

Quantum measurement
for state generation
and information amplification

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Abstract

The theory of quantum measurement is an important element for quantum computation. In this work, I discuss quantum measurement in the quantum information technologies with some results of my doctor research. I have the following two main achievements:

1. proposal and analysis on improved control scheme of adaptive measurement for state preparation;
2. evaluation for weak value amplification.

The first main result is the proposal and the analysis of an improved scheme of adaptive measurement control based on Jacobs' continuous measurement. The continuous measurement is a well known expression of the dynamics caused by the measurement in optical system, whose noise caused by weak measurement has the Markovian property. This noise property allows us to use stochastic differential equation for description of the dynamics. Using the numerical simulation of the stochastic differential equation, I discuss the adaptive control strategy proposed by Jacobs, and propose an improved scheme for state generation. While Jacobs' scheme needs to adaptively change the measurement strength and the measured observable according to the measurement results, my scheme has constant measurement strength which is permitted by the other way to choose the observable. I discuss their performances.

The second result is the evaluation of the weak value amplification (WVA). The WVA technique is based on the weak measurement theory proposed by Aharonov, Albert and Vaidman. It is characterized by the following three features: 1. an indirect measurement model with the von Neumann type Hamiltonian, 2. the weak strength of the interaction Hamiltonian for the 1st order approximation to be valid, 3. the post-selection, which is the most important property of this measurement. The post-selection is the essence of this weak measurement, and it is the selection of data obtained by the post-selective measurements on the measured system. This operation requires us to collect the data only when the right outcome of the post-selective measurements on the probe. If we fail to get the relevant result, we ignore the result of the probe. I focus the weakness of the interaction and the data loss by the post-selection, and give the condition that the WVA is available.

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Chapter 1

Introduction

The quantum measurement is an old and new topic in physics. A fundamental discussion using a mathematical model of quantum measurement was proposed by von Neumann [1]. His idea is as follows; assume that there exists an identical measurement device which always gives accurate results. The state transition with this device is represented by a projection operator. This postulate was improved by Lüders, Holevo, and many mathematicians [2, 3]. They gave the fundamental formalism of the quantum measurement based on the probability and the statistical theory. More precisely, the quantum measurement is characterized as the statistical model on the noncommutative operator space. In these theories, the time evolution of a physical system and its statistics of a measurement are represented by the algebra of linear operators. The state transition caused by a measurement is represented as a map from the linear operator to another linear operator conditioned by the statistical property of physics. Ozawa developed this fundamental formalism of the quantum measurement, and he gave completely positive instrument in Ref. [4].

However, the statistics of quantum measurement has confused physicists and philosophers. Collapse of a wave function was treated as a paradox in the quantum physics [5, 6]. A philosopher Albert stated in his book [7] that the difficulty of the intuitive understanding of the quantum mechanics is attributed to the fact that the quantum mechanics predicts the state transition. He mentioned that only a probability distribution is what we are allowed to have a precise prediction by the quantum mechanics. He also remarked the collapse of a wave function, which is not a natural thing since experiment has always uncertainty.

The statistical study of the quantum mechanics has made the quantum mechanics easy to use. Because of these studies, it has become possible to study technological application, without being bound to the philosophical problem. They do not touch the philosophical aspects of the quantum measurement [8, 9, 10]. The goals of quantum information are the invention of the quantum computer. This machine is expected to take full advantage of several quantum mechanical properties such as the superposition principle, entanglement, reversible unitary time-evolution, and most importantly in our scenario,

the projection measurement [11]. For this engineering, we need to reveal the fundamental theorems of quantum physics. In fact, we have a history that the technological discussions tell the foundation of the quantum physics [11, 12, 13].

The theories of the quantum measurement and the quantum statistics have also been developed with the quantum information [14]. The quantum statistics gives the informational limit on the measurement, and it also improved the fundamental theory of quantum mechanics [15]. The quantum state is considered to be an extension of a statistical distribution function in the quantum statistical and information theory.

I have studied quantum measurement from the view of statistical theory. Through this study, we become to believe that the quantum measurement is a powerful tool for development of the quantum computer. Quantum measurement has a potential to make the high quality processors for quantum computation. We think that there are two expected properties for quantum measurement in the quantum computation processing:

1. Precise back-action of measurement to quantum state,
2. High accuracy of measurement.

Below we would like to describe the examples that show why these two things are required.

First, we show where the precise back-action is required in quantum information processing. As the example of this, we discuss topological quantum computation (TQC). TQC is a promising idea for the realization of the quantum information processing [16, 17]. This method generated the combination method of one way computation and surface-code [18, 19]. One way commutation is a computational method using the measurement back-action, and the surface code is an encoding method of the quantum information, which has error correction with the quantum measurement. For TQC, the quantum measurement is essential. This idea has problem concerning the generation of magic state. The magic states are pure states, and they give a non-trivial processing in TQC [20], but their physical realization is still being discussed. Theoretically, the back-action of measurement stochastically generates any pure states. Thus, the back-action of measurement is an expected technology for the state generation.

Second, we explain why the high accuracy measurement is required for the quantum computation processing. In order to perform a quantum gate, we have to know the physical properties of the processor. The highly accurate measurement assures the high quality processors. As an example of quantum processors, the composite system of flux-qubits and NV-centers is available for the gate operations. In this system, C-NOT gate would be feasible [21]. The physical system is expected to be the most close to practical use. However, the physical properties of flux qubits are different for each device, and the high accurate measurement is desired.

In this work, we discuss the above two properties and give two main results concerning them: the improved method for the state generation by quantum measurement, and the proposal of the evaluation method to clarify the accuracy of the weak value amplification (WVA) technique. This thesis is constructed as

follows.

In Chapter 2, I review the fundamental theory of quantum measurement, which we use in later chapters.

In Chapter 3, we explain the effect of back-action of continuous quantum measurement through the discussion of the state generation by the continuous measurement. The result of this chapter is published in Ref. [22], and this work is based on the idea originally developed by Jacobs [23]. Jacobs showed that the continuous measurement, which adaptively changes the measured observable and the measurement strength depending on the results of previous measurements, generates desired pure states. This scheme is epochal [24, 25], and we expect that this scheme provides high fidelity state generation. From the measurement results, we can monitor the system during the control. The information from the measurement allows us to assure the accuracy of the generated state.

In Chapter 4, we discuss the effects of the post-selection on the weak value amplification (WVA) for the precise measurement. The idea of the WVA is based on the fundamental theory of quantum mechanics proposed by Aharonov and his co-workers [26]. The weak measurement is characterized by the so-called post-selection, which is an operation such that we read out the probe system only when we have a particular result from the measurement on the measured system. When the interaction strength for measurement is enough weak that the first order approximation of the strength is allowed, the probe has larger shift than general measurement. In order to see the pure effect of post-selection, we discuss the shift with full order calculation, and we find that the large shift is allowed only when the strength is small. We also focus on the data loss by the post-selection, and show that this amplification is limited by the success probability of the post-selection. We conclude that the post-selection whose success probability is small amplifies the information of the estimated parameter by choosing such a rare post-selected state when we can repeat the measurement. This result was published in Ref. [27, 28].

Chapter 5 is the summary and the conclusions of this thesis.

Chapter 2

Quantum measurement and its statistics

In this chapter, we review the measurement theory and the estimation theory. The measurement process in quantum system is expressed by statistical and probability theory, and the estimation theory supports physical interpretation of measurement results. We use these theories as the background of Chaps. 3 and 4.

2.1 Quantum states and quantum measurements

In quantum theory, a quantum state is an essential notion rather than a distribution function and the state evolution represents that of statistics of physical values. Then, in this section, we see the state evolution of a quantum state caused by measurement [6, 29].

2.1.1 Projection postulate

The quantum state transition induced by an ideal quantum measurement is represented by a projection operator. This hypothesis was proposed by von Neumann [1]. The philosophy is as follows. We have an ideal and perfect measuring device having no error nor noise, and we measure a quantum system twice with this device. Here, we assume that the time interval of the 1st and 2nd measurement is infinitesimally small, and that the state change caused by the free Hamiltonian is ignorable. Thus, this measuring device surely gets the same results in the 1st and 2nd measurements. This event occurs with probability 1, the statistics is the same and the quantum system is not changed by the 2nd measurement. Since the quantum state is an essential element representing statistics in quantum mechanics, the state must be the same in the 1st and 2nd measurements. Now, assume that we obtain the measurement result j , let $\hat{P}_j$

be the state transition operator with an ideal measurement. Thus, $\hat{P}_j$ satisfies

$$\hat{P}_j|\psi\rangle = \hat{P}_j^2|\psi\rangle$$

for any initial state $|\psi\rangle$. Here, the operator of an ideal measurement satisfies

$$\hat{P}_j = \hat{P}_j^2, \quad (2.1)$$

and this operator is a projection operator. The projection operator whose rank is 1 can be represented by a vector $|a_j\rangle$, as $\hat{P}_j = |a_j\rangle\langle a_j|$, where $|a_j\rangle$ is an eigenvector of $\hat{P}_j$. This discussion is called the *projection postulate*. The state after an ideal measurement to the initial state $|\psi\rangle$ is

$$\hat{P}_j|\psi\rangle. \quad (2.2)$$

2.1.2 von Neumann's indirect measurement model

Now, we extend this discussion to a more realistic and experimental model. von Neumann presented an *indirect measurement model* (See Fig. 2.1). An indirect measurement model has two systems, i.e. the measured system $\mathcal{H}$ and the measuring device or probe system $\mathcal{K}$. These systems have interaction with each other. The measured system $\mathcal{H}$ has a measured observable, and we read out the information about the measured observable from the probe system $\mathcal{K}$ by using the interaction. Next, let us see the process of measuring the observable $\hat{A}^{\mathcal{H}} = \sum_a a|a\rangle^{\mathcal{H}}\langle a|$ in measured system. Let the initial state of measured system $|i\rangle^{\mathcal{H}} = \sum_j \alpha_a |a\rangle^{\mathcal{H}}$ and the initial state of the probe system $|\psi\rangle^{\mathcal{K}} = \int dx \psi(x)|x\rangle^{\mathcal{K}}$. Thus, the initial state of the composite system $\mathcal{H} \otimes \mathcal{K}$ is represented as a tensor product state $|i\rangle^{\mathcal{H}} \otimes |\psi\rangle^{\mathcal{K}}$. Here, we use the interaction Hamiltonian as

$$\hat{A}^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}. \quad (2.3)$$

The Hamiltonian given by Eq. (2.3) is called *von Neumann type* Hamiltonian. From now on, we omit the indexes of systems $\mathcal{H}, \mathcal{K}$ and tensor product. We

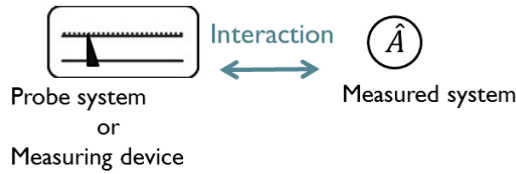


Figure 2.1: Measured system and measuring device. In the indirect measurement, we read out the information about measured system from the probe system, the measuring device.

take $\hbar = 1$ and denote the unitary operator of the interaction as $e^{-i\theta\hat{A}\otimes\hat{p}}$, where $\theta \in \mathbb{R}$. We also have $[\hat{x}, \hat{p}] = i$. The wave function of $\hat{x}$ is represented as a Fourier transform $\psi(x) \propto \int dp \tilde{\psi}(p) e^{ixp}$, and the operator e^{-iap} moves the wave function $\psi(x)$ to $\int dp \tilde{\psi}_i(p) e^{i(x-a)p}$. Thus, the probe shifts $x \rightarrow x - a$. The $e^{-i\theta a \hat{p}}$ is called a transition operator of x . Therefore, the state after interaction is

$$\begin{aligned} \hat{U}|i\rangle_{\mathcal{H}} \otimes |\psi\rangle_{\mathcal{K}} &:= e^{-i\theta\hat{A}\otimes\hat{p}} \sum_a \alpha_a |a\rangle \otimes \int dx \psi_i(x) |x\rangle \\ &= \sum_a \alpha_a \int dx e^{-i\theta a \hat{p}} \psi_i(x) |a\rangle \otimes |x\rangle \\ &= \sum_a \alpha_a \int dx \psi(x - \theta a) |a\rangle \otimes |x\rangle, \end{aligned} \quad (2.4)$$

where $\hat{x} = \sum_x x |x\rangle\langle x|$. In Eq. (2.4), the wave function of the probe system is a superposition of the shifted wave functions $\psi(x - \theta a)$ depending on the eigenvalue a of the measured observable $\hat{A}$ on the measured system. The interaction Hamiltonian transmits the wave function by the eigenvalue of the measured observable. Reading out this shift, we can have the information about $\hat{A}$.

2.1.3 Probability of the measurement result and Born's rule

The Born rule gives the probability that we have a measurement result by a measurement and this rule plays an important role in the quantum measurement theory. This rule gives the probability that we have same state after a measurement in quantum system.

Born's rule

For a quantum state $|\psi\rangle$, the probability to get the result $\hat{A} = a$ by measuring an observable $\hat{A}$ is

$$\text{Pr}[a] = |\langle a|\psi\rangle|^2 = \langle\psi|a\rangle\langle a|\psi\rangle \quad (2.5)$$

$$= \text{Tr} [|a\rangle\langle a|\psi\rangle\langle\psi|] \quad (2.6)$$

where $|a\rangle$ is an eigenvector of obtainable $\hat{A}$. That is, $\hat{A}$ is decomposed as $\hat{A} = \int a |a\rangle\langle a| da$.

Using this rule, we can calculate the 1st moment of an observable $\hat{A}$,

$$\langle\hat{A}\rangle = \int a \text{Pr}[a] da = \int a \langle\psi|a\rangle\langle a|\psi\rangle da \quad (2.7)$$

$$= \langle\psi| \left(\int a |a\rangle\langle a| \right) |\psi\rangle da = \langle\psi|\hat{A}|\psi\rangle. \quad (2.8)$$

We can also have n -th moment by substituting $A^n = \int a^n |a\rangle\langle a| da$ for $\hat{A}$ in Eq. (2.7). A quantum state and its eigenfunction expansion of observables characterize statistics in a quantum system.

2.1.4 Stern-Gerlach's experiment and POVM

Born's rule gives the probability to get a measurement result for a given quantum state. We extend this rule to the interaction model in the Stern-Gerlach example. We also summarize the state transition by measurement.

We discuss the state transition in indirect measurement model with Stern-Gerlach experiment. In the Stern-Gerlach experiment, the beam of silver atoms pass through the inhomogeneous magnetic field, and the orbit of atoms is affected by the spin-orbit interaction of the magnetic field [30]. Because of this interaction, the beam splits into two paths according to their spin state.

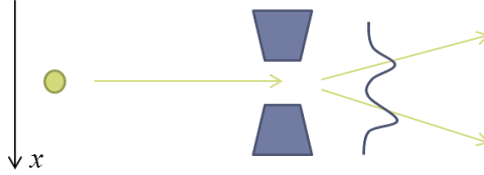


Figure 2.2: Schematic of Stern-Gerlach's experiment. The orbit is affected by inhomogeneous magnetic field.

In this phenomenon, the spin freedom is interpreted as a measured system and the orbit is interpreted as a probe system of the indirect measurement model (See Fig. 2.2). The initial state of the spin system is $|i\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, and the initial wave function of the orbit is $\psi(x)$. The interaction is caused by the inhomogeneous magnetic field, and the time evolution of the composite system is explained by the spin-orbit interaction unitary operator $e^{-i\theta\hat{\sigma}_z}$. Here, θ is a constant which represents the strength of the inhomogeneous interaction and $\hat{\sigma}_z := |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$. The state of the composite system is

$$\begin{aligned} |\Psi\rangle &= e^{-i\theta\hat{\sigma}_z \otimes \hat{p}} |i\rangle \int_{-\infty}^{\infty} \psi(x) |x\rangle dx \\ &= \int_{-\infty}^{\infty} (\alpha e^{-i\theta\hat{p}} \psi(x) |\uparrow\rangle + \beta e^{i\theta\hat{p}} \psi(x) |\downarrow\rangle) |x\rangle dx \\ &= \int_{-\infty}^{\infty} (\alpha \psi(x - \theta) |\uparrow\rangle + \beta \psi(x + \theta) |\downarrow\rangle) |x\rangle dx. \end{aligned} \quad (2.9)$$

Here, assume $\langle \hat{x} \rangle = \int_{-\infty}^{\infty} x \cdot \psi(x) dx = 0$, and we can guess that the spin state is up $\uparrow$ when the atoms are in the region $x \geq 0$, while the spin state is up $\downarrow$ when the atoms in the region $x \leq 0$.

The probability $\Pr[x > 0]$ that the atoms after passing through the magnetic field lies in the positive region is

$$\begin{aligned}\Pr[x > 0] &= \int_0^\infty dx |\langle x | \Psi \rangle| \\ &= \int_0^\infty \text{Tr}[\langle x | e^{-i\theta\hat{\sigma}_z\hat{p}} | \psi \rangle \langle i | \langle i | \langle \psi | e^{i\theta\hat{\sigma}_z\hat{p}} | x \rangle] dx \\ &= \text{Tr}[\hat{M}_+ | i \rangle \langle i |],\end{aligned}\quad (2.10)$$

$$(2.11)$$

and the probability $\Pr[x < 0]$ that the atoms lies in the negative region is

$$\begin{aligned}\Pr[x < 0] &= \int_{-\infty}^0 dx |\langle x | \Psi \rangle| \\ &= \int_{-\infty}^0 dx \text{Tr}[\langle x | e^{-i\theta\hat{\sigma}_z\hat{p}} | \psi \rangle \langle i | \langle i | \langle \psi | e^{i\theta\hat{\sigma}_z\hat{p}} | x \rangle] \\ &= \text{Tr}[\hat{M}_- | i \rangle \langle i |],\end{aligned}\quad (2.12)$$

$$(2.13)$$

where

$$\begin{aligned}\hat{M}_+ &:= \int_0^\infty \langle \psi | e^{i\theta\hat{\sigma}_z\hat{p}} | x \rangle \langle x | e^{-i\theta\hat{\sigma}_z\hat{p}} | \psi \rangle dx \\ &= \int_0^\infty (|\psi(x-\theta)|^2 |\uparrow\rangle\langle\uparrow| + |\psi(x+\theta)|^2 |\downarrow\rangle\langle\downarrow|) dx,\end{aligned}\quad (2.14)$$

$$\begin{aligned}\hat{M}_- &:= \int_{-\infty}^0 \langle \psi | e^{i\theta\hat{\sigma}_z\hat{p}} | x \rangle \langle x | e^{-i\theta\hat{\sigma}_z\hat{p}} | \psi \rangle dx \\ &= \int_{-\infty}^0 (|\psi(x-\theta)|^2 |\uparrow\rangle\langle\uparrow| + |\psi(x+\theta)|^2 |\downarrow\rangle\langle\downarrow|) dx.\end{aligned}\quad (2.15)$$

Thus, we assume that the spin state is $\uparrow$ in the probability $\Pr[x > 0]$ and that the spin state is $\downarrow$ in the probability $\Pr[x < 0]$. From Eqs. (2.6), (2.14) and (2.15), $\hat{M}_\pm$ is an operator representing the probability of $\uparrow$ or $\downarrow$. Using these operators, we do not need to give the initial state of the probe and the interaction Hamiltonian explicitly in order to drive the probabilities from the initial state of the measured system. The set of operators $\hat{\mathbf{M}} = \{\hat{M}_+, \hat{M}_-\}$ is called *POVM* (*positive operator valued measure*), and this operator set fully characterizes the measurement property. The definition of POVM is given as follows [3, 9].

Definition of POVM (positive operator-valued measure)

The operator set $\hat{\mathbf{M}} = \{\hat{M}_i\}_i$ which satisfies

$$0 \leq \hat{M}_i \leq \hat{I}, \quad \sum_i \hat{M}_i = \hat{I} \quad (2.16)$$

is called POVM (positive operator valued measure).

This operator set gives the probability measure in quantum theory.

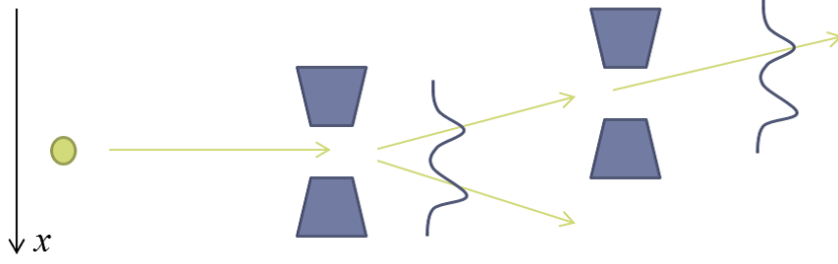


Figure 2.3: The schematic of the measurements in Stern-Gerlach experiment. Here, we measure the spin twice.

We see that the POVM gives the distribution of the measurement results. So, the question arises what happens in the state after measurement. In order to see the state transition caused by measurement, let us consider that we measure the spin twice using the Stern-Gerlach experiment (See Fig. 2.3). Let the initial state of the probe system be

$$|\psi\rangle_{\mathcal{K}} := \int \psi(x)|x\rangle dx.$$

Thus, the state after passing through the 1st magnetic field is

$$|\Psi\rangle = e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |i\rangle_{\mathcal{H}} \otimes |\psi\rangle_{\mathcal{K}}.$$

From the projection postulate, the state after we get the result x_1 from measurement of the probe observable $\hat{x}$ is

$$|x_1\rangle_{\mathcal{K}} \langle x_1 | \Psi \rangle = \langle x_1 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |\psi\rangle_{\mathcal{K}} |i\rangle_{\mathcal{H}} \otimes |x_1\rangle_{\mathcal{K}}.$$

Thus, the state after passing through the 2nd magnetic field is

$$e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |x_1\rangle_{\mathcal{K}} \langle x_1 | \Psi \rangle = e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} \langle x_1 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |\psi\rangle_{\mathcal{K}} |i\rangle_{\mathcal{H}} \otimes |x_1\rangle_{\mathcal{K}},$$

and the state after we get the result x_2 from the 2nd measurement of $\hat{x}$ is

$$\begin{aligned} & |x_2\rangle_{\mathcal{K}} \langle x_2 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |x_1\rangle_{\mathcal{K}} \langle x_1 | \Psi \rangle \\ &= \langle x_2 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |x_1\rangle_{\mathcal{K}} \langle x_1 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |\psi\rangle_{\mathcal{K}} |i\rangle_{\mathcal{H}} \otimes |x_2\rangle_{\mathcal{K}} \rangle \\ &= (\hat{E}_2(x_2, x_1) \hat{E}_1(x_1) |i\rangle_{\mathcal{H}}) \otimes |x_2\rangle_{\mathcal{K}}. \end{aligned} \quad (2.17)$$

Here, we denote

$$\hat{E}_1(x_1) := \langle x_1 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |\psi\rangle_{\mathcal{K}}, \quad (2.18)$$

$$\hat{E}_2(x_2, x_1) := \langle x_2 | e^{-i\theta\hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}} |x_1\rangle_{\mathcal{K}}. \quad (2.19)$$

Both $\hat{E}_1$ and $\hat{E}_2$ are operators on the measured system. The probability that we obtain the result x_1 of the 1st measurement and the result x_2 of the 2nd measurement with the Stern-Gerlach experiment is

$$\begin{aligned} \Pr[x_1, x_2] &= \left| \hat{E}_2(x) \hat{E}_1(x) (|i\rangle_{\mathcal{H}} \otimes |x_2\rangle_{\mathcal{K}}) \right|^2 \\ &= \text{Tr} \left[\hat{E}_2(x) \hat{E}_1(x) |i\rangle_{\mathcal{H}} \langle i| \hat{E}_1(x)^\dagger \hat{E}_2(x)^\dagger \right]. \end{aligned} \quad (2.20)$$

From Eqs. (2.17) and (2.20), $\hat{E}_1(x)$ is an operator representing the state transition by the 1st measurement, and $\hat{E}_2(x)$ is an operator representing the state transition by the 2nd measurement. From Eqs. (2.10), (2.12), (2.14) and (2.15), we have

$$\hat{M}_+ := \int_0^\infty \hat{E}_1(x)^\dagger \hat{E}_1(x) dx, \quad \hat{M}_- := \int_{-\infty}^0 \hat{E}_1(x)^\dagger \hat{E}_1(x) dx. \quad (2.21)$$

We can conclude that the POVM gives the probability that we have particular measurement result, but the POVM is not enough to represent the state transition through measurement. Generally, the state transition of measurement is represented by completely positive instrument [4, 8, 9].

Assume the density operator of initial state $\hat{S}$. When we have j -th measurement result, the state after the measurement is

$$\frac{\hat{E}_j \hat{S} \hat{E}_j^\dagger}{\text{Tr}[\hat{S} \hat{E}_j^\dagger \hat{E}_j]}, \quad (2.22)$$

where

$$\sum_j \hat{E}_j \hat{E}_j^\dagger = \hat{I}. \quad (2.23)$$

When the initial state is denoted by a vector $|\psi\rangle$, the state after the measurement becomes

$$\frac{\hat{E}_j |\psi\rangle}{\sqrt{\text{Tr}[\hat{S} \hat{E}_j^\dagger \hat{E}_j]}}. \quad (2.24)$$

2.1.5 Continuous measurement and its measurement operator

In the previous section, we review the derivation of the POVM formalism from the indirect measurement model. The measurement operator and the POVM can represent any measurement as a mathematical model. In this section, we discuss dynamics of state caused by continuous measurement as an example of POVM [31].

