Title	Graph theoretic concepts and the incidence matrix
Sub Title	
Author	竹中, 淑子(Takenaka, Yoshiko)
Publisher	慶応義塾大学工学部
Publication year	1971
Jtitle	Keio engineering reports Vol.24, No.3 (1971.) ,p.23- 43
JaLC DOI	
Abstract	Several different kinds of matrices such as an incidence matrix S of arcs, an incidence matrix A of edges, a loop matrix B and a path matrix Pαβ and so on are used in current theories of graph in order to obtain insights into various properties of graph. Nevertheless so far as the auther of the present paper is aware of, there has been no systematic use of any of these matrices to establish a unified expression for various fundamental properties of a graph. One of the main purposes of this paper is to appeal to a systematic use of an incidence matrix A of edges in terms of which we shall give a clear and simple expression for each of fundamental properties of a graph. This approach makes is possible to apply various theories of matrices, MARKOV chain, linear algebra, linear programming, dynamic programming, game theory and block design to systematic investigation of graph in a very simple and transparent way. In Section 1 we shall give a systematic use of an edge-edge incidence matix A in terms of which various fundamental notaions in graph theory will be enunciated. Apart from some trivial and elementary translations of these notations there do exist a few enunications which deserve our specific attentions. For instance one of the translations given in Table 1.3 makes us possible to investigate the existence of a kernel of the sum graph or that of the product graph of n given graphs without making use of GRUNDY functions each of which is associated with each of n graphs. Our considerations are applicable not only to a graph without any loop but also to a graph with loops. The second purpose of this paper is to investigate the problem of perfect matching and that of the famous HAMILTONIAN circuit. In Section 2 we shall appeal to a systematic use of the incidence matrix of edges for these classical problems. On the basis of an incidence matrix of edges we shall introduce a transition probability matrix associated with a given graph, which enables us to consider a classification of states in the homogenous MARKOV chain. The necessary and sufficient condition for a graph to have a HAMILTONIAN circuit is enunciated by Theorem 1 in Subsection 2.3 and by Theorem 2 in Subsection 2.4. The classical solution to the famous EULER problem of knight's tour, which has been given for more than one cenutry merely as due to the rule of thumb without any explicit theoretical foundation, is now reduced to a special application of our Theorem 2 given in Subsection 2.4 and verified in Subsection 2.5. That is to say, the existence of the solution for the EULER problem of knight's tour is justified by our Theorem 2. In Section 3 we shall give another approach to the problems discussed in Section 2, with restriction to a graph of even order. The main technique in Section 3 is to appeal to an decomposition of the set of all vertices into two subsets, which gives us a partial graph which is a simple graph and to which the theory of KöNInG and HALL [2] can be applied. Here again the necessary and sufficient condition for a graph of even order to have a perfect matching is given in Theorem 3 in Subsection 3.1, and similarly that for a graph of even order to have a HAMILTONIAN circuit is given in Theorem 4 in Subsection 3.2, which also give us the theoretical foundation for the existence of solution to the EULER problem of knight's tour again.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00240003-0023

KEIO ENGINEERING REPORTS
VOL. 24 NO. 3

GRAPH THEORETIC CONCEPTS AND THE INCIDENCE MATRIX

BY

YOSHIKO TAKENAKA

FACULTY OF ENGINEERING
KEIO UNIVERSITY
YOKOHAMA 1971

KEIO ENGINEERING REPORTS
VOL. 24, NO. 3, pp. 23-43, 1971

GRAPH THEORETIC CONCEPTS AND THE INCIDENCE MATRIX

YOSHIKO TAKENAKA

Dept. of Administration Engineering, Keio University, Yokohama, Japan

(*Received Nov. 10, 1971*)

ABSTRACT

Several different kinds of matrices such as an incidence matrix S of arcs, an incidence matrix A of edges, a loop matrix B and a path matrix $P_{\alpha\beta}$ and so on are used in current theories of graph in order to obtain insights into various properties of graph. Nevertheless so far as the auther of the present paper is aware of, there has been no systematic use of any of these matrices to establish a unified expression for various fundamental properties of a graph. One of the main purposes of this paper is to appeal to a systematic use of an incidence matrix A of edges in terms of which we shall give a clear and simple expression for each of fundamental properties of a graph. This approach makes is possible to apply various theories of matrices, MARKOV chain, linear algebra, linear programming, dynamic programming, game theory and block design to systematic investigation of graph in a very simple and transparent way.

In Section 1 we shall give a systematic use of an edge-edge incidence matix A in terms of which various fundamental notaions in graph theory will be enunciated. Apart from some trivial and elementary translations of these notations there do exist a few enunications which deserve our specific attentions. For instance one of the translations given in Table 1.3 makes us possible to investigate the existence of a kernel of the sum graph or that of the product graph of n given graphs without making use of GRUNDY functions each of which is associated with each of n graphs. Our considerations are applicable not only to a graph without any loop but also to a graph with loops.

The second purpose of this paper is to investigate the problem of perfect matching and that of the famous HAMILTONIAN circuit. In Section 2 we shall appeal to a systematic use of the incidence matrix of edges for these classical problems. On the basis of an incidence matrix of edges we shall introduce a transition probability matrix associated with a given graph, which enables us to consider a classification of states in the homogenous MARKOV chain. The necessary and sufficient condition for a graph to have a HAMILTONIAN circuit is enunciated by Theorem 1 in Subsection 2.3 and by Theorem 2 in Subsection 2.4.

The classical solution to the famous EULER problem of knight's tour, which has been given for more than one cenutry merely as due to the rule of thumb without any explicit

theoretical foundation, is now reduced to a special application of our Theorem 2 given in Subsection 2.4 and verified in Subsection 2.5. That is to say, the existence of the solution for the EULER problem of knight's tour is justified by our Theorem 2.

In Section 3 we shall give another approach to the problems discussed in Section 2, with restriction to a graph of even order. The main technique in Section 3 is to appeal to an decomposition of the set of all vertices into two subsets, which gives us a partial graph which is a simple graph and to which the theory of KÖNING and HALL [2] can be applied. Here again the necessary and sufficient condition for a graph of even order to have a perfect matching is given in Theorem 3 in Subsection 3.1, and similarly that for a graph of even order to have a HAMILTONIAN circuit is given in Theorem 4 in Subsection 3.2, which also give us the theoretical foundation for the existence of solution to the EULER problem of knight's tour again.

SECTION 1

GRAPH THEORETIC CONCEPTS AND THE INCIDENCE MATRIX

1.1. Definitionons

An edge-edge incidence matrix $A=(a_{ij})$ of an oriented graph (X, Γ) is defined by

$$a_{ij}=\begin{cases} 1 : (x_i, x_j) \in U \\ 0 : (x_i, x_j) \notin U, \end{cases} \tag{1. 1}$$

where U is the set of arcs of the graph (X, Γ).

We define the operations $+$ and $\times$ by

$$\begin{aligned} &1+1=1, \quad 0+1=1+0=1, \quad 0+0=0, \\ &1\times1=1, \quad 1\times0=0\times1=0, \quad 0\times0=0, \end{aligned} \tag{1. 2}$$

i.e. the operations $+$ and $\times$ are BOOLEAN addition and BOOLEAN multiplication respectively.

