \Sigma \text{ and an atom } \mathcal{Q} \in \Pi, \text{ if for any } V \in \mathscr{V} \\ \text{there is } \gamma_V \in V \text{ such that } \gamma_V \mathcal{Q} \leqq \mathcal{P}, \text{ we have } \mathcal{Q} \leqq \mathcal{P}.\end{gathered}$$

Proposition 4.1.6. *Suppose Π admits a unique zero element 0. Let Σ be a nonempty subset of Π which is atomistic and has limit inclusion property. Then the local matching topology on Σ is Hausdorff.*

Proof. Take $\mathcal{P}, \mathcal{Q} \in \Sigma$ and suppose $(\mathcal{P}, \mathcal{Q}) \in \mathcal{U}_{K,V}$ for any $K \in \mathcal{C}_0(X)$ and $V \in \mathscr{V}$. We show $\mathcal{P} = \mathcal{Q}$. Take $\mathcal{R} \in A(\mathcal{P})$. Set $K = \operatorname{supp} \mathcal{R}$. For any $V \in \mathscr{V}$ there is $\gamma_V \in V$ such that

$$\mathcal{P} \cap K = (\gamma_V^{-1} \mathcal{Q}) \cap K.$$

This implies that

$$\gamma_V \mathcal{R} \leqq \mathcal{Q},$$

and so by limit inclusion property, we have

$$\mathcal{R} \leqq \mathcal{Q}.$$

Since Σ is atomistic, we have $\mathcal{P} \leqq \mathcal{Q}$. The converse is proved in the same way and we have $\mathcal{Q} \leqq \mathcal{P}$ and so $\mathcal{P} = \mathcal{Q}$. □

Lemma 4.1.7. *Let Y be a non-empty topological space and y_0 be an element of Y. Take a group homomorphism $\phi\colon \Gamma \to \operatorname{Homeo}(Y)$ which is continuous with respect to the compact-open topology and such that $\phi(\gamma) y_0 = y_0$ for each $\gamma \in \Gamma$. Then $C_b(X, Y, y_0) = \{f \in \operatorname{Map}_\phi(X, Y, y_0) \mid$ continuous and bounded$\}$ is atomistic and has limit inclusion property as a subset of the pattern space $\operatorname{Map}_\phi(X, Y, y_0)$ (Definition 3.2.9).*

Proof. For each $x \in X$ and $y \in Y \setminus \{y_0\}$, the function defined by

$$\varphi_x^y(x') = \begin{cases} y & \text{if } x = x' \\ y_0 & \text{if } x \neq x' \end{cases}$$

is an atom of $\operatorname{Map}_\phi(X, Y, y_0)$. Any atom of $\operatorname{Map}_\phi(X, Y, y_0)$ is of this form. For $f \in C_b(X, Y, y_0)$, we have

$$A(f) = \{\varphi_x^{f(x)} \mid x \in X \text{ and } f(x) \neq y_0\}.$$

We see $f = \bigvee A(f)$. We have proved that $C_b(X, Y, y_0)$ is atomistic.

Next we show $C_b(X, Y, y_0)$ has limit inclusion property. Take any $x \in X$ and $y \in Y \setminus \{y_0\}$, and assume that for any $V \in \mathscr{V}$ there is $\gamma_V \in V$ such that $\gamma_V \varphi_x^y \leqq f$. Since $\operatorname{supp} \gamma_V \varphi_x^y = \{\gamma_V x\}$, we have

$$f(\gamma_V x) = (\gamma_V \varphi_x^y)(\gamma_V x) = \phi(\gamma_V)(\varphi_x^y(x)) = \phi(\gamma_V)(y).$$

Since f is continuous and the action $\Gamma \curvearrowright X$ is continuous,

$$f(x) = \lim_V f(\gamma_V x) = \lim_V \phi(\gamma_V)(y) = y,$$

and so $f \geqq \varphi_x^y$. We have shown $C_b(X, Y, y_0)$ has limit inclusion property. □

Corollary 4.1.8. *The relative topology of the local matching topology on* $C_b(X, Y, y_0)$ *is Hausdorff.*

Proof. Clear by Proposition 4.1.6 and Lemma 4.1.7. □

As the following lemma shows, the local matching topology on $\operatorname{Map}_\phi(X, Y, y_0)$ is not necessarily Hausdorff:

Lemma 4.1.9. *On* $\operatorname{Map}(\mathbb{R}, \mathbb{C}, 0)$*, the local matching topology is not Hausdorff.*

Proof. Take $f = 1_{\mathbb{Q}}$ and $g = 1_{\mathbb{Q}+a}$, where a is any irrational number. Then (f, g) belongs to any entourage. □

Lemma 4.1.10. *The pattern space* $\operatorname{Patch}(X)$ *(Definition 3.2.7) over* (X, Γ) *is atomistic and has limit inclusion property.*

Proof. Let T be a tile. Then $\{T\}$ is an atom. Any atom in $\operatorname{Patch}(X)$ is of this form. For any patch $\mathcal{P} \in \operatorname{Patch}(X)$, we have

$$A(\mathcal{P}) = \{\{T\} \mid T \in \mathcal{P}\},$$

and so $\mathcal{P} = \bigcup A(\mathcal{P}) = \bigvee A(\mathcal{P})$. We have shown that $\operatorname{Patch}(X)$ is atomistic.

To prove $\operatorname{Patch}(X)$ satisfies limit inclusion property, take $\mathcal{P} \in \operatorname{Patch}(X)$ and a tile T and assume for any $V \in \mathscr{V}$ there is $\gamma_V \in V$ such that $\gamma_V\{T\} \leqq \mathcal{P}$, that is, $\gamma_V T \in \mathcal{P}$. We show $T \in \mathcal{P}$. There is $V_0 \in \mathscr{V}$ such that if $V_1, V_2 \in \mathscr{V}$ and $V_j \subset V_0$ for each j, then $\gamma_{V_1} T \cap \gamma_{V_2} T \neq \emptyset$. Since $\gamma_{V_j} T$ is in a patch $\mathcal{P}$ for each j, we see $\gamma_{V_1} T = \gamma_{V_2} T$. It suffices to show that $T = \gamma_{V_0} T$ since $\gamma_{V_0} T \in \mathcal{P}$. If $x \in T$, then if $V_1 \in \mathscr{V}$ is small enough we have $V_1 \subset V_0$ and $\gamma_{V_1}^{-1} x \in T$. Since $\gamma_{V_1} T = \gamma_{V_0} T$, we see $x \in \gamma_{V_0} T$. Conversely, if $x \in \gamma_{V_0} T$, then if $V_1 \in \mathscr{V}$ is small enough $\gamma_{V_1} x \in \gamma_{V_0} T = \gamma_{V_1} T$, and so $x \in T$. We have shown $T = \gamma_{V_0} T$. □

Corollary 4.1.11. *The local matching topology on* $\operatorname{Patch}(X)$ *is Hausdorff.*

Proof. Clear by Proposition4.1.6 and Lemma 4.1.10. ☐

Lemma 4.1.12. *The pattern space $\mathcal{C}(X)$ (Example 3.2.8) is atomistic and has limit inclusion property.*

Proof. For any $x \in X$, the one-point set $\{x\}$ is an atom. Any atom in $\mathcal{C}(X)$ is of this form. For $D \in \mathcal{C}(X)$, $A(D) = \{\{x\} \mid x \in D\}$ and $D = \bigvee A(D)$. Thus $\mathcal{C}(X)$ is atomistic.

Take any $x \in X$ and $D \in \mathcal{C}(X)$, and assume for each $V \in \mathscr{V}$ there is $\gamma_V \in V$ such that $\gamma_V\{x\} \leqq D$, that is, $\gamma_V x \in D$. Since D is closed, we see $x = \lim_V \gamma_V x \in D$. ☐

Corollary 4.1.13. *The local matching topology on $\mathrm{UD}_r(X)$ is Hausdorff.*

Proof. Clear by Proposition 4.1.6 and Lemma 4.1.12. ☐

Next we show the group action is continuous.

