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ON THE SPECTRUM OF A GRAPH

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ABSTRACT

In this paper we discuss about the spectrum of a graph. We obtain the relations between a regular graph and its spectrum, and a complete graph and its spectrum, respectively. We obtain a bound for eigenvalues of an oriented graph with loops as a generalization of a non-oriented graph without loops. We prove that the maximum eigenvalue of a graph equals to its uppar bound and its lower bound if and only if the graph is a complete graph and a regular graph, respectively.

1. The spectrum of regular graphs

Let G be a graph whose vertex-set VG is the set $\{v_1, v_2, \cdots, v_n\}$ and whose edge-set EG is the subset of the set of unordered pairs of elements of VG. We call a graph with n vertices and m edges is an (n, m) non-oriented graph. A vertex-subgraph of G is a graph constructed by taking a subset U of VG together with all edges of G which are incident in G only with vertices belonging to U.

The adjacency matrix $A(G)$ of an (n, m) non-oriented graph G is an $n\times n$ symmetric matrix whose entries a_{ij} are given by

$$a_{ij}=\begin{cases}1 & \text{if } \{v_i, v_j\}\in EG \\ 0 & \text{if } \{v_i, v_j\}\notin EG\end{cases} \tag{1.1}$$

The spectrum of an (n, m) graph G, Spec G, is the set of eigenvalues of $A(G)$ together with their multiplicities. Namely, if the distinct eigenvalues of $A(G)$ are $\lambda_1>\lambda_2>\cdots>\lambda_s$, and their multiplicities are $m(\lambda_1), m(\lambda_2), \cdots, m(\lambda_s)$, then we write the spectrum of a graph G by

$$\text{Spec } G = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_s) \end{pmatrix}. \tag{1.2}$$

We also write the maximum and minimum eigenvalues of $A(G)$ by $\lambda_{\max}A(G)$ and $\lambda_{\min}A(G)$, respectively. We use the notation $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$ in place of $\lambda_{\max}A(G)$ and $\lambda_{\min}A(G)$, respectively.

Now let us consider the spectrum of a regular graph. A graph is said to be regular of degree k if each of its vertices has degree k. It is known that a regular graph G of degree k has $\lambda_{\max}(G)=k$, and $m(\lambda_{\max}(G))=1$ if G is connected.

LEMMA 1. An (n, m) graph G which has p connected components is regular of degree $(2m/n)$ if and only if $\lambda_{\max}(G)=(2m/n)$ and $m(\lambda_{\max}(G))=p$.

Proof. $(\Rightarrow)$ By a suitable labelling of the vertices of G, the adjacency matrix $A(G)$ can be written in the partitioned form

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_p \end{bmatrix} \tag{1.3}$$

where submatrices $A_i (i=1, \cdots, p)$ are corresponding to the adjacency matrices of a connected component G_i of G.

As G is a regular connected graph of degree $2m/n$ and $\lambda_{\max}(G_i)=2m/n$ and its multiplicity is 1, i.e. $m(\lambda_{\max}(G_i))=1$. As the eigenvalues of G consist of all eigenvalues of $G_1, G_2, \cdots, G_p$, $\lambda_{\max}(G)=2m/n$ and its multiplicity is p.

$(\Leftarrow)$ For any real $n \times n$ symmetric matrix X and for any real non-zero column n-vector $\boldsymbol{z}$, we call $\{(\boldsymbol{z}, X\boldsymbol{z})/(\boldsymbol{z}, \boldsymbol{z})\}$ be Rayleigh quotient and denote it by $R(X: \boldsymbol{z})$. Here $(\boldsymbol{x}, \boldsymbol{y})$ is the inner product of vector $\boldsymbol{x}$ and $\boldsymbol{y}$. It is known that

$$\lambda_{\max}(X) \geq R(X: \boldsymbol{z}) \geq \lambda_{\min}(X) \quad \text{for } \forall \boldsymbol{z} \neq 0 \tag{1.4}$$

and the equality $R(X: \boldsymbol{z})=\lambda_{\max}(X)$ holds if and only if $\boldsymbol{z}$ is an eigenvector corresponding to the eigenvalue $\lambda_{\max}(X)$.