We define a set of matrices $A^{(n)}=(a_{ij}^{(n)})$ $(n=1, 2, 3, \cdots)$ with $A^{(1)}=A$ by

$$a_{ij}^{(n)}=\sum_{k=1} a_{ik}^{(n-1)}a_{kj} \qquad n=2, 3, \cdots.$$

Consequently, we have

$$A^{(n)}=A^{(n-1)}A \qquad n=1, 2, 3, \cdots \tag{1. 3}$$

Let I and E be an n-vector and an $n\times n$ matrix, respectively, whose elments are all 1. The column vectors of an $(n\times n)$-matrix A are defined by $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots \boldsymbol{a}_n$. and the row vectors are $\boldsymbol{a}^1, \boldsymbol{a}^2, \cdots \boldsymbol{a}^n$. Furthermore, we define the following:

$$A^0=\sum_{i=1}^{n}A^{(i)}$$

$$|A|=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij} \qquad (1.4)$$

$$|\boldsymbol{a}_i|=\sum_{j=1}^{n}a_{ij}.$$

For a set of vertices $X=\{x_1, x_2, \cdots x_n\}$ of a graph we define the index set of X as $J_X=\{1, 2, \cdots n\}$ and for any subset $S=\{x_{i_1}, x_{i_2}, \cdots x_{i_k}\}$ of X we define the index set of S as $J_S=\{i_1, i_2, \cdots i_k\}$.

Let us define by σ any permutation of $(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots \boldsymbol{a}_n)$ such that

$$\sigma=\begin{pmatrix} \boldsymbol{a}_1, & \boldsymbol{a}_2, & \cdots & \boldsymbol{a}_n \\ \boldsymbol{a}_{k_1}, & \boldsymbol{a}_{k_2}, & \cdots & \boldsymbol{a}_{k_n} \end{pmatrix}, \qquad (1.5)$$

and denote by $A(\sigma)$ the incidence matrix whose column vectors are $\boldsymbol{a}_{k_1}, \boldsymbol{a}_{k_2}, \cdots \boldsymbol{a}_{k_n}$.

Let B be a matrix which can be obtained from the incidence matrix A of the graph (X, Γ) by replacing some of the 1's by 0's. Now it is evident that we can find a graph (X, Γ') whose incidence matrix is B. The graph (X, Γ') is called a partial graph of (X, Γ).

Let C be a matrix some of whose column vectors (correspondingly, row vectors) are added and make one column vector (row vector) at the incidence matrix A of the graph (X, Γ). We call this operation shrinkage and the graph (X, Γ'') whose incidence matrix C is called a subgraph of (X, Γ). Let us define a shrinkage matrix of a matrix C by C^*.

1.2. Definitions on the Types, Arcs, Paths and Fundamental Numbers of a Graph

Most notation of graph theory can be related to its incidence matrix and its manipulations.

Table 1.1.

Graph (X, Γ), $\lvert X\rvert=n$	incidence matrix A $(n\times n)$
1. simple	$\exists\sigma: A(\sigma)=\left[\begin{array}{c\|c} 0 & A' \\ \hline 0 & 0 \end{array}\right]$
2. complete	$A+A^T=E$
3. connected	$\sum_{i=1}^{n}A^{(i)}=E$
4. transitive	$\exists k>1: a_{ij}^{(k)}=1 \Rightarrow a_{ij}=1$
5. $\exists$arc (x_i, x_j) / otherwise	$a_{ij}=\begin{cases}1\\0\end{cases}$

6. $\begin{cases} {}^{\exists}\text{path } (x_i, \cdots x_j) \text{ of the length } k \\ \text{otherwise} \end{cases}$	$a_{ij}^{(k)} = \begin{cases} 1 \\ 0 \end{cases}$
7. $\begin{cases} {}^{\exists}\text{circuit } (x_i, \cdots x_i) \text{ of the length } k \\ \text{otherwise} \end{cases}$	$a_{ii}^{(k)} = \begin{cases} 1 \\ 0 \end{cases}$
8. $^{\exists}$perfect matching	$^{\exists}$partial graph (X, Γ'), $B: \|B\|=n$ $\|b_i\|=\|b^j\|=1$ $(i, j=1, 2, \cdots n)$
9. $^{\exists}$Hamiltonian path	$^{\exists}\sigma: a_{i\ i+1}(\sigma)=1$ $(i=1, 2, \cdots n-1)$
10. $^{\exists}$Hamiltonian circuit	$^{\exists}\sigma: a_{i\ i+1}(\sigma)=1$ $(i=1, 2, \cdots n-1)$ $a_{n\ 1}(\sigma)=1$
11. $^{\exists}$arborescence with root x_i	$\sum_{j=1}^{n} a_i^j = [11\cdots1\overset{i}{0}1\cdots1]$
12. connected component number p	$^{\exists}\sigma: A(\sigma)=\begin{bmatrix} E & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{bmatrix}\begin{matrix} 1 \\ 2 \\ \vdots \\ p \end{matrix}$
13. cyclomatic number $V(G)$	$V(G)=\|A\|-n+p$
14. chromatic number p	$^{\exists}C^*(\sigma): c_{ii}^*=0$ $(i=1, 2, \cdots p)$

1.3. Definitions on a Subset S of X

Let Y_1, Y_2, Y_3, Y_4, Y_5 be n-vectors such that the following conditions are satisfied:

Table 1.2.

	at J_S	at $J_S^c(=J_{X-S})$
Y_1	1	0
Y_2	0	1
Y_3	1	*
Y_4	0	*
Y_5	*	1

Table 1.3.

$S(\subset X)$	incidence matrix A
15. internally stable	$AY_1=Y_4$
16. externally stable	$AY_3=Y_5$
17. kernel	$AY_1=Y_2$
18. basis	$A^0Y_2=Y_2$
19. support	$AY_2=Y_1$

Here * means no restriction, that is, it may be either 0 or 1.

1.4. Some Proofs

In this section, we give proofs of the relations 1-19. Since some of them are trivial and many of them are similar, we shall only give a few examples which are sufficient to indicate the proofs for all 19 relations. We shall make use of the notation $A \Longleftrightarrow B$ to show the equivalence of two assertions A and B.

[*proof of* (3)]

A graph (X, Γ) is connected.

$\overset{\text{(def.)}}{\Longleftrightarrow}$

For any i and j, there exist two natural numbers $k_1 \geqq 1$ and $k_2 > 0$ such that $p_{ij}^{(k_1)}=1$ and $p_{ji}^{(k_2)}=1$ are satisfied.

$\Longleftrightarrow$

$$A^{(1)}+A^{(2)}+\cdots+A^{(n)}=\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

$\Longleftrightarrow$

$$\sum_{i=1}^{n} A^{(i)}=E \tag{1. 6}$$

(q.e.d.)

[*proof of* (14)]

A graph (X, Γ) is p-chromatic.

$\overset{\text{(def.)}}{\Longleftrightarrow}$

All vertices of the graph (X, Γ) can be painted with p distinct colours in such a way that no two adjacent vertices are of the same colour.

$\Longleftrightarrow$

There exists σ for which we have

$$A(\sigma)=\overset{1 \cdots\cdots p}{\left[\begin{array}{c:c:c:c} 0 & * & * & * \\ \hdashline * & 0 & * & * \\ \hdashline * & * & 0 & * \\ \hdashline * & * & * & 0 \end{array}\right]}\begin{matrix} 1 \\ 2 \\ \vdots \\ p \end{matrix} \tag{1. 7}$$

here 0 is a matrix whose elements are all 0.