In continuous measurement, the time evolution in the short time interval Δt is represented by

$$\hat{A}(\alpha) := \left(\frac{4k\Delta t}{\pi} \right)^{1/4} \int_{-\infty}^{\infty} e^{-2k\Delta t(x-\alpha)^2} |x\rangle\langle x| dx. \quad (2.25)$$

This operator $\hat{A}$ represents the state transition of measurement and satisfies Eq. (2.23). The operator $\hat{A}(\alpha)$ has the projection operator $|x\rangle\langle x|$ weighted with a Gaussian distribution. The measurement operator (2.25) is derived when the probe wave function is Gauss and the interaction is von Neumann [32]. Assume the measured system observables $\hat{x}, \hat{p}$ denote the state of the measured system $\hat{\rho}$ before the interaction. Assume the probe observables $\hat{x}, \hat{p}$, and denote the state of the probe before the interaction as

$$\langle \hat{x} | \Upsilon \rangle = \frac{1}{(\pi\sigma)^{1/4}} \exp[-\hat{x}^2/2\sigma]. \quad (2.26)$$

Thus, we have the state transition operator

$$\hat{E}(x) = \langle \hat{x} | e^{-i\hat{x}\hat{p}} | \Upsilon \rangle = \left(\frac{1}{\pi\sigma} \right)^{1/4} \exp[-(\hat{x} - x)^2/2\sigma], \quad (2.27)$$

and the state after the non-selective measurement is

$$\int_{-\infty}^{\infty} dx \hat{E}(x) \rho \hat{E}^\dagger(x) \quad (2.28)$$

Therefore, we have the operator (2.25) by substring $\sigma = 1/(4k\Delta t)$ and $\hat{x} = \alpha$. This fact is known numerically and experimentally [33, 34].

The measurement operator (2.25) gives a continuous measurement of position. Here, α is continuous index. If the initial state is $|\psi\rangle = \int \psi(x)|x\rangle dx$, the probability to get the result α becomes

$$\text{Pr}[\alpha] = \text{Tr} [\hat{A}^\dagger(\alpha) \hat{A}(\alpha) |\psi\rangle\langle\psi|] \quad (2.29)$$

$$= \sqrt{\frac{4k\Delta t}{\pi}} \int_{-\infty}^{\infty} |\psi(x)|^2 e^{-4k\Delta t(x-\alpha)^2} dx. \quad (2.30)$$

and the 1st moment α satisfies

$$\langle \alpha \rangle = \int \alpha \text{Pr}(\alpha) d\alpha \quad (2.31)$$

$$= \int_{-\infty}^{\infty} \alpha \text{Tr} [\hat{A}^\dagger(\alpha) \hat{A}(\alpha) |\psi\rangle\langle\psi|] d\alpha \quad (2.32)$$

$$= \sqrt{\frac{4k\Delta t}{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha |\psi(x)|^2 e^{-4k\Delta t(x-\alpha)^2} dx d\alpha \quad (2.33)$$

$$= \int_{-\infty}^{\infty} x |\psi(x)|^2 dx = \langle \hat{x} \rangle. \quad (2.34)$$

Assume that the random variable α can be represented by

$$\alpha = \langle \hat{x} \rangle + \frac{\Delta w}{\sqrt{8k\Delta t}}, \quad (2.35)$$

where Δw is a random variable which follows the Gaussian distribution whose mean value is 0 and variance is Δt . The time evolution of this random variable is continuous although it is not smooth [35, 36]. It is practically known that this formalism represents the experiment such that $\hat{x}$ is continuously measured with an optical probe system [25, 31]. Thus, we have

$$\begin{aligned} |\psi(t + \Delta t)\rangle &\propto \hat{A}(\alpha)|\psi(t)\rangle \\ &\propto e^{-2k\Delta t(\alpha - \langle \hat{x} \rangle)^2} |\psi(t)\rangle \\ &\propto e^{-2k\Delta t(\hat{x} - \langle \hat{x} \rangle)^2 + \sqrt{2k}\Delta w(\hat{x} - \langle \hat{x} \rangle) - 2k} |\psi(t)\rangle \\ &\propto e^{-2k\Delta t(\hat{x} - \langle \hat{x} \rangle)^2 + \sqrt{2k}\Delta w(\hat{x} - \langle \hat{x} \rangle)} |\psi(t)\rangle. \end{aligned} \quad (2.36)$$

Here, we used Eq. (2.35). Expanding the state and take the 1st order of Δt , we have

$$\begin{aligned} |\psi(t + \Delta t)\rangle &\propto \{1 - 2k\Delta t(\hat{x} - \langle \hat{x} \rangle)^2 + \sqrt{2k}\Delta w(\hat{x} - \langle \hat{x} \rangle) + k\Delta t(\hat{x} - \langle \hat{x} \rangle)^2\} |\psi(t)\rangle. \end{aligned} \quad (2.37)$$

Taking the infinite limit $\Delta t \rightarrow dt$, we have

$$|\psi(t + dt)\rangle \propto \{1 - kdt(\hat{x} - \langle \hat{x} \rangle)^2 + \sqrt{2k}dw(\hat{x} - \langle \hat{x} \rangle)\} |\psi(t)\rangle. \quad (2.38)$$

Since $|\psi(t + dt)\rangle = |\psi\rangle + d|\psi\rangle$, we have

$$d|\psi\rangle \propto \{-k(\hat{x} - \langle \hat{x} \rangle)^2 dt + \sqrt{2k}(\hat{x} - \langle \hat{x} \rangle)dw\} |\psi\rangle. \quad (2.39)$$

Equation (2.39) is called the *stochastic Schrödinger equation*, and the trajectory of the solution of this equation is called *quantum trajectory*. Moreover, this equation is extended to the case of general mixed state. From $\hat{\rho}(t + dt) = \hat{\rho}(t) + d\hat{\rho}$, we have the stochastic master equation

$$\begin{aligned} d\hat{\rho} &= (d|\psi\rangle)\langle\psi| + |\psi\rangle(d\langle\psi|) + (d|\psi\rangle)(d\langle\psi|) \\ &= -k[\hat{x}, [\hat{x}, \hat{\rho}]]dt + \sqrt{2k}(\hat{x}\hat{\rho} + \hat{\rho}\hat{x} - 2\langle\hat{x}\rangle\hat{\rho})dw. \end{aligned} \quad (2.40)$$

The continuous measurement is widely known as a useful model to explain the weak measurement experiment such that the optical system works as a probe in infinitesimally short time interval [24, 25]. It also known that this formalism is available when the measured system is two level [31, 33]. This time evolution is derived by using path integral methods [32, 37].

2.1.6 Summary of the section

The fundamental mathematical model of the quantum measurement was proposed by von Neumann; his idea is that an ideal measurement process is represented by an projection operator. This idea is called the projection postulate. The projection postulate and Born rule is theoretical foundation of the quantum measurement. While these theories give basic pictures of the quantum measurement, they are not enough for the detail descriptions of experiments. The indirect measurement model is helpful to explain the experimental setup of quantum measurement more precisely. This model also allows us to expand Born rule, and it gives the POVM which is the probability measure in the quantum theory. We want to remark that any measurement is determined by POVM which has its own indirect measurement models, and this fact is proved by Ozawa [38].

2.2 Statistical inference and estimation

Up to now, we have discussed the state transition and the probability of the quantum measurement. Although these discussions give the pictures of the quantum measurement itself, they do not suggest how to get information from the measurement or which measurement is appropriate for purpose. The classical and quantum theories of statistical inference give us the methods.

Usually, we have a physical experiment with particular aim, and try to build the experimental setup in order to reach the aim. We usually have information about experimental setup before we have the measurement results. Therefore, the parametric model is available in practice. In parametric model, the function type of distribution is given while the parameters characterizing the distribution are unknown. This model is useful in practice and mathematical sense. In this model, the function types of the systems are given, and we infer parameters in the system.

Estimation theory is based on the statistical inference in the parametric model. If we want to infer the parameter values, the estimation theory is used. Using the measurement results $X = \{x_1, x_2, \dots\}$, we create estimator $t(X)$, where the measured value itself is not necessarily to be what we want. That is, the measurement can be indirect. The estimator cannot exactly equal to the true value, but it can be consistent with the true value in the some range of fluctuation. The estimation usually requires enough number of data for generation of it. Since the measurement results are selected in probability, the measurement results $X = \{x_1, x_2, \dots\}$ can be treated as random variables. When the measurement apparatus gives the same statistical process, the results must follow the same probability distribution $f(x|\theta)$, and they are independent and identically distributed (iid). Here, parameter θ characterizes the destitution.

2.2.1 Classical estimation and estimator

Assume we have some samples $X = \{x_1, x_2, \dots\}$ as results of the physical measurements, and a statistic $t = t(X)$ is calculated from the samples. We want to make this statistic t close to the true value θ_0 . The statistic t is called an estimator [39, 40].

There are some important properties of the estimator as follows.

- **Consistency:** The estimator coverages in probability to the estimated quantity as the sample size grows. Suppose $t_n = t(X_1, \dots, X_n)$ is an estimator. A sequence of estimators $\{t_n\}$ is a consistent estimator for the parameter θ if and only if, for $\forall \epsilon > 0$, no matter how small, we have $\lim_{n \rightarrow \infty} \Pr \{|t_n - \theta| < \epsilon\} = 1$.
- **Unbiasedness:** The expected value of an estimator equals to the estimated quantity. If and only if the estimator satisfies $E_\theta[t(X)] = \theta$, the estimator $t(X)$ is called an unbiased estimator.

The accuracy of estimators is usually evaluated with the mean squared error

$$E_\theta [(t(X) - \theta)^2] = \int_{-\infty}^{\infty} dx (t(X) - \theta)^2 f(x|\theta). \quad (2.41)$$

This value is bounded by the Cramér-Rao inequality as follows;

Cramér-Rao inequality for a single parameter

For any estimator $t(X)$, the following inequality holds;

$$\text{Var}_\theta [t(X)] \geq \left(\frac{\partial \varphi}{\partial \theta} \right)^2 I_\theta^{-1}, \quad (2.42)$$

where $\varphi := E_\theta [t(X)] \in \mathbb{R}$, and I_θ is called Fisher information such that

$$I_\theta := E_\theta [\partial_\theta l(\theta, x) \partial_\theta l(\theta, x)] = \int dx \left(\frac{\partial \ln f(x|\theta)}{\partial \theta} \right)^2 f(x|\theta) \quad (2.43)$$

where $l(\theta, x) = \ln f(x|\theta)$. Especially, an unbiased estimator such that

$$E_\theta [t(X)] = \theta \quad (2.44)$$

satisfies the following Cramér-Rao inequality

$$\text{Var}_\theta [t(X)] \geq 1/I_\theta. \quad (2.45)$$

We review proof of the Cramér-Rao inequality when the estimator is unbiased [39, 40, 41].

(Proof)

Since the estimator is unbiased, we have

$$\theta = E_\theta [t(x)].$$

Differentiate both hands by θ ,

$$\begin{aligned} 1 &= \partial_\theta \int t(x) f(x|\theta) dx = \int t(x) \frac{\partial \ln f(x|\theta)}{\partial \theta} f(x|\theta) dx \\ &= E_\theta [t(x) \partial_\theta l(x, \theta)], \end{aligned}$$

and we have

$$E_\theta [\partial_\theta l(\theta, x)] = \int \frac{\partial_\theta \ln f(x|\theta)}{\partial \theta} f(x|\theta) dx = \int \frac{\partial_\theta f(x|\theta)}{\partial \theta} dx = 0.$$

Thus, we have

$$E_\theta [t(x) \partial_\theta l(\theta, x)] = E_\theta [(t(x) - \theta) \partial_\theta l(\theta, x)] = \text{Cov}_\theta [t(x), \partial_\theta l(\theta, x)].$$

Therefore, we have

$$\begin{aligned} 1 &= \text{Cov}_\theta [t(x), \partial_\theta l(\theta, x)] \\ &\leq \text{Var}_\theta [t(x)] \text{Var}_\theta [l(\theta, x)] = \text{Var}_\theta [t(x)] I_\theta, \end{aligned} \quad (2.46)$$

and we have the Cramér-Rao inequality

$$\text{Var}_\theta [t(x)] \geq 1/I_\theta.$$

(QED)

2.2.2 Maximum likelihood estimator and asymptotic aspects of classical estimation theory

In this part, we review the asymptotic aspects of classical estimator tokeinyumon, Rao. Estimators are on general functions of measurement results X . Especially, the maximum likelihood estimator (MLE) is widely used, and its definition is

$$\hat{\theta}^{ML} = \text{argmax}_\theta f(X|\theta). \quad (2.47)$$

Here, argmax means argument for which the given function attains its maximum value, and it defined by $\text{argmax}_\theta f(\theta) = \{\theta | \forall t : f(t) \leq f(\theta)\}$. The MLE $\hat{\theta}^{ML}$ is consistent, and its mean squared error becomes the inverse of the Fisher information at the large number of data. In order to discuss the consistency of MLE, we introduce the Kullback-Libler divergence $I_{KL}(f, g)$;

$$\begin{aligned} I_{KL}(f, g) &:= \int_{-\infty}^{\infty} \ln(f(x)/g(x)) f(x) dx \\ &= E_f [\ln(f(x)/g(x))]. \end{aligned} \quad (2.48)$$

Let us see some properties of the Kullback-Libler divergence. First, any $x \geq 0$ satisfies $\ln x \leq x - 1$, and we have

$$\begin{aligned} I_{\text{KL}}(f, g) &= - \int \ln(g(x)/f(x)) dx \geq \int \left(1 - \frac{g(x)}{f(x)}\right) f(x) dx \\ &= \int f(x) dx - \int g(x) dx = 1 - 1 = 0. \end{aligned} \quad (2.49)$$

Assume that θ_0 is the true value of the estimated parameter θ , the Kullback-Libler divergence has following properties

$$I_{\text{KL}}(f_\theta, f_\theta) = 0, \quad (2.50)$$

$$\left. \frac{\partial}{\partial \theta} I_{\text{KL}}(f_\theta, f_\theta) \right|_{\theta=\theta_0} = 0, \quad (2.51)$$

$$\left. \frac{\partial^2}{\partial \theta^2} I_{\text{KL}}(f_\theta, f_\theta) \right|_{\theta=\theta_0} = I_{\theta_0}, \quad (2.52)$$

where I_{θ_0} is Fisher information.

We define

$$\eta(\theta_0, \theta) = \int \ln(f(x|\theta)) f(x|\theta) dx. \quad (2.53)$$

Especially $-\eta(\theta_0, \theta_0)$ is the so called *Shannon entropy*, and this value is used in information theory. Since the Kullback-Libler divergence is positive, we have

$$\eta(\theta_0, \theta) \leq \eta(\theta_0, \theta_0). \quad (2.54)$$

We introduce the log likelihood function as follows

$$\frac{1}{N} l(\theta) := \frac{1}{N} \sum_{i=1}^N \ln f(x_i|\theta), \quad (2.55)$$

and, for the maximum likelihood estimator $\hat{\theta}_N^{ML}$, we have

$$l(\hat{\theta}_N^{ML}) = \max l(\theta). \quad (2.56)$$

and we have

$$\mathbb{E} \left[\frac{1}{N} l(\theta) \right] = \frac{1}{N} \sum_{i=1}^N \mathbb{E}_{\theta_0} [\ln f(x_i|\theta)]. \quad (2.57)$$

From the law of large numbers, we have

$$l(\hat{\theta}_N^{ML}) \xrightarrow{P} \eta(\theta_0, \theta). \quad (2.58)$$

Note $X_n \xrightarrow{P} X$ means convergence in probability and definition as follows: Suppose that x_n be a sequence of random variables and that X is another random

variable. If $\forall \varepsilon > 0$, $\lim_{n \rightarrow \infty} \Pr[|X_n - X| \geq \varepsilon] = 0$, x_n converges to x in probability and it is also denoted by $X_n \xrightarrow{p} X$ or $\text{plim}_{n \rightarrow \infty} X_n = X$. A sequence X_n of random variables is said to converge in distribution if $\lim_{n \rightarrow \infty} F_n(x) = F(x)$, $\forall x$. Here F_n and F are the distribution functions of X_n and X . It is also denoted by $F_n \xrightarrow{d} F$, $X_n \xrightarrow{d} X$.

From Eq. (2.58), the parameter θ which maximizes $l(\theta)$ and $\eta(\theta_0, \theta)$ seems to converge in probability with some regular condition;

$$\hat{\theta}_N^{ML} \xrightarrow{p} \theta_0 \quad (N \rightarrow \infty).$$

That is, we have the following lemma;

$X_1, \dots, X_n$ is independent and identically distributed (i.i.d.) random variables which distribution is $f(x|\theta)$. The estimate parameter θ satisfies $\theta \in (-\infty, \infty)$ and $\theta_0 = 0$. Here, suppose followings;

- $\eta(0, \theta)$ is a monotonically increasing function of $|\theta|$,
- $g_M(x)$ satisfies $g_M(x) = \sup_{|\theta| > M} \ln f(x|\theta)$ and $E[g_M(X)] = c_g < \eta(0, 0)$ at enough large M ,
- $h_M(X)$ such that $h_M(X) = \sup_{|\theta| \leq M} |\ln f(x|\theta)|$ satisfy $E[h_M(X)] = c_g < \eta(0, 0)$ at enough large M .

If these assumptions hold, the MLE $\hat{\theta}_N^{ML}$ converges to $\theta_0 = 0$ in probability, and t becomes a consistent estimator.

(Proof)

We have

$$\frac{1}{N} l(0) \xrightarrow{p} \eta(0, 0), \quad (2.59)$$

and

$$\frac{1}{n} < \frac{1}{N} \sum_{i=1}^N g_M(x_i) \xrightarrow{p} c_g. \quad (2.60)$$

Since $c_g \leq \eta(0, 0)$, if $|\theta| > M$, the function $l(\theta)$ of θ is not larger than $l(0)$ at $N \rightarrow \infty$. Thus, we have

$$\Pr[|\hat{\theta}_N^{ML}| > M] \rightarrow 0 \quad (N \rightarrow \infty). \quad (2.61)$$

Therefore, the probability that $\hat{\theta}_N^{ML}$ exists in the regain $[-M, M]$ for large N is 0.

Suppose we have a small value ε such that $\varepsilon > 0$, and we have

$$\delta = \min\{\eta(0, 0) - \eta(0, -\varepsilon), \eta(0, 0) - \eta(0, \varepsilon)\} > 0, \quad (2.62)$$

We introduce K such that

$$K > 2Mc_h/\delta. \quad (2.63)$$

Here, we divide the region $[-M, -\varepsilon]$ and $[\varepsilon, M]$ to K small intervals equally. We define $\theta_1 = -M, \theta_2 = -M + (M - \varepsilon)/K, \dots, \theta_{K+1} = -\varepsilon$ and $\theta_{K+2} = -M, \dots, \theta_{2K+2} = M$. For any $i = 1, \dots, 2K + 2$, we have

$$\frac{1}{N} l(\theta) \xrightarrow{P} \eta(0, 0). \quad (2.64)$$

For $j \neq K + 1$, we have

$$\begin{aligned} \sum_{\theta \in [\theta_j, \theta_{j+1}]} \frac{1}{N} |l(\theta) - l(\theta_j)| &= \sup_{\theta \in [\theta_j, \theta_{j+1}]} \left| \frac{\partial}{\partial \theta} l(\theta^*) \right| \\ &\quad (\theta_j \leq \theta^* \leq \theta) \\ &\leq (\theta_{j+1} - \theta_j) \frac{1}{N} \sum_{i=1}^N h_M(x_i) < \frac{M}{K} \frac{1}{N} \sum_{i=1}^N h_M(x_i) \\ &< \frac{\delta}{2c_k} \sum_{i=1}^N h_M(x_i). \end{aligned} \quad (2.65)$$

From the law of large numbers, $\sum_{i=1}^N h_M(X_i)$ converges to c_h and we get

$$\Pr \left[\sum_{\theta \in [\theta_j, \theta_{j+1}]} \frac{1}{N} |l(\theta) - l(\theta_j)| \leq \frac{\delta}{2} \right] \rightarrow 1 \quad (N \rightarrow \infty). \quad (2.66)$$

From Eqs. (2.55) and (2.59), we can say that $l(\theta_j)$ does not give the maximum of $l(\theta)$. From Eqs. (2.64) and (2.66), The MLE does not exist in $[\theta_j, \theta_{j+1}]$, ($j \neq K + 1$) at the limit $N \rightarrow \infty$;

$$\Pr [\hat{\theta}_N^{ML} \in [\theta_j, \theta_{j+1}]] \rightarrow 0 \quad (N \rightarrow \infty). \quad (2.67)$$

Thus, from Eqs. (2.61), (2.67), we have

$$\Pr [|\hat{\theta}_N^{ML}| \geq \varepsilon] \rightarrow 0 \quad (N \rightarrow \infty). \quad (2.68)$$

Therefore, the MLE $\hat{\theta}_N^{ML}$ converges to $\theta_0 = 0$ in probability.

(QED)

Next, we show the asymptotic distribution of MLE is

$$\sqrt{N}(\hat{\theta}_N^{ML} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1/I_{\theta_0}), \quad (2.69)$$

where the normal distribution $\exp[-(x-\mu)^2/2\sigma^2]/\sqrt{2\pi\sigma^2}$ is denoted as $\mathcal{N}(\mu, \sigma^2)$. This equation shows asymptotic character. We denote the differential of likelihood function as l' , and the MLE $\hat{\theta}^{ML}$ is a solution of equation

$$l'(\hat{\theta}) = 0. \quad (2.70)$$

Here, we expand at θ_0 and have

$$0 = l'(\hat{\theta}_N) = l'(\theta_0) + (\hat{\theta}_N^{ML} - \theta_0)l''(\theta^*), \quad (2.71)$$

where θ^* exists between θ_0 and $\hat{\theta}_N^{ML}$. Thus, we have

$$\sqrt{N}(\hat{\theta}_N^{ML} - \theta_0) = -\sqrt{N} \frac{l'(\theta_0)}{l''(\theta^*)} = \frac{\frac{1}{\sqrt{N}}l'(\theta_0)}{-\frac{1}{N}l''(\theta^*)}, \quad (2.72)$$

whose numerator is

$$l'(\theta_0) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{\partial}{\partial \theta} \ln f(X_i, \theta_0), \quad (2.73)$$

where $\frac{\partial}{\partial \theta} \ln f(X_i|\theta_0)$ is iid random valuable. Thus, we can use the central limit theorem and have

$$E_{f(x|\theta)} \left[\frac{\partial}{\partial \theta} \ln f(X_i, \theta_0) \right] = 0, \quad (2.74)$$

$$E_{f(x|\theta)} \left[\left(\frac{\partial}{\partial \theta} \ln f(X_i|\theta_0) \right)^2 \right] = I_{KL}(0, \theta), \quad (2.75)$$

and

$$\frac{1}{\sqrt{N}}l'(\theta_0) \xrightarrow{d} \mathcal{N}(0, I_{\theta_0}). \quad (2.76)$$

2.2.3 Quantum estimator

In the classical estimation, we infer the estimate parameter of a given distribution function type. In the quantum estimation theory, the function type of a quantum state is given, and this state is characterized by the parameter θ . Here, we represent the density operator of state as $\hat{S}_\theta$. We estimate the parameter of the state characterizing it from the measurement result. Because of the Born rule, a state and the POVM give the probability distribution of the measurement results. Thus, we have the freedom to choose the optimal POVM in the quantum estimation process, and we compose a quantum estimator containing the measurement property. That is, we extend an estimator to the self-adjoint operator;

$$(\text{Quantum estimator}) = \sum (\text{estimator}) \cdot (\text{POVM}).$$

Then, it becomes

$$\hat{T} = \sum_i t_i \hat{M}_i, \quad (2.77)$$

where $\{\hat{M}_i\}$ is a POVM. Now, we see the set of the quantum estimators. Let the set of the quantum estimators $\mathcal{T} : \{\hat{T} : \hat{T} = \sum_i t_i \hat{M}_i, \{M_i\} \text{ POVM}, t_i \in \mathbb{R}\}$, then we have $\mathcal{L}_h := \{\hat{A} = \hat{A}^\dagger, \exists N \text{ s.t. } |\hat{A}| < N\}$. Since the POVM operators $\hat{M}_i^\dagger = \hat{M}_i$ are self-adjoint, we have $\hat{T} = \hat{T}^\dagger$. The estimator on the finite Hilbert state $\mathcal{H}$ satisfies $\mathcal{T} \in \mathcal{L}_h$. Conversely, let $\{|\psi_i\rangle\}$ be a set of eigenstates of $\hat{A} \in \mathcal{L}_h$, which is the orthonormal basis of the Hilbert space $\mathcal{H}$. The operator $\hat{A}$ has spectral expansion $\hat{A} = \sum_i a_i \hat{E}_i$, where $\hat{E}_i = |\psi_i\rangle\langle\psi_i|$. Then, the set $\{\hat{E}_i = |\psi_i\rangle\langle\psi_i|\}_i$ can be POVM, and we have $\mathcal{L}_h \in \mathcal{T}$. Therefore, $\mathcal{L}_h = \mathcal{T}$, and we can say that the set of quantum estimators is identical to the set of quantum observables.