Lemma 4.1.14. *The group action $\Gamma \curvearrowright \Pi$ is jointly continuous with respect to the local matching topology.*

Proof. Take $\mathcal{P}_0 \in \Pi$ and $\gamma_0 \in \Gamma$ arbitrarily. We show the map $\Pi \times \Gamma \ni (\mathcal{P}, \gamma) \mapsto \gamma\mathcal{P} \in \Pi$ is continuous at $(\mathcal{P}_0, \gamma_0)$. To prove this take a neighborhood of $\gamma_0\mathcal{P}_0$ arbitrarily. We may assume that this neighborhood is of the form $\mathcal{U}_{K,V}(\gamma_0\mathcal{P}_0)$ for some $K \in \mathcal{C}_0(X)$ and $V \in \mathscr{V}$. Set $K' = \gamma_0^{-1}K$ and take $V' \in \mathscr{V}$ such that if $\xi \in V'$ and $\gamma \in \gamma_0 V'$, we have $\gamma_0\xi\gamma^{-1} \in V$.

If $\mathcal{P} \in \mathcal{U}_{K',V'}(\mathcal{P}_0)$ and $\gamma \in \gamma_0 V'$, there is $\xi \in V'$ such that $\mathcal{P}_0 \cap K' = (\xi\mathcal{P}) \cap K'$. We have

$$\begin{aligned}\gamma_0\mathcal{P}_0 \cap K &= \gamma_0(\mathcal{P}_0 \cap K') \\ &= \gamma_0(\xi\mathcal{P} \cap K') \\ &= (\gamma_0\xi\gamma^{-1}\gamma\mathcal{P}) \cap K,\end{aligned}$$

and so $\gamma\mathcal{P} \in \mathcal{U}_{K,V}(\gamma_0\mathcal{P}_0)$. ☐

Next we prove that under a mild condition the local matching uniform structure on a subshift is complete.

Lemma 4.1.15. *For each $n = 1, 2, \dots$ take $\gamma_n \in \Gamma$ such that $\rho_\Gamma(e, \gamma_n) < \frac{1}{2^n}$. Then the following hold:*

1. $\rho_\Gamma(\gamma_n\gamma_{n-1}\cdots\gamma_m, e) < \frac{1}{2^m}$ *for each $n \geqq m \geqq 1$.*
2. *For any $m \geqq 1$ the sequence $(\gamma_n\gamma_{n-1}\cdots\gamma_m)_{n \geqq m}$ is a Cauchy sequence.*

Proof. 1. We have

$$\begin{aligned}\rho_\Gamma(\gamma_n\cdots\gamma_m, e) &\leqq \sum_{k=m}^{n-1}\rho_\Gamma(\gamma_n\cdots\gamma_k, \gamma_n\cdots\gamma_{k+1}) + \rho_\Gamma(\gamma_n, e)\\ &= \sum_{k=m}^{n}\rho_\Gamma(e, \gamma_{k+1})\\ &< \sum \frac{1}{2^{k+1}}\\ &< \frac{1}{2^m}.\end{aligned}$$

2. For any $\varepsilon > 0$, there is $\delta > 0$ such that if $\gamma, \eta, \zeta \in B(e, 1)$ and $\rho_\Gamma(\gamma, \eta) < \delta$, then $\rho_\Gamma(\gamma\zeta, \eta\zeta) < \varepsilon$. This follows from the fact that $B(e, 1)$ is compact and so the multiplication $B(e, 1) \times B(e, 1) \ni (\gamma, \eta) \mapsto \gamma\eta \in \Gamma$ is uniformly continuous. If $n > k \geqq m$ and k is large enough, by 1.,

$$\rho_\Gamma(\gamma_n\cdots\gamma_{k+1}, e) < \delta.$$

By the definition of δ, we have

$$\rho_\Gamma(\gamma_n\cdots\gamma_m, \gamma_k\cdots\gamma_m) < \varepsilon.$$

Since ε was arbitrary, we see the sequence is Cauchy. □

Proposition 4.1.16. *Suppose* Π *is glueable. Suppose also that there is* $x_0 \in X$ *such that*

$$\rho(\gamma x_0, \eta x_0) \leqq \rho_\Gamma(\gamma, \eta)$$

holds for any $\gamma, \eta \in \Gamma$*. Let* Σ *be a glueable subshift of* Π *which has limit inclusion property and is atomistic. Then the local matching uniform structure on* Σ *is complete.*

Proof. Since on Σ the local matching topology is Hausdorff (Proposition 4.1.6), the local matching uniform structure on Σ is metrizable (Lemma B.0.16). It suffices to show that any Cauchy sequences in Σ converge.

Let $(\mathcal{P}_n)_n$ be a Cauchy sequence in Σ. Set $K_n = B(x_0, n)$ and $V_n = B(e, \frac{1}{2^n}) \subset \Gamma$ for each $n = 1, 2, \ldots$. Since it suffices to show a subsequence of $(\mathcal{P}_n)$ converges, we may assume that $(\mathcal{P}_k, \mathcal{P}_l) \in \mathcal{U}_{K_n, V_n}$ for any $n > 0$ and $k, l \geqq n$. For each $n > 0$ there is $\gamma_n \in V_n$ such that

$$(\gamma_n\mathcal{P}_n) \cap K_n = \mathcal{P}_{n+1} \cap K_n.$$

By Lemma 4.1.15, since Γ is complete, there is a limit

$$\xi_n = \lim_{m\to\infty}\gamma_m\gamma_{m-1}\cdots\gamma_n \in B(e, \frac{1}{2^n})$$

for each $n > 0$. Note that $\xi_n = \xi_{n+1}\gamma_n$ for each n.

If $n < m$, then since

$$\xi_{m+1}K_m = B(\xi_{m+1}x_0, m) \supset B(x_0, m-1) \supset B(x_0, n) = K_n,$$

we have

$$\begin{aligned}(\xi_m\mathcal{P}_m) \cap K_n &= (\xi_{m+1}((\gamma_m\mathcal{P}_m) \cap K_m)) \cap K_n \\ &= (\xi_{m+1}(\mathcal{P}_{m+1} \cap K_m)) \cap K_n \\ &= (\xi_{m+1}\mathcal{P}_{m+1}) \cap K_n.\end{aligned}$$

By induction we have

$$(\xi_m\mathcal{P}_m) \cap K_n = (\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n \tag{4.2}$$

foe each n, m with $m > n$. This means that

$$(\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n \leqq (\xi_{n+2}\mathcal{P}_{n+1}) \cap K_{n+1} \tag{4.3}$$

for any $n > 0$.

Set

$$\mathcal{Q}_k = \bigvee\{(\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n \mid n > k\}$$

for each $k = 1, 2, \ldots$. We need to show that such a supremum exists. To this objective it suffices to show that $\Xi_k = \{(\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n \mid n > k\}$ is locally finite and pairwise compatible. By (4.2), we have

$$(\xi_{m+1}\mathcal{P}_{m+1}) \cap K_m \cap K_n = (\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n$$

for any n, m with $k < n < m$, and so Ξ_k is pairwise compatible. To prove Ξ is locally finite, take a closed ball B. For any sufficiently large n, we have $K_n \supset B$, and so if m is larger than this n we have by (4.2)

$$(\xi_{m+1}\mathcal{P}_{m+1}) \cap K_m \cap B = (\xi_{m+1}\mathcal{P}_{m+1}) \cap K_n \cap B = (\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n \cap B,$$

and so $\Xi \cap B$ is finite. Since B was arbitrary, Ξ is locally finite. Thus $\mathcal{Q}_k$ is well-defined and is in Σ since Σ is glueable.

By $\Xi_1 \supset \Xi_k$, we have $\mathcal{Q}_1 \geqq \mathcal{Q}_k$ for each k. On the other hand, by (4.3) $\mathcal{Q}_k \geqq (\xi_{n+1}\mathcal{P}_{n+1}) \cap K_n$ for any n and so $\mathcal{Q}_k \geqq \mathcal{Q}_1$; we have shown $\mathcal{Q}_1 = \mathcal{Q}_k$ for any $k > 0$.

Finally $\mathcal{Q}_1$ is the limit of $(\mathcal{P}_n)$, since for each $k > 0$ (4.2) implies that

$$\begin{aligned}\mathcal{Q}_1 \cap K_k &= \bigvee\{\xi_{n+1}\mathcal{P}_{n+1} \cap K_k \mid n > k\} \\ &= (\xi_{k+1}\mathcal{P}_{k+1}) \cap K_k\end{aligned}$$

and so $\mathcal{P}_{k+1} \in \mathcal{U}_{K_k,V_k}(\mathcal{Q}_1)$. (Note that $\mathcal{U}_{K_{k+1},V_{k+1}} \subset \mathcal{U}_{K_k,V_k}$.) □

Remark 4.1.17. As corollaries we have Proposition 2.1.25 in this article and Proposition 2.1 in [26] when $\mathfrak{G}$ admits an invariant proper metric compatible with the original topology. Proposition 2.1 in [26] assumes that $\mathfrak{G}$ is commutative, but Proposition 4.1.16 allow $\mathfrak{G}$ to be non-commutative.