$\Longleftrightarrow$

The shrinkage matrix $C^*(\sigma)$ of $A(\sigma)$ can be expressed in the following way:

$$C^*(\sigma)=\overbrace{\begin{bmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{bmatrix}}^{p}\Bigg\} p \tag{1. 8}$$

(q.e.d.)

[*proof of* (17)]

S is a kernel of the graph (X, Γ).

$\overset{\text{(def.)}}{\Longleftrightarrow}$

For any $x \in S$, $\Gamma x \cap S = \phi$ and for any $x \in S^c$, $\Gamma x \cap s \neq \phi$.

$\Longleftrightarrow$

There exists σ for which we have

$$A(\sigma) = \overset{\quad\overbrace{J_s}\ \ \overbrace{J_s^c}}{\left[\begin{array}{c:c} 0 & * \\ \hdashline B & * \end{array}\right]} \begin{array}{l} \big) J_s \\ \big) J_s^c \end{array} \tag{1. 9}$$

where B is a matrix whose every column vector has at least one 1.

$\Longleftrightarrow$

$$A(\sigma) \times \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \big\} J_s \\ \big\} J_s^c \end{array} = \left(\begin{array}{c:c} 0 & * \\ \hdashline B & * \end{array}\right) \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hdashline 1 \\ \vdots \\ 1 \end{pmatrix} \tag{1. 10}$$

$\Longleftrightarrow$

$$AY_1 = Y_2 \tag{1. 11}$$

(q.e.d.)

[*proof of* (18)]

S is a basis of the graph (X, Γ).

$\overset{\text{(def.)}}{\Longleftrightarrow}$

(i) If $b_1, b_2 \in S$ and $b_1 \neq b_2 \Rightarrow b_1 \leq^c b_2$, $b_2 \leq^c b_1$.

(ii) If $x \notin S$, then there exists $b \in S$ such that $x \leq b$, here $x \leq y$ states that there exists a path from the vertex x to the vertex y and $x \leq^c y$ states that there exists no path.

$\Longleftrightarrow$

There exists σ for which we have

$$A^0(\sigma) = \left[\begin{array}{c:c} 0 & * \\ \hdashline B & * \end{array}\right] \begin{array}{l} \big) J_s \\ \big) J_s^c \end{array} \tag{1. 12}$$

$\Longleftrightarrow$

$$A^0(\sigma) \times \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \big\} J_s \\ \big\} J_s^c \end{array} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hdashline 1 \\ \vdots \\ 1 \end{pmatrix} \tag{1. 13}$$

$\Longleftrightarrow$

$$A^0 Y_1 = Y_2 \tag{1. 14}$$

(q.e.d.)

[*proof of* (19)]

S is a support of the non-oriented graph (X, Γ).

(def.)
$\Longleftrightarrow$

Every edge of X has at least one vertex in S.

$\Longleftrightarrow$

There exists σ for which we have

$$A(\sigma)=\begin{array}{c} \overbrace{\quad}^{J_s}\ \overbrace{\quad}^{J_s^c} \\ \left[\begin{array}{c|c} * & B \\ \hline * & 0 \end{array}\right]\begin{array}{l} \}J_s \\ \}J_s^c \end{array}\end{array} \tag{1.15}$$

$\Longleftrightarrow$

$$A(\sigma)\times\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.16}$$

$\Longleftrightarrow$

$$AY_2=Y_1 \tag{1.17}$$

(q.e.d.)

SECTION 2

EXISTENCE OF A HAMILTONIAN CIRCUIT (I)

A transition probability matrix can be associated with an oriented graph (X, Γ), and the classification of states in the homogenous MARKOV chain defined by this transition probability matrix is applied to a graph theory. Several sets of the necessary and sufficient conditions for a graph to have a HAMILTONIAN circuit are obtained by means of the classification of these states.

2.1. Definitions

To each pont x_i belonging to a finite or countable infinite graph (X, Γ) let there be assigned a set of exactly q_i arcs having the point x_i as the starting point, which are designated by (x_i, x_{j1}), (x_i, x_{j2}), $\cdots$ (x_i, x_{ji}) where $1\leq q_i<\infty$, $i=1, 2, \cdots$. With each of these q_i arcs the equal probability $1/q_i$ is associated in the sense that just in the passage of unit time a moving point which exists at the point x_i will move to one of x_{jh} ($h=1, 2, \cdots q_i$) with the probability $1/q_i$. This is the fundamental aspect of the transition probability matrix in which the transition probability p_i is equal to $1/q_i$ ($i=1, 2, \cdots$). However it will be convenient to appeal to the following procedure.

Let a matrix of transition probability $A=(a_{ij})$ associated with a graph (X, Γ) be defined by

$$a_{ij}=\begin{cases} 1 & \text{if} \quad (x_i, x_j)\in U \\ 0 & \text{if} \quad (x_i, x_j)\notin U, \end{cases} \tag{2. 1}$$

where U is a set of all arcs in the graph (X, Γ). In the consequence $a_{ij}=1>0$ if and only if there exists an arc from the point x_i to the point x_j in the graph (X, Γ).

A set of matrices $A^{(n)}=(a_{ij}^{(n)})$ is defined by $A^{(n)}=A^{(n-1)}$ ($n\geq 2$) and $A^{(1)}=A$, where

$$a_{ij}^{(k)}=\begin{cases} 1 : \sum_{l=1}^{n} a_{il}^{(k-1)}a_{lj}>0 \\ \cdots\cdots \\ 0 : \sum_{l=1}^{n} a_{il}^{(k-1)}a_{lj}=0. \end{cases} \qquad (k=2, 3, \cdots) \tag{2. 2}$$

In the consequence $a_{ij}^{(k)}=1>0$ if and only if there exists a path of length k from the point x_i to the point x_j in the graph (X, Γ).

Now a set of matrices of transition probabilities $P^{(n)}=(P_{ij}^{(n)})$ ($n\geq 2$) and $P^{(1)}=P=(p_{ij})$ associated with the graph (X, Γ) be defined by means of the set of $A^{(n)}$ ($n\geq 1$):

$$p_{ij}=\frac{a_{ij}}{q_i} \tag{2. 3}$$

$$p_{ij}^{(k)}=\frac{a_{ij}^{(k)}}{h_i}, \qquad (k=2, 3, \cdots) \tag{2. 4}$$

where h_i is the number of positive elements among the set of $\{a_{ij}^{(k)}\}, =1, 2, \cdots$.

With the transition probabiity matrix P we can define a MARKOV chain in a graph (X, Γ). We call this MARKOV chain a MARKOV chain associated with the graph (X, Γ).

If we concern a HAMILTONIAN path in a graph (X, Γ), it is sufficient to conpider only the matrices $A^{(k)}$ ($k=1, 2, \cdots$) but not the matrices $P^{(k)}$ ($k=1, 2, \cdots$), because it is sufficient to have merely $p_{ij}^{(k)}=0$ or $p_{ij}^{(k)}>0$.

2.2. A Classification of States

In this subsection we will state the well-known results on a classification of states in a homogenous MARKOV chain (cf. [2]).

We write $i\Rightarrow j$ if there exists $n>0$ such that $p_{ij}^{(n)}>0$, and $i\Rightarrow^c j$ if $p_{ij}^{(n)}=0$ for all $n>0$. $i\Leftrightarrow j$ if $i\Rightarrow j$ and $j\Rightarrow i$.