2.2.4 Quantum Cramér-Rao bound

In this section, we discuss the quantum Cramér-Rao bound in the single parameter estimation problem [42, 43]. The quantum estimation problems of multiple parameters are difficult, because the quantum estimators are not always commutative with each other. This noncommutativity prohibits simultaneous projection measurement and estimation of the parameters, and it does not allow the simultaneous measurement. The discussion of multiple estimated parameters in the quantum theory is not so simple [42], and we do not discuss multiple parameter estimation in this thesis. We also assume the Hilbert space of the state is finite for simplification. The state $\hat{S}_\theta$ is characterized the parameter θ .

Here, we derive the quantum Cramér-Rao inequality of the mixed state. For simplification, we assume a density matrix $\hat{S}_\theta$ is a full rank matrix. First, we introduce an operator which is extension of a logarithmic derivative $l'_\theta(X)$. Since quantum states are extensions of distribution functions, the operator replacing the logarithmic derivative is thought to be a kind of derivative. Here, assume the $p_\theta(X)$ is a classical distribution function s.t. $p_\theta(X) \in [0, 1]$, and we have

$$l'_\theta(X) = \frac{\partial \log p_\theta(X)}{\partial \theta} = \frac{\partial_\theta p_\theta(X)}{p_\theta(X)},$$

and

$$\partial_\theta p_\theta(X) = \frac{1}{2} [p_\theta(X) l'_\theta(X) + l'_\theta(X) p_\theta(X)]. \quad (2.78)$$

Thus, we introduce a quantum logarithmic derivative L_θ . This operator is defined as a self-adjoint operator such that

$$\frac{\partial \hat{S}_\theta}{\partial \theta} = \frac{1}{2} (\hat{S}_\theta \hat{L}_\theta + \hat{L}_\theta \hat{S}_\theta). \quad (2.79)$$

The above equation becomes an analogy of Eq. (2.78). If $\hat{L}_\theta$ which satisfies Eq. (2.79) is a self-adjoint operator, $\hat{L}_\theta$ is called the symmetric logarithmic derivative (SLD). There are different kinds of quantum informations [44, 45, 46].

Next, we explain the property of the SLD $\hat{L}_\theta$ and its uniqueness. Assume $\mathcal{M}_\theta$ is the set of all operator satisfying Eq. (2.79), and we have the following;

- (a) For $\lambda \in \mathbb{R}$, $\hat{L}_1, \hat{L}_2 \in \mathcal{M}_\theta$, the liner combination of $\hat{L}_1, \hat{L}_2$ satisfies $\lambda\hat{L}_1 + (1 - \lambda)\hat{L}_2 \in \mathcal{M}_\theta$.
- (b) There exists a unique self-adjoint operator in $\mathcal{M}_\theta$.

We can check (a) by substituting it in Eq. (2.79).

We can show (b) as follows. Let a map $\mathcal{TL}$ from arbitrary operator $\hat{L} \in \mathcal{L}_h$ be

$$\mathcal{TL} : \hat{L} \mapsto \frac{1}{2} (\hat{S}_\theta \hat{L} + \hat{L} \hat{S}_\theta). \quad (2.80)$$

Since a state $\hat{S}_\theta$ is a positive operator on a n -dimensional Hilbert space, all the eigenvalues $\{\lambda_i\}_{1 \leq i \leq n}$ of $\hat{S}_\theta$ are positive semi-definite, i.e. $\lambda_i \geq 0$. Let $|\psi\rangle_i$ be the eigenvector belonging to the eigenvalue λ_i . If $\hat{S}_\theta$ has full rank, then the set of the eigenvectors $\{|\psi_i\rangle\}$ become the basis of the Hilbert space $\mathcal{H}$. Let $\hat{K} \in \text{Ker}[\mathcal{TL}]$, and from Eq. (2.79), we have

$$\frac{1}{2} (\hat{K} \hat{S}_\theta + \hat{S}_\theta \hat{K}) = 0, \quad (2.81)$$

and

$$\begin{aligned} 0 &= \langle \psi_i | (\hat{S}_\theta \hat{K} + \hat{K} \hat{S}_\theta) | \psi_j \rangle = \langle \psi_i | \hat{S}_\theta \hat{K} | \psi_j \rangle + \langle \psi_i | \hat{K} \hat{S}_\theta | \psi_j \rangle \\ &= (\lambda_i + \lambda_j) \langle \psi_i | \hat{K} | \psi_j \rangle. \end{aligned} \quad (2.82)$$

Thus, $\langle \psi_i | \hat{K} | \psi_j \rangle = 0$ for any i, j , and we have $\hat{K} = 0$. Thus, we have $\text{Ker}[\mathcal{TL}] = \{0\}$. Therefore, $\mathcal{TL} : \hat{L} \rightarrow \hat{S}_\theta \hat{L} + \hat{L} \hat{S}_\theta$ is a one to one mapping, and there exists the unique self-adjoint operator $\hat{L}_\theta$ satisfying Eq. (2.79). On the other hand, the SLD-Fisher information becomes uniquely determined when the state is pure [47].

Here, we define an inner product with respect to the state $\hat{S}_\theta$ as

$$(\hat{A}, \hat{B})_{\hat{S}_\theta} := \text{Tr}[\hat{S}_\theta \hat{B} \hat{A}^\dagger]. \quad (2.83)$$

Using the Schwarz inequality, we have

$$\begin{aligned} &(\hat{T} - \theta_0 \hat{I}, \hat{T} - \theta_0 \hat{I})_{\hat{S}_\theta} (\hat{L}_\theta, \hat{L}_\theta)_{\hat{S}_\theta} \geq |(\hat{T} - \theta_0 \hat{I}, \hat{L}_\theta)_{\hat{S}_\theta}|^2 \\ &\geq [\text{Re}(\hat{T} - \theta_0 \hat{I}, \hat{L}_\theta)_{\hat{S}_\theta}]^2 = \left\{ \text{Tr} \left[\frac{1}{2} (\hat{S}_\theta \hat{L}_\theta + \hat{L}_\theta^\dagger \hat{S}_\theta) (\hat{T} - \theta_0 \hat{I}) \right] \right\}^2 \\ &= \left\{ \text{Tr} \left[\frac{\partial \hat{S}_\theta}{\partial \theta} (\hat{T} - \theta_0 \hat{I}) \right] \right\} = \left\{ \text{Tr} \left[\frac{\partial \hat{S}_\theta}{\partial \theta} \hat{T} \right] - \theta_0 \frac{\partial}{\partial \theta} (\text{Tr}[\hat{S}_\theta]) \right\}^2 \\ &= 1. \end{aligned} \quad (2.84)$$

Then, we have the quantum Cramér-Rao inequality as

$$(\hat{T} - \theta_0 \hat{I}, \hat{T} - \theta_0 \hat{I})_{\hat{S}_\theta} = \left\langle (\hat{T} - \theta_0 \hat{I})^2 \right\rangle_{\hat{S}_\theta} \geq 1/I_{\text{SLD}}(\hat{S}_\theta), \quad (2.85)$$

where

$$I_{\text{SLD}}(\hat{S}_\theta) := (\hat{L}_\theta, \hat{L}_\theta)_{\hat{S}_\theta} \quad (2.86)$$

is called SLD-Fisher information. If

$$\hat{T} = \theta_0 \hat{I} + \frac{\hat{L}_\theta}{I_{\text{SLD}}(\hat{S}_\theta)}, \quad (2.87)$$

then the equality in Eq. (2.85) holds. In parametric model, the Cramér-Rao inequality (2.85) gives the best accuracy of the estimate parameter θ that we can obtain when we measure with the optimal POVM. Although there are some other estimators [44, 45, 46], the SLD-Fisher information gives the most tight bound.

Let us review the derivation of uncertainty relations from the quantum Cramér-Rao bound. From the inequality (2.85), we also have the uncertainty relation of time and energy. Assume

$$\hat{S}_t = e^{-it\hat{H}} \hat{S}_0 e^{it\hat{H}}, \quad (2.88)$$

and a logarithmic derivative becomes

$$\hat{L}_t = 2i \left(\hat{H} - \langle \hat{H} \rangle_{\hat{S}_t} \hat{I} \right), \quad (2.89)$$

where $\langle \hat{H} \rangle_{\hat{S}_t} = \text{Tr} [\hat{H} \hat{S}_t]$. Note this $\hat{L}$ is not SLD. The quantum Fisher information becomes

$$\left\langle \hat{L}_t^2 \right\rangle_{\hat{S}_t} = \frac{4}{\hbar^2} \left(\langle \hat{H}^2 \rangle_{\hat{S}_t} - \langle \hat{H} \rangle_{\hat{S}_t}^2 \right). \quad (2.90)$$

Assume $\hat{T}$ is an unbiased estimator,

$$\langle \hat{T} \rangle_{\hat{S}_t} = t.$$

Then, we have the uncertainty relation of time and energy;

$$\left\langle \Delta \hat{T}^2 \right\rangle_{\hat{S}_t} \geq \frac{4}{\hbar^2} \left\langle \Delta \hat{H}^2 \right\rangle_{\hat{S}_t}, \quad (2.91)$$

where

$$\Delta \hat{T} := \hat{T} - \langle \hat{T} \rangle_{\hat{S}_t}, \quad \Delta \hat{H} := \hat{H} - \langle \hat{H} \rangle_{\hat{S}_t}.$$

2.2.5 Convergence of the estimator in quantum theory

Equation (2.87) gives an estimator that achieves the quantum Cramér-Rao bound. The optimal measurement has the POVM which is given as a set of projection operators of the quantum estimator (2.87). However, the estimator (2.87) depends on the unknown estimated parameter, and realization of this measurement seems to be difficult. But, Nagaoka and Fujiwara showed that there is an adaptive measurement scheme that allows achieving the Quantum Cramér-Rao bound [48, 49];

Nagaoka-Fujiwara's scheme

1. Choose arbitrary POVM $\hat{\mathbf{M}} = \{\hat{M}_1(x)\}_x$ and measure the system with it. Calculate the distribution function $f_1(x_1|\theta) = \text{Tr}[S_\theta \hat{M}_1(x)]$ and MLE such that $\theta_1 = \text{argmax}_\theta \ln f_1(x|\theta)$.
2. Substitute $\theta = \theta_1$ into $\hat{S}_\theta$, and calculate SLD $\hat{L}_1$ of $\hat{S}_{\theta_1}$. Measure the system with the PVM $\hat{\mathbf{M}}_2 = \{\hat{M}_2(x)\}_x$ which is determined by the spectral decomposition of $\hat{L}_1 = \sum_x \lambda_x \hat{M}_2(x)$. Use $\hat{\mathbf{M}}_2$ and the result of measurement $\hat{\mathbf{M}}_2$, compute the distribution function and the MLE $\theta_2 = \text{argmax}_\theta \ln f_2(x_2|\theta)$.
3. Repeat 2.

Using this adaptive measurement, we make the estimator converge to the true value. The detailed proof is given in Ref. [49].

Chapter 3

Adaptive measurement for state generation

In Chapter 2, we summarize the state transition caused by continuous measurement. Generally, it is known that the projective measurement creates any pure state in probability, and the quantum state preparation is an important technique for the quantum information technology [24]. It is shown that non-project measurement can also create any pure states on a quantum two-level system with numerical simulations [23, 50]. Ashhab and his co-worker proposed a POVM set that contains the four POVM elements, and they showed that their POVM can generate almost all states on the Bloch surface by the numerical simulation [50]. Jacobs proposed the state preparation method by using continuous measurement [23, 51, 52]. He proposed a continuous measurement scheme for state preparation. In his scheme, the back action of the quantum measurement plays an important role. He got this idea from the convergence of the solution in a Fokker-Planck equation. He showed that the random transition of state caused by measurement allows us to control a physical state.

Here, we propose an improvement of his scheme, and we also evaluate his and our scheme. Our scheme can be used even in the presence of unknown external force, and it is faster and more robust than Jacobs' scheme. We demonstrate the robustness of our scheme by numerical simulation [22].

3.1 Dynamics of Jacobs' state generation by adaptive continuous measurement

First, we review the state preparation scheme using a continuous measurement proposed by Jacobs. We derive an evolution equation for the angular distance between the target and the current state from the statistical master equation. After that, we propose an improvement of Jacobs' proposal, and compare the properties of the two schemes with external force.

3.1.1 The model and the time evolution caused by continuous measurement

The time evolution of the state continuously under the continuous measurement of the observable $\hat{\sigma}$ is given by Eq. (2.40) [23, 51, 52];

$$d\hat{\rho} = -k[\hat{\sigma}, [\hat{\sigma}, \hat{\rho}]]dt + \sqrt{2k}(\hat{\sigma}\hat{\rho} + \hat{\rho}\hat{\sigma} - 2\text{Tr}[\hat{\sigma}\hat{\rho}]\hat{\rho})dw, \quad (3.1)$$

where k denotes the measurement strength and w is a random variable following Wiener process [53, 54]. The small difference dw follows Ito rule;

$$dw^2 = dt, dw dt = 0. \quad (3.2)$$

While dt means the deterministic time evolution, dw represents the time evolution of random noise.

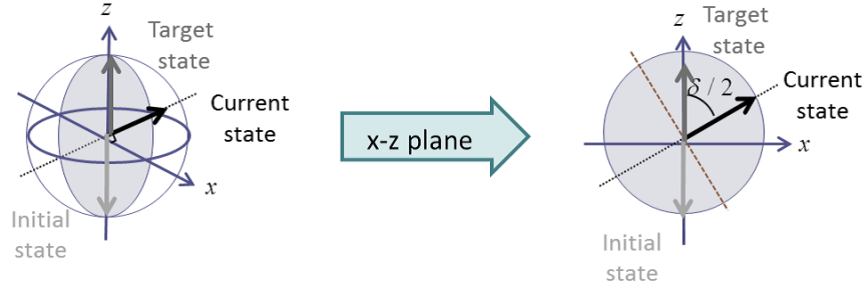


Figure 3.1: The initial and target states on the Bloch sphere. We treat only the pure state on the x - z plane in the Bloch sphere.

Next, we simplify the model in order to make the numerical simulation easier. Since the continuous measurement taken over a long period of time increases purity of the state [55], we focus on pure states on the x - z plane. Then, we denote the state as $|\psi\rangle = \cos(\delta/2)|0\rangle + \sin(\delta/2)|1\rangle$, and its density matrix is

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2} \left(\hat{I} + \hat{\sigma}_z \cos(\delta) + \hat{\sigma}_x \sin(\delta) \right). \quad (3.3)$$

Here, the parameter δ gives the angular distance between the current state $|\psi\rangle$ and the target state $|0\rangle$ (See Fig. 3.1). Without losing of generality, we set the initial state to $|1\rangle$ ($\hat{\sigma}_z|0\rangle = |0\rangle$, $\hat{\sigma}_z|1\rangle = -|1\rangle$). We use the notation of two leveled system, or qubit system which widely is used in the quantum information. In the time evolution Eq. (3.1), the small differential dt causes continuous and smooth state transition, while dw causes continuous but non-smooth transition of $\hat{\rho}$. The random variable w follows the Wiener process,

and its time evolution is continuous. Therefore, it causes the continuous state transition. This continuity allows us to mathematical analysis. That is, we can create any pure state if we can create the state which is orthogonal to the initial state. The orthogonal state is the farthest state, and the continuity of the state transition assures that the system passes through any pure states on the x - z plane. This is reason that we choose the target state as $|0\rangle$.

The state generation scheme proposed by Jacob [23] has the two important elements for control, the measured observable $\hat{\sigma}$ and the strength $k(\delta)$. These values are adaptively changed depending on the preceding measurement result. In Eq. (3.1), we chose the appropriate observable $\hat{\sigma}$ and strength $k(\delta)$ in each time in order to make δ achieving 0. We compute the angular distance δ instead of the state developing with Eq. (3.1). Here, the fidelity of the current state and the target state becomes $\sqrt{\langle 0|\rho|0\rangle} = |\langle 0|\psi\rangle| = \cos(\delta/2)$, and the transition probability becomes $|\langle 0|\psi\rangle|^2 = (1 + \cos(\delta))/2$. Thus, we will make numerical simulation of the time evolution with δ .

By using the time evolutions of the angular distances δ , we will have the detail discussion of Jacobs' state generation and our improved scheme in Sec. 3.1.2 and 3.1.3. We compare the numerical results of these two schemes in Sec. 3.2.1. The difference of the two schemes appears in the robustness against external force (we show it in Sec. 3.2.1). In order to make clear the difference between Jacobs' and our scheme, we consider the time evolution with a Hamiltonian $\hat{H} = \Delta\hat{\sigma}_y$ which disturbs the control by the measurement, and we discuss the following two cases;

1. The strength of the Hamiltonian Δ is known,
2. The strength of the Hamiltonian Δ is unknown.

First, when the strength of the Hamiltonian is given, we simply add the time evolution term $-i[\hat{H}, \hat{\sigma}_y]dt$ of the Hamiltonian to the master equation, and we have

$$d\hat{\rho} = -i\Delta[\hat{\rho}, \hat{\sigma}_y]dt - k[\hat{\sigma}, [\hat{\sigma}, \hat{\rho}]]dt + \sqrt{2k}(\hat{\sigma}\hat{\rho} + \hat{\rho}\hat{\sigma} - 2\text{Tr}[\hat{\sigma}\hat{\rho}]\hat{\rho})dw. \quad (3.4)$$

Next, we introduce the time evolution when the force strength Δ is unknown. The dynamics of this situation is given by the control theory [53, 54]. The control with prediction causes the estrangement between the true physical system and our prediction. The state that we predicted through controlling is called *nominal* state. Here, we consider the case where we do not know the true strength Δ but we have a prediction of it Δ' , and we use Δ' for the prediction of the time evolution. The nominal state $\hat{\rho}'$ determines the observable $\hat{\sigma}$ and the measurement strength k . The time evolution of the true state $\hat{\rho}$ is

$$d\hat{\rho} = -i\Delta'[\hat{\rho}, \hat{\sigma}_y]dt - k[\hat{\sigma}, [\hat{\sigma}, \hat{\rho}]]dt + \sqrt{2k}(\hat{\sigma}\hat{\rho} + \hat{\rho}\hat{\sigma} - 2\text{Tr}[\hat{\sigma}\hat{\rho}]\hat{\rho})dw. \quad (3.5)$$

This equation looks the same to Eq.(3.4), but its transition is different from Eq. (3.4) because $\hat{\sigma}$ depends on the nominal values $\hat{\rho}'$ and Δ' . The time evolution of

the nominal state $\hat{\rho}'$ is

$$d\hat{\rho}' = -i\Delta'[\hat{\rho}', \hat{\sigma}_y]dt - k[\hat{\sigma}, [\hat{\sigma}, \hat{\rho}']]dt + \sqrt{2k}(\hat{\sigma}\hat{\rho}' + \hat{\rho}'\hat{\sigma} - 2\text{Tr}[\hat{\sigma}\hat{\rho}']\hat{\rho}')(dy - \sqrt{2k}\text{Tr}[\hat{\sigma}\hat{\rho}']dt), \quad (3.6)$$

where a measurement result y is a Gaussian random variable which fluctuates around an eigenvalue of $\hat{\sigma}$. The time evolution of y is following [52]:

$$dy = \sqrt{2k}\text{Tr}[\hat{\sigma}\hat{\rho}]dt + dw. \quad (3.7)$$

The difference between the true state $\hat{\rho}$ and the nominal state $\hat{\rho}'$ causes the disorder of the control. There is qualitative difference between Jacobs' and our scheme in these cases. We see this with numerical simulations in the following sections.

3.1.2 Jacobs' adaptive measurement scheme for state preparation

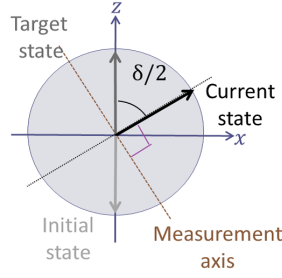


Figure 3.2: The eigenstate of Jacobs' measured observable. The eigenstates of $\hat{\sigma}_J$ are always orthogonal to the current state, and the magnitude of the measurement strength needs to be decreased as the current state approaches to the target.

In Ref. [23], Jacobs' proposed an adaptive measurement scheme for state preparation. He gives the measured observable $\hat{\sigma}$ and the measurement strength k as follows;

Observable and strength of Jacobs' scheme

$$\hat{\sigma}_J = \cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z, \quad (3.8)$$

and the measurement strength k also depending on the distance δ between the target and the state at t ;

$$k = \kappa\delta^2. \quad (3.9)$$

Now, we derive the time evolution equation of δ from Eq. (3.1) when $\Delta = 0$. The difference δ depends on $\hat{\rho}$, and it also depend on t and w . Thus, we have

$$d\delta = A dt + B dw. \quad (3.10)$$

From Eq. (3.3), we have

$$\begin{aligned} d\hat{\rho} &= \hat{\rho}(t + dt) - \hat{\rho}(t) \\ &= \frac{1}{2} \left[\left\{ \cos(\delta)d\delta - \frac{1}{2} \sin(\delta)\hat{\sigma}_x(d\delta)^2 \right\} \hat{\sigma}_x + \left\{ -\sin(\delta)d\delta - \frac{1}{2} \cos(\delta)\hat{\sigma}_z(d\delta)^2 \right\} \hat{\sigma}_z \right] \\ &= \frac{1}{2} \left\{ \left(\cos(\delta)A - \frac{B^2}{2} \sin(\delta) \right) \hat{\sigma}_x + \left(-\sin(\delta)A - \frac{B^2}{2} \cos(\delta) \right) \hat{\sigma}_z \right\} dt \\ &\quad + \frac{B}{2} \{ \cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z \} dw. \end{aligned} \quad (3.11)$$

On the other hand, the coefficient of dt is

$$\begin{aligned} [\hat{\sigma}_J, [\hat{\sigma}_J, \hat{\rho}]] &= 2(\hat{\rho} - \hat{\sigma}_J \hat{\rho} \hat{\sigma}_J) \\ &= \hat{I} + \sin(\delta)\hat{\sigma}_x + \cos(\delta)\hat{\sigma}_z \\ &\quad - \hat{I} - (\cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z)(\sin(\delta)\hat{\sigma}_x + \cos(\delta)\hat{\sigma}_z)(\cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z) \\ &= \sin(\delta)\hat{\sigma}_x - (\cos^2(\delta)\sin(\delta) - \sin^2(\delta)\sin(\delta)^2 - 2\sin(\delta)\cos(\delta)\sin(\delta))\hat{\sigma}_x \\ &\quad + \cos(\delta)\hat{\sigma}_z - (-2\sin(\delta)\cos(\delta)\sin(\delta) - \cos^2(\delta)\cos(\delta) - \sin^2(\delta)\cos(\delta)^2)\hat{\sigma}_z \\ &= (\sin(\delta) + \sin(2\delta - \delta))\hat{\sigma}_x + (\cos(\delta) + \cos(2\delta - \delta))\hat{\sigma}_z \\ &= 2(\sin(\delta)\hat{\sigma}_x + \cos(\delta)\hat{\sigma}_z), \end{aligned} \quad (3.12)$$

the coefficient of dw is

$$\begin{aligned} &\hat{\rho}\hat{\sigma}_J + \hat{\sigma}_J\hat{\rho} - 2\text{Tr}[\hat{\rho}\hat{\sigma}_J]\hat{\rho} \\ &= \hat{\rho}\hat{\sigma}_J + \hat{\sigma}_J\hat{\rho} - \text{Tr}[\hat{\rho}\hat{\sigma}_J + \hat{\sigma}_J\hat{\rho}]\hat{\rho} \\ &= \frac{1}{2} \left\{ \begin{aligned} &(\hat{I} + \sin(\delta)\hat{\sigma}_x + \cos(\delta)\hat{\sigma}_z)(\cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z) \\ &+ (\cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z)(\hat{I} + \sin(\delta)\hat{\sigma}_x + \cos(\delta)\hat{\sigma}_z) \end{aligned} \right\} - \text{Tr}[\hat{\rho}\hat{\sigma}_J + \hat{\sigma}_J\hat{\rho}]\hat{\rho} \\ &= \frac{1}{2} \left(\begin{aligned} &\cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z + \cos^2(\delta)\hat{\sigma}_z\hat{\sigma}_x \\ &- \cos(\delta)\sin(\delta)\hat{I} + \cos(\delta)\sin(\delta)\hat{I} - \sin^2(\delta)\hat{\sigma}_x\hat{\sigma}_z \\ &+ \cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z + \cos^2(\delta)\hat{\sigma}_x\hat{\sigma}_z \\ &- \cos(\delta)\sin(\delta)\hat{I} + \cos(\delta)\sin(\delta)\hat{I} - \sin^2(\delta)\hat{\sigma}_z\hat{\sigma}_x \end{aligned} \right) - 0 \cdot \hat{\rho} \\ &= \cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z. \end{aligned} \quad (3.13)$$

From Eqs. (3.11) and (3.13), we have

$$\begin{aligned}\frac{B}{2} \cos(\delta) &= \sqrt{2k} \cos(\delta), \\ \frac{B}{2} \sin(\delta) &= \sqrt{2k} \sin(\delta),\end{aligned}\tag{3.14}$$

and we obtain $B = 2\sqrt{2k}$. From Eqs. (3.11) and (3.12),

$$\begin{aligned}\frac{1}{2} \left(A \cos(\delta) - \frac{B^2}{2} \sin(\delta) \right) &= -2k \sin(\delta), \\ \frac{1}{2} \left(-A \sin(\delta) - \frac{B^2}{2} \cos(\delta) \right) &= -2k \cos(\delta),\end{aligned}\tag{3.15}$$

thus we have $A = 0$. As a result we have the time evolution equation of δ as follows;

Dynamics of Jacobs' scheme —

$$d\delta = \sqrt{8k(\delta)}dw = \sqrt{8\kappa}\delta dw.\tag{3.16}$$

Here, the coefficient of dw gives the speed of the probabilistic time evolution. We can use it for an index to evaluate the speed of the probabilistic time evolution, and denote $\gamma_{sJ} = \sqrt{8\kappa}$.