Now let us put $\boldsymbol{z}=[\overbrace{11 \cdots 1}^{n}]'$, then we have

$$\lambda_{\max}(G) \geq R(A(G): \boldsymbol{z}) = \frac{2m}{n} \tag{1.5}$$

On the other hand we have $\lambda_{\max}(G)=2m/n$ by the hypothesis. Hence $\boldsymbol{z}$ is an eigenvector corresponding to the eigenvalue $2m/n$, that is to say, $A\boldsymbol{z}=\{(2m/n)\}\boldsymbol{z}$. This implies each row sum of A is $2m/n$ and so G is a regular graph of degree $2m/n$.

Let a graph G has k components, than all k components of G are regular connected graphs of degree $(2m/n)$. Each component of G has $\lambda_{\max}(G)=2m/n$ whose multiplicity is one. Hence we have $k=p$.

2. The spectrum of the complete graph

The complete graph K_n has n vertices and each distinct pair is adjacent. It is known the spectrum of the complete graph K_n is

$$\text{Spec } K_n = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}. \tag{2.1}$$

LEMMA 2. If the spectrum of G is

$$\text{Spec } G = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}, \tag{2.2}$$

then G must be the complete graph K_n.

Proof. Let G be an (n, m) graph and the eigenvalues of $A(G)$ be $\lambda_1, \lambda_2, \cdots, \lambda_n (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$. Then

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A^2) = 2m \tag{2.3}$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = (n-1)^2 + (-1)^2(n-1) \tag{2.4}$$

$$\therefore m = \frac{n(n-1)}{2} \tag{2.5}$$

Hence it follows that G must be the complete graph K_n.

3. The lower and upper bounds for the maximum and minimum eigenvalues for a graph

LEMMA 3. For any (n, m) graph G with $n \geq 2$ and $m \geq 1$, we have

$$\lambda_{\max}(G) \geq 1, \quad -1 \geq \lambda_{\min}(G) \tag{3.1}$$

Proof. Any (n, m) graph G with $n \geq 2$ and $m \geq 1$ has at least one (2, 1) vertex-subgraph G_1. Let A_1 be the adjacency matrix of the vertex-subgraph G_1, then the adjacency matrix A of G can be written in the partitioned form

$$A = \left[\begin{array}{c|c} A_1 & * \\ \hline * & * \end{array}\right], \quad \text{where } A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.2}$$

Let $\boldsymbol{x}_1$ be a 2-vector which satisfies the condition $A_1\boldsymbol{x}_1=\lambda_{\max}(G_1)\boldsymbol{x}_1$ and $\boldsymbol{z}_1$ be a 2-vector which satisfies the condition $A_1\boldsymbol{z}_1=\lambda_{\min}(G_1)\boldsymbol{z}_1$. Let us put $\boldsymbol{x}=[\boldsymbol{x}_1\ \overbrace{0\cdots 0}^{n-2}]$ and $\boldsymbol{z}=[\boldsymbol{z}_1\ \overbrace{0\cdots 0}^{n-2}]$. Then

$$\lambda_{\max}(G_1)=R(A_1:\boldsymbol{x}_1)=R(A:\boldsymbol{x})\leq\lambda_{\max}(G) \tag{3.3}$$

$$\lambda_{\min}(G_1)=R(A_1:\boldsymbol{z}_1)=R(A:\boldsymbol{z})\geq\lambda_{\min}(G) \tag{3.4}$$

As

$$\lambda_{\max}(G_1)=1 \qquad \lambda_{\min}(G_1)=-1\,, \tag{3.5}$$

we have

$$\lambda_{\max}(G)\geq 1 \qquad -1\geq\lambda_{\min}(G)\,. \tag{3.6}$$

LEMMA 4. If a connected (n,m) graph G with $n\geqq 3$ is not the complete graph K_n, then

$$\lambda_{\max}(G)\geq 2^{1/2}, \qquad -2^{1/2}\geq\lambda_{\min}(G) \tag{3.7}$$