4.2 Repetitivity

We begin with an investigation of almost periodicity in an general setting and after that come back to the context of pattern spaces.

Lemma 4.2.1. *Let G be a group with a left-invariant metric ρ_G. For $S \subset G$ the following two conditions are equivalent:*

1. *there is a compact $C \subset G$ such that $SC = G$.*
2. *There is $R > 0$ such that for any $\gamma \in G$ we have $S \cap B(\gamma, R)^\circ \neq \emptyset$.*

Proof. First assume 1. There is a compact $C \subset G$ as in 1. We can take $R > 0$ such that $C \subset B(e, R)^\circ$. If $\gamma \in G$, then there are $\eta \in S$ and $\xi \in C$ such that $\gamma = \eta\xi$. Since ρ_G is left invariant, we see $\rho_\Gamma(\gamma, \eta) < R$, and so $\eta \in S \cap B(\gamma, R)^\circ$. We have proved 2.

Next assume 2. Set $C = B(e, R)$. If $\gamma \in G$ is an arbitrary element, there is $\eta \in S \cap B(\gamma, R)$. We have $\rho_\Gamma(e, \eta^{-1}\gamma) \leqq R$ and so $\gamma = \eta\eta^{-1}\gamma \in SC$. We have proved $G = SC$, and so the proof is completed. □

Definition 4.2.2. A subset $S \subset G$ is said to be relatively dense if the equivalent conditions in Lemma 4.2.1 are satisfied.

Definition 4.2.3. Let Ω be a nonempty uniform space and suppose a group G with a left invariant metric acts on Ω. Take $x \in \Omega$.

1. x is Bohr almost periodic if for each entourage U of Ω, the set
$$\{g \in G \mid (x, g^{-1}x) \in U\}$$
is relatively dense in G.
2. x is Bochner almost periodic if the closure $\overline{\mathcal{O}_x} = \overline{\{gx \mid g \in G\}}$ of the orbit is compact.

Remark 4.2.4. If Ω is the space of uniformly continuous bounded complex-valued functions on a locally compact abelian group G, on which G acts by translation, and the uniform structure is given by the sup norm, the two conditions in Definition 4.2.3 are equivalent. Bohr initiated the investigation of the functions which satisfy these conditions (in the case where $G = \mathbb{R}$) and such functions are now called Bohr almost periodic functions or strongly almost periodic functions.

If we replace the topology of sup norm with weak topology, those f which are Bochner almost periodic are called weak almost periodic functions. Weak almost periodic functions have been actively investigated and are important in the context of aperiodic order.

Lemma 4.2.5. *Let Ω_1, Ω_2 be complete and Hausdorff uniform spaces on which a group G with a left invariant metric acts. Let x_1 (resp. x_2) be an element of Ω_1(resp. Ω_2). Suppose*

$$\mathcal{O}_{x_1} \ni \gamma x_1 \mapsto \gamma x_2 \in \mathcal{O}_{x_2}.$$

is well-defined and uniformly continuous. Then the following hold:

1. *If x_1 is Bohr almost periodic, then so is x_2.*

2. *If x_1 is Bochner almost periodic, then so is x_2.*

Setting 7. In rest of this section Γ is a locally compact group with a left invariant metric ρ_Γ. Assume any closed balls in Γ are compact. Assume also that Γ acts on a proper metric space X as isometries and the action is jointly continuous.

Assume further that there are $x_0 \in X$ and $C_0 > 0$ such that

$$\rho(\gamma x_0, \eta x_0) \leqq \rho_\Gamma(\gamma, \eta) \leqq \rho(\gamma x_0, \eta x_0) + C_0$$

for each $\gamma, \eta \in \Gamma$.

Let Π be a glueable pattern space over (X, Γ) and Σ a glueable subshift which is atomistic and satisfies limit inclusion property.

Definition 4.2.6. Take $\mathcal{P} \in \Pi$. We say

1. $\mathcal{P}$ is weakly repetitive if $\mathcal{P}$ is Bohr almost periodic with respect to the local matching uniform structure, and

2. $\mathcal{P}$ has finite local complexity (FLC) if it is Bochner almost periodic with respect to the local matching uniform structure.

Lemma 4.2.7. *If $\mathcal{P} \in \Sigma$ has FLC, then it is weakly repetitive if and only if the corresponding dynamical system $(X_\mathcal{P}, \Gamma)$ is minimal.*

Proof. Clear by Gottschalk theorem ([2], Chapter 1, Theorem 7). □

Lemma 4.2.8. *Suppose the action $\Gamma \curvearrowright X$ is transitive. For $\mathcal{P} \in \Pi$ the following conditions are equivalent:*

1. *$\mathcal{P}$ is weakly repetitive.*

2. *For any $R > 0$ and $x \in X$ there is $R' > 0$ such that, whenever we take $y \in X$, there is $\gamma \in \Gamma$ with*

   (a) *$\mathcal{P} \cap B(\gamma x, R) = \gamma(\mathcal{P} \cap B(x, R))$, and*

   (b) *$\rho(\gamma x, y) < R'$.*

Proof. 1.⇒2. Take $x \in X$ and $R > 0$ arbitrarily. We can take $R_1 > 0$ such that $B(x, R) \subset B(x_0, R_1)$. For $K = B(x_0, R_1)$ and $V = B(e, 1)$, by condition 1., the set

$$S = \{\gamma \in \Gamma \mid (\mathcal{P}, \gamma^{-1}\mathcal{P}) \in \mathcal{U}_{K,V}\}$$

is relatively dense; in other words, there is $R_2 > 0$ such that $B(\gamma, R_2) \cap S \neq \emptyset$ for any $\gamma \in \Gamma$. Take $y \in X$ arbitrarily. Since the action of Γ on X is transitive, there is $\gamma_0 \in \Gamma$ such that $\gamma_0 x_0 = y$. There is $\gamma_1 \in B(\gamma_0, R_2) \cap S$. By the definition of S, $(\mathcal{P}, \gamma_1^{-1}\mathcal{P}) \in \mathcal{U}_{K,V}$, and so there is $\gamma_2 \in B(e, 1)$ such that $\mathcal{P} \cap B(x_0, R_1) = (\gamma_2\gamma_1^{-1}\mathcal{P}) \cap B(x_0, R_1)$. Then since $B(\gamma_1\gamma_2^{-1}x, R) \subset B(\gamma_1\gamma_2^{-1}x_0, R_1)$, we have $\gamma_1\gamma_2^{-1}(\mathcal{P} \cap B(x, R)) = \mathcal{P} \cap B(\gamma_1\gamma_2^{-1}x, R)$ and

$$\begin{aligned}
\rho(\gamma_1\gamma_2^{-1}x, y) &\leqq \rho(\gamma_1\gamma_2^{-1}x, \gamma_1\gamma_2^{-1}x_0) + \rho_\Gamma(\gamma_1\gamma_2^{-1}, \gamma_0) \\
&\leqq \rho(x, x_0) + \rho(\gamma_1, \gamma_0) + \rho(\gamma_2^{-1}, e) \\
&\leqq R_1 + R_2 + 1.
\end{aligned}$$

The condition 2 holds for constant $R' = R_1 + R_2 + 1$.

2.⇒1. Take a compact $K \subset X$ and a compact neighborhood V of $e \in \Gamma$ arbitrarily. There is $R_0 > 0$ such that $K \subset B(x_0, R_0)$. For x_0 and R_0 there is $R_1 > 0$ as in the condition 2. Set $S = \{\gamma \in \Gamma \mid (\mathcal{P}, \gamma^{-1}\mathcal{P}) \in \mathcal{U}_{K,V}\}$ and we show S is relatively dense. Take $\gamma \in \Gamma$ and set $y = \gamma x_0$. There is $\gamma_0 \in \Gamma$ such that

1. $\mathcal{P} \cap B(\gamma_0 x_0, R_0) = \gamma_0(\mathcal{P} \cap B(x_0, R_0))$, and
2. $\rho(\gamma_0 x_0, y) < R_1$.