Next, we derive the time evolution equation when the external forth Δ is non-zero but it is known. The term to time evolution by the forth becomes

$$-i\Delta[\hat{\sigma}_y, \hat{\rho}] = -\frac{i}{2}(\hat{\sigma}_y + \sin(\delta)\hat{\sigma}_x + \cos(\delta)\hat{\sigma}_z) = \cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z,\tag{3.17}$$

and Eq. (3.14) does not change. Then, Eq. (3.15) becomes

$$\begin{aligned}\frac{1}{2} \left(A \cos(\delta) - \frac{B^2}{2} \sin(\delta) \right) &= -2k \sin(\delta) + \Delta \cos(\delta), \\ \frac{1}{2} \left(-A \sin(\delta) - \frac{B^2}{2} \cos(\delta) \right) &= -2k \cos(\delta) - \Delta \sin(\delta),\end{aligned}\tag{3.18}$$

and we have $A = 2\Delta$. Therefore we have the time evolution equation of δ with the forth whose strength is known as follows;

Dynamics of Jacobs' scheme with external force —

$$d\delta = -2\Delta dt + \sqrt{8\kappa}\delta dw.\tag{3.19}$$

We also derive the time evolution in the case when the strength Δ is unknown. Here, we have two kind of state, the true and the nominal state. The true state means the actual state in physical systems, while the nominal state

is a fictitious state predicted from the measurement results. The time evolution of the true state is

$$\hat{\rho} = \frac{1}{2} \left(\hat{I} + \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \right). \quad (3.20)$$

On the other hand, the time evolution of the nominal state is

$$\hat{\rho}' = \frac{1}{2} \left(\hat{I} + \sin(\delta') \hat{\sigma}_x + \cos(\delta') \hat{\sigma}_z \right), \quad (3.21)$$

where δ' is the nominal distance between the nominal state and the target. Since we can not predict the true state, the observable depending on the state $\hat{\rho}'$ is

$$\hat{\sigma}_J = \cos(\delta') \hat{\sigma}_x - \sin(\delta') \hat{\sigma}_z \quad (3.22)$$

and k_J is

$$k_J = \kappa \delta'. \quad (3.23)$$

Then, the time evolution Eqs. (3.5) and (3.6) become

$$\begin{aligned} d\hat{\rho} &= \Delta (\cos(\delta) \hat{\sigma}_x - \sin(\delta) \hat{\sigma}_z) dt \\ &\quad - k (\{\sin(\delta) + \sin(2\delta' - \delta)\} \hat{\sigma}_x + \{\cos(\delta) + \cos(2\delta' - \delta)\} \hat{\sigma}_z) dt \\ &\quad + \sqrt{2k} (\{\cos(\delta') - \sin(\delta' - \delta) \sin(\delta)\} \hat{\sigma}_x + \{-\sin(\delta') - \sin(\delta' - \delta) \cos(\delta)\} \hat{\sigma}_z) dw, \\ d\hat{\rho}' &= \Delta' (\cos(\delta') \hat{\sigma}_x - \sin(\delta') \hat{\sigma}_z) dt - 2k (\sin(\delta') \hat{\sigma}_x + \cos(\delta') \hat{\sigma}_z) dt \\ &\quad + \sqrt{2k} (\cos(\delta') \hat{\sigma}_x - \sin(\delta')) (dy - \text{Tr}[\hat{\sigma} \hat{\rho}'] dt). \end{aligned} \quad (3.24)$$

Here, we derive the time evolution equations of the distance δ and δ' . These equations are denoted as

$$d\delta = A dt + B dw, \quad (3.26)$$

$$d\delta' = C dt + D (dy - \text{Tr}[\hat{\sigma} \hat{\rho}'] dt). \quad (3.27)$$

The time evolution equations of $\hat{\rho}$ and $\hat{\rho}'$ are

$$\begin{aligned} d\hat{\rho} &= \left(\left\{ A \cos(\delta) - \frac{B^2}{2} \sin(\delta) \right\} \hat{\sigma}_x + \left\{ -A \sin(\delta) - \frac{B^2}{2} \cos(\delta) \right\} \hat{\sigma}_z \right) dt \\ &\quad + B (\cos(\delta) \hat{\sigma}_x - \sin(\delta) \hat{\sigma}_z) dw, \end{aligned} \quad (3.28)$$

$$\begin{aligned} d\hat{\rho}' &= \left(\left\{ A \cos(\delta') - \frac{B^2}{2} \sin(\delta') \right\} \hat{\sigma}_x + \left\{ -A \sin(\delta') - \frac{B^2}{2} \cos(\delta') \right\} \hat{\sigma}_z \right) dt \\ &\quad + B (\cos(\delta') \hat{\sigma}_x - \sin(\delta') \hat{\sigma}_z) (dy - \text{Tr}[\hat{\sigma} \hat{\rho}'] dt). \end{aligned} \quad (3.29)$$

Thus, the time evolution equations of $\hat{\rho}$ and $\hat{\rho}'$ are also denoted with the parameters $\tilde{d}_x, \tilde{d}_z, \tilde{h}_x$, and $\tilde{h}_z$ as follows:

$$d\hat{\rho} = (\tilde{d}_x \hat{\sigma}_x + \tilde{d}_z \hat{\sigma}_z) dt + (\tilde{h}_x \hat{\sigma}_x + \tilde{h}_z \hat{\sigma}_z) dw, \quad (3.30)$$

$$d\hat{\rho}' = (\tilde{d}'_x \hat{\sigma}_x + \tilde{d}'_z \hat{\sigma}_z) dt + (\tilde{h}'_x \hat{\sigma}_x + \tilde{h}'_z \hat{\sigma}_z) (dy - \text{Tr}[\hat{\sigma} \hat{\rho}'] dt). \quad (3.31)$$

Comparing Eqs.(3.28) and (3.29) with Eqs. (3.30) and (3.31) , we have

$$2\tilde{d}_x = A \cos(\delta) - \frac{B^2}{2} \sin(\delta), \quad 2\tilde{h}_x = B \cos(\delta), \quad (3.32)$$

$$2\tilde{d}_z = -A \sin(\delta) - \frac{B^2}{2} \cos(\delta), \quad 2\tilde{h}_z = -B \sin(\delta), \quad (3.33)$$

$$2\tilde{d}'_x = C \cos(\delta') - \frac{D^2}{2} \sin(\delta'), \quad 2\tilde{h}'_x = D \cos(\delta'), \quad (3.34)$$

$$2\tilde{d}'_z = -C \sin(\delta') - \frac{D^2}{2} \cos(\delta'), \quad 2\tilde{h}'_z = -D \sin(\delta'). \quad (3.35)$$

Calculating $\text{Tr}[\hat{\sigma}_x \cdot (3.28)]$ and $\text{Tr}[\hat{\sigma}_x \cdot (3.30)]$, we have Eqs. (3.32). In the same way, we have Eqs. (3.33), (3.34), and (3.35). Solving these equations for A, B, C , and D , we get

$$\begin{aligned} A &= 2 \left(\cos(\delta) \tilde{d}_x - \sin(\delta) \tilde{d}_z \right), \quad B = 2 \left(\cos(\delta) \tilde{h}_x - \sin(\delta) \tilde{h}_z \right), \\ C &= 2 \left(\cos(\delta') \tilde{d}'_x - \sin(\delta') \tilde{d}'_z \right), \quad D = 2 \left(\cos(\delta') \tilde{h}'_x - \sin(\delta') \tilde{h}'_z \right). \end{aligned} \quad (3.36)$$

From Eqs. (3.24), (3.25) and (3.36), we have

$$\begin{aligned} A &= 2 \left[\begin{aligned} &\Delta - k(\sin(\delta) \cos(\delta) - \sin(\delta) \cos(\delta)) \\ &+ \sin(2\delta' - \delta) \cos(\delta) - \sin(2\delta' - \delta) \sin(\delta) \end{aligned} \right] \\ &= 2 [\Delta - k \sin(2\delta' - 2\delta),] \end{aligned} \quad (3.37)$$

$$B = 2\sqrt{2k} [\cos(\delta) \cos(\delta') + \sin(\delta) \sin(\delta')] = 2\sqrt{2k} \cos(\delta - \delta'), \quad (3.38)$$

$$C = 0, \quad (3.39)$$

$$D = 2\sqrt{2k}. \quad (3.40)$$

Thus, we have

$$\begin{aligned} dy - \sqrt{2k} \text{Tr}[\hat{\sigma} \hat{\rho}'] dt &= \sqrt{2k} \text{Tr} [\hat{\sigma} (\hat{\rho} - \hat{\rho}')] dt + dw \\ &= \sqrt{2k} \text{Tr} [(\cos(\delta') \hat{\sigma}_x - \sin(\delta') \hat{\sigma}_z) (\{\sin(\delta) - \sin(\delta')\} \hat{\sigma}_x + \{\cos(\delta) - \cos(\delta')\} \hat{\sigma}_z)] dt \\ &\quad + dw \\ &= \sqrt{2k} \sin(\delta' - \delta) dt + dw. \end{aligned}$$

Therefore, the time evolutions of δ and δ' become as follows;

Dynamics of Jacobs' scheme with unknown external force

$$\begin{aligned} d\delta &= 2\Delta dt - 2\kappa \delta' \sin(2\delta' - 2\delta) dt \\ &\quad + 2\sqrt{2\kappa} \delta' \cos(\delta - \delta') dw, \end{aligned} \quad (3.41)$$

$$d\delta' = 2\Delta' dt + 2\sqrt{2\kappa} \delta' (\sin(\delta' - \delta) dt + dw). \quad (3.42)$$

3.1.3 Improved adaptive measurement scheme

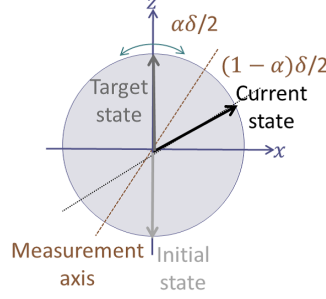


Figure 3.3: The eigenstate of the measured observable in our scheme. The eigenstates of $\hat{\sigma}_R$ divide $\alpha : 1 - \alpha$ into the target state and the current, and continuous measurement of this observable drags the current state to the target state.

We use a different observable from Jacobs' one, and propose a new scheme [22]. The our observable is as follows;

Observable and strength of our scheme

$$\begin{aligned}\hat{\sigma}_R &= \hat{\sigma}_x \sin(\alpha\delta) - \hat{\sigma}_z \cos(\alpha\delta), \\ \beta &:= 1 - \alpha.\end{aligned}\tag{3.43}$$

The eigenvectors of this observable divide the angle between the target and the current state at t in the ratio $(1 - \alpha) : \alpha$. Here, α denotes the strength pulling the system to the target. This measurement scheme can keep the measurement strength constant $k_R = k$ during control, while Jacobs' one cannot. In this sense, our scheme is relatively easier to realize.

We can derive the time evolution equation of δ from Eq. (3.1) in the same

way as Jacobs'. We have

$$\begin{aligned}
 & [\hat{\sigma}_R, [\hat{\sigma}_R, \hat{\rho}]] \\
 &= 2(\hat{\rho} - \hat{\sigma}_R \hat{\rho} \hat{\sigma}_R) \\
 &= \hat{I} + \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \\
 &\quad - \hat{I} - (\sin(\alpha\delta) \hat{\sigma}_x + \cos(\alpha\delta) \hat{\sigma}_z) (\sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z) (\sin(\alpha\delta) \hat{\sigma}_x + \cos(\alpha\delta) \hat{\sigma}_z) \\
 &= \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \\
 &\quad - \sin(\delta) (\sin^2(\alpha\delta) \hat{\sigma}_x - \cos^2(\alpha\delta) \hat{\sigma}_x + 2 \sin(\alpha\delta) \cos(\alpha\delta) \hat{\sigma}_z) \\
 &\quad - \cos(\delta) (-\sin^2(\alpha\delta) \hat{\sigma}_z + \cos^2(\alpha\delta) \hat{\sigma}_z + 2 \sin(\alpha\delta) \cos(\alpha\delta) \hat{\sigma}_x) \\
 &= \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \\
 &\quad - (-\sin(\delta) \cos(\alpha\delta) + \cos(\delta) \sin(\alpha\delta)) \hat{\sigma}_x - (\cos(\delta) \cos(\alpha\delta) + \sin(\delta) \sin(\alpha\delta)) \hat{\sigma}_z \\
 &= (\sin(\delta) - \sin(2\alpha\delta - \delta)) \hat{\sigma}_x + (\cos(\delta) - \cos(2\alpha\delta - \delta)) \hat{\sigma}_z, \tag{3.45}
 \end{aligned}$$

and

$$\begin{aligned}
 & \hat{\rho} \hat{\sigma}_R + \hat{\sigma}_R \hat{\rho} - 2 \text{Tr}[\hat{\sigma}_R \hat{\rho}] \hat{\rho} \\
 &= \frac{1}{2} \left\{ \begin{aligned} & (\sin(\alpha\delta) \hat{\sigma}_x + \cos(\alpha\delta) \hat{\sigma}_z) \left(\hat{I} + \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \right) \\ & + \left(\hat{I} + \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \right) (\sin(\alpha\delta) \hat{\sigma}_x + \cos(\alpha\delta) \hat{\sigma}_z) \end{aligned} \right\} \\
 &\quad - \text{Tr}[\hat{\sigma}_R \hat{\rho} + \hat{\rho} \hat{\sigma}_R] \hat{\rho} \\
 &= (\sin(\alpha\delta) \cos(\delta) + \cos(\alpha\delta) \sin(\delta)) \hat{I} + \sin(\alpha\delta) \hat{\sigma}_x + \cos(\alpha\delta) \hat{\sigma}_z \\
 &\quad - \text{Tr}[\hat{\sigma}_R \hat{\rho} + \hat{\rho} \hat{\sigma}_R] \hat{\rho} \\
 &= \cos(\alpha\delta - \delta) \hat{I} + \sin(\alpha\delta) \hat{\sigma}_x + \cos(\alpha\delta) \hat{\sigma}_z \\
 &\quad - 2 \cos(\alpha\delta - \delta) \frac{1}{2} \left(\hat{I} + \sin(\delta) \hat{\sigma}_x + \cos(\delta) \hat{\sigma}_z \right) \\
 &= (\sin(\alpha\delta) - \sin(\delta) \cos(\alpha\delta - \delta)) \hat{\sigma}_x + (\cos(\alpha\delta) - \cos(\delta) \cos(\alpha\delta - \delta)) \hat{\sigma}_z. \tag{3.46}
 \end{aligned}$$

From Eqs. (3.11) and (3.46), we have

$$\begin{aligned}
 \frac{B}{2} \cos(\delta) &= \sqrt{2k} (\sin(\alpha\delta) - \sin(\delta) \cos(\alpha\delta - \delta)), \\
 -\frac{B}{2} \sin(\delta) &= \sqrt{2k} (\cos(\alpha\delta) - \cos(\delta) \cos(\alpha\delta - \delta)), \tag{3.47}
 \end{aligned}$$

and we have $B = 2\sqrt{2k} \sin(\delta - \alpha\delta)$. Then, From Eqs. (3.11) and (3.45), we have

$$\begin{aligned}
 \frac{1}{2} \left(A \cos(\delta) - \frac{B^2}{2} \sin(\delta) \right) &= -2k (\sin(\alpha\delta) - \sin(2\alpha - \delta)), \\
 \frac{1}{2} \left(-A \sin(\delta) - \frac{B^2}{2} \cos(\delta) \right) &= -2k (\cos(\alpha\delta) - \cos(2\alpha - \delta)), \tag{3.48}
 \end{aligned}$$

and we have $A = k \sin(2\alpha\delta - 2\delta)$. Therefore, we have the time evolution of δ as follows;

Dynamics of our scheme with external force

$$\begin{aligned} d\delta &= -2k \sin(2\beta\delta)dt + \sqrt{8k} \sin(\beta\delta)dw, \\ \beta &:= 1 - \alpha. \end{aligned} \quad (3.49)$$

Here, simplifying this equation by the 1st order approximation of δ , we have

$$d\delta = -2k2\beta\delta dt + \sqrt{8k}\beta\delta dw, \quad (3.50)$$

The coefficient $\gamma_{sR} := \sqrt{8k}\beta$ of dw means the strength of probabilistic time evolution.

Next, we see the time evolution with the external force Δ . Deriving in the same way as Jacobs' scheme, we have following equation;

Dynamics of our scheme with external force

$$d\delta = -2\{\Delta + k \sin(2\beta\delta)\}dt + \sqrt{8k} \sin(\beta\delta)dw. \quad (3.51)$$

Finally, we introduce the time evolution of the case that the strength of Hamiltonian Δ , is unknown. We think it also the same way as Jacobs' scheme. The measured observable becomes

$$\hat{\sigma}_R = \sin(\delta')\hat{\sigma}_x + \cos(\delta')\hat{\sigma}_z, \quad (3.52)$$

while the measurement strength k does not change. Then, we have

$$\begin{aligned} d\hat{\rho} &= \Delta (\cos(\delta)\hat{\sigma}_x - \sin(\delta)\hat{\sigma}_z) dt \\ &\quad - k (\{\sin(\delta) - \sin(2\alpha\delta' - \delta)\}\hat{\sigma}_x + \{\cos(\delta) - \cos(2\alpha\delta' - \delta)\}\hat{\sigma}_z) dt \\ &\quad + \sqrt{2k} \left(\begin{array}{l} \{-\sin(\alpha\delta) + \cos(2\alpha\delta' - \delta)\sin(\delta)\}\hat{\sigma}_x \\ + \{-\cos(\alpha\delta) - \cos(2\alpha\delta' - \delta)\cos(\delta)\}\hat{\sigma}_z \end{array} \right) dw, \end{aligned} \quad (3.53)$$

$$\begin{aligned} d\hat{\rho}' &= \Delta (\cos(\delta')\hat{\sigma}_x - \sin(\delta')\hat{\sigma}_z) dt \\ &\quad - k (\{\sin(\delta') - \sin(2\alpha\delta' - \delta)\}\hat{\sigma}_x + \{\cos(\delta') - \cos(2\alpha\delta' - \delta')\}\hat{\sigma}_z) dt \\ &\quad + \sqrt{2k} \left(\begin{array}{l} \{-\sin(\alpha\delta') + \cos(2\alpha\delta' - \delta')\sin(\delta')\}\hat{\sigma}_x \\ + \{-\cos(\alpha\delta') - \cos(2\alpha\delta' - \delta')\cos(\delta')\}\hat{\sigma}_z \end{array} \right) (dy - \text{Tr}[\hat{\sigma}'\hat{\rho}']dt), \end{aligned} \quad (3.54)$$

and

$$\begin{aligned} &dy - \sqrt{2k}\text{Tr}[\hat{\sigma}\hat{\rho}']dt \\ &= \sqrt{2k}\text{Tr} \left[(\sin(\delta')\hat{\sigma}_x + \cos(\delta')\hat{\sigma}_z) \left(\begin{array}{l} \{\sin(\delta) - \sin(\delta')\}\hat{\sigma}_x \\ + \{\cos(\delta) - \cos(\delta')\}\hat{\sigma}_z \end{array} \right) \right] dt + dw \\ &= \sqrt{2k} \{(\cos(\delta) - \cos(\delta'))\cos(\delta) - (\sin(\delta) - \sin(\delta'))\sin(\delta)\} dt + dw. \end{aligned}$$

Then, the time evolution equations of δ and δ' are given as follows;

Dynamics of our scheme with unknown external force

$$d\delta = 2\Delta dt - 2k \sin(2\delta - 2\alpha\delta')dt + 2\sqrt{k} \sin(\delta - \alpha\delta')dw, \quad (3.55)$$

$$\begin{aligned} d\delta' = & 2\Delta' dt - 2k \sin(2\beta\delta')dt \\ & + 2\sqrt{k} \sin(\beta\delta') \left[\sqrt{2k} \left\{ \begin{array}{c} (\cos(\delta) - \cos(\delta')) \cos(\delta) \\ - (\sin(\delta) - \sin(\delta')) \sin(\delta) \end{array} \right\} dt + dw \right]. \end{aligned} \quad (3.56)$$

3.2 Comparison of Jacobs' and our schemes

Since the distance δ is defined on the Bloch sphere, Eqs. (3.16) and (3.49) have complex boundary condition. It makes difficult to solve the equations. Thus, we numerically solve them. The adaptive measurement uses the back-action of measurement for control, and the large measurement strength k causes the high speed control. This fact can be seen from the results of the numerical simulations. Here, the strength is an important factor for fair comparison. But, Jacobs scheme requires changing the measurement strength while our scheme keeps the strength constant. Therefore it is not easy to compare these two schemes. Here, we compare the scheme under the condition that the random time evolution term $\gamma_{s,J}$ equal to $\gamma_{s,R}$. Then the two schemes have the same randomness, and the fluctuations are considered to be the same.

3.2.1 Numerical results

Since the time evolutions are stochastic process, each of sample paths is different. In order to obtain general properties of these schemes, we calculate the mean value and variance. Here, the sample path $\vec{\delta}$ can denote a sequence of random valuables $\vec{\delta} := (\delta_{t_0}, \delta_{t_1}, \dots, \delta_{t_{N-1}})^T$, where $t_0 < t_1 < \dots < t_{N-1}$. Let $\vec{\delta}(i)$ the i -th sample path, we use the following mean value

$$\begin{aligned} \text{Ave}(\delta) &:= \frac{1}{M} \sum_{i=1}^M \vec{\delta}(i) \\ &= \frac{1}{M} \begin{pmatrix} \sum_{i=1}^M \delta_{t_0}(i) \\ \sum_{i=1}^M \delta_{t_1}(i) \\ \vdots \\ \sum_{i=1}^M \delta_{t_{N-1}}(i) \end{pmatrix} \end{aligned} \quad (3.57)$$

and the variance is

$$\begin{aligned} \sqrt{\text{Var}(\delta)} &:= \sqrt{\frac{1}{M} \sum_{i=1}^M \vec{\delta}^2(i) - \text{Ave}(\delta)^2} \\ &= \begin{pmatrix} \sqrt{\sum_{i=1}^M \delta_{t_0}^2(i)/M - \left(\sum_{j=1}^M \delta_{t_0}(j)/M\right)^2} \\ \sqrt{\sum_{i=1}^M \delta_{t_1}^2(i)/M - \left(\sum_{j=1}^M \delta_{t_1}(j)/M\right)^2} \\ \vdots \\ \sqrt{\sum_{i=1}^M \delta_{t_{N-1}}^2(i)/M - \left(\sum_{j=1}^M \delta_{t_{N-1}}(j)/M\right)^2} \end{pmatrix}. \end{aligned} \quad (3.58)$$

Here, note $\text{Ave}(\delta)$ and $\text{Var}(\delta)$ are N dimension vector. We numerically compute these values. In numerical simulation, we put $\sqrt{8\kappa} = \sqrt{8k\beta}$ and $\kappa = 1$. Here, the relation $\kappa = k\beta^2$ holds.

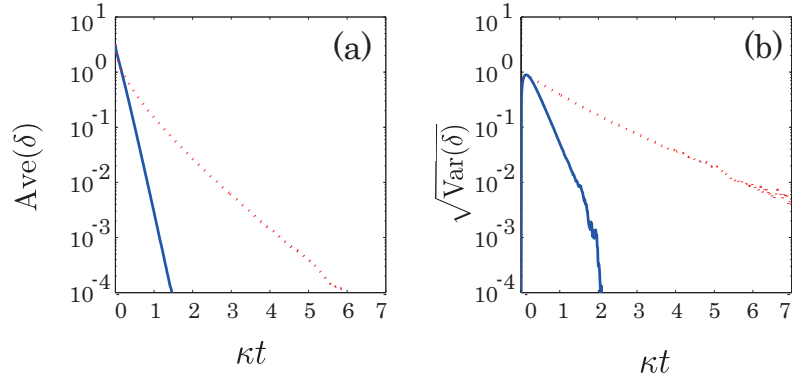


Figure 3.4: (a) The comparison of the two scheme's speed under the same random evolution term. The blue line is our scheme at $\alpha = 0.5$ and the red dashed line is Jacobs scheme. The horizontal axis is time, and the vertex is the mean δ_t of 10^6 sample paths. When lines in the lower region in the graph, it means that the state becomes closer to the target. (b) The comparison of the two scheme's fluctuation under the same random evolution. The blue line is our measurement at $\alpha = 0.5$ and the red dashed line is Jacobs scheme. The horizontal axis is time, and the vertex axis is the $\sqrt{\text{Var}(\delta)}$ of 10^6 data. When lines in the lower region in the graph, it means that the fluctuation is suppressed.

We consider the most simple cases, the solutions of Eqs. (3.16) and (3.49). Figure 3.4 shows the averages (3.57) and the variances (3.58) of the two schemes at $\alpha = 0.5$. We fit the lines of Fig. 3.4 (a) with the function $C_1 e^{-C_2 \kappa t}$, and we have $C_2 = 1.357$ in Jacobs' scheme, and $C_2 = 10.03$ in our scheme. Then, our scheme have about 7.4 times the faster than Jacobs' in this situation. Similarly, we fit the lines of Fig. 3.4 (b) with the function $C_1 e^{-C_2 \kappa t}$. Thus, we have

$C_2 = 0.6895$ in Jacobs' scheme, while $C_2 = 9.6349$ in our scheme and our scheme has about 0.072 times less fluctuation in this situation. Therefore, we can say that our scheme gives higher speed and lower fluctuation.

