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Essays on Mechanism Design and Social Indices

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Abstract

This dissertation consists of five essays on mechanism design and social indices. I study the problems of designing mechanisms and social indices that satisfy “desirable" properties.

Consider a social planner who wishes to implement a social objective. In order for the planner to do this, he/she needs to gather information about individuals' preferences. However, even if the planner directly asks individuals about their preferences, they might not tell him the truth. This is because individuals may have incentives to misrepresent their preferences. Since a proposal based on the misrepresented preferences may fail, designing a mechanism that brings us to the social objective is important. In Chapters 1, 2 and 3, I deal with this problem. I particularly search for mechanisms that satisfy “desirable" properties such as *efficiency*, *individual rationality*, and *strategy-proofness*.

On the other hand, during the stage in which social objectives are decided, an accurate understanding of social conditions is indispensable. However, social conditions are usually determined by complex phenomena such as *polarization*. Using some quantitative indices is a helpful way to perceive such complex phenomena. However, if we use some “ bad ” indices, we may misperceive the complex phenomena, and also possibly misperceive social conditions, failing to plan appropriate social objectives. Therefore, designing "desirable" indices is important. In Chapters 4 and 5 , I deal with this problem. I search for indices that satisfy “desirable" properties to measure the levels of human development and polarization in a society.

In Chapter 1, I study famous Clarke's (1971) pivotal mechanisms. Moulin (1986) characterizes the pivotal mechanisms under the assumption of the full domain of quasi-linear preferences. In this chapter, I provide properties of restricted domains that are necessary and sufficient for Moulin's (1986) characterizations to hold. I also provide simple economic conditions that imply these properties.

In Chapter 2, I search for a mechanism that overcomes a drawback of the pivotal mechanisms: they do not satisfy *individual rationality*. I consider the problem of designing mechanisms that mediate disputes. A specific feature of the problem I examine is that each disputant may have a veto power to the outcomes of mechanisms. Given the specific feature, I impose *individual rationality* on mechanisms so that each disputant voluntarily accepts outcomes of mechanisms. First, I show that on the full domain of disputants' valuations, a mechanism that always forces disputants to continue the dispute uniquely satisfies a weaker version of *efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility*. Second, I

show that on mildly restricted domains of disputants' valuations, there exist well-performed mechanisms that satisfy all four axioms and Pareto-dominates all mechanisms as such.

In Chapter 3, I suggest how to find a boundary between the possibility and impossibility of implementing social choice rules. I introduce a new concept of implementation that uses the planner's "guess" of individual preferences. Given a family of subsets of possible preference profiles, the planner guesses a subset to which individuals' true preference profile belongs. A social choice rule is said to be *G-implementable* if it is implemented in dominant strategies as long as the planner's guess is correct. I apply this implementability concept to public decision and auction problems. In a public decision problem, I characterize a class of social choice rules satisfying *efficiency*, *individual rationality*, *feasibility*, and *G-implementability*. In an auction problem with homogeneous goods, I characterize a class of auction rules that generate more revenue than any other auction rules satisfying *efficiency*, *individual rationality*, and *G-implementability*. I also show that rules in these two classes only require "minimal information" for the planner to guess correctly.

In Chapter 4 (co-authored with Yoko Kawada and Shuhei Otani), we provide an axiomatic foundation of the Human Development Index (HDI). The aggregation formula of HDI was changed to geometric mean in 2010. In this chapter, we search for a theoretical justification for employing this new HDI formula. First, we find a maximal class of index functions, what we call *quasi-geometric means*, that satisfy *symmetry for the characteristics*, *normalization*, and *separability*. Second, we show that power means are the only quasi-geometric means satisfying *homogeneity*. Finally, the new HDI is the only power mean satisfying two local complementability axioms, what we call *minimal lower boundedness* and *sensitivity to lowest-level characteristics*.

In Chapter 5 (co-authored with Yoko Kawada and Keita Sunada), we consider a design problem of polarization measures. Esteban and Ray (1994) formalize an idea of polarization and develop a theory for its measurement. In their main theorem, they claim that a class of polarization measures, called the *Esteban-Ray measures*, is characterized by a set of axioms that capture an idea of polarization. However, we show that the claim does not hold by presenting a counterexample. We amend the main theorem by strengthening their Axiom 1.

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Contents

Chapter 1

The Uniqueness of Pivotal Mechanisms on Restricted Domains

1.1 Introduction

In his seminal work, Moulin (1986) provides three characterizations of Clarke's (1971) pivotal mechanisms. First, he shows that *efficiency, strategy-proofness*, and the *no free ride axiom* characterize the pivotal mechanisms. Second, he shows that the same characterization holds if the no free ride axiom is replaced by *welfare lower boundedness*. Finally, he shows that even if strategy-proofness is dropped, *efficiency*, the *no free ride* axiom, and two mild *monotonicity* axioms characterize the pivotal mechanisms. All of Moulin's (1986) results are shown under the assumption of the full domain. In this chapter, we provide properties of restricted domains that are necessary and sufficient for Moulin's (1986) characterizations to hold. We also provide simple economic conditions that imply these properties.

Related literature

The pivotal mechanisms belong to the class of Groves mechanisms, which have played a prominent role in mechanism design theory (Groves 1973). Green and Laffont (1977) show that if a domain contains all continuous preferences, then Groves mechanisms are the only *efficient* and *strategy-proof* mechanisms. However, Walker (1978) points out that since Green and Laffont's (1977) proof depends crucially on the assumption of a large domain, we cannot directly apply their uniqueness result to some of the important restricted domains (e.g., a domain which consists of all the concave valuation functions). Given this criticism, Holmström (1979) provides a general sufficient condition on domains that ensures the uniqueness of Groves mechanisms. Subsequently, Suijs (1996) and Carbajal (2010) provide necessary and sufficient conditions on domains for the uniqueness of Groves mechanisms.

The uniqueness of Groves mechanisms relates to a *payoff equivalence* of mechanisms.[*1]

[*1]For early literature on payoff (revenue) equivalence, see Myerson (1981), Jehiel, Moldovanu, and Stacchetti (1999), Krishna and Maenner (2001), and Milgrom and Segal (2002).

Payoff equivalence states that each individual's payments under two arbitrary incentive compatible mechanisms with the same decision function differ at most constants with respect to his reported preferences. Chung and Olszewski (2007) and Heydenreich, Müller, and Vohra (2009) provide necessary and sufficient conditions on domains under which strategy-proof mechanisms satisfy payoff equivalence. The presence of *efficiency* is the main difference between the works by Suijs (1996) and Carbajal (2010) and those by Chung and Olszewski (2007) and Heydenreich, Müller, and Vohra (2009). In contrast to their works, we further impose Moulin's (1986) axioms and search for necessary and sufficient domain conditions for the uniqueness of the pivotal mechanisms.

The remainder of this chapter is organized as follows. Section 1.2 introduces our model. Section 1.3 offers necessary and sufficient domain conditions for Moulin's (1986) characterizations to hold. Section 1.4 gives concluding remarks. While some results in Section 1.3 are stated under the assumption of *connectedness* of domains, Appendix A deals with the case that relaxes this assumption. Some proofs are relegated to Appendix B.

1.2 The Model

Let $N = \{1, 2, \ldots, n\}$ be the finite set of *individuals*, and $X = \{x_1, x_2, \ldots, x_m\}$ the finite set of *alternatives*. An *outcome* is a pair $(x, t) \in X \times \mathbb{R}^n$, where $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ is a vector of monetary transfers among individuals. Each individual $i \in N$ has a *valuation function* $v_i : X \longrightarrow \mathbb{R}$. Individual i's *quasi-linear utility* for an outcome $(x, t) \in X \times \mathbb{R}^n$ is $v_i(x) + t_i$. Let $V \equiv \mathbb{R}^X$ be the set of valuation functions. A *valuation profile* is an n-tuple of valuation functions $v = (v_1, \ldots, v_n) \in V^n$. For each $v \in V^n$ and each $N' \subset N$, $v_{N'}$ and $v_{-N'}$ denote $\{v_j\}_{j \in N'}$ and $\{v_j\}_{j \in N \setminus N'}$, respectively. For each $v_i \in V$, let $E(v_i) \subset X$ be the set of alternatives that maximize v_i; that is,

$$E(v_i) = \arg\max_{y \in X} v_i(y).$$

A *domain* is a nonempty subset of the set of valuation profiles $\prod_{j=1}^n D_j \subset V^n$. Given a domain $\prod_{j=1}^n D_j$, a *decision function* is a function $d : \prod_{j=1}^n D_j \longrightarrow X$ that maps each valuation profile $v \in \prod_{j=1}^n D_j$ to an alternative $d(v) \in X$. A *transfer function* is a function $t : \prod_{j=1}^n D_j \longrightarrow \mathbb{R}^n$ that maps each valuation profile $v \in \prod_{j=1}^n D_j$ to a vector $t(v) = \big(t_1(v), \ldots, t_n(v)\big) \in \mathbb{R}^n$. A *mechanism* is a pair of decision and transfer functions (d, t). Throughout this chapter, we focus on *efficient* and *feasible* mechanisms: for each $v \in \prod_{j=1}^n D_j$,

(i) *efficiency*; $d(v) \in E\left(\sum_{j=1}^n v_j\right)$,

(ii) *feasibility*; $\sum_{j=1}^{n} t_j(v) \leq 0$.

Let $\mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ be the set of efficient and feasible mechanisms on a domain $\prod_{j=1}^{n} D_j$. A mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ *Pareto-dominates* a mechanism $(d',t') \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ if for each $i \in N$ and each $v \in \prod_{j=1}^{n} D_j$,

$$v_i\big(d(v)\big) + t_i(v) \geq v_i\big(d'(v)\big) + t'_i(v).$$

A mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ is a *Groves mechanism* if for each $i \in N$, there exists a function $h_i : \prod_{j \neq i} D_j \longrightarrow \mathbb{R}$ such that

$$t_i(v) = \sum_{j \neq i} v_j\big(d(v)\big) - h_i(v_{-i}) \quad \text{for all } v \in \prod_{j=1}^{n} D_j.$$

A mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ is a *Pivotal mechanism* if for each $i \in N$,

$$t_i(v) = \sum_{j \neq i} v_j\big(d(v)\big) - \max_{y \in X} \sum_{j \neq i} v_j(y) \quad \text{for all } v \in \prod_{j=1}^{n} D_j.$$

Note that any pivotal mechanism is a Groves mechanism.

1.3 Main Results

1.3.1 Strategy-proofness and no free ride

Throughout Subsections 1.3.1 and 1.3.2, we focus only on *connected* domains for the sake of simplicity. In Appendix A, we study the case with non-connected domains.

Definition 1. A domain $\prod_{j=1}^{n} D_j \subset V^n$ is *connected* if for each $i \in N$, there exists no disjoint open subsets $T, U \subset V = \mathbb{R}^X$ such that $T \cap D_i \neq \emptyset$, $U \cap D_i \neq \emptyset$, and $D_i \subset T \cup U$.

Strategy-proofness requires that reporting the true valuation function be a weakly dominant strategy for anyone.

Strategy-proofness. For each $i \in N$, each $v \in \prod_{j=1}^{n} D_j$, and each $v'_i \in D_i$,

$$v_i\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}) \geq v_i\big(d(v'_i, v_{-i})\big) + t_i(v'_i, v_{-i}).$$

The no free ride axiom requires that no one can gain by withdrawing from a mechanism.

The no free ride axiom. For each $i \in N$ and each $v \in \prod_{j=1}^{n} D_j$,

$$v_i\big(d(v)\big) + t_i(v) \geq \min_{y \in E(\sum_{j \neq i} v_j)} v_i(y).$$

Note that any pivotal mechanism satisfies strategy-proofness and the no free ride axiom. Moulin (1986) shows that on the full domain V^n, strategy-proofness and the no free ride axiom characterize the pivotal mechanisms.

Moulin's Theorem 1 (1986). *On the full domain $\prod_{j=1}^n D_j = V^n$, a mechanism $(d,t) \in \mathscr{M}(V^n)$ satisfies strategy-proofness and the no free ride axiom if and only if it is a pivotal mechanism.*

We next introduce two properties of domains. Consider any pivotal mechanism $(d^*,t^*) \in \mathscr{M}(\prod_{j=1}^n D_j)$. Then, by feasibility and definition of the pivotal mechanism, for each $v \in \prod_{j=1}^n D_j$,

$$0 \leq -\sum_{j=1}^n t_j^*(v) = \sum_{j=1}^n \max_{y\in X} \sum_{k\neq j} v_k(y) - (n-1)\max_{y\in X}\sum_{j=1}^n v_j(y). \tag{1.1}$$

The next property requires that for each $v_{-i} \in \prod_{j\neq i} D_j$, there exist a valuation function $v_i \in D_i$ such that a budget surplus $-\sum_{j=1}^n t_j^*(v_i, v_{-i})$ of the pivotal mechanism becomes arbitrarily small.

Property 1. For each $i \in N$, each $v_{-i} \in \prod_{j\neq i} D_j$, and each $\varepsilon > 0$, there exists $v_i \in D_i$ such that

$$\sum_{j=1}^n \max_{y\in X} \sum_{k\neq j} v_k(y) - (n-1)\max_{y\in X}\sum_{j=1}^n v_j(y) < \varepsilon.$$

Note that the condition above does not depend on the selection of the pivotal mechanism (d^*,t^*).

Since a pivotal mechanism $(d^*,t^*) \in \mathscr{M}(\prod_{j=1}^n D_j)$ satisfies the no free ride axiom, for each $v \in \prod_{j=1}^n D_j$,

$$0 \leq v_i\left(d^*(v)\right) + t_i^*(v) - \min_{y\in E(\sum_{j\neq i} v_j)} v_i(y) = \max_{y\in X}\sum_{j=1}^n v_j(y) - \max_{y\in X}\sum_{j\neq i} v_j(y) - \min_{y\in E(\sum_{j\neq i} v_j)} v_i(y).$$

The following property requires that for each $v_{-i} \in \prod_{j\neq i} D_j$, there exist a valuation function $v_i \in D_i$ such that the difference between individual i's final utility under the pivotal mechanism and his minimum utility of withdrawing from the mechanism becomes arbitrarily small.

Property 2. For each $i \in N$, each $v_{-i} \in \prod_{j\neq i} D_j$, and each $\varepsilon > 0$, there exists $v_i \in D_i$ such that

$$\max_{y\in X}\sum_{j=1}^n v_j(y) - \max_{y\in X}\sum_{j\neq i} v_j(y) - \min_{y\in E(\sum_{j\neq i} v_j)} v_i(y) < \varepsilon.$$

Theorem 1 states that strategy-proofness and the no free ride axiom characterize the pivotal mechanisms on a domain $\prod_{j=1}^n D_j$ if and only if the domain satisfies Properties 1 and 2.

Theorem 1. *On any connected domain $\prod_{j=1}^{n} D_j$, the following statements are equivalent:*

(i) all strategy-proof and no free ride mechanisms on $\prod_{j=1}^{n} D_j$ are pivotal mechanisms;

(ii) $\prod_{j=1}^{n} D_j$ satisfies Properties 1 and 2.

Proof. See Appendix B. ■

The following example shows that on an auction domain, there exists a strategy-proof and no free ride mechanism other than the pivotal mechanisms since it violates Property 1.

Example 1. Consider a single-item auction, that is, $X = N$ and a domain $\prod_{j=1}^{n} D_j^A$ is such that $v_i \in D_i^A \subset V$ if and only if there exists a real number $a_i \in \mathbb{R}_+$ for which

$$v_i(j) = \begin{cases} a_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

For each $v \in \prod_{j=1}^{n} D_j^A$, denoted by $v[k]$ and $v_{-i}[k]$ the k-th highest real numbers among $\{v_j(j)\}_{j\in N}$ and $\{v_j(j)\}_{j\neq i}$, respectively. On this auction domain $\prod_{j=1}^{n} D_j^A$, any pivotal mechanism $(d^*, t^*) \in \mathscr{M}(\prod_{j=1}^{n} D_j^A)$ is equivalent to a second price auction because decision and transfer functions (d^*, t^*) of the pivotal mechanism can be written as

$$d^*(v) \in \arg\max_{j\in N} v_j(j),$$
$$t_i^*(v) = \begin{cases} -v[2] & \text{if } d^*(v) = i, \\ 0 & \text{otherwise.} \end{cases}$$

However, the pivotal mechanisms are not the only mechanisms that satisfy strategy-proofness and the no free ride axiom on this domain. To see this, let $(\bar{d}, \bar{t}) \in \mathscr{M}(\prod_{j=1}^{n} D_j^A)$ be such that

$$\bar{d}(v) \in \arg\max_{j\in N} v_j(j),$$
$$\bar{t}(v) = \begin{cases} -v[2] + \frac{1}{n}v_{-i}[2] & \text{if } \bar{d}(v) = i, \\ \frac{1}{n}v_{-i}[2] & \text{otherwise.} \end{cases}$$

This mechanism gives a subsidy $\frac{1}{n}v_{-i}[2]$ to each individual in addition to his payment under a second price auction. Thus, the mechanism $(\bar{d}, \bar{t})$ differs from the pivotal mechanisms, but it can be shown that the mechanism $(\bar{d}, \bar{t})$ satisfies strategy-proofness and the no free ride axiom. Therefore, the uniqueness of the pivotal mechanisms does not hold on this auction domain. This is because $\prod_{j=1}^{n} D_j^A$ violates Property 1.

One may consider that Properties 1 and 2 are a bit complicated. Therefore, we provide simple domain conditions that imply these properties. Property 1* requires that given $v_{-i} \in \prod_{j\neq i} D_j$, a domain admit a valuation function $v_i \in D_i$ such that for some $x \in E(\sum_{k\neq i} v_k)$, $v_i(x)$ is large enough to guarantee that the alternative x becomes efficient for each population $N \setminus \{j\}$ of removing an individual $j \in N$. Obviously, the full domain V^n satisfies Property 1*. [*2]

Property 1*. For each $i \in N$ and each $v_{-i} \in \prod_{j\neq i} D_j$, there exist $v_i \in D_i$ and $x \in E(\sum_{k\neq i} v_k)$ such that $x \in E(\sum_{k\neq j} v_k)$ for all $j \in N$.

Lemma 1. *If a domain $\prod_{j=1}^n D_j$ satisfies Property 1*, then it also satisfies Property 1.*

Proof. See Appendix B. ■

Property 2* requires that a domain admit a constant valuation function for each individual. Obviously, the full domain V^n satisfies Property 2*.

Property 2*. For each $i \in N$, there exist $v_i \in D_i$ and a constant $C \in \mathbb{R}$ such that $v_i(x) = C$ for all $x \in X$.

Lemma 2. *If a domain $\prod_{j=1}^n D_j$ satisfies Property 2*, then it also satisfies Property 2.*

Proof. See Appendix B. ■

The following corollary gives a sufficient domain condition for the uniqueness of pivotal mechanisms among all strategy-proof and no free ride mechanisms.

Corollary 1. *Suppose that a domain $\prod_{j=1}^n D_j$ is connected. If $\prod_{j=1}^n D_j$ satisfies Properties 1* and 2*, then any strategy-proof and no free ride mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ must be a pivotal mechanism.*

Proof. Immediately follows from Theorem 1 and Lemmas 1 and 2. ■

1.3.2 Strategy-proofness and welfare lower boundedness

Welfare lower boundedness requires that each individual's final utility under a mechanism not be lower than that under his least preferred alternative with zero-transfer.

[*2]Moulin (1986) invokes this to deduce the uniqueness of pivotal mechanisms among all no free ride Groves mechanisms on the full domain.

Welfare lower boundedness. For each $i \in N$ and each $v \in \prod_{j=1}^{n} D_j$,

$$v_i\big(d(v)\big) + t_i(v) \geq \min_{y \in X} v_i(y).$$

Any pivotal mechanism satisfies welfare lower boundedness. Moulin (1986) shows that on the full domain V^n, strategy-proofness and welfare lower boundedness characterize the pivotal mechanisms.

Moulin's Theorem 3 (1986). *On the full domain $\prod_{j=1}^{n} D_j = V^n$, a mechanism $(d,t) \in \mathscr{M}(V^n)$ satisfies strategy-proofness and welfare lower boundedness if and only if it is a pivotal mechanism.*

Consider any pivotal mechanism $(d^*,t^*) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$. Since the pivotal mechanism (d^*,t^*) satisfies welfare lower boundedness, for each $v \in \prod_{j=1}^{n} D_j$,

$$0 \leq v_i\big(d^*(v)\big) + t_i^*(v) - \min_{y \in X} v_i(y) = \max_{y \in X} \sum_{j=1}^{n} v_j(y) - \max_{y \in X} \sum_{j \neq i} v_j(y) - \min_{y \in X} v_i(y).$$

The next property requires that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} D_j$, there exist a valuation function $v_i \in D_i$ such that the difference between individual i's final utility under the pivotal mechanism and that under his least preferred alternative with zero-transfer becomes arbitrarily small.

Property 3. For each $i \in N$, each $v_{-i} \in \prod_{j \neq i} D_j$, and each $\varepsilon > 0$, there exists $v_i \in D_i$ such that

$$\max_{y \in X} \sum_{j=1}^{n} v_j(y) - \max_{y \in X} \sum_{j \neq i} v_j(y) - \min_{y \in X} v_i(y) < \varepsilon.$$

Property 3 is similar to Property 2 in the sense that the term $-\min_{y \in E(\sum_{j \neq i} v_j)} v_i(y)$ in Property 2 is replaced by $-\min_{y \in X} v_i(y)$ in Property 3. In fact, if a domain satisfies Property 3, then it also satisfies Property 2.

Theorem 2 states that strategy-proofness and welfare lower boundedness characterize the pivotal mechanisms on a domain $\prod_{j=1}^{n} D_j$ if and only if the domain satisfies Properties 1 and 3.[*3]

Theorem 2. *On any connected domain $\prod_{j=1}^{n} D_j$, the following statements are equivalent:*

[*3] Note that the mechanism $(\bar{d},\bar{t})$ in Example 1 (Subection 1.3.1) also satisfies welfare lower boundedness. Therefore, the uniqueness result by Moulin (1986; Theorem 3) no longer holds on the auction domain $\prod_{j=1}^{n} D_j^A$. This is because the auction domain violates Property 1.

(i) all strategy-proof and welfare lower bounded mechanisms on $\prod_{j=1}^{n} D_j$ are pivotal mechanisms;

(ii) $\prod_{j=1}^{n} D_j$ satisfies Properties 1 and 3.

Proof. See Appendix B. ■

By a similar argument to Lemma 2, we obtain the following lemma.

Lemma 3. *If a domain $\prod_{j=1}^{n} D_j$ satisfies Property 2*, then it also satisfies Property 3.*

Corollary 2 gives simple sufficient conditions on domains under which Moulin's characterization in his Theorem 3 holds.

Corollary 2. *Suppose that a domain $\prod_{j=1}^{n} D_j$ is connected. If $\prod_{j=1}^{n} D_j$ satisfies Properties 1* and 2*, then any strategy-proof and welfare lower bounded mechanism $(d, t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ must be a pivotal mechanism.*

Proof. Immediately follows from Theorem 2, and Lemmas 1 and 3. ■

1.3.3 No free ride, no disposal of utility and distribution

We introduce two mild monotonicity axioms. *No disposal of utility* requires that if individual i's valuation weakly increases for all alternatives, then his final utility also increases weakly.

No disposal of utility. For each $i \in N$, each $v \in \prod_{j=1}^{n} D_j$, and each $v'_i \in D_i$, if $v'_i(x) \geq v_i(x)$ for all $x \in X$, then

$$v'_i\big(d(v'_i, v_{-i})\big) + t_i(v'_i, v_{-i}) \geq v_i\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}).$$

Distribution requires that given a valuation profile, if individual i's valuation increases for a unique efficient alternative, then no other individuals suffer from this change.

Distribution. For each $i \in N$, each $v \in \prod_{j=1}^{n} D_j$, and each $v'_i \in D_i$, if $E(\sum_{j=1}^{n} v_j) = \{z\}$ and

$$v'_i(z) > v_i(z), \;\; v'_i(x) = v_i(x) \;\text{ for all } x \neq z,$$

then

$$v_j\big(d(v'_i, v_{-i})\big) + t_j(v'_i, v_{-i}) \geq v_j\big(d(v_i, v_{-i})\big) + t_j(v_i, v_{-i}) \quad \text{for all } j \neq i.$$

Moulin (1986) shows that on the full domain V^n, the no free ride axiom, no disposal of utility and distribution characterize the pivotal mechanisms.

Moulin's Theorem 4 (1986). *Suppose* $|N| \geq 3$*. On the full domain* $\prod_{j=1}^{n} D_j = V^n$*, any pivotal mechanism* $(d^*, t^*) \in \mathscr{M}(V^n)$ *Pareto-dominates all mechanisms* $(d,t) \in \mathscr{M}(V^n)$ *satisfying the no free ride axiom, no disposal of utility and distribution.*

To introduce Property 4, we employ additional notation. For each $i \in N$ and each $v, v' \in \prod_{j=1}^{n} D_j$, we say that v' is a *monotonic transformation of* v *with respect to* $i \in N$ and write $v \underset{i}{\to} v'$ if there exists a finite sequence $v^{(0)}, v^{(1)}, \ldots, v^{(h)} \in \prod_{j=1}^{n} D_j$ for which $v^{(0)} = v$, $v^{(h)} = v'$, and for each $\ell \in \{1, \ldots, h\}$, one of the following two conditions holds:

(i) $v_i^{(\ell)} \geq v_i^{(\ell-1)}$ and $v_j^{(\ell)} = v_j^{(\ell-1)}$ for all $j \neq i$,

(ii) $E(\sum_{j=1}^{n} v_j^{(\ell-1)}) = \{z\}$ for some $z \in X$ and there exists $j \neq i$ such that

$$\begin{aligned} &v_j^{(\ell)}(z) > v_j^{(\ell-1)}(z), \;\; v_j^{(\ell)}(x) = v_j^{(\ell-1)}(x) \;\text{ for all } x \neq z, \\ &v_k^{(\ell)} = v_k^{(\ell-1)} \quad \text{for all } k \neq j. \end{aligned}$$

Note that if a mechanism satisfies no disposal of utility and distribution, then $v \underset{i}{\to} v'$ implies that

$$v_i'\big(d(v')\big) + t_i(v') \geq v_i\big(d(v)\big) + t_i(v).$$

Consider any individual $i \in N$ and any valuation profiles $v, v' \in \prod_{j=1}^{n} D_j$ with $v \underset{i}{\to} v'$. Let $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ be a mechanism that satisfies the no free ride axiom. Then, since (d,t) satisfies the no free ride axiom,

$$v_j'\big(d(v')\big) + t_j(v') \geq \min_{y \in E(\sum_{k \neq j} v_k')} v_j'(y) \quad \text{for all } j \neq i.$$

Moreover, by feasibility of (d,t), we have

$$v_i'\big(d(v')\big) + t_i(v') \leq \max_{x \in X} \sum_{j=1}^{n} v_j'(x) - \sum_{j \neq i} \big\{v_j'(d(v')) + t_j(v')\big\}.$$

Therefore, by combining these two inequalities,

$$v_i'\big(d(v')\big) + t_i(v') \leq \max_{y \in X} \sum_{j=1}^{n} v_j'(y) - \sum_{j \neq i} \min_{y \in E(\sum_{k \neq j} v_k')} v_j'(y), \tag{1.2}$$

that is, individual i's final utility at v' under the mechanism (d,t) is at most the right hand side of equation (1.2). On the other hand, individual i's final utility at v under a pivotal mechanism is

$$\max_{y \in X} \sum_{j=1}^{n} v_j(y) - \max_{y \in X} \sum_{j \neq i} v_j(y). \tag{1.3}$$

The following property requires that for each $i \in N$ and each $v \in \prod_{j=1}^n D_j$, there exist $v' \in \prod_{j=1}^n D_j$ with $v \underset{i}{\rightarrow} v'$ such that the difference between the right hand side of equation (1.2) and equation (1.3) becomes arbitrarily small.

Property 4. For each $i \in N$, each $v \in \prod_{j=1}^n D_j$, and each $\varepsilon > 0$, there exists $v' \in \prod_{j=1}^n D_j$ with $v \underset{i}{\rightarrow} v'$ such that

$$\left\{\max_{y\in X}\sum_{j=1}^n v'_j(y) - \sum_{j\neq i}\min_{y\in E(\sum_{k\neq j} v'_k)} v'_j(y)\right\} - \left\{\max_{y\in X}\sum_{j=1}^n v_j(y) - \max_{y\in X}\sum_{j\neq i} v_j(y)\right\} < \varepsilon.$$

Theorem 3 states that the no free ride axiom, no disposal of utility and distribution characterize the pivotal mechanisms on a domain $\prod_{j=1}^n D_j$ if and only if the domain satisfies Property 4. Note that in Theorem 3, connectedness is not imposed.

Theorem 3. *Suppose $|N| \geq 3$. The following statements are equivalent:*

(i) any pivotal mechanism $(d^*, t^*) \in \mathscr{M}(\prod_{j=1}^n D_j)$ Pareto-dominates all mechanisms $(d, t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ satisfying the no free ride axiom, no disposal of utility, and distribution;

(ii) $\prod_{j=1}^n D_j$ satisfies Property 4.

Proof. See Appendix B. ■

The following is an example of domains that violate Property 4.

Example 2. Consider a domain that is bounded from above by some valuation function $\bar{u} \in V$; for each $i \in N$, let

$$D_i^{\bar{u}} \equiv \left\{v_i \in V; v_i \leq \bar{u}\right\}.$$

Then, the domain $\prod_{j=1}^n D_j^{\bar{u}}$ does not satisfy Property 4. To see this, consider a simple case where $N = \{1, 2, 3\}$, $X = \{a, b, c\}$, and $\bar{u} \in V$ is as illustrated in Table 1. There, $\bar{u}(a)$ is 100 and $\bar{u}(b)$ is 95, for example. Let $v_1 = \bar{u}$, and v_2 and v_3 be as in Table 1. Then, $E\left(\sum_{j=1}^3 v_j\right) = \{a, b\}$. Therefore, by definition of a monotonic transformation $\underset{i}{\rightarrow}$, there exists no $v' \in \prod_{j=1}^3 D_j^{\bar{u}}$ with $v' \neq v$ such that $v \underset{1}{\rightarrow} v'$. Moreover,

$$\max_{y\in X}\sum_{j=1}^3 v_j(y) - \min_{y\in E(\sum_{j\neq 2} v_j)} v_2(y) - \min_{y\in E(\sum_{j\neq 3} v_j)} v_3(y) = 120 - 10 - 10 = 100,$$

$$\max_{y\in X}\sum_{j=1}^3 v_j(y) - \max_{y\in X}\sum_{j\neq 1} v_j(y) = 120 - 25 = 95.$$

	a	b	c
$\bar{u} = v_1$	100	95	25
v_2	10	10	20
v_3	10	15	5
$v_1 + v_2$	110	105	45
$v_1 + v_3$	110	110	30
$v_2 + v_3$	20	25	25
$v_1 + v_2 + v_3$	120	120	50

Table 1: Individuals' valuation profile

Thus, the domain $D_{\bar{u}}$ violates Property 4. Then, by Theorem 3, we know that there exists a mechanism that satisfies the no free ride axiom, no disposal of utility and distribution, but that is not Pareto-dominated by a pivotal mechanism. [*4]

Next, we introduce a simple sufficient condition for Property 4. Property 4* requires that if a domain admits a valuation function v_i, i.e., $v_i \in D_i$, then it also admit all valuation functions greater than v_i with respect to the vector inequalities.

Property 4*. For each $i \in N$, each $v_i \in D_i$, and each $v'_i \in V$, if $v_i \leq v'_i$, then $v'_i \in D_i$.

Lemma 4. *Suppose $|N| \geq 3$. If a domain $\prod_{j=1}^n D_j$ satisfies Property 4*, then it also satisfies Property 4.*

Proof. See Appendix B. ■

Corollary 3 states that Property 4* is sufficient for the uniqueness of pivotal mechanisms among all mechanisms satisfying the no free ride and two monotonicity axioms.

Corollary 3. *Suppose $|N| \geq 3$. If a domain $\prod_{j=1}^n D_j$ satisfies Property 4*, then any pivotal mechanism $(d^*, t^*) \in \mathscr{M}(\prod_{j=1}^n D_j)$ Pareto-dominates all mechanisms $(d, t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ satisfying the no free ride axiom, no disposal of utility, and distribution.*

Proof. Immediately follows from Theorem 3 and Lemma 4. ■

[*4] For the construction of such a mechanism, see equation (2) in Appendix B; Proof of Theorem 3.

1.4 Conclusion

We introduced properties of restricted domains and showed that these properties are necessary and sufficient for Moulin's (1986) characterizations to hold. First, we showed that any strategy-proof and no free ride mechanism is a pivotal mechanism if and only if its domain satisfies Properties 1 and 2. Second, we confirmed that any strategy-proof and welfare lower bounded mechanism is a pivotal mechanism if and only if its domain satisfies Properties 1 and 3. Finally, we showed that any pivotal mechanism Pareto-dominates all mechanisms satisfying the no free ride axiom, no disposal of utility and distribution if and only if its domain satisfies Property 4. These results generalize the usefulness of Moulin's (1986) characterizations. Finding necessary and sufficient conditions without connectedness when there are infinitely many alternatives is left to future research.

Appendix A: Domain characterization without connectedness

We provide a necessary and sufficient condition for the uniqueness of pivotal mechanisms without assuming connectedness of domains. To simplify our discussion here, we focus only on Moulin's characterization imposing strategy-proofness and welfare lower boundedness. However, the result below can be easily extended to the case of with the no free ride axiom (just replace $\min_{y \in X} v_i(y)$ with $\min_{y \in E(\sum_{j \neq i} v_j)} v_i(y)$ in the subsequent arguement).

Given $\varepsilon > 0$, given $v_{-i} \in \prod_{j \neq i} D_j$, and given disjoint subsets $Y, Z \subset X$, let

$$A_i(\varepsilon, v_{-i}) = \left\{ v_i \in V : (n-1) \max_{x \in X} \sum_{j=1}^{n} v_j(x) - \sum_{i=1}^{n} \max_{x \in X} \sum_{j \neq i} v_j(x) < -\varepsilon \right\}, \tag{1.4}$$

$$B_i(\varepsilon, v_{-i}) = \left\{ v_i \in V : \max_{x \in X} \sum_{j=1}^{n} v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) > \min_{x \in X} v_i(x) + \varepsilon \right\}, \tag{1.5}$$

$$S_i(\varepsilon, v_{-i}, Y, Z) = \bigcup_{y \in Y} \left\{ v_i \in V : \ \forall z \in Z \ \ \sum_{j=1}^{n} v_j(y) > \sum_{j=1}^{n} v_j(z) + \varepsilon \right\}, \tag{1.6}$$

$$W_i(v_{-i}, Y, Z) = \bigcup_{z \in Z} \left\{ v_i \in V : \ \forall y \in Y \ \ \sum_{j=1}^{n} v_j(z) \geq \sum_{j=1}^{n} v_j(y) \right\}. \tag{1.7}$$

$A_i(\varepsilon, v_{-i})$ is the set of individual i's valuation functions at which the total transfer under a pivotal mechanism becomes strictly less than $-\varepsilon$ (see equation 1.1). $B_i(\varepsilon, v_{-i})$ is the set of individual i's valuation functions at which his final utility under a pivotal mechanism becomes strictly greater than that under his worst alternative with the positive transfer ε. $S_i(\varepsilon, v_{-i}, Y, Z)$

is the set of individual i's valuation functions at which for some alternative $y \in Y$, the total valuation becomes strictly greater than that for any alternative $z \in Z$ added by ε. $W_i(v_{-i}, Y, Z)$ is the set of individual i's valuation functions at which for some alternative $z \in Z$, the total valuation becomes weakly greater than that for any alternative $y \in Y$. To avoid notational complexity, denote $A_i \equiv A_i(\varepsilon, v_{-i})$, $B_i \equiv B_i(\varepsilon, v_{-i})$, $S_i \equiv S_i(\varepsilon, v_{-i}, Y, Z)$ and $W_i \equiv W_i(v_{-i}, Y, Z)$ when $\varepsilon > 0$, $v_{-i} \in \prod_{j \neq i} D_j$, and disjoint subsets $Y, Z \subset X$ are identified.

Property 5 is a slightly weaker condition than connectedness of domains. It states that each D_i cannot be divided into a union of two disjoint, non-empty sets of a particular form. A similar condition to Property 5 is considered by Chung and Olszewski (2007; Theorem 4) to provide a necessary and sufficient condition for *payoff (revenue) equivalence*.

Property 5. For each $i \in N$, each $\varepsilon > 0$, each $v_{-i} \in \prod_{j \neq i} D_j$, and each disjoint subsets $Y, Z \subset X$ such that $(Y \cup Z) \cap E(\sum_{j=1}^{n} v_j) \neq \emptyset$ for all $v_i \in D_i$, neither of the following two conditions hold:

(i) $S_i \cap D_i \neq \emptyset$, $(W_i \cap A_i) \cap D_i \neq \emptyset$, and $D_i \subset S_i \cup (W_i \cap A_i)$.

(ii) $(S_i \cap B_i) \cap D_i \neq \emptyset$, $W_i \cap D_i \neq \emptyset$, and $D_i \subset (S_i \cap B_i) \cup W_i$.

Lemma 5 states that Property 5 is actually weaker than connectedness.

Lemma 5. *If a domain $\prod_{j=1}^{n} D_j$ is connected, then it satisfies Property 5.*

Proof. See Appendix B. ■

Theorem 2* states that even if we do not impose connectedness of domains, strategy-proofness and welfare lower boundedness characterize the pivotal mechanisms on a domain $\prod_{j=1}^{n} D_j$ if and only if the domain satisfies Properties 1, 3 and 5.

Theorem 2*. *On any domain $\prod_{j=1}^{n} D_j$, the following statements are equivalent:*

(i) all strategy-proof and welfare lower bounded mechanisms are pivotal mechanisms;

(ii) $\prod_{j=1}^{n} D_j$ satisfies Properties 1, 3, and 5.

Proof. See Appendix B. ■

Appendix B: Proofs of theorems and Lemmas

Proof of Theorem 1: (i) ⇒ (ii). We show that if $\prod_{j=1}^{n} D_j$ violates Property 1 or 2, then there exists a non-pivotal mechanism that satisfies strategy-proofness and the no free ride axiom.

Case 1. Consider the case where $\prod_{j=1}^{n} D_j$ violates Property 1. Then, there exist $i \in N$ and $v_{-i} \in \prod_{j\neq i} D_j$ such that

$$a \equiv \inf_{v_i \in D_i} \Big\{ \sum_{j=1}^{n} \max_{x\in X} \sum_{k\neq j} v_k(x) - (n-1) \max_{x\in X} \sum_{j=1}^{n} v_j(x) \Big\} > 0.$$

Define a mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ to satisfy that for each $u \in \prod_{j=1}^{n} D_j$,

$$d(u) \in E\big(\sum_{j=1}^{n} u_j\big),$$

$$t_i(u) = \begin{cases} t_i^*(u) + a & \text{if } u_{-i} = v_{-i}, \\ t_i^*(u) & \text{otherwise,} \end{cases}$$

$$t_j(u) = t_j^*(u) \quad \text{for all } j \neq i,$$

where for each $j \in N$, t_j^* is a transfer function of a pivotal mechanism;

$$t_j^*(u) = \sum_{k\neq j} u_k\big(d(u)\big) - \max_{x\in X} \sum_{k\neq j} u_k(x).$$

One can easily check that (d,t) satisfies efficiency, strategy-proofness and the no free ride axiom. Moreover, (d,t) is not a pivotal mechanism.

Let us confirm that (d,t) is feasible. Take any $u \in \prod_{j=1}^{n} D_j$. If $u_{-i} \neq v_{-i}$, then $\sum_{j=1}^{n} t_j(u) = \sum_{j=1}^{n} t_j^*(u) \leq 0$. Suppose that $u_{-i} = v_{-i}$. Then,

$$\begin{aligned}
\sum_{j=1}^{n} t_j(u) &= \sum_{j=1}^{n} \Big\{ \sum_{k\neq j} u_k\big(d(u)\big) - \max_{x\in X} \sum_{k\neq j} u_k(x) \Big\} + a \\
&= a - \sum_{j=1}^{n} \Big\{ \max_{x\in X} \sum_{k\neq j} u_k(x) - \sum_{k\neq j} u_k(d(u)) \Big\} \\
&= a - \sum_{j=1}^{n} \max_{x\in X} \sum_{k\neq j} u_k(x) - \sum_{j=1}^{n} \sum_{k\neq j} u_k(d(u)) \\
&= a - \sum_{j=1}^{n} \max_{x\in X} \sum_{k\neq j} u_k(x) - (n-1) \max_{x\in X} \sum_{j=1}^{n} u_j(x) \leq 0,
\end{aligned}$$

where the last inequality follows from $u_{-i} = v_{-i}$ and definition of a.

Case 2. Consider the case where $\prod_{j=1}^n D_j$ violates Property 2. Then, there exist $i \in N$ and $v_{-i} \in \prod_{j \neq i} D_j$ such that

$$a \equiv \inf_{v_i \in D} \left\{ \max_{x \in X} \sum_{j=1}^n v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in E(\sum_{j \neq i} v_j)} v_i(x) \right\} > 0.$$

Define a mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ to satisfy that for each $u \in \prod_{j=1}^n D_j$,

$$d(u) \in E\big(\sum_{j=1}^n u_j\big),$$

$$t_i(u) = \begin{cases} t_i^*(u) - a & \text{if } u_{-i} = v_{-i}, \\ t_i^*(u) & \text{otherwise,} \end{cases}$$

$$t_j(u) = t_j^*(u) \quad \text{for all } j \neq i,$$

where for each $j \in N$, t_j^* is a transfer function of a pivotal mechanism;

$$t_j^*(u) = \sum_{k \neq j} u_k\big(d(u)\big) - \max_{x \in X} \sum_{k \neq j} u_k(x).$$

One can easily check that (d,t) satisfies efficiency, strategy-proofness, and feasibility. Obviously, (d,t) is not a pivotal mechanism.

Let us show that (d,t) satisfies the no free ride axiom. Take any $u \in \prod_{j=1}^n D_j$. If $u_{-i} \neq v_{-i}$, then,

$$u_i\big(d(u)\big) + t_i(u) = u_i\big(d(u)\big) + t_i^*(u) \geq \min_{x \in E(\sum_{j \neq i} u_j)} u_i(x).$$

Suppose that $u_{-i} = v_{-i}$. Then, by definitions of t_i and a,

$$\begin{aligned} & u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) - \min_{x \in E(\sum_{j \neq i} v_j)} u_i(x) \\ =& u_i\big(d(u_i, v_{-i})\big) + \sum_{j \neq i} v_j\big(d(u_i, v_{-i})\big) - \max_{x \in X} \sum_{j \neq i} v_j(x) - a - \min_{x \in E(\sum_{j \neq i} v_j)} u_i(x). \\ =& \left\{ \max_{x \in X} \left(u_i(x) + \sum_{j \neq i} v_j(x) \right) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in E(\sum_{j \neq i} v_j)} u_i(x) \right\} - a \geq 0. \end{aligned}$$

Hence (d,t) satisfies the no free ride axiom.

(ii) ⇒ (i). Suppose that Statement (ii) holds. Take any strategy-pfroof and no free ride mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$. Since a domain $\prod_{j=1} D_j$ is connected, from Suijs (1996;

Theorem 3.2 and Proposition 3.3), we know that (d, t) must be a Groves mechanism, that is, for each $i \in N$, there exists a function $h_i : \prod_{j \neq i} D_j \to \mathbb{R}$ such that

$$t_i(v) = \sum_{j \neq i} v_j\big(d(v)\big) - h_i(v_{-i}) \quad \text{for all } v \in \prod_{j=1}^{n} D_j.$$

It suffices to show that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} D_j$, $h_i(v_{-i}) = \max_{x \in X} \sum_{j \neq i} v_j(x)$.

Step 1: $\forall i \in N$, $\forall v_{-i} \in \prod_{j \neq i} D_j, h_i(v_{-i}) \leq \max_{x \in X} \sum_{j \neq i} v_j(x)$. Suppose, by contradiction, that there exist $i \in N$ and $v_{-i} \in \prod_{j \neq i} D_j$ such that

$$h_i(v_{-i}) > \max_{x \in X} \sum_{j \neq i} v_j(x).$$

Since $\prod_{j=1}^{n} D_j$ satisfies Property 2, there exists $v_i \in D_i$ such that

$$\max_{x \in X} \sum_{j=1}^{n} v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in E(\sum_{j \neq i} v_j)} v_i(x) < h_i(v_{-i}) - \max_{x \in X} \sum_{j \neq i} v_j(x).$$

Then,

$$\begin{aligned}
&v_i\big(d(v)\big) + t_i(v) - \min_{x \in E(\sum_{j \neq i} v_j)} v_i(x) \\
&= v_i\big(d(v)\big) + \sum_{j \neq i} v_j\big(d(v)\big) - h_i(v_{-i}) - \min_{x \in E(\sum_{j \neq i} v_j)} v_i(x) \\
&= \max_{x \in X} \sum_{j=1}^{n} v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in E(\sum_{j \neq i} v_j)} v_i(x) - h_i(v_{-i}) + \max_{x \in X} \sum_{j \neq i} v_j(x) \\
&< 0,
\end{aligned}$$

a contradiction to the assumption that (d, t) satisfies the no free ride axiom.

Step 2: $\forall i \in N$, $\forall v_{-i} \in \prod_{j \neq i} D_j, h_i(v_{-i}) = \max_{x \in X} \sum_{j \neq i} v_j(x)$. Suppose, by contradiction, that there exist $i \in N$ and $v_{-i} \in \prod_{j \neq i} D_j$ such that $h_i(v_{-i}) \neq \max_{x \in X} \sum_{j \neq i} v_j(x)$. Then, by Step 1,

$$h_i(v_{-i}) < \max_{x \in X} \sum_{j \neq i} v_j(x). \tag{1.8}$$

By feasibility of (d, t), for any $v_i \in D_i$,

$$0 \geq \sum_{j=1}^{n} t_j(v) = \sum_{j=1}^{n} \Big(\sum_{k \neq j} v_k\big(d(v)\big) - h_j(v_{-j}) \Big),$$

and hence

$$\sum_{j\neq i} h_j(v_{-j}) - \sum_{j=1}^{n} \sum_{k\neq j} v_k\big(d(v)\big) \geq -h_i(v_{-i}). \tag{1.9}$$

Then, by Step 1 and equations (1.9) and (1.8), for any $v_i \in D_i$,

$$\begin{aligned}
&\sum_{j=1}^{n} \max_{x\in X} \sum_{k\neq j} v_k(x) - (n-1) \max_{x\in X} \sum_{j=1}^{n} v_j(x) \\
&= \sum_{j=1}^{n} \max_{x\in X} \sum_{k\neq j} v_k(x) - \sum_{j=1}^{n} \sum_{k\neq j} v_k\big(d(v)\big) \\
&= \max_{x\in X} \sum_{k\neq i} v_k(x) + \sum_{j\neq i} \max_{x\in X} \sum_{k\neq j} v_k(x) - \sum_{j=1}^{n} \sum_{k\neq j} v_k\big(d(v)\big) \\
&\geq \max_{x\in X} \sum_{k\neq i} v_k(x) + \sum_{j\neq i} h_j(v_{-j}) - \sum_{j=1}^{n} \sum_{k\neq j} v_k\big(d(v)\big) \\
&\geq \max_{x\in X} \sum_{j\neq i} v_j(x) - h_i(v_{-i}) > 0,
\end{aligned}$$

a contradiction to the assumption that $\prod_{j=1} D_j$ satisfies Property 1.

□

Proof of Lemma 1: Suppose that a domain $\prod_{j=1}^{n} D_j$ satisfies Property 1*. Take any $i \in N$ and any $v_{-i} \in \prod_{j\neq i} D_j$. Since $\prod_{j=1}^{n} D_j$ satisfies Property 1*, there exist $v_i \in D_i$ and $x \in E(\sum_{j\neq i} v_j)$ such that $x \in E\left(\sum_{k\neq j} v_k\right)$ for all $j \in N$. Then, obviously $x \in E(\sum_{j=1}^{n} v_j)$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{n} \max_{y\in X} \sum_{k\neq j} v_k(y) - (n-1) \max_{y\in X} \sum_{j=1}^{n} v_j(y) &= \sum_{j=1}^{n} \sum_{k\neq j} v_k(x) - (n-1) \sum_{j=1}^{n} v_j(x) \\
&= (n-1) \sum_{j=1}^{n} v_j(x) - (n-1) \sum_{j=1}^{n} v_j(x) = 0.
\end{aligned}$$

Hence $\prod_{j=1}^{n} D_j$ satisfies Property 1. □

Proof of Lemma 2: Suppose that a domain $\prod_{j=1}^{n} D_j$ satisfies Property 2*. Take any $i \in N$ and any $v_{-i} \in \prod_{j\neq i} D_j$. Since a domain $\prod_{j=1}^{n} D_j$ satisfies Property 2*, there exists $v_i \in D_i$ such that $v_i(x) = C$ for all $x \in X$. Then,

$$\max_{y\in X} \sum_{j=1}^{n} v_j(y) - \max_{y\in X} \sum_{j\neq i} v_j(y) - \min_{y\in E(\sum_{j\neq i} v_j)} v_i(y) = \max_{y\in X} \sum_{j\neq i} v_j(y) + C - \max_{y\in X} \sum_{j\neq i} v_j(y) - C = 0.$$

Therefore, $\prod_{j=1}^{n} D_j$ satisfies Property 2.

□

Proof of Theorem 2: The proof is similar to that of Theorem 1.

(i) ⇒ (ii). We show that if a domain $\prod_{j=1}^{n} D_j$ violates Property 1 or 3, then there exists a non-pivotal mechanism that satisfies strategy-proofness and welfare lower boundedness. If $\prod_{j=1}^{n} D_j$ violates Property 1, then such a mechanism can be defined by the same way as that of Proof of Theorem 1 (Case 1 of "(i) ⇒ (ii) part").

Let us consider the case where $\prod_{j=1}^{n} D_j$ violates Property 3. Then, there exist $i \in N$ and $v_{-i} \in \prod_{j \neq i} D_j$ such that

$$a \equiv \inf_{v_i \in V} \left\{ \max_{x \in X} \sum_{j=1}^{n} v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in X} v_i(x) \right\} > 0.$$

Define a mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ to satisfy that for each $u \in \prod_{j=1}^{n} D_j$,

$$\begin{aligned} & d(u) \in E\big(\sum_{j=1}^{n} u_j\big), \\ & t_i(u) = \begin{cases} t_i^*(u) - a & \text{if } u_{-i} = v_{-i}, \\ t_i^*(u) & \text{otherwise,} \end{cases} \\ & t_j(u) = t_j^*(u) \quad \text{for all } j \neq i, \end{aligned}$$

where for each $j \in N$, $t_j^*(u) = \sum_{k \neq j} u_k\big(d(u)\big) - \max_{x \in X} \sum_{k \neq j} u_k(x)$. Then, (d,t) is not a pivotal mechanism and satisfies efficiency, strategy-proofness and feasibility.