Then we have

$$(\gamma_0^{-1}\mathcal{P}) \cap B(x_0, R_0) = \mathcal{P} \cap B(x_0, R_0),$$

and by the definition of R_0 and S, $\gamma_0 \in S$. Since we have

$$\rho_\Gamma(\gamma_0, \gamma) \leqq \rho(\gamma_0 x_0, \gamma x_0) + C_0 < R_1 + C_0,$$

we see $S \cap B(\gamma, R_1 + C_0) \neq \emptyset$ and so S is relatively dense. □

Remark 4.2.9. The second condition in Lemma 4.2.8 means that, whenever we take a patch of the form $\mathcal{P} \cap B(x, R)$, the copies of such patch appear infinitely often in $\mathcal{P}$ with bounded gap.

Next we investigate relations between almost periodicity and local derivability.

Lemma 4.2.10. *Let X_1, X_2 be nonempty proper metric spaces on which the group Γ acts as isometries. Take a glueable pattern space Π_j over (X_j, Γ) for each $j = 1, 2$. Take also an abstract pattern $\mathcal{P}_j \in \Pi_j$ for each $j = 1, 2$. If $\mathcal{P}_1 \xrightarrow{\Gamma} \mathcal{P}_2$ and $\mathcal{P}_2$ consists of bounded components, then the map*

$$\mathcal{O}_{\mathcal{P}_1} \ni \gamma\mathcal{P}_1 \mapsto \gamma\mathcal{P}_2 \in \mathcal{O}_{\mathcal{P}_2}$$

is well-defined and is uniformly continuous.

Proof. That the map is well-defined follows from Lemma 3.2.21. Take $x_1 \in X_1$ and $x_2 \in X_2$ arbitrarily. There is $R_0 > 0$ as in Lemma 3.2.15 with respect to x_1 and x_2. Take a compact $K \subset X_2$ and $V \in \mathscr{V}$ arbitrarily. We can take $L > 0$ such that $K \subset B(x_2, L)$. If $\gamma, \eta \in \Gamma$ and $(\gamma\mathcal{P}_1, \eta\mathcal{P}_2) \in \mathcal{U}_{B(x_1,R_0+L),V}$, then there is $\xi \in V$ such that

$$(\gamma\mathcal{P}_1) \cap B(x_1, R_0 + L) = (\xi\eta\mathcal{P}_1) \cap B(x_1, R_0 + L).$$

By the definition of R_0, we have

$$(\gamma\mathcal{P}_2) \cap B(x_2, L) = (\xi\eta\mathcal{P}_2) \cap B(x_2, L),$$

and so

$$(\gamma\mathcal{P}_2) \cap K = (\xi\eta\mathcal{P}_2) \cap K,$$

which implies that $(\gamma\mathcal{P}_2, \eta\mathcal{P}_2) \in \mathcal{U}_{K,V}$. ☐

Proposition 4.2.11. *Let $(X_j, \rho_j)(j = 1, 2)$ be proper metric spaces on which Γ acts as isometries. Suppose there are $x_j \in X_j$ for each j such that*

$$\rho_j(\gamma x_j, \eta x_j) \leqq \rho_\Gamma(\gamma, \eta)$$

for each $\gamma, \eta \in \Gamma$ and $j = 1, 2$. Take a glueable pattern space Π_j over (X_j, Γ) for each $j = 1, 2$. Let Σ_j be a glueable subshift of Π_j which is atomistic and has limit inclusion property, for each $j = 1, 2$. If $\mathcal{P}_j \in \Sigma_j$ $(j = 1, 2)$, $\mathcal{P}_2$ consists of bounded components and $\mathcal{P}_1 \xrightarrow{\Gamma} \mathcal{P}_2$, then the following hold:

1. *if $\mathcal{P}_1$ is weakly repetitive, then so is $\mathcal{P}_2$.*
2. *if $\mathcal{P}_2$ has FLC, then so does $\mathcal{P}_2$.*

Proof. Clear by Lemma 4.2.10 and Lemma 4.2.5, since Σ_1 and Σ_2 are complete and Hausdorff by Proposition 4.1.6 and Proposition 4.1.16. ☐

4.3 Stripe structures

Recall we endowed a metric $\rho_{\mathbb{T}}$ on $\mathbb{T}$ in Definition 2.3.1.

Definition 4.3.1. Take two positive real numbers R_1, R_2. Let Π be a pattern space over $(\mathbb{R}^d, \mathbb{R}^d)$. An abstract pattern $\mathcal{P} \in \Pi$ is said to admit (R_1, R_2)-stripe structure if there is $a \in \mathbb{R}^d$ with $\|a\| = 1$ and $R > 0$ such that, for any $x \in \mathbb{R}^d$, the set

$$\{y \in \mathbb{R}^d \mid (\mathcal{P} - x) \cap B(0, R) = (\mathcal{P} - y) \cap B(0, R)\}$$

is contained in $S(a, x, R_1, R_2)$ (Definition 2.3.2).

In what follows we study relations between the stripe structure of an abstract pattern $\mathcal{P}$ in a pattern space over $(\mathbb{R}^d, \mathbb{R}^d)$ and the properties of the corresponding dynamical system $(X_{\mathcal{P}}, \mathbb{R}^d)$.

Lemma 4.3.2. *Let G be a (not necessarily closed) subgroup of $\mathbb{R}^d$. Set*

$$V = \bigcap_{r>0} \overline{\operatorname{span}_{\mathbb{Z}} G \cap B(0,r)}. \tag{4.4}$$

Then V is a vector subspace of $\mathbb{R}^d$.

Proof. First, $0 \in V$.

Second, if $x, y \in V$, then $x + y \in V$. Indeed, for any $r > 0$ and $\varepsilon > 0$ there are $x', y' \in \operatorname{span}_{\mathbb{Z}} G \cap B(0,r)$ such that $\|x - x'\| < \varepsilon$ and $\|y - y'\| < \varepsilon$. Since $x' + y' \in \operatorname{span}_{\mathbb{Z}} G \cap B(0,r)$ and $\|x + y - (x' + y')\| < 2\varepsilon$, we see $x + y \in \overline{\operatorname{span}_{\mathbb{Z}} G \cap B(0,r)}$.

Third, we show that if $x \in V$ and $n \in \mathbb{Z}_{>0}$, then $\frac{1}{n}x \in V$. For any $m \in \mathbb{Z}_{>0}$, let $\mathcal{B}$ be a maximal linear independent subset of $G \cap B(0, 1/m)$. We may take $\lambda_b \in \mathbb{R}$ for each $b \in \mathcal{B}$ such that $x = \sum_{b \in \mathcal{B}} \lambda_b b$, since $x \in \overline{\operatorname{span}_{\mathbb{Z}} G \cap B(0, 1/m)} \subset \operatorname{span}_{\mathbb{R}} \mathcal{B}$. We may take $l_b \in \mathbb{Z}$ for each $b \in \mathcal{B}$ such that $|\lambda_b - n l_b| < n$ for each b. Set $x_m = \sum_{b \in \mathcal{B}} l_b b$. We have

$$\begin{aligned}
\|x - n x_m\| &= \| \sum_{b \in \mathcal{B}} \lambda_b b - \sum_{b \in \mathcal{B}} n l_b b \| \\
&\leqq \sum |\lambda_b - n l_b| \|b\| \\
&\leqq \frac{nd}{m},
\end{aligned}$$

and so

$$\| \frac{1}{n} x - x_m \| \leqq \frac{d}{m}.$$

For any $r > 0$ and $\varepsilon > 0$, if m is large enough, $x_m \in \operatorname{span}_{\mathbb{Z}} G \cap B(0,r)$ and $\|\frac{1}{n}x - x_m\| < \varepsilon$. This shows $\frac{1}{n}x \in V$.