Proof. If a connected (n,m) graph G with $n\geqq 3$ is not the complete graph K_n, G has at least one (3, 2) vertex-subgraph G_1. A can be partioned as follows:

$$A=\left[\begin{array}{c:c} A_1 & * \\ \hdashline * & * \end{array}\right], \qquad \text{where } A_1=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{3.8}$$

As

$$\lambda_{\max}(G)\geq\lambda_{\max}(G_1) \tag{3.9}$$

$$\lambda_{\min}(G)\leq\lambda_{\min}(G_1) \tag{3.10}$$

and

$$\lambda_{\max}(G_1)=2^{1/2} \qquad \lambda_{\min}(G_1)=-2^{1/2}. \tag{3.11}$$

Then we have

$$\lambda_{\max}(G)\geq 2^{1/2} \qquad \lambda_{\min}(G)\leq -2^{1/2}. \tag{3.12}$$

4. A bound for the eigenvalues of an oriented graph

In this section, we consider an (n,m) oriented graph G with loops whose eigenvalues of the adjacency matrix A are all real numbers. The difference between non-oriented graph and an oriented graph is only its adjacency matrix A is symmetric or not.

THEOREM 1. Let an (n,m) oriented graph G has t loops and c cycles whose length are 2, then

$$\frac{t-\sqrt{(n-1)(2cn+nt-t^2)}}{n}\leq\lambda_i\leq\frac{t+\sqrt{(n-1)(2cn+nt-t^2)}}{n} \tag{4.1}$$

$$(i=1,2,\cdots,n)$$

Proof. By the hypotheses we have

$$\sum_{i=1}^{n}\lambda_i=tr(A)=t \tag{4.2}$$

$$\sum_{i=1}^{n}\lambda_i^2=tr(A^2)=2c+t \tag{4.3}$$

Let us put $x=[\lambda_2,\lambda_3,\cdots,\lambda_n]'$ and $y=[\overbrace{1,1,\cdots,1}^{n-1}]'$. Then

$$|(\boldsymbol{x},\boldsymbol{y})|=\left|\sum_{i=2}^{n}\lambda_i\right|=|t-\lambda_1| \tag{4.4}$$

$$||\boldsymbol{x}||=\sqrt{\sum_{i=2}^{n}\lambda_i^2}=\sqrt{(2c+t)-\lambda_1^2} \tag{4.5}$$

$$||\boldsymbol{y}||=\sqrt{\sum_{i=2}^{n}1^2}=\sqrt{n-1} \tag{4.6}$$

For these $\boldsymbol{x}$ and $\boldsymbol{y}$, applying the Cauchy-Schwarz inequality we have

$$|t-\lambda_1|\leq\sqrt{(2c+t)-\lambda_1^2}\sqrt{n-1} \tag{4.7}$$

$$\therefore\frac{t-\sqrt{(n-1)(2nc+nt-t^2)}}{n}\leq\lambda_1\leq\frac{t+\sqrt{(n-1)(2nc+nt-t^2)}}{n}. \tag{4.8}$$

In a similar fashion, we can show that the inequality (4.8) holds for any $\lambda_i (i=2,3,\cdots,n)$.

This theorem can be reduced to a non-oriented graph with loops. The number of cycles whose length are 2 in an oriented graph equals to the number of edges in a non-oriented graph, i.e. $c=m$. By putting $c=m$ and $t=0$ in (4.8), we have

$$-\sqrt{\frac{2m(n-1)}{n}}\leq\lambda_i\leq\sqrt{\frac{2m(n-1)}{n}} \qquad (i=1,2,\cdots,n) \tag{4.9}$$

Especially

$$\lambda_{\max}(G)\leq\sqrt{\frac{2m(n-1)}{n}}, \tag{4.10}$$

and this result coincides with the result which has already established.