Let us show that (d,t) also satisfies welfare lower boundedness. Take any $u \in \prod_{j=1}^{n} D_j$. If $u_{-i} \neq v_{-i}$, then

$$u_i\big(d(u)\big) + t_i(u) = u_i\big(d(u)\big) + t_i^*(u) \geq \min_{x \in X} u_i(x).$$

Suppose that $u_{-i} = v_{-i}$. Then, by definitions of t_i and a,

$$\begin{aligned} & u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) - \min_{x \in X} u_i(x) \\ =& u_i\big(d(u_i, v_{-i})\big) + \sum_{j \neq i} v_j\big(d(u_i, v_{-i})\big) - \max_{x \in X} \sum_{j \neq i} v_j(x) - a - \min_{x \in X} u_i(x). \\ =& \left\{ \max_{x \in X} \left(u_i(x) + \sum_{j \neq i} v_j(x) \right) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in X} u_i(x) \right\} - a \geq 0. \end{aligned}$$

Therefore, (d,t) satisfies welfare lower boundedness.

(ii) ⇒ (i). Take any mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ that satisfies strategy-proofness and welfare lower boundedness. Since a domain $\prod_{j=1} D_j$ is connected, from Suijs (1996; Theorem 3.2 and Proposition 3.3) we know that (d,t) must be a Groves mechanism, that is, for each $i \in N$, there exists a function $h_i : \prod_{j \neq i} D_j \to \mathbb{R}$ such that

$$t_i(v) = \sum_{j \neq i} v_j\big(d(v)\big) - h_i(v_{-i}) \quad \text{for all } v \in \prod_{j=1}^n D_j.$$

Let us show that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} D_j$, $h_i(v_{-i}) \geq \max_{x \in X} \sum_{j \neq i} v_j(x)$. Suppose, by contradiction, that there exist $i \in N$ and $v_{-i} \in \prod_{j \neq i} D_j$ such that

$$\max_{x \in X} \sum_{j \neq i} v_j(x) < h_i(v_{-i}).$$

Since $\prod_{j=1}^n D_j$ satisfies Property 3, there exists $v_i \in D_i$ such that

$$\max_{x \in X} \sum_{j=1}^n v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in X} v_i(x) < h_i(v_{-i}) - \max_{x \in X} \sum_{j \neq i} v_j(x). \tag{1.10}$$

Then, by equation (1.10),

$$\begin{aligned}
& v_i\big(d(v)\big) + t_i(v) - \min_{x \in X} v_i(x) \\
&= v_i\big(d(v)\big) + \sum_{j \neq i} v_j\big(d(v)\big) - h_i(v_{-i}) - \min_{x \in X} v_i(x) \\
&= \sum_{j=1}^n v_j\big(d(v)\big) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in X} v_i(x) - h_i(v_{-i}) + \max_{x \in X} \sum_{j \neq i} v_j(x) \\
&= \left\{ \sum_{j=1}^n v_j\big(d(v)\big) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \min_{x \in X} v_i(x) \right\} - \left\{ h_i(v_{-i}) - \max_{x \in X} \sum_{j \neq i} v_j(x) \right\} \\
&< 0,
\end{aligned}$$

a contradiction to the assumption that (d,t) satisfies welfare lower boundedness.

Then, by the same argument as Step 2 of "(ii) ⇒ (i) part" in Proof of Theorem 1, for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} D_j$, $h_i(v_{-i}) = \max_{x \in X} \sum_{j \neq i} v_j(x)$.

□

Proof of Theorem 3: (i) ⇒ (ii). Let us show that if a domain $\prod_{j=1}^n D_j$ violates Property 4, then there exists a mechanism that satisfies the three axioms and is not Pareto-dominated by a pivotal mechanism. Suppose that $\prod_{j=1}^n D_j$ violates Property 4. Let $(d^*, t^*) \in \mathscr{M}(\prod_{j=1}^n D_j)$

be a pivotal mechanism. Since $\prod_{j=1}^n D_j$ violates Property 4, there exist $i \in N$, $v \in \prod_{j=1}^n D_j$ and $\varepsilon > 0$ such that for any $v' \in \prod_{j=1}^n D_j$ with $v \underset{i}{\to} v'$,

$$\max_{x\in X}\sum_{j=1}^n v'_j(x) - \sum_{j\neq i}\min_{x\in E(\sum_{k\neq j} v'_k)} v'_j(x) - \Big(v_i\big(d^*(v)\big) + t_i^*(v)\Big) \geq \varepsilon. \tag{1.11}$$

Let $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ be such that for each $u \in \prod_{j=1}^n D_j$,

$$d(u) \in E\big(\sum_{j=1}^n u_j\big), \tag{1.12}$$

$$t_i(u) = -u_i\big(d(u)\big) + \max\Bigg\{u_i\big(d^*(u)\big) + t_i^*(u), \inf_{u \underset{i}{\to} v'}\Big\{\max_{x\in X}\sum_{j=1}^n v'_j(x) - \sum_{j\neq i}\min_{x\in E(\sum_{k\neq j} v'_k)} v'_j(x)\Big\}\Bigg\},$$

$$t_j(u) = -u_j\big(d(u)\big) + \min_{x\in E(\sum_{k\neq j} u_k)} u_j(x) \quad \forall j \neq i.$$

For each $j \in N$ and each $u \in \prod_{j=1}^n D_j$, let $S_j(u) \equiv u_j\big(d(u)\big) + t_j(u)$ and $S_j^*(u) \equiv u_j\big(d^*(u)\big) + t_j^*(u)$. Note that by equation (1.11), $S_i(v) \geq S_i^*(v) + \varepsilon > S_i^*(v)$. Therefore, (d,t) is not Pareto dominated by a pivotal mechanism. In addition, we can easily check that (d,t) satisfies the no free ride axiom. Let us show that (d,t) satisfies feasibility, no disposal of utility, and distribution.

Feasibility. Take any $u \in \prod_{j=1}^n D_j$. We consider two cases. First, consider the case with $S_i(u) = S_i^*(u)$. Then,

$$\sum_{j=1}^n S_j(u) \leq \sum_{j=1}^n S_j^*(u) \leq \max_{x\in X}\sum_{j=1}^n u_j(x).$$

This inequality implies that $\sum_{j=1}^n t_j(u) \leq 0$. Second, consider the case with

$$S_i(u) = \inf_{u \underset{i}{\to} v'}\Big\{\max_{x\in X}\sum_{j=1}^n v'_j(x) - \sum_{j\neq i}\min_{x\in E(\sum_{k\neq j} v'_k)} v'_j(x)\Big\}.$$

Then, since $u \underset{i}{\to} u$,

$$\begin{aligned}&\max_{x\in X}\sum_{j=1}^n u_j(x) - \sum_{j=1}^n S_j(u)\\ &= \Big\{\max_{x\in X}\sum_{j=1}^n u_j(x) - \sum_{j\neq i}\min_{x\in E(\sum_{k\neq j} u_k)} u_j(x)\Big\} - \inf_{u \underset{i}{\to} v'}\Big\{\max_{x\in X}\sum_{j=1}^n v'_j(x) - \sum_{j\neq i}\min_{x\in E(\sum_{k\neq j} v'_k)} v'_j(x)\Big\}\end{aligned}$$

$\geq 0.$

Therefore, $\sum_{j=1}^{n} t_j(u) \leq 0$.

No disposal of utility. Take any $j \neq i$, any $u \in \prod_{j=1}^{n} D_j$ and any $u'_j \in D_j$ with $u_j \leq u'_j$. Then,

$$S_j(u_j, u_{-j}) = \min_{x \in E(\sum_{k \neq j} u_k)} u_j(x) \leq \min_{x \in E(\sum_{k \neq j} u_k)} u'_j(x) = S_j(u'_j, u_{-j}).$$

Next, take any $u \in \prod_{j=1}^{n} D_j$ and any $u'_i \in D_i$ with $u_i \leq u'_i$. Then,

$$S_i^*(u_i, u_{-i}) \leq S_i^*(u'_i, u_{-i}),$$

and since $(u_i, u_{-i}) \underset{i}{\rightarrow} (u'_i, u_{-i})$,

$$\begin{aligned} &\inf_{(u_i, u_{-i}) \underset{i}{\rightarrow} v'} \Big\{ \max_{x \in X} \sum_{j=1}^{n} v'_j(x) - \sum_{j \neq i} \min_{x \in E(\sum_{k \neq j} v'_k)} v'_j(x) \Big\} \\ \leq &\inf_{(u'_i, u_{-i}) \underset{i}{\rightarrow} v'} \Big\{ \max_{x \in X} \sum_{j=1}^{n} v'_j(x) - \sum_{j \neq i} \min_{x \in E(\sum_{k \neq j} v'_k)} v'_j(x) \Big\}. \end{aligned}$$

Therefore,

$$S_i(u_i, u_{-i}) \leq S_i(u'_i, u_{-i}).$$

Distribution. It suffices to show that for any distinct $k, \ell \in N$, any $u \in \prod_{j=1}^{n} D_j$ with $\{z\} = E(\sum_{j=1}^{n} u_j)$, and any $u'_k \in D_k$ such that

$$u'_k(z) > u_k(z) \text{ and } u'_k(x) = u_k(x) \text{ for all } x \in X \setminus \{z\},$$

we have

$$S_\ell(u'_k, u_{-k}) \geq S_\ell(u_k, u_{-k}).$$

Take any distinct $k, \ell \in N$.

Let us consider the case with $\ell \neq i$. Take any $u \in \prod_{j=1}^{n} D_j$ such that for some $z \in X$, $\{z\} = E(\sum_{j=1}^{n} u_j)$. Let us first show that

$$\min_{x \in E(\sum_{j \neq \ell} u_j)} u_\ell(x) \leq u_\ell(z). \tag{1.13}$$

Suppose, by contradiction, that

$$\min_{x \in E(\sum_{j \neq \ell} u_j)} u_\ell(x) > u_\ell(z).$$

Then,

$$\min_{x \in E(\sum_{j \neq \ell} u_j)} u_\ell(x) + \max_{x \in X} \sum_{j \neq \ell} u_j(x) > \sum_{j=1}^{n} u_j(z),$$

a contradiction to $\{z\} = E(\sum_{j=1}^n u_j)$. Therefore, equation (1.13) holds. Now, take any $u'_k \in D_k$ such that $u'_k(z) > u_k(z)$ and $u'_k(x) = u_k(x)$ for all $x \in X \setminus \{z\}$. Then,

$$E(u'_k + \sum_{j \neq k,\ell} u_j) \subset E(\sum_{j \neq \ell} u_j) \cup \{z\}. \tag{1.14}$$

Therefore, by $\ell \neq i$ and equations (1.14) and (1.13),

$$\begin{aligned} S_\ell(u'_k, u_{-k}) &= \min_{x \in E(u'_k + \sum_{j \neq k,\ell} u_j)} u_\ell(x) \geq \min_{x \in E(\sum_{j \neq \ell} u_j) \cup \{z\}} u_\ell(x) \\ &= \min_{x \in E(\sum_{j \neq \ell} u_j)} u_\ell(x) = S_\ell(u_k, u_{-k}). \end{aligned}$$

Next, consider the case with $\ell = i$. Take any $u \in \prod_{j=1}^n D_j$ with $\{z\} = E(\sum_{j=1}^n u_j)$ and any $u'_k \in D_k$ such that $u'_k(z) > u_k(z)$ and $u'_k(x) = u_k(x)$ for all $x \in X \setminus \{z\}$. Then,

$$S_i^*(u'_k, u_{-k}) \geq S_i^*(u_k, u_{-k}),$$

and since $(u_k, u_{-k}) \overrightarrow{_i} (u'_k, u_{-k})$,

$$\begin{aligned} &\inf_{(u_k, u_{-k}) \overrightarrow{_i} v'} \Big\{ \max_{x \in X} \sum_{j=1}^{n} v'_j(x) - \sum_{j \neq i} \min_{x \in E(\sum_{j' \neq j} v'_{j'})} v'_j(x) \Big\} \\ \leq &\inf_{(u'_k, u_{-k}) \overrightarrow{_i} v'} \Big\{ \max_{x \in X} \sum_{j=1}^{n} v'_j(x) - \sum_{j \neq i} \min_{x \in E(\sum_{j' \neq j} v'_{j'})} v'_j(x) \Big\}. \end{aligned}$$

Therefore,

$$S_i(u'_k, u_{-k}) \geq S_i(u_k, u_{-k}).$$

(ii) ⇒ (i). Suppose that a domain $\prod_{j=1}^n D_j$ satisfies Property 4. Take any mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ that satisfies the no free ride axiom, no disposal of utility and distribution. Let $(d^*, t^*) \in \mathscr{M}(\prod_{j=1}^n D_j)$ be a pivotal mechanism. Let us show that for each $i \in N$ and each $v \in \prod_{j=1}^n D_j$,

$$v_i\big(d(v)\big) + t_i(v) \leq v_i\big(d^*(v)\big) + t_i^*(v).$$

Take any $i \in N$, any $v \in \prod_{j=1}^n D_j$, and any $\varepsilon > 0$. Since $\prod_{j=1}^n D_j$ satisfies Property 4, there exists $v' \in \prod_{j=1}^n D_j$ with $v \overrightarrow{_i} v'$ such that

$$\max_{x \in X} \sum_{j=1}^{n} v'_j(x) - \sum_{j \neq i} \min_{x \in E(\sum_{k \neq j} v'_k)} v'_j(x) - \Big(v_i\big(d^*(v)\big) + t_i^*(v) \Big) < \varepsilon. \tag{1.15}$$

On the other hand, by feasibility of (d,t),

$$\sum_{j=1}^{n}\Big(v'_j\big(d(v')\big)+t_j(v')\Big)\leq \max_{x\in X}\sum_{j=1}^{n}v'_j(x).$$

Therefore,

$$v'_i\big(d(v')\big)+t_i(v')\leq \max_{x\in X}\sum_{j=1}^{n}v'_j(x)-\sum_{j\neq i}\Big(v'_j\big(d(v')\big)+t_j(v')\Big). \tag{1.16}$$

Moreover, since (d,t) satisfies the no free ride axiom,

$$\max_{x\in X}\sum_{j=1}^{n}v'_j(x)-\sum_{j\neq i}\Big(v'_j\big(d(v')\big)+t_j(v')\Big)\leq \max_{x\in X}\sum_{j=1}^{n}v'_j(x)-\sum_{j\neq i}\min_{x\in E(\sum_{k\neq j}v'_k)}v'_j(x). \tag{1.17}$$

Then, by combining equations (1.16), (1.17), and (1.15), we have

$$v'_i\big(d(v')\big)+t_i(v')\leq \max_{x\in X}\sum_{j=1}^{n}v'_j(x)-\sum_{j\neq i}\min_{x\in E(\sum_{j\neq i}v'_j)}v'_j(x)<v_i\big(d^*(v)\big)+t^*_i(v)+\varepsilon.$$

Since $v \overset{\rightarrow}{_i} v'$, no disposal of utility and distribution of (d,t) implies that

$$v_i\big(d(v)\big)+t_i(v)\leq v'_i\big(d(v')\big)+t_i(v')<v_i\big(d^*(v)\big)+t^*_i(v)+\varepsilon$$

Since ε was taken arbitrarily,

$$v_i\big(d(v)\big)+t_i(v)\leq v_i\big(d^*(v)\big)+t^*_i(v).$$

□

Proof of Lemma 4: Suppose that a domain $\prod_{j=1}^{n}D_j$ satisfies Property 4*. Take any $i\in N$, any $v\in\prod_{j=1}^{n}D_j$, and any $\varepsilon>0$. Fix some $z\in E(\sum_{j\neq i}v_j)$. Let $v'_i\in V$ be such that for each $x\in X$,

$$v'_i(x)=\begin{cases}\max_{y\in X}\sum_{j=1}^{n}v_j(y)-\sum_{j\neq i}v_j(z)+\frac{1}{2}\varepsilon & \text{if } x=z,\\ v_i(x) & \text{otherwise.}\end{cases}$$

Then, $v'_i\geq v_i$, and hence $v \overset{\rightarrow}{_i} (v'_i,v_{-i})$. Moreover, $E(v'_i+\sum_{j\neq i}v_j)=\{z\}$.

For each $k\in N\setminus\{i\}$, Let v'_k be such that for each $x\in X$,

$$v'_k(x)=\begin{cases}v_k(z)+\delta & \text{if } x=z,\\ v_k(x) & \text{otherwise,}\end{cases}$$

where $\delta > 0$ is large enough to guarantee that $E(\sum_{k\neq j} v'_k) = \{z\}$ for all $j \in N$. Note that since $|N| \geq 3$, such δ exists. Then, $v \underset{i}{\to} v'$ and $E(\sum_{j=1}^n v'_j) = \{z\}$. Therefore, by definition of v'_i and by $z \in E(\sum_{j\neq i} v_j)$,

$$\begin{aligned}
\max_{y\in X}\sum_{j=1}^n v'_j(y) - \sum_{j\neq i}\min_{y\in E(\sum_{k\neq j} v'_k)} v'_j(y) &= v'_i(z) + \sum_{j\neq i} v'_j(z) - \sum_{j\neq i} v'_j(z)\\
&= v'_i(z)\\
&= \max_{y\in X}\sum_{j=1}^n v_j(y) - \sum_{j\neq i} v_j(z) + \frac{1}{2}\varepsilon\\
&= \max_{y\in X}\sum_{j=1}^n v_j(y) - \max_{y\in X}\sum_{j\neq i} v_j(y) + \frac{1}{2}\varepsilon.
\end{aligned}$$

Thus,

$$\left\{\max_{y\in X}\sum_{j=1}^n v'_j(y) - \sum_{j\neq i}\min_{y\in E(\sum_{k\neq j} v'_k)} v'_j(y)\right\} - \left\{\max_{y\in X}\sum_{j=1}^n v_j(y) - \max_{y\in X}\sum_{j\neq i} v_j(y)\right\} = \frac{1}{2}\varepsilon < \varepsilon,$$

and hence the domain $\prod_{j=1}^n D_j$ satisfies Property 4. □

Proof of Lemma 5: Let us show that if a domain $\prod_{j=1}^n D_j$ violates Property 5, then $\prod_{j=1}^n D_j$ is not connected. Since $\prod_{j=1}^n D_j$ violates Property 5, there exist $i \in N$, $\varepsilon > 0$, $v_{-i} \in \prod_{j\neq i} D_j$, and disjoint sets $Y, Z \subset X$ such that for each $v_i \in D_i$, $(Y \cup Z) \cap E(\sum_{j=1}^n v_j) \neq \emptyset$ and one of the two conditions in statement of Property 5 holds. Consider the case where condition (i) holds. Then,

$$D_i \subset S_i \cup \big(W_i \cap A_i\big).$$

Let

$$W'_i = \bigcup_{z\in Z}\left\{v_i \in V : \ \forall y \in Y \quad \sum_{j=1}^n v_j(z) + \varepsilon > \sum_{j=1}^n v_j(y)\right\}.$$

Then, $W_i \subset W'_i$. Therefore,

$$D_i \subset S_i \cup \big(W'_i \cap A_i\big).$$

In addition, S_i and W'_i are disjoint, and S_i, W'_i and A_i are open. Thus, $\prod_{j=1}^n D_j$ is not connected. A similar argument to above shows the case where condition (ii) of statement in Property 5 holds. □

Proof of Theorem 2*: Our proof is based on that by Chung and Olszewski (2007; Theorem 4).

(i) ⇒ (ii). Let us show that if a domain $\prod_{j=1}^n D_j$ violates property 5, then there exists a strategy-proof and welfare lower bounded mechanism other than the pivotal mechanisms. Suppose that $\prod_{j=1}^n D_j$ violates property 5. Then, there exist $i \in N$, $\varepsilon > 0$, $v_{-i} \in \prod_{j \neq i} D_j$, and disjoint sets $Y, Z \subset X$ such that for each $v_i \in D_i$, $\big(Y \cup Z\big) \cap E(\sum_{j=1}^n v_j) \neq \emptyset$ and one of the two conditions in statement of Property 5 is satisfied.

Case 1. Consider the case where condition (ii) of statement in Property 5 holds;

$$S_i \cap D_i \neq \emptyset, \ \ (W_i \cap A_i) \cap D_i \neq \emptyset, \text{ and } D_i \subset S_i \cup (W_i \cap A_i). \tag{1.18}$$

Let $(d,t) \in \mathscr{M}(\prod_{j=1}^n D_j)$ be such that for each $i \in N$ and each $u \in \prod_{j=1}^n D_j$,

$$d(u) \in \begin{cases} Z \cap E(\sum_{j=1}^n u_j) & \text{if } u_{-i} = v_{-i} \text{ and } Z \cap E(\sum_{j=1}^n u_j) \neq \emptyset, \\ Y \cap E(\sum_{j=1}^n u_j) & \text{if } u_{-i} = v_{-i} \text{ and } Z \cap E(\sum_{j=1}^n u_j) = \emptyset, \\ E(\sum_{j=1}^n u_j) & \text{otherwise,} \end{cases}$$

$$t_i(u) = \begin{cases} t_i^*(u) + \varepsilon & \text{if } u_{-i} = v_{-i} \text{ and } u_i \in W_i \cap A_i, \\ t_i^*(u) & \text{otherwise,} \end{cases}$$

$$t_j(u) = t_j^*(u),$$

where for each $j \in N$, $t_j^*(u) = \sum_{k \neq j} u_k\big(d(u)\big) - \max_{x \in X} \sum_{k \neq j} u_k(x)$. We can easily check that (d,t) satisfies efficiency and welfare lower boundedness. Let us show that (d,t) also satisfies strategy-proofness and feasibility.

Strategy-proofness. Take any $u \in \prod_{j=1}^n D_j$ and any $u_i' \in D_i$. If $u_{-i} = v_{-i}$ and $u_i \in W_i \cap A_i$, then by definition of (d,t),

$$\begin{aligned} u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) &= u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i}) + \varepsilon \\ &\geq u_i\big(d(u_i', v_{-i})\big) + t_i^*(u_i', v_{-i}) + \varepsilon \geq u_i\big(d(u_i', v_{-i})\big) + t_i(u_i', v_{-i}). \end{aligned}$$

Moreover, if $u_{-i} \neq v_{-i}$, then

$$\begin{aligned} u_i\big(d(u_i, u_{-i})\big) + t_i(u_i, u_{-i}) &= u_i\big(d(u_i, u_{-i})\big) + t_i^*(u_i, u_{-i}) \\ &\geq u_i\big(d(u_i', u_{-i})\big) + t_i^*(u_i', u_{-i}) = u_i\big(d(u_i', u_{-i})\big) + t_i(u_i', u_{-i}). \end{aligned}$$

Therefore, it suffices to consider the case with $u_{-i} = v_{-i}$ and $u_i \notin W_i \cap A_i$. Note that in this case $u_i \in S_i$ by the assumption (1.18).

If $u_i' \notin W_i \cap A_i$, then

$$u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) = u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i})$$

$$\geq u_i\big(d(u_i', v_{-i})\big) + t_i^*(u_i', v_{-i}) = u_i\big(d(u_i', v_{-i})\big) + t_i(u_i', v_{-i}).$$

Suppose that $u_i' \in W_i \cap A_i$. Then, by definition of W_i, there exists $z \in Z$ such that

$$u_i'(z) + \sum_{j \neq i} v_j(z) \geq u_i'(y) + \sum_{j \neq i} v_j(y) \text{ for all } y \in Y.$$

Hence by the assumption that $\big(Y \cup Z\big) \cap E(\sum_{j=1}^n v_j) \neq \emptyset$ for all $v_i \in D_i$,

$$Z \cap E(u_i' + \sum_{j \neq i} v_j) \neq \emptyset.$$

Then, by definition of d, $d(u_i', v_{-i}) \in Z$. On the other hand, since $u_i \in S_i$, by definition of S_i, there exists $y \in Y$ such that

$$u_i(y) + \sum_{j \neq i} v_j(y) > u_i(z) + \sum_{j \neq i} v_j(z) + \varepsilon \text{ for all } z \in Z. \tag{1.19}$$

Therefore, $Z \cap E(u_i + \sum_{j \neq i} v_j) = \emptyset$. Then, by definition of d, $d(u_i, v_{-i}) \in Y$.

Let $y^* \equiv d(u_i, v_{-i}) \in Y$and $z^* \equiv d(u_i', v_{-i}) \in Z$. Note that by (1.19),

$$u_i(y^*) + \sum_{j \neq i} v_j(y^*) > u_i(z^*) + \sum_{j \neq i} v_j(z^*) + \varepsilon. \tag{1.20}$$

Hence by equation (1.20),

$$\begin{aligned} u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) &= u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i}) \\ &= u_i(y^*) + \sum_{j \neq i} v_j(y^*) - \max_{x \in X} \sum_{j \neq i} v_j(x) \\ &> u_i(z^*) + \sum_{j \neq i} v_j(z^*) + \varepsilon - \max_{x \in X} \sum_{j \neq i} v_j(x) \\ &= u_i\big(d(u_i', v_{-i})\big) + t_i^*(u_i', v_{-i}) + \varepsilon = u_i\big(d(u_i', v_{-i})\big) + t_i(u_i', v_{-i}). \end{aligned}$$

Therefore, (d, t) satisfies strategy-proofness.

Feasibility. Take any $u \in \prod_{j=1}^n D_j$. If $u_{-i} \neq v_{-i}$ or $u_i \notin W_i \cap A_i$, then

$$\sum_{j=1}^n t_j(u) = \sum_{j=1}^n t_j^*(u) \leq 0.$$

Suppose that $u_{-i} = v_{-i}$ and $u_i \in W_i \cap A_i$. Then, by definition of A_i,

$$\sum_{j=1}^n t_j^*(u_i, v_{-i}) < -\varepsilon.$$

Therefore,

$$\sum_{j=1}^{n} t_j(u_i, v_{-i}) = \sum_{j=1}^{n} t_j^*(u_i, v_{-i}) + \varepsilon < -\varepsilon + \varepsilon = 0.$$

Thus, (d, t) satisfies feasibility.

Case 2. Consider the case where condition (ii) of statement in Property 5 holds;

$$(S_i \cap B_i) \cap D_i \neq \emptyset, \;\; W_i \cap D_i \neq \emptyset, \text{ and } D_i \subset (S_i \cap B_i) \cup W_i. \tag{1.21}$$

Let $(d, t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ be such that for each $i \in N$ and each $u \in \prod_{j=1}^{n} D_j$,

$$d(u) \in \begin{cases} Z \cap E(\sum_{j=1}^{n} u_j) & \text{if } u_{-i} = v_{-i} \text{ and } Z \cap E(\sum_{j=1}^{n} u_j) \neq \emptyset, \\ Y \cap E(\sum_{j=1}^{n} u_j) & \text{if } u_{-i} = v_{-i} \text{ and } Z \cap E(\sum_{j=1}^{n} u_j) = \emptyset, \\ E(\sum_{j=1}^{n} u_j) & \text{otherwise,} \end{cases}$$

$$t_i(u) = \begin{cases} t_i^*(u) - \varepsilon & \text{if } u_{-i} = v_{-i} \text{ and } u_i \in S_i \cap B_i, \\ t_i^*(u) & \text{otherwise,} \end{cases}$$

$$t_j(u) = t_j^*(u),$$

where for each $j \in N$, $t_j^*(u) = \sum_{k \neq j} u_k\big(d(u)\big) - \max_{x \in X} \sum_{k \neq j} u_k(x)$. One can easily check that (d, t) satisfies efficiency and feasibility. Let us confirm that (d, t) also satisfies strategy-proofness and welfare lower boundedness.

Strategy-proofness. Take any $u \in \prod_{j=1}^{n} D_j$ and any $u_i' \in D_i$. If $u_{-i} \neq v_{-i}$ or $u_i \notin S_i \cap B_i$, then

$$\begin{aligned} u_i\big(d(u_i, u_{-i})\big) + t_i(u_i, u_{-i}) &= u_i\big(d(u_i, u_{-i})\big) + t_i^*(u_i, u_{-i}) \\ &\geq u_i\big(d(u_i', u_{-i})\big) + t_i^*(u_i', u_{-i}) \geq u_i\big(d(u_i', u_{-i})\big) + t_i(u_i', u_{-i}). \end{aligned}$$

Consider the case with $u_{-i} = v_{-i}$ and $u_i \in S_i \cap B_i$. If $u_i' \in S_i \cap B_i$, then

$$\begin{aligned} u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) &= u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i}) - \varepsilon \\ &\geq u_i\big(d(u_i', v_{-i})\big) + t_i^*(u_i', v_{-i}) - \varepsilon = u_i\big(d(u_i', v_{-i})\big) + t_i(u_i', v_{-i}). \end{aligned}$$

Suppose that $u_i' \notin S_i \cap B_i$, and hence $u_i' \in W_i$ by the assumption (1.21). Then, by definition of W_i, there exists $z \in Z$ such that

$$u_i'(z) + \sum_{j \neq i} v_j(z) \geq u_i'(y) + \sum_{j \neq i} v_j(y) \;\; \text{for all } y \in Y.$$

Thus, by the assumption that $\big(Y \cup Z\big) \cap E(\sum_{j=1}^{n} v_j) \neq \emptyset$ for all $v_i \in D_i$,

$$Z \cap E(u_i' + \sum_{j \neq i} v_j) \neq \emptyset.$$

Therefore, by definition of d, $d(u_i', v_{-i}) \in Z$. On the other hand, since $u_i \in S_i$, by definition of S_i, there exists $y \in Y$ such that

$$u_i(y) + \sum_{j \neq i} v_j(y) > u_i(z) + \sum_{j \neq i} v_j(z) + \varepsilon \text{ for all } z \in Z. \tag{1.22}$$

Hence $Z \cap E(u_i + \sum_{j \neq i} v_j) = \emptyset$. Then, by definition of d, $d(u_i, v_{-i}) \in Y$.

Let $y^* \equiv d(u_i, v_{-i}) \in Y$ and $z^* \equiv d(u_i', v_{-i}) \in Z$. Note that by equation (1.22) and $y^* \in E(u_i' + \sum_{j \neq i} v_j)$,

$$u_i(y^*) + \sum_{j \neq i} v_j(y^*) > u_i(z^*) + \sum_{j \neq i} v_j(z^*) + \varepsilon. \tag{1.23}$$

Hence by equation (1.23),

$$\begin{aligned}
u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) &= u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i}) - \varepsilon \\
&= u_i(y^*) + \sum_{j \neq i} v_j(y^*) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \varepsilon \\
&> u_i(z^*) + \sum_{j \neq i} v_j(z^*) + \varepsilon - \max_{x \in X} \sum_{j \neq i} v_j(x) - \varepsilon \\
&= u_i\big(d(u_i', v_{-i})\big) + t_i^*(u_i', v_{-i}) = u_i\big(d(u_i', v_{-i})\big) + t_i(u_i', v_{-i}).
\end{aligned}$$

Therefore, (d, t) satisfies strategy-proofness.

Welfare lower boundedness. Let us show that for each $u \in \prod_{j=1}^{n} D_j$,

$$u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) \geq \min_{x \in X} u_i(x).$$

Take any $u \in \prod_{j=1}^{n} D_j$. If $u_{-i} \neq v_{-i}$ or $u_i \notin S_i \cap B_i$, then

$$u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) = u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i}) \geq \min_{x \in X} v_i(x).$$

Suppose that $u_{-i} = v_{-i}$ and $u_i \in S_i \cap B_i$. Then, by definition of B_i,

$$\max_{x \in X} \big(u_i(x) + \sum_{j \neq i} v_j(x)\big) - \max_{x \in X} \sum_{j \neq i} v_j(x) - \varepsilon > \min_{x \in X} u_i(x).$$

Therefore,

$$u_i\big(d(u_i, v_{-i})\big) + t_i(u_i, v_{-i}) = u_i\big(d(u_i, v_{-i})\big) + t_i^*(u_i, v_{-i}) - \varepsilon$$

$$= \max_{x\in X} \big(u_i(x) + \sum_{j\neq i} v_j(x)\big) - \max_{x\in X} \sum_{j\neq i} v_j(x) - \varepsilon > \min_{x\in X} u_i(x).$$

(ii) ⇒ (i). In Proof of Theorem 2, we showed that if a domain satisfies Properties 1 and 3, then any Groves mechanism must be a pivotal mechanism. Therefore, it suffices to show that if a domain satisfies Properties 1, 3 and 5, then any strategy-proof and welfare lower bounded mechanism must be a Groves mechanism. Suppose that a domain $\prod_{j=1}^{n} D_j$ satisfies Properties 1, 3 and 5. Take any mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ that satisfies strategy-proofness and welfare lower boundedness. Let $(d^*,t^*) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ be a pivotal mechanism such that $d^* = d$. To show that (d,t) is a Groves mechanism, it suffices to show that for each $i \in N$ and each $v_{-i} \in \prod_{j\neq i} D_j$,

$$t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) = t_i(v_i', v_{-i}) - t_i^*(v_i', v_{-i}) \ \text{ for all } v_i, v_i' \in D_i.$$

Step 1. Let us show that for each $i \in N$, each $v_{-i} \in \prod_{j\neq i} D_j$ and each $v_i, v_i' \in D_i$, if $d(v_i, v_{-i}) = d(v_i', v_{-i})$, then

$$t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) = t_i(v_i', v_{-i}) - t_i^*(v_i', v_{-i}). \tag{1.24}$$

Take any $i \in N$, any $v_{-i} \in \prod_{j\neq i} D_j$, and any $v_i, v_i' \in D_i$ with $d(v_i, v_{-i}) = d(v_i', v_{-i})$. Let $x \equiv d(v_i, v_{-i}) = d(v_i', v_{-i})$. Then, by strategy-proofness of (d,t),

$$\begin{aligned} t_i(v_i, v_{-i}) &= v_i\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}) - v_i(x) \\ &\geq v_i\big(d(v_i', v_{-i})\big) + t_i(v_i', v_{-i}) - v_i(x) = t_i(v_i', v_{-i}), \end{aligned}$$

and

$$\begin{aligned} t_i(v_i', v_{-i}) &= v_i'\big(d(v_i', v_{-i})\big) + t_i(v_i', v_{-i}) - v_i(x) \\ &\geq v_i'\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}) - v_i(x) = t_i(v_i, v_{-i}). \end{aligned}$$

Therefore, $t_i(v_i, v_{-i}) = t_i(v_i', v_{-i})$. Similarly, we can show that $t_i^*(v_i, v_{-i}) = t_i^*(v_i', v_{-i})$. Therefore, equation (1.24) holds.

Step 2. Let us show that for each $i \in N$ and each $v_{-i} \in \prod_{j\neq i} D_j$,

$$t_i^*(v_i, v_{-i}) \leq t_i(v_i, v_{-i}) \ \text{ for all } v_i \in D_i. \tag{1.25}$$

Take any $i \in N$ and any $v_{-i} \in \prod_{j\neq i} D_j$. We consider two cases.

Case 2-1. Consider the case with

$$t_i(u_i, v_{-i}) - t_i^*(u_i, v_{-i}) = t_i(u_i', v_{-i}) - t_i^*(u_i', v_{-i}) \text{ for all } u_i, u_i' \in D_i.$$

Then, there exists $h_i(v_{-i}) \in \mathbb{R}$ such that for each $v_i \in D_i$, $t_i(v_i, v_{-i}) = \sum_{j \neq i} v_j\big(d(v_i, v_{-i})\big) - h_i(v_{-i})$. Hence by the same argument as Step 1 of Proof of Theorem 2, it follows that $t_i^*(v_i, v_{-i}) \leq t_i(v_i, v_{-i})$ for all $v_i \in D_i$.

Case 2-2. Consider the case with

$$t_i(u_i, v_{-i}) - t_i^*(u_i, v_{-i}) \neq t_i(u_i', v_{-i}) - t_i^*(u_i', v_{-i}) \text{ for some } u_i, u_i' \in D_i.$$

Then, by Step 1 and finiteness of X, $t_i(\,\cdot\,, v_{-i}) - t_i^*(\,\cdot\,, v_{-i})$ takes at least two and only a finite number of values. Suppose, by contradiction, that there exists $w_i \in D_i$ such that $t_i(w_i, v_{-i}) < t_i^*(w_i, v_{-i})$. Then, by $t_i(w_i, v_{-i}) - t_i^*(w_i, v_{-i}) < 0$, there exists $\varepsilon > 0$ such that the values of $t_i(\,\cdot\,, v_{-i}) - t_i^*(\,\cdot\,, v_{-i})$ belong to the union of two intervals $(-\infty, -2\varepsilon)$ and $(-\varepsilon, \infty)$, and each interval contains at least one value.

Let $Y, Z \subset X$ be such that

$$\begin{aligned} Y &\equiv \left\{x \in X : \exists v_i \in D_i,\ \ d(v_i, v_{-i}) = x \ \text{ and } \ t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) < -2\varepsilon\right\}, \\ Z &\equiv \left\{x \in X : \exists v_i \in D_i,\ \ d(v_i, v_{-i}) = x \ \text{ and } \ -\varepsilon < t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i})\right\}. \end{aligned}$$

By Step 1, Y and Z are disjoint, and $(Y \cup Z) \cap E(\sum_{j=1}^n v_j) \neq \emptyset$ for all $v_i \in D_i$. Then, we can define A_i, B_i, S_i and W_i as in equations (1.4) to (1.7) in Appendix A. Let us show that condition (ii) of statement in Property 5 holds, and hence Property 5 is violated. Note that for each $v_i \in D_i$, $d(v_i, v_{-i}) \in Y$ or $d(v_i, v_{-i}) \in Z$.

Substep 2-2-1. Let us show that for each $v_i \in D_i$, if $d(v_i, v_{-i}) \in Y$, then $v_i \in S_i$. Take any $v_i \in D_i$ with $d(v_i, v_{-i}) \in Y$. Consider any $z \in Z$. We shall show that $\sum_{j=1}^n v_j\big(d(v)\big) > \sum_{j=1}^n v_j(z) + \varepsilon$. By $d(v_i, v_{-i}) \in Y$, there exists $v_i' \in D_i$ such that $d(v_i, v_{-i}) = d(v_i', v_{-i})$ and $t_i(v_i', v_{-i}) - t_i^*(v_i', v_{-i}) < -2\varepsilon$. Hence by Step 1,

$$t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) < -2\varepsilon. \tag{1.26}$$

Since $z \in Z$, there exists $v_i'' \in D_i$ such that $d(v_i'', v_{-i}) = z$ and

$$-\varepsilon < t_i(v_i'', v_{-i}) - t_i^*(v_i'', v_{-i}).$$

Therefore, by combining these two inequalities,

$$t_i^*(v_i'', v_{-i}) - t_i^*(v_i, v_{-i}) + \varepsilon < t_i(v_i'', v_{-i}) - t_i(v_i, v_{-i}). \tag{1.27}$$

On the other hand, by strategy-proofness of (d, t),

$$v_i\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}) \geq v_i\big(d(v_i'', v_{-i})\big) + t_i(v_i'', v_{-i}),$$

or

$$v_i\big(d(v_i, v_{-i})\big) - v_i\big(d(v_i'', v_{-i})\big) \geq t_i(v_i'', v_{-i}) - t_i(v_i, v_{-i}). \tag{1.28}$$

Then, by equations (1.27) and (1.28),

$$v_i\big(d(v_i, v_{-i})\big) - v_i\big(d(v_i'', v_{-i})\big) > t_i^*(v_i'', v_{-i}) - t_i^*(v_i, v_{-i}) + \varepsilon,$$

and hence

$$v_i\big(d(v_i, v_{-i})\big) + t_i^*(v_i, v_{-i}) > v_i\big(d(v_i'', v_{-i})\big) + t_i^*(v_i'', v_{-i}) + \varepsilon.$$

Therefore, $\sum_{j=1}^n v_j\big(d(v)\big) > \sum_{j=1}^n v_j(z) + \varepsilon$ by definition of t_i^* and $d(v_i'', v_{-i}) = z$. Thus, $v_i \in S_i$.

Substep 2-2-2. Let us show that for each $v_i \in D_i$, if $d(v_i, v_{-i}) \in Y$, then $v_i \in B_i$. Take any $v_i \in D_i$ with $d(v_i, v_{-i}) \in Y$. Since $d(v_i, v_{-i}) \in Y$, by the same argument as equation (1.26),

$$t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) < -2\varepsilon.$$

Then, by welfare lower boundedness of (d, t),

$$\begin{aligned}\max_{x \in X} \sum_{j=1}^n v_j(x) - \max_{x \in X} \sum_{j \neq i} v_j(x) &= v_i\big(d(v_i, v_{-i})\big) + t_i^*(v_i, v_{-i}) \\ &> v_i\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}) + 2\varepsilon > \min_{x \in X} v_i(x) + \varepsilon.\end{aligned}$$

Therefore, $v_i \in B_i$.

Substep 2-2-3. Let us show that for each $v_i \in D_i$, if $d(v_i, v_{-i}) \in Z$, then $v_i \in W_i$. Take any $v_i \in D_i$ with $d(v_i, v_{-i}) \in Z$. Since (d, t) is efficient, for each $y \in Y$,

$$\sum_{j=1}^n v_j\big(d(v)\big) \geq \sum_{j=1}^n v_j(y).$$

Therefore, $v_i \in W_i$.

Then, by Substeps 2-2-1 to 2-2-3, $D_i \subset (S_i \cap B_i) \cup W_i$. Moreover, by constructions of Y and Z, $(S_i \cap B_i) \cap D_i \neq \emptyset \neq W_i \cap D_i$. Thus, a domain $\prod_{j=1}^n D_j$ violates Property 5, a contradiction to the assumption that $\prod_{j=1}^n D_j$ satisfies Property 5.

Step 3. Let us show that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} D_j$,

$$t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) = t_i(v_i', v_{-i}) - t_i^*(v_i', v_{-i}) \text{ for all } v_i, v_i' \in D_i.$$

Take any $i \in N$ and any $v_{-i} \in \prod_{j \neq i} D_j$. Suppose, by contradiction, that

$$t_i(u_i, v_{-i}) - t_i^*(u_i, v_{-i}) \neq t_i(u_i', v_{-i}) - t_i^*(u_i', v_{-i}) \text{ for some } u_i, u_i' \in D_i.$$

Then, by Step 1 and finiteness of X, $t_i(\,\cdot\,, v_{-i}) - t_i^*(\,\cdot\,, v_{-i})$ takes at least two and only a finite number of values. Moreover, by Step 2, $t_i^*(v_i, v_{-i}) \leq t_i(v_i, v_{-i})$ for all $v_i \in D_i$. Therefore, there exists $\varepsilon > 0$ such that the values of $t_i(\,\cdot\,, v_{-i}) - t_i^*(\,\cdot\,, v_{-i})$ belong to the union of two intervals $(-\infty, \varepsilon)$ and $(2\varepsilon, \infty)$, and each interval contains at least one value.

Let $Y, Z \subset X$ be such that

$$Y \equiv \left\{x \in X : \exists v_i \in D_i,\ \ d(v_i, v_{-i}) = x \ \text{ and } \ t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) < \varepsilon\right\},$$
$$Z \equiv \left\{x \in X : \exists v_i \in D_i,\ \ d(v_i, v_{-i}) = x \ \text{ and } \ 2\varepsilon < t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i})\right\}.$$

By Step 1, Y and Z are disjoint, and $(Y \cup Z) \cap E(\sum_{j=1}^n v_j) \neq \emptyset$ for all $v_i \in D_i$. Let us show that condition (i) of statement in Property 5 holds. Note that for each $v_i \in D_i$, $d(v_i, v_{-i}) \in Y$ or $d(v_i, v_{-i}) \in Z$.

Substep 3-1. Let us show that for each $v_i \in D_i$, if $d(v_i, v_{-i}) \in Y$, then $v_i \in S_i$. Take any $v_i \in D_i$ with $d(v_i, v_{-i}) \in Y$ and any $z \in Z$. It suffices to show that $\sum_{j=1}^n v_j\big(d(v)\big) > \sum_{j=1}^n v_j(z) + \varepsilon$. Since $d(v_i, v_{-i}) \in Y$, by Step 1 and definition of Y, we have

$$t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}) < \varepsilon.$$

Moreover, by $z \in Z$, there exists $v_i' \in D_i$ such that $d(v_i', v_{-i}) = z$ and

$$2\varepsilon < t_i(v_i', v_{-i}) - t_i^*(v_i', v_{-i}).$$

Therefore, by combining these two inequalities, we have

$$t_i^*(v_i', v_{-i}) - t_i^*(v_i, v_{-i}) + \varepsilon < t_i(v_i', v_{-i}) - t_i(v_i, v_{-i}). \tag{1.29}$$

On the other hand, by strategy-proofness of (d, t),

$$v_i\big(d(v_i, v_{-i})\big) + t_i(v_i, v_{-i}) \geq v_i\big(d(v_i', v_{-i})\big) + t_i(v_i', v_{-i}),$$

or

$$v_i\big(d(v_i, v_{-i})\big) - v_i\big(d(v_i', v_{-i})\big) \geq t_i(v_i', v_{-i}) - t_i(v_i, v_{-i}). \tag{1.30}$$

Then, by equations (1.29) and (1.30),

$$v_i\big(d(v_i, v_{-i})\big) - v_i\big(d(v_i', v_{-i})\big) > t_i^*(v_i', v_{-i}) - t_i^*(v_i, v_{-i}) + \varepsilon,$$

and hence

$$v_i\big(d(v_i, v_{-i})\big) + t_i^*(v_i, v_{-i}) > v_i\big(d(v_i', v_{-i})\big) + t_i^*(v_i', v_{-i}) + \varepsilon.$$

Therefore, $\sum_{j=1}^n v_j\big(d(v)\big) > \sum_{j=1}^n v_j(z) + \varepsilon$ by definition of t_i^* and $d(v_i', v_{-i}) = z$. Thus, $v_i \in S_i$.

Substep 3-2. Let us show that for each $v_i \in D_i$, if $d(v_i, v_{-i}) \in Z$, then $v_i \in W_i$. Take any $v_i \in D_i$ with $d(v_i, v_{-i}) \in Z$. Since the decision function d chooses an efficient alternative, for each $y \in Y$,

$$\sum_{j=1}^n v_j\big(d(v)\big) \geq \sum_{j=1}^n v_j(y).$$

Therefore, $v_i \in W_i$.

Substep 3-3. Let us show that for each $v_i \in D_i$, if $d(v_i, v_{-i}) \in Z$, then $v_i \in A_i$. Take any $v_i \in D_i$ with $d(v_i, v_{-i}) \in Z$. Since $d(v_i, v_{-i}) \in Z$, by Step 1 and definition of Z,

$$2\varepsilon < t_i(v_i, v_{-i}) - t_i^*(v_i, v_{-i}).$$

Remember that by Step 2, $t_j^*(v_i, v_{-i}) \leq t_j(v_i, v_{-i})$ for all $j \neq i$. Then, by feasibility of (d, t),

$$\begin{aligned}(n-1)\max_{x\in X}\sum_{j=1}^n v_j(x) - \sum_{j=1}^n \max_{x\in X}\sum_{k\neq j} v_k(x) &= t_i^*(v_i, v_{-i}) + \sum_{j\neq i} t_j^*(v_i, v_{-i})\\ &< t_i(v_i, v_{-i}) - 2\varepsilon + \sum_{j\neq i} t_j^*(v_i, v_{-i})\\ &\leq t_i(v_i, v_{-i}) - 2\varepsilon + \sum_{j\neq i} t_j(v_i, v_{-i}) \leq -2\varepsilon < -\varepsilon.\end{aligned}$$

Therefore, $v_i \in A_i$.

Then, by Substeps 3-1 to 3-3, $D_i \subset S_i \cup (W_i \cap A_i)$. Moreover, by constructions of Y and Z, $S_i \cap D_i \neq \emptyset \neq (W_i \cap A_i) \cap D_i$. Thus, a domain $\prod_{j=1}^n D_j$ violates Property 5, a contradiction to the assumption that $\prod_{j=1}^n D_j$ satisfies Property 5.

□

Chapter 2

Dispute Mediation Mechanisms

2.1 Introduction

We consider the problem of designing mechanisms that mediate disputes, such as border disputes, commercial disputes, or civil disputes. Since disputes arise from differences in disputants' preferred social states, finding a compromise that benefits for all disputant is important when we think of the resolution of the dispute. However, even if such a compromise exists, individuals often misrepresent their preferences, and proposals based on the misrepresented preferences may fail. We interpret this as an incentive problem for mechanisms that aggregate disputants' preferences.

A specific feature of the problem we examine is that each disputant may have a veto power to the outcomes of mechanisms. This feature is embedded in real-world disputes in which each disputant can reject an agreement by continuing the dispute. In fact, in many dispute resolution processes outside courts (e.g., the *mediation*in alternative dispute resolution processes), disputants have the right to reject an agreement (Muthoo 1999). Since we allow the veto power, we search for mechanisms that satisfy *individual rationality* so that each disputant voluntarily accepts the outcomes of mechanisms.

We consider an environment in which all disputants have quasi-linear preferences over finitely many social states and money. The set of social states includes the status quo in which the unresolved dispute continues. Each mechanism chooses a social state and determines monetary transfers among disputants. Since the seminal work by Groves (1973), it has been well-known that Groves mechanisms are the only mechanisms that satisfy *efficiency* and *strategy-proofness* (Green and Laffont 1977, Holmström, 1979, Carbajal, 2010). However, it is also known that no Groves mechanism satisfies both *individual rationality* and *feasibility* (Subsection 3.9; Jackson 2003). In other words, there exists no mechanism that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility*. Given the negative result about the existence of "desirable" mechanisms, we need to give up at least one of the four properties. Since *strategy-proofness* and *individual rationality* are central to our analysis, we relax *efficiency*. In particular, we introduce *weak efficiency*, which requires that a mechanism choose an efficient social state whenever it chooses a social state other than the status quo.

First, we consider the full valuation domain. On this domain, *the status quo mechanism* is the unique mechanism that satisfies *weak efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility* (Theorem 1). The *status quo mechanism* is a mechanism that always chooses the status quo without monetary transfers among disputants. This result is negative because the unresolved dispute always persists under the status quo mechanism.

Second, given the negative result, we consider a situation where the planner (or mediator) can restrict the valuation domain by estimating an upper bound of each disputant's valuations. For example, if the mediator knows that each disputant's valuation of obtaining some disputed item is less than $10,000, then he can restrict a valuation domain by using such an information. This admits us to focus on a mildly restricted domain in which each disputant's valuations are bounded from above. We show that on this mildly restricted domain, the *mediation mechanisms* satisfy *weak efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility* (Theorem 2 (i)). A *mediation mechanism* chooses an efficient alternative whenever the maximal social surplus among disputants exceeds an endogenous amount, and otherwise it chooses the status quo. We also show that *mediation mechanisms* Pareto-dominate all mechanisms satisfying *weak efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility* (Theorem 2 (ii)). These results imply that the mediation mechanisms exhibit fairly good performances in our dispute resolution problem whenever the valuation domain can be restricted by finding upper bounds of disputants' valuations.[*5] We also observe that performances of the mediation mechanisms increases as the mediator selects the lower upper bound of each disputant's valuations when he restrict the valuation domain.