Finally, by the second and the third part of this proof, if λ is an rational number, then $\lambda x \in V$. Since V is closed, this holds even if λ is irrational. $\square$

Lemma 4.3.3. *Let G be a subgroup of $\mathbb{R}^d$ and define V by (4.4). If $0 \in \mathbb{R}^d$ is a limit point of G, then the dimension of V is more than 0.*

Proof. For each integer $n > 0$ there is $x_n \in G$ such that $0 < \|x_n\| < 1/n$. We may find $k_n \in \mathbb{Z}$ such that $1/2 < \|k_n x_n\| < 3/2$. The sequence $(k_n x_n)$ admits a limit point x. Then $x \neq 0$. Moreover, $x \in \overline{\operatorname{span}_{\mathbb{Z}} G \cap B(0,r)}$ for each $r > 0$ since $k_n x_n \in \operatorname{span}_{\mathbb{Z}} G \cap B(0,r)$ for large n, and so $x \in V$. $\square$

Lemma 4.3.4 ([12], Lemma 4.1)**.** *Let D be an FLC Delone set in $\mathbb{R}^d$ and χ a continuous character of $\mathbb{R}^d$. Then χ is an eigenvalue for the topological dynamical system $(X_D, \mathbb{R}^d)$ if and only if χ is a weakly D-equivariant function, that is, for any $\varepsilon > 0$ there is $R > 0$ such that*

$$\rho_{\mathbb{T}}(\chi(x), \chi(y)) < \varepsilon$$

for any $x, y \in \mathbb{R}^d$ with

$$(D - x) \cap B(0, R) = (D - y) \cap B(0, R).$$

Lemma 4.3.5. *Let D be a Delone set of $\mathbb{R}^d$ which has FLC. Suppose that 0 is a limit point of the set of all topological eigenvalues for $(X_D, \mathbb{R}^d)$. Then for any $L_1, L_2 > 0$ and $\varepsilon > 0$, there are $R_1, R_2 > 0$ such that*

1. *$|R_j - L_j| < \varepsilon$ for each $j = 1, 2$, and*
2. *D has (R_1, R_2)-stripe structure.*

Proof. By Lemma 4.3.2 and Lemma 4.3.3, we can take an eigenvalue a such that $|\frac{1}{\|a\|} - L_1| < \varepsilon$. Take $r > 0$ such that $\frac{r}{\|a\|} = L_2$. We set $R_1 = \frac{1}{\|a\|}$ and $R_2 = \frac{r}{\|a\|}$.

Since the character χ_a is weakly D-equivariant, there is $R > 0$ such that $x, y \in \mathbb{R}^d$ and

$$(D - x) \cap B(0, R) = (D - y) \cap B(0, R) \tag{4.5}$$

imply

$$\rho_{\mathbb{T}}(\chi_a(x), \chi_a(y)) \leqq r.$$

We will show that this R satisfies the condition in Definition 4.3.1. Take $x \in \mathbb{R}^d$ and fix it. If $y \in \mathbb{R}^d$ and (4.5) holds, then we have

$$|\langle y - x, a\rangle - n| \leqq r.$$

for some $n \in \mathbb{Z}$. We obtain

$$\langle y - x, a\rangle \in \mathbb{Z} + [-r, r],$$

and so $y \in S(\frac{1}{\|a\|}a, x, R_1, R_2)$. We have proved

$$\{y \in \mathbb{R}^d \mid (D - y) \cap B(0, R) = (D - x) \cap B(0, R)\}$$

is contained in $S(\frac{1}{\|a\|}a, x, R_1, R_2)$, and so D has (R_1, R_2)-stripe structure. □

Lemma 4.3.6. *Let Π_1, Π_2 be pattern spaces over $(\mathbb{R}^d, \mathbb{R}^d)$, R_1 and R_2 positive real numbers. Take $\mathcal{P}_1 \in \Pi_1$ and $\mathcal{P}_2 \in \Pi_2$ and assume $\mathcal{P}_2 \overset{\mathbb{R}^d}{\to} \mathcal{P}_1$. If $\mathcal{P}_1$ has (R_1, R_2)-stripe structure, then $\mathcal{P}_2$ has (R_2, R_2)-stripe structure.*

Proof. There is $R > 0$ and a as in Definition 4.3.1. In other words, $x, y \in \mathbb{R}^d$ and

$$(\mathcal{P}_1 - x) \cap B(0, R) = (\mathcal{P}_1 - y) \cap B(0, R) \tag{4.6}$$

imply $y \in S(a, x, R_1, R_2)$.

Since $\mathcal{P}_2 \overset{\mathbb{R}^d}{\to} \mathcal{P}_1$, we can take a constant $R_0 > 0$ as in Definition 3.2.16 with respect to $x_0 = y_0 = 0$. If $x, y \in \mathbb{R}^d$ and

$$(\mathcal{P}_2 - x) \cap B(0, R + R_0) = (\mathcal{P}_2 - y) \cap B(0, R + R_0),$$

then (4.6) holds, and so $y \in S(a, x, R_1, R_2)$. We have proved $\mathcal{P}_2$ has (R_1, R_2)-stripe structure with respect to $R + R_0$. □

Theorem 4.3.7. *Let Π be a glueable pattern space over $(\mathbb{R}^d, \mathbb{R}^d)$. Take an abstract pattern $\mathcal{P} \in \Pi$ and assume that it is Delone-deriving, consists of bounded components and has FLC. For example, take an FLC tiling of $\mathbb{R}^d$ of finite tile type. Suppose that $0 \in \mathbb{R}^d$ is a limit point of the set of topological eigenvalues of the corresponding dynamical system $(X_{\mathcal{P}}, \mathbb{R}^d)$. Then for any $R_1, R_2, \varepsilon > 0$, there are $L_1, L_2 > 0$ such that*

1. *$|R_j - L_j| < \varepsilon$ for each $j = 1, 2$, and*
2. *$\mathcal{P}$ has (L_1, L_2)-stripe structure.*

Proof. By Theorem 3.3.1, there is a Delone set D such that $\mathcal{P} \overset{\mathbb{R}^d}{\leftrightarrow} D$. The set of eigenvalues of the dynamical system $(X_D, \mathbb{R}^d)$ is the same as the one of $(X_{\mathcal{P}}, \mathbb{R}^d)$; D has (L_1, L_2)-stripe structure, where $|R_j - L_j| < \varepsilon$ by Lemma 4.3.5; by Lemma 4.3.6, $\mathcal{P}$ also has (L_1, L_2)-stripe structure. □

Remark 4.3.8. In plain language, Theorem 4.3.7 says that, inside an abstract pattern $\mathcal{P}$, given information of the appearance of an abstract pattern $\mathcal{Q}$ which is large enough, there is a "forbidden area" of the appearance of translates of $\mathcal{Q}$. In other words, if we find a translate of $\mathcal{Q}$ inside $\mathcal{P}$, there is a region relative to that translate of $\mathcal{Q}$ where other translates of $\mathcal{Q}$ will never happen. Such "forbidden area" consists of "bands" and is periodic (see Figure 2.2 in page 41).

In what follows we prove the converse of Theorem 4.3.7 under the assumption of weak repetitivity.

Lemma 4.3.9. *Let D be a weakly repetitive $(R, r)-$Delone set in $\mathbb{R}^d$. Take $x_0 \in \mathbb{R}^d$ and $R_0 > R$ arbitrarily. Set*

$$E = \{x \in \mathbb{R}^d \mid (D - x_0) \cap B(0, R_0) = (D - x) \cap B(0, R_0)\}.$$

Then E is a Delone set and $D \overset{\mathbb{R}^d}{\to} E$.

Proof. Take x and y from E. Since R_0 is greater than R, the set

$$(D - x) \cap B(0, R_0) = (D - y) \cap B(0, R_0)$$

is not empty. Take z from this set. We see $x+z, y+z \in D$ and $\|x-y\| = \|x+z-(y+z)\|$ is either 0 or greater than r. This shows E is uniformly discrete with respect to r.

Next, since D is weakly repetitive, by Lemma 4.2.8, there is $R > 0$ such that for any $x \in \mathbb{R}^d$ there is $y \in \mathbb{R}^d$ with

1. $\|x - (x_0 + y)\| < R$, and
2. $D \cap B(x_0 + y, R_0) = (D \cap B(x_0, R_0)) + y$.

Let $x \in \mathbb{R}^d$ be an arbitrary element and $y \in \mathbb{R}^d$ satisfy the above two conditions. Then

$$(D - (x_0 + y)) \cap B(0, R_0) = (D - x_0) \cap B(0, R_0),$$

and so $x_0 + y \in E$. We have shown that $E \cap B(x, R) \neq \emptyset$ and E is relatively dense with respect to R.