DEFINITION 1. A state i_0 is called a stable state if $j \Rightarrow i_0$ for all j which satisfies $i_0 \Rightarrow j$. A state i_0 is called an isolated state if $j \Rightarrow^c i_0$ for all j.

The relation $\Leftrightarrow$ satisfies the following three equivalent conditions;

(i) If $i \Leftrightarrow j$ and $j \Leftrightarrow k$, then $j \Leftrightarrow k$.

(ii) If $i \Leftrightarrow j$, then $j \Leftrightarrow i$.

(iii) If for each i there exists at least one j such that $i \Leftrightarrow j$, then $i \Leftrightarrow i$.

The class of all isolated states in X is denoted by C_0, and $X-C_0$ be classified into the family of disjoint sets C_1, C_2, $\cdots$ in the following way: $i \Leftrightarrow j$ if $i, j \in C_\alpha$ (≥ 1), $i \Leftrightarrow^c j$ if $i \in C_\alpha$, $j \in C_\beta$ $(\alpha \neq \beta)$.

PROPOSITION 1. The set of all states in C_α $(\alpha \geq 1)$ is a set of stable states or a set of unstable state.

For any $i \in C_\alpha$ $(\alpha \geq 1)$ let M_i be the set of n such that $P_{ii}^{(n)} > 0$ and let $d(i)$ be the G.C.M. of M. We say $d(i)$ is a period of i.

PROPOSITION 2. $d(i) = d(j) = d$ for any $i, j \in C_\alpha$ $(\alpha \geq 1)$.

PROPOSITION 3. If $P_{ij}^{(l)} > 0$, $P_{ij}^{(m)} > 0$ for $i, j \in C_\alpha$ $(\alpha \geq 1)$, then $l = m$ (mod. d), where d is a period of C_α.

Let us divide the set C_α $(\alpha \geq 1)$ into d classes $C_i(1)$, $C_i(2)$, $\cdots$ $C_i(d)$ by means of mod. d to which the remainder $j (j = 1, 2, \cdots d)$ of $l (= kd + j)$ with $P_{ij}^{(l)} > 0$ belongs:

$$C_\alpha = C_i(1) + C_i(2) + \cdots + C_i(d) \tag{2. 5}$$

for any $i \in C_\alpha$. Here $j \in C_i(m_0)$ means that l with $P_{ij}^{(l)} > 0$ satisfies $l = m_0$ (mod. d).

PROPOSITION 4. If $j_0 \in C_i(\alpha)$, $j_1 \in C_i(\beta)$, $P_{j_0 j_1}^{(l)} > 0$, then $l = \beta - \alpha$ (mod. d).

PROPOSITION 5. If $n = r$ (mod. d), then

$$P(x_n \in C_i(r) | x_0 = i) = \sum_{j \in C_i(r)} P_{ij}(n) = 1. \tag{2. 6}$$

2.3. A Connected Graph with a Period d

We consider the graph (X, Γ) which is connected and has a period d. In this case, as the graph (X, Γ) is connected,

$$C_0 = \phi, \quad C_2 = C_3 = \cdots = C_n = \phi \tag{2. 7}$$

is satisfied and as the graph (X, Γ) has a period d, C_1 is composed of d subsets $C_i(1)$, $C_i(2)$, $\cdots$ $C_i(d)$ for any $x_i \in C_1$. This means X can be divided into $C_i(1)$, $C_i(2)$, $\cdots$ $C_i(d)$ in the graph (X, Γ);

$$X = \sum_{j=1}^{d} C_i(j) \tag{2. 8}$$

LEMMA 2.1. In a connected graph (X, Γ), $|x| = b \times d$ with a period d, a set X can be divided into b subsets as b paths of length d, each point of which belongs to exactly one of the d subsets $C_i(1)$, $C_i(2)$, $\cdots$, $C_i(d)$ respectively.

$\Longleftrightarrow$

The following two conditions (i), (ii) are satisfied. Here $|A|$ is the number of elements of a set A.

(i) $$|C_i(1)|=|C_i(2)|=\cdots=|C_i(d)|=b \tag{2.9}$$

(ii) For any subset $S(\subset X)$,

$$|S|\leq|\Gamma S| \tag{2.10}$$

Proof. ($\Longrightarrow$)

$$C_i(k)\cap C_i(j)=\phi \qquad (k\neq j) \tag{2.11}$$

and if $x\in C_i(d)$, then $\Gamma x\in C_i(1)$. If $x\in C_i(j)$, then $\Gamma x\in C_i(j+1)$ for $j=1, 2, \cdots, (d-1)$.

In a simple graph $(C_i(1), C_i(2), \Gamma)$, for any subset $S(\subset C_i(1))$, $|\Gamma S|\geq|S|$ is satisfied. Consequently according to the theory of KÖNIG and HALL [1] there is a matching from $C_i(1)$ to $C_i(2)$. Let us write this matching

$$\{(x_{11}, x_{12}), (x_{21}, x_{22}), \cdots, (x_{b1}, x_{b2})\}. \tag{2.12}$$

Similarly in simple graph $(C_i(2), C_i(3), \Gamma)$, $\cdots$, $(C_i(d-1), C_i(d), \Gamma)$, there are matchings

$$\begin{gathered}\{(x_{12}, x_{13}), \cdots, (x_{b2}, x_{b3})\}\\ \{(x_{13}, x_{14}), \cdots, (x_{b3}, x_{b4})\}\\ \cdots\cdots\cdots\\ \{(x_{1d-1}, x_{1d}), \cdots, (x_{bd-1}, x_{bd})\}.\end{gathered} \tag{2.13}$$

So there are b paths of length d;

$$\begin{gathered}u_1=(x_{11}, x_{12}, \cdots, x_{1d})\\ u_2=(x_{21}, x_{22}, \cdots, x_{2d})\\ \cdots\cdots\cdots\\ u_b=(x_{b1}, x_{b2}, \cdots, x_{bd})\end{gathered} \tag{2.14}$$

here $x_{kj}\in C_i(j)$, $k=1, 2, \cdots, b$.

($\Longleftarrow$) obvious.

LEMMA 2.2. Every edge of the connected graph (X, Γ), $|x|=b\times d$ with a period d is given a new index of the form $cd+j$ such that $X_{cd+j}\in C_i(j)$, $c=0, 1, 2, \cdots b-1$ and $X=\{x_1, x_2, \cdots x_{bd}\}$ are satisfied.

$\Longleftrightarrow$

The following three conditions are satisfied;

(i) $$|C_i(1)|=|C_i(2)|=\cdots=|C_i(d)|=b \tag{2.15}$$

(ii) For any subset $S(\subset X)$,

$$|S|\leq|\Gamma S| \tag{2.16}$$

(iii) When we consider b paths of length $(d+1)$ which are composed of b paths u_1 u_2, $\cdots$ u_b of b arcs of the matching from $C_i(d)$ to $C_i(1)$, there exists a matching from $C_i(d)$ to $C_i(1)$ such that a permutation of these b arcs is a cyclic permutation of $(x_{11}, x_{21}, \cdots u_{b1})$.

Proof. From the condition (iii) we can obtain a simple path of length bd tieing up the b paths and then we can give an index to every element of the set X such that $x_{cd+j} \in C_i(j)$ is satisfied. (q.e.d.)

Then we obtain the following theorem.

THEOREM 1. A connected graph (X, Γ) with a period d has a HAMILTONIAN circuit if and only if the conditions (i), (ii) and (iii) of Lemma 2 are satisfied.