Our theorems parallel those of Roberts (1979), Mishra and Sen (2012), and Carbajal, McLennan, and Tourky (2013). Among them, Roberts (1979) shows that any strategy-proof mechanism that may not be efficient belongs to a class of mechanisms that are called *affine maximizers*. Mediation mechanisms are in fact a subclass of affine maximizers. Subsequently, Mishra and Sen (2012) and Carbajal, McLennan, and Tourky (2013) generalize Roberts' theorem to some of restricted valuation domains. In the proof of Theorem 1, we can apply Roberts' theorem because there we consider the full valuation domain. On the other hand, in the proof of Theorem 2, none of their results can be directly applicable since our restricted

[*5] Although our model covers a wide range of public decision problems, the mediation mechanisms seem to mostly applicable to dispute resolution problems. This is because the mediation mechanisms exhibit relatively good performances when the number of individuals involved in a problem is only a few, consistently with the situations of the most of dispute resolution problems. We discuss this in Subsection 2.4.1. For the literature on mechanism design specifically focusing on dispute resolution problems, see for example Fey and Ramsay (2007; 2009), Hörner, Morelli, and Squintani (2015), and Miljkovic and Gomez (2013).

domains are outside the valuation domains they analyze. Therefore, we present a proof that directly characterizes *weak efficient* and *strategy-proof* mechanisms.

The rest of this chapter is organized as follows. Section 2.2 introduces our model. Section 2.3 offers our main results. Section 2.4 discusses performances of mediation mechanisms, and Section 2.5 gives concluding remarks. All the proofs are relegated to Appendix.

2.2 The Model

Let $N = \{1, 2, \ldots, n\}$ be the finite set of *disputants*, and $X = \{\phi, x_1, \ldots, x_m\}$ the finite set of *social states*, where ϕ denotes the *status quo* in which the unresolved dispute continues. Each individual $i \in N$ has a *valuation function* $v_i : X \to \mathbb{R}$. Let V be the set of valuation functions. For future convenience, we normalize each v_i as $v_i(\phi) = 0$.[*6] A *valuation profile* is an n-tuple of valuation functions $v = (v_1, \ldots, v_n) \in V^n$. For each $v \in V^n$ and $N' \subset N$, $v_{N'}$ and $v_{-N'}$ denote $\{v_j\}_{j \in N'}$ and $\{v_j\}_{j \in N \setminus N'}$, respectively. Given any $v_i \in V$, its associated utility function $U(\cdot, \cdot; v_i) : X \times \mathbb{R} \to \mathbb{R}$ is defined by

$$U(x, t_i;\, v_i) = v_i(x) + t_i,$$

where $t_i \in \mathbb{R}$ denotes the amount of money disputant i receives.

We consider two types of valuation domains. In Subsection 2.3.1, we consider the full valuation domain V^n. On the other hand, in Subsection 2.3.2, we consider a situation where the mediator can estimate an upper bound of each disputant's valuations. In other words, we focus on a mildly restricted domain in which each disputants' valuations are bounded from above. For each positive number $r_i \in \mathbb{R}_{++}$, let

$$V_{r_i} \equiv \{v_i \in V : v_i(x) < r_i \ \text{ for all } x \in X\}.$$

The subset $V_{r_i} \subset V$ is the set of valuation functions that are bounded from above by a positive number $r_i \in \mathbb{R}_{++}$. Then, a restricted domain we focus on in Subsection 2.3.2 is of the form $\prod_{j=1}^{n} V_{r_j}$.[*7] Throughout this section and Section 2.3, we fix an arbitrary n-dimensional vector $\boldsymbol{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n_{++}$. We interpret this fixed $\boldsymbol{r}$ as the mediator's estimations of upper

[*6] All of our results hold without this assumption.

[*7] Since each V_{r_i} is bounded from above, our valuation domain $\prod_{j=1}^{n} V_{r_j}$ is outside the domains analyzed by Carbajal, McLennan, and Tourky (2013). In addition, since we normalize each $v_i(\phi)$ as zero, each V_{r_i} is not a *finite dimensional open interval domain*. Therefore, our valuation domain $\prod_{j=1}^{n} V_{r_j}$ is outside the domains analyzed by Mishra and Sen (2012).

bounds of disputants' valuations. When the valuation domain is not specified, we generally denote it by $\prod_{j=1}^{n} D_j \subset \Big\{ \prod_{j=1}^{n} V_{r_j},\ V^n \Big\}$.

A *decision function* is a function $d : \prod_{j=1}^{n} D_j \to X$ that maps each valuation profile $v \in \prod_{j=1}^{n} D_j$ to a social state $d(v) \in X$. A *transfer function* is a function $t : \prod_{j=1}^{n} D_j \to \mathbb{R}^n$ that maps each valuation profile $v \in \prod_{j=1}^{n} D_j$ to a vector of monetary transfers $t(v) = (t_1(v), \ldots, t_n(v)) \in \mathbb{R}^n$. A *mechanism* is a pair of decision and transfer functions (d, t). Let $\mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ be the set of mechanisms on $\prod_{j=1}^{n} D_j$.

We introduce four axioms on mechanisms. *Efficiency* requires that a mechanism choose a social state that maximizes the disputants' social surplus.

Axiom 1 (Efficiency)**.** A mechanism $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ satisfies *efficiency* if for each $v \in \prod_{j=1}^{n} D_j$,

$$d(v) \in \operatorname*{arg\,max}_{x \in X \cup \{\phi\}} \Big\{ \sum_{j=1}^{n} v_j(x) \Big\}.$$

Strategy-proofness requires that revealing the true valuation function be a weakly-dominant strategy for each disputant.

Axiom 2 (Strategy-proofness)**.** A mechanism $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ satisfies *strategy-proofness* if for each $i \in N$, each $v \in \prod_{j=1}^{n} D_j$ and each $v_i' \in D_i$,

$$U\big(d(v_i, v_{-i}), t_i(v_i, v_{-i});\ v_i\big) \geq U\big(d(v_i', v_{-i}), t_i(v_i', v_{-i});\ v_i\big).$$

Individual rationality requires that no disputant's final utility become worse than that at the status quo.

Axiom 3 (Individual rationality)**.** A mechanism $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ satisfies *individual rationality* if for each $i \in N$ and each $v \in \prod_{j=1}^{n} D_j$,

$$U\big(d(v_i, v_{-i}), t_i(v_i, v_{-i});\ v_i\big) \geq U\big(\phi, 0;\ v_i\big).$$

Feasibility requires that net transfer be non-positive. Note that money may flaw out from disputants.[*8]

Axiom 4 (Feasibility)**.** A mechanism $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ satisfies *feasibility* if for each $v \in \prod_{j=1}^{n} D_j$,

$$\sum_{j=1}^{n} t_j(v) \leq 0$$

[*8] In our setting, no strategy-proof mechanism can be *budget balanced* even in the restricted domains.

Although these four axioms seem to be "desirable", no mechanism satisfies all of them. Therefore, we must relax at least one of the four axioms.

Impossibility Result (Green and Laffont 1977; Holmström 1979). *For each* $\prod_{j=1}^{n} D_j \subset \left\{\prod_{j=1}^{n} V_{r_j},\ V^n\right\}$*, no mechanism* $(d,t) \in \mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ *satisfies efficiency, strategy-proofness, individual rationality, and feasibility.*

In Appendix A, we show that the above impossibility result holds true even on the restricted domain $\prod_{j=1}^{n} V_{r_j}$.

2.3 Main Results

Given the impossibility result, we relax efficiency and design mechanisms that satisfy a weaker version of efficiency and the other three axioms. *Weak efficiency* requires that a mechanism choose an efficient outcome whenever it chooses a social state other than the status quo. Sakai (2013) established a similar axiom to characterize second price auctions with reserve prices. Obviously, weak efficiency is implied by efficiency.

Axiom 5 (Weak Efficiency)**.** A mechanism $(d,t) \in \mathscr{M}\big(\prod_{j=1}^{n} D_j\big)$ satisfies *weak efficiency* if for each $v \in \prod_{j=1}^{n} D_j$,

$$d(v) \in \arg\max_{x \in X} \left\{ \sum_{j=1}^{n} v_j(x) \right\},$$

whenever $d(v) \neq \phi$.

If a mechanism satisfies weak efficiency and chooses a non-status quo social state, then the grand coalition I does not has an incentive to change the social state after the decision by the mechanism; that is, for each $v \in \prod_{j=1}^{n} D_j$, if $d(v) \neq \phi$, then there exists no outcome $(x, p_1, \ldots, p_n) \in X \times \mathbb{R}^n$ such that $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} t_j(v)$ and

$$U\big(x, p_i;\ v_i\big) \geq U\big(d(v), t_i(v);\ v_i\big) \text{ for all } i \in I,$$

strict inequality holding for some $j \in I$.

2.3.1 Full valuation domain

On the full valuation domain, a mechanism that always chooses the status quo uniquely satisfies weak efficiency, strategy-proofness, individual rationality, and feasibility. A mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ is the *status quo mechanism* if for each $v \in \prod_{j=1}^{n} D_j$ and each $i \in N$, $d(v) = \emptyset$ and $t_i(v) = 0$.

Theorem 1. *Suppose $|X| \geq 3$ and $\prod_{j=1}^{n} D_j = V^n$. Then, the status quo mechanism is the unique mechanism that satisfies weak efficiency, strategy-proofness, individual rationality, and feasibility.*

Proof. See Appendix C. ∎

Theorem 1 implies that when the valuation domain cannot be restricted, no mechanism satisfies the four axioms resolving the dispute, i.e., choosing a non-status quo social state. When there are only two social states, i.e., $|X| = 2$, a mechanism conducting the unanimous voting without monetary transfers satisfies all the four axioms.

2.3.2 Restricted valuation domain

In this subsection, we focus on the restricted valuation domain $\prod_{j=1}^{n} D_j = \prod_{j=1}^{n} V_{r_j}$, where upper bounds of disputants' valuations are estimated by the mediator. We now introduce a class of mechanisms important to this study. Let $\mathbf{1}_{\phi} : X \to \{0, 1\}$ be an indicator function of ϕ. That is, for each $x \in X$, $\mathbf{1}_{\phi}(x) = 1$ if and only if $x = \phi$.

Definition 1. A mechanism $(d^m, t^m) \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ is a *mediation mechanism* if for each $v \in \prod_{i=1}^{n} V_{r_j}$ and each $i \in N$,

(i) $d^m(v) \in \arg\max_{x \in X} \Big\{ \sum_{j=1}^{n} v_j(x) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(x) \Big\}$,

(ii) $t_i^m(v) = \sum_{j \neq i} v_j(d(v)) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(d(v)) - \frac{(n-1)}{n} \sum_{j=1}^{n} r_j$,

(iii) $\Big[\exists x \in X \setminus \{\phi\},\ x \in \arg\max_{x \in X} \Big\{ \sum_{j=1}^{n} v_j(x) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(x) \Big\} \Big] \Longrightarrow d(v) \neq \phi.$

Remember that n is the number of disputants. Any mediation mechanism chooses a non-status quo social state that maximize the social surplus if the maximal social surplus is greater than $\frac{(n-1)}{n} \sum_{j=1}^{n} r_j$. Otherwise, the mediation mechanism chooses the status quo. To break ties, the mediation mechanism always chooses a social state other than the status quo. All mediation mechanisms are welfare equivalent. Therefore, in welfare terms, we can consider that there is essentially only one mediation mechanism. This class of mechanisms belongs to the class of *affine maximizers* (Roberts 1979), but not to the class of Vickrey-Clarke-Groves mechanisms.

We give a numerical example of mediation mechanisms.

	x	y	z	ϕ
v_1	-40	20	25	0
v_2	15	10	-40	0
$v_1 + v_2$	-25	35	-10	0

Table 1: Valuation profile of two individuals

Example 1. Consider a situation in which there are two disputants and four social states including the status quo. Each disputant $i \in \{1, 2\}$ has a valuation function v_i as illustrated in Table 1. There, disputant 1's valuation of x is -40, for example. Then, the social surplus of $v_1 + v_2$ is maximized at y, and hence, it is the efficient social state. Now, suppose that $\boldsymbol{r} = (30, 20) \in \mathbb{R}^2_{++}$. Then, $v \in V_{30} \times V_{20}$. In addition, since $v_1(y) + v_2(y) = 35 > \frac{1}{2}\big(r_1 + r_2\big) = \frac{1}{2}\big(30 + 20\big) = 25$, a mediation mechanism assigns y to this valuation profile. Disputant 1 receives $v_2(y) - \frac{1}{2}\big(r_1 + r_2\big) = -15$ and Disputant 2 receives $v_1(y) - \frac{1}{2}\big(r_1 + r_2\big) = -5$. Overall, disputant 1's final utility is $20 - 15 = 5$ and that of disputant 2 is $10 - 5 = 5$. Therefore, both disputants end up with the higher utilities than the status quo.

We say that a mechanism $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ *Pareto-dominates* another mechanism $(d', t') \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ if for each $v \in \prod_{j=1}^{n} V_{r_j}$ and each $i \in N$,

$$U\big(d(v), t_i(v);\ v_i\big) \geq U\big(d'(v), t'_i(v);\ v_i\big).$$

Theorem 2 is our main result and gives rationalizes on the desirability of the mediation mechanisms.

Theorem 2. *Suppose $|X| \geq 3$ and $\prod_{j=1}^{n} D_j = \prod_{j=1}^{n} V_{r_j}$. Then, for each mediation mechanism $(d^m, t^m) \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$,*

(i) (d^m, t^m) satisfies weak efficiency, strategy-proofness, individual rationality, and feasibility,

(ii) (d^m, t^m) Pareto dominates all mechanisms $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ satisfying weak efficiency, strategy-proofness, individual rationality, and feasibility.

Proof. See Appendix D. ■

In Appendix E, we give an example that shows the existence of a mechanism that is not a mediation mechanism satisfying all four axioms.

Corollary 1 states that the mediation mechanisms have an additional property of maximizing the opportunity for dispute resolution; that is, they choose an non-status quo social state more frequently than any other mechanisms satisfying the four axioms.

Corollary 1. *Suppose* $|X| \geq 3$ *and* $\prod_{j=1}^{n} D_j = \prod_{j=1}^{n} V_{r_j}$. *For any mediation mechanism* $(d^m, t^m) \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ *and any mechanism* $(d, t) \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ *that is not a mediation mechanism, if* (d,t) *satisfies weak efficiency, strategy-proofness, individual rationality and feasibility, then*

(i) $d(v) \neq \phi \Longrightarrow d^m(v) \neq \phi$ *for all* $v \in \prod_{i=1}^{n} V_{r_i}$,

(ii) there exists $v \in \prod_{i=1}^{n} V_{r_i}$ *such that* $d(v) = \phi$ *and* $d^m(v) \neq \phi$.

Proof. Immediately follows from Lemma 7 and Step 3 of Proof of Theorem 2(ii) in Appendix D (pp. 72 and pp. 74). ■

2.4 Discussions on Mediation Mechanisms

2.4.1 Other Properties

We explain two other properties of mediation mechanisms. First, under a mediation mechanism, disputants' final utilities are equivalent. That is, for each $i, j \in N$ and each $v \in \prod_{j=1} V_{r_j}$,

$$U\big(d^m(v), t_i^m(v);\ v_i\big) = U\big(d^m(v), t_j^m(v);\ v_j\big).$$

This is because for each $i, j \in N$ and each $v \in \prod_{j=1} V_{r_j}$, by definition of mediation mechanism,

$$\begin{aligned} U\big(d^m(v), t_i^m(v);\ v_i\big) &= \max_{x \in X} \Big\{ \sum_{j=1}^{n} v_j(x) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(x) \Big\} - \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \\ &= U\big(d^m(v), t_j^m(v);\ v_j\big). \end{aligned}$$

Therefore, the mediation mechanisms are fair in a sense that all disputants have identical final utilities.[*9]

Second, the mediation mechanisms exhibit relatively good performances when only a few disputants are involved. To see this, we compare two cases where $n = 2$ and $n = 10$, for example. Recall that for each $v \in \prod_{j=1}^{n} V_{r_j}$,

$$d^m(v) \in \arg\max_{x \in X} \Big\{ \sum_{j=1}^{n} v_j(x) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(x) \Big\}.$$

[*9] For instance, the mediation mechanisms satisfy *equal treatment of equals*.

Then, a mediation mechanism chooses an efficient social state if and only if

$$\frac{(n-1)}{n}\sum_{j=1}^{n} r_j \leq \max_{x \in X} \sum_{j=1}^{n} v_j(x).$$

In addition, for each $v \in \prod_{j=1}^{n} V_{r_j}$, by definitions of $V_{r_1}, \ldots, V_{r_n}$,

$$\max_{x \in X} \sum_{j=1}^{n} v_j(x) < \sum_{j=1}^{n} r_j.$$

Therefore, when $n = 2$, a mediation mechanism chooses an efficient social state if and only if

$$\frac{1}{2}\sum_{j=1}^{n} r_j \leq \max_{x \in X} \sum_{j=1}^{n} v_j(x) < \sum_{j=1}^{n} r_j.$$

On the other hand, when $n = 10$, a mediation mechanism chooses an efficient social state if and only if

$$\frac{9}{10}\sum_{j=1}^{n} r_j \leq \max_{x \in X} \sum_{j=1}^{n} v_j(x) < \sum_{j=1}^{n} r_j.$$

Hence the mediation mechanisms seem to more often choose an efficient social state when $n = 2$ rather than when $n = 10$.[*10]

2.4.2 Importance of selecting upper valuation bounds $\boldsymbol{r} \in \mathbb{R}^n_{++}$

To apply the mediation mechanisms realistically, we must select upper bounds $\boldsymbol{r} \in \mathbb{R}^n_{++}$ of disputants' valuations. Note that for each upper bound $\boldsymbol{r} \in \mathbb{R}^n$, we can define the mediation mechanisms with upper bounds $\boldsymbol{r} \in \mathbb{R}^n_{++}$. However, for a given valuation profile, selection of $\boldsymbol{r}$ directly affects the outcome. Consider the same situation as Example 1 in Subsection 2.3.2. First, consider a mediation mechanism with upper bounds $\boldsymbol{r} = (30, 20) \in \mathbb{R}^2_{++}$. Then, by the same argument as that in Example 1, this mediation mechanism chooses a social state y. Next, consider a mediation mechanism with upper bounds $\boldsymbol{r} = (40, 40) \in \mathbb{R}^2_{++}$. Note that $v \in V_{40} \times V_{40}$ still holds. Then, since $v_1(y) + v_2(y) = 35 < \frac{1}{2}(r_1 + r_2) = \frac{1}{2}(40 + 40) = 40$, this mediation mechanism chooses the status quo ϕ. Remembering that y is the efficient outcome, the former mechanism yields a more efficient outcome. More generally, a mediation mechanism with lower $\boldsymbol{r} \in \mathbb{R}^n_{++}$ yields a more efficient outcome for a given valuation profile.[*11]

[*10] For a precise discussion, we need to define a measure on $\prod_{j=1}^{n} V_{r_j}$.

[*11] Note that for a given valuation profile $v \in V^n$, mechanism designers need to select upper bounds $\boldsymbol{r} \in \mathbb{R}^n_{++}$ so as to $v \in \prod_{j=1}^{n} V_{r_j}$. Otherwise, the analysis in Subsection 2.3.2 cannot be applied, and the mediation mechanisms may violate feasibility.

This argument shows the importance of the selection of the valuation upper bounds $\boldsymbol{r} \in \mathbb{R}^n_{++}$ in the real-use situation of our mechanisms.

2.5 Conclusion

We have considered the problem of designing mechanisms to mediate disputes. First, we showed that on the full valuation domain, the status quo mechanism uniquely satisfies weak efficiency, strategy-proofness, individual rationality and feasibility. Second, we showed that on a mildly restricted valuation domain, the mediation mechanisms satisfy all four axioms and Pareto-dominate all mechanisms satisfying the set of axioms. These results theoretically justify the use of the mediation mechanisms when the mediator can restrict the valuation domain by finding upper bounds of disputants' maximal valuations. Finding alternate way to relax efficiency is left to the future research.

Appendix A: Proof of the impossibility result

Proof. Let us consider the case with $\prod_{j=1}^n D_j = \prod_{j=1}^n V_{r_j}$. Suppose, by contradiction, that there exists a mechanism $(d,t) \in \mathscr{M}(\prod_{j=1}^n V_{r_j})$ that satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility*. Since $\prod_{j=1}^n V_{r_j}$ is a convex domain, by Holmström (1979; Theorem 1), for each $i \in N$, there exists $h_i : \prod_{j \neq i} V_{r_j} \to \mathbb{R}$ such that for each $v \in \prod_{j=1}^n V_{r_j}$,

$$d(v) \in \arg\max_{y \in X} \sum_{j=1}^n v_j(y), \tag{2.31}$$

and

$$t_i(v) = \sum_{i \neq j} v_j(d(v)) + h_i(v_{-i}). \tag{2.32}$$

Step 1. Let us show that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} V_{r_j}$, $h_i\big(v_{-i}\big) \geq 0$. Take any $i \in N$ and any $v_{-i} \in \prod_{j \neq i} V_{r_j}$. Suppose, by contradiction, that $h_i\big(v_{-i}\big) < 0$. Let $v_i \in V_{r_i}$ be such that

$$v_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ \min\left\{r_i, h_i(v_{-i}) - \sum_{j \neq i} v_j(y)\right\} - 1 & \text{if } y \in X \setminus \{\phi\}. \end{cases}$$

Then, for each $x \in X \setminus \{\phi\}$,

$$\sum_{j=1}^n v_j(x) \leq h_i(v_{-i}) - 1 < h_i(v_{-i}) < 0.$$

Hence $d(v) = \phi$. However,

$$v_i\big(d(v)\big) + t_i(v) = \sum_{j=1}^{n} v_j(\phi) + h_i(v_{-i}) = h_i(v_{-i}) < 0.$$

This is a contradiction to individual rationality.

Step 2 (Derive a contradiction). Let us show that there exist $v \in \prod_{j=1}^{n} V_{r_j}$ such that $\sum_{j=1}^{n} t_j(v) > 0$. Take some $x \in X \setminus \{\phi\}$. For each $i \in N$, let $v_i \in V_{r_i}$ be such that

$$v_i(y) = \begin{cases} r_i - \frac{1}{n}\big(\frac{n-1}{n}\sum_{j=1} r_j\big) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{j=1}^{n} v_j(x) = \sum_{j=1}^{n}\Big(r_j - \frac{1}{n}\big(\frac{n-1}{n}\sum_{k=1} r_k\big)\Big) = \sum_{j=1}^{n} r_j - \frac{n-1}{n}\sum_{j=1} r_j = \frac{1}{n}\sum_{j=1}^{n} r_j > 0. \tag{2.33}$$

Therefore, $d(v) = x \neq \phi$. Then, by Step 1 and equation (2.33),

$$\begin{aligned} \sum_{j=1}^{n} t_j(v) = \sum_{j=1}^{n}\left(\sum_{k\neq j} v_k(x) + h_j(v_{-i})\right) &= (n-1)\sum_{j=1}^{n} v_j(x) + \sum_{j=1}^{n} h_j(v_{-j}) \\ &\geq (n-1)\sum_{j=1}^{n} v_j(x) = \frac{(n-1)}{n}\sum_{j=1}^{n} r_j > 0. \end{aligned}$$

This is a contradiction to feasibility. Therefore, no mechanism satisfies efficiency, strategy-proofness, individual rationality, and feasibility. ■

Appendix B: Preparation for Proofs of Theorems 1 and 2

In this Appendix, we prepare a lemma used in Proofs of Theorems 1 and 2.

Lemma 1. *Consider any strategy-proof mechanism* $(d, t) \in \mathscr{M}(\prod_{j=1}^{n} D_j)$ *such that there exists* $\alpha \in \mathbb{R}$ *for which*

$$d(v) \in \arg\max_{y\in X}\left\{\sum_{j=1}^{n} v_j(y) + \alpha\mathbf{1}_{\phi}(y)\right\} \text{ for all } v \in \prod_{j=1}^{n} D_j.$$

Then, for each $i \in N$*, there exists* $h_i : \prod_{j\neq i} V_{r_j} \to \mathbb{R}$ *such that*

$$t_i(v) = \sum_{j\neq i} v_j\big(d(v)\big) + \alpha\mathbf{1}_{\phi}\big(d(v)\big) + h_i\big(v_{-i}\big) \text{ for all } v \in \prod_{j=1}^{n} D_j.$$

Proof. It suffices to consider the case with $\prod_{j=1}^n D_j = \prod_{j=1}^n V_{r_j}$. The proof parallels to that of Holmström (1979; Theorem 1). For each $i \in N$, let $h_i : \prod_{j=1}^n V_{r_j} \to \mathbb{R}$ be such that for each $v \in \prod_{j=1}^n V_{r_j}$,

$$t_i(v) = \sum_{j \neq i} v_j\big(d(v)\big) + \alpha \mathbf{1}_\phi\big(d(v)\big) + h_i\big(v\big).$$

Take any $i \in N$, any $v \in \prod_{j=1}^n V_{r_j}$ and any $v_i' \in V_{r_i}$. Let us show that $h_i(v_i, v_{-i}) = h_i(v_i', v_{-i})$.

Step 1. Let us show that there exists a sequence of valuation functions $\{v_i(\,\cdot\,; s)\}_{s \in [0,1]} \subset V_{r_i}$ such that for each $x \in X$,

(i) $v_i(x; 0) = v_i(x)$,

(ii) $v_i(x; 1) = v_i'(x)$,

(iii) $\frac{\partial v_i(x;s)}{\partial s}$ exists for all $s \in [0,1]$,

and, moreover, for each $x \in \big\{y \in X : y = d\big(v_i(\,\cdot\,; t), v_{-i}\big) \ \exists t \in [0,1]\big\}$ and each $s \in [0,1]$,

(iv) $\left|\frac{\partial v_i(x;s)}{\partial s}\right| \leq C$ for some $0 < C < \infty$.

For each $x \in X$ and each $s \in [0,1]$, let

$$v_i(x; s) = (1 - s) \cdot v_i(x) + s \cdot v_i'(x).$$

Then, for each $x \in X$ and each $s \in [0,1]$, $v_i(x; s) \in V_{r_i}$. Moreover, $\{v_i(\cdot; s)\}_{s \in [0,1]}$ obviously satisfies (i), (ii) and (iii). In fact, $\frac{\partial v_i(x;s)}{\partial s} = v_i'(x) - v_i(x)$ for all $x \in X$ and all $s \in [0,1]$. Then, (iv) is followed from finiteness of X.

Step 2. For each $t \in [0,1]$, let

$$\bar{d}(t) = d\big(v_i(\,\cdot\,; t), v_{-i}\big),$$
$$\bar{h}_i(t) = h_i\big(v_i(\,\cdot\,; t), v_{-i}\big).$$

Then, by definition of d and strategy-proofness, it follows that for each $s \in [0,1]$,

$$s \in \arg\max_{t \in [0,1]} \sum_{j \neq i} v_j\big(\bar{d}(t)\big) + v_i\big(\bar{d}(t); s\big) + \alpha \mathbf{1}_\phi(\bar{d}(t)),$$
$$s \in \arg\max_{t \in [0,1]} \sum_{j \neq i} v_j\big(\bar{d}(t)\big) + v_i\big(\bar{d}(t); s\big) + \alpha \mathbf{1}_\phi(\bar{d}(t)) + \bar{h}_i(t).$$

Therefore, by the lemma of Holmström (1979; Appendix), $\bar{h}_i(0) = \bar{h}_i(1)$. Hence $h_i(v_i, v_{-i}) = h_i(v_i', v_{-i})$ using the definitions of $\bar{h}_i$ and $v_i(\,\cdot\,; s)$. ■

Appendix C: Proof of Theorem 1

Since "if" part is obvious, we show "only if" part. Take any mechanism $(d,t) \in \mathscr{M}(V^n)$ that satisfies *weak efficiency*, *strategy-proofness*, *individual rationality*, and *feasibility*. Since (d,t) satisfies *strategy-proofness*, by Carbajal, McLennan and Tourky (2013; Theorem 1), one of the following three cases occurs[*12]:

Case 1. there exist $\sigma \in \mathbb{R}^n_+$ with $\sigma_1 + \cdots + \sigma_n = 1$ and $q : X \to \mathbb{R}$ such that for each $v \in V^n$,

$$d(v) \in \arg\max_{y \in X} \sum_{j=1}^{n} \sigma_j v_j(y) + q(y),$$

Case 2. there exist distinct $x, y \in X$ with $d(V^n) = \{x, y\}$ such that for each $i \in N$ and each $v_{-i} \in V^{n-1}$, there exists $\delta_i(v_{-i}) \in \mathbb{R} \cup \{-\infty, +\infty\}$ for which for each $v_i \in V$,

(a) $v_i(x) - v_i(y) > \delta_i(v_{-i}) \Longrightarrow d(v) = x$,

(b) $v_i(x) - v_i(y) < \delta_i(v_{-i}) \Longrightarrow d(v) = y$,

Case 3. d is a constant function.

Let us show that cases 1 and 2 lead to contradictions.

Case 1. Suppose that the statement in Case 1 holds.

Step 1-1. Let us show that $q(x) = q(y)$ for all $x, y \in X \setminus \{\phi\}$. Suppose, by contradiction, that there exist $x, y \in X \setminus \{\phi\}$ such that $q(x) \neq q(y)$. Without loss of generality, suppose that $q(x) > q(y)$. For each $i \in N$, let $v_i \in V$ be such that

$$v_i(z) = \begin{cases} \frac{1}{n}\Big(q(\phi) - \frac{1}{2}\big(q(x) + q(y)\big)\Big) & \text{if } z = x, \\ \frac{1}{n}\big(q(\phi) - q(z)\big) & \text{otherwise.} \end{cases}$$

Then, by $\sigma_1 + \cdots + \sigma_n = 1$,

$$\sum_{j=1}^{n} \sigma_j v_j(x) + q(x) = q(\phi) + \frac{1}{2}\big(q(x) - q(y)\big),$$

[*12]Case 1 is followed from the fact that any lexicographic affine maximizer is also an affine maximizer.

and for each $z \in X \setminus \{x\}$,

$$\sum_{j=1}^{n} \sigma_j v_j(z) + q(z) = q(\phi).$$

Therefore, since $q(x) > q(y)$, $d(v) = x$. However,

$$\sum_{j=1}^{n} v_j(x) = q(\phi) - q(y) - \frac{1}{2}\big(q(x) - q(y)\big) < q(\phi) - q(y) = \sum_{j=1}^{n} v_j(y),$$

a contradiction to weak efficiency. Hence $q(x) = q(y)$ for all $x, y \in X \setminus \{\phi\}$.

Step 1-2. Let us show that $\sigma_i = \sigma_j$ for all $i, j \in N$. Suppose, by contradiction, that there exist $i, j \in N$ such that $\sigma_i \neq \sigma_j$. Without loss of generality, suppose that $\sigma_i > \sigma_j$. Take some $x, y \in X \setminus \{\phi\}$. Let $v_i, v_j \in V$ be such that

$$v_i(z) = \begin{cases} \sigma_j + 1 + \frac{1}{n}\big(q(\phi) - q(x)\big) & \text{if } z = x, \\ \frac{1}{n}\big(q(\phi) - q(z)\big) & \text{otherwise,} \end{cases}$$

$$v_j(z) = \begin{cases} \sigma_i + 1 + \frac{1}{n}\big(q(\phi) - q(y)\big) & \text{if } z = y, \\ \frac{1}{n}\big(q(\phi) - q(z)\big) & \text{otherwise.} \end{cases}$$

For each $k \in N \setminus \{i, j\}$, let $v_k \in V$ be such that $v_k(z) = \frac{1}{n}\big(q(\phi) - q(z)\big)$ for all $z \in X$. Then, by $\sigma_1 + \cdots + \sigma_n = 1$,

$$\sum_{j=1}^{n} \sigma_j v_j(x) + q(x) = \sigma_i\sigma_j + \sigma_i + q(\phi),$$

$$\sum_{j=1}^{n} \sigma_j v_j(y) + q(y) = \sigma_i\sigma_j + \sigma_j + q(\phi),$$

$$\sum_{j=1}^{n} \sigma_j v_j(z) + q(z) = q(\phi) \quad \text{for all } z \in X \setminus \{x, y\}.$$

Therefore, since $\sigma_i > \sigma_j$, $d(v) = x$. However, by Step 1 and $\sigma_i > \sigma_j$,

$$\sum_{j=1}^{n} v_j(x) = \sigma_j + 1 + q(\phi) - q(x) < \sigma_i + 1 + q(\phi) - q(y) = \sum_{j=1}^{n} v_j(y).$$

This is a contradiction to weak efficiency. Hence $\sigma_i = \sigma_j$ for all $i, j \in N$.

Step 1-3. By Steps 1-1 and 1-2, there exists $\alpha \in \mathbb{R}$ such that for each $v \in V^n$,

$$d(v) \in \arg\max_{y \in X} \sum_{j=1}^{n} v_j(y) + \alpha \cdot \mathbf{1}_{\phi}(y). \tag{2.34}$$

Then, by Lemma 1 in Appendix B, for each $v \in V^n$,

$$t_i(v) = \sum_{i \neq j} v_j(d(v)) + \alpha \mathbf{1}_{\phi}(d(v)) + h_i(d(v_{-i})). \tag{2.35}$$

Step 1-4. Let us show that for each $i \in N$ and each $v_{-i} \in V^{n-1}$,

$$h_i\big(v_{-i}\big) \geq -\alpha. \tag{2.36}$$

Take any $i \in N$ and any $v_{-i} \in V^{n-1}$. Suppose, by contradiction, that $h_i\big(v_{-i}\big) < -\alpha$. Let $v_i \in V$ be such that

$$v_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ \alpha - \sum_{j \neq i} v_j(y) - 1 & \text{if } y \in X \setminus \{\phi\}. \end{cases}$$

Then, for each $x \in X \setminus \{\phi\}$,

$$\sum_{j=1}^{n} v_j(x) = \alpha - 1 < \alpha.$$

Hence by Step 1-3, $d(v) = \phi$. However,

$$v_i\big(d(v)\big) + t_i(v) = \sum_{j=1}^{n} v_j(\phi) + \alpha + h_i(v_{-i}) = \alpha + h_i(v_{-i}) < 0.$$

This is a contradiction to individual rationality. Therefore, equation (2.36) holds.

Step 1-5. Let us show that there exists $v \in V^n$ such that $\sum_{j=1}^{n} t_j(v) > 0$. Take some $x \in X \setminus \{\phi\}$. For each $i \in N$, let $v_j \in V$ be such that

$$v_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ \frac{1}{n-1}\big(\alpha + 1\big) & \text{if } y = x, \\ \frac{\alpha}{n-1} & \text{otherwise.} \end{cases}$$

Then, by Step 1-4, $d(v) = x \neq \phi$. Therefore, by Steps 1-4 and 1-5,

$$\begin{aligned} \sum_{j=1}^{n} t_j(v) &= (n-1)\sum_{j=1}^{n} v_j(x) + \sum_{j=1}^{n} h_j(v_{-j}) \geq (n-1)\sum_{j=1}^{n} v_j(x) - n\alpha \\ &= (n-1)\sum_{j=1}^{n}\Big(\frac{1}{n-1}\big(\alpha+1\big)\Big) - n\alpha = n(\alpha+1) - n\alpha = n > 0. \end{aligned}$$

This is a contradiction to feasibility.

Case 2. Suppose that the statement in Case 1 holds. Note that $x \neq \phi$ or $y \neq \phi$. Without loss of generality, suppose that $x \neq \phi$. Take any $i \in N$. Then, since $d(V^n) = \{x, y\}$, there exists $v_{-i} \in V^{n-1}$ such that $\delta_i(v_{-i}) < \infty$. We consider three subcases.

Subcase 2-1. Let us consider the case with $\delta_i(v_{-i}) = -\infty$. Let $v_i \in V$ be such that

$$v_i(z) = \begin{cases} -\sum_{j\neq i} v_j(x) - 1 & \text{if } z = x, \\ 0 & \text{otherwise,} \end{cases}$$

Then,

$$v_i(x) - v_i(y) > -\infty = \delta(v_{-i}).$$

Therefore, $d(v) = x$. Moreover, by feasibility,

$$\sum_{j=1}^{n} \big(v_j(d(v)) + t_j(v)\big) = \sum_{j=1}^{n} \big(v_j(x) + t_j(v)\big) \leq -1 < 0.$$

Thus, there exists $j \in N$ such that

$$v_j(d(v)) + t_j(v) < 0,$$

a contradiction to individual rationality.

Subcase 2-2. Let us consider the case with $\delta_i(v_{-i}) \neq -\infty$ and $y \neq \phi$. Let $v_i \in V$ be such that

$$v_i(z) = \begin{cases} -\sum_{j\neq i} v_j(x) - 1 & \text{if } z = x, \\ -\sum_{j\neq i} v_j(x) - \delta_i(v_{-i}) - 2 & \text{if } z = y, \\ 0 & \text{otherwise,} \end{cases}$$

Then,

$$v_i(x) - v_i(y) = \delta_i(v_{-i}) + 1 > \delta_i(v_{-i}).$$

Therefore, $d(v) = x$. Moreover, by feasibility,

$$\sum_{j=1}^{n} \big(v_j(d(v)) + t_j(v)\big) = \sum_{j=1}^{n} \big(v_j(x) + t_j(v)\big) \leq -1 < 0.$$

Thus, similarly to Subcase 2-1, there exists $j \in N$ such that

$$v_j(d(v)) + t_j(v) = v_j(x) + t_j(v) < 0,$$

contradicting to individual rationality.

Subcase 2-3. Let us consider the case with $\delta_i(v_{-i}) \neq -\infty$ and $y = \phi$. Let $v_i \in V$ be such that

$$v_i(z) = \begin{cases} 0 & \text{if } z = \phi, \\ \delta_i(v_{-i}) + 1 & \text{if } z = x, \\ \delta_i(v_{-i}) + 2 + \sum_{j \neq i} v_j(x) - \sum_{j \neq i} v_j(z) & \text{otherwise.} \end{cases}$$

Then,

$$v_i(x) - v_i(y) = v_i(x) - v_i(\phi) = \delta_i(v_{-i}) + 1 > \delta_i(v_{-i}).$$

Therefore, $d(v) = x$. However, for each $z \in X \setminus \{x\}$,

$$\sum_{j=1}^{n} v_j(z) = \delta_i(v_{-i}) + 2 + \sum_{j \neq i} v_j(x) > \delta_i(v_{-i}) + 1 + \sum_{j \neq i} v_j(x) = \sum_{j=1}^{n} v_j(x).$$

This is a contradiction to weak efficiency.

Therefore, Case 3 must be hold, and hence d is a constant function. In addition, by weak efficiency, we have $d(v) = \phi$ for all $v \in V^n$. Then, by individual rationality and feasibility, $t_i(v) = 0$ for all $i \in N$ and all $v \in V^n$. □

Appendix D: Proof of Theorem 2

Since Proof of Statement (i) is straightforward, we only show Statement (ii).

Lemma 2. *Suppose that a mechanism* $(d,t) \in \mathscr{M}$ *satisfies* weak efficiency *and* strategy-proofness. *For each* $i \in N$*, each* $v \in \prod_{j=1}^{n} V_{r_j}$ *with* $d(v) = \phi$*, and each* $v_i' \in V_{r_i}$*, if* $v_i'(x) < v_i(x)$ *for all* $x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \left\{ v_i'(y) + \sum_{j \neq i} v_j(y) \right\}$*, then* $d(v_i', v_{-i}) = \phi$.

Proof. Take any $i \in N$, any $v \in \prod_{j=1}^{n} V_{r_j}$ with $d(v) = \phi$, and any $v_i' \in V_{r_i}$. Suppose that

$$v_i'(x) < v_i(x) \quad \text{for all } x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \left\{ v_i'(y) + \sum_{j \neq i} v_j(y) \right\}. \tag{2.37}$$

Suppose, by contradiction, that $d(v_i', v_{-i}) \neq \phi$. By strategy-proofness,

$$\begin{aligned} v_i'(d(v_i', v_{-i})) + t_i(v_i', v_{-i}) &\geq v_i'(d(v_i, v_{-i})) + t_i(v_i, v_{-i}) \\ &= v_i'(\phi) + t_i(v_i, v_{-i}) = t_i(v_i, v_{-i}). \end{aligned}$$

Then,

$$t_i(v_i', v_{-i}) \geq -v_i'(d(v_i', v_{-i})) + t_i(v_i, v_{-i}). \tag{2.38}$$

By weak efficiency,

$$d(v_i', v_{-i}) \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \left\{ v_i'(y) + \sum_{j \neq i} v_j(y) \right\}.$$

Then, by (2.37)

$$-v_i'(d(v_i', v_{-i})) > -v_i(d(v_i', v_{-i})).$$

Hence by (2.38),

$$t_i(v_i', v_{-i}) > -v_i(d(v_i', v_{-i})) + t_i(v_i, v_{-i}).$$

Therefore,

$$v_i(d(v_i', v_{-i})) + t_i(v_i', v_{-i}) > t_i(v_i, v_{-i}) = v_i(d(v_i, v_{-i})) + t_i(v_i, v_{-i}),$$

a contradiction to strategy proofness. ■

Define a function $s : \prod_{j=1}^n V_{r_j} \to \mathbb{R}$ by

$$s(v) = \max_{y \in X \setminus \{\phi\}} \sum_{j=1}^n v_j(y).$$

Lemma 3. *Suppose that a mechanism* $(d, t) \in \mathscr{M}$ *satisfies* weak efficiency *and* strategy-proofness. *For each* $i \in N$*, each* $v \in \prod_{j=1}^n V_{r_j}$ *with* $d(\phi) = \phi$*, and each* $v_i' \in V_{r_i}$*, if* $s(v_i', v_{-i}) < s(v_i, v_{-i})$*, then* $d(v_i', v_{-i}) = \phi$*.*

Proof. Take any $i \in N$, any $v \in \prod_{j=1}^n V_{r_j}$ with $d(\phi) = \phi$, and any $v_i' \in V_{r_i}$. Suppose that $s(v_i', v_{-i}) < s(v_i, v_{-i})$. Let $v_i'' \in V_{r_i}$ be such that for each $x \in X \setminus \{\phi\}$,

(i) If $x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \{v_i'(y) + \sum_{j \neq i} v_j(y)\}$ and $x \notin \underset{y \in X \setminus \{\phi\}}{\arg\max} \sum_{j=1}^n v_j(y)$, then

$$v_i''(x) = \min \left\{ \max_{y \in X \setminus \{\phi\}} \sum_{j=1}^n v_j(y) - \sum_{j \neq i} v_j(x) - \varepsilon,\ r_i - \varepsilon \right\}.$$

(ii) Otherwise, $v_i''(x) = v_i(x) - \frac{1}{2}\varepsilon$, where $\varepsilon > 0$ is such that

$$\varepsilon < \min\{r_i - v_i'(x), s(v_i, v_{-i}) - s(v_i', v_{-i})\}. \tag{2.39}$$

Note that for each $x \in X \setminus \{\phi\}$, $v_i''(x) < r_i$, i.e., v_i'' is well-defined.

Step 1. Let us show that $d(v_i'', v_{-i}) = \phi$. We first show that for each $x \in \arg\max_{y \in X\setminus\{\phi\}}\{v_i''(y) + \sum_{j\neq i} v_j(y)\}$, $v_i''(x) < v_i(x)$. Take any $x \in \arg\max_{y \in X\setminus\{\phi\}}\{v_i''(y) + \sum_{j\neq i} v_j(y)\}$. We shall show $v_i''(x) = v_i(x) - \frac{1}{2}\varepsilon$. Suppose, by contradiction, that $v_i''(x) \neq v_i(x) - \frac{1}{2}\varepsilon$. Then,

$$v_i''(x) = \min\left\{\max_{y \in X\setminus\{\phi\}} \sum_{j=1}^{n} v_j(y) - \sum_{j\neq i} v_j(x) - \varepsilon,\ r_i - \varepsilon\right\}.$$

Take some $z \in \arg\max_{y \in X\setminus\{\phi\}} \sum_{j=1}^{n} v_j(y)$. Then,

$$\begin{aligned} v_i''(x) + \sum_{j\neq i} v_j(x) &\leq \max_{y \in X\setminus\{\phi\}} \sum_{j=1}^{n} v_j(y) - \sum_{j\neq i} v_j(x) - \varepsilon + \sum_{j\neq i} v_j(x) = \max_{y \in X\setminus\{\phi\}} \sum_{j=1}^{n} v_j(y) - \varepsilon \\ &= \sum_{j=1}^{n} v_j(z) - \varepsilon < v_i(z) + \sum_{j\neq i} v_j(z) - \frac{1}{2}\varepsilon = v_i''(z) + \sum_{j\neq i} v_j(z), \end{aligned}$$

a contradiction to $x \in \arg\max_{y \in X\setminus\{\phi\}}\{v_i''(y) + \sum_{j\neq i} v_j(y)\}$. Therefore, $v_i''(x) = v_i(x) - \frac{1}{2}\varepsilon$. Then,

$$v_i''(x) = v_i(x) - \frac{1}{2}\varepsilon < v_i(x).$$

Hence by Lemma 2, $d(v_i'', v_{-i}) = \phi$.

Step 2. Let us show that $d(v_i', v_{-i}) = \phi$. By Lemma 2 and Step 1, it suffices to show that for each $x \in \arg\max_{y \in X\setminus\{\phi\}}\{v_i'(y) + \sum_{j\neq i} v_j(y)\}$, $v_i'(x) < v_i''(x)$. Take any $x \in \arg\max_{y \in X\setminus\{\phi\}}\{v_i'(y) + \sum_{j\neq i} v_j(y)\}$. We consider three cases.

Case 1. Consider the case with $v_i''(x) = r_i - \varepsilon$. Then, by (2.39)

$$v_i'(x) < r_i - \varepsilon = v_i''(x).$$

Case 2. Consider the case with $v_i''(x) = \max_{y \in X\setminus\{\phi\}}\{\sum_{j=1}^{n} v_j(y)\} - \sum_{j\neq i} v_j(x) - \varepsilon$. Then, it follows that

$$v_i''(x) = \max_{y \in X\setminus\{\phi\}} \sum_{j=1}^{n} v_j(y) - \sum_{j\neq i} v_j(x) - \varepsilon = s(v_i, v_{-i}) - \varepsilon - \sum_{j\neq i} v_j(x) > s(v_i', v_{-i}) - \sum_{j\neq i} v_j(x)$$

$$= \max_{y \in X \setminus \{\phi\}} \left\{ v_i'(y) + \sum_{j \neq i} v_j(y) \right\} - \sum_{j \neq i} v_j(x) = v_i'(x) + \sum_{j \neq i} v_j(x) - \sum_{j \neq i} v_j(x) = v_i'(x),$$

where the third inequality follows from (2.39).

Case 3. Consider the case with $v_i''(x) = v_i(x) - \frac{1}{2}\varepsilon$. By Cases 1 and 2, and $x \in \arg\max_{y \in X \setminus \{\phi\}} \{v_i'(y) + \sum_{j \neq i} v_j(y)\}$, it suffices to consider the case with $x \in \arg\max_{y \in X \setminus \{\phi\}} \sum_{j=1}^n v_j(y)$. Then,

$$\begin{aligned} v_i''(x) &= v_i(x) - \frac{1}{2}\varepsilon = \sum_{j=1}^n v_j(x) - \frac{1}{2}\varepsilon - \sum_{j \neq i} v_j(x) = s(v_i, v_{-i}) - \frac{1}{2}\varepsilon - \sum_{j \neq i} v_j(x) \\ &> s(v_i', v_{-i}) - \sum_{j \neq i} v_j(x) = \max_{y \in X \setminus \{\phi\}} \left\{ v_i'(y) + \sum_{j \neq i} v_j(y) \right\} - \sum_{j \neq i} v_j(x) \\ &= \left\{ v_i'(x) + \sum_{j \neq i} v_j(x) \right\} - \sum_{j \neq i} v_j(x) = v_i'(x), \end{aligned}$$

where the forth inequality follows from (2.39) and $s(v_i, v_{-i}) > s(v_i', v_{-i})$. ■

Lemma 4. *Suppose that a mechanism $(d, t) \in \mathscr{M}$ satisfies* weak efficiency *and* strategy-proofness. *For each $K \subset \{1, 2, \ldots, n\}$, each $v \in \prod_{j=1}^n V_{r_j}$ with $d(\phi) = \phi$, and each $v' \in \prod_{j=1}^n V_{r_j}$, if the following two conditions are satisfied, then $d(v_K', v_{-K}) = \phi$:*

(i) $s(v_K', v_{-K}) < s(v_K, v_{-K})$,

(ii) there exists no $w \in X \setminus \{\phi\}$ such that

$$w \in \arg\max_{y \in X \setminus \{\phi\}} \left\{ \sum_{j \in K} v_j'(y) + \sum_{j \notin K} v_j(y) \right\} \cap \arg\max_{y \in X \setminus \{\phi\}} \sum_{j=1}^n v_j(y).$$

Proof. We show this by induction. Consider any $k \geq 2$. Suppose that for any $L \subset \{1, 2, \ldots, n\}$ with $|L| = k - 1$, the statement in Lemma 4 holds. Take any $K \subset \{1, 2, \ldots, n\}$ with $|K| = k$, any $v \in \prod_{j=1}^n V_{r_j}$ with $d(v) = \phi$, and any $v' \in \prod_{j=1}^n V_{r_j}$. Without loss of generality, suppose that $K = \{1, 2, \ldots, k\}$. Suppose also that conditions (i) and (ii) hold.

First, we show that for each $x \in \arg\max_{y \in X \setminus \{\phi\}} \sum_{j=1}^n v_j(y)$,

$$\sum_{j \in K} v_j'(x) < \sum_{j \in K} v_j(x). \tag{2.40}$$

Take any $x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \sum_{j=1}^{n} v_j(y)$. Suppose, by contradiction, that $\sum_{j \in K} v'_j(x) \geq \sum_{j \in K} v_j(x)$. Then,

$$s(v'_K, v_{-K}) \geq \sum_{j \in K} v'_j(x) + \sum_{j \notin K} v_j(x) \geq \sum_{j=1}^{n} v_j(x) = s(v_K, v_{-K}),$$

a contradiction to $s(v'_K, v_{-K}) < s(v_K, v_{-K})$. Thus, (2.40) holds.

Second, we show that for each $x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \sum_{j=1}^{n} v_j(y)$, there exists $L \subset K$ with $|L| = k-1$ such that

$$\sum_{j \in L} v'_j(x) < \sum_{j \in L} v_j(x). \tag{2.41}$$

Take any $x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \sum_{j=1}^{n} v_j(y)$. Suppose, by contradiction, that for each $L \subset K$ with $|L| = k - 1$, $\sum_{j \in L} v'_j(x) \geq \sum_{j \in L} v_j(x)$. Then,

$$\begin{aligned}(k-1) \sum_{j \in K} v'_j(x) = \sum_{\substack{L \subset K \\ |L| = k-1}} \sum_{j \in L} v'_j(x) &\geq \sum_{\substack{L \subset K \\ |L| = k-1}} \sum_{j \in L} v_j(x) \\ &= (k-1) \sum_{j \in K} v_j(x),\end{aligned}$$

a contradiction to (2.40). Thus, (2.41) holds. Without loss of generality, suppose that $L = \{1, 2, \ldots, k-1\}$.

Take any $x \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \sum_{j=1}^{n} v_j(y)$ and any $z \in X \setminus \{\phi, x\}$ with $z \in \underset{y \in X \setminus \{\phi\}}{\arg\max} \{\sum_{j \in K} v'_j(y) + \sum_{j \notin K} v_j(y)\}$. Condition (ii) of our assumption implies that such z exists. We now consider two cases.