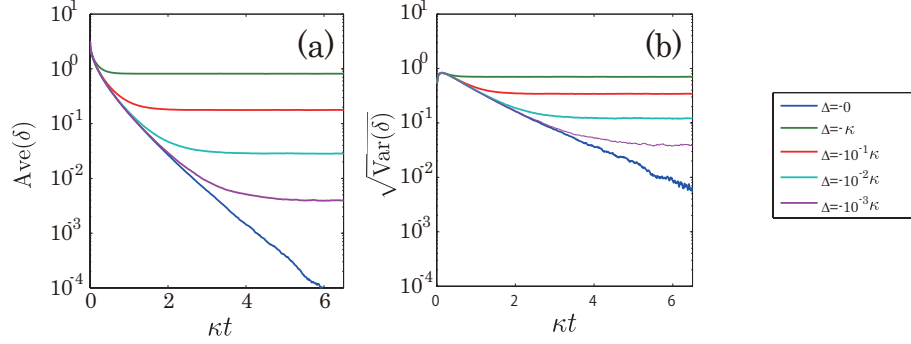


Figure 3.5: The numerical solution of Jacobs' scheme with fixed κ . The blue, green, red, light blue, purple lines correspond to $\Delta = 0, -\kappa, -10^{-1}\kappa, -10^{-2}\kappa, -10^{-3}\kappa$.

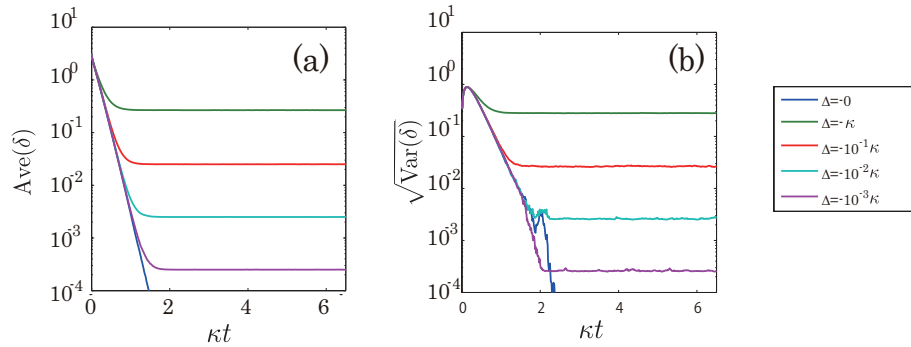


Figure 3.6: The numerical solution of our scheme with fixed κ . The blue, green, red, light blue, purple lines correspond to $\Delta = 0, -\kappa, -10^{-1}\kappa, -10^{-2}\kappa, -10^{-3}\kappa$.

Next, we compare the two schemes when the known external force exists. Figures 3.5 and 3.6 show the solutions of Eqs. (3.19) and (3.51). In the Figs. 3.5 and 3.6, the strengths k_R and κ are fixed, while each path has different $\Delta = 0, -\kappa, -10^{-1}\kappa, -10^{-2}\kappa, -10^{-3}\kappa$. First, we compare the averages of Fig. 3.5 (a) and Fig. 3.6 (a). The lines in Fig. 3.5 (a) have the same slope at large δ , but their convergence destinations are different.

From Fig. 3.6 (a), the slopes of each lines are similar to each other. The lines converge to some value, and there remain some errors at all paths δ . The convergence destinations of δ depend on Δ . Let the convergence destinations be C_3 , and we have the relation $C_3 = 0.2544 \times \Delta/\kappa$. This value is close to

$\Delta/(2k\beta)$. For a fixed value of Δ , the lines in our scheme are lower than those in Jacobs' scheme. That is, our scheme allows more accurate control with the external force Δ .

Figures 3.5 (b) and 3.6 (b) show fluctuations. The slopes of the lines in Fig. 3.5 are almost the same at large δ , but they become different when the distance δ are small. Figure 3.6 (b) shows the fluctuation, and the slope of each line is similar and the fluctuations converges a constant value. Therefore, we can say that the qualitative proprieties are similar. But our scheme is faster and the more robust against the external force.

3.2.2 Discussion on the numerical solutions

We cannot analytically solve Eqs. (3.19) and (3.51), since of the boundary condition . We also can not have the analytical comparison of Jacobs' and our scheme. Taking the expected value of Eq. (3.19), we have

$$(3.19) \rightarrow dE[\delta(t)] = -2\Delta dt, \quad (3.59)$$

and we have the time evolution of the above expected value as

$$E[\delta_t] = \pi - \Delta t. \quad (3.60)$$

On the other hand, the time evolution of our scheme is given by Eq. (3.51). This equation is non-linear and it can not be solved easily even if we ignore the boundary condition. Then, using the 1st approximation of δ of Eq. (3.51), we have

$$d\delta \sim -2\{\Delta + 2k\beta\delta\}dt + \sqrt{8k\beta}\delta dw.. \quad (3.61)$$

From Eq. (3.61), we have

$$(3.61) \rightarrow dE[\delta(t)] = -2\{\Delta + 2k\beta E[\delta(t)]\}dt, \quad (3.62)$$

and the solution of Eq. (3.62) as

$$E[\delta_t] = \left(\pi + \frac{\Delta}{2k\beta t}\right) e^{-4k\beta} - \frac{\Delta}{2k\beta}. \quad (3.63)$$

From Fig. 3.5, we see that Eq. (3.60) fails to describe the numerical solution. On the other hand, we find that Eq.(3.63) roughly explains the trajectory of Eq. (3.51) from the numerical solution in Fig. 3.6. The second term $-\frac{\Delta}{2k}$ of Eq. (3.63) explains the points in which the numerical solution converges at enough large t . Since $k_R\beta^2 = \kappa$ and $\beta = 1 - \alpha = 0.5$, we have $\frac{\Delta}{2k\beta} = 0.25\frac{\Delta}{\kappa}$, while the numerical solution converges in $C_3 = 0.2544 \times \Delta/\kappa$. Therefore, the time evolution of Eq. (3.51) can be explained by Eq. (3.63). The deterministic time evolution in Eq. (3.51) is stronger than the random time evolution. Note $\sin(1) = 0.8418$, and the sin function gives good approximation of x at the large t limit for $\delta \ll 1$.

These differences seem to be caused by the differences of the boundary condition effect. The fluctuation in Jacobs' scheme Eq. (3.19) is large, since there is no deterministic control term. Thus, the boundary condition becomes critical, and we cannot ignore it. On the other hand, our scheme Eq. (3.63) has the deterministic time evolution term, and it suppresses the fluctuation. Thus, the discussion without the boundary condition can be reasonable in this sense, and expected value can explain the numerical simulation result. We can discuss the fluctuation of the fluctuation continuation of the two schemes are thought to be equal. Here, the difference between the coefficients of the deterministic terms is $\Delta\gamma_d = |\gamma_{dR}| - |\gamma_{dJ}| = 4k\beta^2 \geq 0$, and our scheme seems to have high speed if they have the same fluctuation. If γ_d is the same, $\kappa = k\beta(1 + \beta)$ and the mean square error of the randomness is $\Delta\gamma_s^2 = \gamma_{sR}^2 - \gamma_{sJ}^2 = -8k\beta$. Then our scheme seems to have less fluctuation.

3.2.3 Robustness against unknown external force

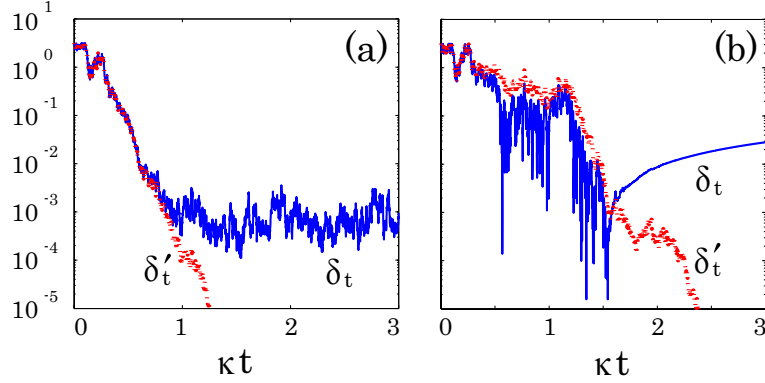


Figure 3.7: Sample paths of the true variable δ_t (solid blue lines) and the nominal one δ'_t (dashed red lines) at $\Delta = 1/10^2$ and $\Delta' = 0$. (a) is our scheme and (b) is Jacobs'.

Now, we compare the two schemes when Δ is unknown. That is, we compare the solutions of Eqs. (3.41), (3.42), (3.55), and (3.56). Figure 3.7 shows the sample paths at $\Delta = 1/10^2\kappa$ and $\Delta' = 0$. Figure 3.7(a) also shows the time evolution of our scheme, and the red line gives the time evolution of the nominal δ' . Here, the external force is $\Delta' = 0$, and the nominal path looks similar to the blue line in Fig. 3.4. The sample path of δ is represented by the red line of Fig. 3.7(a). This line looks similar to the one in Fig. 3.6(a). On the other hand, Fig. 3.7(b) shows the time evolution of δ and δ' in the Jacobs' scheme. From this figure, the control does not work at the small δ' , and the true δ finally becomes to obey the unitary time evolution caused by the external force. The reason of this is that the strength k becomes small depending on δ' . When δ' is small, the measurement strength $k = \kappa\delta^2$ also becomes small, and

the control does not work. Then the unitary time evolution becomes dominant. We can see the same effects in Fig. 3.8(a) and (b).

Figure 3.8 shows the sample paths of Δ and Δ' in Jacobs' and our scheme: (a) and (b) show the evolutions of $\Delta > \Delta'$, (c) and (d) show the evolutions of $\Delta = \Delta'$, and (e) and (f) show the evolutions of $\Delta < \Delta'$. Figure 3.8(c) and (d) shows that the prediction is perfect if $\Delta = \Delta'$, while Fig. (e) and (f) show the prediction itself is not so bad if $\Delta > \Delta'$. Therefore, we can say that our scheme suppresses the fluctuation.

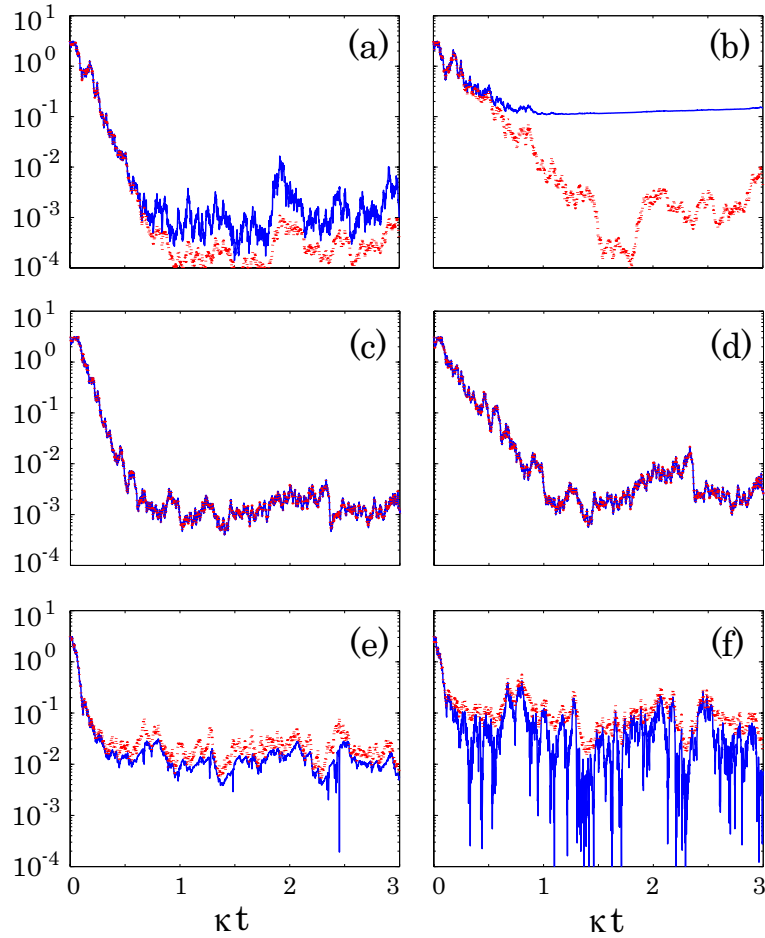


Figure 3.8: Sample paths of the true variable δ_t (solid blue lines) and the nominal one δ'_t (dashed red lines) for (a), (c), (e) the presented adaptive measurement scheme and (b), (d), (f) Jacobs' scheme. For all cases, the true external force is set to $\Delta = 1/10^2 \times \kappa$, while several nominal values of Δ' are examined: (a, b) represent the case when $\Delta' = 1/10^3 \times \kappa$, (c, d) for the case $\Delta' = 1/10^2 \times \kappa$, and (e, f) for the case $\Delta' = 1/10 \times \kappa$.

3.3 Summary and discussions of the Chapter

We propose a new adaptive measurement scheme. This improvement is derived from different choice of the measured observables. In our scheme, an eigenstate of the measured observable becomes the intermediate point of the target and the current state. This scheme gives the drift term in the dynamics. This drift term allows us to have more stable control in the adaptive measurement than Jacobs' scheme. Our scheme has robustness against the unknown external force although we have discussion about the robustness against other noises, like decoherence.

We want to remark the differences of this adaptive measurement control from general unitary state generation in the robustness and accuracy of the state. It is known that the unitary time evolution rotates the state on the surface of the Bloch sphere. When we want to generate a particular state, we apply an appropriate Hamiltonian in appropriate time. The state generation by unitary time evolution requires precise pulse controlling. The strength of the Hamiltonian must be stable, and the time must be accurate. This state generation is also sensitive to noise and other unknown force. On the other hand, our scheme is tolerant. The state generation using adaptive measurement keeps the state to the target state after the system reaches the target state. Continuing the measurement control does not break the state and increasing the fidelity of the target state.

Finally, I want to note future prospection of this scheme. We think this scheme suggest how to choose the measured observable. Since we get that target state by choosing the observable whose eigenvector is an intermediate point of the target state and the current state, we think the same idea is available for multi-qubit systems. I hope this idea works for the two-qubit system and generate the meaningful state like Bell state. Although the state generation using measurement is limited to theoretical proposal, we hope that this scheme is realized using optical cavities [24].

Chapter 4

Post-selection on the weak measurement

Form scholastic sense for non-reversibility on quantum measurement, Aharonov, Albert and Vaidman found a measurement way such that spin 1/2 particle can turn out 100%, which is called *weak measurement* based on the two-state vector formalism[26]. This measurement result is called the *weak value*, and it depends on both pre- and post-selected states. The weak value can become infinitely large when the post-selected state is orthogonal to the pre-selected state. By using this property, the weak measurement has been applied to the amplification technique, called *weak value amplification*.

In this section, we reevaluate the weak value amplification technique based on the AAV weak measurement theory in two ways; the moment with full order calculation [27] and the quantum Fisher information of the states [28]. From these discussion, we found that the post-selection causes the amplification when the interaction strength is small, but the amplification requires the rare post-selected state. Then, the weak value amplification is not efficient or not effective if we take into account of the loss by the post-selection.

4.1 Aharonov-Albert-Vaidman weak measurement

In this section, we review the basic idea of Aharonov-Albert-Vaidman (AAV) measurement and weak value. We also see how to measure the weak value, which is complex number, in physical system.

4.1.1 The basic idea of AAV weak measurement

Weak measurement proposed by Aharonov, Albert, and Vaidman (AAV weak measurement) has the following three specific features [26]:

1. von Neumann type indirect measurement $\hat{H}_{\text{int}} := \theta \hat{A} \otimes \hat{p}$,

2. the small interaction strength θ enough that the 1st-order approximation in θ to be valid,
3. the given initial and final state, called the pre-selected and post-selected state,

The last property is important and essential in the weak measurement. This operation decides the state after interaction. We perform a projection measurement on the state of the measured system after the interaction, and read out the probe only if the post-selection is successful. If it fails, we ignore the result on the probe, and assume that the AAV weak measurement itself fails.

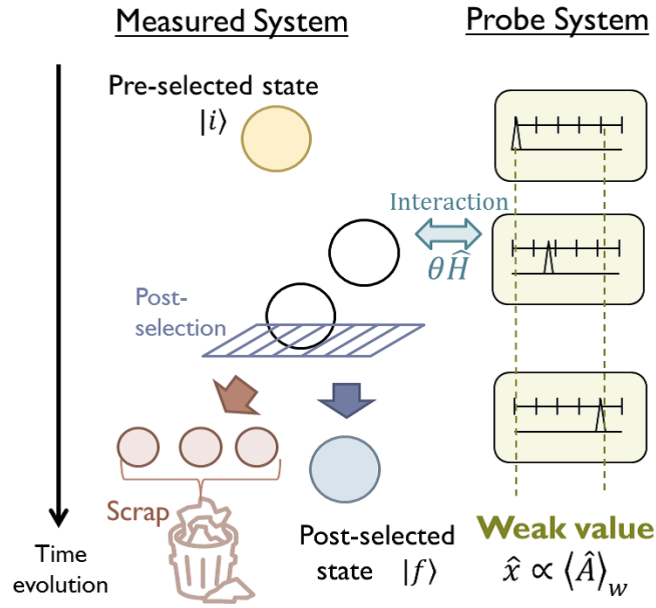


Figure 4.1: Schematic of the weak measurement process. The weak measurement based on the indirect measurement model. The initial and final states of the measured system are given, and these states are called the pre- and post-selected states. The post-selection is realized by measuring the measured system. If the post-selection is successful, the probe shift becomes proportional to the weak value within the range where the 1st-order approximation in the interaction strength θ is valid.

The measurement having above properties is realized by the following process (see Fig. 4.1):

1. interact the measured system and the probe system,

2. measure particular observable on the measured system for the post-selection,
3. read out the probe if the post-selection is successful,

We also denote the pre- and post-selected states, the initial state of the probe, and the interaction as follows:

- the pre-selected state $|i\rangle = \sum_j \alpha_j |a_j\rangle$ on the measured system,
- the post-selected state $|f\rangle = \sum_j \beta_j |a_j\rangle$ on the measured system,
- the interaction unitary operator $\hat{U} = e^{-i\theta \hat{A} \hat{p}}$ on the combined system,
- the initial state $|\psi\rangle = \int dx \psi(x) |x\rangle$ on the probe system.

Therefore, the initial state of the composite system is

$$|i\rangle \otimes |\psi\rangle, \quad (4.1)$$

and, after the interaction, the state becomes

$$\hat{U}|i\rangle \otimes |\psi\rangle = e^{-i\theta \hat{A} \hat{p}}|i\rangle \otimes |\psi\rangle. \quad (4.2)$$

Here, the real-valued parameter θ determines the strength of the interaction. The state of the composite system after the post-selection is

$$|f\rangle \langle f| \hat{U}(|i\rangle \otimes |\psi\rangle) = |f\rangle \otimes (\langle f| \hat{U} |i\rangle |\psi\rangle), \quad (4.3)$$

where this state is unnormalized. Because this state is separable and the state $|f\rangle$ of the measured system does not change any more, we need to pay attention only to the probe state. Assume that the interaction θ is enough small, and we expand the state after the post-selection and use the 1st-order approximation of θ to the probe state. Therefore, we have

$$\begin{aligned} \langle f| \hat{U} |i\rangle \otimes |\psi\rangle &= \langle f| e^{-i\theta \hat{A} \hat{p}} |i\rangle \otimes |\psi\rangle = \sum_n \langle f| \frac{(-i\theta)^n (\hat{A})^n \otimes (\hat{p})^n}{n!} |i\rangle |\psi\rangle \\ &\simeq \langle f| \left(1 - i\theta \hat{A} \otimes \hat{p} \right) |i\rangle \otimes |\psi\rangle = \langle f|i\rangle (1 - i\theta \langle A \rangle_w \hat{p}) \otimes |\psi\rangle \\ &\simeq \langle f|i\rangle e^{-i\theta \langle A \rangle_w \hat{p}} |\psi\rangle = \langle f|i\rangle \int dx \psi(x - \theta \langle A \rangle_w) |x\rangle, \end{aligned}$$

and the probe state the after post-selection is

$$|\psi'\rangle = \langle f|i\rangle \int dx \psi(x - \theta \langle A \rangle_w) |x\rangle, \quad (4.4)$$

where this state is unnormalized. We used the fact that the operator $e^{-ia\hat{p}}$ shifts the wave function $\psi(x)$ by a . Then the wave function moves by $\theta \langle A \rangle_w$. Here, we define the weak value as follows:

$$\langle A \rangle_w := \frac{\langle f| \hat{A} |i\rangle}{\langle f|i\rangle}. \quad (4.5)$$

This value becomes complex number, and becomes infinitely large when the pre- and post-selected states become orthogonal.

4.1.2 Weak value as an expected value

From Eq. (4.5), the weak value $\langle A \rangle_w$ is a complex number, and it is doubtful that this value is truly measured in experiments because the measurement result of quantum system is a real number. Jozsa showed where the weak value appears in experiments [56]. The weak value is derived in the indirect measurement model, and the measurement result of the observable $\hat{A}$ is seen through some other observables in the probe system. Jozsa showed that the real part of the weak value appears in the position shift x and that the imaginary part of the weak value appears in the moment shift p when the Hamiltonian of the system is $\hat{H} = \hat{p}^2/2m + V(\hat{x})$.

Assume that the initial state of the probe system is denoted by $|\psi\rangle$. Using 1st order approximation with respect to θ , we have the unnormalized state $|\psi'\rangle$ as

$$|\psi'\rangle = \langle f|e^{-i\theta\hat{A}\hat{p}}|i\rangle|\psi\rangle = \langle f|i\rangle(I - i\theta\langle A \rangle_w\hat{p})|\psi\rangle + O(\theta^2). \quad (4.6)$$

The expected value of an observable $\hat{o}$ after AAV weak measurement becomes

$$\begin{aligned} \langle \hat{o} \rangle &:= \frac{\langle \psi'|\hat{o}|\psi' \rangle}{\langle \psi'|\psi' \rangle} = \frac{\langle \psi|\hat{o}|\psi \rangle + i\theta\langle A \rangle_w^* \langle \psi|\hat{p}\hat{o}|\psi \rangle - i\theta\langle A \rangle_w \langle \psi|\hat{o}\hat{p}|\psi \rangle}{\langle \psi|\psi \rangle + i\theta\langle A \rangle_w^* \langle \psi|\hat{p}|\psi \rangle - i\theta\langle A \rangle_w \langle \psi|\hat{p}|\psi \rangle} \\ &= \frac{\langle \hat{o} \rangle + i\theta\langle A \rangle_w^* \langle \hat{p}\hat{o} \rangle - i\theta\langle A \rangle_w \langle \hat{o}\hat{p} \rangle}{1 + i\theta\langle A \rangle_w^* \langle \hat{p} \rangle - i\theta\langle A \rangle_w \langle \hat{p} \rangle} \\ &= \langle \hat{o} \rangle + i\theta\text{Re}(\langle A \rangle_w \langle \hat{p}\hat{o} - \hat{o}\hat{p} \rangle) + \theta\text{Im}(\langle A \rangle_w \langle \hat{p}\hat{o} + \hat{o}\hat{p} \rangle) \\ &\quad - 2\theta\text{Im}(\langle A \rangle_w \langle \hat{p} \rangle \langle \hat{o} \rangle) + O(\theta^2), \end{aligned} \quad (4.7)$$

where

$$\langle \hat{o} \rangle := \langle \psi|\hat{o}|\psi \rangle + O(\theta^2).$$

Substituting $\hat{p}$ into $\hat{o}$ of Eq. (4.7), we have

$$\begin{aligned} \frac{\langle \psi'|\hat{p}|\psi' \rangle}{\langle \psi'|\psi' \rangle} &= \langle \hat{p} \rangle + 2\theta\text{Im}(\langle A \rangle_w \langle \hat{p}^2 \rangle) - 2\theta\text{Im}(\langle A \rangle_w \langle \hat{p} \rangle^2) + O(\theta^2) \\ &= \langle \hat{p} \rangle + 2\theta\text{Im}(\langle A \rangle_w \langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle) + O(\theta^2). \end{aligned} \quad (4.8)$$

On the other hand, substituting $\hat{x}$ into $\hat{o}$ of Eq. (4.7) and using the relation $[\hat{p}, \hat{x}] = -i$, we have

$$\begin{aligned} \frac{\langle \psi'|\hat{x}|\psi' \rangle}{\langle \psi'|\psi' \rangle} &= \langle \hat{x} \rangle + i\theta\text{Re}(\langle A \rangle_w \langle \hat{p}\hat{x} - \hat{x}\hat{p} \rangle) + \theta\text{Im}(\langle A \rangle_w \langle \hat{p}\hat{x} + \hat{x}\hat{p} \rangle) \\ &\quad - 2\theta\text{Im}(\langle A \rangle_w \langle \hat{p} \rangle \langle \hat{x} \rangle) + O(\theta^2) \\ &= \langle \hat{x} \rangle + \theta\text{Re}(\langle A \rangle_w) + \theta\text{Im}(\langle A \rangle_w) m \frac{d}{dt} \langle \hat{x}^2 \rangle - 2\theta\text{Im}(\langle A \rangle_w \langle \hat{x} \rangle) m \frac{d}{dt} \langle \hat{x} \rangle + O(\theta^2) \\ &= \langle \hat{x} \rangle + \theta\text{Re}(\langle A \rangle_w) + \theta\text{Im}(\langle A \rangle_w) \left[m \frac{d}{dt} \langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle \right] + O(\theta^2), \end{aligned} \quad (4.9)$$

where m is mass of the probe system. Here, we used

$$i \frac{d}{dt} \langle \hat{x} \rangle = \langle [\hat{x}, \hat{H}] \rangle = i \frac{\langle \hat{p} \rangle}{m}, \quad (4.10)$$

$$i \frac{d}{dt} \langle \hat{x}^2 \rangle = \langle [\hat{x}^2, \hat{H}] \rangle = i \frac{\langle \hat{x} \hat{p} + \hat{p} \hat{x} \rangle}{m}, \quad (4.11)$$

since $\hat{H} = \hat{p}^2/2m + V(\hat{x})$.