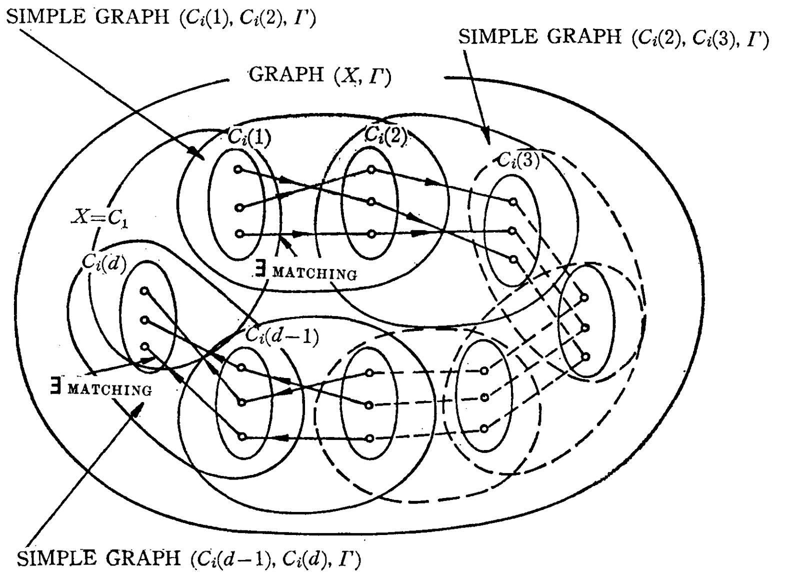

Fig. 2.1.

2.4 A Graph with some Conditions

Let $\boldsymbol{a}, \boldsymbol{b}, \cdots$ be column vectors with its components are all 0 or 1 and $|\boldsymbol{a}|, |\boldsymbol{b}|, \cdots$ be the number of components which are 1 of $\boldsymbol{a}, \boldsymbol{b}, \cdots$.

Let us denote I be a n-vector all elements are 1 and E be a $(n \times n)$-matrix all element are 1.

LEMMA 2.3. If integers k and p $(1<k+p\leq n)$ exist such that $A^{(k)}, A^{(k+1)}, \cdots, A^{(k+p-1)}$ are composed of p vectors $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_{p-1}$ satisfying the conditions

$$\sum_{i=0}^{p-1} \boldsymbol{a}_i = I \tag{2. 17}$$

and

$$\sum_{i=0}^{p-1} A(k+i)=E \tag{2.18}$$

then

$$\begin{aligned} A^{(k+pi)} &= A^{(k)} \\ A^{(k+1+pi)} &= A^{(k+1)} \\ &\cdots\cdots\cdots\cdots \qquad i=1,\ 2,\ \cdots \\ A^{(k+p-1+pi)} &= A^{(k+p-1)}, \end{aligned} \tag{2.19}$$

and consequently

$$\sum_{n=0}^{p-1} A^{(k+pi+n)}=E, \qquad i=1,\ 2,\ \cdots \tag{2.20}$$

Proof. For the sake of simplicity, we shall prove the assertion to Lemma 3 only for $p=2$, but it is evident for all p this lemma is proved similarly.

If the first column vector of $A^{(k)}$ is $\boldsymbol{a}$, the first column vector of $A^{(k+1)}$ must be $\boldsymbol{b}$ and if the second column vector of $A^{(k)}$ is $\boldsymbol{b}$, the second column vector of $A^{(k+1)}$ must be $\boldsymbol{a}$ from (2. 18).

$$\boldsymbol{a}A=\boldsymbol{b} \qquad \boldsymbol{b}A=\boldsymbol{a}$$

$$\boldsymbol{a}A^2=\boldsymbol{b}A=\boldsymbol{a} \qquad \boldsymbol{b}A^2=\boldsymbol{a}A=\boldsymbol{b}$$

Therefore the first column vector of $A^{(k+2)}$ is $\boldsymbol{a}$ and the second column vector of $A^{(k+2)}$ is $\boldsymbol{b}$. And so on we obtain (2.20), we were to prove.

THEOREM 2. If a finite graph (X, Γ) satisfies the following two conditions (i) and (ii), then the graph (X, Γ) has a HAMILTONIAN circuit.

(i) $$|S|\leq|\Gamma S| \tag{2.21}$$

for all subsets S of X.

(ii) Integers k and p $(1<k+p\leq n)$ exist such that $A^{(k)}$, $A^{(k+1)}$, $\cdots$ $A^{(k+p-1)}$ are composed of p vetors $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\cdots$, $\boldsymbol{a}_{p-1}$ satisfying the conditions

$$\sum_{i=0}^{p-1} \boldsymbol{a}_i=I \tag{2.22}$$

$$|\boldsymbol{a}_i|=n/p, \qquad i=1,\ 2,\ \cdots,\ p-1 \tag{2.23}$$

and

$$\sum_{i=0}^{p-1} A^{(k+i)}=E, \tag{2.24}$$

where p is a divisor of n.

Proof. Let us prove this theorem for only $p=2$ as the same reason as the Lemma 1. (2. 24) mean that any point in X which can be attained through a path of length k can be attained through paths of length $(k+1)$ starting at any point in X. From Lemm 2. 1 any point which is attained through a path of length k can be attained through paths of length $(k+i)$, $i=2,\ 4,\ \cdots$ and any point which can be attained through a path of length $(k+1)$ can be attained through paths of length $(k+i)$, $i=3, 5, \cdots$.

Let $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{(n/2)}\}$ be a set of X which can be attained through paths of length k and $\{b_1, \cdots, b_{(n/2)}\}$ be a set of X which can be attained through paths of length $(k+1)$. From (i) the graph (X, Γ) has at least one matching and then it must be written:

$$(a_1, b_{k1}), (a_2, b_{k2}), \cdots, (a_{(n/2)}, b_{k(n/2)}) \tag{2. 25}$$

$$(b_1, a_{l1}), (b_2, a_{l2}), \cdots, (b_{(n/2)}, a_{l(n/2)}) \tag{2. 26}$$

where $(k_1, k_2, \cdots, k_{(n/2)})$ and $(l_1, l_2, \cdots, l_{(n/2)})$ are permutations of $(1, 2, \cdots n/2)$ and $k_i \neq l_i$ $(i=1, 2, \cdots n)$. In (2. 25) and (2. 26) we can choose numbers such that

$$k_i=l_j, \qquad k_j=l_h, \cdots \tag{2. 27}$$

and we arrange the n arcs:

$$(a_i, b_{ki}), (b_{ij}, a_j), (a_j, b_{kj}), (a_{lh}, a_h), (a_h, *) \cdots \tag{2. 28}$$

If $a_j=a_i$, b_{kj} must be equal to b_{ki} and then (a_i, b_{ki}) is equal to (a_j, b_{kj}). It means $a_j \neq a_i$. If $b_{kj}=b_{ki}$ $(=b_{ij})$, a_j must be eqnal to a_i and then (a_j, b_{kj}) is equal to (a_j, b_{ki}). It means $b_{kj} \neq b_{ki}$. And so on a_i, a_j, a_h, $\cdots$ are all different from each other. From this reason

$$\mu=[a_i, b_{ki}, a_j, b_{kj}, a_h, \cdots] \tag{2. 29}$$

is a HAMILTONIAN circuit of this graph (X, Γ). (q.e.d.)