Case 1. Consider the case with $\sum_{j=1}^{k-1} v_j(z) \leq \sum_{j=1}^{k-1} v'_j(z)$. Let $\varepsilon > 0$ be such that

$$\varepsilon < \min \left\{ \frac{1}{2}\big(s(v_K, v_{-K}) - s(v'_K, v_{-K})\big), \; \frac{1}{2} \sum_{j=1}^{k-1} \big(v_j(x) - v'_j(x)\big), \; r_k - v'_k(z) \right\}. \tag{2.42}$$

Define $v''_k \in V_{r_k}$ by

$$v''_k(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v_k(x) + s(v'_K, v_{-K}) - s(v_K, v_{-K}) + 2\varepsilon & \text{if } y = x, \\ v'_k(z) + \varepsilon & \text{if } y = z, \\ -\delta & \text{otherwise,} \end{cases}$$

where $-\delta$ is a sufficiently small number under which

$$\underset{y \in X \setminus \{\phi\}}{\arg\max} \left\{ v''_k(y) + \sum_{j \neq k} v_j(y) \right\} \subset \{x, z\},$$

and

$$\underset{y\in X\setminus\{\phi\}}{\arg\max}\left\{\sum_{j=1}^{k-1} v'_j(y) + v''_k(y) + \sum_{j=k+1}^{n} v_j(y)\right\} \subset \{x, z\}.$$

Note that v''_k is well-defined by definition of ε.

Step 1-1. Let us show that $d(v''_k, v_{-k}) = \phi$. Since $x \in \underset{y\in X\setminus\{\phi\}}{\arg\max} \sum_{j=1}^{n} v_j(y)$ and $z \in \underset{y\in X\setminus\{\phi\}}{\arg\max}\{\sum_{j\in K} v'_j(y) + \sum_{j\notin K} v_j(y)\}$, it follows that

$$\begin{aligned}
v''_k(x) + \sum_{j\neq k} v_j(x) &= v_k(x) + s(v'_K, v_{-K}) - s(v_K, v_{-K}) + 2\varepsilon + \sum_{j\neq k} v_j(x) \\
&= s(v'_K, v_{-K}) + 2\varepsilon > s(v'_K, v_{-K}) + \varepsilon = v'_k(z) + \varepsilon + \sum_{j=1}^{k-1} v'_j(z) + \sum_{j=k+1}^{n} v_j(z) \\
&\geq v'_k(z) + \varepsilon + \sum_{j=1}^{k-1} v_j(z) + \sum_{j=k+1}^{n} v_j(z) = v''_k(z) + \sum_{j\neq k} v_j(z),
\end{aligned}$$

where the fifth inequality follows from the assumption of Case 1. Therefore,

$$s(v''_k, v_{-k}) = v''_k(x) + \sum_{j\neq k} v_j(x). \tag{2.43}$$

Then, by definitions of v''_k and ε,

$$s(v''_k, v_{-k}) = v''_k(x) + \sum_{j\neq k} v_j(x) = s(v'_K, v_{-K}) + 2\varepsilon < s(v_K, v_{-K}). \tag{2.44}$$

Hence by Lemma 3, $d(v''_k, v_{-k}) = \phi$.

Step 1-2. Let us show that $d(v'_L, v''_k, v_{-K}) = \phi$. Note that by definitions of ε and v''_k,

$$\begin{aligned}
\sum_{j=1}^{k-1} v'_j(x) + v''_k(x) + \sum_{j=k+1}^{n} v_j(x) &< \sum_{j=1}^{k-1} v_j(x) + v''_k(x) + \sum_{j=k+1}^{n} v_j(x) - 2\varepsilon = s(v'_K, v_{-K}) \\
&< s(v'_K, v_{-K}) + \varepsilon = \sum_{j=1}^{k-1} v'_j(z) + v''_k(z) + \sum_{j=k+1}^{n} v_j(z).
\end{aligned}$$

Thus, we have shown that

$$\underset{y\in X\setminus\{\phi\}}{\arg\max}\left\{\sum_{j=1}^{k-1} v'_j(y) + v''_i(y) + \sum_{j=k+1}^{n} v_j(y)\right\} = \{z\}.$$

On the other hand, by (2.43),

$$\operatorname*{arg\,max}_{y\in X\setminus\{\phi\}}\left\{\sum_{j\neq k} v_j(y)+v''_k(y)\right\}=\{x\}.$$

In addition, by definition of v''_k and the first two equations of (2.44),

$$\begin{aligned}s(v'_L, v''_k, v_{-K}) &= \sum_{j=1}^{k-1} v'_j(z)+v''_k(z)+\sum_{j=k+1}^{n} v_j(z) = s(v'_K, v_{-K})+\varepsilon\\ &< s(v'_K, v_{-K})+2\varepsilon = s(v''_k, v_{-k}) = s(v_L, v''_k, v_{-K}).\end{aligned}$$

Therefore, by the assumption of the mathematical induction, $d(v'_L, v''_i, v_{-K})=\phi$.

Step 1-3. Let us show that $d(v'_K, v_{-K})=\phi$. Note that by definition of v''_k,

$$\begin{aligned}s(v'_L, v'_k, v_{-K}) &= \sum_{j=1}^{k} v'_j(z)+\sum_{j=k+1}^{n} v_j(z) < \sum_{j=1}^{k-1} v'_j(z)+v'_k(z)+\varepsilon+\sum_{j=k+1}^{n} v_j(z)\\ &= \sum_{j=1}^{k-1} v'_j(z)+v''_k(z)+\sum_{j=k+1}^{n} v_j(z) = s(v'_L, v''_k, v_{-K}).\end{aligned}$$

Then, by Lemma 3, $d(v'_K, v_{-K})=\phi$.

Case 2. We next consider the case with $\sum_{j=1}^{k-1} v_j(z) > \sum_{j=1}^{k-1} v'_j(z)$. Let $\varepsilon>0$ be such that

$$\varepsilon<\min\left\{\frac{1}{2(k-2)}\sum_{j=1}^{n}\big(v_j(x)-v_j(z)\big),\ \frac{1}{2k-1}\big(s(v_K, v_{-K})-s(v'_K, v_{-K})\big),\ \min_{j\in K}\{r_j-v'_j(z)\}\right\}. \tag{2.45}$$

Note that by assumption (ii) of the mathematical induction $\sum_{j=1}^{n} v_j(x)-\sum_{j=1}^{n} v_j(z)>0$. For each $j\in L$, let $v''_j\in V_r$ be such that

$$v''_j(y)=\begin{cases}0 & \text{if } y=\phi,\\ v_j(x)-\varepsilon & \text{if } y=x,\\ v'_j(z)+\varepsilon & \text{if } y=z,\\ -\delta & \text{otherwise,}\end{cases}$$

where $-\delta$ is a sufficiently small number under which

$$\operatorname*{arg\,max}_{y\in X\setminus\{\phi\}}\left\{v''_k(y)+\sum_{j\neq k} v_j(y)\right\}\subset\{x, z\},$$

and

$$\mathop{\arg\max}_{y\in X\setminus\{\phi\}}\left\{\sum_{j=1}^{k-1} v'_j(y)+v''_k(y)+\sum_{j=k+1}^{n} v_j(y)\right\}\subset\{x,z\}.$$

Note that each v''_j is well-defined.

Step 2-1. Let $M\subset L$ be such that $M\equiv\{j\in L: v'_j(z)\le v_j(z)\}$. Let us show that $d(v''_M, v_{-M})=\phi$.

Substep 2-1-1. Take any $j_1\in M$. Let us show that $d(v''_{j_1}, v_{-j_1})=\phi$. By $j_1\in M$ and (2.45),

$$\begin{aligned} v''_{j_1}(z)+\sum_{j\neq j_1} v_j(z) = v'_{j_1}(z)+\sum_{j\neq j_1} v_j(z)+\varepsilon \le v_{j_1}(z)+\sum_{j\neq j_1} v_j(z)+\varepsilon \\ < v_{j_1}(x)+\sum_{j\neq j_1} v_j(x)-\varepsilon = v''_{j_1}(x)+\sum_{j\neq j_1} v_j(x). \end{aligned}$$

Hence

$$s(v''_{j_1}, v_{-j_1}) = v''_{j_1}(x)+\sum_{j\neq j_1} v_j(x). \tag{2.46}$$

In addition,

$$\begin{aligned} s(v''_{j_1}, v_{-j_1}) = v''_{j_1}(x)+\sum_{j\neq j_1} v_j(x) = v_{j_1}(x)-\varepsilon+\sum_{j\neq j_1} v_j(x) \\ < v_{j_1}(x)+\sum_{j\neq j_1} v_j(x) = s(v_{j_1}, v_{-j_1}). \end{aligned}$$

Then, by Lemma 3, $d(v''_{j_1}, v_{-j_1})=\phi$.

Substep 2-1-2. Take any $j_2\in M\setminus\{j_1\}$. Note that if such j_2 exists, then $k\ge 3$. Let us show that $d(v''_{j_1}, v''_{j_2}, v_{-j_1\cup j_2})=\phi$. We first confirm that

$$s(v''_{j_1\cup j_2}, v_{-j_1\cup j_2}) = v''_{j_1}(x)+v''_{j_2}(x)+\sum_{j\neq j_1,j_2} v_j(x). \tag{2.47}$$

By $j_1, j_2\in M$ and (2.45),

$$\begin{aligned} & v''_{j_1}(z)+v''_{j_2}(z)+\sum_{j\neq j_1,j_2} v_j(z) = v'_{j_1}(z)+v'_{j_2}(z)+\sum_{j\neq j_1,j_2} v_j(z)+2\varepsilon \\ & \le v_{j_1}(z)+v_{j_2}(z)+\sum_{j\neq j_1,j_2} v_j(z)+2\varepsilon < v_{j_1}(x)+v_{j_2}(x)+\sum_{j\neq j_1,j_2} v_j(x)-2\varepsilon \\ & = v''_{j_1}(x)+v''_{j_2}(x)+\sum_{j\neq j_1,j_2} v_j(x). \end{aligned}$$

Hence (2.47) holds. Then, similarly to Substep 2-1-1,

$$s(v''_{j_1}, v''_{j_2}, v_{-j_1 \cup j_2}) = v''_{j_1}(x) + v''_{j_2}(x) + \sum_{j \neq j_1, j_2} v_j(x) = v_{j_1}(x) + v_{j_2}(x) + \sum_{j \neq j_1, j_2} v_j(x) - 2\varepsilon$$
$$< v_{j_1}(x) + \sum_{j \neq j_1} v_j(x) - \varepsilon = v''_{j_1}(x) + \sum_{j \neq j_1} v_j(x) = s(v''_{j_1}, v_{-j_1}).$$

Then, by Lemma 3, $d(v''_{j_1}, v''_{j_2}, v_{-j_1 \cup j_2}) = \phi$.

Substep 2-1-3. Repeating the similar arguments to Substep 2-1-2 show that $d(v''_M, v_{-M}) = \phi$. This is the end of Step 2-1

Step 2-2. Let us show that $d(v''_L, v_{-L}) = \phi$.

Substep 2-2-1. Take any $j_1 \in L \setminus M$. We shall show that

$$s(v''_M, v''_{j_1}, v_{-M \cup \{j_1\}}) = \sum_{j \in M} v''_j(x) + v''_{j_1}(x) + \sum_{j \notin M \cup \{j_1\}} v_j(x). \tag{2.48}$$

To show this, we confirm that

$$\sum_{j \in M} v'_j(z) + v'_{j_1}(z) < \sum_{j \in M} v_j(z) + v_{j_1}(z). \tag{2.49}$$

Suppose, by contradiction, that

$$\sum_{j \in M} v'_j(z) + v'_{j_1}(z) \geq \sum_{j \in M} v_j(z) + v_{j_1}(z).$$

Then, by definition of M,

$$\sum_{j \in M} v'_j(z) + v'_{j_1}(z) + \sum_{j \in L \setminus (M \cup \{j_1\})} v'_j(z) \geq \sum_{j \in M} v_j(z) + v_{j_1}(z) + \sum_{j \in L \setminus (M \cup \{j_1\})} v'_j(z)$$
$$\geq \sum_{j \in M} v_j(z) + v_{j_1}(z) + \sum_{j \in L \setminus (M \cup \{j_1\})} v_j(z),$$

a contradiction to the assumption of Case 2. Hence (2.49) holds.

Then, by (2.49), (2.45), and definition of ε,

$$\sum_{j \in M} v''_j(z) + v''_{j_1}(z) + \sum_{j \notin M \cup \{j_1\}} v_j(z) = \sum_{j \in M} (v'_j(z) + \varepsilon) + v'_{j_1}(z) + \varepsilon + \sum_{j \notin M \cup \{j_1\}} v_j(z)$$
$$< \sum_{j \in M} v_j(z) + v_{j_1}(z) + \sum_{j \notin M \cup \{j_1\}} v_j(z) + (|M| + 1)\varepsilon$$
$$< \sum_{j \in M} v_j(x) + v_{j_1}(x) + \sum_{j \notin M \cup \{j_1\}} v_j(x) - (|M| + 1)\varepsilon = \sum_{j \in M} v''_j(x) + v''_{j_1}(x) + \sum_{j \notin M \cup \{j_1\}} v_j(x).$$

Hence (2.48) holds. Then,

$$\begin{aligned} s(v''_M, v''_{j_1}, v_{-M\cup j_1}) &= \sum_{j\in M} v''_j(x) + v''_{j_1}(x) + \sum_{j\notin M\cup\{j_1\}} v_j(x) \\ &= \sum_{j\in M} v_j(x) + v_{j_1}(x) + \sum_{j\notin M\cup\{j_1\}} v_j(x) - (|M|+1)\varepsilon \\ &< \sum_{j\in M} v_j(x) + v_{j_1}(x) + \sum_{j\notin M\cup\{j_1\}} v_j(x) - |M|\varepsilon = s(v''_M, v_{-M}). \end{aligned}$$

Therefore, by Lemma 3, $d(v''_M, v''_{j_1}, v_{-M\cup j_1}) = \phi$.

Substep 2-2-2. Repeating the similar arguments to Substep 2-2-1 show that $d(v''_L, v_{-L}) = \phi$. This is the end of Step 2-2.

Let $v''_k \in V_r$ be such that

$$v''_k(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v'_k(z) + \varepsilon & \text{if } y = z, \\ -\delta & \text{otherwise,} \end{cases}$$

where ε is defined by (2.45) and $-\delta$ is a sufficiently small number under which all $s(v''_k, \cdot)$ in the subsequent argument take the values at z.

Step 2-3. Let us show that $d(v''_L, v''_k, v_{-K}) = \phi$. Note that by definition of ε,

$$\begin{aligned} s(v''_L, v''_k, v_{-K}) &= \sum_{j=1}^{k-1} v''_j(z) + v''_k(z) + \sum_{j=k+1}^{n} v_j(z) = \sum_{j=1}^{k-1}(v'_j(z)+\varepsilon) + v'_k(z) + \varepsilon + \sum_{j=k+1}^{n} v_j(z) \\ &= s(v'_K, v_{-K}) + k\varepsilon < s(v_K, v_{-K}) - (k-1)\varepsilon = \sum_{j=1}^{k-1}(v_j(x) - \varepsilon) + \sum_{j=k}^{n} v_j(x) \\ &= s(v''_L, v_{-L}). \end{aligned}$$

Then, by Lemma 3, $d(v''_L, v''_k, v_{-K}) = \phi$.

Step 2-4. Let us show that $d(v'_L, v''_k, v_{-K}) = \phi$.

Substep 2-4-1. Let us show that $d(v'_1, v''_{K\setminus\{1\}} v_{-K}) = \phi$. It follows that

$$s(v'_1, v''_{K\setminus\{1\}}, v_{-K}) = v'_1(z) + \sum_{j=2}^{k} v''_j(z) + \sum_{j=k+1}^{n} v_j(z) < v'_1(z) + \varepsilon + \sum_{j=2}^{k} v''_j(z) + \sum_{j=k+1}^{n} v_j(z)$$

$$= \sum_{j=1}^{k} v_j''(z) + \sum_{j=k+1}^{n} v_j(z) = s(v_K'', v_{-K}).$$

Hence by Lemma 3, $d(v_1', v_{K\setminus\{1\}}'' v_{-K}) = \phi$.

Substep 2-4-2. Let us show that if $k \geq 3$, then $d(v_1', v_2', v_{K\setminus\{1\}\setminus\{2\}}'', v_{-K}) = \phi$. Similarly to Substep 2-4-1, it follows that

$$\begin{aligned} s(v_1', v_2', v_{K\setminus\{1\}\setminus\{2\}}'', v_{-K}) &= v_1'(z) + v_2'(z) + \sum_{j=3}^{k} v_j''(z) + \sum_{j=k+1}^{n} v_j(z) \\ &< v_1'(z) + v_2'(z) + \varepsilon + \sum_{j=3}^{k} v_j''(z) + \sum_{j=k+1}^{n} v_j(z) \\ &= v_1'(z) + \sum_{j=2}^{k} v_j''(z) + \sum_{j=k+1}^{n} v_j(z) = s(v_1', v_{K\setminus\{1\}}'', v_{-K}). \end{aligned}$$

Hence, by Lemma 3, $d(v_1', v_2', v_{K\setminus\{1\}\setminus\{2\}}'', v_{-K}) = \phi$.

Substep 2-4-3. Repeating the similar arguments to Substep 2-4-2 yields that $d(v_L', v_k'', v_{-K}) = \phi$. This is the end of Step 2-4.

Step 2-5. Let us show that $d(v_K', v_{-K}) = \phi$. It follows that

$$\begin{aligned} s(v_K', v_{-K}) &= \sum_{j=1}^{k} v_j'(z) + \sum_{j=k+1}^{n} v_j(z) < \sum_{j=1}^{k-1} v_j'(z) + v_k'(z) + \varepsilon + \sum_{j=k+1}^{n} v_j(z) \\ &= \sum_{j=1}^{k-1} v_j'(z) + v_k''(z) + \sum_{j=k+1}^{n} v_j(z) = s(v_L', v_k'', v_{-N}). \end{aligned}$$

Therefore, by Lemma 3, $d(v_K', v_{-K}) = \phi$. ■

Lemma 5. *Suppose that a mechanism* $(d,t) \in \mathscr{M}$ *satisfies* weak efficiency *and* strategy-proofness. *For each* $K \subset \{1, 2, \ldots, n\}$*, each* $v \in \prod_{j=1}^{n} V_{r_j}$ *with* $d(\phi) = \phi$*, and each* $v' \in \prod_{j=1}^{n} V_{r_j}$*, if the following two conditions are satisfied, then* $d(v_K', v_{-K}) = \phi$*:*

(i) $s(v_K', v_{-K}) < s(v_K, v_{-K})$*,*

(ii) there exists $x \in X \setminus \{\phi\}$ *such that*

$$x \in \arg\max_{y \in X\setminus\{\phi\}} \left\{ \sum_{j\in K} v_j'(y) + \sum_{j \notin K} v_j(y) \right\} \cap \in \arg\max_{y\in X\setminus\{\phi\}} \sum_{j=1}^{n} v_j(y).$$

Proof. Let us show by induction. Consider any $k \geq 2$. Suppose that for each $L \subset \{1, 2, \ldots, n\}$ with $|L| = k - 1$, the statement in Lemma 5 holds. Take any $K \subset \{1, 2, \ldots, n\}$ with $|K| = k$, any $v \in \prod_{j=1}^{n} V_{r_j}$ with $d(v) = \phi$, and any $v' \in \prod_{j=1}^{n} V_{r_j}$. Without loss of generality, we write $K = \{1, 2, \ldots, k\}$. Suppose that conditions (i) and (ii) holds.

Since $s(v'_K, v_{-K}) < s(v_K, v_{-K})$, by condition (ii),

$$\sum_{j=1}^{k} v'_j(x) + \sum_{j=k+1}^{n} v_j(x) = s(v'_K, v_{-K}) < s(v_K, v_{-K}) = \sum_{j=1}^{n} v_j(x).$$

Therefore,

$$\sum_{j=1}^{k} v'_j(x) < \sum_{j=1}^{k} v_j(x). \tag{2.50}$$

Step 1. Let us show that there exists $L \subset K$ with $|L| = k - 1$ such that

$$\sum_{j \in L} v'_j(x) < \sum_{j \in L} v_j(x). \tag{2.51}$$

Suppose, by contradiction, that for each $L \subset K$ with $|L| = k - 1$,

$$\sum_{j \in L} v'_j(x) \geq \sum_{j \in L} v_j(x).$$

Then,

$$\begin{aligned}(k-1)\sum_{j \in K} v'_j(x) = \sum_{\substack{L \subset K \\ |L| = k-1}} \sum_{j \in L} v'_j(x) &\geq \sum_{\substack{L \subset K \\ |L| = k-1}} \sum_{j \in L} v_j(x) \\ &= (k-1)\sum_{j \in K} v_j(x),\end{aligned}$$

a contradiction to (2.50). Thus, (2.51) holds. Without loss of generality, suppose that $L = \{1, 2, \ldots, k-1\}$.

Step 2. We define $v''_{-K} \in \prod_{j=k+1}^{n} V_{r_j}$ as follows. Take some $z \in X \setminus \{\phi\} \setminus \{x\}$. Let $\varepsilon > 0$ be such that

$$\begin{aligned}\varepsilon < \min \Big\{ &\min_{j \in K}\{r_j - v'_j(x)\}, \min_{j \in N \setminus K}\{r_j - v_j(z)\}, \frac{1}{n+k}\big(r_k - v_k(z)\big), \\ &\sum_{j=1}^{k-1}\big(v_j(x) - v'_j(x)\big), \frac{1}{2n}\big(r_k - v'_k(x)\big), \frac{1}{n+2k}\sum_{j=1}^{k}\big(v_j(x) - v'_j(x)\big)\Big\}.\end{aligned} \tag{2.52}$$

Define $v''_{k+1} \in V_{r_{k+1}}$ to satisfy that if

$$v_{k+1}(z) \geq \sum_{j \neq k+1} (v_j(x) - v_j(z)) + v_{k+1}(x) - \varepsilon,$$

then

$$v''_{k+1}(y) = v_{k+1}(y) \text{ for all } y \in X \setminus \{\phi\},$$

and otherwise,

$$v''_{k+1}(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v_{k+1}(x) - \varepsilon & \text{if } y = x, \\ \min\left\{r_{k+1} - \varepsilon,\ \sum_{j \neq k+1}(v_j(x) - v_j(z)) + v_{k+1}(x) - \varepsilon\right\} & \text{if } y = z, \\ -\delta_{k+1} & \text{otherwise,} \end{cases}$$

where $-\delta_{k+1}$ is a sufficiently small number under which

$$\operatorname*{arg\,max}_{y \in X \setminus \{\phi\}} \left\{ v''_{k+1}(y) + \sum_{j \neq k+1} v_j(y) \right\} \subset \{x, z\}.$$

For each $i \in \{k+1, \ldots, n\}$, let $J_i \subset N$ be such that $J_i = \{k+1, k+2, \ldots, i\}$, and let $J_k = \emptyset$. For each $i \in \{k+1, \ldots, n\}$, define $v''_i \in V_{r_i}$ recursively to satisfy that if

$$v_i(z) \geq \sum_{j \notin J_i} (v_j(x) - v_j(z)) + v_i(x) + \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) - \varepsilon,$$

then

$$v''_i(y) = v_i(y) \text{ for all } y \in X \setminus \{\phi\},$$

and otherwise,

$$v''_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v_i(x) - \varepsilon & \text{if } y = x, \\ \min\left\{r_i - \varepsilon,\ \sum_{j \notin J_i}\left(v_j(x) - v_j(z)\right) + v_i(x) + \sum_{j \in J_{i-1}}\left(v''_j(x) - v''_j(z)\right) - \varepsilon\right\} & \text{if } y = z, \\ -\delta_i & \text{otherwise,} \end{cases}$$

where $-\delta_i$ is a sufficiently small number under which

$$\operatorname*{arg\,max}_{y \in X \setminus \{\phi\}} \left\{ \sum_{j \in J_i} v''_j(y) + \sum_{j \notin J_i} v_j(y) \right\} \subset \{x, z\}.$$

Step 3. Let us show that $d(v_K, v''_{-K}) = \phi$. We shall confirm that for each $i \in \{k+1, \dots, n\}$ with $v''_i \neq v_i$,

$$s(v''_{J_i}, v_{-J_i}) = \sum_{j \in J_i} v''_j(x) + \sum_{j \notin J_i} v_j(x). \tag{2.53}$$

Take any $i \in \{k+1, \dots, n\}$ with $v''_i \neq v_i$. Then,

$$\begin{aligned}
&\sum_{j \in J_i} v''_j(z) + \sum_{j \notin J_i} v_j(z) = v''_i(z) + \sum_{j \in J_{i-1}} v''_j(z) + \sum_{j \notin J_i} v_j(z) \\
&\leq \sum_{j \notin J_i} (v_j(x) - v_j(z)) + v_i(x) + \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) - \varepsilon + \sum_{j \in J_{i-1}} v''_j(z) + \sum_{j \notin J_i} v_j(z) \\
&= \sum_{j \notin J_i} v_j(x) + v_i(x) - \varepsilon + \sum_{j \in J_{i-1}} v''_j(x) = \sum_{j \in J_i} v''_j(x) + \sum_{j \notin J_i} v_j(x).
\end{aligned}$$

Hence (2.53) holds.

Let us show, by induction, that for each $i \in \{k+1, \dots, n\}$, $d(v''_{J_i}, v_{-J_i}) = \phi$. Suppose that $d(v''_{J_{i-1}}, v_{-J_{i-1}}) = \phi$. If $v''_i = v_i$, then obviously $d(v''_{J_i}, v_{-J_i}) = \phi$. Suppose that $v''_i \neq v_i$. Then, by (2.53) and condition (ii),

$$\begin{aligned}
s(v''_{J_i}, v_{-J_i}) &= \sum_{j \in J_i} v''_j(x) + \sum_{j \notin J_i} v_j(x) = \sum_{j \in J_{i-1}} (v''_j(x)) + v_i(x) - \varepsilon + \sum_{j \notin J_i} v_j(x) \\
&< \sum_{j \in J_{i-1}} (v''_j(x)) + v_i(x) + \sum_{j \notin J_i} v_j(x) = s(v''_{J_{i-1}}, v_i, v_{-J_i}).
\end{aligned}$$

Hence by Lemma 3, $d(v''_{J_i}, v_{-J_i}) = \phi$. Therefore, $d(v_K, v''_{-K}) = \phi$.

Step 4. Let $v''_k \in V_{r_k}$ be such that

$$v''_k(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v_k(x) - \varepsilon & \text{if } y = x, \\ \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n} v''_j(z) - n\varepsilon & \text{if } y = z, \\ -\delta_k & \text{otherwise,} \end{cases}$$

where $-\delta_k$ is a sufficiently small number under which

$$\operatorname*{arg\,max}_{y \in X \setminus \{\phi\}} \left\{ \sum_{j=1}^{k-1} v_j(y) + v''_k(y) + \sum_{j=k+1}^{n} v''_j(y) \right\} \subset \{x, z\}.$$

Let us show that v''_k is well-defined. It suffices to show that $v''_k(z) < r_k$.

Substep 4-1. Let us show that for each $i \in \{k+1, \dots, n\}$, if

$$v''_i(z) \geq \sum_{j \notin J_i} (v_j(x) - v_j(z)) + v_i(x) + \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) - \varepsilon, \tag{2.54}$$

then

$$v''_{i+1}(z) \geq \sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) + v_{i+1}(x) + \sum_{j \in J_i} (v''_j(x) - v''_j(z)) - \varepsilon. \tag{2.55}$$

Take any $i \in \{k+1, \ldots, n\}$. Suppose that (2.54) holds. Then,

$$\begin{aligned}
&\sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) + v_{i+1}(x) + \sum_{j \in J_i} (v''_j(x) - v''_j(z)) - \varepsilon \\
&= \sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) + v_{i+1}(x) + v''_i(x) - v''_i(z) + \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) - \varepsilon \\
&\leq \sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) + v_{i+1}(x) - \sum_{j \notin J_i} (v_j(x) - v_j(z)) \\
&\qquad - \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) + \varepsilon + \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) - \varepsilon \\
&= \sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) + v_{i+1}(x) - \sum_{j \notin J_i} (v_j(x) - v_j(z)) \\
&= \sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) + v_{i+1}(x) - (v_{i+1}(x) - v_{i+1}(z)) - \sum_{j \notin J_{i+1}} (v_j(x) - v_j(z)) \\
&= v_{i+1}(x) - (v_{i+1}(x) - v_{i+1}(z)) = v_{i+1}(z).
\end{aligned}$$

Then, by definition of v''_{i+1}, it follows that $v''_{i+1} = v_{i+1}$. Therefore, by $v''_{i+1}(z) = v_{i+1}(z)$ and by the above inequality, (2.55) holds.

Substep 4-2. Let us show that if

$$v''_n(z) < \sum_{j \notin J_n} (v_j(x) - v_j(z)) + v_n(x) + \sum_{j \in J_{n-1}} (v''_j(x) - v''_j(z)) - \varepsilon, \tag{2.56}$$

then $v''_k(z) < r_k$. Suppose that (2.56) holds. Then, by taking the contraposition of the claim in Substep 4-1 (equations 2.54 and 2.55), for each $i \in \{k+1, \ldots, n\}$,

$$v''_i(z) < \sum_{j \notin J_i} (v_j(x) - v_j(z)) + v_i(x) + \sum_{j \in J_{i-1}} (v''_j(x) - v''_j(z)) - \varepsilon.$$

Therefore by definition of v''_i, it follows that for each $i \in \{k+1, \ldots, n\}$, $v''_i(z) = r_i - \varepsilon$. Thus,

$$\begin{aligned}
v''_k(z) &= \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n} v''_j(z) - n\varepsilon = \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n} (r_j - \varepsilon) - n\varepsilon \\
&= \sum_{j=1}^{n} v_j(x) - \sum_{j \neq k} r_j - k\varepsilon < \sum_{j=1}^{n} v_j(x) - \sum_{j \neq k} r_j \leq v_k(x) < r_k.
\end{aligned}$$

Substep 4-3. Let us show that if

$$v_n''(z) \geq \sum_{j \notin J_n} (v_j(x) - v_j(z)) + v_n(x) + \sum_{j \in J_{n-1}} (v_j''(x) - v_j''(z)) - \varepsilon, \tag{2.57}$$

then $v_k''(z) < r_k$. Suppose that (2.57) holds. Then,

$$\begin{aligned}
v_k''(z) &= \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n} v_j''(z) - n\varepsilon \\
&= \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n-1} v_j''(z) - v_n''(z) - n\varepsilon \\
&\leq \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n-1} v_j''(z) - \sum_{j=1}^{k} (v_j(x) - v_j(z)) \\
&\qquad - v_n(x) - \sum_{j=k+1}^{n-1} (v_j''(x) - v_j''(z)) + \varepsilon - n\varepsilon \\
&= \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=1}^{k} (v_j(x) - v_j(z)) - v_n(x) - \sum_{j=k+1}^{n-1} v_j''(x) - (n-1)\varepsilon \\
&= \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=1}^{k} (v_j(x) - v_j(z)) - \sum_{j=k+1}^{n} v_j(x) - k\varepsilon \\
&\leq \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=1}^{k} (v_j(x) - v_j(z)) - \sum_{j=k+1}^{n} v_j(x) \\
&= \sum_{j=1}^{k} v_j(z) - \sum_{j=1}^{k-1} r_j = v_k(z) + \sum_{j=1}^{k-1} \big(v_j(z) - r_j\big) < v_k(z) < r_k.
\end{aligned}$$

Then, by Substeps 4-2 and 4-3, v_k'' is well-defined.

Step 5. Let us show that $d(v_L, v_k'', v_{-K}'') = \phi$. Note that since $v_j(z) < r_j$ for all $j \in N$,

$$\begin{aligned}
&\sum_{j=1}^{k-1} v_j(z) + v_k''(z) + \sum_{j=k+1}^{n} v_j''(z) \\
&= \sum_{j=1}^{k-1} v_j(z) + \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n} v_j''(z) - n\varepsilon + \sum_{j=k+1}^{n} v_j''(z) \\
&= \sum_{j=1}^{k-1} v_j(z) - \sum_{j=1}^{k-1} r_j + \sum_{j=1}^{n} v_j(x) - n\varepsilon < \sum_{j=1}^{n} v_j(x) - (n-k+1)\varepsilon
\end{aligned}$$

$$= \sum_{j=1}^{k-1} v_j(x) + v_k(x) - \varepsilon + \sum_{j=k+1}^{n} (v_j(x) - \varepsilon) \leq \sum_{j=1}^{k-1} v_j(x) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x).$$

Therefore,

$$s(v_L, v_k'', v_{-K}'') = \sum_{j=1}^{k-1} v_j(x) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x). \tag{2.58}$$

Then,

$$\begin{aligned} s(v_L, v_k'', v_{-K}'') &= \sum_{j=1}^{k-1} v_j(x) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x) = \sum_{j=1}^{k-1} v_j(x) + v_k(x) - \varepsilon + \sum_{j=k+1}^{n} v_j''(x) \\ &< \sum_{j=1}^{k-1} v_j(x) + v_k(x) + \sum_{j=k+1}^{n} v_j''(x) = \sum_{j=1}^{k-1} v_j(x) + v_k(x) + \sum_{j=k+1}^{n} v_j''(x) \\ &= s(v_L, v_k, v_{-K}''). \end{aligned}$$

Thus, by Lemma 3, $d(v_L, v_k'', v_{-K}'') = \phi$.

Step 6. For each $i \in L$, let $v_i'' \in V_{r_i}$ be such that

$$v_i''(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v_i'(x) + \varepsilon & \text{if } y = x, \\ r_j - \varepsilon & \text{if } y = z, \\ -\delta_j & \text{otherwise,} \end{cases}$$

where $-\delta_j$ is a sufficiently small number under which

$$s(v_L'', v_k'', v_{-K}'') = \sum_{j=1}^{k-1} v_j''(x) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x), \tag{2.59}$$

or

$$s(v_L'', v_k'', v_{-K}'') = \sum_{j=1}^{k-1} v_j''(z) + v_k''(z) + \sum_{j=k+1}^{n} v_j''(z). \tag{2.60}$$

Then, by definition of ε (pp. 61), v_i'' is well-defined. Let us show that $d(v_L'', v_k'', v_{-K}'') = \phi$.

Case 6-1. Consider the case with (2.59). Then, by definition of ε,

$$\begin{aligned} s(v_L'', v_k'', v_{-K}'') &= \sum_{j=1}^{k-1} v_j''(x) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x) = \sum_{j=1}^{k-1} (v_j'(x) + \varepsilon) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x) \\ &< \sum_{j=1}^{k-1} v_j(x) + v_k''(x) + \sum_{j=k+1}^{n} v_j''(x) = s(v_L, v_k'', v_{-K}''). \end{aligned}$$

Then, by the assumption of the mathematical induction, $d(v''_L, v''_k, v''_{-K}) = \phi$.

Case 6-2. Consider the case where only (2.60) holds. Then,

$$\begin{aligned}
s(v''_L, v''_k, v''_{-K}) &= \sum_{j=1}^{k-1} v''_j(z) + v''_k(z) + \sum_{j=k+1}^{n} v''_j(z) \\
&= \sum_{j=1}^{k-1}(r_j - \varepsilon) + \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - \sum_{j=k+1}^{n} v''_j(z) - n\varepsilon + \sum_{j=k+1}^{n} v''_j(z) \\
&= \sum_{j=1}^{k-1} r_j - (k-1)\varepsilon + \sum_{j=1}^{n} v_j(x) - \sum_{j=1}^{k-1} r_j - n\varepsilon = \sum_{j=1}^{n} v_j(x) - (n+k-1)\varepsilon \\
&< \sum_{j=1}^{n} v_j(x) - (n-k+1)\varepsilon = \sum_{j=1}^{k-1} v_j(x) + v''_k(x) + \sum_{j=k+1}^{n} v''_j(x) \\
&= s(v_L, v''_k, v''_{-K}). \qquad (2.61)
\end{aligned}$$

Moreover, in Step 5, we showed that $\arg\max_{y \in X\setminus\{\phi\}}\{\sum_{j=1}^{k-1} v_j(y) + v''_k(y) + \sum_{j=k+1}^{n} v''_j(y)\} = \{x\}$. Hence, by Lemma 4, $d(v''_L, v''_k, v''_{-K}) = \phi$.

Step 7. Let $\ell = \left|\left\{j \notin K; v''_j \neq v_j\right\}\right|$ and $\varepsilon' = \min\left\{\frac{1}{2}\sum_{j=k+1}^{n}\left(v''_j(z) - v_j(z)\right), \varepsilon\right\}$. Note that by definition of ε (pp. 61), for each $j \in \{k+1, \ldots, n\}$, $v_j(z) < r_j - \varepsilon$. Thus, by definition of v''_j, $v_j(z) \leq v''_j(z)$ for all $j \in \{k+1, \ldots, n\}$. Therefore, $\varepsilon' \geq 0$. In addition, if there exists $j \in \{k+1, \ldots, n\}$ such that $v''_j \neq v_j$, then $v''_j(z) > v_j(z)$. Therefore, in such a case, $\varepsilon' > 0$.

Let $v_k^{(3)} \in V_{r_k}$ be such that

$$v_k^{(3)}(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v'_k(x) + \varepsilon & \text{if } y = x, \\ v'_k(x) + \sum_{j \neq k}(v''_j(x) - v''_j(z)) + (\ell+1)\varepsilon + \varepsilon' & \text{if } y = z \\ -\delta'_k & \text{otherwise,} \end{cases}$$

where $-\delta'_k$ is a sufficiently small number under which

$$\arg\max_{y \in X\setminus\{\phi\}}\left\{\sum_{j=1}^{k-1} v''_j(y) + v_i^{(3)}(y) + \sum_{j=k+1}^{n} v_j(y)\right\} \subset \{x, z\}.$$

Let us show that $v_k^{(3)}$ is well-defined. By definition of ε, it suffices to show that $v_k^{(3)}(z) < r_k$.

Case 7-1. Consider the case with

$$v''_n(z) \geq \sum_{j=1}^{k}(v_j(x) - v_j(z)) + v_n(x) + \sum_{j=k+1}^{n-1}(v''_j(x) - v''_j(z)) - \varepsilon.$$

Then,

$$\sum_{j=k+1}^{n}(v''_j(x)-v''_j(z)) = \sum_{j=k+1}^{n} v''_j(x) - \sum_{j=k+1}^{n-1} v''_j(z) - v''_n(z)$$
$$\leq \sum_{j=k+1}^{n} v''_j(x) - \sum_{j=k+1}^{n-1} v''_j(z) - \sum_{j=1}^{k}(v_j(x)-v_j(z)) - v_n(x) - \sum_{j=k+1}^{n-1}(v''_j(x)-v''_j(z)) + \varepsilon$$
$$= \sum_{j=k+1}^{n} v''_j(x) - \sum_{j=1}^{k}(v_j(x)-v_j(z)) - v_n(x) - \sum_{j=k+1}^{n-1} v''_j(x) + \varepsilon = -\sum_{j=1}^{k}(v_j(x)-v_j(z)). \quad (2.62)$$

Thus, by (2.62), (2.50) (in pp. 61), $2k+\ell \leq n+k$, and by definitions of ε and ε',

$$v_k^{(3)}(z) = v'_k(x) + \sum_{j\neq k}(v''_j(x)-v''_j(z)) + (\ell+1)\varepsilon + \varepsilon'$$
$$= \sum_{j=1}^{k-1}(v'_j(x) - r_j + 2\varepsilon) + v'_k(x) + \sum_{j=k+1}^{n}(v''_j(x)-v''_j(z)) + (\ell+1)\varepsilon + \varepsilon'$$
$$\leq \sum_{j=1}^{k-1}(v'_j(x) - r_j + 2\varepsilon) + v'_k(x) - \sum_{j=1}^{k}(v_j(x)-v_j(z)) + (\ell+2)\varepsilon$$
$$= \sum_{j=1}^{k} v'_j(x) - \sum_{j=1}^{k} v_j(x) + v_k(z) + \sum_{j=1}^{k-1} v_j(z) - \sum_{j=1}^{k-1} r_k + (2k+\ell)\varepsilon$$
$$< v_k(z) + (2k+\ell)\varepsilon < r_k.$$

Case 7-2. Consider the case with

$$v''_n(z) < \sum_{j=1}^{k}(v_j(x)-v_j(z)) + v_n(x) + \sum_{j=k+1}^{n-1}(v''_j(x)-v''_j(z)) - \varepsilon.$$

Then, by Substep 4-1, for each $i \in \{k+1,\dots,n\}$, $v''_i(z) = r_i - \varepsilon$. Thus, by definitions of v''_j and V_{r_j},

$$v_k^{(3)}(z) = v'_k(x) + \sum_{j\neq k}(v''_j(x)-v''_j(z)) + (\ell+1)\varepsilon + \varepsilon'$$
$$\leq \sum_{j=1}^{k-1}(v'_j(x) - r_j + 2\varepsilon) + v'_k(x) + \sum_{j=k+1}^{n}(v''_j(x) - r_j + \varepsilon) + (\ell+2)\varepsilon$$
$$< v'_k(x) + (n+k+\ell)\varepsilon < r_k,$$

where the last inequality follows from $n+K+\ell \leq 2n$ and from definition of ε. Hence $v_k^{(3)}$ is well-defined.

Step 8. Let us show that $d(v''_L, v^{(3)}_k, v''_{-K}) = \phi$. It follows that

$$\begin{aligned}
&\sum_{j=1}^{k-1} v''_j(z) + v^{(3)}_k(z) + \sum_{j=k+1}^{n} v''_j(z) \\
&= \sum_{j=1}^{k-1} v''_j(z) + v'_k(x) + \sum_{j\neq k}(v''_j(x) - v''_j(z)) + (\ell+1)\varepsilon + \varepsilon' + \sum_{j=k+1}^{n} v''_j(z) \\
&= \sum_{j=1}^{k-1} v''_j(x) + v'_k(x) + (\ell+1)\varepsilon + \varepsilon' + \sum_{j=k+1}^{n} v''_j(x) \geq \sum_{j=1}^{k-1} v''_j(x) + v'_k(x) + \varepsilon + \sum_{j=k+1}^{n} v''_j(x) \\
&= \sum_{j=1}^{k-1} v''_j(x) + v^{(3)}_k(x) + \sum_{j=k+1}^{n} v''_j(x). \qquad (2.63)
\end{aligned}$$

Therefore,

$$s(v''_L, v^{(3)}_k, v''_{-K}) = \sum_{j=1}^{k-1} v''_j(z) + v^{(3)}_k(z) + \sum_{j=k+1}^{n} v''_j(z).$$

Then, by definitions of ℓ and ε, and by the second, third, and fourth equations in (2.61),

$$\begin{aligned}
s(v''_L, v^{(3)}_k, v''_{-K}) &= \sum_{j=1}^{k-1} v''_j(z) + v^{(3)}_k(z) + \sum_{j=k+1}^{n} v''_j(z) \\
&= \sum_{j=1}^{k-1} v''_j(x) + v'_k(x) + (\ell+1)\varepsilon + \varepsilon' + \sum_{j=k+1}^{n} v''_j(x) \\
&\leq \sum_{j=1}^{k-1}(v'_j(x) + \varepsilon) + v'_k(x) + \varepsilon + \varepsilon' + \sum_{j=k+1}^{n} v_j(x) \\
&\leq \sum_{j=1}^{k} v'_j(x) + (k+1)\varepsilon + \sum_{j=k+1}^{n} v_j(x) < \sum_{j=1}^{n} v_j(x) - (n+k-1)\varepsilon \\
&= \sum_{j=1}^{k-1} v''_j(z) + v''_k(z) + \sum_{j=k+1}^{n} v''_j(z) \leq s(v''_L, v''_k, v''_{-K}).
\end{aligned}$$

Hence by Lemma 3, $d(v''_L, v^{(3)}_k, v''_{-K}) = \phi$.

Step 9. Let us show that $d(v''_L, v^{(3)}_k, v_{-K}) = \phi$. If for each $j \in \{k+1, \dots, n\}$, $v''_j = v_j$, then clearly $d(v''_L, v^{(3)}_k, v_{-K}) = \phi$. Suppose that there exists $i \in \{k+1, \dots, n\}$ such that $v''_i \neq v_i$. Then, by definition of ϵ', $\epsilon' > 0$, In addition, by definitions of ε' and ℓ,

$$\sum_{j=1}^{k-1} v''_j(z) + v^{(3)}_k(z) + \sum_{j=k+1}^{n} v_j(z) < \sum_{j=1}^{k-1} v''_j(z) + v^{(3)}_k(z) + \sum_{j=k+1}^{n} v''_j(z) - 2\varepsilon'$$

$$
\begin{aligned}
&< \sum_{j=1}^{k-1} v_j''(z) + v_k^{(3)}(z) + \sum_{j=k+1}^{n} v_j''(z) - \varepsilon' \\
&= \sum_{j=1}^{k-1} v_j''(z) + v_k'(x) + \sum_{j\neq k}(v_j''(x) - v_j''(z)) + (\ell+1)\varepsilon + \varepsilon' + \sum_{j=k+1}^{n} v_j''(z) - \varepsilon' \\
&= \sum_{j=1}^{k-1} v_j''(x) + v_k'(x) + \varepsilon + \sum_{j=k+1}^{n} v_j(x) = \sum_{j=1}^{k-1} v_j''(x) + v_k^{(3)}(x) + \sum_{j=k+1}^{n} v_j(x). \qquad (2.64)
\end{aligned}
$$

Hence

$$s(v_L'', v_k^{(3)}, v_{-K}) = \sum_{j=1}^{k-1} v_j''(x) + v_k^{(3)}(x) + \sum_{j=k+1}^{n} v_j(x).$$

Moreover, by (2.64),

$$
\begin{aligned}
s(v_L'', v_k^{(3)}, v_{-K}) &= \sum_{j=1}^{k-1} v_j''(x) + v_k^{(3)}(x) + \sum_{j=k+1}^{n} v_j(x) = \sum_{j=1}^{k-1} v_j''(z) + v_k^{(3)}(z) + \sum_{j=k+1}^{n} v_j''(z) - \varepsilon' \\
&< \sum_{j=1}^{k-1} v_j''(z) + v_k^{(3)}(z) + \sum_{j=k+1}^{n} v_j''(z) \leq s(v_L'', v_k^{(3)}, v_{-K}'').
\end{aligned}
$$

Note that by (2.63) and $\varepsilon' > 0$, there exists no $w \in X \setminus \{\phi\}$ such that

$$w \in \operatorname*{arg\,max}_{y\in X\setminus\{\phi\}} \left\{ \sum_{j=1}^{k-1} v_j''(y) + v_k^{(3)}(y) + \sum_{j=k+1}^{n} v_j''(y) \right\} = \{z\},$$

and

$$w \in \operatorname*{arg\,max}_{y\in X\setminus\{\phi\}} \left\{ \sum_{j=1}^{k-1} v_j''(y) + v_k^{(3)}(y) + \sum_{j=k+1}^{n} v_j(y) \right\} = \{x\}.$$

Hence by Lemma 4, $d(v_L'', v_k^{(3)}, v_{-K}) = \phi$.

Step 10. Let $v_k^{(4)} \in V_r$ be such that

$$v_k^{(4)}(y) = \begin{cases} 0 & \text{if } y = \phi, \\ v_k'(x) + \frac{1}{2}\varepsilon & \text{if } y = x, \\ -\delta_k'' & \text{otherwise,} \end{cases}$$

where $-\delta_k''$ is a sufficiently small number under which the following two conditions are satisfied:

(a) $\operatorname*{arg\,max}_{y\in X\setminus\{\phi\}} \left\{ \sum_{j=1}^{k-1} v_j''(y) + v_k^{(4)}(y) + \sum_{j=k+1}^{n} v_j(y) \right\} = \{x\}$,

(b) $\underset{y \in X \setminus \{\phi\}}{\arg\max} \left\{ \sum_{j=1}^{k-1} v'_j(y) + v_k^{(4)}(y) + \sum_{j=k+1}^{n} v_j(y) \right\} = \{x\}$.

Then, by condition (a),

$$\begin{aligned} s(v''_L, v_k^{(4)}, v_{-K}) &= \sum_{j=1}^{k-1} v''_j(x) + v_k^{(4)}(x) + \sum_{j=k+1}^{n} v_j(x) = \sum_{j=1}^{k-1} v''_j(x) + v'_k(x) + \frac{1}{2}\varepsilon + \sum_{j=k+1}^{n} v_j(x) \\ &< \sum_{j=1}^{k-1} v''_j(x) + v'_k(x) + \varepsilon + \sum_{j=k+1}^{n} v_j(x) = \sum_{j=1}^{k-1} v''_j(x) + v_k^{(3)}(x) + \sum_{j=k+1}^{n} v_j(x) \\ &= s(v''_L, v_k^{(3)}, v_{-K}) \end{aligned}$$

Hence by Lemma 3, $d(v''_L, v_k^{(4)}, v_{-K}) = \phi$.

Step 11. Let us show that $d(v'_L, v_k^{(4)}, v_{-K}) = \phi$. Note that by condition (b)

$$\begin{aligned} s(v'_L, v_k^{(4)}, v_{-K}) &= \sum_{j=1}^{k-1} v'_j(x) + v_k^{(4)}(x) + \sum_{j=k+1}^{n} v_j(x) < \sum_{j=1}^{k-1} (v'_j(x) + \varepsilon) + v_k^{(4)}(x) + \sum_{j=k+1}^{n} v_j(x) \\ &= \sum_{j=1}^{k-1} v''_j(x) + v_k^{(4)}(x) + \sum_{j=k+1}^{n} v_j(x) = s(v''_L, v_k^{(4)}, v_{-K}). \end{aligned}$$

Then, by conditions (a) and (b) of the definition of $v_k^{(4)}$ and by the assumption of the mathematical induction, it follows that $d(v'_L, v_k^{(4)}, v_{-K}) = \phi$.

Step 12. Finally, let us show that $d(v'_L, v'_k, v_{-K}) = \phi$. Note that by the assumption (ii),

$$\begin{aligned} s(v'_L, v'_k, v_{-K}) &= \sum_{j=1}^{k-1} v'_j(x) + v'_k(x) + \sum_{j=k+1}^{n} v_j(x) < \sum_{j=1}^{k-1} v'_j(x) + v'_k(x) + \frac{1}{2}\varepsilon + \sum_{j=k+1}^{n} v_j(x) \\ &= \sum_{j=1}^{k-1} v'_j(x) + v_k^{(4)}(x) + \sum_{j=k+1}^{n} v_j(x) = s(v'_L, v_k^{(4)}, v_{-K}). \end{aligned}$$

Then, by Lemma 3, $d(v'_L, v'_k, v_{-K}) = \phi$. This completes the proof of Lemma 5. ■

Lemma 6. *Suppose that a mechanism* $(d, t) \in \mathscr{M}$ *satisfies* weak efficiency *and* strategy-proofness. *For each* $v \in \prod_{j=1}^{n} V_{r_j}$ *with* $d(v) = \phi$ *and each* $v' \in \prod_{j=1}^{n} V_{r_j}$, *if* $s(v') < s(v)$, *then* $d(v') = \phi$.

Proof. Immediately follows from Lemmas 4 and 5. ■

Lemma 7. *If a mechanism* $(d,t) \in \mathscr{M}$ *satisfies* weak efficiency *and* strategy-proofness*, then one of the following two conditions holds:*

(i) there exists $\alpha \in \mathbb{R}$ *such that for each* $v \in \prod_{j=1}^{n} V_{r_j}$,

$$d(v) \in \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\},$$

(ii) for each $v \in \prod_{j=1}^{n} V_{r_j}$, $d(v) = \phi$.