Finally, we show that $D \overset{\mathbb{R}^d}{\to} E$. Take $x, y \in \mathbb{R}^d$ and $L > 0$ arbitrarily and assume

$$(D - x) \cap B(0, R_0 + L) = (D - y) \cap B(0, R_0 + L).$$

To prove

$$(E - x) \cap B(0, L) = (E - y) \cap B(0, L), \tag{4.7}$$

we take $z \in E$ such that $z - x \in B(0, L)$. Then

$$\begin{aligned}(D - x_0) \cap B(0, R_0) =&(D - z) \cap B(0.R_0)\\ =&((D - x) \cap B(0, R_0 + L) \cap B(z - x, R_0)) + x - z\\ =&((D - y) \cap B(0, R_0 + L) \cap B(z - x, R_0)) + x - z\\ =&(D + x - y - z) \cap B(0, R_0).\end{aligned}$$

This implies that $z + y - x \in E$ and so $z - x \in E - y$. We have shown

$$(E - x) \cap B(0, L) \subset (E - y) \cap B(0, L),$$

and since the proof for the reverse inclusion is the same, we have (4.7). □

Lemma 4.3.10. *Let D be an (R,r)-Delone set in $\mathbb{R}^d$ and assume D is weakly repetitive. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a (not necessarily continuous) bounded function such that $a,b,c,d \in D$ and $a-b=c-d$ imply $f(a)-f(b)=f(c)-f(d)$. Then for any $\varepsilon>0$ there is a Delone D_ε in $\mathbb{R}^d$ such that*

1. *$D \supset D_\varepsilon$,*
2. *$D \overset{\mathbb{R}^d}{\to} D_\varepsilon$, and*
3. *if $a,b \in D_\varepsilon$, then $|f(a)-f(b)|<\varepsilon$.*

Proof. We may replace f with $f+C$ for some constant $C \in \mathbb{R}$ so that we may assume

$$M = \sup_{a \in D} f(a) = -\inf_{a \in D} f(a).$$

For any $\varepsilon > 0$ there are a_0 and b_0 in D such that $f(a_0) > M - \varepsilon/2$ and $f(b_0) < -M + \varepsilon/2$. Take $c_0 \in D$ and fix it. If $R_0 > R$ is sufficiently large, we have $a_0, b_0 \in B(c_0, R_0)$. Set

$$D_\varepsilon = \{x \in \mathbb{R}^d \mid (D - c_0) \cap B(0,R_0) = (D-x) \cap B(0,R_0)\}.$$

Then $D_\varepsilon \subset D$ and by Lemma 4.3.9, D_ε is Delone and $D \overset{\mathbb{R}^d}{\to} D_\varepsilon$.

Next, take $a \in D_\varepsilon$ arbitrarily and we show $|f(a)-f(c_0)| < \varepsilon/2$. Since

$$(D \cap B(c_0,R_0)) - c_0 = (D-a) \cap B(0,R_0),$$

by definition of R_0 we see $a_0 - c_0 + a \in D$ and $b_0 - c_0 + a \in D$. Since

$$f(a_0 - c_0 + a) - f(b_0 - c_0 + a) = f(a_0) - f(b_0) > 2M - \varepsilon,$$

we have either $f(a_0 - c_0 + a) > M - \varepsilon/2$ or $f(b_0 - c_0 + a) < -M + \varepsilon/2$. In the latter case, we see

$$\begin{aligned} f(a) - f(c_0) =& f(a + a_0 - c_0) - f(a_0) + f(a) - f(a + a_0 - c_0) - f(c_0) + f(a_0) \\ =& f(a + a_0 - c_0) - f(a_0) \\ \in& (-\varepsilon/2, \varepsilon/2), \end{aligned}$$

since

$$f(a) - f(a + a_0 - c_0) = f(c_0) - f(a_0)$$

by the assumption on f. Similarly in the latter case

$$f(a) - f(c_0) \in (\varepsilon/2, \varepsilon/2).$$

Finally, if $a, b \in D_\varepsilon$, then by the previous paragraph

$$|f(a) - f(b)| \leqq |f(a) - f(c_0)| + |f(b) - f(c_0)| < \varepsilon,$$

which completes the proof. □

Lemma 4.3.11. *Let D be a weakly repetitive Delone set in $\mathbb{R}^d$. Let a_0 and b_0 be elements of $\mathbb{R}^d$. Assume if $x \in D$ we have*

$$|\langle x - b_0, a_0\rangle - n| < 1/4$$

for some $n \in \mathbb{Z}$. Then the character $\chi_{a_0} \colon \mathbb{R}^d \ni x \mapsto e^{2\pi i\langle x, a_0\rangle} \in \mathbb{T}$, is weakly D-equivariant.

Proof. We may take $\theta \colon \mathbb{R}^d \to [-\pi, \pi)$ such that for any $x \in \mathbb{R}^d$

$$2\pi\langle x - b_0, a_0\rangle = \theta(x) + 2n\pi$$

for some $n \in \mathbb{Z}$. For any $a, b, c, d \in D$ such that $a - b = c - d$, we have

$$\begin{aligned} e^{i(\theta(a)-\theta(b))} =& e^{2\pi i(\langle a-b_0,a_0\rangle - 2\pi\langle b-b_0,a_0\rangle)} \\ =& e^{2\pi i\langle a-b,a_0\rangle} \\ =& e^{2\pi i\langle c-d,a_0\rangle} \\ =& e^{i\theta(c)-\theta(d)}. \end{aligned}$$

Since $\theta(a), \theta(b), \theta(c)$ and $\theta(d)$ are in $(-\pi/2, \pi/2)$ by the assumption, we see $\theta(a) - \theta(b) = \theta(c) - \theta(d)$. By Lemma 4.3.10, for each $\varepsilon > 0$ there is a Delone $D_\varepsilon \subset D$ such that $D \overset{\mathbb{R}^d}{\to} D_\varepsilon$ and $|\theta(a) - \theta(b)| < \varepsilon$ for any $a, b \in D_\varepsilon$. Let R_1 be a constant for the local derivability $D \overset{\mathbb{R}^d}{\to} D_\varepsilon$ and $R_2 > 0$ be such that D_ε is relatively dense with respect to R_2. If $x, y \in \mathbb{R}^d$ and

$$(D - x) \cap B(0, R_1 + R_2) = (D - y) \cap B(0, R_1 + R_2) \tag{4.8}$$

then

$$(D_\varepsilon - x) \cap B(0, R_2) = (D_\varepsilon - y) \cap B(0, R_2).$$

Take $z \in D_\varepsilon$ such that $z - x \in B(0, R_2)$. Then $z - x + y \in D_\varepsilon$ and $|\theta(z) - \theta(z - x + y)| < \varepsilon$. Using

$$\begin{aligned} |e^{2\pi i\langle x,a_0\rangle} - e^{2\pi i\langle y,a_0\rangle}| =& |e^{2\pi i\langle z-b_0,a_0\rangle} - e^{2\pi i\langle z-x+y-b_0,a_0\rangle}| \\ =& |e^{i\theta(z)} - e^{i\theta(z-x+y)}|, \end{aligned}$$

we see that for any $\eta > 0$, if $\varepsilon > 0$ is small enough, the equation (4.8) implies that

$$\rho_{\mathbb{T}}(\chi_{a_0}(x), \chi_{a_0}(y)) < \eta.$$

□

Lemma 4.3.12. *Let D be an FLC and weakly repetitive Delone set in $\mathbb{R}^d$. Suppose for any $R_1, R_2 > 0$ and $\varepsilon > 0$, there are $L_1, L_2 > 0$ such that*

1. $|L_j - R_j| < \varepsilon$ *for each* $j = 1, 2$, *and*

2. D *has* (L_1, L_2)*-stripe structure.*

Then 0 *is a limit point of the group of all topological eigenvalues for* $(X_D, \mathbb{R}^d)$.

Proof. For any $R_1, R_2, \varepsilon > 0$ we take L_1 and L_2 as in the assumption. By the definition of stripe structure (Definition 4.3.1), there are $a_0 \in \mathbb{R}^d$ with $\|a_0\| = 1$ and $R > 0$ such that

$$E = \{y \in \mathbb{R}^d \mid (D - y) \cap B(0, R) = (D - x) \cap B(0, R)\} \subset S(a, x, L_1, L_2)$$

for each $x \in \mathbb{R}^d$. Since we can take arbitrarily large R, by Lemma 4.3.9 E is Delone and $D \overset{\mathbb{R}^d}{\to} E$.