Therefore, the expected values of the position $\hat{x}$ and the momentum $\hat{p}$ of the probe after the post-selection are as follows:

$$\langle \hat{x} \rangle' := \frac{\langle \psi' | \hat{x} | \psi' \rangle}{\langle \psi' | \psi' \rangle} = \langle \hat{x} \rangle + \theta \text{Re} \langle A \rangle_w + \theta \text{Im} \langle A \rangle_w \left[m \frac{d}{dt} \langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle \right] + O(\theta^2), \quad (4.12)$$

$$\langle \hat{p} \rangle' := \frac{\langle \psi' | \hat{p} | \psi' \rangle}{\langle \psi' | \psi' \rangle} = \langle \hat{p} \rangle + 2\theta \text{Im} \langle A \rangle_w \langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle + O(\theta^2). \quad (4.13)$$

Then, we read out the weak value from the probe which is observed in experiments. That is,

- the real part of weak value $\text{Re} \langle \hat{A} \rangle_w$ appears in the shift of the probe observable $\hat{x}$,
- the imaginary part of weak value $\text{Im} \langle \hat{A} \rangle_w$ appears in the shift of the probe observable $\hat{p}$, which is the conjugate of $\hat{x}$.

4.2 WVA with full-order calculation

From previous section, with the 1st order approximation of θ , we see that Aharonov and his coworker showed that the probe displacement in the weak measurement is proportional to the weak value Eq. (4.5);

$$\langle \hat{x} \rangle \propto \theta \text{Re} \langle \hat{A} \rangle_w.$$

The weak value takes infinitely large if the pre- and post-selected states are orthogonal, i.e. $\langle f | i \rangle = 0$, and the probe shift $\langle \hat{x} \rangle$ also is thought to become large. From this property, the weak value amplification technique was proposed by Hosten and his coworker [57]. They used this property and amplified the intercalation strength θ and directed the spin hall effect of light. Dixon and his coworker also used this technique for detecting the small tilt of the Piezo mirror in Sagnac interferometer [58].

Despite of these facts, AAV theorem has inconsistency. They assume the interaction strength is small enough and the 1st-order approximation in the measurement strength θ to be valid. Since the weak value $\langle \hat{A}_w \rangle_w$ can be infinitely large if the pre- and post-selected states are orthogonal, the approximation conflicts with the assumption that the higher orders of the interaction are ignorable. In order to resolve this inconsistency, Wu and Li computed $\langle x \rangle$ containing the

higher order of θ [59]. They showed that the displacement of the probe $\langle x \rangle$ is not proportional to $\langle A \rangle_w$, and the displacement cannot become infinitely large. They also showed the effect of the 2nd order of θ for the 1st moment of probe shift x . In this section, we discuss the effect of the higher order terms to the 1st moment of the probe observable through the precise description of Dixon experiment [58]. That is, we calculate the probe shift $\langle x \rangle$ in full order, and this calculation shows the physics of the post-selection in the indirect measurement. Choosing the measured system to be the two-level system, we can make calculate the expected value in full order. By discussing the experiment [58], we discuss the effect of the higher order terms for the shift of the probe.

4.2.1 Review of WVA experiment with Sagnac interferometer

Before discussing the effect of the higher order terms for the probe shift, we review the experiment [58]. This experiment demonstrates the weak value amplification technique.

4.2.1-a Experimental setup and the probe shift derived from AAV weak measurement theorem

First, we review the experiment [58]. This measured system has the two-level measured system [58]. Figure 4.2 shows the experiment setup. The aim of this experiment is measuring the tilt of the Piezo mirror in the Sagnac interferometer. In the experiment, the measured system is composed of two paths of the light, clockwise rotation path $|\odot\rangle$ and counterclockwise rotation path $|\oslash\rangle$. The probe system is the orbit x of the light. That is the position of the optical axis. The light state change in the following

1. The light goes through the lens, the 1/2-wave plate, the 1/4-wave plate, and the polarization beam splitter (PBS), and it is adjusted to the proper polarization and the distribution.
2. The light beam goes through a 50/50 beam splitter (BS), the light enters into the interferometer, and it is splitted into the two paths. i.e. clockwise and counterclockwise path.
3. The Piezo mirror with the small tilt makes interaction between the path and the orbit of the light. The small tilt of Piezo mirror displace the optical axis, and that causes the interaction of the orbit and optical paths. .
4. The Soliel-Babinet composer (SBC) creates the phase difference ϕ between the clockwise and the counterclockwise paths. SBC generates phase difference depending on the polarization of light. Because of 1/2-wave plate, the light in the clockwise path has vertical polarization and the light in the counterclockwise path has horizontal polarization at SBC. Then, the phase difference is generated.

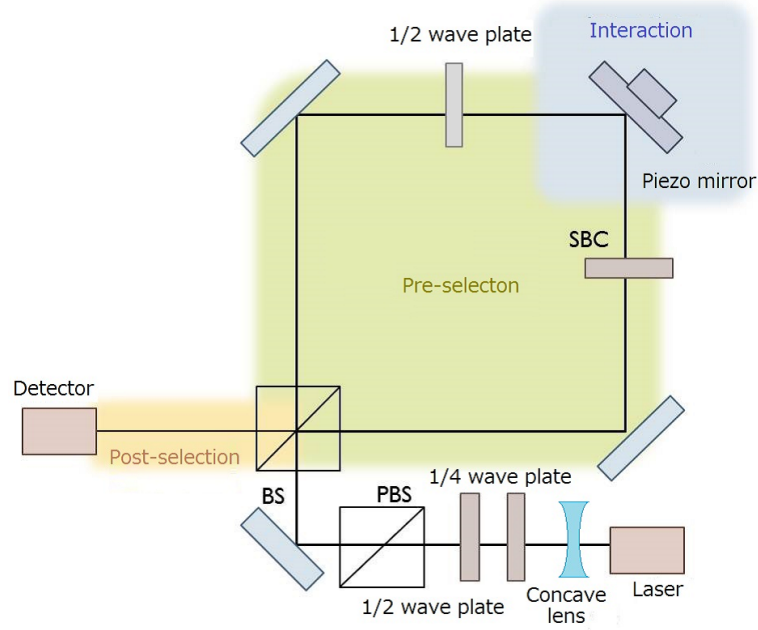


Figure 4.2: The Sagnac interferometer [58]. The wave plates and PBS manipulate the proper polarization. BS splits the light to the two paths, clockwise and counterclockwise. SBC creates phase difference ϕ , and Piezo mirror creates the interaction of path and orbit. The light beam is split, and the light goes back to the laser and the detector. Putting the detector along one side of the two paths serves as the post-selection.

5. Both the clockwise and the counterclockwise optical path are split by the 50/50 BS to the two directions, i.e. to the detector and laser light source.
6. Putting the detector on the one port of the two directions works as the post-selection.
7. Using the detector, measure the shift x of the optical axis.

Manipulating the lens, they changed the initial beam width σ and measured probe shift. They examined the three cases. That is, they measured the probe shift with three different phase differences $\phi = 7.2^\circ, 11.5^\circ$, and 17.3° by manipulating the SBC. The maximum beam diameter is $\sigma = 1240 \pm 50 \mu\text{m}$, which allows the weak value amplification with this system.

Table 4.1: Physical values in the Sagnac interferometer experiment

a		the beam diameter after the concave lens
σ	$= a \frac{l_{im}+l_{md}}{s}$	beam diameter before the detector
s		the focal length of the of the concave lens
l_{im}	$= s + l_{lm}$	the distance between the concave lens and Piezo mirror
l_{md}	114 cm	the distance between Piezo mirror to the detector
l_{lm}	48 cm	the distance between lens to the Piezo mirror
k_0		the wavenumber when the wave length $\lambda = 780$ nm

4.2.1-b Correspondence to AAV weak measurement and the experimental results

Dixon and his co-workers used following setups

- the pre-selected state $|i\rangle = \frac{1}{\sqrt{2}} (e^{-i\phi/2} |\circ\rangle + ie^{i\phi/2} |\oslash\rangle)$
- the post-selected state $|f\rangle = \frac{1}{\sqrt{2}} (i |\circ\rangle + |\oslash\rangle)$,
- the interaction unitary $\hat{U} = e^{-ik\hat{\sigma}_z\hat{x}}$,
- the initial state of probe $|\psi\rangle = \int dx \psi(x)|x\rangle$, $\psi(x) = \sqrt{\frac{1}{\sqrt{2\pi}} \frac{1}{a}} e^{-\frac{x^2}{4a^2}}$,

where

$$\hat{\sigma}_z |\circ\rangle = |\circ\rangle, \quad \hat{\sigma}_z |\oslash\rangle = -|\oslash\rangle.$$

In this experiment, we must not forget the fact that the Hamiltonian of von Neumann interaction is

$$\hat{\sigma}_z \otimes \hat{x},$$

while usual von Neumann interaction is $\hat{A}\hat{p}$. That is, the position operator x shifts the momentum p in this experiment. Then, from Eq. (4.12), not from Eq. (4.13), we have

$$\langle \hat{x} \rangle = 2k \text{Im} \langle \hat{\sigma}_z \rangle_w \langle \hat{x}^2 \rangle = 2ka^2 \text{Im} \langle \hat{\sigma}_z \rangle_w = 2ka^2 \cot(\phi/2). \quad (4.14)$$

Here, we used

$$\langle \hat{\sigma}_z \rangle_w = i \cot(\phi/2), \quad \text{Im} \langle \sigma_z \rangle_w = |\langle \sigma_z \rangle_w| \quad (4.15)$$

The physical values of the experiment are shown in the Table 4.1. Since the light beam diameter becomes wider through the interferometer, we take into account of this and multiply the following correction factor to the 1st moment:

$$a^2 \rightarrow \frac{\sigma^2 l_{lm} + \sigma a l_{md}}{l_{lm} + l_{md}}. \quad (4.16)$$

Note that we show the derivation of the above relation in Sec. 4.2.2. Thus, multiplying this we have the probe shift as

$$\langle \hat{x} \rangle = 2k |\langle \sigma_z \rangle_w| \frac{\sigma^2 l_{lm} + \sigma a l_{md}}{l_{lm} + l_{md}} = 2k \cot(\phi/2) \frac{\sigma^2 l_{lm} + \sigma a l_{md}}{l_{lm} + l_{md}}. \quad (4.17)$$

The parameter k determines the tilt of the Piezo mirror. The tilt δ of the Piezo mirror is

$$\delta = k l_{lm} / k_0, \quad (4.18)$$

and the amplification gain is defined as

$$\mathcal{A} = \langle x \rangle / \delta. \quad (4.19)$$

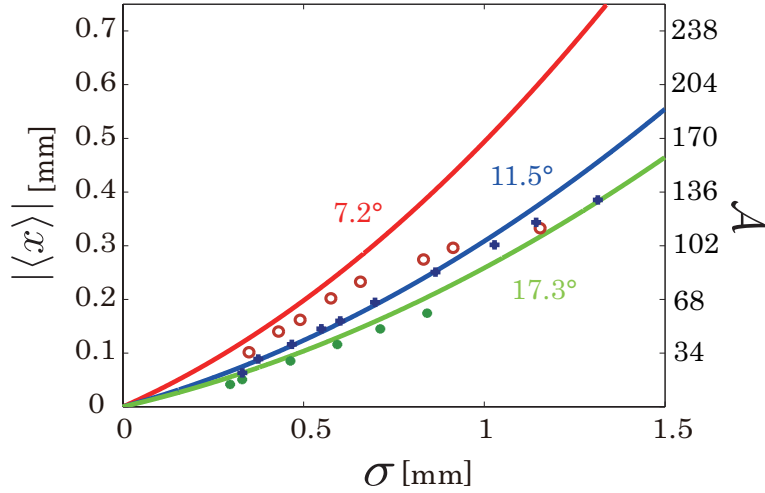


Figure 4.3: The experimental result of [58]. The horizontal axis represents the beam diameter, and vertical axis represents of the shift. Each point indicates the measurement result. The solid curves are theoretical predictions from Eq. (4.17). The red, blue, green curves are corresponds to $\phi = 7.2^\circ, 11.5^\circ$ and 17.3° respective.

They changed the initial beam width and measured the measured probe shift. They got amplification gain 86 when the mean size of the detector was $\sigma = 1240 \pm 50 \mu\text{m}$. Their results of this experiment are shown in Fig. 4.3. The points with error bar are experimental results and the solid curves are the theoretical predictions from AAV theorem based on Eq. (4.17). When the beam radius is small, the theoretical curves give good predictions. But, the prediction does not work well when the beam radius is large. Based on the above setup, Dixon and coworkers have Fig. 4.3. From the discussion of Dixon and coworkers, the stray light causes the gap between the experimental result

and theoretical prediction of 7.2° as seen in Fig. 4.3. The stray light is caused by the experimental setting and it disturbs the accuracy of the experiment [58]. We will consider this gap later with the full order discussion of the weak value amplification technique.

4.2.2 Full-order calculation to analyze the experiment [58] based on wave optics theory

In this section, we try to explain the the experiment [58] from the view point of the wave optics theory without the 1st-order approximation of the interaction strength [60]. By using wave optics theory, we can avoid to use the AAV theorem, and it is expected that we have more precise description. In order to take the widening beam radiation into account, we derive the 1st moment of probe shift $\langle \hat{x} \rangle$ without any approximations.

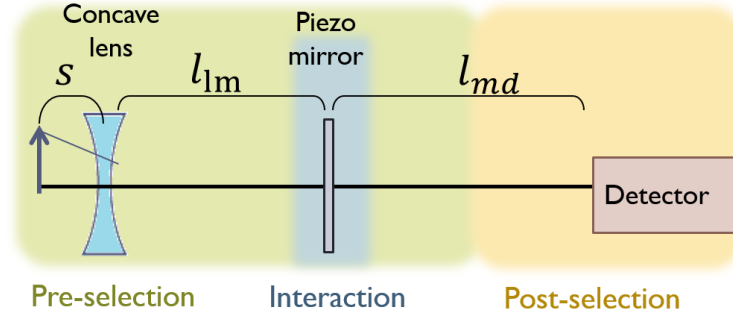


Figure 4.4: The schematic model of the experimental setup shown in Fig.4.2.

We include the effect of free evolution in the model, and we follow the process shown in Fig. 4.4 and calculate the wave function affected by the free evolution. The experiment was done with following process

1. Operate the phase transition $e^{\frac{ik_I}{2s}\hat{x}^2}$ on the wave function.
2. Operate the free time evolution $e^{-\frac{il_{lm}}{2k_I}\hat{k}^2}$ on the wave function in k -representation to travel through the distance l_{lm} for the k representation of wave function.
3. Operate the phase transition $e^{i\hat{x}\otimes\hat{A}}$ by the Piezo mirror.
4. Operate the free evolution $e^{-\frac{il_{md}}{2k_I}\hat{k}^2}$ on the wave function in k -representation to travel through the distance l_{md} . Manipulating the operator to the k representation wave function.

5. Manipulating the post-selection operator $|f\rangle\langle f|$ on the state of the composite system.

Here, only the Piezo mirror affects on the measured system of the light path. Thus, we can operate the operator of the post-selection whenever after the light through the Piezo mirror. The Fourier transform changes the representation x to the representation k and the inverse Fourier transform changes the representation $k \rightarrow x$.

Let us see the time evolution of weak measurement process and the time transition of the states. From now on, we derive the wave function and the 1st moment of position x after the post-selection, and the results is given by Eq. (4.35). If you want to skip the derivation, go to Eq. (4.35). The initial state of the composite system is

$$|\Psi\rangle = \int \sum_j \alpha_j dx \psi(x) |x\rangle |a_j\rangle \quad (4.20)$$

The initial state of the path space is set to $|i\rangle = \sum_j \alpha_j |a_j\rangle$. Assume the probe wave function as the Gaussian wave function $\psi(x) = \exp(-\frac{1}{4a^2}x^2)$. The lens affects only on the wave function of the probe system. Manipulating the operator of lens $e^{\frac{ik_I}{2s}\hat{x}^2}$ on the wave function of probe [60], we have

$$\psi(x) = \exp\left(-\frac{1}{4a^2}x^2\right) \rightarrow \psi(x) = \exp\left\{\left(\frac{-1}{4a^2} + \frac{i}{2s/k_I}\right)x^2\right\}, \quad (4.21)$$

where the parameter s is focal length of concave lens.

Since $\int dk |k\rangle\langle k|$ is the identity operator, we multiply this operator on the wave function in order to translate the x represent wave function into k representation. Here, we use $\langle k|x\rangle = e^{ikx}$, and the Fourier transform of e^{-ax^2} is given by $e^{-\frac{1}{4a^2}k^2}$ (see Appendix A). The state after the composite system after the concave lens is

$$\begin{aligned} |\Psi\rangle &= \int dx \int dk \sum_j \alpha_j \exp\left[\frac{-1}{2}\left(\frac{1}{2a^2} + \frac{-i}{s/k_I}\right)x^2\right] |k\rangle\langle k|x\rangle |a_j\rangle \\ &= \int dk \sum_j \alpha_j \left\{ \int dx \exp\left[\frac{-1}{2}\frac{s/k_I - 2ia^2}{2a^2s/k_I}x^2\right] \exp(ikx) \right\} |k\rangle |a_j\rangle \\ &= \int dk \sum_j \alpha_j \left\{ \int dx \exp\left[\frac{i}{2}\frac{is/2k_I + a^2}{a^2s/k_I}x^2\right] \exp(ikx) \right\} |k\rangle |a_j\rangle \\ &= \int dk \sum_j \alpha_j \exp\left(\frac{-1}{4i}\frac{2a^2s/k_I}{a^2 + is/2k_I}k^2\right) |k\rangle |a_j\rangle \\ &= \int dk \sum_j \alpha_j \exp\left(\frac{-i}{2k_I}\frac{a^2s}{a^2 + is/2k_I}k^2\right) |k\rangle |a_j\rangle \\ &= \int dx \sum_j \alpha_j \exp\left(-\frac{i}{2}\frac{\alpha}{k_I}k^2\right) |k\rangle |a_j\rangle, \end{aligned} \quad (4.22)$$

where we define

$$\alpha := \frac{a^2 s}{a^2 + is/2k_I} = \frac{a^2 s \left(a^2 - \frac{is}{2k_I}\right)}{a^4 + \frac{s^2}{4k_I^2}} = \frac{a^4 s - \frac{ia^2 s^2}{2k_I}}{a^4 + \frac{s^2}{4k_I^2}}. \quad (4.23)$$

The operator that the light passing the distance l_{lm} with the free evolution is $\exp\left\{\left(-\frac{i}{2}\frac{l_{lm}}{k_I}\right)\hat{k}^2\right\}$ (see Appendix B). Thus, the state of the composite system is

$$|\Psi'\rangle = \int dx \sum_j \alpha_j \exp\left\{\frac{-i}{2}\left(\frac{\alpha + l_{lm}}{k_I}\right)k^2\right\} |k\rangle |a_j\rangle. \quad (4.24)$$

The interaction caused by the Piezo mirror is $e^{i\hat{x}\hat{A}} = e^{-\partial_k \hat{A}}$. Here, the translate operator of the wave function in k representation is $e^{i\hat{x}a}$, and it satisfies $e^{i\hat{x}a}\psi(k) = \psi(k - a)$. The state of the composite system after the Piezo mirror is

$$\begin{aligned} |\Psi'\rangle &= \int dx \sum_j \alpha_j e^{i\hat{x}\hat{A}} \exp\left\{\frac{-i}{2}\left(\frac{\alpha + l_{lm}}{k_I}\right)k^2\right\} |k\rangle |a_j\rangle \\ &= \int dx \sum_j \alpha_j e^{i\hat{x}a_j} \exp\left\{\frac{-i}{2}\left(\frac{\alpha + l_{lm}}{k_I}\right)k^2\right\} |k\rangle |a_j\rangle \\ &= \int dx \sum_j \alpha_j \exp\left\{\frac{-i}{2}\left(\frac{\alpha + l_{lm}}{k_I}\right)(k - a_j)^2\right\} |k\rangle |a_j\rangle. \end{aligned} \quad (4.25)$$

Manipulating the operator $\exp\left(-\frac{i}{2}\frac{l_{md}\hat{k}^2}{k_I}\right)$, we have the state after the light passes the distance l_{md} is

$$\begin{aligned} |\Psi'\rangle &= C' \int dx \sum_j \alpha_j \exp\left\{\begin{aligned} &-\frac{i}{2}\left(\frac{\alpha + l_{lm} + l_{md}}{k_I}\right)k^2 \\ &-\frac{i}{2}\left(\frac{\alpha + l_{lm}}{k_I}\right)(-2ka_j + a_j^2) \end{aligned}\right\} |k\rangle |a_j\rangle \\ &= C' \int dx \sum_j \alpha_j \exp\left\{-\frac{i}{2}\left(\frac{\alpha + l_{lm} + l_{md}}{k_I}\right)\left[k - \frac{\frac{\alpha + l_{lm}}{k_I}}{\frac{\alpha + l_{lm} + l_{md}}{k_I}}a_j\right]^2\right\} \\ &\quad \times \exp\left\{\frac{i}{2}\left[-\frac{\left(\frac{\alpha + l_{lm}}{k_I}\right)^2}{\frac{\alpha + l_{lm} + l_{md}}{k_I}} + \frac{\alpha + l_{lm}}{k_I}\right]a_j^2\right\} |k\rangle |a_j\rangle \\ &= C' \int dx \sum_j \alpha_j \exp\left\{-\frac{i}{2}\frac{\alpha + l_{lm} + l_{md}}{k_I}\left(k - \frac{\alpha + l_{lm}}{\alpha + l_{lm} + l_{md}}a_j\right)^2\right\} \\ &\quad \times \exp\left\{\frac{i}{2}\frac{\alpha + l_{lm}}{k_I}\frac{l_{md}}{\alpha + l_{lm} + l_{md}}a_j^2\right\} |k\rangle |a_j\rangle. \end{aligned} \quad (4.26)$$

The Fourier transform of the above wave function becomes

$$\begin{aligned}
 |\Psi'\rangle &= \left(\int dx |x\rangle \langle x| \right) (|\Psi'\rangle) \\
 &= C'' \int dx \sum_j \alpha_j \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} + \frac{i(\alpha + l_{lm}) x a_j}{\alpha + l_{lm} + l_{md}} \right] \\
 &\quad \exp \left\{ \frac{i}{2} \frac{\alpha + l_{lm}}{k_I} \frac{l_{md}}{\alpha + l_{lm} + l_{md}} a_j^2 \right\} |x\rangle |a_j\rangle, \quad (4.27)
 \end{aligned}$$

where the normalization factor C'' is

$$\begin{aligned}
 C''^{-2} &= \sqrt{\frac{\pi |\alpha + l_{lm} + l_{md}|^2}{-k_0 \text{Im} \alpha}} \exp \left[\frac{-l_{md}^2 \text{Im} \alpha}{k_0 |\alpha + l_{lm} + l_{md}|^2} \right] \\
 &\quad \times \exp \left\{ l_{md} \frac{-\text{Im} \alpha a_j^2}{|\alpha + l_{lm} + l_2|^2} \right\}. \quad (4.28)
 \end{aligned}$$

Therefore, from Eq. (4.27), the 1st moment without the post-selection is

$$\langle \Psi | \hat{x} \otimes \hat{I} | \Psi \rangle = 0. \quad (4.29)$$

Next, we derive the 1st moment with the post-selection. We operate the projection operator $|f\rangle \langle f|$ ($|f\rangle := \sum_j \beta_j |a_j\rangle$) of the post-selection to Eq. (4.27). Taking the free evolution of the light into account, we have the state after the post-selection

$$\begin{aligned}
 |\psi'\rangle &= C''' \int dx \sum_j \beta_j^* \alpha_j \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} + \frac{i(\alpha + l_{lm}) x a_j}{\alpha + l_{lm} + l_{md}} \right] \\
 &\quad \times \exp \left\{ \frac{i}{2} \frac{\alpha + l_{lm}}{k_I} \frac{l_{md}}{\alpha + l_{lm} + l_{md}} a_j^2 \right\} |x\rangle \\
 &= C''' \int dx \sum_j \beta_j^* \alpha_j \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} + \frac{i(\alpha + l_{lm}) x a_j}{\alpha + l_{lm} + l_{md}} \right] \\
 &\quad \times \exp \left\{ \frac{i}{2} \frac{\alpha + l_{lm}}{k_I} \frac{l_{md}}{\alpha + l_{lm} + l_{md}} a_j^2 \right\} |x\rangle. \quad (4.30)
 \end{aligned}$$