Proof. Consider the case with $\sup\Big\{ s(v) \in \mathbb{R} : v \in \prod_{j=1}^{n} V_{r_j}, d(v) = \phi \Big\} = \infty$. Let us show that Condition (ii) holds. Take any $v \in \prod_{j=1}^{n} V_{r_j}$. Then, since $s(v) \in \mathbb{R}$, there exists $v' \in \prod_{j=1}^{n} V_{r_j}$ with $d(v') = \phi$ such that $s(v) < s(v')$. Therefore, by Lemma 6, $d(v) = \phi$.

We next consider the case with $\sup\Big\{ s(v) \in \mathbb{R} : v \in \prod_{j=1}^{n} V_{r_j}, s(v) = \phi \Big\} < \infty$. Let

$$\alpha = \sup\Big\{ s(v) \in \mathbb{R} : v \in \prod_{j=1}^{n} V_{r_j}, d(v) = \phi \Big\}, \tag{2.65}$$

and take any $v \in \prod_{j=1}^{n} V_{r_j}$. Let us show that $d(v) \in \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}$.

First, we consider the case with $\phi \notin \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}$. Then, $s(v) > \alpha$. We shall show that $d(v) \neq \phi$. If $d(v) = \phi$, then $d(v) = \phi$ and $s(v) > \alpha$, a contradiction to (2.65). Therefore, $d(v) \neq \phi$. Then, by weak efficiency and $s(v) > \alpha$,

$$d(v) \in \arg\max_{y \in X \setminus \{\phi\}} \Big\{ \sum_{j=1}^{n} v_j(y) \Big\} = \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}.$$

Second, we consider the case with $\{\phi\} = \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}$. Then,

$$\alpha = \sum_{j=1}^{n} v_j(\phi) + \alpha > s(v).$$

Therefore, by (2.65), there exists $v' \in \prod_{j=1}^{n} V_{r_j}$ such that $s(v') = \phi$ and $s(v') > s(v)$. Thus, by Lemma 6,

$$d(v) = \phi \in \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}.$$

Finally, we consider the case where $\phi \in \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}$ and $\{\phi\} \neq \arg\max_{y \in X} \Big\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \Big\}$. If $d(v) = \phi$, then the desired condition holds. Suppose that $d(v) \neq \phi$. Then, $s(v) > \alpha$. Otherwise, there exists $v' \in \prod_{j=1}^n V_{r_j}$ such that $s(v') = \phi$ and $s(v') > s(v)$, and hence $d(v) = \phi$, a contradiction. Thus, by weak efficiency,

$$d(v) \in \arg\max_{y \in X \setminus \{\phi\}} \Big\{ \sum_{j=1}^n v_j(y) \Big\} \subset \arg\max_{y \in X} \Big\{ \sum_{j=1}^n v_j(y) + \alpha \mathbf{1}_\phi(y) \Big\}.$$

■

Proof of Statement (ii): Take any mediation mechanism $(d^m, t^m) \in \mathscr{M}$ and any mechanism $(d, t) \in \mathscr{M}$ that satisfies *weak efficiency*, *strategy-proofness*, *individual rationality* and *feasibility*. Then, one of the two conditions in Lemma 7 holds for (d, t). Let us show that for each $i \in N$ and each $v \in \prod_{j=1}^n V_{r_j}$,

$$v_i\big(d^m(v)\big) + t_i^m(v) \geq v_i\big(d(v)\big) + t_i(v). \tag{2.66}$$

Consider the case with $d(v) = \phi$ for all $v \in \prod_{j=1}^n V_{r_j}$. Then by *individual rationality*, *feasibility*, $t_i(v) = 0$ for all $i \in N$ and all $v \in \prod_{j=1}^n V_{r_j}$. Thus, (2.66) holds from the fact that (d^m, t^m) satisfies individual rationality.

Next, consider the case where there exists $\alpha \in \mathbb{R}$ such that for each $v \in \prod_{j=1}^n V_{r_j}$,

$$d(v) \in \arg\max_{y \in X} \Big\{ \sum_{j=1}^n v_j(y) + \alpha \mathbf{1}_\phi(y) \Big\}. \tag{2.67}$$

Then, by Lemma 1 in Appendix B, for each $i \in N$, there exists $h_i : \prod_{j \neq i} V_{r_j} \to \mathbb{R}$ such that

$$t_i(v) = \sum_{j \neq i} v_j\big(d(v)\big) + \alpha \mathbf{1}_\phi\big(d(v)\big) + h_i\big(v_{-i}\big) \text{ for all } v \in \prod_{j=1}^n V_{r_j}. \tag{2.68}$$

Step 1. Let us show that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} V_{r_j}$, $h_i\big(v_{-i}\big) \geq -\alpha$. Take any $i \in N$ and any $v_{-i} \in \prod_{j \neq i} V_{r_j}$. Suppose, by contradiction, that

$$h_i\big(v_{-i}\big) < -\alpha.$$

Let $v_i \in V_{r_i}$ be such that

$$v_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ \min\Big\{r_i, \alpha - \sum_{j \neq i} v_j(y)\Big\} - 1 & \text{if } y \in X \setminus \{\phi\}. \end{cases}$$

Then, for each $x \in X \setminus \{\phi\}$,

$$\sum_{j=1}^{n} v_j(x) \leq \alpha - 1 < \alpha.$$

Hence by (2.67), $d(v) = \phi$. However, by (2.68),

$$v_i\big(d(v)\big) + t_i(v) = \sum_{j=1}^{n} v_j(\phi) + \alpha + h_i(v_{-i}) = \alpha + h_i(v_{-i}) < 0.$$

This is a contradiction to individual rationality.

Step 2. Let us show that for each $i \in N$ and each $v_{-i} \in \prod_{j \neq i} V_{r_j}$, $h_i\big(v_{-i}\big) = -\alpha$. Take any $i \in N$ and any $v_{-i} \in \prod_{j \neq i} V_{r_j}$. Note that by Step 1, $h_i\big(v_{-i}\big) \geq -\alpha$. Suppose, by contradiction, that

$$h_i\big(v_{-i}\big) > -\alpha.$$

Let $v_i \in V_{r_i}$ be such that

$$v_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ \min\Big\{r_i, \alpha - \sum_{j \neq i} v_j(y)\Big\} - 1 & \text{if } x \in X \setminus \{\phi\}. \end{cases}$$

Then, for each $x \in X \setminus \{\phi\}$,

$$\sum_{j=1}^{n} v_j(x) < \alpha.$$

Hence by (2.67), $d(v) = \phi$. Then, by (2.68) and Step 1,

$$\sum_{j=1}^{n} t_j(v) = \alpha + h_i(v_{-i}) + \sum_{j \neq i} \big(\alpha + h_j(v_{-j})\big) \geq \alpha + h_i(v_{-i}) > 0,$$

a contradiction to feasibility.

Step 3. Let us show that $\alpha \geq \frac{(n-1)}{n} \sum_{j=1}^{n} r_j$. Suppose, by contradiction, that $\alpha < \frac{(n-1)}{n} \sum_{j=1}^{n} r_j$. Take some $x \in X \setminus \{\phi\}$. For each $i \in N$, let $v_i \in V_{r_i}$ be such that

$$v_i(y) = \begin{cases} 0 & \text{if } y = \phi, \\ r_i - \frac{1}{n}\big(\frac{n-1}{n} \sum_{j=1} r_j - \alpha\big) & \text{otherwise.} \end{cases}$$

Then, for each $x \in X \setminus \{\phi\}$,

$$\sum_{j=1}^{n} v_j(x) = \sum_{j=1}^{n}\Big(r_j - \frac{1}{n}\big(\frac{n-1}{n} \sum_{k=1} r_k - \alpha\big)\Big)$$

$$= \sum_{j=1}^{n} r_j - \frac{n-1}{n} \sum_{j=1}^{n} r_j + \alpha = \frac{1}{n} \sum_{j=1}^{n} r_j + \alpha > \alpha. \tag{2.69}$$

Therefore, by (2.67), $d(v_i, v_{-i}) \neq \phi$. Then, by Step 2, (2.68), (2.69), and by $\alpha < \frac{(n-1)}{n} \sum_{j=1}^{n} r_j$,

$$\sum_{j=1}^{n} t_j(v) = (n-1) \sum_{j=1}^{n} v_j(d(v)) + \sum_{j=1}^{n} h_j(v_{-j}) = (n-1) \sum_{j=1}^{n} v_j(d(v)) - n\alpha$$
$$= \frac{(n-1)}{n} \sum_{j=1}^{n} r_j + (n-1)\alpha - n\alpha = \frac{(n-1)}{n} \sum_{j=1}^{n} r_j - \alpha > 0,$$

a contradiction to feasibility.

Step 4. Let us show that for each $i \in N$ and each $v \in \prod_{j=1}^{n} V_{r_j}$, $v_i\big(d^m(v)\big) + t_i^m(v) \geq v_i\big(d(v)\big) + t_i(v)$. Note that for each $\alpha \geq \frac{(n-1)}{n} \sum_{j=1}^{n} r_j$,

$$\max_{y \in X} \left\{ \sum_{j=1}^{n} v_j(y) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(y) \right\} - \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \geq \max_{y \in X} \left\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \right\} - \alpha.$$

Then, by (2.67), (2.68), Steps 2 and 3, and by definition of the mediation mechanisms, for each $i \in N$ and each $v \in \prod_{j=1}^{n} V_{r_j}$,

$$v_i\big(d^m(v)\big) + t_i^m(v) = \max_{y \in X} \left\{ \sum_{j=1}^{n} v_j(y) + \frac{(n-1)}{n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(y) \right\} - \frac{(n-1)}{n} \sum_{j=1}^{n} r_j$$
$$\geq \max_{y \in X} \left\{ \sum_{j=1}^{n} v_j(y) + \alpha \mathbf{1}_{\phi}(y) \right\} - \alpha = v_i\big(d(v)\big) + t_i(v).$$

□

Appendix E: Example of a mechanism satisfying the four axioms

Let $(d', t') \in \mathscr{M}\big(\prod_{j=1}^{n} V_{r_j}\big)$ be such that for each $v \in \prod_{j=1}^{n} V_{r_i}$ and each $i \in N$,

(i) $d'(v) \in \arg\max_{x \in X} \Big\{ \sum_{j=1}^{n} v_j(x) + \frac{(2n-1)}{2n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(x) \Big\}$,

(ii) $t_i'(v) = \sum_{j \neq i} \big(v_j(d'(v)) - v_j(\phi)\big) + \frac{(2n-1)}{2n} \sum_{j=1}^{n} r_j \cdot \mathbf{1}_{\phi}(d'(v)) - \frac{(2n-1)}{2n} \sum_{j=1}^{n} r_j$,

with an arbitrary tie-breaking rule. Then, (d', t') satisfies weak efficiency, strategy-proofness, individual rationality and feasibility. The form of (d', t') resembles those of the mediation mechanisms because (d', t') belongs to a class of mechanisms characterized in Lemma 7 of Appendix C by using weak efficiency and strategy-proofness.

Chapter 3

Mechanism Design with a Guess

3.1 Introduction

Even after social objectives are embedded in a social choice rule, there still remains a problem of how to implement that rule. If the social planner fully knows individual preferences, then any social choice rule is trivially implemented because the planner also knows the rule's outcome. However, in many real-life situations, the planner does not have full information about individual preferences. On the other hand, if the planner does not know individual preferences at all, then the problem of implementation arises. Mechanism design theory has focused on this problem and succeeded in designing mechanisms that implement "attractive" social choice rules (e.g., Maskin 1999, Groves and Ledyard 1977, Hurwicz 1979, Schmeidler 1980, Vickrey 1961, Clarke 1971, Groves 1973). However, plenty of impossibility results about the implementability of social choice rules have also been recognized since the seminal works by Hurwicz (1972), Gibbard (1973), Satterthwaite (1975), and Muller and Satterthwaite (1977). Therefore, without information about individual preferences, the planner may face difficulty. This fact motivates us to consider a "middle" situation in which the planner "partially" knows individual preferences.

Any study of mechanism design theory associates a player with a preference space. For example, in exchange economies, a player's preferences are restricted to be *monotone*, *convex*, *continuous*, etc. This means that in a model, the planner is assumed to know certain information about individual preferences without verification. Alternatively, we can consider that the planner has some guesses of the individual preference types. In this chapter, we make this point explicit and analyze the possibility of using the planner's guess for implementation. In particular, we introduce a new idea of implementation using the planner's "guess" of individual preferences and apply this concept to two quasi-linear mechanism design problems: a public decision problem and an auction problem with homogeneous goods.

We consider the following process of implementation of social choice rules, illustrated by Figure 1. First, the planner is equipped with a family $\{G_\lambda\}_{\lambda\in\Lambda}$ consisting of subsets of possible preference profiles $\mathscr{R}_I \equiv \mathscr{R}_1 \times \cdots \times \mathscr{R}_n$ such that $\bigcup_{\lambda\in\Lambda} G_\lambda = \mathscr{R}_I$. Second, the planner selects a subset $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ before running a mechanism. The planner does not know which preference profile $\succsim = (\succsim_1, \dots, \succsim_n) \in \mathscr{R}_I$ is true, but he selects G_λ hoping that the

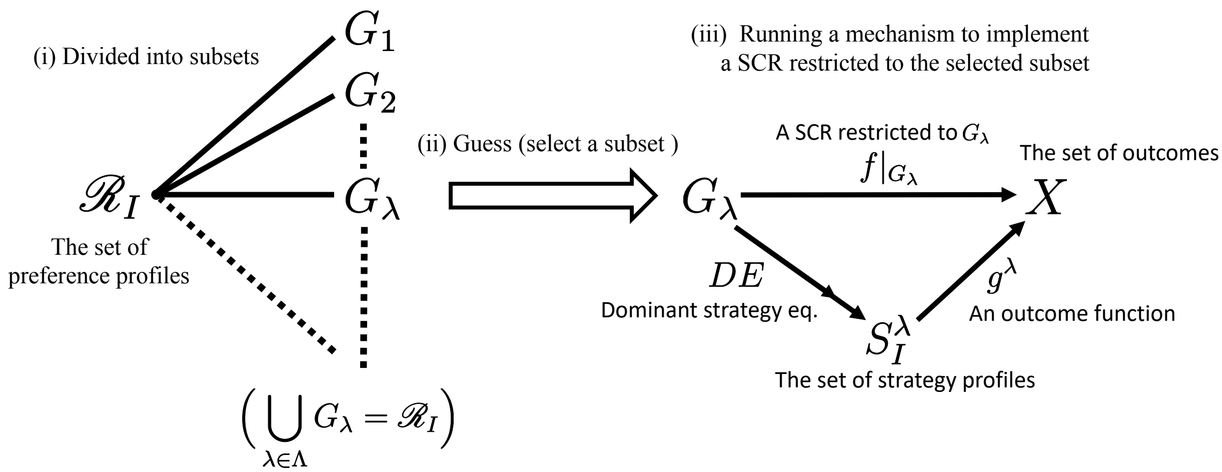


Figure 1: Mechanism design with a guess

true profile belongs to it, i.e., $\succsim \in G_\lambda$. In other words, the planner "guesses" $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ to which the true preference profile belongs. In this sense, each element $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ is called a *guess* under $\{G_\lambda\}_{\lambda\in\Lambda}$, and the family $\{G_\lambda\}_{\lambda\in\Lambda}$ is called *a list of guesses*. Since the planner does not know which profile is true, the true profile may or may not belong to the planner's particular guess G_λ. A guess G_λ is *correct* if the true profile belongs to it. Otherwise it is *incorrect*.[*13] Finally, the planner announces a mechanism M^λ (depending on the guess), and individuals play a game form determined by the announced mechanism. A social choice rule is *G-implementable* with respect to a list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$ if for each $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$, there exists a mechanism M^λ that dominant strategy implements the social choice rule restricted to G_λ. If a social choice rule is G-implementable, then its outcome can be a dominant strategy outcome as long as the planner's guess is correct.

A simple example of a list of guesses is the family, $\big\{\{\succsim\}\big\}_{\succsim\in\mathscr{R}_I}$, consisting of all singletons of the set of preference profiles. This list of guesses is called the *finest list*. Under the finest list, the planner needs full information about individual preferences for his guess to be correct. Obviously, any social choice rule is G-implementable with respect to the finest list. Another simple example of a list of guesses is the family, $\{\mathscr{R}_I\}$, consisting of only the set of preference profiles itself. This list of guesses is called the *trivial list*. Under the trivial list, the planner does not need to know anything about the true profile for his guess to be correct. Since

[*13]We can also interpret each $\{G_\lambda\}_{\lambda\in\Lambda}$ as the planner's "structure of knowledge" about individual preferences rather than a "list of guesses." That is, we can also consider a situation where given $\{G_\lambda\}_{\lambda\in\Lambda}$, the planner always knows $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ to which the true preference profile belongs. Nevertheless, we use the terminology "list of guesses" because one of our purposes is to design $\{G_\lambda\}_{\lambda\in\Lambda}$, which requires "minimal information" about individual preferences of the planner. It is uncommon to say that we design the planner's structure of knowledge.

$\mathscr{R}_I$ is the unique element of the trivial list, *G-implementability with respect to the trivial list* is equivalent to the standard *dominant strategy implementability*. Therefore, whenever we face an impossibility of implementation of social choice rule in dominant strategies, we must consider a non-trivial list to G-implement that rule. In each design problem, we search for a list of guesses that requires “minimal" information about individual preferences for the planner to guess correctly. A boundary between the possibility and impossibility of implementing social choice rules can be found by identifying such a “minimal" list of guesses.

Our purpose in the public decision problem is to overcome the following well-known impossibility (Subsection 3.9; Jackson 2003): there exists no social choice rule that satisfies *efficiency*, *individual rationality*, *feasibility*, and *dominant strategy implementability*, or equivalently *G-implementability with respect to the trivial list*.[*14] This impossibility forces us to consider a list of guesses other than the trivial list. In this chapter, we consider a list of guesses dividing the set of preference profiles into intervals of the maximal social surplus – the maximum of the total valuation gain from the status-quo. Under our list of guesses, the planner only needs to guess an interval to which the maximal social surplus in the true state belongs. Thus, our list of guesses requires much less information than the trivial list. Our first main theorem states that (i) a social choice rule is *efficient*, *individually rational*, *feasible*, and *G-implementable with respect to our list* if and only if it is a *mediation rule*, and that (ii) our list of guesses requires minimal information about individual preferences among a class of lists of guesses such that some G-implementable social choice rule satisfying the set of properties exists. A mediation rule is defined with our interval-based list of guesses and is a kind of patchwork consisting of Vickrey-Clarke-Groves rules defined on different intervals.

Our purpose in the auction problem is to design a revenue maximizing auction rule. From the seminal work by Myerson (1981), it is well-known that, in the single-object case, second price auctions with a suitable reserve price maximize the revenue. However, there are at least two obstacles to apply Myerson’s result to real-life auctions: (1) the revenue-maximizing auction rules require knowledge of prior distributions of bidders’ valuations; (2) generalizing his result to the multiple-objects case is difficult because of complicated prior-based maximizations (Armstrong 1996, 2000). In contrast to Myerson’s approach,

[*14]The logic behind this impossibility is as follows: Green and Laffont (1977) and Holmström (1979) show that *Groves mechanisms* (Groves 1973) are the only mechanisms that satisfy *efficiency* and *dominant strategy implementability*. Subsequently, Moulin (1986) shows that Clarke’s (1971) *pivotal mechanisms* are the only Groves mechanisms that satisfy *feasibility* and *welfare lower boundedness*, which is a weaker condition than *individual rationality*. However, since the pivotal mechanisms do not satisfy *individual rationality*, the above impossibility holds.

our approach does not require any knowledge of the prior, but does require the planner's (or auctioneer's) guess be correct in an ex-post sense. In addition, to avoid complicated prior-based maximizations, we impose the following three axioms on auction rules[*15]: (i) *efficiency*, (ii) *(ex-post) individual rationality*, and (iii) *G-implementability*. We design an ex-post revenue maximizing auction rule among the class of auction rules satisfying these three axioms.[*16]

Since the seminal work by Vickrey (1961), it has been recognized that Vickrey auctions generate higher revenue than all auction rules satisfying *efficiency*, *individual rationality*, and *dominant strategy implementability* (Holmström 1979). However, as pointed out by Ausubel and Cramton (2004) and by even Vickrey himself, Vickrey auctions may yield low revenues if competition is weak and the bidders are asymmetric. An effective way to avoid such low revenues is to focus on *G-implementability with respect to a non-trivial list*.

In the single-object case, we consider a list of guesses consisting of only two types of guesses. One includes all valuation profiles such that the highest valuation among all bidders exceeds a given positive number $r \in \mathbb{R}_{++}$. The other includes profiles where the highest valuation is below r. Under this list of guesses, the auctioneer only needs to guess whether the highest valuation in the true state is above the given positive number r or not. This situation seems to be often observed in many real-life auctions. In fact, in a real-life auction with a reserve price, the true highest valuation would be believed as above the reserve price. Otherwise, no object would be sold, and the auctioneer only suffers from the cost for holding the auction.

In the multiple ($L \geq 2$) objects case, we consider a list of guesses generalizing an idea of the single-object case. Given an L-dimensional vector $\boldsymbol{r} = (r_1, r_2, \ldots, r_L) \in \mathbb{R}^L_+$, the auctioneer guesses whether it is the case that each k-th highest valuation in the true state is above the k-th coordinate r_k of the given vector $\boldsymbol{r}$ or not. Our second main theorem states that (i) *Vickrey auctions with a cutoff reserve price* are ex-post revenue maximizing among all auction rules satisfying *G-implementability with respect to our list* and the other two axioms, and that (ii) our lists of guesses require minimal information about bidders' valuations among a class of lists of guesses such that some auction rule satisfying the three properties generates at least

[*15]Kazumura, Mishra, and Serizawa (2017) takes a similar approach without the assumption of quasi-linearity.

[*16]One may consider that imposing *efficiency* is unsuitable for our purpose of finding a revenue-maximizing auction rule. Nevertheless, Ausubel and Cramton (1999) show that when the seller cannot prevent resale among bidders after the auction, and the seller cannot commit to not sell the withheld objects after the auction, it is optimal to assign objects efficiently. Therefore, *efficiency* seems to be an appropriate requirement in many real-life auction problems.

the same revenue as our auction rules. A *Vickrey auction with a cutoff reserve price* imposes a reserve price if each k-th highest valuation in the true state is above the k-th coordinate r_k of the given vector $\boldsymbol{r}$. Otherwise it chooses the same outcome as a Vickrey auction. Our result justifies the use of reserve pricing when the auctioneer can correctly guess the first L-th highest valuations.

Finally, we note that our approach of using the planner's guesses differs from usual *domain restriction approaches* (e.g., Black 1948, Suijs 1996, Mitra 2001, Pápai 2003, Ohseto 2004). In a usual domain restriction approach, social choice rules are defined on a restricted domain, and the implementability of these rules is analyzed. Therefore, if the true preference profile is outside the restricted domain, then the result in this approach says nothing. However, in our "guess approach", social choice rules are defined on the full domain. Thus, our results about G-implementability of social choice rules are meaningful whatever the true preference profile is.

The rest of this chapter is organized as follows. Section 3.2 introduces our model. Section 3.3 states our main result about the public decision problem, and Section 3.4 states that about the auction problem. Section 3.5 offers some discussion, and Section 3.6 gives concluding remarks. Omitted proofs are relegated to Appendix.

3.2 The model

3.2.1 Preliminaries

Let $I = \{1, 2, \ldots, n\}$ ($n \geq 2$) be the finite set of individuals, and A the finite set of *alternatives*. An *outcome* is a pair $(a, p) \in A \times \mathbb{R}^n$, where $p \in \mathbb{R}^n$ is a vector of monetary transfers. A *valuation function* is a function $v_i : A \rightarrow \mathbb{R}$. Individual i's *quasi-linear utility* is $u(a, p; v_i) = v_i(a) - p_i$. Let V_i be the set of possible valuation functions for individual i. A *valuation profile* is an n-tuple of valuation functions $v = (v_1, \ldots, v_n) \in V_I$, where $V_I \equiv V_1 \times \cdots \times V_n$ is the set of valuation profiles. For each $v \in V_I$ and each $I' \subset I$, let $v_{I'} \equiv \{v_j\}_{j \in I'}$ and $v_{-I'} \equiv \{v_j\}_{j \in I \setminus I'}$.

We consider two classes of design problems. In a public decision problem, the set of alternatives contains the *status quo* $\phi \in A$. The set of possible valuation functions V_i consists of all the valuation functions on A. When we consider this public decision problem, we write V_P instead of V_i for each $i \in I$. For simplicity, we normalize each $v_i \in V_P$ as $v_i(\phi) = 0$.[*17]

In an auction with L homogeneous goods, the set of alternatives A consists of all the *assignment vectors*; $a \equiv (a_1, a_2, \ldots, a_n) \in \{0, 1, \ldots, L\}^n$ such that $\sum_{j=1}^n a_j \leq L$, where a_i is

[*17] All of our results hold without this assumption.

the number of objects individual i obtains. A *marginal valuation vector* is an L-dimensional vector $(v_{i1}, v_{i2}, \ldots, v_{iL}) \in \mathbb{R}^L_+$ such that

$$v_{i1} \geq v_{i2} \geq \cdots \geq v_{iL}.$$

The set of possible valuation functions V_i consists of all the valuation functions $v_i : A \to \mathbb{R}$ such that there exists a marginal valuation vector $(v_{i1}, v_{i2}, \ldots, v_{iL})$ for which

$$v_i(a) = \sum_{\ell=1}^{a_i} v_{i\ell} \quad \text{for all } a \in A. \tag{3.70}$$

When we consider this auction problem, we write V_A instead of V_i for all $i \in I$. Since each valuation function in V_A is identified with a marginal valuation vector, we regard V_A as the set of marginal valuation vectors with a typical element $v_i = (v_{i1}, v_{i2}, \ldots, v_{iL}) \in V_A$. Let

$$\mathbf{0}_L \equiv (\underbrace{0, 0, \ldots, 0}_{L \text{ times}}) \in V_A,$$

$$\mathbf{0} \equiv (\underbrace{\mathbf{0}_L, \mathbf{0}_L, \ldots, \mathbf{0}_L}_{n \text{ times}}) \in V_A^n.$$

For each $v \in V_A^n$, denote by $v[k]$ and $v_{-i}[k]$ the k-th highest real numbers among v and v_{-i}, respectively.

A *social choice rule* is a function $f : V_I \to A \times \mathbb{R}^n$ that maps each valuation profile $v \in V_I$ to an outcome $f(v) \equiv (a(v), p_1(v), \ldots, p_n(v)) \in A \times \mathbb{R}^n$. We introduce three axioms on social choice rules.

Efficiency. For each $v \in V_I$, $a(v) \in \arg\max_{b \in A} \left\{ \sum_{j=1}^n v_j(b) \right\}$.

Individual Rationality. For each $i \in I$ and each $v \in V_I$, $u\big(f(v);\ v_i\big) \geq 0$.

Feasibility. For each $v \in V_I$, $\sum_{j=1}^n p_j(v) \geq 0$.

For each $i \in I$, let S_i be the strategy space of individual i, and let $S_I \equiv S_1 \times \cdots \times S_n$. A *strategy profile* is an n-tuple of strategies $s = (s_1, \ldots, s_n) \in S_I$. A *type-strategy* $\boldsymbol{s}_i(\cdot)$ assigns a strategy $\boldsymbol{s}_i(v_i) \in S_i$ to each valuation function (type) $v_i \in V_i$ of individual i.[*18] A *type-strategy profile* is an n-tuple of type-strategies $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$.

[*18] Type-strategies are employed to preserve the analogy with the famous *revelation principle* for strategy-proof social choice rules (Dasgupta, Hammond, and Maskin 1979) in our definition of G-implementability. See Definition 2 in Subsection 3.2.2 and Appendix B.

An *outcome function* is a function $g : S_I \to A \times \mathbb{R}^n$ that maps each strategy profile $s \in S_I$ to an outcome $g(s) \in A \times \mathbb{R}^n$. A *mechanism* is a pair $M = (S_I, g)$ of strategy spaces and an outcome function. A mechanism $M = (S_I, g)$ is a *revelation mechanism* if $S_I = V_I$. Under a revelation mechanism (V_I, g), each individual only needs to report a valuation function possible for him. Given a mechanism $M = (S_I, g)$ and given a valuation function $v_i \in V_i$, a strategy $s_i \in S_i$ is a *dominant strategy* at (M, v_i) if for each $s_{-i} \in \prod_{j \neq i} S_j$ and each $s_i' \in S_i$,

$$u\left(g(s_i, s_{-i}); v_i\right) \geq u\left(g(s_i', s_{-i}); v_i\right).$$

The following definition summaries the standard notion of *dominant strategy implementability*.

Definition 1. A mechanism and a type-strategy profile $(M, \boldsymbol{s})$ *implements f in dominant strategies* if for each $v \in V_I$,

(i) $\boldsymbol{s}_i(v_i)$ is a dominant strategy at (M, v_i) for all $i \in I$,

(ii) $f(v) = g\Big((\boldsymbol{s}_i(v_i))_{i \in I} \Big)$.

A social choice rule f is *dominant strategy implementable* if there exists $(M, \boldsymbol{s})$ that implements f in dominant strategies.

3.2.2 Mechanism Design with a Guess

We introduce a concept of implementation that uses the planner's "guess" of individual valuations. A family $\{G_\lambda\}_{\lambda \in \Lambda}$ of subsets of V_I is a *list of guesses* if $\bigcup_{\lambda \in \Lambda} G_\lambda = V_I$. Given a list of guesses $\{G_\lambda\}_{\lambda \in \Lambda}$, the planner "guesses" to which $G_\lambda \in \{G_\lambda\}_{\lambda \in \Lambda}$ the true valuation profile belongs. In this sense, we call each $G_\lambda \in \{G_\lambda\}_{\lambda \in \Lambda}$ a *guess* under a list $\{G_\lambda\}_{\lambda \in \Lambda}$. A guess $G_\lambda \in \{G_\lambda\}_{\lambda \in \Lambda}$ is *correct* for a valuation profile $v \in V_I$ if $v \in G_\lambda$, otherwise, the guess G_λ is *incorrect* for v.

Example 1. The followings are examples of lists of guesses.

- $\{V_I\}$ is a list of guesses, and called the *trivial list*. Under the trivial list, the planner needs no information about individual valuations for his guess to be correct.
- $\{\{v\}\}_{v \in V_I}$ is a list of guesses, and called the *finest list*. Under the finest list, the planner needs the full information about individual valuations for his guess to be correct.

- Consider the auction problem $V_I = V_A^n$. Let $\overline{G} = \{v \in V_A^n : v[1] \geq 100\}$ and $\underline{G} = \{v \in V_A^n : v[1] < 100\}$. Then, $\{\overline{G}, \underline{G}\}$ is a list of guesses. Under $\{\overline{G}, \underline{G}\}$, the planner only needs to know whether the highest valuation $v[1]$ is greater than 100 or not.

Given a list of guesses G_λ, our process proceeds as follows: First, the planner determines his guess G_λ from a given list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$. The guess G_λ may or may not be correct for the true valuation profile, but the planner tries to do so. Second, the planner announces a mechanism (depending on his guess), and individuals play a game form determined by the mechanism. We say that a social choice rule is *G-implementable* if its outcome becomes a dominant strategy outcome of a mechanism whenever the planner's guess is correct.[*19]

Definition 2. Given a list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$, a set of mechanisms and type-strategy profiles $\{(M^\lambda, \boldsymbol{s}^\lambda)\}_{\lambda\in\Lambda}$ *G-implements f with respect to a list of guesses* $\{G_\lambda\}_{\lambda\in\Lambda}$ if for each $\lambda \in \Lambda$ and each $v \in G_\lambda$,

(i) $\boldsymbol{s}^\lambda(v_i)$ is a dominant strategy at (M^λ, v_i) for all $i \in I$,

(ii) $f(v) = g^\lambda\left(\left(\boldsymbol{s}_i^\lambda(v_i)\right)_{i\in I}\right)$.

A set of mechanisms $\{M^\lambda\}_{\lambda\in\Lambda}$ *G-implements f with respect to a list of guesses* $\{G_\lambda\}_{\lambda\in\Lambda}$ if there exists $\{\boldsymbol{s}^\lambda\}_{\lambda\in\Lambda}$ such that $\{(M^\lambda, \boldsymbol{s}^\lambda)\}_{\lambda\in\Lambda}$ G-implements f with respect to a list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$. A social choice rule f is *G-implementable with respect to* $\{G_\lambda\}_{\lambda\in\Lambda}$ if there exists $\{M^\lambda\}_{\lambda\in\Lambda}$ that G-implements f with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$.[*20]

To understand the meaning of G-implementation, consider the case where the true valuation profile is $v \in V_I$. Suppose that $\{(M^\lambda, \boldsymbol{s}^\lambda)\}_{\lambda\in\Lambda}$ G-implements f with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$, and that a guess $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ is correct for v, i.e., $v \in G_\lambda$. Then, using a mechanism $M^\lambda \in \{M^\lambda\}_{\lambda\in\Lambda}$, whose index is the same as that of G_λ, the equilibrium outcome of $(M^\lambda, \boldsymbol{s}^\lambda)$ at v becomes $f(v)$ by definition of G-implementation. Therefore, whenever the planner's guess is correct, the outcome of the social choice rule f can be an equilibrium outcome by using a corresponding mechanism to the correct guess. This is the precise meaning of G-implementation. Once the planner knows that $\{(M^\lambda, \boldsymbol{s}^\lambda)\}_{\lambda\in\Lambda}$ G-implements f with respect

[*19]Since each $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ may not be a Cartesian product of subsets of V_i, we need to consider indirect mechanisms. In fact, we cannot define *strategy-proofness* for social choice rules restricted to G_λ.

[*20]Note that our interest is in the weak implementation. Nevertheless, all the results in Section 3.3 holds even if we focus on the full implementation.

to $\{G_\lambda\}_{\lambda\in\Lambda}$, he is only required to determine his guess G_λ and announce a mechanism with an index the same as G_λ.

The following two facts are the starting point of this study.

Fact 1. For each social choice rule f, the following statements are equivalent:

(i) f is dominant strategy implementable,

(ii) f is G-implementable with respect to the trivial list $\{V_I\}$.

Fact 2. Any social choice rule f is G-implementable with respect to the finest list $\{\{v\}\}_{v\in V_I}$.

In Sections 3.3 and 3.4, we introduce two impossibility results about dominant strategy implementability of social choice rules. If we face such impossibilities, Fact 1 forces us to consider a list of guesses other than the trivial list. On the other hand, from Fact 2, we know that any social choice rule f is G-implementable with respect to the finest list $\{\{v\}\}_{v\in V_I}$. However, under the finest list, the planner needs full information about individual valuations, which is quite difficult to obtain in many real-life situations. Therefore, we search for a *non-trivial list* that requiring less information than the *finest list*.

Given two lists of guesses $\{H_\kappa\}_{\kappa\in K}$ and $\{G_\lambda\}_{\lambda\in\Lambda}$, we denote $\{H_\kappa\}_{\kappa\in K} \subset \{G_\lambda\}_{\lambda\in\Lambda}$ if for each $\kappa \in K$, there exists $\lambda(\kappa) \in \Lambda$ such that $H_\kappa \subset G_{\lambda(\kappa)}$, and denote $\{H_\kappa\}_{\kappa\in K} \subsetneq \{G_\lambda\}_{\lambda\in\Lambda}$ if $H_{\kappa'} \subsetneq G_{\lambda(\kappa')}$ holds for some $\kappa' \in K$. If $\{H_\kappa\}_{\kappa\in K} \subsetneq \{G_\lambda\}_{\lambda\in\Lambda}$, then the planner's guess is easier under $\{G_\lambda\}_{\lambda\in\Lambda}$ rather than under $\{H_\kappa\}_{\kappa\in K}$. This is because given a valuation profile $v \in V_I$, if the planner knows that $v \in H_\kappa$, then he automatically knows that $v \in G_{\lambda(\kappa)}$. However, even if the planner knows that $v \in G_{\lambda(\kappa)}$, it may not follow that $v \in H_\kappa$. Hence he may not know which guess is correct under $\{H_\kappa\}_{\kappa\in K}$. Therefore, a guess under $\{G_\lambda\}_{\lambda\in\Lambda}$ requires less information than $\{H_\kappa\}_{\kappa\in K}$ about individual valuations when $\{H_\kappa\}_{\kappa\in K} \subsetneq \{G_\lambda\}_{\lambda\in\Lambda}$. One of our purposes in Sections 3.3 and 3.4 is to find a list of guesses requiring minimal information about individual valuations among a class of lists of guesses.

Definition 3. A list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$ *requires minimal information about individual valuations among a class of lists of guesses* if there exits no list of guesses $\{H_\kappa\}_{\kappa\in K}$ in this class such that $\{G_\lambda\}_{\lambda\in\Lambda} \subsetneq \{H_\kappa\}_{\kappa\in K}$.

We have admitted a list of guesses that is not a partition of V_I, but instead covers it. This is because there would be a case where a non-partitioned list of guesses requires minimal information about individual valuations among a class of list of guesses. In fact, in Sections 3.3 and 3.4, we provide non-partitioned lists of guesses requiring minimal informations about individual valuations.

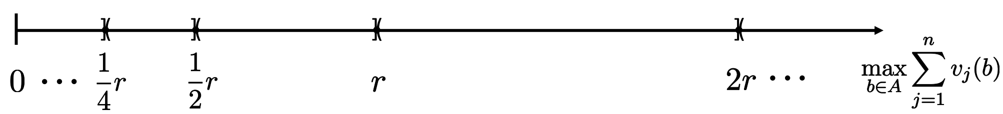


Figure 2: Intervals of the maximal social surplus ($n = 2$)

3.3 Public Decision Problem

In this section, we consider the public decision problem where $V_I = V_P^n$. The purpose of this section is to overcome the following impossibility.

Impossibility Result (Green and Laffont 1977). *There exists no social choice rule that is efficient, individually rational, feasible, and dominant strategy implementable.*

We introduce an important class of lists of guesses that divide V_P^n into intervals of the maximal social surplus. Given a valuation profile $v \in V_P^n$, *the maximal social surplus* at v is defined by $\max_{b \in A} \sum_{j=1}^{n} \big(v_j(b) - v_j(\phi)\big)$. Since we have normalized each $v_i \in V_P^n$ as $v_i(\phi) = 0$, we can rewrite this by $\max_{b \in A} \sum_{j=1}^{n} v_j(b)$. Note that by the same reason, the maximal social surplus is always non-negative since $\phi \in A$. For each positive number $r \in \mathbb{R}_{++}$ and each integer $\ell \in \mathbb{Z}$, let

$$G_\ell^r = \left\{ v \in V_P^n : \max_{b \in A} \sum_{j=1}^{n} v_j(b) = 0 \;\text{ or }\; r\left(\frac{n}{n-1}\right)^{\ell-1} < \max_{b \in A} \sum_{j=1}^{n} v_j(b) \leq r\left(\frac{n}{n-1}\right)^{\ell} \right\}.$$

Remember that n denotes the number of individuals. For each $r \in \mathbb{R}_{++}$ and each $\ell \in \mathbb{Z}$, G_ℓ^r is the set of valuation profiles whose maximal social surpluses are zero or in the half-open interval $\big(r(\frac{n}{n-1})^{\ell-1}, r(\frac{n}{n-1})^{\ell}\big]$. Figure 2 illustrates such intervals when there are only two individuals, i.e., $n = 2$. One can easily check that for each positive number $r \in \mathbb{R}_{++}$, $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$ is a list of guesses, i.e., $\bigcup_{\ell \in \mathbb{Z}} G_\ell^r = V_P^n$. Under a list of guesses $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$, the planner only needs to know to which half-open interval the maximal social surplus belongs. In this sense, under $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$, the planner is required much “less" information than that under the *finest list* $\big\{\{v\}\big\}_{v \in V_P^n}$. Note that the lengths of the intervals of $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$ increase as the number of individuals decreases, and hence the planner's guess becomes “easiest" when $n = 2$. In Section 3.6, we discuss that a lot of *dispute resolution problems* are compatible with such a situation.

We now define a social choice rule that is *efficient*, *individually rational*, *feasible*, and *G-implementable with respect to our list of guesses* $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$. Let $\mathbf{1}_\phi : A \to \{0,1\}$ be an indicator function of $\phi \in A$, that is, $\mathbf{1}_\phi(\phi) = 1$, and $\mathbf{1}_\phi(b) = 0$ for all $b \in A \setminus \{\phi\}$.

Definition 4. A social choice rule $f^r = (a^r, p^r_1, \ldots, p^r_n)$ is a *mediation rule with a basis* $r \in \mathbb{R}_{++}$ if for each $v \in V^n_P$ and each $i \in N$,

$$\text{(i) } a^r(v) \in \arg\max_{b\in A} \sum_{j=1}^{n} v_j(b),$$

$$\text{(ii) } p^r_i(v) = -\sum_{j\neq i} v_j\big(a^r(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a^r(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \quad \text{if } v \in G^r_\ell,$$

$$\text{(iii) } \Big[\max_{b\in A} \sum_{j=1}^{n} v_j(b) = 0\Big] \Longrightarrow a^r(v) = \phi.$$

Each mediation rule chooses an alternative that maximizes the social surplus. The payment rule of a mediation rule is equivalent to those of different VCG rules for different intervals of $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$. Note that condition (iii) determines the way to break ties especially when the maximal social surplus is zero. All mediation rules with the same basis are welfare equivalent. Under a mediation rule $f^r(v)$, each individual's final utility $u\big(f^r(v);\, v_i\big)$ becomes a function of the only maximal social surplus, and it is illustrated by Figure 3.

Theorem 1 states that (i) mediation rules are the only social choice rules that are *efficient*, *individually rational*, *feasible*, and *G-implementable with respect to our list* $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$, and that (ii) a list of guesses $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$ requires minimal information about individual valuations among a class of lists of guesses such that some G-implementable social choice rule satisfying the three properties exists.

Theorem 1. *For each positive number* $r \in \mathbb{R}_{++}$*,*

(i) a social choice rule f *is efficient, individually rational, feasible, and G-implementable with respect to* $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$ *if and only if* f *is a mediation rule with the basis* r*,*

(ii) if $\{G^r_\ell\}_{\ell\in\mathbb{Z}} \subsetneq \{G_\lambda\}_{\lambda\in\Lambda}$*, then no social choice rule is efficient, individually rational, feasible, and G-implementable with respect to* $\{G_\lambda\}_{\lambda\in\Lambda}$*.*

Proof. See Appendix C. ■

A mediation rule f^r is G-implemented by a set of *revelation mechanisms* under which reporting the true valuation function is the unique dominant strategy for each individual. In

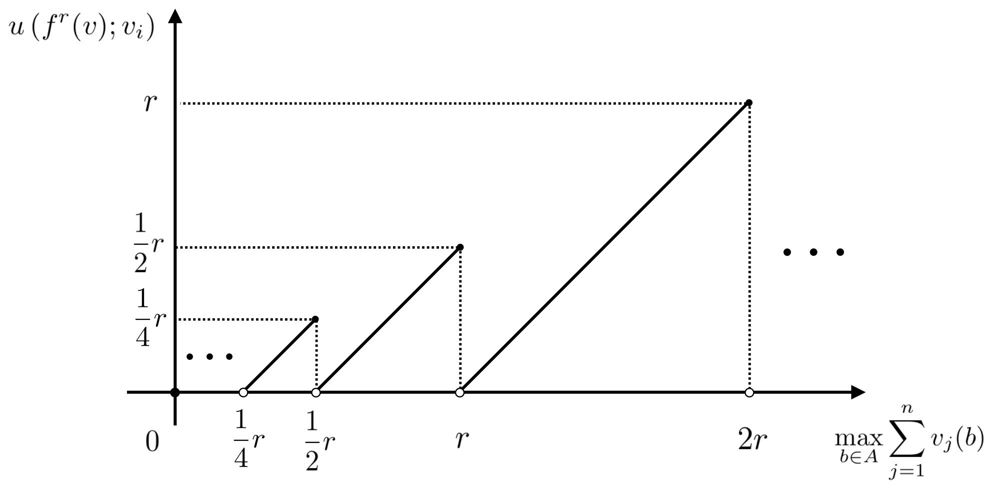


Figure 3: Individual i's final utility under a mediation rule when $n = 2$

Appendix A, we define such a set of mechanisms, and observe what happens if the planner's guess is incorrect.

Let us sketch the proof of Statement (ii) by focusing on the case with $n = 2$. Without loss of generality, suppose that there exists $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ such that $G_0^r \subsetneq G_\lambda$, that is, G_λ is a super set of valuation profiles whose social surplusrs are zero or in the interval $(\frac{1}{2}r, r]$. Then, for each social choice rule f that is *efficient*, *individually rational*, *feasible*, and *G-implementable with respect to* $\{G_\lambda\}_{\lambda\in\Lambda}$, we can show that all individual final utilities at $v \in G_\lambda$ are the same and obtained by Line A of Figure 4 (see Lemma 3; Appendix B). In addition, a social choice rule f is feasible if and only if each individual's final utility is below Line B.[*21] Then, if there exists $v \in G_\lambda$ whose maximal social surplus is q, each individual's final utility becomes less than zero, contradicting to the aussumption that f is *individually rational*. On the other hand, if there exists $v \in G_\lambda$ whose social surplus is q', then each individual's final utility is above

[*21]This is because by *efficiency* of f, it follows that

$$\sum_{j=1}^{n}\big(v_j(a(v)) + p_j(v)\big) = \max_{b\in A}\sum_{j=1}^{n} v_j(b) + \sum_{j=1}^{n} p_j(v).$$

Then, since individual final utilities are symmetric, a social choice rule f is feasible if and only if each individual's final utility is below $\frac{1}{n}\max_{b\in A}\sum_{j=1}^{n} v_j(b)$.

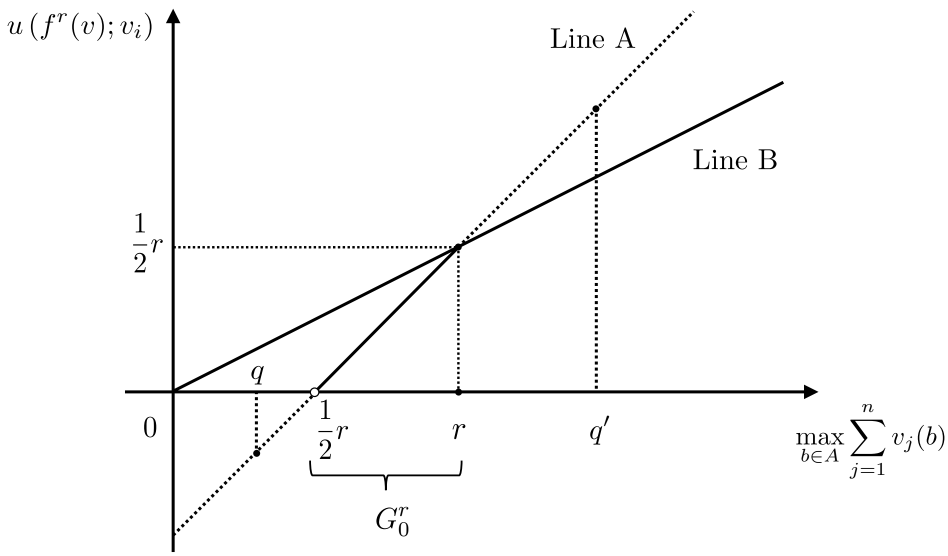


Figure 4: Sketch of the proof when $n = 2$

Line (b), contradicting to the assumption that f is *feasible*.

For each $\ell \in \mathbb{Z}$, let $\overline{G_\ell^r}$ be the closure of G_ℓ^r, that is,

$$\overline{G_\ell^r} = \left\{ v \in V_P^n : \max_{b\in A} \sum_{j=1}^{n} v_j(b) = 0 \text{ or } r\left(\frac{n}{n-1}\right)^{\ell-1} \leq \max_{b\in A} \sum_{j=1}^{n} v_j(b) \leq r\left(\frac{n}{n-1}\right)^{\ell} \right\}.$$

Corollary 1 states that each $G_\ell^r \in \{G_\ell^r\}_{\ell\in\mathbb{Z}}$ is large enough to guarantee that no list of guesses can include superset of $\overline{G_\ell^r}$ in order to G-implement a social choice rule that is *efficient*, *individually rational*, and *feasible*.

Corollary 1. *For each positive number $r \in \mathbb{R}_{++}$ and each list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$, if there exists $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ such that $\overline{G_\ell^r} \subsetneq G_\lambda$ for some $\ell \in \mathbb{Z}$, then no social choice rule is efficient, individually rational, feasible, and G-implementable with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$.*

Proof. Immediately follows from Proof of Statement (ii) of Theorem 1 in Appendix C. ■

Corollary 2 shows the reason we consider the half-open intervals of the maximal social surplus in our list of guesses $\{G_\ell^r\}_{\ell\in\mathbb{Z}}$. It states that if we add the left-hand endpoint of the half-open interval $\left(r(\frac{n}{n-1})^{\ell-1}, r(\frac{n}{n-1})^{\ell}\right]$ to G_ℓ^r, then each individual's final utility must decrease.

Corollary 2. *For each positive number $r \in \mathbb{R}_{++}$ and each list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$, if there exists $G_\lambda \in \{G_\lambda\}_{\lambda\in\Lambda}$ such that $G^r_\ell \subsetneq G_\lambda \subset \overline{G^r_\ell}$ for some $\ell \in \mathbb{Z}$, then for each mediation rule f^r and each social choice rule f satisfying efficiency, individual rationality, feasibility, and G-implementability with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$, the following statement holds: for each $i \in I$ and each $v \in G_\lambda$,*

$$u\big(f^r(v);\, v_i\big) \geq u\big(f(v);\, v_i\big),$$

strict inequality holding for all $v' \in G_\lambda \setminus G^r_\ell$.

Proof. Immediately follows from Lemma 3 in Appendix C. ■

Theorem 1 and Corollaries 1 and 2 have both positive and negative implications. On the one hand, these three results imply that if the planner can obtain enough information about the maximal social surplus, then mediation rules can be implemented in dominant strategies. On the other hand, these results also imply that if the planner cannot obtain such information, then he cannot implement social choice rules with the set of desirable properties. Therefore, our results determine the boundary between the possibility and impossibility of implementing social choice rules.

3.4 Auction Problem with Homogeneous Goods

In this section, we consider the auction problem where $V_I = V^n_A$. To begin with, we introduce some definitions.

Definition 5. A social choice rule $f = (a, p)$ *generates a higher revenue* than another social choice rule $f' = (a', p')$ if

$$\sum_{j=1}^{n} p_j(v) \geq \sum_{j=1}^{n} p'_j(v) \;\text{ for all } v \in V^n_A,$$

strict inequality holding for at least one $v' \in V^n_A$, and a *lower revenue* if the opposite holds.