5. The upper and lower bounds for the maximum eigenvalue of a graph.

Now let us consider a non-oriented graph again.

THEOREM 2. In an (n, m) graph G with $n \geq 1$ and without loops, we have

$$\frac{2m}{n} \underset{(1)}{\leq} \lambda_{\max}(G) \underset{(2)}{\leq} \sqrt{\frac{2m(n-1)}{n}}, \tag{5.1}$$

where the equality (1) holds if and only if G is a regular graph of degree $(2m/n)$ and the equality (2) holds if and only if G is the complete graph K_n.

Proof. The equality of (1) is clear from Lemma 1.

We will prove that the equality of the Cauchy-Schwarz inequality

$$|(\boldsymbol{x}, \boldsymbol{y})| \leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \tag{5.2}$$

holds if and only if $\lambda_{\max}(G) = \sqrt{2m(n-1)/n}$ is satisfied.

Let us put $\boldsymbol{x} = [\lambda_2, \lambda_3, \cdots, \lambda_n]'$ and $\boldsymbol{y} = [\overbrace{1, 1, \cdots, 1}^{n-1}]'$, then the equality of (5,2) holds if and only if $\boldsymbol{y} = a\boldsymbol{x}$ is satisfied, that is when $\lambda_2 = \lambda_3 = \cdots = \lambda_n (\equiv \lambda)$ is satisfied. As the graph G has no loops,

$$0 = trA = \lambda_1 + (n-1)\lambda = \lambda_{\max}(G) + (n-1)\lambda \tag{5.3}$$

$$\therefore \lambda_{\max}(G) = -(n-1)\lambda, \tag{5.4}$$

we have

$$|(\boldsymbol{x}, \boldsymbol{y})| = \left| \sum_{i=2}^{n} \lambda_i \right| = |(n-1)\lambda| = |-\lambda_{\max}(G)|. \tag{5.5}$$

As the graph G has m edges,

$$2m = tr(A^2) = \sum_{i=1}^{n} \lambda_i^2 = \lambda_1^2 + (n-1)\lambda^2 = \lambda^2_{\max}(G) + (n-1)\lambda^2 \tag{5.6}$$

$$\therefore (n-1)\lambda^2 = 2m - \lambda^2_{\max}(G), \tag{5.7}$$

we have

$$||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| = \sqrt{(n-1)\lambda^2} \cdot \sqrt{n-1} = \sqrt{2m - \lambda^2_{\max}(G)}\sqrt{n-1}. \tag{5.8}$$

From (5.5) and (5.8),

$$|-\lambda_{\max}(G)| = \sqrt{2m - \lambda^2_{\max}(G)}\sqrt{n-1} \tag{5.9}$$

$$\therefore \lambda_{\max}(G) = \sqrt{\frac{2(n-1)m}{n}} \tag{5.10}$$

$$\therefore \lambda = -\sqrt{\frac{2m}{(n-1)n}} \tag{5.11}$$

Hence the spectrum of G must be

$$\text{Spec } G=\begin{pmatrix} \sqrt{\dfrac{2m(n-1)}{n}} & -\sqrt{\dfrac{2m}{n(n-1)}} \\ 1 & n-1 \end{pmatrix}. \tag{5.12}$$

As $m \leq n(n-1)/2$, it must

$$-\sqrt{\frac{2m}{n(n-1)}} \geq -1. \tag{5.13}$$

On the other hand, by Lemma 3,

$$-1 \geq \lambda_{\min}(G) = -\sqrt{\frac{2m}{n(n-1)}}\,. \tag{5.14}$$

It must

$$-\sqrt{\frac{2m}{n(n-1)}} = -1 \tag{5.15}$$

$$\therefore m = \frac{n(n-1)}{2}\,, \tag{5.16}$$

and the graph G must be the complete graph K_n. Hence we can conclude that the equality of (2) holds if and only if G is the complete graph K_n.

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