Note that the post-selection is realized with the measurement and that the normalization factor C''' is not equal to C'' . In the experiment [58], the parameters satisfy as follows; $\alpha_0 = e^{-i\phi/2}$, $\alpha_1 = ie^{i\phi/2}$, $\beta_0 = \frac{i}{\sqrt{2}}$, $\beta_1 = \frac{1}{\sqrt{2}}$, $a_0 = k_0$, and

$a_1 = -k_0$. Thus, we have

$$\begin{aligned} |\psi'\rangle &= C''' \int dx \left(-i \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} + \frac{i(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} - i \frac{\phi}{2} \right] \right. \\ &\quad \left. + i \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} - \frac{i(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} + i \frac{\phi}{2} \right] \right) \\ &\quad \times \exp \left\{ \frac{i}{2} \frac{\alpha + l_{lm}}{k_I} \frac{l_{md}}{\alpha + l_{lm} + l_{md}} k_0^2 \right\} |x\rangle \\ &= C''' \int dx \left(\exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} + \frac{i(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} - i \frac{\phi}{2} \right] \right. \\ &\quad \left. - \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} - \frac{i(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} + i \frac{\phi}{2} \right] \right) |x\rangle. \end{aligned}$$

The normalization factor C''' satisfies

$$\begin{aligned} C'''^{-2} &= \int dx \left| \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} + \frac{i(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} - i \frac{\phi}{2} \right] \right. \\ &\quad \left. - \exp \left[-i \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} - \frac{i(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} + i \frac{\phi}{2} \right] \right|^2 \\ &= \int dx \exp \left[\text{Im} \frac{k_I x^2}{2(\alpha + l_{lm} + l_{md})} \right] \\ &\quad \times 2 \left[\cosh \left(2 \text{Im} \frac{(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} \right) - \cos \left(2 \text{Re} \frac{(\alpha + l_{lm})xk_0}{\alpha + l_{lm} + l_{md}} - \phi \right) \right] \\ &= \sqrt{\frac{\pi |\alpha + l_{lm} + l_{md}|^2}{-k_I \text{Im} \alpha}} \left(\exp \left\{ \frac{\text{Im} \alpha k_0^2}{|\alpha + l_{lm} + l_{md}|^2} \right\} \right. \\ &\quad \left. - \exp \left\{ \frac{(|\alpha + l_{lm}|^2 + l_{md} \text{Re} \alpha + l_{lm} l_{md})^2 k_0^2}{|\alpha + l_{lm} + l_{md}|^2 \text{Im} \alpha} \right\} \cos \phi \right). \end{aligned} \quad (4.31)$$

Then, the 1st moment $\langle \hat{x} \rangle$ after the post-selection is

$$\begin{aligned} \langle x \rangle &= \langle \psi' | x | \psi' \rangle = - \frac{\frac{2 \text{Re}[(\alpha + l_{lm})(\alpha + l_{lm} + l_2)]}{|\alpha + l_{lm} + l_{md}|^2} k_0 \exp \left(\frac{|\alpha + l_{lm}|^2}{-k_I \text{Im} \alpha} k_0^2 \right) \sin \phi}{\frac{-\text{Im} k_0 \alpha}{|\alpha + l_{lm} + l_{md}|^2} \left(1 + \exp \left(\frac{|\alpha + l_{lm}|^2}{-k_I \text{Im} \alpha} k_0^2 \right) \cos \phi \right)} \\ &= \frac{\text{Re}[(\alpha + l_{lm})(\alpha + l_{lm} + l_2)] \exp \left(\frac{|\alpha + l_{lm}|^2}{-k_I \text{Im} \alpha} k_0^2 \right) \sin \phi}{\text{Im} \alpha \left(1 + \exp \left(\frac{|\alpha + l_{lm}|^2}{-k_I \text{Im} \alpha} k_0^2 \right) \cos \phi \right)}. \end{aligned} \quad (4.32)$$

Here, we assume $\frac{s}{2k_I} \ll a^2$ and we have

$$\begin{aligned} \alpha + l_{lm} &\rightarrow l_{im}, \\ \text{Im} \alpha &\rightarrow -\frac{s^2}{2a^2 k_I}. \end{aligned}$$

Then,

$$\begin{aligned} \langle \hat{x} \rangle &= \frac{l_{im}(l_{im} + l_{md}) \exp \left(-\frac{2l_{im}^2 a^2}{s^2} k_0^2 \right) \sin \phi}{\frac{s^2}{2a^2} \left[1 - \exp \left(-\frac{2l_{im}^2 a^2}{s^2} k_0^2 \right) \cos \phi \right]} \\ &= 2a^2 \frac{l_{im}(l_{im} + l_{md}) \exp \left(-\frac{2l_{im}^2 a^2}{s^2} k_0^2 \right) \sin \phi}{s^2 \left[1 - \exp \left(-\frac{2l_{im}^2 a^2}{s^2} k_0^2 \right) \cos \phi \right]}. \end{aligned} \quad (4.33)$$

Taking the free evolution into account, we have the 1st moment of $\hat{x}$ as

$$\langle \hat{x} \rangle = 2a^2 \frac{l_{im}(l_{im} + l_{md}) \sin \phi}{s^2 \left[\exp \left(\frac{2l_{im}^2 a^2}{s^2} k^2 \right) - \cos \phi \right]}. \quad (4.34)$$

From the Table. 4.1, substituting $\sigma = a \frac{l_{im} + l_{md}}{s}$ and $a_m := a \frac{l_{im}}{s}$ into above equation, we have the position shift of the probe as follows;

The position shift after post-selection

$$\langle \hat{x} \rangle = \frac{2k_0 a_m \sigma \sin \phi}{\exp(2a_m^2 k_0^2) - \cos \phi}. \quad (4.35)$$

4.2.1-c The comparison of experimental results and theoretical prediction with full-order calculation

Using the result from the above discussion, we discuss the weak value amplification in the Sagnac interferometer. Taking the free evolution in the Sagnac interferometer into account, we have the expected value $\langle \hat{x} \rangle$

$$\langle \hat{x} \rangle = \frac{2k_0 a_m \sigma \sin \phi}{\exp(2a_m^2 k_0^2) - \cos \phi}. \quad (4.35)$$

From Eq. (4.35) the maximum value of x with respect to ϕ is given by

$$|\langle \hat{x} \rangle|_{\max \phi} = \frac{2a^2}{s^2} \frac{l_{im}(l_{im} + l_{md})}{\sqrt{\exp \left(\frac{4l_{im}^2 a^2}{s^2} k_0^2 \right) - 1}}, \quad (4.36)$$

where $\phi_{\max}$ is

$$\cos \phi_{\max} = \exp \left(\frac{-2l_{im}^2 a^2}{s^2} k^2 \right). \quad (4.37)$$

Using the result of Eqs. (4.17) and (4.35), we have Fig. 4.5 whose horizontal axis is the diameter σ . From this figure, the probe shift $\langle \hat{x} \rangle$ including the higher order terms improve a little. That is, the behaviors of Eqs. (4.17) and (4.35) are smiler to each other at small σ . In Fig. 4.5, we assume $ak_0 \approx 0.01$. The exponential term $\exp(2l_{im}^2 a^2 k_0^2 / s^2)$ becomes dominant when $2l_{im}^2 a^2 k_0^2 / s^2$ is enough large. Since $\sigma = a(l_{im} + l_{md})/s$, the large σ makes $2l_{im}^2 a^2 k_0^2 / s^2$ large, and the effect of the higher order terms appears. Here, a and k are fixed and σ depending on the focal length s . Note that a is also changed thought the experiment.

The time evolution of free evolution appears in the beam diameter σ , and the shift $\langle \hat{x} \rangle$ equals to the shift without the time evolution if $\sigma = a$. Figure 4.6 shows k_0 and $\langle x \rangle$ given by Eqs. (4.35) and (4.45) at $a = 640 \mu\text{m}$. We compare the results of Eqs. (4.14) and (4.45) with $\phi = 7.2^\circ, 11.5^\circ$, and 17.3° . From Fig. 4.6, the region where the 1st-order approximation is available is limited.

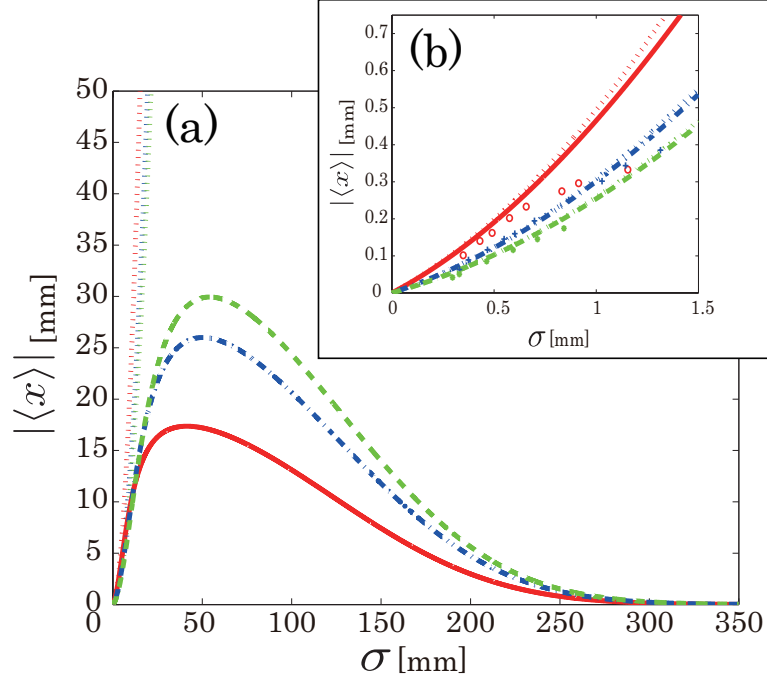


Figure 4.5: The beam diameter and the gain of the experimental results [58] and the theoretical line. Both (a) and (b) show the same results, and (b) is enlarged view of (a). The vertical axis is $\langle \hat{x} \rangle$, and the horizontal line is the beam diameter σ . The solid red, the chain blue and the dashed green curves correspond to $\phi = 7.2^\circ, 11.5^\circ$ and 17.3° of Eq. (4.35). The dotted lines correspond to $\phi = 7.2^\circ, 11.5^\circ$, and 17.3° of Eq. (4.17) with the 1-st order approximation. Here, $ak_0 = 20.8 \text{ m}^{-1}$, and $a = 640 \text{ } \mu\text{m}$. The red $\circ$, the blue $+$ and the green $*$ correspond to $\phi = 7.2^\circ, 11.5^\circ$, and 17.3° in Ref. [58].

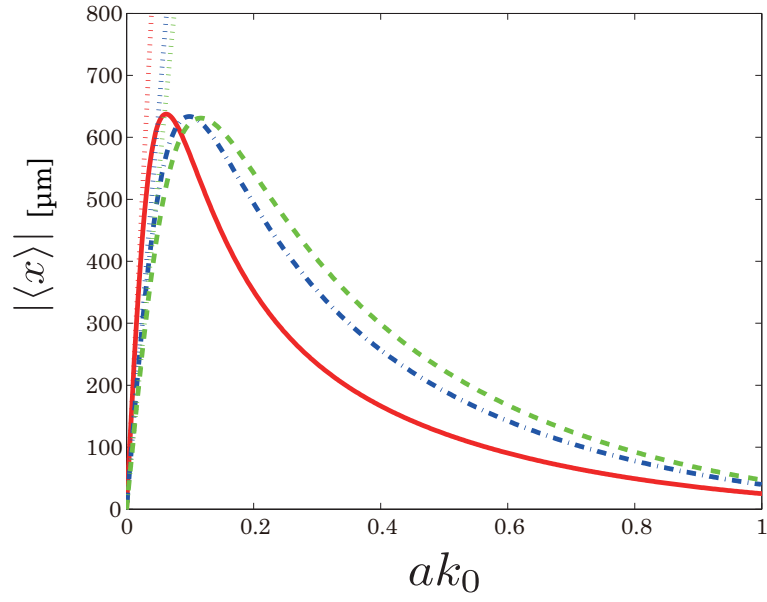


Figure 4.6: The gain by weak value amplification. The vertical line is the probe value $\langle x \rangle$, and the horizontal is the interaction strength which we want amplify. Here, we used the dimensionless value ak_0 for k_0 , and $a = 640 \mu\text{m}$. From the inside to the outside, the red, the blue, and the green curves correspond to $\phi = 7.2^\circ, 11.5^\circ$, and 17.3° in Eq.(4.45). Each dotted lines are Eq.(4.14) with the 1-st order approximation at $\phi = 7.2^\circ, 11.5^\circ$, and 17.3° .

4.2.3 WVA on the two-level measured system

In this section, we derive the m -th moment of full order calculation [27, 61]. When the measured system is a two-level system, the pre- and post-selected states are given by

$$\begin{aligned} |i\rangle &= \cos(t_1/2)|0\rangle + e^{i\phi_1} \sin(t_1/2)|1\rangle, \\ |f\rangle &= \cos(t_2/2)|0\rangle + e^{i\phi_2} \sin(t_2/2)|1\rangle, \\ \phi &= \phi_2 - \phi_1 \in [0, 2\pi). \end{aligned} \quad (4.38)$$

Without losing generality we denote $\hat{A} = \hat{\sigma}_z$. Here, we substitute $\hat{x}$ to $\hat{p}$ and $\theta = k_0$ in order to match this theory to the experimental results, and the interaction Hamiltonian can be denoted as $\hat{\sigma}_z \otimes \hat{x}$. These setting is the same as the one of Sec. 4.2.1.

The probe state $|\psi'\rangle$ after the post-selection is

$$|\psi'\rangle_{\mathcal{K}} = \frac{\hat{B}|\psi\rangle_{\mathcal{K}}}{|\hat{B}|\psi\rangle_{\mathcal{K}}|}, \quad (4.39)$$

where

$$\hat{B} := \langle f|e^{-ik_0\hat{\sigma}_z\hat{x}}|i\rangle_{\mathcal{H}}. \quad (4.40)$$

Since

$$\hat{\sigma}_z^{2n+1} = \hat{\sigma}_z, \quad \hat{\sigma}_z^{2n} = \hat{I}, \quad (4.41)$$

we have

$$\begin{aligned} \hat{B} &= \langle f| \left(\sum_{n=0}^{\infty} \frac{(k_0\hat{\sigma}_z\hat{x})^n}{n!} \right) |i\rangle_{\mathcal{H}} \\ &= \langle f| \left(\sum_{m=0}^{\infty} \frac{(k_0\hat{\sigma}_z\hat{x})^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(k_0\hat{\sigma}_z\hat{x})^{2m+1}}{(2m+1)!} \right) |i\rangle_{\mathcal{H}} \\ &= \langle f|i\rangle_{\mathcal{H}} (\cos(k_0\hat{x}) - i\langle\sigma_z\rangle_w \sin(k_0\hat{x})). \end{aligned} \quad (4.42)$$

Generally, the m -th moment becomes

$$\begin{aligned} \langle \hat{x}^n \rangle' &:= \langle \psi'|\hat{x}^n|\psi' \rangle = \frac{\langle \psi|\hat{B}^\dagger \hat{x}^n \hat{B}|\psi \rangle}{\langle \psi|\hat{B}^\dagger \hat{B}|\psi \rangle} \\ &= \frac{(1 + |\langle\sigma_z\rangle_w|^2) \langle x^n \rangle + (1 - |\langle\sigma_z\rangle_w|^2) \langle x^n \cos(2k_0x) \rangle + 2\text{Im}\langle\sigma_z\rangle_w \langle x^n \sin(2k_0x) \rangle}{(1 + |\langle\sigma_z\rangle_w|^2) + (1 - |\langle\sigma_z\rangle_w|^2) \langle \cos(2k_0x) \rangle + 2\text{Im}\langle\sigma_z\rangle_w \langle \sin(2k_0x) \rangle}, \end{aligned} \quad (4.43)$$

where $\langle \hat{o} \rangle_i = \langle \psi|o|\psi \rangle$ and $\langle o \rangle' = \langle \psi'|o|\psi' \rangle$.

Assume the pre- and pose-selected states equal to the experiment [58]

$$\begin{aligned} |i\rangle &= \frac{1}{\sqrt{2}} \left(e^{-i\phi/2} |0\rangle + e^{i\phi/2} |1\rangle \right), \\ |f\rangle &= \frac{1}{\sqrt{2}} (i|0\rangle + |1\rangle) \end{aligned}$$

and the initial wave function of the probe system to be Gaussian

$$|\psi\rangle = \sqrt{\frac{1}{\sqrt{2\pi}a}} e^{-\frac{x^2}{4a^2}},$$

and weak value is

$$\langle \hat{\sigma}_z \rangle = -i \cot \frac{\phi}{2}. \quad (4.44)$$

The 1st moment of $\hat{x}$ is

$$\langle \hat{x} \rangle' = 2a^2 k_0 \frac{\sin \phi}{e^{2a^2 k_0^2} - \cos \phi}. \quad (4.45)$$

The maximum of Eq. (4.45) with respect to ϕ is

$$|\langle \hat{x} \rangle'|_{\max \phi} = \frac{2a^2 k_0}{\sqrt{e^{4a^2 k_0^2} - 1}} \quad (4.46)$$

which is given at $\phi_{\max}$ satisfying

$$\cos \phi_{\max} = \langle \cos 2k_0 x \rangle, \quad (4.47)$$

and the interaction strength $k_{0\max}$ which gives the maximum of x becomes

$$\cos \phi = (1 - 4a^2 k_{0\max}^2) e^{2a^2 k_{0\max}^2}. \quad (4.48)$$

Comparing Eqs. (4.35) and (4.45), we have the correlation factor Eq. (4.16):

$$a^2 \rightarrow \frac{\sigma^2 l_m + \sigma a l_{md}}{l_m + l_{md}}.$$

Here, we derived the m -th moment of the observable $\hat{x}$ on the probe system. Especially, we compare the 1st moment theoretically and experimentally derived.

4.3 Information change by the weak value amplification

We discussed the effect of the post-selection on the moments by using the full-order calculation in Sec 4.2. Actually there are some works [27, 59, 61, 62, 63, 64]

that describe the weak measurement theory in full order. These works describe the physics in the case where the interaction strength θ is not weak, and show the properties of the post-selection. These works show that the weak value amplification is the phenomena limited to the weak interaction case. Especially Lee and Tsutsui have shown the reason for the error caused when the finite data is available [64].

The probe expected value does not tell the all the measurement result. Generally, the weak value amplification technique requires large fluctuation of a wave function, and the technique may increase the noise in the probe shift. Thus, we need to consider the relation of the noise and the shift. In this section, we use the statistical estimation theory for this problem. In order to evaluate the accuracy of the estimated value of θ , we use the Fisher information in Sec. 2.2.3. We discussed the accuracy of the estimated value of θ with the information. The Maximal likelihood estimator (MLE) converges to the true value in the large amount of data limit, and the fluctuation of the estimator is bounded by the inverse of Fisher information, which is easy to calculate [42, 43, 47, 48, 49].

Although the post-selection is the essential manipulation of the weak value amplification, the success of the post-selection depends on luck and this technique does not always work. In this section, we take into account the loss by post-selection, and re-evaluate this amplification by using quantum estimation theory. Fisher-information gives us accuracy of the estimation in the quantum system, Moreover, the asymptotic theory of estimation suggests that the SLD-Fisher information including the success probability of the post-selection shows the speed to achieve Cramér-Rao bound [28]. Note that there are other researches discussing the loss by post-selection [65, 66, 67], and they have the same results.

4.3.1 The state transition through the AAV weak measurement

Since a quantum state collapses to a pure state by a projective measurement, the post-selection is realized by a measurement on the measured system. This post-selective measurement caused state transition in probability and the success of the post-selection is probabilistic. When the post-selected state is a pure state $|f\rangle_{\mathcal{H}}$, the POVM of is $\{|f\rangle_{\mathcal{H}}\langle f|, \hat{I}^{\mathcal{H}} - |f\rangle_{\mathcal{H}}\langle f|\}$. The success probability that we have a pure post-selected state is ¹

$$\Pr(f) := \Pr \left[f \middle| \hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}} \right] = \text{Tr} \left[(\hat{\pi}_f^{\mathcal{H}} \otimes \hat{I}^{\mathcal{K}}) \hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}} \right]. \quad (4.49)$$

And the probe state after the post-selection is

$$\hat{\rho}_{\text{PPS}}^{\mathcal{K}} = \frac{\text{Tr}_{\mathcal{H}} \left[(\hat{\pi}_f^{\mathcal{H}} \otimes \hat{I}^{\mathcal{K}}) \hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}} \right]}{\Pr(f)}. \quad (4.50)$$

¹Note using POVM measurement we have a mixed post-selected state. But here we limit the discussion of the pure post-selected state and use a projective measurement for the post-selection.

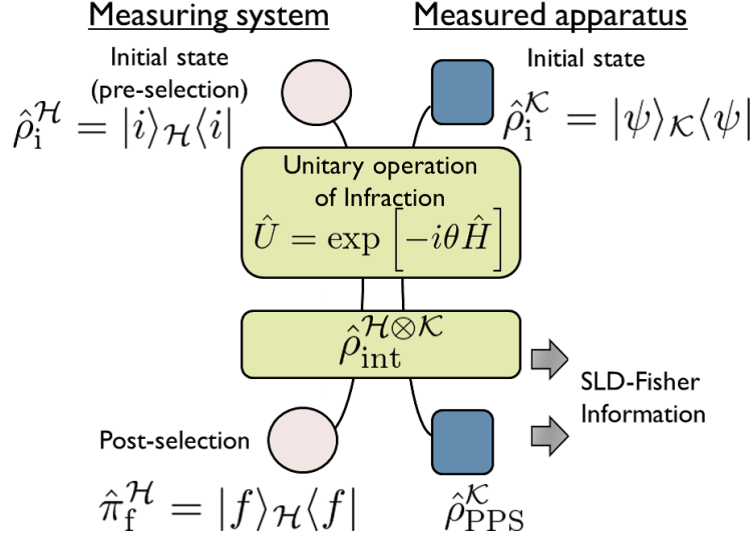


Figure 4.7: The process of AAV-weak measurement. Through this process, the probe state $\hat{\rho}_i^{\mathcal{K}}$ changes to $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$. We compare the Fisher informations of the state $\hat{\rho}_i^{\mathcal{H}\otimes\mathcal{K}}$ and $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$.

Here, we introduce the POVM on $\mathcal{K}$,

$$\hat{\mathbf{N}}_X^{\mathcal{K}} = \{\hat{N}^{\mathcal{K}}(x) : \hat{N}^{\mathcal{K}}(x) \text{ is an operator which measurement result is } x\}.$$

Measuring the probe system with this POVM, the probability to get the result x is

$$\begin{aligned}
 \text{Tr} [\hat{N}^{\mathcal{K}}(x) \hat{\rho}_{\text{PPS}}^{\mathcal{K}}] &= \frac{\text{Tr} \left[\left(\hat{\pi}_f^{\mathcal{H}} \otimes \hat{N}^{\mathcal{K}}(x) \right) \hat{\rho}_{\text{int}}^{\mathcal{H}\otimes\mathcal{K}} \right]}{\text{Tr} \left[\left(\hat{\pi}_f^{\mathcal{H}} \otimes \hat{I}^{\mathcal{K}} \right) \hat{\rho}_{\text{int}}^{\mathcal{H}\otimes\mathcal{K}} \right]} \\
 &= \frac{\text{Pr} [f, x | \hat{\rho}_{\text{int}}^{\mathcal{H}\otimes\mathcal{K}}]}{\text{Pr} [f | \hat{\rho}_{\text{int}}^{\mathcal{H}\otimes\mathcal{K}}]} = \text{Pr} [x | f, \hat{\rho}_{\text{int}}^{\mathcal{H}\otimes\mathcal{K}}].
 \end{aligned} \tag{4.51}$$

Therefore, the state $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$ is a conditional state on $\mathcal{K}$ conditioned on the post-selected state f on $\mathcal{H}$.

The initial state of the measured system is a pure state $\hat{\rho}_i^{\mathcal{H}} = |i\rangle_{\mathcal{H}}\langle i|$. Here, we introduce an operator $\hat{B}$ as follows:

$$\hat{B} = \langle f | \hat{U} | i \rangle_{\mathcal{H}} \in \mathcal{L}(\mathcal{K}), \tag{4.52}$$

and the state transition through the weak measurement of the probe system is

$$\hat{\rho}_i^{\mathcal{K}} \rightarrow \hat{\rho}_{\text{PPS}}^{\mathcal{K}} = \frac{\hat{\sigma}^{\mathcal{K}}}{\text{Pr}(f)} = \frac{\hat{B} \hat{\rho}_i^{\mathcal{K}} \hat{B}^\dagger}{\text{Tr} [\hat{B} \hat{\rho}_i^{\mathcal{K}} \hat{B}^\dagger]}, \tag{4.53}$$

where this transition occurs with probability $\text{Pr}(f)$.

4.3.2 SLD-Fisher information and weak value

Let us recall that the pre-selected state is pure and the unitary operator of the interaction is $\hat{U} = \exp[-i\theta\hat{H}]$. We assume the initial state $\hat{\rho}_i^{\mathcal{K}}$ is given and the interaction strength parameter θ is unknown, and we want to estimate the value θ . This model becomes generalization of the experiments [57, 58] and the theories [27, 61, 62, 63].