Definition 6. A social choice rule $f = (a, p)$ is a *Vickrey auction* if

(i) for each $v \in V^n_A$ and each $i \in I$, $a_i(v) > 0$ implies that $v_{ia_i(v)} \geq v[L]$,

(ii) for each $v \in V^n_A$ and each $\ell \leq L$, $v[\ell] > 0$ implies that $\sum_{j=1}^n a_j(v) \geq \ell$,

(iii) for each $v \in V^n_A$ and each $i \in I$,

$$p_i(v) = \sum_{\ell=L-a_i(v)+1}^{L} v_{-i}[\ell].$$

Note that all Vickrey auctions generate the same revenue. The purpose of this section is to find a social choice rule that generates higher revenue than those of Vickrey auctions. In other words, our purpose is to overcome the following impossibility, which we reinterpret results by Vickrey (1961) and Holmström (1978).

Impossibility Result (Vickrey 1961, Holmström 1978). *All auction rules satsfying efficiency, individual rationality, and dominant strategy implementability generate lower revenue than those of Vickrey auctions.*

For each L-dimensional vector $\boldsymbol{r} = (r_1, \ldots, r_L) \in V_A$, let

$$G_0^{\boldsymbol{r}} \equiv \left\{v \in V_A^n : v[1] \geq r_1, v[2] \geq r_2, \ldots, v[L] \geq r_L\right\},$$
$$G_1^{\boldsymbol{r}} \equiv V_A^n \setminus G_0^{\boldsymbol{r}}.$$

Here, $G_0^{\boldsymbol{r}}$ is the set of valuation profiles whose first L-th highest valuations are grater than those of a given vector $\boldsymbol{r}$, and $G_1^{\boldsymbol{r}}$ is the complement of it. Then, $\left\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\right\}$ is a list of guesses. Under $\left\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\right\}$, the planner (or auctioneer) only needs to guess whether each k-th highest valuation $v[k]$ exceeds the k-th coordinate r_k of the given vector $\boldsymbol{r}$.

We next introduce an important class of social choice rules, called *Vickrey auctions with cutoff reserve prices*. For each reserve price $\boldsymbol{r} = (r_1, \ldots, r_L) \in V_A$ and each $v_{-i} \in V_A^{n-1}$, we say that $v_{-i}[1]$ *blocks the k-th coordinate* r_k of $\boldsymbol{r}$ if r_k is the first coordinate such that $v_{-i}[1] \geq r_k$, and $v_{-i}[2]$ *blocks the k-th coordinate* r_k if r_k is the first coordinate not blocked by $v_{-i}[1]$ such that $v_{-i}[2] \geq r_k$. More generally, we say that $v_{-i}[\ell]$ *blocks the k-th coordinate* r_k if r_k is the first coordinate not blocked by any $v_{-i}[\ell']$ with $\ell' < \ell$ such that $v_{-i}[\ell] \geq r_k$. Each coordinate $r_k \in \{r_1, \ldots, r_L\}$ is said to be *effective at* $v_{-i} \in V_A^{n-1}$ if r_k is not blocked by any $v_{-i}[\ell]$ with $\ell \leq L$. Let $E^{\boldsymbol{r}}(v_{-i}) \subset \{r_1, \ldots, r_L\}$ be the set of effective coordinates of $\boldsymbol{r} \in V_A$ at $v_{-i} \in V_A^{n-1}$. For example, if $L = 3$ and $r_1 > v_{-i}[1] > v_{-i}[2] = v_{-i}[3] > r_2 > r_3$, then $v_{-i}[1]$ blocks r_2, $v_{-i}[2]$ blocks r_3, $v_{-i}[3]$ blocks nothing, and hence $E^{\boldsymbol{r}}(v_{-i}) = \{r_1\}$. Let $E^{\boldsymbol{r}}(v_{-i})[k]$ be the k-th highest element among $E^{\boldsymbol{r}}(v_{-i})$.

Definition 7. A social choice rule $f = (a, p)$ is a *Vickrey auction with a cutoff reserve price* $r \in V_A$ if

(i) for each $v \in V_A^n$ and each $i \in I$, $a_i(v) > 0$ implies that $v_{ia_i(v)} \geq v[L]$,

(ii) for each $v \in V_A^n$ and each $\ell \leq L$, $v[\ell] > 0$ implies that $\sum_{j=1}^n a_j(v) \geq \ell$,

	v_{i1}	v_{i2}	v_{i3}	v_{i4}	v_{i5}
$i = 1$	**200**	**150**	90	50	20
$i = 2$	**140**	**100**	60	30	10
$i = 3$	**100**	60	30	20	10

Table 1: Marginal valuation vectors of the three bidders

(iii) for each $v \in V_A^n$ and each $i \in I$,

$$p_i(v) = \begin{cases} \sum_{\ell=L-a_i(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell & \text{if } v \in G_0^{\boldsymbol{r}}, \\ \sum_{\ell=L-a_i(v)+1}^{L} v_{-i}[\ell] & \text{if } v \in G_1^{\boldsymbol{r}}. \end{cases}$$

A Vickrey auction with a cutoff reserve price imposes a reserve price to a valuation profile if the valuation profile belongs to $G_0^{\boldsymbol{r}}$, and otherwise it is equivalent to a Vickrey auction.

The next example shows that Vickrey auctions with cutoff reserve prices may generate much higher revenues than those of Vickrey auctions.

Example 2. Consider a situation in which there are three bidders and five objects to be allocated. Each bidder $i \in \{1,2,3\}$ has a marginal valuation vector $v_i = (v_{i1}, \ldots, v_{i5})$ as illustrated in Table 1. For example, bidder 1's marginal valuation for the first object is $v_{11} = 200$, and that for the second object is $v_{12} = 150$. Then, under an efficient social choice rule, bidders 1 and 2 obtain two objects, and bidder 3 obtains one object. For this valuation profile, a Vickrey auction $f = (a, p)$ generates a revenue $p_1(v) + p_2(v) + p_3(v) = 120 + 150 + 90 = 360$. Now, let us compute the revenue of a Vickrey auction $f^{\boldsymbol{r}} = (a^{\boldsymbol{r}}, p^{\boldsymbol{r}})$ with a cutoff reserve price $\boldsymbol{r} = (170, 130, 120, 80, 80)$.[*22] Since $v_{-1}[1] = 140$, $v_{-1}[2] = 100$ and $v_{-1}[3] = 100$ block $r_2 = 130$, $r_4 = 80$ and $r_5 = 80$, respectively, and the others block nothing, the set of effective coordinates of $\boldsymbol{r}$ at v_{-1} is $E^{\boldsymbol{r}}(v_{-1}) = \{r_1, r_3\} = \{170, 120\}$. In addition, since $v[k] \geq r_k$ for all $k \leq L$, $v = (v_1, v_2, v_3) \in G_0^{\boldsymbol{r}}$. Thus, bidder 1's payment under $f^{\boldsymbol{r}}$ is $p_1^{\boldsymbol{r}}(v) = r_1 + r_3 = 170 + 120 = 290$. By a similar argument, we can check that $E^{\boldsymbol{r}}(v_{-2}) = \{r_3\} = \{120\}$, $E^{\boldsymbol{r}}(v_{-3}) = \emptyset$, and hence $p_2^{\boldsymbol{r}}(v) = r_3 + v_{-2}[4] = 120 + 90 = 210$, $p_3^{\boldsymbol{r}}(v) = v_{-3}[5] = 90$. Therefore, the Vickrey auction $f^{\boldsymbol{r}}$ with the cutoff reserve price $\boldsymbol{r}$ generates a revenue $p_1^{\boldsymbol{r}}(v) + p_2^{\boldsymbol{r}}(v) + p_3^{\boldsymbol{r}}(v) = 590$. This revenue is much higher than those of Vickrey auctions.

The following Lemma is a preliminary result to our second main theorem.

[*22] In Section 3.5, we discuss how to choose a cutoff reserve price.

Lemma 1. *For each Vickrey auction $f^{\boldsymbol{r}}$ with a cutoff reserve price $\boldsymbol{r} \in V_A$,*

(i) $f^{\boldsymbol{r}}$ is efficient, individually rational, and G-implementable with respect to $\left\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\right\}$,

(ii) $f^{\boldsymbol{r}}$ generates a higher revenue than any other social choice rule satisfying efficiency, individual rationality, and G-implementability with respect to $\left\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\right\}$.

Proof. See Appendix. ■

Consider any Vickrey auction $f^{\boldsymbol{r}}$ with a cutoff reserve price $\boldsymbol{r} \in V_A$. Then, by Lemma 1, $f^{\boldsymbol{r}}$ is G-implementable with respect to $\left\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\right\}$. However, when $L \geq 2$, there still exists a list of guesses under which the planner is required the less information than $\left\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\right\}$ to G-implement $f^{\boldsymbol{r}}$. Let $T_i^{\boldsymbol{r}} \subset V_A^n$ be such that $v \in T_i^{\boldsymbol{r}}$ if and only if $v \in G_0^{\boldsymbol{r}}$ and for each $v_i' \in V_A$ with $(v_i', v_{-i}) \in G_1^{\boldsymbol{r}}$,

$$u\left(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i\right) \geq u\left(f^{\boldsymbol{r}}(v_i', v_{-i}); v_i\right).$$

If $L = 1$, $T_i^{\boldsymbol{r}}$ is empty, otherwise it is non-empty. Let $T^{\boldsymbol{r}} \equiv \bigcap_{i=1}^n T_i^{\boldsymbol{r}}$, $\hat{G}_0^{\boldsymbol{r}} \equiv G_0^{\boldsymbol{r}} \cup \{\boldsymbol{0}\}$, and $\hat{G}_1^{\boldsymbol{r}} \equiv G_1^{\boldsymbol{r}} \cup T^{\boldsymbol{r}}$.[*23] Then, $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$ is a list of guesses. Note that since $T^{\boldsymbol{r}} \subset \hat{G}_0^{\boldsymbol{r}}$ and $T^{\boldsymbol{r}} \subset \hat{G}_1^{\boldsymbol{r}}$, if the true profile v is in $T^{\boldsymbol{r}}$, the planner's guess is always correct under $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$. Theorem 2 states that a list of guesses $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$ requires minimal information about individual valuations among a class of lists of guesses such that some social choice rule satisfying the three properties generates at least the same revenue as that of $f^{\boldsymbol{r}}$.

Theorem 2. *For each Vickrey auction $f^{\boldsymbol{r}}$ with a cutoff reserve price $\boldsymbol{r} \in V_A$,*

(i) $f^{\boldsymbol{r}}$ is efficient, individually rational, G-implementable with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$,

(ii) $f^{\boldsymbol{r}}$ generates a higher revenue than any other social choice rule satisfying efficiency, individual rationality, and G-implementability with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$,

(iii) if $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\} \subsetneq \{G_\lambda\}_{\lambda \in \Lambda}$, then $f^{\boldsymbol{r}}$ is no longer G-implementable with respect to $\{G_\lambda\}_{\lambda \in \Lambda}$.

Proof. See Appendix. ■

A Vickrey auction with a cutoff reserve price is G-implemented by a set of *revelation mechanisms* under which reporting the true valuation function is a dominant strategy for each individual. In Appendix A, we define such a set of mechanisms.

[*23]Definitions of $T_i^{\boldsymbol{r}}$ and $T^{\boldsymbol{r}}$ are independent of the selection of Vickrey auction $f^{\boldsymbol{r}}$ with a cutoff reserve price $\boldsymbol{r} \in V_A$. In other words, we can explicitly define $T_i^{\boldsymbol{r}}$ and $T^{\boldsymbol{r}}$ without using $f^{\boldsymbol{r}}$. Here, we adopt the above implicit definition to avoid complexity.

3.5 Discussion

3.5.1 How to Choose a Basis $r \in \mathbb{R}_{++}$ in Public Decision Problem

In Section 3.3, we showed that for each basis $r \in \mathbb{R}_{++}$, mediation rules with the basis r are the only social choice rules that are *efficient*, *individually rational*, *feasible*, and *G-implementable with respect to our list* $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$. However, in a real-use situation of such social choice rules, the planner needs to determine a basis $r \in \mathbb{R}_{++}$ of those rules. In this subsection, we introduce a practical way to choose such $r \in \mathbb{R}_{++}$. Consider a situation in which the true valuation profile is $v \in V^n_P$. We focus on a case where the maximal social surplus at v is positive, i.e., $\max_{b\in A}\sum_{j=1}^n v_j(b) > 0$, since the choice of $r \in \mathbb{R}_{++}$ is irrelevant to the outcomes of mediation rules when the maximal social surplus is 0. First, we note that a simple way for the planner to choose a positive number $r \in \mathbb{R}_{++}$ would be trying to choose $r \in \mathbb{R}_{++}$ so that the following inequality holds:

$$\frac{n-1}{n}r < \max_{b\in A}\sum_{j=1}^n v_j(b) \leq r. \tag{3.71}$$

This is because if the planner chooses such $r \in \mathbb{R}_{++}$ successfully, then the guess $G^r_0 \in \{G^r_\ell\}_{\ell\in\mathbb{Z}}$ becomes correct one. Therefore, in such a case, the planner's guess can be automatically correct by selecting G^r_0 as his guess.[*24] In other words, we can replace the planner's "guess problem" with the choice problem of $r \in \mathbb{R}_{++}$ satisfyiung equation (3.71).

Next, we compare performances of mediation rules with different bases satisfying equation (3.71). Consider a basis $r \in \mathbb{R}_{++}$ satisfying equation (3.71), and let f^r be a mediation rule with the basis r. Then, by definition of f^r, each individual's final utility under f^r at the true valuation profile v is

$$\max_{b\in A}\sum_{j=1}^n v_j(b) - \frac{n-1}{n}r.$$

Therefore, within a class of bases satisfying equation (3.71), everyone's final utility increases as the basis r gets smaller and is maximized at $r = \max_{b\in A}\sum_{j=1}^n v_j(b)$. This argument implies that even if the planner does not know the true valuation profile v, he should try to choose a basis $r \in V_A$ as close as possible to the maximal social surplus $\max_{b\in A}\sum_{j=1}^n v_j(b)$, while guaranteeing equation (3.71).

[*24]Note that for each basis $r \in \mathbb{R}_{++}$, there exists a basis $r' \in \mathbb{R}_{++}$ such that equation (3.71) holds and mediation rules with the basis r are equivalent to those with r'. Similarly, any list of guesses $\{G^r_\ell\}_{\ell\in\mathbb{Z}}$ is equivalent to some $\{G^{r'}_\ell\}_{\ell\in\mathbb{Z}}$ with a basis $r' \in \mathbb{R}_{++}$ satisfying equation (3.71) (see Figure 2).

3.5.2 How to Choose a Reserve Price $\boldsymbol{r} \in V_A$ in Auction Problem

In Section 3.4, we showed that for each reserve price $\boldsymbol{r} \in V_A$, a Vickrey auction $f^{\boldsymbol{r}}$ with the cutoff reserve price $\boldsymbol{r}$ generates the highest revenue among a class of social choice rules that are G-implementable with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$. Now, let us compare performances of Vickrey auctions with different reserve prices. Suppose that $v \in V_A^n$ is the true valuation profile. First, consider the case where the planner chooses a reserve price $\boldsymbol{r} \in V_A$ such that $v[k] \geq r_k$ for all $k \leq L$. Then, by definition of a Vickrey auction $f^{\boldsymbol{r}}$ with the cutoff reserve price $\boldsymbol{r}$, its revenue at v becomes weakly higher than that of Vickrey auctions. Next, consider the case where the auctioneer chooses a reserve price $\boldsymbol{r} \in V_A$ such that $v[k] < r_k$ for some $k \leq L$. Then, the revenue of Vickrey auctions with the cutoff reserve price $\boldsymbol{r}$ at the true valuation profile v is the same as those of Vickrey auctions. Overall, we can conclude that the revenue at the former case is higher than that at the latter case. In addition, focusing on the former case, the revenue of a Vickrey auction with a cutoff reserve price at v increases as its reserve price increases, and the revenue is maximized when the reserve price is $\boldsymbol{r} = (v[1], v[2], \ldots, v[L])$. Therefore, even if the auctioneer does not know the true valuation profile v, he is required to choose a reserve price $\boldsymbol{r} \in V_A$ as high as possible, while guaranteeing that $v[k] \geq r_k$ for all $k \leq L$.

3.6 Conclusion

We have considered the problem of implementing social choice rules by using the planner's "guess" of individual preferences. We introduced the concept of G-implementability and applied it to public decision and auction problems. For both problems, we characterized social choice rules satisfying some desirable properties, and found lists of guesses requiring minimal information about individual preferences. Our result about the public decision problem (Theorem 1) characterizes a class of social choice rules satisfying *efficiency*, *individual rationality*, *feasibility*, and *G-implementability*. This result seems mostly applicable to dispute resolutions, such as border disputes, commercial disputes or civil disputes. There are essentially two reasons for this. The first reason is related to the importance of *individual rationality*. *Individual rationality* would be less important if the planner could force individuals to follow his decision and not need to consider any participation constraint. However, in many dispute resolution processes outside the courts (e.g., the *mediation* in alternative dispute resolution processes), disputants often have the right to reject an agreement. Therefore, *individual rationality* is rather important so that everyone voluntarily accepts the outcomes of the social choice rule. The second reason is related to the number of individuals included in the public decision problem. In the case of dispute resolution problems, it is natural to consider that the number of individuals is small. Then, the planner's guess becomes relatively "easy" under our interval-based list of guesses because each of our list's interval length increases as the number of individuals decreases. Our result about the auction problem (Theorem 1) provides the ex-post revenue maximizing auction rules that are *efficient*, *individually rational*, and *G-implementable*. This result is applicable to many real-life auctions (e.g., treasury bills or radio spectrum). Studying an environment other than ours remains an area for future research.

Appendix A: Mechanisms that G-implement Our Social Choice Rules

Even if a social choice rule is G-implementable, one may wonder what kind of mechanisms G-implement that rule. If the strategy space S_I of a mechanism $M = (S_I, g)$ has some undesirable feature, such as infinite-dimensionality, then it is difficult to justify the use of such a mechanism (Dutta, Sen, and Vohra 1995, Saijo, Tatamitani and Yamato 1996). In addition, it is also important in our model to confirm what outcome is chosen when the planner's guess is incorrect. Therefore, in this Appendix we introduce *revelation mechanisms*; $S_I = V_I$, with which dominant strategy outcomes become not so "bad" even when the planner's guess is incorrect.

Public Decision Problem

First, we introduce a set of revelation mechanisms that G-implements a mediation rule $f^r = (a^r, p_1^r, \dots, p_n^r)$. For each $\ell \in \mathbb{Z}$, let $M^\ell = (V_P^n, g^\ell)$ be a revelation mechanism such that for each $v \in V_P^n$ and each $i \in N$, the outcome function $g^\ell = (d^\ell, t_1^\ell, \dots, t_n^\ell)$ takes the form of

$$d^\ell(v) \in \underset{b \in \{a^r(v), \phi\}}{\arg\max} \Big\{ \sum_{j=1}^{n} v_j(b) + r\big(\frac{n}{n-1}\big)^{\ell-1} \cdot 1_\phi(b) \Big\},$$
$$t_i^\ell(v) = -\sum_{j \neq i} v_j\big(d^\ell(v)\big) - r\big(\frac{n}{n-1}\big)^{\ell-1} \cdot 1_\phi\big(d^\ell(v)\big) + r\big(\frac{n}{n-1}\big)^{\ell-1}. \tag{3.72}$$

Note that under a revelation mechanism (V_P^n, g^ℓ), each individual can report any valuation function $v_i \in V_P$. An outcome function g^ℓ chooses the alternative $a^r(v)$ if the social surplus at $a^r(v)$ exceeds $r\big(\frac{n}{n-1}\big)^{\ell-1}$.[*25] A way to breaking ties is determined arbitrarily. In Proof of Theorem 1, we show that a set of revelation mechanisms $\{M^\ell\}_{\ell \in \mathbb{Z}}$ G-implements the mediation rule f^r with respect to $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$.

[*25] Remember that we normalize each $v_i(\phi)$ as zero.

Next, we observe what alternative is chosen when the planner's guesses is incorrect. Consider a situation in which the true valuation profile is $v \in V_P^n$, and the planner's guess is $G_\ell^r \in \{G_\ell^r\}_{\ell\in\mathbb{Z}}$. Under the revelation mechanism M^ℓ, reporting the true valuation function is always the unique dominant strategy for each individual regardless of whether the guess is correct or incorrect. Now suppose that the planner's guess is incorrect, i.e., $v \notin G_\ell^r$. Then, there are two cases. First case is that the true maximal social surplus is less than $r\left(\frac{n}{n-1}\right)^{\ell-1}$, that is,

$$0 < \max_{b\in A} \sum_{j=1}^{n} v_j(b) \;\le\; r\left(\frac{n}{n-1}\right)^{\ell-1}.$$

Then, the mechanism M^ℓ may assign the status quo ϕ, which is inefficient to v, but its outcome is always individually rational and feasible one. Second case is that the true maximal social surplus exceeds $r\left(\frac{n}{n-1}\right)^{\ell}$, that is,

$$r\left(\frac{n}{n-1}\right)^{\ell} \;<\; \max_{b\in A} \sum_{j=1}^{n} v_j(b).$$

Then, the mechanism M^ℓ assigns an individually rational and efficient, but infeasible outcome to v.

Auction Problem

We define a set of revelation mechanisms that G-implements a Vickrey auction $f^{\boldsymbol{r}} = (a^{\boldsymbol{r}}, p_1^{\boldsymbol{r}}, \ldots, p_n^{\boldsymbol{r}})$ with a reserve price $\boldsymbol{r} \in V_A$.

For each $v \in V_A^n$, each $p_i \in \mathbb{R}$, and each $k \le L$, let

$$\hat{u}\,(k, p_i;\, v_i) \equiv \sum_{\ell=1}^{k} v_{i\ell} - p_i.$$

Here, $\hat{u}\,(k, p_i;\, v_i)$ denotes individual i's utility of obtaining k objects with the payment p_i. Let $g^0 = (d^0, t_1^0, \ldots, t_n^0)$ be an outcome function $g^0 : V_A^n \to A \times \mathbb{R}^n$ such that for each $i \in I$ and each $v \in V_A^n$,

(i) $\sum_{j=1}^{n} d_j^0(v) \le L$,

(ii) if $v \in \hat{G}_0^{\boldsymbol{r}}$, then $d_i^0(v) = a_i^{\boldsymbol{r}}(v)$,

(iii) if $v \in \hat{G}_1^{\boldsymbol{r}}$, then $d_i^0(v) \in \min\left\{\arg\max_{k\le L}\; \hat{u}\left(k,\, \hat{t}_i^0(k, v_{-i});\, v_i\right)\right\}$,

	v_{i1}	v_{i2}	v_{i3}
$i = 1$	60	50	25
$i = 2$	40	35	15

Table 2: Marginal valuation vectors of two bidders

(iv) $t_i^0(v) = \hat{t}_i^0\big(d_i^0(v), v_{-i}\big)$,

where

$$\hat{t}_i^0(k, v_{-i}) \equiv \begin{cases} \sum_{\ell=|E^{\boldsymbol{r}}(v_{-i})|-k+1}^{|E^{\boldsymbol{r}}(v_{-i})|} E^{\boldsymbol{r}}(v_{-i})[\ell] & \text{if } k \le |E^{\boldsymbol{r}}(v_{-i})|, \\ \sum_{\ell=L-k+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell & \text{if } |E^{\boldsymbol{r}}(v_{-i})| < k. \end{cases}$$

In the above definition, $\hat{t}_i^0(k, v_{-i})$ denotes individual i's payment under g^0 when he obtains k objects. If $v \in \hat{G}_0^{\boldsymbol{r}}$, then the outcome function g^0 chooses the same outcome as that of $f^{\boldsymbol{r}}$. If $v \in \hat{G}_1^{\boldsymbol{r}}$, then the outcome function g^0 assigns to each individual a number of objects that maximizes his utility given a payment rule $\hat{t}_i^0$. In Proof of Theorem 2, we show that g^0 is a well-defined outcome function.[*26]

Example 3. Consider a situation in which there are two bidders and three objects to be allocated. Each bidder $i \in \{1, 2\}$ has a marginal valuation vector $v_i = (v_{i1}, v_{i2}, v_{i3})$ as illustrated in Table 2. Suppose that $\boldsymbol{r} = (70, 30, 10)$. Then, $E^{\boldsymbol{r}}(v_{-1}) = E^{\boldsymbol{r}}(v_{-2}) = \{70\}$ and $v \in \hat{G}_1^{\boldsymbol{r}}$. Let us compute Bidder 1's assignment under g^0. We can confirm that $\hat{t}_i^0(0, v_{-1}) = 0$, $\hat{t}_i^0(1, v_{-1}) = 70$, $\hat{t}_i^0(2, v_{-1}) = 35 + 70 = 105$, and $\hat{t}_i^0(3, v_{-1}) = 40 + 35 + 70 = 145$. Thus, $\hat{u}\big(0, \hat{t}_i^0(0, v_{-1}); v_i\big) = 0$, $\hat{u}\big(1, \hat{t}_i^0(1, v_{-1}); v_i\big) = 60 - 70 = -10$, $\hat{u}\big(2, \hat{t}_i^0(2, v_{-1}); v_i\big) = 60 + 50 - 105 = 5$, and $\hat{u}\big(3, \hat{t}_i^0(3, v_{-1}); v_i\big) = 60 + 50 + 25 - 145 = -10$. Among them, $\hat{u}$ is maximized when Bidder 1 obtains two objects. Therefore, he obtains two objects and pays $\hat{t}_i^0(2, v_{-1}) = 105$. Bidder 2's assignment under g^0 can be similarly computed; since $\hat{u}\big(0, \hat{t}_i^0(0, v_{-2}); v_i\big) = 0$, $\hat{u}\big(1, \hat{t}_i^0(1, v_{-2}); v_i\big) = 40 - 70 = -30$, $\hat{u}\big(2, \hat{t}_i^0(2, v_{-2}); v_i\big) = 40 + 35 - 120 = -45$, and $\hat{u}\big(3, \hat{t}_i^0(3, v_{-1}); v_i\big) = 40 + 35 + 25 - 180 = -80$, Bidder 2 obtains nothing.

Let $g^1 = (d^1, t_1^1, \ldots, t_n^1)$ be an outcome function $g^1 : V_A^n \to A \times \mathbb{R}^n$ such that for each $i \in I$ and each $v \in V_A^n$,

$$\text{(i)} \quad d_i^1(v) = \begin{cases} a_i^{\boldsymbol{r}}(v) & \text{if } v \in G_1^{\boldsymbol{r}} \cup T_i^{\boldsymbol{r}}, \\ \min\Big\{a_i^{\boldsymbol{r}}(v), \max\{a_i^{\boldsymbol{r}}(v_i', v_{-i}) : (v_i', v_{-i}) \in G_1^{\boldsymbol{r}}\}\Big\} & \text{otherwise,} \end{cases}$$

[*26]To show this, it suffices to confirm that Conditions (i), (ii), and (iii) are compatible. We also show that g_0 has a property such that for each individual $i \in I$ with $d_i^0(v) > 0$, $v_{id_i^0(v)} \ge v[L]$.

(ii) $t_i^1(v) = \begin{cases} p_i^{\boldsymbol{r}}(v) & \text{if } v \in G_1^{\boldsymbol{r}} \cup T_i^{\boldsymbol{r}} \\ \sum_{\ell=L-d_i^1(v)+1}^{L} v_{-i}[\ell] & \text{otherwise,} \end{cases}$

By definition of $T_i^{\boldsymbol{r}}$, if $v \notin T_i^{\boldsymbol{r}}$, then $E^{\boldsymbol{r}}(v_{-i}) \neq \phi$, and hence there exists $v_i' \in V_A$ such that $(v_i', v_{-i}) \in G_1^{\boldsymbol{r}}$. Therefore, the function d_i^1 is well-defined. Since $T^{\boldsymbol{r}} \subset T_i^{\boldsymbol{r}}$, if $v \in \hat{G}_1^{\boldsymbol{r}}$, then the outcome function g^1 chooses the same outcome as that of $f^{\boldsymbol{r}}$. Otherwise, g^1 assigns to each individual a number of objects so that she does not have an incentive to misrepresenting her valuation as $v_i' \in V_A$ with $(v_i', v_{-i}) \in G_1^{\boldsymbol{r}}$. In Proof of Theorem 2, we show that a set of revelation mechanisms $\{(V_L^n, g^0), (V_L^n, g^1)\}$ G-implements $f^{\boldsymbol{r}}$ with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$.

Example 4. Consider the situation of Example 3. Suppose that $\boldsymbol{r} = (40, 35, 35)$. Then, $E^{\boldsymbol{r}}(v_{-1}) = E^{\boldsymbol{r}}(v_{-2}) = \{35\}$ and $v \in G_0^{\boldsymbol{r}}$. Let us compute Bidder 1's assignment under g^1. First, let us confirm that $v \notin T_1^{\boldsymbol{r}}$. Suppose that Bidder 1 misreports his valuation as $v_1' \in V_A$ such that $35 > v_{11}' > 15$. Then, by definition of $G_1^{\boldsymbol{r}}$, $(v_1', v_2) \in G_1^{\boldsymbol{r}}$. In addition, by *efficiecy* of $f^{\boldsymbol{r}}$, $a_i^{\boldsymbol{r}}(v_1', v_2) = 1$. Since $(v_1', v_2) \in G_1^{\boldsymbol{r}}$, $p_i^{\boldsymbol{r}}(v_1', v_2)$ becomes the Vickrey payment 15. Thus, Bidder 1's final utility at $f^{\boldsymbol{r}}(v_1', v_2)$ becomes $60 - 15 = 45$. On the other hand, under truth telling, Bidder 1's final utility at $f^{\boldsymbol{r}}(v_1, v_2)$ is $60 + 50 - (35 + 35) = 40$. Therefore, $f^{\boldsymbol{r}}(v_1', v_2) > f^{\boldsymbol{r}}(v_1, v_2)$, and hence $v \notin T_1^{\boldsymbol{r}}$. Next, consider any $v_1' \in V_A$ such that $(v_1', v_2) \in G_1^{\boldsymbol{r}}$. Then, by definition of $G_1^{\boldsymbol{r}}$, $35 > v_{11}'$. Furthermore, since $f^{\boldsymbol{r}}$ is *efficient*, $a_i^{\boldsymbol{r}}(v_i', v_{-i})$ is maximized to be 1 when $v_{11}' > 15$. Therefore, under g^1, Bidder 1 obtains only one object and pays 15. Similarly, we can confirm that under g^1, Bidder 2 obtains one object and pays 25.

Let us observe what outcome is chosen when the planner's guesses is incorrect. Consider a situation where the true valuation profile is $v \in V_A^n$, and the planner's guess is $\hat{G}_h^{\boldsymbol{r}} \in \{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$. Then, it can be shown that under the revelation mechanism (V_L^n, g^h), reporting the true valuation function is a dominant strategy for each individual regardless of whether the guess is correct or not. In addition, even when the planner's guess is incorrect, i.e., $v \notin \hat{G}_h^{\boldsymbol{r}}$, the dominant strategy outcome under (V_L^n, g^h) becomes individually rational one. In this sense, even when the planner's guess is incorrect, the dominant strategy outcomes of above revelation mechanisms $\{(V_L^n, g^0), (V_L^n, g^1)\}$ are not so bad.

Appendix B: Revelation Principle for G-implementable Social Choice Rules

Here, we develop an analogous result to the revelation principle for strategy-proof social choice rules.

Lemma 2. *If a social choice rule $f : V_I \to A \times \mathbb{R}^n$ is G-implementable with respect to a list of guesses $\{G_\lambda\}_{\lambda\in\Lambda}$, then for any $i \in N$, any $\lambda \in \Lambda$, any $v \in G_\lambda$, and any $v'_i \in V_I$ with $(v'_i, v_{-i}) \in G_\lambda$,*

$$u\left(f(v_i, v_{-i}); v_i\right) \geq u\left(f(v'_i, v_{-i}); v_i\right).$$

Proof. Suppose that $f = (a, p) : V_I \to A \times \mathbb{R}^n$ is G-implementable with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$. Then, there exists $\left\{\left(M^\lambda, \boldsymbol{s}^\lambda\right)\right\}_{\lambda\in\Lambda}$ that G-implements f with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$. Take any $i \in N$, any $\lambda \in \Lambda$, any $v \in G_\lambda$, and any $v'_i \in V_I$ with $(v'_i, v_{-i}) \in G_\lambda$. Note that by definition of G-implementation, $\boldsymbol{s}^\lambda_i(v_i)$ is a dominant strategy at (M^λ, v_i). Then,

$$\begin{aligned} u\left(f(v_i, v_{-i}); v_i\right) &= u\left(g\left(\boldsymbol{s}^\lambda_i(v_i), (\boldsymbol{s}^\lambda_j(v_j))_{j\neq i}\right); v_i\right) \\ &\geq u\left(g\left(\boldsymbol{s}^\lambda_i(v'_i), (\boldsymbol{s}^\lambda_j(v_j))_{j\neq i}\right); v_i\right) = u\left(f(v'_i, v_{-i}); v_i\right). \end{aligned}$$

■

Appendix C: Proof of Theorem 1

Let

$$H_0 = \left\{ v \in V_P^n : \max_{b\in A} \sum_{j=1}^{n} v_j(b) = 0 \right\}.$$

Before starting, we prepare the following lemma.

Lemma 3. *Consider a list of guesseses $\{G_\lambda\}_{\lambda\in\Lambda}$ such that $\{G^r_\ell\}_{\ell\in\mathbb{Z}} \subset \{G_\lambda\}_{\lambda\in\Lambda}$. If a social choice rule $f = (a, p)$ is efficient, individually rational, feasible, and G-implementable with respect to $\{G_\lambda\}_{\lambda\in\Lambda}$, then for each $G_{\lambda(\ell)} \in \{G_\lambda\}_{\lambda\in\Lambda}$ such that $G^r_\ell \subset G_{\lambda(\ell)}$ for some $\ell \in \mathbb{Z}$,*

$$p_i(v) = -\sum_{j\neq i} v_j\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \quad \textit{for all } i \in I \textit{ and } v \in G_{\lambda(\ell)} \setminus H_0.$$

Proof. Suppose that a social choice rule $f = (a, p)$ is *efficient*, *individually rational*, *feasible*, and *G-implementable with respect to* $\{G_\lambda\}_{\lambda\in\Lambda}$ such that $\{G^r_\ell\}_{\ell\in\mathbb{Z}} \subset \{G_\lambda\}_{\lambda\in\Lambda}$. Take any $G_{\lambda(\ell)} \in \{G_\lambda\}_{\lambda\in\Lambda}$ such that $G^r_\ell \subset G_{\lambda(\ell)}$ for some $\ell \in \mathbb{Z}$.

Step 1: Let us show that for any $i \in N$, any $v \in G_{\lambda(\ell)}$, and any $v'_i \in V_P$ with $(v'_i, v_{-i}) \in G_{\lambda(\ell)}$, if $a(v_i, v_{-i}) = a(v'_i, v_{-i})$, then $p_i(v_i, v_{-i}) = p_i(v'_i, v_{-i})$. Take any $i \in N$. Suppose, by contradiction, that there exists $v \in G_{\lambda(\ell)}$ and $v'_i \in V_P$ with $(v'_i, v_{-i}) \in G_{\lambda(\ell)}$ such that $a(v_i, v_{-i}) = a(v'_i, v_{-i})$ and $p_i(v_i, v_{-i}) \neq p_i(v'_i, v_{-i})$. Without loss of generality, we consider the case with $p_i(v_i, v_{-i}) > p_i(v'_i, v_{-i})$. Then,

$$u\left(f(v_i, v_{-i}); v_i\right) = v_i\big(a(v_i, v_{-i})\big) - p_i(v_i, v_{-i}) < v_i\big(a(v'_i, v_{-i})\big) - p_i(v'_i, v_{-i}) = u\left(f(v'_i, v_{-i}); v_i\right),$$

a contradiction to Lemma 2.

Step 2. Let us show that for any $i \in N$ and any $v \in G_{\lambda(\ell)} \setminus H_0$,

$$p_i(v) \leq -\sum_{j \neq i} v_j\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}.$$

Take any $i \in N$. For each $v \in G_{\lambda(\ell)} \setminus H_0$, let

$$h_i(v) \equiv -p_i(v) - \sum_{j \neq i} v_j\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}.$$

It suffices to show that for any $v \in G_{\lambda(\ell)} \setminus H_0$,

$$h_i(v) \geq 0.$$

Suppose, by contradiction, that there exists $v \in G_{\lambda(\ell)} \setminus H_0$ such that

$$h_i(v) < 0.$$

Since $v \notin H_0$, by *efficiency* of f, $a(v) \neq \phi$. Let $\epsilon > 0$ be such that

$$\epsilon = \min\Big\{ -\frac{1}{2}h_i(v), r\Big(\frac{n}{n-1}\Big)^{\ell} - r\Big(\frac{n}{n-1}\Big)^{\ell-1}\Big\}.$$

Let $v_i' \in V_P$ be such that

$$v_i'(b) = \begin{cases} 0 & \text{if } b = \phi, \\ -\sum_{j \neq i} v_j\big(a(v)\big) + r\big(\frac{n}{n-1}\big)^{\ell-1} + \epsilon & \text{if } b = a(v), \\ -\sum_{j \neq i} v_j(b) & \text{otherwise.} \end{cases}$$

Then,

$$v_i'(b) + \sum_{j \neq i} v_j(b) = \begin{cases} r\big(\frac{n}{n-1}\big)^{\ell-1} + \epsilon & \text{if } b = a(v), \\ 0 & \text{if } b \neq a(v). \end{cases}$$

Note that by definition ϵ, $(v_i', v_{-i}) \in G_\ell^r \setminus H_0 \subset G_{\lambda(\ell)} \setminus H_0$. Moreover, by *efficiency* of f,

$$a(v_i', v_{-i}) = a(v).$$

Therefore, by Step 1, $p_i(v_i', v_{-i}) = p_i(v_i, v_{-i})$, and hence

$$h_i(v_i', v_{-i}) = h_i(v_i, v_{-i}) < 0.$$

Then,

$$\begin{aligned}
v_i'\big(a(v_i',v_{-i})\big) - p_i(v_i',v_{-i}) &= v_i'\big(a(v_i',v_{-i})\big) + \sum_{j\neq i} v_j\big(a(v_i',v_{-i})\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} + h_i(v_i',v_{-i}) \\
&= r\Big(\frac{n}{n-1}\Big)^{\ell-1} - r\Big(\frac{n}{n-1}\Big)^{\ell-1} + h_i(v_i',v_{-i}) + \epsilon \\
&= h_i(v_i',v_{-i}) + \epsilon \\
&\leq h_i(v_i',v_{-i}) - \frac{1}{2}h_i(v_i,v_{-i}) = \frac{1}{2}h_i(v_i,v_{-i}) < 0,
\end{aligned}$$

a contradiction to *individual rationality* of f.

Step 3. Let us show that for any $i \in N$ and any $v \in G_{\lambda(\ell)} \setminus H_0$,

$$p_i(v) = -\sum_{j\neq i} v_j\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}.$$

Take any $i \in N$. It suffices to show that for any $v \in G_{\lambda(\ell)}$,

$$h_i(v) = 0.$$

Suppose, by contradiction, that there exists $v \in G_{\lambda(\ell)} \setminus H_0$ such that

$$h_i(v) \neq 0.$$

Then, by Step 2, $h_i(v) > 0$.

Since $v \notin H_0$, by *efficiency* of f, $a(v) \neq \phi$. Let $v_i' \in V$ be such that

$$v_i'(b) = \begin{cases} 0 & \text{if } b = \phi, \\ -\sum_{j\neq i}^{n} v_j\big(a(v)\big) + r\big(\frac{n}{n-1}\big)^{\ell} & \text{if } b = a(v), \\ -\sum_{j\neq i}^{n} v_j(b) & \text{otherwise.} \end{cases}$$

Then, by definition of v_i', $(v_i', v_{-i}) \in G_\ell^r \setminus H_0 \subset G_{\lambda(\ell)} \setminus H_0$. Moreover, by *efficiency* of f,

$$a(v_i', v_{-i}) = a(v).$$

Therefore, by Step 1, $p_i(v_i', v_{-i}) = p_i(v_i, v_{-i})$, and hence

$$h_i(v_i', v_{-i}) = h_i(v_i, v_{-i}) > 0.$$

Noting that, from Step 2, $h_j(v_i', v_{-i}) \geq 0$ for all $j \in N$,

$$\sum_{j=1}^{n} p_j(v_i', v_{-i})$$

$$= -(n-1)\Big(v_i'\big(a(v_i', v_{-i})\big) + \sum_{j \neq i}^{n} v_j\big(a(v_i', v_{-i})\big)\Big) + nr\big(\frac{n}{n-1}\big)^{\ell-1} - h_i(v_i', v_{-i}) - \sum_{j \neq i} h_j(v_i', v_{-i})$$
$$\leq -(n-1)\Big(v_i'\big(a(v_i', v_{-i})\big) + \sum_{j \neq i}^{n} v_j\big(a(v_i', v_{-i})\big)\Big) + nr\big(\frac{n}{n-1}\big)^{\ell-1} - h_i(v_i', v_{-i})$$
$$= -(n-1)\Big(v_i'\big(a(v_i, v_{-i})\big) + \sum_{j \neq i}^{n} v_j\big(a(v_i, v_{-i})\big)\Big) + nr\big(\frac{n}{n-1}\big)^{\ell-1} - h_i(v_i', v_{-i})$$
$$< -(n-1)r\big(\frac{n}{n-1}\big)^{\ell} + nr\big(\frac{n}{n-1}\big)^{\ell-1} - h_i(v_i', v_{-i})$$
$$= -h_i(v_i', v_{-i}) < 0,$$

a contradiction to *feasibility* of f. ■

Proof of Statement (i) of Theorem 1:

"If" part. Let $f^r = (a^r, p^r)$ be a VCG rule with a basis $r \in \mathbb{R}_{++}$. Obviously, f^r satisfies *efficiency*. Let us show that f^r satisfies *individual rationality*, *feasibility*, *G-implementability with respect to* $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$.

***Individual rationality*.** Take any $v \in V_P^n$. Then, there exists $\ell \in \mathbb{Z}$ such that $v \in G_\ell^r$, that is,

$$\max_{b \in A} \sum_{j=1}^{n} v_j(b) = 0 \quad \text{or} \quad r\big(\frac{n}{n-1}\big)^{\ell-1} < \max_{b \in A} \sum_{j=1}^{n} v_j(b) \leq r\big(\frac{n}{n-1}\big)^{\ell}.$$

Consider the case with $\max_{b \in A} \sum_{j=1}^{n} v_j(b) = 0$. Then, by *definition* of f^r, $a^r(v) = \phi$, and hence

$$u(f^r(v);\, v_i) = v_i\big(a^r(v)\big) - p_i^r(v) = \sum_{j=1}^{n} v_j\big(a^r(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a^r(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1}$$
$$= \max_{b \in A} \sum_{j=1}^{n} v_j(b) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} - r\Big(\frac{n}{n-1}\Big)^{\ell-1} = 0.$$

Next, consider the case with $r\big(\frac{n}{n-1}\big)^{\ell-1} < \max_{b \in A} \sum_{j=1}^{n} v_j(b) \leq r\big(\frac{n}{n-1}\big)^{\ell}$. Then, by *efficiency* of f^r, $a^r(v) \neq \phi$, and hence

$$u(f^r(v);\, v_i) = v_i\big(a^r(v)\big) - p_i^r(v) = \sum_{j=1}^{n} v_j\big(a^r(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a^r(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1}$$
$$= \max_{b \in A} \sum_{j=1}^{n} v_j(b) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} > 0.$$

***Feasibility*.** Take any $v \in V_P^n$. Then, there exists $\ell \in \mathbb{Z}$ such that $v \in G_\ell^r$. Consider the case with $\max_{b \in A} \sum_{j=1}^n v_j(b) = 0$. Then, by *definition* of f^r, $a^r(v) = \phi$, and hence

$$\begin{aligned}\sum_{j=1}^n p_j\big(a^r(v)\big) &= \sum_{j=1}^n \left(-\sum_{k \neq j} v_k\big(a^r(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a^r(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \right) \\ &= -(n-1)\max_{b \in A} \sum_{j=1}^n v_j(b) + \sum_{j=1}^n \left(-r\Big(\frac{n}{n-1}\Big)^{\ell-1} + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \right) = 0.\end{aligned}$$

Next, consider the case with $r\big(\frac{n}{n-1}\big)^{\ell-1} < \max_{b \in A} \sum_{j=1}^n v_j(b) \leq r\big(\frac{n}{n-1}\big)^{\ell}$. Then, by *efficiency* of f^r, $a^r(v) \neq \phi$, and hence

$$\begin{aligned}\sum_{j=1}^n p_j\big(a^r(v)\big) &= \sum_{j=1}^n \left(-\sum_{k \neq j} v_k\big(a^r(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a^r(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1} \right) \\ &= -(n-1)\max_{b \in A} \sum_{j=1}^n v_j(b) + nr\Big(\frac{n}{n-1}\Big)^{\ell-1} \\ &= (n-1)\left(-\max_{b \in A} \sum_{j=1}^n v_j(b) + r\Big(\frac{n}{n-1}\Big)^{\ell} \right) \geq 0.\end{aligned}$$

***G-implementability*.** For each $\ell \in \mathbb{Z}$, let $g^\ell = (d^\ell, t^\ell)$ be an outcome function defined in Appendix A; for each $v \in V_P^n$ and each $i \in N$,

$$\begin{aligned} d^\ell(v) &\in \underset{b \in \{a^r(v), \phi\}}{\arg\max} \Big\{ \sum_{j=1}^n v_j(b) + r\big(\frac{n}{n-1}\big)^{\ell-1} \cdot 1_\phi(b) \Big\}, \\ t_i^\ell(v) &= -\sum_{j \neq i} v_j\big(d^\ell(v)\big) - r\big(\frac{n}{n-1}\big)^{\ell-1} \cdot 1_\phi\big(d^\ell(v)\big) + r\big(\frac{n}{n-1}\big)^{\ell-1}. \end{aligned}$$

Let us show that a set of mechanisms $\{M^\ell\}_{\ell \in \mathbb{Z}} \equiv \{(V_P^n, g^\ell)\}_{\ell \in \mathbb{Z}}$ G-implements f^r with respect to a list of guesses $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$.

Step 1. For each $i \in I$, let $\boldsymbol{s}_i^*(\cdot)$ be a type-strategy such that $\boldsymbol{s}_i^*(v_i) = v_i$. Let us show that for each $\ell \in \mathbb{Z}$ and each $v \in G_\ell^r$, $\boldsymbol{s}_i^*(v_i)$ is a dominant strategy at (M^ℓ, v_i). Take any $\ell \in \mathbb{Z}$. It suffices to show that for each $i \in I$, each $v \in V_P^n$, and each $v_i' \in V_P$.

$$u\left(g^\ell(v_i, v_{-i}); v_i\right) \geq u\left(g^\ell(v_i', v_{-i}); v_i\right). \tag{3.73}$$

By definitions of d^ℓ and a^r,

$$v_i\big(d^\ell(v_i, v_{-i})\big) - t_i^\ell(v_i, v_{-i})$$

$$= \max\left\{r\left(\frac{n}{n-1}\right)^{\ell-1},\ v_i\big(a^r(v_i, v_{-i})\big) + \sum_{j\neq i} v_j\big(a^r(v_i, v_{-i})\big) - r\left(\frac{n}{n-1}\right)^{\ell-1}\right\}$$

$$= \max\left\{r\left(\frac{n}{n-1}\right)^{\ell-1},\ \max_{b\in A}\sum_{j=1}^{n} v_j(b) - r\left(\frac{n}{n-1}\right)^{\ell-1}\right\}.$$

On the other hand,

$$v_i\big(d^\ell(v_i', v_{-i})\big) - t_i^\ell(v_i', v_{-i}) = \max\left\{r\left(\frac{n}{n-1}\right)^{\ell-1},\ \sum_{j=1}^{n} v_j\big(a^r(v_i', v_{-i})\big) - r\left(\frac{n}{n-1}\right)^{\ell-1}\right\}.$$

Moreover,

$$\max_{b\in A}\sum_{j=1}^{n} v_j(b) \geq \sum_{j=1}^{n} v_j\big(a^r(v_i', v_{-i})\big).$$

Then,

$$v_i\big(d^\ell(v_i, v_{-i})\big) - t_i^\ell(v_i, v_{-i}) \geq v_i\big(d^\ell(v_i', v_{-i})\big) - t_i^\ell(v_i', v_{-i}),$$

and hence equation (3.73) holds.

Step 2. Let us show that for any $\ell \in \mathbb{Z}$, $g^\ell(v) = f^r(v)$ for all $v \in G_\ell^r$. Take any $v \in G_\ell^r$. Since $v \in G_\ell^r$,

$$\max_{b\in A}\sum_{j=1}^{n} v_j(b) = 0, \tag{3.74}$$

or

$$r\left(\frac{n}{n-1}\right)^{\ell-1} < \max_{b\in A}\sum_{j=1}^{n} v_j(b) \leq r\left(\frac{n}{n-1}\right)^{\ell}. \tag{3.75}$$

Suppose that equation (3.74) holds. Then, by definition of d^ℓ, $d^\ell(v) = \phi = a^r(v)$. Moreover, $t_i^\ell(v) = 0 = p_i^*(v)$ for all $i \in N$. Hence $g^\ell(v) = f^r(v)$.

Next, suppose that equation (3.75) holds. Then, by definition of a^r,

$$\sum_{j=1}^{n} v_j\big(a^r(v)\big) = \max_{b\in A}\sum_{j=1}^{n} v_j(b).$$

Hence by equation (3.75),

$$r\left(\frac{n}{n-1}\right)^{\ell-1} < \sum_{j=1}^{n} v_j\big(a^r(v)\big).$$

Thus, by definition of d^ℓ, $d^\ell(v) = a^r(v)$. Moreover, since $v \in G_\ell^r$, for any $i \in N$,

$$t_i^\ell(v) = -\sum_{j\neq i} v_j\big(d^\ell(v)\big) - r\left(\frac{n}{n-1}\right)^{\ell-1}\cdot 1_\phi\big(d^\ell(v)\big) + r\left(\frac{n}{n-1}\right)^{\ell-1}$$

$$= -\sum_{j\neq i} v_j\big(a^r(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a^r(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}$$
$$= p_i^r(v).$$

Therefore, $g^\ell(v) = f^r(v)$.

Then, by Steps 1 and 2, $\{M^\ell\}_{\ell\in\mathbb{Z}}$ G-implements f^r with respect to $\{G_\ell^r\}_{\ell\in\mathbb{Z}}$. □

"Only if" part. Take any social choice rule $f = (a, p)$ that is efficient, individually rational, feasible, and G-implementable with respect to $\{G_\ell^r\}_{\ell\in\mathbb{Z}}$. Since f is efficient, $a(v) \in \arg\max_{b\in A} \sum_{j=1}^n v_j(b)$. Let us show that conditions (ii) and (iii) in Definition 4 holds.

Step 1. Let us show that for any $i \in N$ and any $v \in H_0$, $p_i(v) = -\sum_{j\neq i} v_j\big(a(v)\big)$. First, we shall show that

$$p_i(v) \leq -\sum_{j\neq i} v_j\big(a(v)\big) \quad \text{for all } i \in N \text{ and } v \in H_0. \tag{3.76}$$

Suppose, by contradiction, that there exists $i \in N$ and $v \in H_0$ such that $p_i(v) > -\sum_{j\neq i} v_j\big(a(v)\big)$. Since $v \in H_0$, it follows that $\max_{b\in A} \sum_{j=1}^n v_j(b) = 0$. Therefore, by $p_i(v) > -\sum_{j\neq i} v_j\big(a(v)\big)$ and *efficiency* of f,

$$v_i\big(a(v)\big) - p_i(v) < v_i\big(a(v)\big) + \sum_{j\neq i} v_j\big(a(v)\big) = \sum_{j=1}^n v_j\big(a(v)\big) = \max_{b\in A} \sum_{j=1}^n v_j(b) = 0,$$

a contradiction to *individual rationality* of f.