2.5 An Example. The Euler Problem of Knigh's Tour

The problem consists of moving a knight on a chessboard in such a way that he goes once and once only through every square on the board. For this problem innumerable methods have been used. Here let us use the preceding results for this problem. A number is given on each square of the chessboard. (cf. Fig. 2.2)

1	9	17	25	33	41	49	57
2	10	18	26	34	42	50	58
3	11	19	27	35	43	51	59
4	12	20	28	36	44	52	60
5	13	21	29	37	45	53	61
6	14	22	30	38	46	54	62
7	15	23	31	39	47	55	63
8	16	24	32	40	48	56	64

Fig. 2.2.

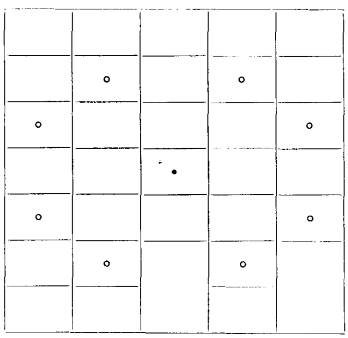

Fig. 2.3.

Let $X=\{1, 2, \cdots 64\}$ and Γx the set of the square's numbers to which he can go from $x \in X$ with a step. As the knight on the square which has • mark can go to the squares which has ∘ mark with one step (cf. Fig. 2.3).

$$\begin{aligned} &\Gamma(1)=[11, 18] \\ &\Gamma(2)=[12, 17, 19] \\ &\vdots \end{aligned} \tag{2. 30}$$

Therefore we can define a graph (X, Γ) and the associated matrix A can be setted. Then the EULER problem of the knight's tour is equivalent to the problem of finding a HAMILTONIAN path in the graph (X, Γ). We find that $A^{(5)}$ and $A^{(6)}$ is composed of 32 vectors $\boldsymbol{a}$ and 32 vectors $\boldsymbol{b}$ such that

$$\begin{aligned} \boldsymbol{a}=(&0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \\ &1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0 \\ &1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0) \end{aligned} \tag{2. 31}$$

$$\boldsymbol{b}=\boldsymbol{I}-\boldsymbol{a}$$

and

$$A^{(5)}+A^{(6)}=E \tag{2. 32}$$

is satisfied.

The column vector's numbers of $A^{(5)}$ composed of $\boldsymbol{a}$ is

$$\begin{aligned} &2,\ 4,\ 6,\ 8,\ 9,\ 11,\ 13,\ 15,\ 18,\ 20,\ 22,\ 24,\ 25,\ 27,\ 29,\ 31,\ 34, \\ &36,\ 38,\ 40,\ 41,\ 43,\ 45,\ 47,\ 50,\ 52,\ 54,\ 56,\ 57,\ 59,\ 61,\ 63 \end{aligned} \tag{2. 33}$$

and composed of $\boldsymbol{b}$ is

$$\begin{aligned} &1,\ 3,\ 5,\ 7,\ 10,\ 12,\ 14,\ 16,\ 17,\ 19,\ 21,\ 23,\ 26,\ 28,\ 30,\ 33,\ 35, \\ &37,\ 39,\ 42,\ 44,\ 46,\ 48,\ 49,\ 51,\ 53,\ 55,\ 58,\ 60,\ 62,\ 64 \end{aligned} \tag{2. 34}$$

There exists a matching between (2. 33) and (2. 34):

$$\begin{aligned} &(1, 11)\ (2, 19)\ (3, 9)\ (4, 14)\ (1, 15)\ (6, 12)\ (7, 13)\ (8, 23)\ (9, 26)\ (10, 4) \\ &(11, 17)\ (12, 2)\ (13, 3)\ (14, 8)\ (15, 32)\ (16, 6)\ (17, 27)\ (18, 1)\ (9, 25) \\ &(20, 5)\ (21, 31)\ (22, 28)\ (23, 29)\ (24, 7)\ (25, 10)\ (26, 20)\ (27, 21)\ (28, 18) \\ &(29, 35)\ (30, 21)\ (31, 16)\ (32, 22)\ (33, 43)\ (34, 49)\ (35, 41)\ (36, 30) \\ &(37, 47)\ (38, 44)\ (39, 45)\ (40, 55)\ (41, 58)\ (42, 36)\ (43, 37)\ (44, 34) \\ &(45, 60)\ (46, 40)\ (47, 64)\ (48, 38)\ (49, 59)\ (50, 33)\ (51, 42)\ (52, 62) \\ &(53, 63)\ (54, 48)\ (55, 61)\ (56, 39)\ (57, 42)\ (58, 52)\ (59, 53)\ (60, 50) \\ &(61, 51)\ (62, 56)\ (63, 46)\ (64, 54) \end{aligned} \tag{2. 35}$$

According to the Theorem 2 we obtain a HAMILTONIAN path

$$\mu=[1, 11, 17, 27, 21, 31, 16, 6, 12, 2, 19, 25, 10, 4, 14, 8, 23, 29, 35, 41, 58, 52, 62, 56, 39, 45, 60, 50, 33, 43, 37, 47, 64, 54, 48, 38, 44, 34, 49, 59, 53, 63, 46, 40, 55, 61, 51, 57, 42, 36, 30, 24, 7, 13, 3, 9, 26, 20, 5, 15, 32, 22, 28, 18] \quad (2.36)$$

SECTION 3

EXISTENCE OF A HAMILTONIAN CIRCUIT (II)

Some fundamental properties of a graph are defined in terms of the edge-edge incidence matrix associated with the graph. By virtue of these definitions we obtain a necessary and sufficient condition for that an oriented graph $G(X, \Gamma)$ of even order has a perfect matching and similarly for that a HAMILTONIAN circuit. Not only for a simple graph but also for any oriented graph of even order, each of these necessary and sufficient condition is useful for knowing the existence of perfect matching and a HAMILTONIAN circuit, respectively.

3.1. Perfect Matching

In this section we consider a graph of even order ($|X|=2n$). Here $|S|$ is the number of elements of the set S. $\boldsymbol{Z}_i$ $(i=1, 2, \cdots)$ are $2n$-vectors each of whose elements is either 1 or 0, and by $|\boldsymbol{Z}_i|$ we denote the number of the elements whose values are 1. $\mathbf{1}$ and $\mathbf{0}$ are $2n$-vectors such that $|\mathbf{1}|=2n$ and $|\mathbf{0}|=0$ respectively.

THEOREM 3. The graph $G(X, \Gamma)$, $|X|=2n$, has a perfect matching, if and only if the following conditions (i) and (ii) are satisfied.

(i) There exist two vectors $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ such that

$$A\boldsymbol{Z}_1=\boldsymbol{Z}_2 \quad (3.1)$$

$$\boldsymbol{Z}_1+\boldsymbol{Z}_2=\mathbf{1} \quad (3.2)$$

$$|\boldsymbol{Z}_1|=n \quad (3.3)$$

(ii) For any $S\subset X$

$$|\Gamma S\cap X_2|\leqq|S\cap X_1| \quad (3.4)$$

where X_1 is a subset of X corresponding to 0 in $\boldsymbol{Z}_1$ and X_2 is corresponding to 1.

Proof. In the graph $G(X, \Gamma)$ we divide the set X into two subsets X_1 and X_2, and neglecting some arcs we make a simple graph $G'(X_1, X_2, \Gamma')$. For this simple graph $G'(X_1, X_2,\Gamma')$ we use the theoem of KÖNING and HALL [1].