Since the Born rule determines the probability distribution with a measurement, the quantum state is interpreted as a mathematical extinction of a distribution function [3]. As we discussed in Sec. 2.2.3 and 2.2.4, a quantum estimator contains measurement operators, and we have a freedom in choosing the measurement. Including the measurement, we represent a quantum estimator as a self-adjoint operator $\hat{T}$. Here, a good estimation may have a small mean squared error $(\hat{T} - \theta\hat{I})^2$. If there are n copies of the state $\hat{S}_\theta$, the mean square error is bounded by the following quantum Cramér-Rao inequality

$$\langle(\hat{T} - \theta\hat{I})^2\rangle_{\hat{S}_\theta} = \text{Tr} \left[\hat{S}_\theta (\hat{T} - \theta\hat{I})^2 \right] \geq 1 / \left(n \cdot I_{\text{SLD}}(\hat{S}_\theta) \right), \quad (4.54)$$

where

$$\partial_\theta \hat{S}_\theta = \frac{1}{2} \left(\hat{S}_\theta \hat{L}_\theta + \hat{L}_\theta \hat{S}_\theta \right), \quad (4.55)$$

$$I_{\text{SLD}}(\hat{S}_\theta) := \langle \hat{L}_\theta^2 \rangle_{\hat{S}_\theta} = \text{Tr} \left[\hat{L}_\theta^2 \hat{S}_\theta \right]. \quad (4.56)$$

The operator $\hat{L}_\theta$ defined with Eq. (4.55) is a self-adjoint operator called symmetric logarithmic derivative (SLD), and the SLD-Fisher information $I_{\text{SLD}}(\hat{S}_\theta)$ is defined with Eq. (4.56) [48]. If the state $\hat{S}_\theta$ is full rank, the SLD is uniquely determined. Moreover, if the state is pure, the SLD-Fisher information is uniquely determined [47]. The Fisher information of the pure state $\hat{S}_\theta = |\chi_\theta\rangle\langle\chi_\theta|$ is

$$\langle \hat{L}^2 \rangle_{\hat{S}_\theta} = 4 \left(\langle \partial_\theta \chi_\theta | \partial_\theta \chi_\theta \rangle - |\langle \partial_\theta \chi_\theta | \chi_\theta \rangle|^2 \right). \quad (4.57)$$

Although there exists an unbiased estimator $\hat{T}_{\text{ub}}(\theta)$ which be attained in the equality (4.54), this estimator depends on the estimate parameter θ . Since the optimal measurement is derived from the quantum estimator $\hat{T}_{\text{ub}}(\theta)$, it seems to difficult to achieve the bound. But, it is known that the equality can be attained with the adaptive measurement [48, 49].

Since the state of the composite system after interaction is $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$, its SLD-Fisher information is

$$\begin{aligned} I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}) &= 4 \left[\langle i, \psi | \hat{H}^2 | i, \psi \rangle - \langle i, \psi | \hat{H} | i, \psi \rangle^2 \right] \\ &= 4(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2). \end{aligned} \quad (4.58)$$

Since the state of device after the post-selection is $\hat{\rho}_{\text{PS}}^{\mathcal{K}}$, its SLD-Fisher infor-

mation is

$$I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}}) = \frac{4}{\langle \psi | \hat{B}^\dagger \hat{B} | \psi \rangle^2} \cdot \begin{pmatrix} \langle \psi | \hat{B}^\dagger \hat{B} | \psi \rangle \langle \psi | \partial_\theta \hat{B}^\dagger \partial_\theta \hat{B} | \psi \rangle \\ - \langle \psi | \partial_\theta \hat{B}^\dagger \hat{B} | \psi \rangle \langle \psi | \hat{B}^\dagger \partial_\theta \hat{B} | \psi \rangle \end{pmatrix} \quad (4.59)$$

$$= \frac{4}{\text{Pr}(f)} \left(\langle \psi | \partial_\theta \hat{B}^\dagger \partial_\theta \hat{B} | \psi \rangle - \frac{|\langle \psi | \partial_\theta \hat{B}^\dagger \hat{B} | \psi \rangle|^2}{\langle \psi | \hat{B}^\dagger \hat{B} | \psi \rangle} \right). \quad (4.60)$$

Here, we used $\langle \psi | \hat{B}^\dagger \hat{B} | \psi \rangle = \text{Pr}(f)$. From Eq. (4.60), $I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ is proportional to $\text{Pr}(f)^{-1}$. A rare post-selected state increases the SLD-Fisher information of θ . That is, the rare event causes large information. This result corresponds to the Shannon's philosophy about information [68], although the Fisher information and the Shannon entropy are different metrics. Concerning about weak measurement, we can conclude that the information that the measured system $\mathcal{H}$ has a rare event increases the information of the state on the probe system $\mathcal{K}$.

When the interaction Hamiltonian is von Neumann type $\hat{H} = \hat{A}^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}$, we derive the relationship between $I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ and the weak value $\langle \hat{A} \rangle_w$. From Eq.(4.59), taking the weak limit of θ , we have

$$\lim_{\theta \rightarrow 0} I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}}) = |\langle \hat{A} \rangle_w|^2 \cdot \langle \psi | (\hat{p}^2 - \langle \hat{p} \rangle^2) | \psi \rangle. \quad (4.61)$$

Therefore, we find that the SLD-Fisher information becomes proportional to the weak value in the weak limit. Hofman showed that the classical Fisher information also becomes proportional to the weak value in the weak limit [69].

4.3.3 Evaluation of the post-selection with SLD-Fisher information

Next, we compare the SLD-Fisher information $I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}})$ and $I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$. For simplicity, we discuss the situation such that

- the measured system is a two-level system and the probe system is a continuous variable system
- the unitary operator of interaction is von Neumann type and its function type is $\exp[-i\theta \hat{H}] = \exp[-i\theta \hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}]$.

Since the measured system is the two-level, the observable on the measured system can be denoted by $\hat{A}^{\mathcal{K}} = a\hat{\sigma}_z$, thus a substitution $\theta \cdot a \rightarrow \theta$ is not essential. Then, this unitary becomes $\exp[-i\theta \hat{\sigma}_z^{\mathcal{H}} \otimes \hat{p}^{\mathcal{K}}]$. The pre-selected state can be represented as $|i\rangle_{\mathcal{H}} = \cos t_1 |0\rangle_{\mathcal{H}} + e^{is_1} \sin t_1 |1\rangle_{\mathcal{H}}$, and the post-selected state can be represented as $|f\rangle_{\mathcal{H}} = \cos t_2 |0\rangle_{\mathcal{H}} + e^{is_2} \sin t_2 |1\rangle_{\mathcal{H}}$. Then, we have

$$\langle f | i \rangle = \cos t_1 \cos t_2 + \sin t_1 \sin t_2 e^{i(s_1 - s_2)}, \quad (4.62)$$

$$\langle f | \hat{\sigma}_z | i \rangle = \cos t_1 \cos t_2 - \sin t_1 \sin t_2 e^{i(s_1 - s_2)}. \quad (4.63)$$

Here, the three parameter t_1, t_2 and $s := s_1 - s_2$ determine the pre- and post-selection, and they satisfy $t_1, t_2 \in [0, \pi]$, $s \in [-2\pi, 2\pi]$. The estimate parameter is single, and we have only one equation by differentiating the information with the estimate parameter for the maximization of the information. The pre- and post-selection is characterize with the three parameters, and the optimization problem of the pre- and post-selection cannot be solved.

Let the initial wave function be the Gaussian function

$$\langle p|\psi\rangle = (2\sigma^2/\pi)^{1/4} \exp[-\sigma^2 p^2],$$

where $|p\rangle_{\mathcal{K}}$ is an eigenvector of the operable $\hat{p}^{\mathcal{K}}$. The SLD-Fisher information of the probe system is

$$I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}}) = \frac{1}{\sigma^2} \frac{w_p^2 + \frac{\theta^2}{\sigma^2} w_p w_m e^{-\frac{\theta^2}{2\sigma^2}} - w_m^2 e^{-\frac{\theta^2}{\sigma^2}}}{\left(w_p + w_m e^{-\frac{\theta^2}{2\sigma^2}}\right)^2}, \quad (4.64)$$

$$w_p = (|\langle f|i\rangle|^2 + |\langle f|\hat{\sigma}_z|i\rangle|^2)/2,$$

$$w_m = (|\langle f|i\rangle|^2 - |\langle f|\hat{\sigma}_z|i\rangle|^2)/2,$$

while the SLD-Fisher information of combined system is

$$I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}) = 1/\sigma^2. \quad (4.65)$$

Here, we calculate the classical Fisher information of the probe system with the projection operators $\hat{\Pi}_x^{\mathcal{K}} := \{|x\rangle_{\mathcal{K}}\langle x| : \hat{x}^{\mathcal{K}}|x\rangle_{\mathcal{K}} = x|x\rangle_{\mathcal{K}}, x \in \mathbb{R}\}$ which is used in the measurement of the observable $\hat{x}^{\mathcal{K}}$. Note the observables $\hat{x}$ and $\hat{p}$ satisfy $[\hat{x}^{\mathcal{K}}, \hat{p}^{\mathcal{K}}] = i$. If the parameters of the pre- and post-selected states are $\cos^2 t_1 = \cos^2 t_2 = \sin^2 t_1 = \sin^2 t_2 = 1/2$ and $\cos s = \pm 1 := c$, we have the distribution of the observable $\hat{x}^{\mathcal{K}}$ with the state $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$ as

$$f_{pps}(x) = \frac{e^{-\frac{(x-\theta)^2}{2\sigma^2}} + e^{-\frac{(x+\theta)^2}{2\sigma^2}} + 2ce^{-\frac{x^2+\theta^2}{2\sigma^2}}}{2\sqrt{2\pi\sigma^2} \left(1 + ce^{-\frac{\theta^2}{2\sigma^2}}\right)}. \quad (4.66)$$

Thus, we can analytically calculate the classical Fisher information of $f_{\text{PPS}}(x)$ as

$$I(f(x)) := \int_{-\infty}^{\infty} dx \left\{ \partial_{\theta} \log f(x) \right\}^2 f(x)$$

$$= \frac{1}{\sigma^2} \frac{1 + c \frac{\theta^2}{\sigma^2} e^{-\frac{\theta^2}{2\sigma^2}} - e^{-\frac{\theta^2}{\sigma^2}}}{\left(1 + e^{-\frac{\theta^2}{2\sigma^2}}\right)^2}. \quad (4.67)$$

Figure 4.8 is the Fisher-information with θ/σ . The blue area represents $\sigma^2 \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ with changing parameters t_1, t_2, s determining the pre- and post-selection, and the yellow dash line is $\sigma^2 \cdot I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{K}})$. Since there is an area such

that $\sigma^2 \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ takes the value above the yellow dash line, the SLD-Fisher information is increased if we take proper pre- and post-selection. We find that the $I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ becomes infinite in the limit $\theta \rightarrow 0$. From the Fig. 4.9 (a), this divergence is caused when the success probability of the post-selection is small. Note that both the dimension of the Hilbert space of measured system $\mathcal{H}$ and the one of the probe system $\mathcal{K}$ are finite.

The sign of $I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}) - I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ depends on the pre-selected state $|i\rangle_{\mathcal{H}}$ and the post-selected state $|f\rangle_{\mathcal{H}}$. There are cases such that the Fisher information of the state $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$ with post-selection is larger than that of the state $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$ without the post-selection.

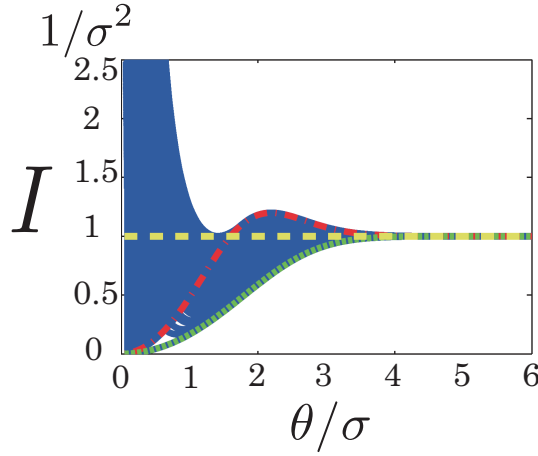


Figure 4.8: Fisher informations versus the parameter θ . The blue region represents the set of curves of $I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$, while the yellow dashed line indicates $I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}})$. The red dashed-dotted and green dotted curves show the classical Fisher information $I_c(f_{\text{pps}}(x))$ with $c = 1$ and $c = -1$, respectively.

4.3.4 Inefficiency caused by post-selection

The post-selection is made by the measurement, which causes probabilistic state transition. So, success of the post selection depends on luck. We need extra amount of states $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$ to reach the lower bound of the inequality (4.54).

We consider the case when we have n copies of states $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$. Through the post-selection, we asymptotically have $n \cdot \Pr(f)$ copies of the states $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$ after the post-selection. The difference of the number of the states is caused by the post-selection. This loss make it difficult that the estimator achieve the true value. That is, when the fluctuation of the estimator $\hat{T}_1(\theta)$ made of the state $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$ becomes $I_{\text{SLD}}^{-1}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}})$, the fluctuation of the estimator $\hat{T}_2(\theta)$ made of $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$ becomes $[\Pr(f) \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})]^{-1}$.

On the other hand, the estimator $\hat{T}_1$ is composed of the state $(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}})^{\otimes n}$,

and the estimator $\hat{T}_2$ is composed by the state $(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})^{\otimes n \cdot \Pr(f)}$. The mean square error of the estimator of n copies of the states $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$ is bounded as $\langle (\hat{T}_1 - \theta \hat{I}^{\mathcal{H} \otimes \mathcal{K}})^2 \rangle_{\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}} \geq 1/(n \cdot I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}))$, while the mean squared error of the estimator of $n \cdot \Pr(f)$ copies of the states $\hat{\rho}_{\text{PPS}}^{\mathcal{K}}$ is bounded as

$$\langle (\hat{T}_2 - \theta \hat{I}^{\mathcal{K}})^2 \rangle_{\hat{\rho}_{\text{PPS}}^{\mathcal{K}}} \geq \frac{1}{n \cdot \Pr(f) \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})}.$$

Thus, we can evaluate the accuracy of the estimators by comparing the SLD-Fisher information the system has before the post-selection $I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}})$ and the SLD-Fisher information taken convergence seed into account is $\Pr(f) \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$. Since $|f\rangle_{\mathcal{H}}\langle f| \leq \hat{I}^{\mathcal{H} \otimes \mathcal{K}}$, we have

$$\begin{aligned} & \Pr[f] \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}}) \\ &= 4 \left(\langle \psi | \partial_{\theta} \tilde{B}^{\dagger} \partial_{\theta} \tilde{B} | \psi \rangle - \frac{|\langle \psi | \partial_{\theta} \tilde{B}^{\dagger} \tilde{B} | \psi \rangle|^2}{\langle \psi | \tilde{B}^{\dagger} \tilde{B} | \psi \rangle} \right). \end{aligned} \quad (4.68)$$

Therefore, we have

$$\begin{aligned} & \Pr[f] \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})/4 \\ &= {}_{\mathcal{K}} \langle \psi | {}_{\mathcal{H}} \langle i | (\hat{H} - \langle \hat{H} \rangle) e^{-i\theta(\hat{H} - \langle \hat{H} \rangle)} | f \rangle_{\mathcal{H}} \langle f | (\hat{H} - \langle \hat{H} \rangle) e^{i\theta(\hat{H} - \langle \hat{H} \rangle)} | i \rangle_{\mathcal{H}} | \psi \rangle_{\mathcal{K}} \\ &\leq \langle i, \psi | (\hat{H} - \langle \hat{H} \rangle)^2 e^{-i\theta(\hat{H} - \langle \hat{H} \rangle)} e^{i\theta(\hat{H} - \langle \hat{H} \rangle)} | i, \psi \rangle \\ &= \langle i, \psi | \hat{U}^{\dagger} (\hat{H} - \langle \hat{H} \rangle)^2 \hat{U} | i, \psi \rangle = I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}})/4, \end{aligned} \quad (4.69)$$

and we have

$$\Pr[f] \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}}) \leq I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}). \quad (4.70)$$

This inequality benefit that the SLD-Fisher never increases with the post-selection, and that post-selection is inefficient when the the same number of $\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}$ is available. That is, there is no benefit to use the weak amplification technique if we can access the combined system freely. Figure 4.9 shows $\Pr(f) \cdot I_{\text{SLD}}(\hat{\rho}_{\text{PPS}}^{\mathcal{K}})$ (green region) and $I_{\text{SLD}}(\hat{\rho}_{\text{int}}^{\mathcal{H} \otimes \mathcal{K}}) = 1/\sigma^2$ (yellow dotted line). This figure shows that the inequality (4.70) holds when the measured system is the two-level system and the probe is continuous variable.

4.3.5 Summary and conclusion of the section

The main result which we have obtained in this section is that, the signal amplification technique based on the WVA or more broadly the post-selection is inefficient for person measurement in the statistics sense. The post-selection conditions the probe state and its statistics. The post-selection whose success probability is small increases the SLD-Fisher information of the parameter θ . This means that rare condition caused by the post-selection increases the information from the measurement result in the probe. We also show the inequality

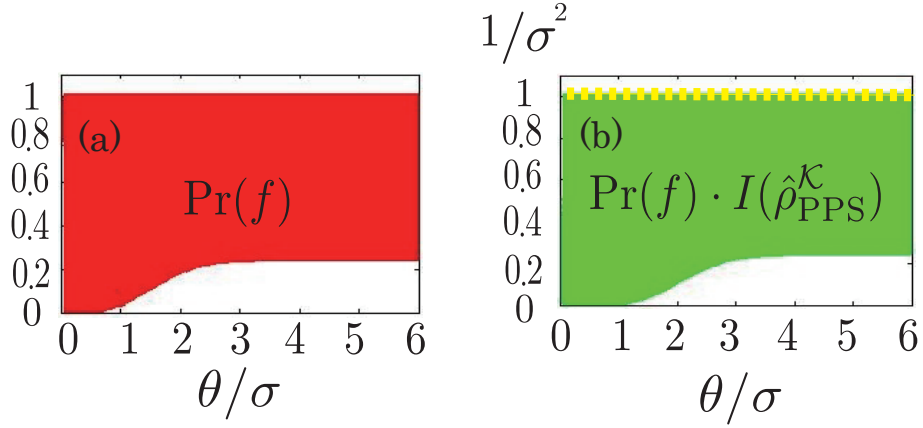


Figure 4.9: (a) The success probability $\Pr(f)$ and (b) Fisher information normalized by $\Pr(f)$, as functions of θ/σ . The regions represent the set of curves of $\Pr(f)$ and $\Pr(f)I(\hat{\rho}_{\text{ps}}^{\mathcal{K}})$ generated with the parameter $(t_1, t_2, s_1 - s_2)$.

(4.70) that means the inefficiency of the WVA. Taking into account the loss by the post-selection, we find that the SLD-Fisher information of the estimate parameter cannot increase, and the loss by the post-selection makes it difficult to reach the quantum Cramér -Rao bound. That is, the Fisher-information of the estimated parameter is never generated by the post-selection, and loss of the post-selection causes the less speed to achieve the true value.

Note that we obtained the inequality (4.70) limiting the case that the interaction Hamiltonian linearly depends on the estimate parameter θ i.e. $\theta\hat{H}$, and that the same inequality is held if the interaction is $f(\theta)\hat{H}$. We also remark that our result does not mean weak value amplification is useless. Especially, the weak value amplification can be helpful to measure the system with systematic errors [70, 71].

Chapter 5

Conclusions and discussions

We discussed the properties of quantum measurement for computation, back-action on the state and preciseness of a result. These properties are significant for accurate quantum computation. These properties were shown in this thesis with the examples, i.e., the theoretical results concerning the continuous state preparation and with the weak value amplification with the AAV weak measurement.

In Chapter 2, we reviewed the basic theories of the quantum measurement and estimation. We review the fundamental mathematical descriptions of quantum measurement, and introduced statistical inference, especially estimation theory, that suggests how to use the measurement results.

In Chapter 3, we showed the first result of our research, the proposal of an improved adaptive measurement scheme originally proposed by Jacobs. This scheme allows more stable and faster scheme than Jacobs'. Although we still need discussion for noises, like decoherence, our scheme has robustness against the external force even when its strength is unknown. Our discussion gave good suggestions for future. The controlling with measurement is thought to give the high fidelity for the state, since the measurement guarantees the accuracy. Especially our scheme gives an intuitive understanding of the state generation by continuous measurement. This scheme suggests how to choose the measured observable. It points out an appropriate way is making the measured observable whose eigenstate exists between the target and the current states. We expect that the present study can be extended for the higher dimensional system for straightforwardly.

In Chapter 4, we gave the second result of our research. We reevaluated the weak value amplification technique in two ways; the full order calculation of the probe shift on the probe system, and the quantum Fisher information. From the full order calculation of the expected value of the probe, we found that the post-selection causes the amplification effect when the interaction strength is small. On the other hand, from the discussion of the quantum Fisher information, we found that the rare post-selected state gives the large amplification. For the evaluation of the rarity, we focused on the data loss caused by the post-

selection and assessed the accuracy by the quantum Fisher information of the states with and without post-selection. We compared the information of the states, and understand that there is tradeoff between the estimation accuracy and data number in the post-selection. This work suggests that the weak value amplification is essentially not suitable for precise measurement in the sense of the estimation, although the weak value amplification has benefits for practical use.

In this thesis, we discussed two worthy properties of the quantum measurement, the back-action and the accuracy, and showed the theoretical improvements of these properties for the quantum computation. I know that we need further study in order to use our results for the computation. I believe that the optical quantum systems become a key role for improvement of our discussions.

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Appendix A

p -representation of the operator $\hat{x}$

For the calculation of the 1st moment of x and p , the following equations are useful:

$$\hat{x} = i \frac{\partial}{\partial p}, \quad (\text{A.1})$$

$$-i \frac{\partial}{\partial x} = \hat{p}. \quad (\text{A.2})$$

Here, we show that the observable $\hat{x}$ is equal to $i\partial_p$ in p -representation. From the canonical commutation relation $[\hat{x}, \hat{p}] = i$, we have

$$\begin{aligned} \langle p' | [\hat{x}, \hat{p}] | p \rangle &= i \langle p' | p \rangle \\ \Leftrightarrow \langle p' | (\hat{x}\hat{p} - \hat{p}\hat{x}) | p \rangle &= i\delta(p - p') \\ \Leftrightarrow (p - p') \langle p' | \hat{x} | p \rangle &= i\delta(p - p'). \end{aligned} \quad (\text{A.3})$$

Since δ function satisfies

$$x \frac{\partial \delta(x)}{\partial x} = -\delta(x), \quad (\text{A.4})$$

we have

$$\langle p' | \hat{x} | p \rangle = -i \frac{\partial \delta(p - p')}{\partial p}, \quad (\text{A.5})$$

and

$$\begin{aligned}\langle p'|\hat{x}|\psi\rangle &= \int dp \langle p'|\hat{x}|p\rangle \langle p|\psi\rangle \\ &= \int dp \{-i\delta'(p-p')\} \psi(p)\end{aligned}\tag{A.6}$$

$$\begin{aligned}&= -i[\delta(p-p')\psi(p)]_{-\infty}^{\infty} + i \int dp \delta(p-p')\psi'(p) \\ &= i\psi'(p') = i\frac{\partial}{\partial p'}\psi(p'),\end{aligned}\tag{A.7}$$

and

$$\begin{aligned}\hat{x}|\phi\rangle &:= \hat{x} \int dp \phi(p)|p\rangle \\ &= \int dp |p\rangle \langle p|\hat{x}|\phi\rangle \\ &= \int dp |p\rangle i\frac{\partial}{\partial p}\phi(p).\end{aligned}\tag{A.8}$$

Then, we have the p representation of $\hat{x}$ as $i\frac{\partial}{\partial p}$. Exchanging x and p and replacing the sign $i \rightarrow -i$. The same discussion holds with p in x representation. The operator $\hat{p}$ becomes $-i\frac{\partial}{\partial x}$ in x -representation.

Appendix B

Evolution of light by free energy

We derive the free Hamiltonian for the light traveling the distance l . Here, we assume $[\hat{x}, \hat{p}] = i\hbar$. The free time evolution of a particle with mass m is described by

$$\exp\left[-i\frac{p^2 t}{2m}\right], \quad (\text{B.1})$$

but the mass of the light equals to 0. According to the theory of the relativity, we have

$$E = mc^2, \quad (\text{B.2})$$

and we obtain $m = E/c^2$. We have $E = h\nu$ from quantum theory, and we substitute it into the exponential of Eq. (B.1). Then, we have

$$t\frac{p^2}{2m} = t\frac{p^2 c^2}{2E} = t\frac{p^2 c^2}{2h\nu}. \quad (\text{B.3})$$

From the momentum of light, we have $p = \hbar k$. The speed of light is c and the time for light traveling the distance l is $t = \frac{l}{c}$. We have

$$\frac{p^2 t}{2m\hbar} = \frac{\hbar k^2 c^2 t}{2E} = \frac{k^2 c^2 t}{2\pi\nu\hbar} = \frac{k^2 ct}{2\frac{2\pi\nu}{c}} = \frac{k^2 l}{4\pi\nu k_I} = \frac{k^2 l}{2k_I}. \quad (\text{B.4})$$

Therefore, the unitary operator when the light travels the distance l is $e^{-i\frac{\hbar k^2 l}{2k_I}}$. Here, we use the wave numbers of light $k_I = \frac{2\pi\nu}{c}$ and the distance $ct = l$ that the wave runs. This free evolution can be derived with the classical electromagnetism and it is known to be $e^{-i\frac{p^2 l}{2k_I}}$ [60].

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