Next, let us how that for any $i \in N$ and any $v \in H_0$, $p_i(v) \geq \sum_{j\neq i} v_j\big(a(v)\big)$. Suppose, by contradiction, that there exists $i \in N$ and $v \in H_0$ such that $p_i(v) < -\sum_{j\neq i} v_j\big(a(v)\big)$. Then, by equation (3.76),

$$\sum_{j=1}^n p_j(v) < -\sum_{j=1}^n \left\{\sum_{k\neq j} v_k\big(a(v)\big)\right\} = -(n-1)\sum_{j=1}^n v_j\big(a(v)\big) = -(n-1)\max_{b\in A} \sum_{j=1}^n v_j(b) = 0,$$

a contradiction to *feasibility* of f.

Step 2. Let us show that for any $v \in H_0$, $a(v) = \phi$. Fix any $i \in N$.

Substep 2-1. We shall show that for any $v_i' \in V_P$ with $a(v_i', v_{-i}) = a(v)$,

$$p_i(v_i', v_{-i}) \leq -\sum_{j\neq i} v_j\big(a(v)\big).$$

Take any $v_i' \in V_P$ with $a(v_i', v_{-i}) = a(v)$. Since $\{G_\ell^r\}_{\ell \in \mathbb{Z}}$ is a list of guesses, there exists $\ell \in \mathbb{Z}$ such that $(v_i', v_{-i}) \in G_\ell^r$. Then, by $v \in H_0 \subset G_\ell^r$ and Lemma 2,

$$v_i'\big(a(v_i', v_{-i})\big) - p_i(v_i', v_{-i}) = u\left(f(v_i', v_{-i}); v_i'\right) \geq u\left(f(v_i, v_{-i}); v_i'\right) = v_i'\big(a(v_i, v_{-i})\big) - p_i(v_i, v_{-i}).$$

Then, by $a(v_i', v_{-i}) = a(v)$ and Step 1,

$$p_i(v_i', v_{-i}) \leq p_i(v_i, v_{-i}) = -\sum_{j \neq i} v_j\big(a(v)\big).$$

Substep 2-2. Suppose, by contradiction, that there exists $v \in H_0$ such that $a(v) \neq \phi$. Let $v_i' \in V_P$ be such that

$$v_i'(b) = \begin{cases} 0 & \text{if } b = \phi, \\ -\sum_{j \neq i}^n v_j\big(a(v)\big) + r\left(\frac{n}{n-1}\right) & \text{if } b = a(v), \\ -\sum_{j \neq i}^n v_j(b) & \text{otherwise.} \end{cases}$$

Then, by *efficiency* of f, it follows that $a(v_i', v_{-i}) = a(v)$. Therefore, by Substep 2-1,

$$p_i(v_i', v_{-i}) \leq -\sum_{j \neq i} v_j\big(a(v)\big). \tag{3.77}$$

On the other hand, since $(v_i', v_{-i}) \in G_1^r \setminus H_0$, by Lemma 3,

$$p_i(v_i', v_{-i}) = -\sum_{j \neq i} v_j\big(a(v_i', v_{-i})\big) + r\left(\frac{n}{n-1}\right)^0 = -\sum_{j \neq i} v_j\big(a(v)\big) + r > \sum_{j \neq i} v_j\big(a(v)\big),$$

a contradiction to equation (3.77).

Step 3. Let us show that for any $i \in N$, any $\ell \in \mathbb{Z}$, and any $v \in G_\ell^r$,

$$p_i(v) = -\sum_{j \neq i} v_j\big(a(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}.$$

Take any $i \in N$, any $\ell \in \mathbb{Z}$, and any $v \in G_\ell^r$. Consider the case with $v \in G_\ell^r \setminus H_0$. Then, by *efficiency* of f, $a(v) \neq \phi$. Therefore, by Lemma 3,

$$\begin{aligned} p_i(v) &= -\sum_{j \neq i} v_j\big(a(v))\big) + r\big(\frac{n}{n-1}\big)^{\ell-1} \\ &= -\sum_{j \neq i} v_j\big(a(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}. \end{aligned}$$

We next consider the case with $v \in H_0$. Then, by Step 2, $a(v) = \phi$, and hence by Step 1,

$$p_i(v) = -\sum_{j \neq i} v_j\big(a(v)\big) = -\sum_{j \neq i} v_j\big(a(v)\big) - r\Big(\frac{n}{n-1}\Big)^{\ell-1} \cdot 1_\phi\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}.$$

By Step 3, condition (ii) in Definition 4 holds. Condition (iii) in Definition 4 immediately follows from Step 2. □

Proof of Statement (ii) of Theorem 1: Consider any list of guesses $\{G_\lambda\}_{\lambda \in \Lambda}$ such that $\{G_\ell^r\}_{\ell \in \mathbb{Z}} \subsetneq \{G_\lambda\}_{\lambda \in \Lambda}$. Suppose, by contradiction, that there exists a social choice rule $f = (a, p)$ that is efficient, individually rational, feasible and G-implementable with respect to $\{G_\lambda\}_{\lambda \in \Lambda}$. Since $\{G_\ell^r\}_{\ell \in \mathbb{Z}} \subsetneq \{G_\lambda\}_{\lambda \in \Lambda}$, for each $\ell \in \mathbb{Z}$, there exists $\lambda(\ell) \in \Lambda$ such that $G_\ell^r \subset G_{\lambda(\ell)}$. Since $\{G_\ell^r\}_{\ell \in \mathbb{Z}} \subsetneq \{G_\lambda\}_{\lambda \in \Lambda}$, there exists $\ell \in \mathbb{Z}$ and $\lambda(\ell) \in \Lambda$ such that $G_\ell^r \subsetneq G_{\lambda(\ell)}$. Then, there exists $v \in G_{\lambda(\ell)} \setminus G_\ell^r$ such that one of the following three conditions holds:

(i) $0 < \max_{b \in A} \sum_{j=1}^n v_j(b) < r\big(\frac{n}{n-1}\big)^{\ell-1}$,

(ii) $\max_{b \in A} \sum_{j=1}^n v_j(b) = r\big(\frac{n}{n-1}\big)^{\ell-1}$,

(iii) $r\big(\frac{n}{n-1}\big)^{\ell} < \max_{b \in A} \sum_{j=1}^n v_j(b)$.

Suppose that condition (i) holds. Then, by $v \in G_{\lambda(\ell)} \setminus H_0$ and Lemma 3,

$$\begin{aligned}
v_i\big(a(v)\big) - p_i(v) &= v_i\big(a(v)\big) + \sum_{j \neq i} v_j\big(a(v)\big) - r\big(\frac{n}{n-1}\big)^{\ell-1} \\
&= \sum_{j=1}^n v_j\big(a(v)\big) - r\big(\frac{n}{n-1}\big)^{\ell-1} \\
&= \max_{b \in A} \sum_{j=1}^n v_j(b) - r\big(\frac{n}{n-1}\big)^{\ell-1} \\
&< r\big(\frac{n}{n-1}\big)^{\ell-1} - r\big(\frac{n}{n-1}\big)^{\ell-1} = 0,
\end{aligned}$$

a contradiction to *individual rationality* of f.

Suppose that condition (ii) holds. Then, $v \in G_{\ell-1}^r \setminus H_0 \subset G_{\lambda(\ell-1)} \setminus H_0$. Therefore, by Lemma 3,

$$p_i(v) = -\sum_{j \neq i} v_j\big(a(v)\big) + r\big(\frac{n}{n-1}\big)^{\ell-2}.$$

On the other hand, since $v \in G_{\lambda(\ell)} \setminus H_0$, by Lemma 3,

$$p_i(v) = -\sum_{j \neq i} v_j\big(a(v)\big) + r\Big(\frac{n}{n-1}\Big)^{\ell-1}.$$

This is a contradiction.

Finally, suppose that condition (iii) holds. Then, by $v \in G_{\lambda(\ell)} \setminus H_0$ and Lemma 3,

$$\begin{aligned}\sum_{i=1}^{n} p_j(v) &= -(n-1)\sum_{j=1}^{n} v_j\big(a(v)\big) + nr\Big(\frac{n}{n-1}\Big)^{\ell-1} \\ &= -(n-1)\max_{b \in A}\sum_{j=1}^{n} v_j(b) + nr\Big(\frac{n}{n-1}\Big)^{\ell-1} \\ &< -(n-1)r\Big(\frac{n}{n-1}\Big)^{\ell} + nr\Big(\frac{n}{n-1}\Big)^{\ell-1} = 0,\end{aligned}$$

a contradiction to *feasibility* of f. □

Appendix D: Proof of Lemma 1

Take any Vickrey auction $f^{\boldsymbol{r}} = (a^{\boldsymbol{r}}, p^{\boldsymbol{r}})$ with a cutoff reserve price $\boldsymbol{r} \in V_A$.

Proof of Statement (i) of Lemma 1. Note that since $\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\} \subsetneq \{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$, if $f^{\boldsymbol{r}}$ is G-implementable with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$, then it is also G-implementable with respect to $\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\}$. Therefore, Statement (i) of Theorem 2 implies *G-implementability* of $f^{\boldsymbol{r}}$ with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$. In addition, one can easily check that $f^{\boldsymbol{r}}$ satisfies *efficiency*. So, we only show that $f^{\boldsymbol{r}}$ satisfies *individual rationality*. Take any $i \in I$ and any $v \in V_A^n$. Consider the case with $v \in G_0^{\boldsymbol{r}}$. Then, for each $\ell \leq L$, $v_{i\ell} \geq E^{\boldsymbol{r}}(v_{-i})[\ell]$, since otherwise $v \notin G_0^{\boldsymbol{r}}$. In addition, by *definition* of $f^{\boldsymbol{r}}$, for each $\ell \leq a_i^{\boldsymbol{r}}(v)$,

$$v_{i\ell} \geq v_{ia_i^{\boldsymbol{r}}(v)} \geq v[L] \geq v_{-i}[L - a_i^{\boldsymbol{r}}(v) + 1].$$

Therefore,

$$\begin{aligned}u\left(f^{\boldsymbol{r}}(v);\, v_i\right) &= \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell \\ &= \sum_{\ell=1}^{|E^{\boldsymbol{r}}(v_{-i})|} v_{i\ell} + \sum_{\ell=|E^{\boldsymbol{r}}(v_{-i})|+1}^{a_i^{\boldsymbol{r}}(v)} v_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell\end{aligned}$$

$$\geq \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell + \sum_{\ell=|E^{\boldsymbol{r}}(v_{-i})|+1}^{a_i^{\boldsymbol{r}}(v)} v[L] - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell \geq 0.$$

Next, consider the case with $v \in G_1^{\boldsymbol{r}}$. Then, since $v_{i\ell} \geq v[L] \geq v_{-i}[L - a_i^{\boldsymbol{r}}(v) + 1]$ for all $\ell \leq a_i^{\boldsymbol{r}}$,

$$u\left(f^{\boldsymbol{r}}(v);\, v_i\right) = \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L} v_{-i}[\ell] \geq \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v[L] - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L} v_{-i}[\ell] \geq 0.$$

Therefore, $f^{\boldsymbol{r}}$ satisfies *individual rationality*. □

Proof of Statement (ii) of Lemma 1. Take any social choice rule $f = (a, p)$ that is *efficient*, *individually rational*, and *G-implementable with respect to* $\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\}$. Let us show that $f^{\boldsymbol{r}}$ generates strictly higher revenue than that of f. Since all Vickrey auctions with the same cutoff reserve price $\boldsymbol{r}$ generates the same revenue, we assume that $a^{\boldsymbol{r}} = a$ without loss of generality. It suffices to show that for each $i \in I$ and each $v \in V_A^n$, $p_i^{\boldsymbol{r}}(v) \geq p_i(v)$.

Step 1. Let us show that for each $i \in I$ and each $v \in V_A^n$, if $v \in G_0^{\boldsymbol{r}}$, then $p_i^{\boldsymbol{r}}(v) \geq p_i(v)$. Take any $i \in I$ and any $v \in V_A^n$. Suppose that $v \in G_0^{\boldsymbol{r}}$. Let

$$W_i^0(v_{-i}) \equiv \{w_i \in V_A : (w_i, v_{-i}) \in G_0^{\boldsymbol{r}}\}.$$

Then, since $W_i^0(v_{-i}) \subset \mathbb{R}_+^L$ is a *convex* set, it is *connected* on $\mathbb{R}_+^L$. In addition, since $f^{\boldsymbol{r}}$ and f are G-implementable with respect t to $\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\}$, by Lemma 2, for each $w_i, w_i' \in W_i^0(v_{-i})$,

$$u\left(f^{\boldsymbol{r}}(w_i, v_{-i}); w_i\right) \geq u\left(f^{\boldsymbol{r}}(w_i', v_{-i}); w_i\right) \text{ and } u\left(f(w_i, v_{-i}); w_i\right) \geq u\left(f(w_i', v_{-i}); w_i\right).$$

Therefore, by Chung and Olszewski (2007; Theorem 4), $f^{\boldsymbol{r}}$ and f satisfy the *payoff (revenue) equivalence property*, that is, there exists a constant $C \in \mathbb{R}$ such that

$$p_i^{\boldsymbol{r}}(w_i, v_{-i}) = p_i(w_i, v_{-i}) + C \text{ for all } w_i \in W_i^0(v_{-i}).$$

We shall show $C \geq 0$. Suppose, by contradiction, that $C < 0$. Let $w_i \in W_i^0(v_{-i})$ be such that

$$w_i = (E^{\boldsymbol{r}}(v_{-i})[1], \ldots, E^{\boldsymbol{r}}(v_{-i})[k], 0 \ldots, 0) \in V_A,$$

where $k = |E^{\boldsymbol{r}}(v_{-i})|$. Note that w_i is well defined. Then, by $C < 0$,

$$p_i^{\boldsymbol{r}}(w_i, v_{-i}) < p_i(w_i, v_{-i}).$$

In addition, by *efficiency* of f and $f^{\boldsymbol{r}}$, $a_i(w_i, v_{-i}) = a_i^{\boldsymbol{r}}(w_i, v_{-i} \geq k$, and hence $p_i^{\boldsymbol{r}}(w_i, v_{-i}) = \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell$. Therefore,

$$u\left(f(w_i, v_{-i}); w_i\right) = \sum_{\ell=1}^{a_i(w_i, v_{-i})} w_{i\ell} - p_i(w_i, v_{-i}) < \sum_{\ell=1}^{a_i(w_i, v_{-i})} w_{i\ell} - p_i^{\boldsymbol{r}}(w_i, v_{-i})$$
$$= \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell = 0,$$

a contradiction to *individual rationality* of f. Hence $C \geq 0$. It in turn implies that

$$p_i^{\boldsymbol{r}}(v_i, v_{-i}) \geq p_i(v_i, v_{-i}).$$

Step 2. Let us show that for each $i \in I$ and each $v \in V_A^n$, if $v \in G_1^{\boldsymbol{r}}$, then $p_i^{\boldsymbol{r}}(v) \geq p_i(v)$. Take any $i \in I$ and any $v \in V_A^n$. Suppose that $v \in G_1^{\boldsymbol{r}}$. Let

$$W_i^1(v_{-i}) \equiv \{w_i \in V_A : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\}.$$

Then, since $W_i^1(v_{-i}) \subset \mathbb{R}_+^L$ is a *connected* on $\mathbb{R}_+^L$, by the same argument as Step 1, there exists a constant $C \in \mathbb{R}$ such that

$$p_i^{\boldsymbol{r}}(w_i, v_{-i}) = p_i(w_i, v_{-i}) + C \;\text{ for all } w_i \in W_i^1(v_{-i}).$$

Note that by definition of $f^{\boldsymbol{r}}$ and *individual rationality* of f,

$$p_i^{\boldsymbol{r}}(\mathbf{0}_L, v_{-i}) = 0 \geq p_i(\mathbf{0}_L, v_{-i}).$$

Then, since $\mathbf{0}_L \in W_i^1(v_{-i})$, $C \geq 0$. Therefore, $p_i^{\boldsymbol{r}}(v_i, v_{-i}) \geq p_i(v_i, v_{-i})$.

Then, by Steps 1 and 2, $f^{\boldsymbol{r}}$ generates a higher revenue than that of f. □

Appendix E: Proof of Theorem 2

Consider any Vickrey auction $f^{\boldsymbol{r}} = (a^{\boldsymbol{r}}, p^{\boldsymbol{r}})$ with a cutoff reserve price $\boldsymbol{r} \in V_A$. We prepare two lemmas.

Lemma 4. *Let $f = (a, p)$ be an efficient social choice rule. For each each $i \in I$, each $v \in V_A^n$, and each $k, k' \leq L$, if $a_i(v) \geq k \geq k'$ or $k' \geq k \geq a_i(v)$, then*

$$\sum_{\ell=1}^{k} v_{i\ell} - \sum_{\ell=L-k+1}^{L} v_{-i}[\ell] \geq \sum_{\ell=1}^{k'} v_{i\ell} - \sum_{\ell=L-k'+1}^{L} v_{-i}[\ell].$$

Proof. Take any $i \in I$, any $v \in V_A^n$, and any $k, k' \leq L$. Let us show the case with $a_i(v) \geq k \geq k'$. By *efficiency* of f, for each $\ell \leq a_i(v)$,

$$v_{i\ell} \geq v_{-i}[L - \ell + 1],$$

since otherwise $\ell > a_i(v)$, which is a contradiction. Then,

$$\begin{aligned}\sum_{\ell=1}^{k} v_{i\ell} - \sum_{\ell=L-k+1}^{L} v_{-i}[\ell] - \left(\sum_{\ell=1}^{k'} v_{i\ell} - \sum_{\ell=L-k'+1}^{L} v_{-i}[\ell]\right) &= \sum_{\ell=k'+1}^{k} v_{i\ell} - \sum_{\ell=L-k+1}^{L-k'} v_{-i}[\ell] \\ &= \sum_{\ell=k'+1}^{k} v_{i\ell} - \sum_{\ell=k'+1}^{k} v_{-i}[L - \ell + 1] \geq 0.\end{aligned}$$

We can similarly show the case with $k' \geq k \geq a_i(v)$. ■

Lemma 5. *For each $v_{-i} \in V_A^{n-1}$ and each $r_\ell \in E^{\boldsymbol{r}}(v_{-i})$, $r_\ell > v_{-i}\big[L - |E^{\boldsymbol{r}}(v_{-i})|\big]$. In addition, if $E^{\boldsymbol{r}}(v_{-i}) \neq \emptyset$, then*

$$\sum_{\ell=1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell \geq \sum_{\ell=1}^{L} v_{-i}[\ell].$$

Proof. Immediately follows from definition of $E^{\boldsymbol{r}}(v_{-i})$. ■

Proof of Statement (i) of Theorem 2. Let us show that $f^{\boldsymbol{r}}$ is *G-implementable with respect to* $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$.

Step 1. Let $g^0 = (d^0, t_1^0, \ldots, t_n^0)$ be an outcome function defined in Appendix A; for each $v \in V_A^n$,

(i) $\sum_{j=1}^n d_j^0(v) \leq L$,

(ii) if $v \in \hat{G}_0^{\boldsymbol{r}}$, then $d_i^0(v) = a_i^{\boldsymbol{r}}(v)$,

(iii) if $v \in \hat{G}_1^{\boldsymbol{r}}$, then $d_i^0(v) \in \min \arg\max_{k \leq L} \hat{u}\left(k, \hat{t}_i^0(k, v_{-i});\, v_i\right)$,

(iv) $t_i^0(v) = \hat{t}_i^0\big(d_i^0(v), v_{-i}\big)$ for all $i \in I$,

where for each $p_i \in \mathbb{R}$ and each $k \leq L$,

$$\hat{u}\,(k,\, p_i;\, v_i) = u\left((0, \ldots, 0, \underbrace{k}_{i\text{-th}}, 0, \ldots, 0),\, p_i;\, v_i\right),$$

$$\hat{t}_i^0(k, v_{-i}) = \begin{cases} \sum_{\ell=|E^{\boldsymbol{r}}(v_{-i})|-k+1}^{|E^{\boldsymbol{r}}(v_{-i})|} E^{\boldsymbol{r}}(v_{-i})[\ell] & \text{if } k \le |E^{\boldsymbol{r}}(v_{-i})|, \\ \sum_{\ell=L-k+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell & \text{if } |E^{\boldsymbol{r}}(v_{-i})| < k. \end{cases}$$

Remember that the domain of g^0 is V_A^n. One can easily check that for each $v \in \hat{G}_0^{\boldsymbol{r}}$, $g^0(v) = f^{\boldsymbol{r}}(v)$. Let us show that g^0 is a well-defined outcome function and that under a revelation mechanism (V_A^n, g^0), reporting the true valuation function is a dominant strategy for each individual.

To show that g^0 is well-defined, it suffices to show that for each $v \in G_1^{\boldsymbol{r}}$, $\sum_{j=1}^n d_j^0(v) \le L$. Suppose, by contradiction, that there exists $v \in G_1^{\boldsymbol{r}}$ such that $\sum_{j=1}^n d_j^0(v) > L$. Then, there exists $i \in N$ such that $d_i^0(v) > 0$ and $v_{jd_j^0(v)} \ge v_{id_i^0(v)}$ for all $j \in I$ with $d_j^0(v) > 0$. Thus,

$$v_{-i}\left[\sum_{j \neq i} d_j^0(v)\right] \ge v_{id_i^0(v)}. \tag{3.78}$$

We consider two cases. First, consider the case with $1 \le d_i^0(v) \le |E^{\boldsymbol{r}}(v_{-i})|$. By definitions of $\hat{t}_i^0$ and d_i^0, we have

$$\begin{aligned} \sum_{\ell=1}^{d_i^0(v)} v_{i\ell} - \sum_{\ell=|E^{\boldsymbol{r}}(v_{-i})|-d_i^0(v)+1}^{|E^{\boldsymbol{r}}(v_{-i})|} E^{\boldsymbol{r}}(v_{-i})[\ell] &= \hat{u}\left(d_i^0(v),\, \hat{t}_i^0\big(d_i^0(v), v_{-i}\big);\ v_i\right) \\ &> \hat{u}\left(d_i^0(v)-1,\, \hat{t}_i^0\big(d_i^0(v)-1, v_{-i}\big);\ v_i\right) \\ &= \sum_{\ell=1}^{d_i^0(v)-1} v_{i\ell} - \sum_{\ell=|E^{\boldsymbol{r}}(v_{-i})|-d_i^0(v)+2}^{|E^{\boldsymbol{r}}(v_{-i})|} E^{\boldsymbol{r}}(v_{-i})[\ell]. \end{aligned}$$

It in turn implies that

$$v_{id_i^0(v)} > E^{\boldsymbol{r}}(v_{-i})\Big[|E^{\boldsymbol{r}}(v_{-i})| - d_i^0(v) + 1\Big] \ge E^{\boldsymbol{r}}(v_{-i})\Big[|E^{\boldsymbol{r}}(v_{-i})|\Big]. \tag{3.79}$$

Let $k \le L$ be such that

$$r_k = E^{\boldsymbol{r}}(v_{-i})\Big[|E^{\boldsymbol{r}}(v_{-i})|\Big].$$

Then, the number of coordinates of $\boldsymbol{r}$ that are blocked by some $v_{-i}[\cdot]$ is

$$\Big|\big\{r_\ell \in \{r_1, \ldots, r_k\} : r_\ell \notin E^{\boldsymbol{r}}(v_{-i})\big\}\Big| = k - |E^{\boldsymbol{r}}(v_{-i})|.$$

Since r_k is not blocked by any $v_{-i}[\cdot]$, i.e., $r_k \in E^{\boldsymbol{r}}(v_{-i})$, by definition of $E^{\boldsymbol{r}}(v_{-i})$,

$$E^{\boldsymbol{r}}(v_{-i})\Big[|E^{\boldsymbol{r}}(v_{-i})|\Big] = r_k > v_{-i}\Big[k - |E^{\boldsymbol{r}}(v_{-i})| + 1\Big]. \tag{3.80}$$

In addition, by $d_i^0(v) \leq |E^r(v_{-i})|$,

$$v_{-i}\Big[k - |E^r(v_{-i})| + 1\Big] \geq v_{-i}\left[L - d_i^0(v) + 1\right]. \tag{3.81}$$

Thus, by equations (3.79), (3.80), and (3.81),

$$v_{id_i^0(v)} > v_{-i}[L - d_i^0(v) + 1].$$

Since $\sum_{j=1}^n d_j^0(v) > L$, $\sum_{j\neq i} d_j^0(v) \geq L - d_i^0(v) + 1$, and hence

$$v_{id_i^0(v)} > v_{-i}[L - d_i^0(v) + 1] \geq v_{-i}\left[\sum_{j\neq i} d_j^0(v)\right],$$

a contradiction to equation (3.78).

Next, consider the case with $d_i^0(v) > |E^r(v_{-i})|$. Then, by definition of d_i^0,

$$\begin{aligned}\sum_{\ell=1}^{d_i^0(v)} v_{i\ell} - \sum_{\ell=L-d_i^0(v)+1}^{L-|E^r(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell\in E^r(v_{-i})} r_\ell &= \hat{u}\left(d_i^0(v), \hat{t_i^1}(d_i^0(v), v_{-i});\ v_i\right)\\ &> \hat{u}\left(d_i^0(v)-1, \hat{t_i^0}(d_i^0(v)-1, v_{-i});\ v_i\right)\\ &= \sum_{\ell=1}^{d_i^0(v)-1} v_{i\ell} - \sum_{\ell=L-d_i^0(v)+2}^{L-|E^r(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell\in E^r(v_{-i})} r_\ell.\end{aligned}$$

Therefore, $v_{id_i^0(v)} > v_{-i}[L - d_i^0(v) + 1]$. Then, since $\sum_{j\neq i} d_j^0(v) \geq L - d_i^0(v) + 1$,

$$v_{id_i^0(v)} > v_{-i}[L - d_i^0(v) + 1] \geq v_{-i}\left[\sum_{j\neq i} d_j^0(v)\right],$$

a contradiction to equation (3.78).

Finally, we show that under a mechanism (V_A^n, g^0), reporting the true valuation function is a dominant strategy for each individual. Take any $i \in I$, any $v \in V_A^n$, and any $v_i' \in V_A$. We shall show that

$$u\left(g^0(v_i, v_{-i}); v_i\right) \geq u\left(g^0(v_i', v_{-i}); v_i\right). \tag{3.82}$$

Note that by *efficiency* of f^r, even if $(v_i, v_{-i}) \in G_0^r$,

$$d_i^0(v) = a_i^r(v) \in \arg\max_{k\leq L}\ \hat{u}\left(k, \hat{t_i^0}(k, v_{-i});\ v_i\right).$$

Then, by definition of $g^0 = (d^0, t_1^0, \ldots, t_n^1)$,

$$u\big(g^0(v_i, v_{-i}); v_i\big) = \hat{u}\left(d_i^0(v_i, v_{-i}), \hat{t}_i^0(d_i^0(v_i, v_{-i}), v_{-i});\ v_i\right)$$
$$\geq \hat{u}\left(d_i^0(v_i', v_{-i}), \hat{t}_i^0(d_i^0(v_i', v_{-i}), v_{-i});\ v_i\right) = u\big(g^0(v_i', v_{-i}); v_i\big).$$

Step 2. Let $g^1 = (d^1, t_1^1, \ldots, t_n^1)$ be an outcome function defined in Appendix A; for each $i \in I$ and each $v \in V_A^n$,

$$\text{(i)} \quad d_i^1(v) = \begin{cases} a_i^{\boldsymbol{r}}(v) & \text{if } v \in G_1^{\boldsymbol{r}} \cup T_i^{\boldsymbol{r}}, \\ \min\Big\{a_i^{\boldsymbol{r}}(v), \max\{a_i^{\boldsymbol{r}}(v_i', v_{-i}) : (v_i', v_{-i}) \in G_1^{\boldsymbol{r}}\}\Big\} & \text{otherwise,} \end{cases}$$

$$\text{(ii)} \quad t_i^1(v) = \begin{cases} p_i^{\boldsymbol{r}}(v) & \text{if } v \in G_1^{\boldsymbol{r}} \cup T_i^{\boldsymbol{r}} \\ \sum_{\ell=L-d_i^1(v)+1}^{L} v_{-i}[\ell] & \text{otherwise,} \end{cases}$$

Obviously, for each $v \in \hat{G}_1^{\boldsymbol{r}}$, $g^1(v) = f^{\boldsymbol{r}}(v)$. Let us show that under a revelation mechanism (V_A^n, g^1), reporting the true valuation function is a dominant strategy for each individual.

Take any $i \in I$, any $v \in V_A^n$, and any $v_i' \in V_A$. Let us show that

$$u\left(g^1(v_i, v_{-i}); v_i\right) \geq u\left(g^1(v_i', v_{-i}); v_i\right). \tag{3.83}$$

We divide into three cases.

Case 1. Consider the case with $(v_i, v_{-i}) \in G_1^{\boldsymbol{r}}$. In this case, by definition of $f^{\boldsymbol{r}}$,

$$d_i^1(v_i, v_{-i}) = a_i^r(v),$$
$$t_i^1(v_i, v_{-i}) = \sum_{\ell=L-d_i^1(v)+1}^{L} v_{-i}[\ell].$$

Note that by definition of $(d^1, t_1^1, \ldots, t_n^1)$,

$$t_i^1(v_i', v_{-i}) = \sum_{\ell=L-d_i^1(v_i', v_{-i})+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell,$$

or

$$t_i^1(v_i', v_{-i}) = \sum_{\ell=L-d_i^1(v_i', v_{-i})+1}^{L} v_{-i}[\ell].$$

In addition, by Lemma 5,

$$\sum_{\ell=L-d_i^1(v_i',v_{-i})+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell \geq \sum_{\ell=L-d_i^1(v_i',v_{-i})+1}^{L} v_{-i}[\ell].$$

Then, by definition of $(d^1, t_1^1, \ldots, t_n^1)$, this equation implies that

$$t_i^1(v_i', v_{-i}) \geq \sum_{\ell=L-d_i^1(v_i',v_{-i})+1}^{L} v_{-i}[\ell].$$

Therefore, by Lemma 4,

$$\begin{aligned}
u\big(g^1(v_i, v_{-i}); v_i\big) &= \sum_{\ell=1}^{a_i^r(v)} v_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L} v_{-i}[\ell] \\
&\geq \sum_{\ell=1}^{d_i^1(v_i',v_{-i})} v_{i\ell} - \sum_{\ell=L-d_i^1(v_i',v_{-i})+1}^{L} v_{-i}[\ell] \\
&\geq \sum_{\ell=1}^{d_i^1(v_i',v_{-i})} v_{i\ell} - t_i^1(v_i', v_{-i}) = u\big(g^1(v_i', v_{-i}); v_i\big).
\end{aligned}$$

Case 2. Consider the case with $(v_i, v_{-i}) \in T_i^{\boldsymbol{r}}$. In this case,

$$\begin{aligned}
d_i^1(v_i, v_{-i}) &= a_i^r(v), \\
t_i^1(v_i, v_{-i}) &= \sum_{\ell=L-a_i^r(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell.
\end{aligned}$$

Suppose that $(v_i', v_{-i}) \in G_1^{\boldsymbol{r}}$. Then, $g^1(v_i', v_{-i}) = f^{\boldsymbol{r}}(v_i', v_{-i})$. Hence by $(v_i, v_{-i}) \in T_i^{\boldsymbol{r}}$ and definition of $T_i^{\boldsymbol{r}}$,

$$u\big(g^1(v_i, v_{-i}); v_i\big) = u\left(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i\right) \geq u\left(f^{\boldsymbol{r}}(v_i', v_{-i}); v_i\right) = u\left(g^1(v_i', v_{-i}); v_i\right).$$

Next, suppose that $(v_i', v_{-i}) \subset T_i^{\boldsymbol{r}}$. Then,

$$\begin{aligned}
d_i^1(v_i', v_{-i}) &= a_i^r(v_i', v_{-i}), \\
t_i^1(v_i', v_{-i}) &= \sum_{\ell=L-a_i^r(v_i',v_{-i})+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell.
\end{aligned}$$

Thus, by Lemma 4,

$$u\big(g^1(v_i, v_{-i}); v_i\big) = \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v_{i\ell} - \sum_{\ell=L-a_i^r(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell$$
$$\geq \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v_i', v_{-i})} v_{i\ell} - \sum_{\ell=L-a_i^r(v_i', v_{-i})+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell = u\big(g^1(v_i', v_{-i}); v_i\big).$$

Finally, suppose that $(v_i', v_{-i}) \in G_0^{\boldsymbol{r}}$ and $(v_i', v_{-i}) \notin T_i^{\boldsymbol{r}}$. Then,

$$d_i^1(v_i', v_{-i}) = \min\Big\{a_i^{\boldsymbol{r}}(v_i', v_{-i}), \max_{w_i \in V_A}\{a_i^{\boldsymbol{r}}(w_i, v_{-i}) : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\}\Big\},$$
$$t_i^1(v_i', v_{-i}) = \sum_{\ell=L-d_i^1(v_i', v_{-i})+1}^{L} v_{-i}[\ell].$$

Note that in particular, $d_i^1(v_i', v_{-i}) \leq \max_{w_i \in V_A}\{a_i^{\boldsymbol{r}}(w_i, v_{-i}) : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\}$. Then, there exists $w_i \in V_A$ such that $(w_i, v_{-i}) \in G_1^{\boldsymbol{r}}$ and $a_i^{\boldsymbol{r}}(w_i, v_{-i}) \geq d_i^1(v_i', v_{-i})$. For each $k \leq L$, let

$$w_i^{(k)} = (w_{i1}, \ldots, w_{ik}, 0, \ldots, 0) \in V_A.$$

Then, for each $k \leq L$, $(w_i^{(k)}, v_{-i}) \in G_1^{\boldsymbol{r}}$. Moreover, by $a_i^{\boldsymbol{r}}(w_i, v_{-i}) \geq d_i^1(v_i', v_{-i})$ and by *efficiency* of $f^{\boldsymbol{r}}$, there exists $k \leq L$ such that

$$u\left(f^{\boldsymbol{r}}(w_i^{(k)}, v_{-i}); v_i\right) = u\left(g^1(v_i', v_{-i}); v_i\right).$$

Since $(v_i, v_{-i}) \in T_i^{\boldsymbol{r}}$, by definition of $T_i^{\boldsymbol{r}}$,

$$u\big(g^1(v_i, v_{-i}); v_i\big) = u\left(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i\right) \geq u\left(f^{\boldsymbol{r}}(w_i^{(k)}, v_{-i}); v_i\right) = u\left(g^1(v_i', v_{-i}); v_i\right).$$

Case 3. Consider the case with $(v_i, v_{-i}) \in G_0^{\boldsymbol{r}}$ and $(v_i, v_{-i}) \notin T_i^{\boldsymbol{r}}$. In this case,

$$d_i^1(v_i, v_{-i}) = \min\Big\{a_i^{\boldsymbol{r}}(v), \max_{w_i \in V_A}\{a_i^{\boldsymbol{r}}(w_i, v_{-i}) : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\}\Big\},$$
$$t_i^1(v_i, v_{-i}) = \sum_{\ell=L-d_i^1(v)+1}^{L} v_{-i}[\ell].$$

If $d_i^1(v_i, v_{-i}) = a_i^{\boldsymbol{r}}(v_i, v_{-i})$, then by the same argument as Case 1, the condition holds true. Suppose that $d_i^1(v_i, v_{-i}) = \max_{w_i \in V_A}\{a_i^{\boldsymbol{r}}(w_i, v_{-i}) : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\}$. We consider two subcases.

Subcase 3-1. Consider the case with $(v_i', v_{-i}) \in G_1^{\boldsymbol{r}}$ or $(v_i', v_{-i}) \in G_0^{\boldsymbol{r}} \setminus T_i^{\boldsymbol{r}}$. Then,

$$d_i^1(v_i', v_{-i}) \leq \max_{w_i \in V_A} \{a_i^{\boldsymbol{r}}(w_i, v_{-i}) : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\} = d_i^1(v_i, v_{-i}) \leq a_i^{\boldsymbol{r}}(v_i, v_{-i}).$$

In addition,

$$t_i^1(v_i', v_{-i}) = \sum_{\ell = L - d_i^1(v_i', v_{-i}) + 1}^{L} v_{-i}[\ell].$$

Then, by Lemma 4,

$$\begin{aligned} u\big(g^1(v_i, v_{-i}); v_i\big) &= \sum_{\ell=1}^{d_i^1(v_i, v_{-i})} v_{i\ell} - \sum_{\ell = L - d_i^1(v_i, v_{-i}) + 1}^{L} v_{-i}[\ell] \\ &\geq \sum_{\ell=1}^{d_i^1(v_i', v_{-i})} v_{i\ell} - \sum_{\ell = L - d_i^1(v_i', v_{-i}) + 1}^{L} v_{-i}[\ell] = u\big(g^1(v_i', v_{-i}); v_i\big). \end{aligned}$$

Subcase 3-2. Consider the case with $(v_i', v_{-i}) \in T_i^{\boldsymbol{r}}$. Then,

$$\begin{aligned} d_i^1(v_i', v_{-i}) &= a_i^{r}(v_i', v_{-i}), \\ t_i^1(v_i', v_{-i}) &= \sum_{\ell = L - a_i^{r}(v_i', v_{-i}) + 1}^{L - |E^r(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^r(v_{-i})} r_\ell. \end{aligned}$$

Since $d_i^1(v_i, v_{-i}) = \max_{w_i \in V_A} \{a_i^{\boldsymbol{r}}(w_i, v_{-i}) : (w_i, v_{-i}) \in G_1^{\boldsymbol{r}}\}$, there exists $w_i \in V_A$ such that $(w_i, v_{-i}) \in G_1^{\boldsymbol{r}}$ and $d_i^1(v_i, v_{-i}) = a_i^{\boldsymbol{r}}(w_i, v_{-i})$. Then, $t_i^1(v_i, v_{-i}) = p_i^{\boldsymbol{r}}(w_i, v_{-i})$, and hence

$$u\big(g^1(v_i, v_{-i}); v_i\big) = u\left(f^{\boldsymbol{r}}(w_i, v_{-i}); v_i\right). \tag{3.84}$$

In addition, for each $w_i' \in V_A$ with $(w_i', v_{-i}) \in G_1^{\boldsymbol{r}}$, since $a_i^{\boldsymbol{r}}(v_i, v_{-i}) \geq d_i^1(v_i, v_{-i}) = a_i^{\boldsymbol{r}}(w_i, v_{-i}) \geq a_i^{\boldsymbol{r}}(w_i', v_{-i})$, by Lemma 4,

$$\begin{aligned} u\left(f^{\boldsymbol{r}}(w_i, v_{-i}); v_i\right) &= \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(w_i, v_{-i})} v_{i\ell} - \sum_{\ell = L - a_i^{\boldsymbol{r}}(w_i, v_{-i}) + 1}^{L} v_{-i}[\ell] \\ &\geq \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(w_i', v_{-i})} v_{i\ell} - \sum_{\ell = L - a_i^{\boldsymbol{r}}(w_i', v_{-i}) + 1}^{L} v_{-i}[\ell] = u\left(f^{\boldsymbol{r}}(w_i', v_{-i}); v_i\right). \end{aligned} \tag{3.85}$$

Since $(v_i, v_{-i}) \in G_0^{\boldsymbol{r}}$ and $(v_i, v_{-i}) \notin T_i^{\boldsymbol{r}}$, by equation (3.85) and definition of $T_i^{\boldsymbol{r}}$,

$$u\left(f^{\boldsymbol{r}}(w_i, v_{-i}); v_i\right) > u\left(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i\right). \tag{3.86}$$

Therefore, by equations (3.84) and (3.86), and by Lemma 4,

$$
\begin{aligned}
u\big(g^1(v_i, v_{-i}); v_i\big) = u\left(f^{\boldsymbol{r}}(w_i, v_{-i}); v_i\right) > u\left(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i\right) & \\
&= \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell \\
&\geq \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v_i', v_{-i})} v_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v_i', v_{-i})+1}^{L-|E^{\boldsymbol{r}}(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^{\boldsymbol{r}}(v_{-i})} r_\ell \\
&= u\big(g^1(v_i', v_{-i}); v_i\big).
\end{aligned}
$$

Then, by Steps 1 and 2, a set of revelation mechanisms $\big\{(V_A^n, g^0), (V_A^n, g^1)\big\}$ G-implements $f^{\boldsymbol{r}}$ with respect to $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$. □

Proof of Statement (ii) of Theorem 2. Let $f = (a, p)$ be a social choice rule that is *efficient*, *individually rational*, and *G-implementable with respect to* $\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$. Note that since $\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\} \subset \{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\}$, f is also G-implementable with respect to $\{G_0^{\boldsymbol{r}}, G_1^{\boldsymbol{r}}\}$. Then, by Statement (ii) of Lemma 1, $f^{\boldsymbol{r}}$ generates a higher revenue than that of f. □

Proof of Statement (iii) of Theorem 2. Take any $\{G_\lambda\}_{\lambda \in \Lambda}$ with $\big\{\hat{G}_0^{\boldsymbol{r}}, \hat{G}_1^{\boldsymbol{r}}\big\} \subsetneq \{G_\lambda\}_{\lambda \in \Lambda}$. Then, there exists $G \in \{G_\lambda\}_{\lambda \in \Lambda}$ such that $\hat{G}_0^{\boldsymbol{r}} \subsetneq G$ or $\hat{G}_1^{\boldsymbol{r}} \subsetneq G$. Let us show that $f^{\boldsymbol{r}}$ is no longer G-implementable with respect to $\{G_\lambda\}_{\lambda \in \Lambda}$. We consider two cases.

Case 1. Let us consider the case with $\hat{G}_1^{\boldsymbol{r}} \subsetneq G$. Then, there exists $v \in G$ such that $v \notin \hat{G}_1^{\boldsymbol{r}} = G_1^{\boldsymbol{r}} \cup T^{\boldsymbol{r}}$. That is, $v \in G_0^{\boldsymbol{r}}$ and $v \notin T^{\boldsymbol{r}}$. Since $T^{\boldsymbol{r}} = \bigcap_{i \in I} T_i^{\boldsymbol{r}}$, there exists $i \in N$ such that $v \notin T_i^{\boldsymbol{r}}$. Then, by definition of $T_i^{\boldsymbol{r}}$, there exists $v_i' \in V_A$ such that $(v_i', v_i) \in G_1^{\boldsymbol{r}} \subset G$ and $u\left(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i\right) < u\left(f^{\boldsymbol{r}}(v_i', v_{-i}); v_i\right)$. This is a contradiction to Lemma 2.

Case 2. Let us consider the case with $\hat{G}_0^{\boldsymbol{r}} \subsetneq G$. Then, there exists $v \in G$ such that $v \notin \hat{G}_0^{\boldsymbol{r}} = G_0^{\boldsymbol{r}} \cup \{\mathbf{0}\}$. That is, $v \in G_1^{\boldsymbol{r}}$ and $v \neq \mathbf{0}$. By *efficiency* of $f^{\boldsymbol{r}}$, there exists $i \in N$ such that $a_i^{\boldsymbol{r}}(v) \geq 1$. Moreover, by $v \in G_1^{\boldsymbol{r}}$, $E^{\boldsymbol{r}}(v_{-i}) \neq \emptyset$, since otherwise $(v_i, v_{-i}) \in G_0^{\boldsymbol{r}}$. We first consider the case with $|E^{\boldsymbol{r}}(v_{-i})| \leq a_i^{\boldsymbol{r}}(v)$. Let $v_i' \in V_A$ be such that

$$
v_{i\ell}' = \begin{cases} \max\{v[1], r_1\} + \varepsilon & \text{if } \ell \leq a_i^{\boldsymbol{r}}(v), \\ 0 & \text{otherwise,} \end{cases}
$$

where ε is an arbitrary positive number. Then, $(v_i', v_{-i}) \in G_0^{\boldsymbol{r}} \subset G$. Moreover, by *efficiency* of $f^{\boldsymbol{r}}$ at (v_i', v_{-i}), $a_i^{\boldsymbol{r}}(v_i', v_{-i}) = a_i^{\boldsymbol{r}}(v)$. Therefore,

$$\begin{aligned} u\big(f^{\boldsymbol{r}}(v_i', v_{-i}); v_i'\big) &= \sum_{\ell=1}^{a_i^r(v_i', v_{-i})} v_{i\ell}' - \sum_{\ell=L-a_i^r(v_i',v_{-i})+1}^{L-|E^r(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^r(v_{-i})} r_\ell \\ &= \sum_{\ell=1}^{a_i^r(v)} v_{i\ell}' - \sum_{\ell=L-a_i^r(v)+1}^{L-|E^r(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^r(v_{-i})} r_\ell. \end{aligned}$$

On the other hand, since $(v_i, v_{-i}) \in G_1^{\boldsymbol{r}}$,

$$u\big(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i'\big) = \sum_{\ell=1}^{a_i^r(v)} v_{i\ell}' - \sum_{\ell=L-a_i^r(v)+1}^{L} v_{-i}[\ell].$$

Note that by Lemma 5,

$$\sum_{\ell=L-a_i^r(v)+1}^{L-|E^r(v_{-i})|} v_{-i}[\ell] + \sum_{r_\ell \in E^r(v_{-i})} r_\ell > \sum_{\ell=L-a_i^r(v)+1}^{L} v_{-i}[\ell].$$

Therefore,

$$u\big(f^{\boldsymbol{r}}(v_i', v_{-i}); v_i'\big) < u\big(f^{\boldsymbol{r}}(v_i, v_{-i}); v_i'\big),$$

a contradiction to Lemma 2.

We next consider the case with $|E^{\boldsymbol{r}}(v_{-i})| > a_i^{\boldsymbol{r}}(v)$. Let $v_i' \in V_A$ be such that

$$v_{i\ell}' = \begin{cases} \max\{v[1], r_1\} + \varepsilon & \text{if } \ell \leq a_i^{\boldsymbol{r}}(v), \\ E^{\boldsymbol{r}}(v_{-i})[\ell] & \text{if } a_i^{\boldsymbol{r}}(v) < \ell \leq |E^{\boldsymbol{r}}(v_{-i})|, \\ 0 & \text{otherwise,} \end{cases}$$

where ε is an arbitrary positive number. Then, $(v_i', v_{-i}) \in G_0^{\boldsymbol{r}} \subset G$. Moreover, by *efficiency* of $f^{\boldsymbol{r}}$ at (v_i', v_{-i}), $a_i^{\boldsymbol{r}}(v_i', v_{-i}) = |E^{\boldsymbol{r}}(v_{-i})|$. Therefore, by using definition of v_i',

$$\begin{aligned} u\big(f^{\boldsymbol{r}}(v_i', v_{-i}); v_i'\big) &= \sum_{\ell=1}^{a_i^r(v_i', v_{-i})} v_{i\ell}' - \sum_{\ell=L-a_i^r(v_i',v_{-i})+1}^{L-|E^r(v_{-i})|} v_{-i}[\ell] - \sum_{r_\ell \in E^r(v_{-i})} r_\ell \\ &= \sum_{\ell=1}^{|E^r(v_{-i})|} v_{i\ell}' - \sum_{r_\ell \in E^r(v_{-i})} r_\ell \\ &= \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v_{i\ell}' + \sum_{\ell=a_i^{\boldsymbol{r}}(v)+1}^{|E^r(v_{-i})|} E^{\boldsymbol{r}}(v_{-i})[\ell] - \sum_{r_\ell \in E^r(v_{-i})} r_\ell \end{aligned}$$

$$= \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v'_{i\ell} - \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} E^{\boldsymbol{r}}(v_{-i})[\ell].$$

On the other hand, since $(v_i, v_{-i}) \in G_1^{\boldsymbol{r}}$,

$$u\big(f^{\boldsymbol{r}}(v_i, v_{-i}); v'_i\big) = \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} v'_{i\ell} - \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L} v_{-i}[\ell].$$

Note that by definition of $E^{\boldsymbol{r}}(v_{-i})$ and $|E^{\boldsymbol{r}}(v_{-i})| > a_i^{\boldsymbol{r}}(v)$, for each $r_\ell \in E^{\boldsymbol{r}}(v_{-i})$,

$$r_\ell > v_{-i}\left[L - |E^{\boldsymbol{r}}(v_{-i})| + 1\right] \geq v_{-i}[L - a_i^{\boldsymbol{r}}(v) + 1].$$

Therefore,

$$\sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} E^{\boldsymbol{r}}(v_{-i})[\ell] > \sum_{\ell=1}^{a_i^{\boldsymbol{r}}(v)} \Big(v_{-i}[L - a_i^{\boldsymbol{r}}(v) + 1]\Big) \geq \sum_{\ell=L-a_i^{\boldsymbol{r}}(v)+1}^{L} v_{-i}[\ell].$$

Hence

$$u\big(f^{\boldsymbol{r}}(v'_i, v_{-i}); v'_i\big) < u\big(f^{\boldsymbol{r}}(v_i, v_{-i}); v'_i\big),$$

a contradiction to Lemma 2.

□

Chapter 4

An Axiomatic Foundation of the Multiplicative Human Development Index

*27

4.1 Introduction

Building on Sen's (1985) idea of capabilities, the Human Development Index (HDI) measures well-being of a society by aggregating the degrees of achievements in three characteristics: health, education, and income. However, over the two decades since its introduction, it had been pointed out that the aggregation formula has a serious drawback: the three characteristics are treated as completely substitutable (e.g., Desai, 1991; Sagar and Najam, 1998; Herrero, Martínez, and Villar, 2010). For example, no matter how bad the state of health is, it can be compensated by further education or additional income. Since achievements in each of the different characteristics contribute to different functionings, their measurements are not in fact completely substitutable. To limit the possibility of such substitutability, Herrero, Martínez, and Villar (2010) defined *minimal lower boundedness* and explored index functions that satisfy this property and other standard axioms: *symmetry for the characteristics*, *normalization*, and *separability*. In their main result, they claimed that a class of multiplicative index functions can be characterized by those axioms.

In 2010, the United Nations Development Programme revised the aggregation formula for HDI by replacing arithmetic mean with geometric mean in its definition. This new HDI belongs to the class of multiplicative index functions by Herrero, Martínez, and Villar. Therefore, their result seems to provide a rationale for the revision. In fact, Zambrano (2014) used their result as a key ingredient in providing a rationale for the revision.

Nevertheless, we show that Herrero, Martinez, and Villar's claim does not hold. We provide examples of non-multiplicative index functions satisfying all their axioms. This means, in particular, that the rationale provided by Zambrano (2014) for the new HDI needs to be fixed.

*27 This chapter is co-authored with Yoko Kawada and Shuhei Otani, and based on Kawada, Nakamura, and Otani (2018).

The purpose of this chapter is to find index functions that treat characteristics as non-substitutable, following Herrero, Martínez, and Villar. However, we focus on the case where achievements have already been aggregated across individuals; so, as in practice, an index function only aggregates them across characteristics. We thus abstract away from considerations regarding the distribution of human development among individuals (e.g., Foster, López-Calva, and Székely, 2005; Seth, 2013).