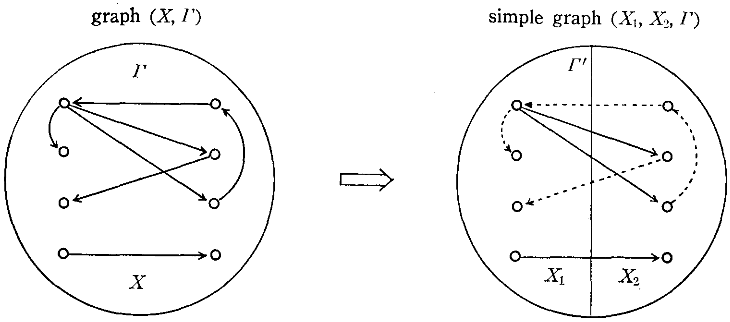

Fig. 3.1.

Ad; The sufficiency. From (3. 2), (3. 3), it follows that $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ can be expressed by giving a new index

$$\boldsymbol{Z}_1=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hdashline 1 \\ \vdots \\ 1 \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \end{matrix} \qquad \boldsymbol{Z}_2=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline * \\ \vdots \\ * \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \end{matrix} \tag{3. 5}$$

For these expressions of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$, (3. 1) implies that the incidence matrix A can be expressed by giving a new index

$$A=\overset{\quad\overbrace{}^{n}\ \overbrace{}^{n}}{\begin{bmatrix} C & B \\ C & C \end{bmatrix}}\begin{matrix} \}\, n \\ \}\, n \end{matrix} \tag{3. 6}$$

where B is a matrix whose every column vetors has at least one element whose value is 1 and C is an any matrix whose elements are either 1 or 0.

Now let X_1 be a subset of X which corresponds to 0 in $\boldsymbol{Z}_1$ and let X_2 be that which corresponds to 1. Then $|X_1|=|X_2|=n$, and for any $x_1 \in X_1$ there exists at least one $x_2 \in X_2$ such that $\Gamma x_1 = x_2$. Let us define a partial graph $G'(X, \Gamma')$ of $G(X, \Gamma)$ by

$$\begin{aligned} &X = X_1 \cup X_2 \\ &\Gamma' : X_1 \longrightarrow X_2. \end{aligned} \tag{3. 7}$$

This means that the graph $G'(X, \Gamma')$ is obtained by removing the following set from the graph $G(X, \Gamma)$;

$$\begin{aligned} \{(x_i, x_j);\ (x_i \in X_1 \text{ and } x_j \in X_1) \cup (x_i \in X_2 \text{ and } x_j \in X_1) \\ \cup (x_i \in X_2 \text{ and } x_j \in X_2\}\} \end{aligned} \tag{3. 8}$$

Since $X_1 \cap X_2 = \phi$, the graph $G'(X, \Gamma')$ is a simple graph and it must be written $G'(X_1, X_2, \Gamma')$. According to the condition (ii), the following inequality is satisfied:

$$|S| \leq |\Gamma S| \tag{3. 9}$$

for any $S \subset X$ in the graph $G'(X_1, X_2, \Gamma')$. By virtue of the theorem of KÖNING and HALL [1] this simple graph $G'(X, X, \Gamma')$ has a perfect matching and so the graph $G(X, \Gamma)$ has a perfect matching.

Ad; The necessity. Le the graph $G(X, \Gamma)$ has a perfect matching

$$(x_1, x_{i_1}),\ (x_2, x_{i_2}),\ \cdots,\ (x_n, x_{i_n}),$$

and let us write

$$X_1 = \{x_1, x_2, \cdots, x_n\}$$

$$X_2 = \{x_{i_1}, x_{i_2}, \cdots x_{i_n}\}.$$

Then for any $x_j \in X_1$ there exists at least one $x_{ij} \in X_2$ such that $\Gamma x_j = x_{ij}$ $(j=1, 2, \cdots n)$. By giving a new index the incidence matrix A can be expressed

$$A = \left[\begin{array}{c|c} C & B \\ \hline C & C \end{array}\right]. \tag{3. 10}$$

Hence for

$$\boldsymbol{Z}_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \end{matrix} \qquad \boldsymbol{Z}_2 = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hline * \\ \vdots \\ * \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n \end{matrix}, \tag{3. 11}$$

we have a following relation

$$\left[\begin{array}{c|c} C & B \\ \hline C & C \end{array}\right] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hline * \\ \vdots \\ * \end{pmatrix}. \tag{3. 12}$$

Let us define a partial graph $G'(X, \Gamma')$ of the graph $G(X, \Gamma)$ by

$$\begin{aligned} X &= X_1 \cup X_2 \\ \Gamma' &: X_1 \longrightarrow X_2 \end{aligned} \tag{3. 13}$$

Then the graph $G'(X, \Gamma')$ has a perfect matching and the following inequality is satisfied:

$$|S| \leq |\Gamma S|. \tag{3. 14}$$

For any $S \subset X_1$ this means that in a graph $G(X, \Gamma)$, the relation

$$|\Gamma S \cap X_2| \geq |S \cap X_1| \tag{3.15}$$

is satisfied for any $S \cup X$.

3.2. Hamiltonian Circuit

THEOREM 4. A graph $G(X, \Gamma)$ of even order (i.e. $|X|=2n$) has a HAMILTONIAN circuit if and only if the following conditions (i), (ii), and (iii) are satisfied.

(i) There exist $2n$-vectors $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$, $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_4$ such that

$$A\boldsymbol{Z}_1=\boldsymbol{Z}_2 \tag{3.16}$$

$$A\boldsymbol{Z}_3=\boldsymbol{Z}_4 \tag{3.17}$$

$$|\boldsymbol{Z}_1|=n \tag{3.18}$$

$$\boldsymbol{Z}_1+\boldsymbol{Z}_2=\mathbf{1} \tag{3.19}$$

$$\boldsymbol{Z}_1+\boldsymbol{Z}_3=\mathbf{1} \tag{3.20}$$

$$\boldsymbol{Z}_1\times\boldsymbol{Z}_3'=0 \quad \text{(scaler)} \tag{3.21}$$

$$\boldsymbol{Z}_3+\boldsymbol{Z}_4=\mathbf{1}. \tag{3.22}$$

(ii) For any $S\subset X$,

$$|S\cap X_1|\leq|\Gamma S\cap X_2| \tag{3.23}$$

$$|S\cap X_2|\leq|\Gamma S\cap X_1|. \tag{3.24}$$

Here X_1 is a subset of X corresponds to 0 in $\boldsymbol{Z}_1$ and X_2 is corresponds to 1.

(iii) Let two sets of matching whose existence are assured from (i) and (ii) be

$$\begin{matrix} (x_1, x_{i_1}), (x_2, x_{i_2}), \cdots (x_n, x_{i_n}) \\ (x_{i_1}, x_{j_1}), (x_{i_2}, x_{j_2}), \cdots (x_{i_n}, x_{j_n}) \end{matrix} \tag{3.25}$$

then $(j_1, j_2, \cdots j_n)$ is a cyclic permutation of $(1, 2, \cdots n)$.