If $R_1 > 4R_2$ and ε is small enough, then $L_1 > 4L_2$. Then if $y \in E$, we have

$$\langle y - x, a_0 \rangle \in L_1\mathbb{Z} + [-L_2, L_2],$$

and so

$$\langle y - x, \frac{1}{L_1} a_0 \rangle \in \mathbb{Z} + (-1/4, 1/4).$$

Since E is weakly repetitive and has FLC by Proposition 4.2.11, using Lemma 4.3.11, we see $\chi_{(1/L_1)a_0}$ is weakly E-equivariant. By Lemma 4.3.4, we see $\chi_{(1/L_1)a_0}$ is a topological eigenvalue for $(X_E, \mathbb{R}^d)$. Since $(X_E, \mathbb{R}^d)$ is a factor of $(X_D, \mathbb{R}^d)$ by the fact that $D \overset{\mathbb{R}^d}{\to} E$ and Lemma 4.2.10, we see it is a topological eigenvalue for $(X_D, \mathbb{R}^d)$. Since L_1 may be arbitrarily large, we see 0 is a limit point of the set of topological eigenvalues for $(X_D, \mathbb{R}^d)$. □

We prove the converse of Theorem 4.3.7.

Theorem 4.3.13. *Let* Π *be a glueable pattern space over* $(\mathbb{R}^d, \mathbb{R}^d)$ *and* $\mathcal{P}$ *an element of* Π *which is weakly repetitive, has FLC, consists of bounded components and is Delone-deriving. Suppose for any* $R_1, R_2 > 0$ *and* $\varepsilon > 0$ *there are* $L_1, L_2 > 0$ *such that*

1. $|R_j - L_j| < \varepsilon$ *for each* $j = 1, 2$, *and*

2. $\mathcal{P}$ *has* (L_1, L_2)*-stripe structure.*

Then 0 *is a limit point of the set of all topological eigenvalues for* $(X_{\mathcal{P}}, \mathbb{R}^d)$.

Proof. By Theorem 3.3.1 there is a Delone set D of $\mathbb{R}^d$ such that $\mathcal{P} \overset{\mathbb{R}^d}{\leftrightarrow} D$. By Lemma 4.3.6 and Lemma 4.3.12, we see 0 is a limit point of the set of topological eigenvalues for $(X_D, \mathbb{R}^d)$. Since $(X_{\mathcal{P}}, \mathbb{R}^d)$ and $(X_D, \mathbb{R}^d)$ are topologically conjugate, we obtain the conclusion. □

Chapter 5

Further research

In this chapter we comment on the possible directions of the further research.

5.1 The relation that corresponds to an isomorphism for the spaces of pattern-equivariant functions

By Theorem 3.4.10, under an assumption, two Delone-deriving abstract patterns $\mathcal{P}$ and $\mathcal{Q}$ that consist of bounded components are MLD if and only if the spaces $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{Q}}$ of pattern-equivariant functions are the same. (Here, the ambient space X is the Euclidean space $\mathbb{R}^d$ and the group Γ is a closed subgroup of $\mathrm{E}(d)$ that contains $\mathbb{R}^d$.) The word "same" means that they are equal, that is, $\mathcal{A}_{\mathcal{P}} = \mathcal{A}_{\mathcal{Q}}$. It is natural to ask what is the relation between $\mathcal{P}$ and $\mathcal{Q}$ if $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{Q}}$ are just isomorphic in a certain sense. For example, the spaces $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{Q}}$ can be regarded as topological vector spaces with Γ actions. We should ask what is the relation between $\mathcal{P}$ and $\mathcal{Q}$ if the spaces $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{Q}}$ are isomorphic as topological vector spaces with group actions. This problem is reminiscent of the theorem on crystallographic tilings, which showed that if the symmetry groups of two crystallographic tilings are isomorphic, then the isomorphism is given by a conjugation of an affine map, and thus one of the original tilings is MLD with the other after applying the affine map. It may be that we may generalize this result on crystallographic tilings by replacing symmetry group with the space of pattern-equivariant functions, because for a crystallographic tiling, its space of pattern-equivariant functions contains the information on the symmetry group of that tiling.

5.2 Topologlcal local derivability

Take two complex-valued functions f, g on $\mathbb{R}$. We say g is topologically locally derivable from f if, whenever we take $\varepsilon > 0$, there are $R_0 \geqq 0$ and $\delta > 0$ such that,

$$x, y \in \mathbb{R}, R > 0 \text{ and } |f(x+z) - f(y+z)| < \delta \text{ for each } z \in B(0, R+R_0) \\ \Rightarrow |g(x+z) - g(y+z)| < \varepsilon \text{ for each } z \in B(0, R).$$

This means that, if near two points of $\mathbb{R}$ the behaviors of f are "close", then the behaviors of g near those two points are "close". This "closeness" makes sense because we can gauge the distance of "local structures" of functions: the local structures of a function is described by the value of each point; we can gauge the distance of the values of two points by the standard metric on $\mathbb{C}$.

If the "local structures" and the distances of local structures make sense, we can define topological local derivability for more general pattern spaces. For example, the local structure of a Delone set D on $\mathbb{R}^d$ at a point $x \in \mathbb{R}^d$ is described by the position of points in D near x, relative to x. If D is $2r$-uniformly discrete, the intersection $D \cap B(x, r)^\circ$ is either a one-point set or the emptyset; in the former case the position of the point in $D \cap B(x, r)^\circ$ relative to x is an element of $B(0, r)^\circ$; combined with the latter case, the local structure of D is described by an element of $B(0, r)^\circ \cup \{\emptyset\}$, that is, an element of d-dimensional sphere S^d. For FLC and FTT tilings, the local structures are described by an element of Anderson-Putnam complex ([1]). We may axiomatize the properties of these local structures and obtain the notion of pattern space with local structures. We then define topological local derivability as above.

The merit of defining topological local derivability is that it will enable us to define several types of almost periodicity and discuss relations between such almost periodicities of two different abstract patterns. We can ask if for two abstract patterns $\mathcal{P}$ and $\mathcal{Q}$ that are topologically mutually locally derivable (topologically locally derivable in both directions), an almost periodicity of $\mathcal{P}$ is equivalent to the almost periodicity of the same type of $\mathcal{Q}$.

As an application, take a Bohr almost periodic function f on $\mathbb{R}^d$ and a finite subgroup K of $\mathrm{O}(d)$. Then $g = \sum_{A \in K} f \circ A$ is also a Bohr almost periodic function. The function g has a symmetry of K. If we can "translate" this g by constructing an abstract pattern $\mathcal{P}$ which is MLD with g and topologically mutually locally derivable with g, then $\mathcal{P}$ has the symmetry of K and the almost-periodicity inherited from g. Thus it may be possible to construct abstract patterns such as Delone sets or tilings that are almost periodic and have arbitrary symmetry. By cut and project construction, we may construct Delone sets with arbitrary rotational symmetry. The above method may enable us to construct abstract patterns with arbitrary symmetry (although such abstract patterns are not likely to have FLC). This may be seen as a next step from crystallographic restriction, which says that for crystallographic tilings in a dimension 2, if it has n-fold rotational symmetry, then $n = 1, 2, 3, 4$ or 6.

Likewise it may be possible to construct almost periodic Delone sets and tilings on a Riemannian manifold M from an almost periodic functions on the group Γ of isometries on M. Except for the case of $M = \mathbb{R}^d$, there are few known almost periodic tilings and Delone sets on M. However there are many almost periodic functions on Γ, and if the above procedure succeeds we obtain many almost periodic tilings and Delone sets on M.

5.3 Analogy with geometry

Given an FLC Delone set D, we may construct a de Rham complex, by taking the space of differential forms on $\mathbb{R}^d$ of which coefficients are smooth D-equivariant functions. It can be shown that the de Rham cohomoloogy corresponding to this de Rham complex is isomorphic to the Cech cohomology of the continuous hull X_D ([11]). By this fact we may see there is an analogy between the space of smooth pattern-equivariant functions and the space $C^\infty(M)$ of smooth functions on a smooth manifold M. We may further think that being MLD is similar to being diffeomorphic because (1) $\mathcal{P}$ and $\mathcal{Q}$ are MLD if and only if their spaces of smooth pattern-equivariant functions are the same and (2) two smooth manifolds are diffeomorphic if and only if their spaces of smooth functions are isomorphic.

If we assume this analogy, there are several problems. Let $\mathcal{P}$ be an abstract pattern. We may ask the following questions:

1. is there a "Morse function" in the space of smooth $\mathcal{P}$-equivariant functions of which derivatives contain information on properties of $X_\mathcal{P}$?