Proof. Ad; The sufficiency. From (3.18), by giving a new index $\boldsymbol{Z}_1$ can be expressed

$$\boldsymbol{Z}_1=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}\right\} n \end{matrix}. \tag{3.26}$$

From (3.20) it must be

$$Z_3=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline * \\ \vdots \\ * \end{pmatrix}\begin{matrix} \Big\} n \\ \\ \Big\} n \end{matrix} \tag{3. 27}$$

and from (3. 21) it must be

$$Z_3=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline 0 \\ \vdots \\ 0 \end{pmatrix}\begin{matrix} \Big\} n \\ \\ \Big\} n \end{matrix}. \tag{3. 28}$$

Also from (3. 19) it must be

$$Z_2=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hdashline * \\ \vdots \\ * \end{pmatrix}\begin{matrix} \Big\} n \\ \\ \Big\} n \end{matrix} \tag{3. 29}$$

and from (3. 22) it must be

$$Z_4=\begin{pmatrix} * \\ \vdots \\ * \\ \hdashline 1 \\ \vdots \\ 1 \end{pmatrix}\begin{matrix} \Big\} n \\ \\ \Big\} n \end{matrix}. \tag{3. 30}$$

Now let X_1 be a subset of X which corresponds to 0 in $\boldsymbol{Z}_1$ and X_2 be corresponds to 1. Then from (3. 16), for any $x_1 \in X_1$ there exists at least one x_2 such that $x_2=\Gamma x_1$. From (3. 17) there exists at least one x_1 such that $x_1=\Gamma x_2$ for any $x_2 \in X_2$. From the condition (3. 23) in (ii), there exists a perfect matching from X_1 to X_2. Also from the condition (3. 24) in (ii), there exists a perfect matching from X_2 to X_1. For these two sets of matchings the condition (iii) is satisfied and so these arcs must be perfect matching. (q.e.d.)

Ad; The necessity. As there exists a HAMILTONIAN circuit, let us define it by

$$\mu=(x_1, x_2, \cdots x_{2n}). \tag{3. 31}$$

This means that there exist $2n$ arcs

$$(x_1, x_2), (x_2, x_3), \cdots (x_{2n-1}, x_{2n}), (x_{2n}, x_1). \tag{3. 32}$$

Let us define the sets X_1 and X_2 by

$$X_1=\{x_1, x_3, \cdots x_{2n-1}\}$$

$$X_2=\{x_2, x_4, \cdots x_{2n}\}$$

then there exits at least one $x_j \in X_2$ such that $\Gamma x_i = x_j$ for any $x_i \in X_1$. This means that there exist $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ such that

$$A\boldsymbol{Z}_1 = \boldsymbol{Z}_2. \tag{3. 33}$$

Similarly there exists at least one $x_j \in X_1$ such that $\Gamma x_i = x_j$ for any $x_i \in X_2$. This means that there exist $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_4$ such that

$$A\boldsymbol{Z}_3 = \boldsymbol{Z}_4. \tag{3. 34}$$

$\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$, $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_4$ satisfy the relations (3. 18), (3. 19), (3. 20), (3. 21) and (3. 22).

There exist two sets of matching from X_1 to X_2 and from X_2 to X_1. Hence the relation (ii) is satisfied. The $2n$ arcs of (3. 32) is a HAMILTONIAN circuit and so the relation (iii) is satisfied.

EXAMPLE 3.1. Consider again EULER problem of the KNIGHT's tour mentioned in subsection 2. 5. We can find the vectors $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$, $\boldsymbol{Z}_3$, $\boldsymbol{Z}_4$ and the sets X_1, X_2;

$$\begin{aligned}\boldsymbol{Z}_1 = (&0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\\ &0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\\ &1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0)\end{aligned}$$

$$\begin{aligned}\boldsymbol{Z}_2 = (&1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\\ &1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\\ &0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1)\end{aligned} \tag{3. 36}$$

$$\boldsymbol{Z}_3 = \boldsymbol{Z}_2 \tag{3. 37}$$

$$\boldsymbol{Z}_4 = \boldsymbol{Z}_1 \tag{3. 38}$$

$$\begin{aligned}X_1 = \{&2, 4, 6, 8, 9, 11, 13, 15, 18, 20, 22, 24, 25, 27, 29, 31, 34, 36,\\ &38, 40, 41, 43, 45, 47, 50, 52, 54, 56, 57, 59, 61, 63\}\end{aligned} \tag{3. 39}$$

$$\begin{aligned}X_2 = \{&1, 3, 5, 7, 10, 12, 14, 16, 17, 19, 21, 23, 26, 28, 30, 32, 33,\\ &35, 37, 49, 42, 44, 46, 48, 49, 51, 53, 55, 58, 60, 62, 64\}\end{aligned} \tag{3. 40}$$

For these X_1 and X_2, (3. 23), (3. 24) are satisfied. Two sets of matching whose existence are assured from this are

$$\begin{aligned}&(1, 11), (3, 9), (5, 15), (7, 13), (10, 4), (12, 2), (14, 8), (16, 6), (17, 27),\\ &(19, 25), (21, 31), (23, 29), (26, 20), (28, 18), (30, 21), (32, 22), (33, 43),\\ &(35, 41), (37, 47), (39, 45), (42, 36), (44, 34), (46, 40), (48, 38), (49, 59),\\ &(51, 57), (53, 63), (55, 61), (58, 52), (60, 50), (62, 59), (64, 54)\end{aligned} \tag{3. 41}$$

and

$$(2, 19), (4, 14), (6, 12), (8, 23), (9, 26), (11, 17), (13, 3), (15, 32), (18, 1),$$

$$(20, 5), (22, 28), (24, 7), (25, 10), (27, 21), (29, 35), (31, 16) \ (34, 49), (36, 30), (38, 44), (40, 55), (41, 58), (43, 37), (45, 60), (47, 64), (50, 33), (52, 62), (54, 48), (56, 39), (57, 42), (59, 53), (61, 51), (68, 46). \tag{3. 42}$$

For these matching the relation (iii) is satisfied. According to Theorem 4 we conclude that the graph (X, Γ) has a HAMILTONIAN circuit;

$$\mu=[1, 11, 17, 27, 21, 31, 16, 6, 12, 2, 19, 25, 10, 4, 14, 8, 23, 29, 35, 41, 58, 52, 62, 56, 39, 45, 60, 50, 33, 43, 37, 47, 64, 54, 48, 38, 44, 34, 49, 59, 53, 63, 46, 40, 55, 61, 51, 57, 42, 36, 30, 24, 7, 13, 3, 9, 26, 20, 5, 15, 32, 22, 28, 18]. \tag{3. 43}$$

REFERENCES

BERGE, C., (1962): *The theory of graphs and its applications*, Methuem & Co. Ltd. and John Wiley & Sons Inc.

BERGE, C. and GHOUILA-HOURI, A., (1962): *Programming, games and transportatson networks*, Methuem & Co. Ltd.

BUSACKEN, R. G. and SAATY, T. L., (1965): *Finite graphs and networks*, McGraw-Hill Book Co.

ERDOS, P. and KATONA, G., (1968): Theory of graphs, Academic Press.

KIN, W. H. and CHIEN, R. T., (1962): *Topological analysis and synthesis of communication networks*, Columbia Univ. Press.

MARUYAMA, G., (1957): *Probability theory*, Kyoritu Syuppan Co.

TAKENAKA, Y. (1968): On the existence of Hamiltonian path in a graph, *Information and Control*, Vol. 13, No. 6, 555.

TAKENAKA, Y., (1970): Graph theoretic concepts and the incidence matrix, *Information and Control*, Vol. 17, No. 2, 113

TAKENAKA, Y., (1971): On the existence of a Hamiltonian path in a graph II, (In preparation).