2. does the spectrum of the Laplacian $\sum(\frac{\partial}{\partial x_i})^2$ contain information of $\mathcal{P}$ or $X_\mathcal{P}$?

3. how many MLD classes of abstract patterns $\mathcal{Q}$ are there such that $X_\mathcal{P}$ and $X_\mathcal{Q}$ are homeomorphic?

The last question is an analogy to the question on discrepancies between being diffeomorphic and being homeomorphic.

Appendix A

Generalities of dynamical systems

Definition A.0.1. If X is a compact space, G a locally compact abelian group and $\alpha\colon G \curvearrowright X$ is a continuous action, then the triple (X, G, α) (or simply the pair (X, G)) is called a topological dynamical system.

We often suppress α and simply write the image of $x \in X$ by $g \in G$ by $g \cdot x$. Recall a character of G is a homomorphism $\chi\colon G \to \mathbb{T}$ where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Definition A.0.2. Let (X, G) be a topological dynamical system. A non-zero continuous function $f\colon X \to \mathbb{C}$ is called a topological eigenfunction if there is a continuous character $\chi\colon G \to \mathbb{T}$ such that $f(g \cdot x) = \chi(g) f(x)$ for any $g \in G$ and $x \in X$. The character χ is called the eigenvalue for the eigenfunction f.

Remark A.0.3. A non-zero constant function is always a topological eigenfunction.

Definition A.0.4. A topological dynamical system (X, G) is said to be weakly mixing if it admits no topological eigenfunctions other than constants.

Definition A.0.5. A measure-preserving system is a quintuplet $(X, \mathcal{F}, \mu, G, \alpha)$ where $(X, \mathcal{F}, \mu)$ is a probability space, G a locally compact abelian group and $\alpha\colon G \curvearrowright X$ is a measure-preserving action, that is, for each $g \in G$ the map $\alpha_g\colon X \to X$ preserves measurability and measure.

Definition A.0.6. Let $(X, \mathcal{F}, \mu, G, \alpha)$ be a measure-preserving system. An element $f \in \mathcal{L}^2(\mu) \setminus \{0\}$ is called a measurable eigenfunction if there is a continuous character χ such that two functions $x \mapsto f(g \cdot x)$ and $x \mapsto \chi(g) f(x)$ coincide almost everywhere for any $g \in G$. The character χ is called the eigenvalue for the eigenfunction f.

Definition A.0.7. A measure-preserving system $(X, \mathcal{F}, \mu, G, \alpha)$ is said to be weakly mixing if there is no measurable eigenfunction other than constants.

Remark A.0.8. In both topological and measurable cases, if $G = \mathbb{R}^d$, we identify $\hat{\mathbb{R}^d}$ and $\mathbb{R}^d$ and say $\xi \in \mathbb{R}^d$ is an eigenvalue if the character $x \mapsto e^{2\pi i \langle \xi, x \rangle}$ is an eigenvalue for some eigenfunction.

We say a sequence $g_1, g_2, \ldots$ of G converges to infinity if for any compact $K \subset G$, we have $g_n \notin K$ eventually.

Definition A.0.9. Let $(X, \mathcal{F}, \mu, G, \alpha)$ be a measure-preserving system. We say the system is mixing if whenever we take $E, F \in \mathcal{F}$ and a sequence $g_1, g_2, \ldots$ in G that converges to infinity, we have $\mu(E \cap (g_n \cdot F)) \to \mu(E)\mu(F)$.

Appendix B

Uniform structure

Uniform structure is a general framework by which we can discuss uniform continuity, total boundedness and so on. For example, metric spaces admit uniform structure. A set endowed with a uniform structure is called a uniform space. For details see [6].

Definition B.0.10. Let X be a set. A set $\mathfrak{U}$ of subsets of $X \times X$ is called a uniform structure on X if the following conditions are satisfied:

1. If $\mathcal{U} \in \mathfrak{U}$ and $\mathcal{U} \subset \mathcal{V} \subset X \times X$, then we have $\mathcal{V} \in \mathfrak{U}$.
2. The intersection of finitely many elements of $\mathfrak{U}$ is in $\mathfrak{U}$.
3. For any $\mathcal{U} \in \mathfrak{U}$, we have $\{(x, x) \mid x \in X\} \subset \mathcal{U}$.
4. For any $\mathcal{U} \in \mathfrak{U}$, we have $\mathcal{U}^{-1} = \{(y, x) \mid (x, y) \in \mathcal{U}\} \in \mathfrak{U}$.
5. For any $\mathcal{U} \in \mathfrak{U}$ there is $\mathcal{V} \in \mathfrak{U}$ such that

$$\mathcal{V}^2 = \{(x, z) \in X \times X \mid \text{there is } y \in X \text{ such that } (x, y), (y, z) \in \mathcal{V}\} \subset \mathcal{U}.$$

The elements of $\mathfrak{U}$ are called entourages of X.

Definition B.0.11. Let $(X, \mathfrak{U})$ be a uniform space and $\mathfrak{U}_0$ be a nonempty subset of $\mathfrak{U}$. Suppose for any $\mathcal{U} \in \mathfrak{U}$ there is $\mathcal{V} \in \mathfrak{U}_0$ such that $\mathcal{U} \supset \mathcal{V}$. Then we call $\mathfrak{U}_0$ a fundamental system of entourages.

Lemma B.0.12. *Let X be a nonempty set and $\mathfrak{U}_0$ a set of subsets of $X \times X$. Suppose $\mathfrak{U}_0$ satisfies the following conditions:*

1. *$\mathfrak{U}_0$ is nonempty.*
2. *$\{(x, x) \mid x \in X\} \subset \mathcal{U}$ for each $\mathcal{U} \in \mathfrak{U}_0$.*

3. *For any two* $\mathcal{U}_1, \mathcal{U}_2 \in \mathfrak{U}_0$ *there is* $\mathcal{U} \in \mathfrak{U}_0$ *such that* $\mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_2$.

4. *For any* $\mathcal{U} \in \mathcal{U}_0$ *there is* $\mathcal{V} \in \mathcal{U}_0$ *such that* $\mathcal{V} \subset \mathcal{U}^{-1}$.

5. *For any* $\mathcal{U} \in \mathfrak{U}_0$ *there is* $\mathcal{V} \in \mathfrak{U}_0$ *such that* $\mathcal{V}^2 \subset \mathcal{U}$.

Then there is a unique uniform structure on X *for which* $\mathfrak{U}_0$ *is a fundamental system of entourages.*

Lemma B.0.13. *Let* (X, ρ) *be a metric space and* ε *a positive real number. Set*

$$\mathcal{U}_\varepsilon = \{(x, y) \in X \times X \mid \rho(x, y) < \varepsilon\}.$$

Then there is a unique uniform structure such that $\{\mathcal{U}_\varepsilon \mid \varepsilon > 0\}$ *is a fundamental system of entourages.*

Definition B.0.14. Let $(X, \mathfrak{U})$ be a uniform space and ρ a metric on X. If the unique uniform structure in Lemma B.0.13 coincides with $\mathfrak{U}$, then we say the uniform space $(X, \mathfrak{U})$ is metrizable.

Definition B.0.15. Let X be a nonempty set and $\mathfrak{U}$ a uniform structure on X. For each $x \in X$ the set of sets of the form

$$\mathcal{U}(x) = \{y \in X \mid (x, y) \in \mathcal{U}\}$$

where $\mathcal{U}$ runs through $\mathfrak{U}$, satisfies the axiom of neighborhood basis. The topology defined by this is called the topology induced by $\mathfrak{U}$.

Lemma B.0.16 ([7],§2.4, Theorem 1)**.** *A uniform space is metrizable if and only if it admits a countable fundamental system of entourages and the induced topology is Hausdorff.*

Acknowledgements

This research was carried out when I was a graduate student of Keio University, School of Fundamental Science and Technology, under the supervision of Associate Professor Takeshi Katsura. I thank Takeshi Katsura for the discussions we had and advice which he gave me. I also thank Hitoshi Nakada and Hiroki Takahashi for comments on this thesis. I am grateful to Shigeki Akiyama for various discussions and helpful comments on my research. I am also grateful to Katsuhiro Ota, who assisted my visit to University of Copenhagen from 2012 to 2013, and Masato Kurihara, who assisted my visit to Bielefeld University in 2016. Finally I thank Robert Moody, Michael Baake and Nicolae Strungaru for their advice.

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