In this setting, we introduce a class of index functions, what we call *quasi-geometric means*. A quasi-geometric mean has a common function for all characteristics, based on which the index function takes the inverse of the geometric mean across characteristics. We first show that quasi-geometric means are the only index functions satisfying *symmetry for the characteristics*, *normalization*, and *separability*. Second, we prove that power means are the only quasi-geometric means satisfying *homogeneity*. Finally, it is shown that the new HDI (geometric mean) is the only power mean satisfying two complementability axioms; *minimal lower boundedness* and *sensitivity to lowest-level characteristics*, while the old HDI (arithmetic mean) is the only one satisfying *local substitutability*. Therefore, they can be interpreted as the opposite extremes in the class of power means in terms of complementability and substitutability. This contrast provides a theoretical justification for using the new HDI.

The rest of this chapter is organized as follows. Section 4.2 presents the model of Herrero, Martínez, and Villar. Section 4.3 introduces their main result and provides counterexamples. In Section 4.4, we provide characterizations, the proofs of which are relegated to Appendix 4.5. Section 4.5 concludes the discussion. Appendix 4.5 provides examples showing the tightness of the axioms in our theorems, and in Appendix 4.5 we present the model of Zambrano and a rationale for the new HDI in his setting, which builds on our Theorem 1 below.

4.2 The Model

A *society* consists of a finite number of *individuals* $N \equiv \{1, 2, \ldots, n\}$ ($n \geq 1$). Let $K \equiv \{1, 2, \ldots, k\}$ ($k \geq 1$) be a finite set of *characteristics*. In the case of the new HDI, the characteristics are health, education, and income.[*28]

For each $i \in N$ and each $j \in K$, a *measurement of i's achievement for j* is a value $y_{ij} \in [0, 1]$. Note that y_{ij}'s are normalized so that they are comparable independently of the

[*28] In practice, achievements in health, education, and income are measured by life expectancy at birth, mean years and expected years of schooling, and gross national income per capita, respectively.

units in which they are originally measured.[*29] A *measurement vector for $j \in K$* is a vector

$$\boldsymbol{y}_j \equiv \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{nj} \end{pmatrix} \in [0,1]^n.$$

Then a *social state* is a matrix

$$Y \equiv (\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_k) = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1k} \\ \vdots & \vdots & \cdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nk} \end{bmatrix} \in \Omega \equiv [0,1]^{nk}.$$

An *index function* is a continuous function $I : \Omega \to \mathbb{R}$ that assigns each social state $Y \in \Omega$ to an *index* $I(Y) \in \mathbb{R}$. The higher the index is, the better the social state is.

4.3 Counterexamples to Herrero, Martínez, and Villar's (2010) Theorem

Herrero, Martínez, and Villar (2010) claim that a class of multiplicative index functions can be characterized by the following five axioms. *Monotonicity* requires that in any social state, if all the measurements increase, then its index also increases.

Monotonicity. For each $X, Y \in \Omega$, if $X \gg Y$, then $I(X) > I(Y)$.[*30]

For each $Y \in \Omega$ and each permutation π on K, let $\pi(Y)$ be the social state that is obtained by arranging Y's columns according to π. *Symmetry for the characteristics* requires that an index function be independent of the labels of the characteristics.

Symmetry for the characteristics. For each $Y \in \Omega$ and each π, $I\big(\pi(Y)\big) = I(Y)$.

For convenience, define

$$\mathbf{1}_n \equiv \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{n \text{ rows}} \quad \text{and} \quad \mathbf{0}_n \equiv \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{n \text{ rows}}.$$

[*29] It is worth noting that careful thought has to be put when normalizing data to values on the unit interval because the choice of normalization method affects the ordering over the values (Zambrano 2014).

[*30] For each $X, Y \in \Omega$, $X \gg Y$ means that $x_{ij} > y_{ij}$ for all $i \in N$ and all $j \in K$.

In addition, define

$$\mathbf{1} \equiv [\underbrace{\mathbf{1}_n, \mathbf{1}_n, \ldots, \mathbf{1}_n}_{k \text{ columns}}] \quad \text{and} \quad \mathbf{0} \equiv [\underbrace{\mathbf{0}_n, \mathbf{0}_n, \ldots, \mathbf{0}_n}_{k \text{ columns}}].$$

Normalization requires that in any social state, if all the measurements take the same value, then its index also takes the value.

Normalization. For each $\alpha \in [0, 1]$, $I(\alpha \cdot \mathbf{1}) = \alpha$.

For each $Y \in \Omega$ and each $j \in K$, let

$$Y_{-j} \equiv (\boldsymbol{y}_i)_{i \in K \setminus \{j\}} \in [0, 1]^{n(k-1)}.$$

Minimal lower boundedness requires that in any social state, if there exists a characteristic for which the measurement vector is at the lowest level, then its index is not more than that of any other social state.

Minimal lower boundedness. For each $X, Y \in \Omega$ and each $j \in K$, $I(X) \geq I(Y_{-j}, \mathbf{0}_n(j))$.

Consider two arbitrary social states with a common measurement vector for some characteristic. *Separability* requires that if this common measurement vector is replaced with another one, then an index function preserves the order between the two social states.

Separability. For each $X, Y \in \Omega$ with $X, Y \gg \mathbf{0}$ and each $j \in K$,

$$I(X_{-j}, \boldsymbol{x}_j) \geq I(Y_{-j}, \boldsymbol{x}_j) \Longrightarrow I(X_{-j}, \boldsymbol{y}_j) \geq I(Y_{-j}, \boldsymbol{y}_j).$$

Consider any index function $I : \Omega \to \mathbb{R}$ satisfying the aforementioned five axioms. Herrero, Martínez, and Villar (2010) define the *egalitarian equivalent value function* $\xi_j : \Omega \to \mathbb{R}$ *for each* $j \in K$ implicitly by such an I. That is, for each $Y \in \Omega$,

$$I(Y) = I\left(Y_{-j}, \xi_j(Y_{-j}, \boldsymbol{y}_j) \cdot \mathbf{1}_n\right).$$

For each $j \in K$, if $\xi_j : \Omega \to \mathbb{R}$ is independent of Y_{-j}, i.e.,

$$\forall \boldsymbol{y}_j \in [0, 1]^n,\ \forall X_{-j}, Y_{-j} \in [0, 1]^{n(k-1)},\ \xi_j(X_{-j}, \boldsymbol{y}_j) = \xi_j(Y_{-j}, \boldsymbol{y}_j),$$

then $\xi_j(Y_{-j}, \boldsymbol{y}_j)$ is simply denoted by $\xi_j(\boldsymbol{y}_j)$. In addition, if all the egalitarian equivalent value functions are independent of their characteristic, i.e.,

$$\forall j, \ell \in K,\ \xi_j = \xi_\ell,$$

then the *common egalitarian equivalent value function* ξ_j is simply denoted by $\xi : [0,1]^n \to \mathbb{R}$.

Herrero, Martínez, and Villar (2010, Theorem) claim that an index function satisfies the set of the five axioms if and only if it takes the multiplicative form of a common egalitarian equivalent value function, which contains the new HDI.

Claim 1 (Herrero, Martínez, and Villar, 2010, Theorem)**.** *For each index function* $I : \Omega \to \mathbb{R}$*, the following statements (i) and (ii) are equivalent:*

(i) $I : \Omega \to \mathbb{R}$ *satisfies monotonicity, symmetry for the characteristics, normalization, minimal lower boundedness, and separability;*

(ii) there exists $\xi : [0,1]^n \to \mathbb{R}$ *such that for each* $Y \in \Omega$*,*

$$I(Y) = \prod_{j \in K} \xi(\boldsymbol{y}_j)^{\frac{1}{k}}.$$

We provide two counterexamples to this claim which show that (i) does not imply (ii) for any $k \geq 2$. We assume $n = 1$ for simplicity, but the ideas of these counterexamples work for any $n \geq 2$.

Example 1 ($k = 2$)**.** Let $\hat{I} : [0,1]^2 \to \mathbb{R}$ be an index function such that for each $(y_1, y_2) \in [0,1]^2$,

$$\hat{I}(y_1, y_2) \equiv \frac{1}{2} y_1^{\frac{2}{3}} y_2^{\frac{1}{3}} + \frac{1}{2} y_1^{\frac{1}{3}} y_2^{\frac{2}{3}}.$$

Then $\hat{I}$ satisfies (i) but violates (ii). ◊

Proof. **Step 1: $\hat{I}$ satisfies (i).** One can easily show that $\hat{I}$ satisfies *monotonicity*, *symmetry for the characteristics*, *normalization*, and *minimal lower boundedness*. To show *separability*, take any $(x_1, x_2), (y_1, y_2) \gg \mathbf{0}$. We need to prove that

$$\hat{I}(x_1, x_2) \geq \hat{I}(y_1, x_2) \Longrightarrow \hat{I}(x_1, y_2) \geq \hat{I}(y_1, y_2); \text{ and}$$
$$\hat{I}(x_1, x_2) \geq \hat{I}(x_1, y_2) \Longrightarrow \hat{I}(y_1, x_2) \geq \hat{I}(y_1, y_2).$$

We only offer a proof for the first equation since the second can be proven in a similar way. Suppose $\hat{I}(x_1, x_2) \geq \hat{I}(y_1, x_2)$. Then

$$\begin{aligned} 0 &\leq \hat{I}(x_1, x_2) - \hat{I}(y_1, x_2) \\ &= \left[\frac{1}{2} x_1^{\frac{2}{3}} x_2^{\frac{1}{3}} + \frac{1}{2} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}\right] - \left[\frac{1}{2} y_1^{\frac{2}{3}} x_2^{\frac{1}{3}} + \frac{1}{2} y_1^{\frac{1}{3}} x_2^{\frac{2}{3}}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}x_2^{\frac{1}{3}}\left[(x_1^{\frac{2}{3}} - y_1^{\frac{2}{3}}) + (x_1^{\frac{1}{3}} - y_1^{\frac{1}{3}})x_2^{\frac{1}{3}}\right] \\
&= \frac{1}{2}x_2^{\frac{1}{3}}\left[(x_1^{\frac{1}{3}} + y_1^{\frac{1}{3}})(x_1^{\frac{1}{3}} - y_1^{\frac{1}{3}}) + (x_1^{\frac{1}{3}} - y_1^{\frac{1}{3}})x_2^{\frac{1}{3}}\right] \\
&= \frac{1}{2}x_2^{\frac{1}{3}}(x_1^{\frac{1}{3}} + y_1^{\frac{1}{3}} + x_2^{\frac{1}{3}})(x_1^{\frac{1}{3}} - y_1^{\frac{1}{3}}).
\end{aligned}
$$

Hence, by $x_1, x_2, y_1 > 0$, $x_1^{\frac{1}{3}} \geq y_1^{\frac{1}{3}}$. Therefore,

$$
\hat{I}(x_1, y_2) - \hat{I}(y_1, y_2) = \frac{1}{2}y_2^{\frac{1}{3}}(x_1^{\frac{1}{3}} + y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}})(x_1^{\frac{1}{3}} - y_1^{\frac{1}{3}}) \geq 0,
$$

meaning that $\hat{I}(x_1, y_2) \geq \hat{I}(y_1, y_2)$.

Step 2: $\hat{I}$ violates (ii). Suppose, by contradiction, that there exists $\xi : [0, 1] \to \mathbb{R}$ such that for each $(y_1, y_2) \in [0, 1]^2$,

$$
\hat{I}(y_1, y_2) = \xi(y_1)^{\frac{1}{2}}\xi(y_2)^{\frac{1}{2}}. \tag{4.87}
$$

Then for each $\alpha \in [0, 1]$,

$$
\alpha = \hat{I}(\alpha, \alpha) = \xi(\alpha)^{\frac{1}{2}}\xi(\alpha)^{\frac{1}{2}} = \xi(\alpha). \tag{4.88}
$$

By computation,

$$
\begin{aligned}
\hat{I}\left(1, \frac{1}{64}\right) &= \frac{1}{2} \cdot 1^{\frac{2}{3}} \cdot \left(\frac{1}{64}\right)^{\frac{1}{3}} + \frac{1}{2} \cdot 1^{\frac{1}{3}} \cdot \left(\frac{1}{64}\right)^{\frac{2}{3}} \\
&= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{16} \\
&= \frac{5}{32},
\end{aligned}
$$

and thus, by (4.87),

$$
\xi(1)^{\frac{1}{2}}\xi\left(\frac{1}{64}\right)^{\frac{1}{2}} = \frac{5}{32}.
$$

However, by (4.88),

$$
\xi(1)^{\frac{1}{2}}\xi\left(\frac{1}{64}\right)^{\frac{1}{2}} = 1^{\frac{1}{2}} \cdot \left(\frac{1}{64}\right)^{\frac{1}{2}} = \frac{1}{8},
$$

a contradiction. ∎

A natural extension of $\hat{I}$ to the case $k = 3$ is that for each $(y_1, y_2, y_3) \in [0, 1]^3$,

$$\hat{I}(y_1, y_2, y_3) \equiv \frac{1}{3} y_1^{\frac{2}{4}} y_2^{\frac{1}{4}} y_3^{\frac{1}{4}} + \frac{1}{3} y_1^{\frac{1}{4}} y_2^{\frac{2}{4}} y_3^{\frac{1}{4}} + \frac{1}{3} y_1^{\frac{1}{4}} y_2^{\frac{1}{4}} y_3^{\frac{2}{4}}.$$

However, this violates *separability* since

$$\hat{I}(1, 0.082, 0.01) \fallingdotseq 0.1044 > 0.1037 \fallingdotseq \hat{I}(0.3, 0.3, 0.01),$$
$$\hat{I}(1, 0.082, 1) \fallingdotseq 0.4522 < 0.4528 \fallingdotseq \hat{I}(0.3, 0.3, 1),$$

meaning that the counterexample works only for $k = 2$. Therefore, we provide another counterexample for $k = 3$, which can be modified to be a counterexample to Zambrano's (2014) Theorem 1. This counterexample works for any $k \geq 2$ with straightforward generalization.

Example 2 ($k = 3$)**.** Let $\tilde{I} : [0, 1]^3 \to [0, 1]$ be an index function such that for each $(y_1, y_2, y_3) \in [0, 1]^3$,

$$\tilde{I}(y_1, y_2, y_3) \equiv \log\left[(e^{y_1} - 1)^{\frac{1}{3}}(e^{y_2} - 1)^{\frac{1}{3}}(e^{y_3} - 1)^{\frac{1}{3}} + 1\right].$$

Then $\tilde{I}$ satisfies (i) but violates (ii). ◊

Proof. **Step 1: $\tilde{I}$ satisfies (i).** One can easily show that $\tilde{I}$ satisfies *monotonicity*, *symmetry for the characteristics*, *normalization*, and *minimal lower boundedness*. To show *separability*, take any $(x_1, x_2, x_3), (y_1, y_2, y_3) \gg \mathbf{0}$. We need to prove that

$$\hat{I}(x_1, x_2, x_3) \geq \hat{I}(y_1, y_2, x_3) \Longrightarrow \hat{I}(x_1, x_2, y_3) \geq \hat{I}(y_1, y_2, y_3);$$
$$\hat{I}(x_1, x_2, x_3) \geq \hat{I}(y_1, x_2, y_3) \Longrightarrow \hat{I}(x_1, y_2, x_3) \geq \hat{I}(y_1, y_2, y_3); \text{ and}$$
$$\hat{I}(x_1, x_2, x_3) \geq \hat{I}(x_1, y_2, y_3) \Longrightarrow \hat{I}(y_1, x_2, x_3) \geq \hat{I}(y_1, y_2, y_3).$$

We only offer a proof for the first equation since the second and third can be proven in a similar way. Suppose $\tilde{I}(x_1, x_2, x_3) \geq \tilde{I}(y_1, y_2, x_3)$. Then

$$\log\left[(e^{x_1} - 1)^{\frac{1}{3}}(e^{x_2} - 1)^{\frac{1}{3}}(e^{x_3} - 1)^{\frac{1}{3}} + 1\right] \geq \log\left[(e^{y_1} - 1)^{\frac{1}{3}}(e^{y_2} - 1)^{\frac{1}{3}}(e^{x_3} - 1)^{\frac{1}{3}} + 1\right].$$

Since the base e is not less than 1,

$$(e^{x_1} - 1)^{\frac{1}{3}}(e^{x_2} - 1)^{\frac{1}{3}}(e^{x_3} - 1)^{\frac{1}{3}} \geq (e^{y_1} - 1)^{\frac{1}{3}}(e^{y_2} - 1)^{\frac{1}{3}}(e^{x_3} - 1)^{\frac{1}{3}}.$$

Since $e^{x_3} - 1 > 0$ by $x_3 > 0$,

$$(e^{x_1} - 1)(e^{x_2} - 1) \geq (e^{y_1} - 1)(e^{y_2} - 1).$$

Hence,

$$\begin{aligned}
&\tilde{I}(x_1, x_2, y_3) - \tilde{I}(y_1, y_2, y_3) \\
&= \log\left[(e^{x_1} - 1)^{\frac{1}{3}}(e^{x_2} - 1)^{\frac{1}{3}}(e^{y_3} - 1)^{\frac{1}{3}} + 1\right] - \log\left[(e^{y_1} - 1)^{\frac{1}{3}}(e^{y_2} - 1)^{\frac{1}{3}}(e^{y_3} - 1)^{\frac{1}{3}} + 1\right] \\
&\geq \log\left[(e^{y_1} - 1)^{\frac{1}{3}}(e^{y_2} - 1)^{\frac{1}{3}}(e^{y_3} - 1)^{\frac{1}{3}} + 1\right] - \log\left[(e^{y_1} - 1)^{\frac{1}{3}}(e^{y_2} - 1)^{\frac{1}{3}}(e^{y_3} - 1)^{\frac{1}{3}} + 1\right] \\
&= 0.
\end{aligned}$$

Therefore, $\tilde{I}(x_1, x_2, y_3) \geq \tilde{I}(y_1, y_2, y_3)$.

Step 2: $\tilde{I}$ violates (ii). Suppose, by contradiction, that there exists $\xi : [0, 1] \to \mathbb{R}$ such that for each $(y_1, y_2, y_3) \in [0, 1]^3$,

$$\tilde{I}(y_1, y_2, y_3) = \xi(y_1)^{\frac{1}{3}}\xi(y_2)^{\frac{1}{3}}\xi(y_3)^{\frac{1}{3}}. \tag{4.89}$$

Then for each $\alpha \in [0, 1]$,

$$\alpha = \tilde{I}(\alpha, \alpha, \alpha) = \xi(\alpha)^{\frac{1}{3}}\xi(\alpha)^{\frac{1}{3}}\xi(\alpha)^{\frac{1}{3}} = \xi(\alpha). \tag{4.90}$$

By (4.89) and computation,

$$\xi(0.1)^{\frac{1}{3}}\xi(0.5)^{\frac{1}{3}}\xi(0.9)^{\frac{1}{3}} = \tilde{I}(0.1, 0.5, 0.9) \risingdotseq 0.3808.$$

However, by (4.90),

$$\xi(0.1)^{\frac{1}{3}}\xi(0.5)^{\frac{1}{3}}\xi(0.9)^{\frac{1}{3}} = (0.1)^{\frac{1}{3}} \cdot (0.5)^{\frac{1}{3}} \cdot (0.9)^{\frac{1}{3}} \risingdotseq 0.3557,$$

a contradiction. ■

4.4 Characterizations

In this section, we search for an axiomatic foundation of the new HDI. To that end, we focus on the real-use situation by assuming $n = 1$ and $k \geq 3$. Assumption $n = 1$ simplifies a social state to a vector $\boldsymbol{y} \equiv (y_1, \ldots, y_k) \in [0, 1]^k$, which consists of the *aggregated measurements* in each characteristic as in practice.[*31]

In aggregation theory, a number of studies have investigated the class of quasi-arithmetic means, which was introduced by Aczél (1948) in a fixed characteristic model.

[*31] Even when $n \geq 2$, we can aggregate all individuals' measurements for each characteristic by using *egalitarian equivalent value functions* defined in Section 4.3, and then we obtain the situation $n = 1$. Therefore, the assumption $n = 1$ is not restrictive in the real-use situation.

Definition 1. An index function $I : [0,1]^k \to [0,1]$ is a **quasi-arithmetic mean** if there exists a continuous and strictly increasing function $\eta : [0,1] \to \mathbb{R}$ such that for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) \equiv \eta^{-1}\Big(\frac{1}{k}\sum_{j\in K}\eta(y_j)\Big).$$

On the one hand, it is worth noting that the quasi-arithmetic mean with $\eta(y_j) = y_j$ turns out to be arithmetic mean. On the other hand, as long as the domain is restricted to $(0,1]^k$, the quasi-arithmetic mean with $\eta(y_j) = \log y_j$ turns out to be geometric mean. However, since the domain includes 0 in our model, geometric mean does not belong to the class of quasi-arithmetic means.

So, as a counterpart in our model, we define quasi-geometric means. It is then shown that they are the only index functions satisfying *symmetry for the characteristics*, *normalization*, and *separability*.[*32]

Definition 2. An index function $I : [0,1]^k \to [0,1]$ is a **quasi-geometric mean** if there exists a continuous and strictly increasing function $\eta : [0,1] \to \mathbb{R}$ such that for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) \equiv \eta^{-1}\Big(\prod_{j\in K}\eta(y_j)^{\frac{1}{k}}\Big).$$

Proposition 1. *Suppose $n = 1$ and $k \geq 3$. Then quasi-geometric means are the only index functions satisfying symmetry for the characteristics, normalization, and separability.*

Proof. See Appendix 4.5. ■

Let us sketch the proof for the uniqueness in our first proposition. Consider any index function I satisfying the trio of axioms. First, we generate a continuous and complete preordering from I. It inherits *symmetry for the characteristics* and *separability* of I, which enables us to apply Debreu's representation theorem (1959, Theorem 3): the preordering can be represented by an additively separable function. By transforming this function monotonically, we obtain a quasi-geometric mean, which is ordinally equivalent to I. Finally, by continuity and *normalization*, it can be shown that this quasi-geometric mean is in fact cardinally equivalent to I.

[*32] Characterizations of quasi-arithmetic means are often applied to constructing social indices, such as poverty measures or inequality measures (e.g., Foster and Shorrocks, 1991; Shorrocks, 1980). In this context, population is usually treated as *variable* so that *subgroup consistency* or *subgroup decomposability* can be imposed. However, in our model, these axioms require characteristics to be variable, which seems quite strange. So, we impose *separability* instead of the axioms, but it plays a role similar to them.

The index function $\tilde{I}$ in Example 2 is a quasi-geometric mean where $\eta(y_j) \equiv e^{y_j} - 1$ for each $y_j \in [0,1]$. Thus, it satisfies all the three axioms in Proposition 1. However, it violates *homogeneity*.[*33] This axiom requires that if all achievements grow by the same fraction of magnitude, then an index function also varies by the fraction of magnitude.

Homogeneity. For each $\boldsymbol{y} \in [0,1]^k$ and each $\lambda > 0$ with $\lambda \cdot \boldsymbol{y} \in [0,1]^k$,

$$I(\lambda \cdot \boldsymbol{y}) = \lambda \cdot I(\boldsymbol{y}).$$

By adding *homogeneity* to the set of axioms in Proposition 1, the class of power means can be characterized.[*34]

Definition 3. An index function $I : [0,1]^k \to \mathbb{R}$ is a **power mean with exponent** $p \in \mathbb{R}$ if for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) = \begin{cases} \prod_{j \in K} y_j^{\frac{1}{k}} & \text{if } p = 0, \\ \left(\frac{1}{k} \sum_{j \in K} y_j^p\right)^{\frac{1}{p}} & \text{if } p \neq 0. \end{cases}$$

Proposition 2. *Suppose $n = 1$ and $k \geq 3$. Then power means are the only index functions satisfying symmetry for the characteristics, normalization, separability, and homogeneity.*

Proof. See Appendix 4.5. ■

Both geometric mean ($p = 0$) and arithmetic mean ($p = 1$) belong to the class of power means, but they are opposite extremes in the class in terms of complementability and substitutability. First, recall that *minimal lower boundedness* requires that a poor achievement in a characteristic should not be compensated by a good achievement in other characteristics. However, this turns out to be too weak a local complementability axiom to pin down the new HDI. Hence, we impose an Inada-like boundary condition. *Sensitivity to lowest-level characteristics* requires that the marginal contribution of the improvement in a lowest-level characteristic be locally extremely large and thus cannot be compensated with good achievements in other characteristics.

Sensitivity to lowest-level characteristics. For each $i \in K$ and each $\boldsymbol{y}_{-i} \in (0,1]^{k-1}$,

$$\lim_{h \to +0} \frac{I(h, \boldsymbol{y}_{-i}) - I(0, \boldsymbol{y}_{-i})}{h} = +\infty.$$

[*33] The authors would like to thank an anonymous referee for suggesting this axiom.

[*34] Power means are also known as generalized means or Hölder means.

On the other hand, we consider *local substitutability* as defined below. It requires that a state exist in which achievements in some two characteristics are fully substitutable.

Local Substitutability. There exist $\boldsymbol{y} \in (0,1]^k$, $i, j \in K$ and $t > 0$ such that $y_i \neq y_j + t \in [0,1]$ and $y_j \neq y_i + t \in [0,1]$ for which

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

Note that *local substitutability* is weaker than the following notion of *full substitutability*: for each $\boldsymbol{y} \in [0,1]^k$, each $i, j \in K$, and each $t > 0$, if $y_i - t, y_j + t \in [0,1]$, then

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

Arithmetic mean (old HDI) satisfies both local and full substitutabilities. However, *local substitutability* is sufficient to pin it down within the class of power means.

Summarizing these observations, the next theorem contrasts the new and old HDIs within the class of power means: the new HDI is the only power mean satisfying *minimal lower boundedness* and *sensitivity to lowest-level characteristics*, while the old HDI is the only one satisfying *local substitutability*. Therefore, these aggregation formulas are opposite extremes in the class of power means in terms of complementability and substitutability. This contrast provides a theoretical justification for the use of the new HDI.

Theorem 1. *Suppose $n = 1$ and $k \geq 3$.*

(i) The new HDI (geometric mean) is the only index function satisfying symmetry for the characteristics, normalization, separability, homogeneity, minimal lower boundedness*, and* sensitivity to lowest-level characteristics*.*

(ii) The old HDI (arithmetic mean) is the only index function satisfying symmetry for the characteristics, normalization, separability, homogeneity, and local substitutability.

Proof. (i) Take any index function $I : [0,1]^k \to \mathbb{R}$ satisfying *symmetry for the characteristics*, *normalization*, *separability*, *homogeneity*, *minimal lower boundedness*, and *sensitivity to lowest-level characteristics*. Then by Proposition 2, I is a power mean with some exponent $p \in \mathbb{R}$. First, if the exponent were positive ($p > 0$), then

$$I(0, \underbrace{1, 1, \ldots, 1}_{k-1}) = \left(\frac{k-1}{k}\right)^{\frac{1}{p}} > 0,$$

which contradicts *minimal lower boundedness*. Second, suppose by contradiction that the exponent were negative ($p < 0$). Take any $i \in K$ and $\boldsymbol{y}_{-i} \in (0,1]^{k-1}$. Since $I(0, \boldsymbol{y}_{-i}) = 0$ by *minimal lower boundedness*, for all $h > 0$,

$$\begin{aligned}\frac{I(h, \boldsymbol{y}_{-i}) - I(0, \boldsymbol{y}_{-i})}{h} &= \frac{1}{h}\left(\frac{1}{k}(h^p + \sum_{j \neq i} y_j^p)\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{k}\left(1 + \sum_{j \neq i}\left(\frac{y_j}{h}\right)^p\right)\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{k}\left(1 + \sum_{j \neq i}\left(\frac{h}{y_j}\right)^{-p}\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Since $p < 0$, by sending h to 0 from the right,

$$\lim_{h \to +0} \frac{I(h, \boldsymbol{y}_{-i}) - I(0, \boldsymbol{y}_{-i})}{h} = \left(\frac{1}{k}\left(1 + \sum_{j \neq i} \underbrace{\lim_{h \to +0}\left(\frac{h}{y_j}\right)^{-p}}_{=0}\right)\right)^{\frac{1}{p}} = \left(\frac{1}{k}\right)^{\frac{1}{p}} < \infty,$$

which contradictions *sensitivity to lowest-level characteristics*. Therefore, $p = 0$.

(ii) Take any index function $I : [0,1]^k \to \mathbb{R}$ satisfying *symmetry for the characteristics*, *normalization*, *separability*, *homogeneity*, and *local substitutability*. Then by Proposition 2, I is a power mean with some exponent $p \in \mathbb{R}$.

Step 1: there exist $\boldsymbol{y} \in (0,1]^k$, $i, j \in K$ and $t \in \mathbb{R} \setminus \{0\}$ such that $y_i \geq y_j$ and $1 \geq y_i - t \geq y_j + t \geq 0$ for which

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

By *local substitutability*, there exist $\boldsymbol{y} \in (0,1]^k$, $i, j \in K$ and $t > 0$ with $y_i \neq y_j + t \in [0,1]$ and $y_j \neq y_i + t \in [0,1]$ such that

$$I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

Consider the case witht $y_i \geq y_j$. If $y_i - t \geq y_j + t$, then the claim holds. Thus, suppose that $y_i - t < y_j + t$. Let

$$t' = y_i - t - y_j.$$

Since $y_i \neq y_j + t$, $t' \in \mathbb{R} \setminus \{0\}$. Moreover,

$$y_j + t' = y_j + (y_i - t - y_j) = y_i - t,$$

$$y_i + t' = y_i - (y_i - t - y_j) = y_j + t.$$

Therefore, by *symmetry for characteristics*,

$$I(y_i - t', y_j + t', \boldsymbol{y}_{-i,j}) = I(y_j + t', y_i - t', \boldsymbol{y}_{-i,j}) = I(y_i - t, y_j + t, \boldsymbol{y}_{-i,j}) = I(\boldsymbol{y}).$$

By a similar argument, we can show the case with $y_j \geq y_i$.

Step 2 (show that $p \neq 0$)**.** Suppose, by contradiction, that $p = 0$. Then, by Step 1

$$\big(y_i - t\big)\big(y_j + t\big) = y_i y_j. \tag{4.91}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be such that for each $s \in \mathbb{R}$,

$$f(s) = \big(y_i - s\big)\big(y_j + s\big).$$

By differentiating this function for s, we have

$$f'(s) = \big(y_i - s\big) - \big(y_j + s\big).$$

Then, for any $s \in \mathbb{R}$ with $y_i - s > y_j + s$, $f'(s) > 0$, that is, f is strictly increasing on $\big(-\infty, \frac{y_i - y_j}{2}\big]$. Therefore, since $t \in \big(-\infty, \frac{y_i - y_j}{2}\big]$ and $t \neq 0$,

$$\big(y_i - t\big)\big(y_j + t\big) \neq y_i y_j,$$

a contradiction to equation (4.91).

Step 3 ($p = 1$)**.** Suppose, by contradiction, that $p \neq 1$. Then, by Steps 1 and 2,

$$\big(y_i - t\big)^p + \big(y_j + t\big)^p = \big(y_i\big)^p + \big(y_j\big)^p. \tag{4.92}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be such that for each $s \in \mathbb{R}$,

$$g(s) = \big(y_i - s\big)^p + \big(y_j + s\big)^p.$$

By differentiating this function for s, we have

$$g'(s) = -p\big(y_i - s\big)^{p-1} + p\big(y_j + s\big)^{p-1}.$$

Hence, for any $s \in \mathbb{R}$ with $y_i - s > y_j + s$, if $p > 1$, then $g'(s) < 0$ so g is strictly decreasing on $\big(-\infty, \frac{y_i - y_j}{2}\big]$; on the other hand, if $p < 1$, then $g'(s) > 0$ so g is strictly increasing on $\big(-\infty, \frac{y_i - y_j}{2}\big]$. In either case, since $t \in \big(-\infty, \frac{y_i - y_j}{2}\big]$ and $t \neq 0$, equation (4.92) never holds, a contradiction. ∎

4.5 Concluding Remarks

We considered the problem of designing an aggregation formula used in HDI. First, we showed that quasi-geometric means are the only index functions satisfying *symmetry for the characteristics*, *normalization*, and *separability*. Second, we showed that power means are the only index functions satisfying *homogeneity* as well as the trio of axioms. Although both the new HDI (geometric mean) and the old HDI (arithmetic mean) belong to the class of power means, they can be interpreted as the opposite extremes in terms of complementability and substitutability: that is, the new HDI is the only power mean satisfying *minimal lower boundedness* and *sensitivity to lowest-level characteristics* while the old HDI is the only one satisfying *local substitutability*. This contrast provides a theoretical justification for the use of the new HDI.

We also proved that Herrero, Martínez, and Villar's axiomatization (2010, Theorem) does not hold. Consequently, one might question which index functions satisfy their set of axioms: *monotonicity*, *symmetry for the characteristics*, *normalization*, *separability*, and *minimal lower boundedness*. Let us conclude the present chapter by answering this question in our setting $n = 1$ and $k \geq 3$. Note that as seen in the proof of Proposition 1 (Appendix 4.5), under $n = 1$, *monotonicity* is implied by the trio of *symmetry for the characteristics*, *normalization*, and *separability*.[*35] Thus, *monotonicity* can be dropped. It then follows from Proposition 1 that by adding *minimal lower boundedness* to the trio of the axioms therein, quasi-geometric means with $\eta(0) = 0$ can be characterized. This observation clarifies the importance of *homogeneity* in Theorem 1(i).

Corollary 1. *Suppose* $n = 1$ *and* $k \geq 3$. *Then quasi-geometric means with* $\eta(0) = 0$ *are the only index functions satisfying symmetry for the characteristics, normalization, separability, and minimal lower boundedness.*

Proof. Let us only show the uniqueness. Consider any index function $I : [0,1]^k \to \mathbb{R}$ satisfying *symmetry for the characteristics*, *normalization*, *separability*, and *minimal lower boundedness*. Then by Proposition 1, there exists a continuous and strictly increasing function $\eta : [0,1] \to \mathbb{R}_+$ such that for each $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) = \eta^{-1}\Big(\prod_{j\in K} \eta(y_j)^{\frac{1}{k}}\Big). \tag{4.93}$$

[*35] Any index function satisfying the trio of axioms is a quasi-geometric mean with a strictly increasing function $\eta : [0,1] \to \mathbb{R}$. Hence, the inverse function η^{-1} is also strictly increasing, and in turn the quasi-geometric mean satisfies *monotonicity*.

Take any $x \in (0,1]$. By (4.93) and by using *minimal lower boundedness* twice and *normalization*,

$$\eta^{-1}\left(\eta(0)^{\frac{1}{k}}\eta(x)^{\frac{k-1}{k}}\right) = I(0, x, \dots, x) = I(\mathbf{0}) = 0.$$

Hence,

$$\eta(0)^{\frac{1}{k}}\eta(x)^{\frac{k-1}{k}} = \eta(0),$$

meaning that

$$\eta(0)^{\frac{1}{k}}\left(\eta(x)^{\frac{k-1}{k}} - \eta(0)^{\frac{k-1}{k}}\right) = 0.$$

Note that since η is strictly increasing, $x > 0$ implies $\eta(x)^{\frac{k-1}{k}} > \eta(0)^{\frac{k-1}{k}}$. Therefore,

$$\eta(0) = 0. \qquad \blacksquare$$

Appendix A: Omitted Proofs in Section 4.4

In our proof of Proposition 1, we apply Debreu's (1959) representation theorem of a preference on a separable set of variables. To introduce this powerful theorem, let us first set up relevant definitions.

Debreu considers a continuous and complete preordering $\succsim$ on a commodity-bundle space $S \subset \mathbb{R}^{\ell}$. Suppose that this space can be decomposed into m subspaces $S_1, \dots, S_m$ ($m \leq \ell$), that is,

$$S = \times_{i=1}^{m} S_i.^{*36}$$

The factors $1, \dots, m$ are *independent* if for all $i \in \{1, \dots, m\}$, all $\boldsymbol{x}_i, \boldsymbol{y}_i \in S_i$, and all $\boldsymbol{x}_{-i}, \boldsymbol{y}_{-i} \in \times_{j \neq i} S_j$,

$$(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \succsim (\boldsymbol{y}_i, \boldsymbol{x}_{-i}) \iff (\boldsymbol{x}_i, \boldsymbol{y}_{-i}) \succsim (\boldsymbol{y}_i, \boldsymbol{y}_{-i}).$$

A factor $i \in \{1, \dots, m\}$ is *essential* if there exist $\boldsymbol{x}_{-i} \in \times_{j \neq i} S_j$ and $\boldsymbol{x}_i, \boldsymbol{y}_i \in S_i$ such that

$$(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \succ (\boldsymbol{y}_i, \boldsymbol{x}_{-i}).$$

Debreu's Representation Theorem (Debreu, 1959, Theorem 3)**.** *Suppose that the factors $1, \dots, m$ are independent and that at least three of them are essential. If S_i is connected for each $i \in \{1, \dots, m\}$, then a continuous and complete preordering $\succsim$ on $S = \times_{i=1}^{m} S_i$ can be represented by an additively separable utility function: that is, for each $i \in \{1, \dots, m\}$, there exists a continuous function $U_i : S_i \to \mathbb{R}$, and for each $\boldsymbol{x}, \boldsymbol{y} \in S$,*

$$\boldsymbol{x} \succsim \boldsymbol{y} \iff \sum_{i=1}^{m} U_i(\boldsymbol{x}_i) \geq \sum_{i=1}^{m} U_i(\boldsymbol{y}_i).$$

[*36] If $m < \ell$, then each S_i might be a vector space. This is why we denote a generic element of S_i by $\boldsymbol{x}_i$ instead of x_i.

A.1 Proof of Proposition 1

One can easily check that quasi-geometric means satisfy *symmetry for the characteristics*, *normalization*, and *separability*. Conversely, let us prove the uniqueness. Consider any index function $I : [0,1]^k \to \mathbb{R}$ satisfying the trio of axioms.

Step 1 (Find a function that is ordinally equivalent to I on $(0,1]^k$)**.** Note that *separability* of I is defined on $(0,1]^k$. Hence, to use Debreu's Representation Theorem, let $\hat{I}$ be the restriction of I to $(0,1]^k$. Then $\hat{I} : (0,1]^k \to \mathbb{R}$ is continuous on $(0,1]^k$ and satisfies *symmetry for the characteristics*, *normalization*, and *separability* on $(0,1]^k$.

Generate a continuous and complete preordering $\succsim$ on $(0,1]^k$ from $\hat{I}$ according to Euclidean distance $\geq$ on $\mathbb{R}$: for each $\boldsymbol{x}, \boldsymbol{y} \in (0,1]^k$,

$$\boldsymbol{x} \succsim \boldsymbol{y} \iff \hat{I}(\boldsymbol{x}) \geq \hat{I}(\boldsymbol{y}). \tag{4.94}$$

Then the space $(0,1]$ for each characteristic is connected.

First, we claim that the characteristics $1, \ldots, k$ are independent. For each $j \in \{1, \ldots, k\}$, each $x_j, y_j \in (0,1]$, and each $\boldsymbol{x}_{-j}, \boldsymbol{y}_{-j} \in (0,1]^{k-1}$, by (4.94) and *separability* of $\hat{I}$,

$$\begin{aligned}(x_j, \boldsymbol{x}_{-j}) \succsim (y_j, \boldsymbol{x}_{-j}) &\iff \hat{I}(x_j, \boldsymbol{x}_{-j}) \succsim \hat{I}(y_j, \boldsymbol{x}_{-j}) \\ &\iff \hat{I}(x_j, \boldsymbol{y}_{-j}) \succsim \hat{I}(y_j, \boldsymbol{y}_{-j}) \\ &\iff (x_j, \boldsymbol{y}_{-j}) \succsim (y_j, \boldsymbol{y}_{-j}).\end{aligned}$$

Second, we claim that all characteristics $j \in \{1, \ldots, k\}$ are essential. Suppose, by contradiction, that there exists an inessential characteristic. Then by *symmetry for the characteristics* of $\hat{I}$, all characteristics are inessential. Hence, for each distinct $\alpha, \beta \in (0,1]$, by *normalization* and inessentiality of all characteristics,

$$\alpha = \hat{I}(\alpha, \ldots, \alpha) = \hat{I}(\beta, \ldots, \beta) = \beta,$$

a contradiction to $\alpha \neq \beta$.

Therefore, Debreu's Representation Theorem can be applied to $\succsim$ on $(0,1]^k$. That is, for each $j \in \{1, \ldots, k\}$ there exists a continuous function $\hat{g}_j : (0,1] \to \mathbb{R}$, and for each $\boldsymbol{x}, \boldsymbol{y} \in (0,1]^k$

$$\boldsymbol{x} \succsim \boldsymbol{y} \iff \sum_{j=1}^{k} \hat{g}_j(x_j) \geq \sum_{j=1}^{k} \hat{g}_j(y_j). \tag{4.95}$$

Since $\hat{I} : (0,1]^k \to \mathbb{R}$ satisfies *symmetry for the characteristics*, by (4.94) and (4.95),

$$\forall j, \ell \in \{1, \ldots, k\},\ \hat{g}_j = \hat{g}_\ell.$$

Let $\hat{g} \equiv \hat{g}_j$ for all $j \in \{1, \ldots, k\}$. Then, by *normalization* and (4.95), $\hat{g}$ is strictly increasing.

Step 2 (Monotonic transformation of $\hat{g}$)**.** Define $\hat{\eta} : (0,1] \to \mathbb{R}_{++}$ by

$$\forall c \in (0,1],\ \hat{\eta}(c) = e^{\hat{g}(c)} > 0. \tag{4.96}$$

Then since $\hat{g}$ is continuous and strictly increasing on $(0,1]$, so is $\hat{\eta}$. Hence, $\hat{\eta}^{-1} : \hat{\eta}\big((0,1]\big) \to (0,1]$ exists and is strictly increasing.

For each $\boldsymbol{x} \in (0,1]^k$,

$$\prod_{j=1}^{k} \hat{\eta}(x_j)^{\frac{1}{k}} = \prod_{j=1}^{k} \Big(e^{\hat{g}(x_j)}\Big)^{\frac{1}{k}} = e^{\frac{1}{k}\sum_{j=1}^{k} \hat{g}(x_j)},$$

and hence, taking the natural logarithm on the first and last equations,

$$\log\Big[\prod_{j=1}^{k} \hat{\eta}(x_j)^{\frac{1}{k}}\Big] = \frac{1}{k}\sum_{j=1}^{k} \hat{g}(x_j). \tag{4.97}$$

Therefore, by (4.94), (4.95) and (4.97), and since $\log(\cdot)$ and $\hat{\eta}^{-1}$ are strictly increasing, for each $\boldsymbol{x}, \boldsymbol{y} \in (0,1]^k$,

$$\begin{aligned}
\hat{I}(\boldsymbol{x}) \geq \hat{I}(\boldsymbol{y}) \iff\ & \boldsymbol{x} \succsim \boldsymbol{y} \\
\iff\ & \log\Big[\prod_{j=1}^{k} \hat{\eta}(x_j)^{\frac{1}{k}}\Big] = \frac{1}{k}\sum_{j=1}^{k} \hat{g}(x_j) \geq \frac{1}{k}\sum_{j=1}^{k} \hat{g}(y_j) = \log\Big[\prod_{j=1}^{k} \hat{\eta}(y_j)^{\frac{1}{k}}\Big] \\
\iff\ & \prod_{j=1}^{k} \hat{\eta}(x_j)^{\frac{1}{k}} \geq \prod_{j=1}^{k} \hat{\eta}(y_j)^{\frac{1}{k}} \\
\iff\ & \hat{\eta}^{-1}\Big(\prod_{j\in K} \hat{\eta}(x_j)^{\frac{1}{k}}\Big) \geq \hat{\eta}^{-1}\Big(\prod_{j\in K} \hat{\eta}(y_j)^{\frac{1}{k}}\Big).
\end{aligned} \tag{4.98}$$

Step 3 (Quasi-geometric mean with $\hat{\eta}$ is cardinally equivalent to $\hat{I}$ on $(0,1]^k$)**.** Take any $\boldsymbol{y} \in (0,1]^k$. We first claim $0 < \hat{I}(\boldsymbol{y}) \leq 1$. Since $y_j \leq 1$ for each $j \in K$, by strict increasingness of $\hat{\eta}$,

$$\prod_{j\in K} \hat{\eta}(y_j)^{\frac{1}{k}} \leq \prod_{j\in K} \hat{\eta}(1)^{\frac{1}{k}},$$

and by strict increasingness of $\hat{\eta}^{-1}$,

$$\hat{\eta}^{-1}\Big(\prod_{j\in K} \hat{\eta}(y_j)^{\frac{1}{k}}\Big) \leq \hat{\eta}^{-1}\Big(\prod_{j\in K} \hat{\eta}(1)^{\frac{1}{k}}\Big).$$

Therefore, (4.98) and *normalization* of $\hat{I}$ together imply

$$\hat{I}(\boldsymbol{y}) \leq \hat{I}(\mathbf{1}) = 1.$$

On the other hand, since $\boldsymbol{y} \gg \mathbf{0}$, there exists $c > 0$ such that $\boldsymbol{y} \gg c \cdot \mathbf{1} \gg \mathbf{0}$. Since $\hat{\eta}$ and $\hat{\eta}^{-1}$ are strictly increasing,

$$\hat{\eta}^{-1}\Big(\prod_{j \in K} \hat{\eta}(y_j)^{\frac{1}{k}}\Big) > \hat{\eta}^{-1}\Big(\prod_{j \in K} \hat{\eta}(\beta)^{\frac{1}{k}}\Big)$$

Therefore, by *normalization* of $\hat{I}$ and (4.98),

$$\hat{I}(\boldsymbol{y}) > \hat{I}(c \cdot \mathbf{1}) = c > 0.$$

Let $\alpha \equiv \hat{I}(\boldsymbol{y}) \in (0, 1]$. Then by *normalization* of $\hat{I}$ at $\alpha \cdot \mathbf{1}$,

$$\hat{I}(\boldsymbol{y}) = \alpha = \hat{I}(\alpha \cdot \mathbf{1}).$$

Hence, by (4.98),

$$\hat{\eta}^{-1}\Big(\prod_{j \in K} \hat{\eta}(y_j)^{\frac{1}{k}}\Big) = \hat{\eta}^{-1}\Big(\prod_{j \in K} \hat{\eta}(\alpha)^{\frac{1}{k}}\Big) = \alpha = \hat{I}(\boldsymbol{y}).$$

We have shown that for each $\boldsymbol{y} \in (0, 1]^k$,

$$\hat{I}(\boldsymbol{y}) = \hat{\eta}^{-1}\Big(\prod_{j \in K} \hat{\eta}(y_j)^{\frac{1}{k}}\Big). \tag{4.99}$$

Step 4 (Extend $\hat{\eta}$ to $[0, 1]$)**.** Recall the definition (4.96) of $\hat{\eta}$. We can extend $\hat{\eta} : (0, 1]^k \to \mathbb{R}$ to $\eta : [0, 1]^k \to \mathbb{R}$ by defining

$$\eta(0) \equiv \lim_{c \searrow 0} e^{\hat{g}(c)} \in \mathbb{R}_+.$$

Since $\hat{g}$ is continuous and strictly increasing on $(0, 1]$, so is this extension η on $[0, 1]$.

On the other hand, recall that $I : [0, 1]^k \to \mathbb{R}$ is continuous on $[0, 1]^k$. Therefore, the equivalence (4.99) remains to hold even if the domain is extended to $[0, 1]^k$: for each $\boldsymbol{y} \in [0, 1]^k$,

$$I(\boldsymbol{y}) = \eta^{-1}\Big(\prod_{j \in K} \eta(y_j)^{\frac{1}{k}}\Big),$$

completing the proof. ■

It is worth noting that $k \geq 3$ are necessary for (ii) to imply (i). If $k = 2$, then the index function in Example 1 satisfies all the axioms. If $k = 1$, then the identity function

$$\forall y \in [0, 1],\ I(y) = y$$

is the only index function satisfying all the axioms.

A.2 Proof of Proposition 2

It is not difficult to check that power means satisfy *symmetry for the characteristics*, *normalization*, *separability*, and *homogeneity*.

Conversely, to show the uniqueness, consider any index function $I : [0,1]^k \to \mathbb{R}$ satisfying the set of axioms. Then by Steps 1–3 in the proof of Proposition 1, the restriction $\hat{I} : (0,1]^k \to \mathbb{R}$ can be represented by the continuous and strictly increasing function $\hat{\eta} : (0,1] \to \mathbb{R}$ defined in (4.96). Note that for each $y_j \in (0,1]$, since $\hat{\eta}(y_j) > 0$, we can define

$$\hat{\xi}(y_j) \equiv \log \hat{\eta}(y_j) \in \mathbb{R}.$$

Then by (4.99),

$$\begin{aligned}
\hat{I}(\boldsymbol{y}) &= \hat{\eta}^{-1}\Big(\prod_{j\in K} \hat{\eta}(y_j)^{\frac{1}{k}}\Big) \\
&= \hat{\xi}^{-1}\Big(\log\Big[\Big(\prod_{j\in K} \exp^{\hat{\xi}(y_j)}\Big)^{\frac{1}{k}}\Big]\Big) \\
&= \hat{\xi}^{-1}\Big(\log\Big[\exp^{\frac{1}{k}\sum_{j\in K}\hat{\xi}(y_j)}\Big]\Big) \\
&= \hat{\xi}^{-1}\Big(\frac{1}{k}\sum_{j\in K}\hat{\xi}(y_j)\Big).
\end{aligned}$$

Hence, $\hat{I}$ is a quasi-arithmetic mean with $\hat{\xi}$ on $(0,1]^k$.

By Aczél's Corollary 6 (1987, p.131), power means are the only quasi-arithmetic means satisfying *homogeneity*. So there exists an exponent $p \in \mathbb{R}$ such that for all $\boldsymbol{y} \in (0,1]^k$,

$$\hat{I}(\boldsymbol{y}) = \begin{cases} \prod_{j\in K} y_j^{\frac{1}{k}} & \text{if } p = 0, \\ \Big(\frac{1}{k}\sum_{j\in K} y_j^p\Big)^{\frac{1}{p}} & \text{if } p \neq 0. \end{cases}$$

Since both I and power means are continuous on $[0,1]^k$, by extending the domain from $(0,1]^k$ to $[0,1]^k$, this equivalence remains to hold: for all $\boldsymbol{y} \in [0,1]^k$,

$$I(\boldsymbol{y}) = \begin{cases} \prod_{j\in K} y_j^{\frac{1}{k}} & \text{if } p = 0, \\ \Big(\frac{1}{k}\sum_{j\in K} y_j^p\Big)^{\frac{1}{p}} & \text{if } p \neq 0, \end{cases}$$

completing the proof. ■

Appendix B: Tightness of the axioms

For each $\boldsymbol{y} \in [0,1]^k$, denote by $\boldsymbol{y}[j]$ the j-th highest measurement in $\boldsymbol{y} = (y_1, \ldots, y_k)$.

- Let $I_1 : [0,1]^k \to \mathbb{R}$ be an asymmetrically weighted product function

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_1(\boldsymbol{y}) = \prod_{j=1}^{k} (y_j^{\alpha_j}),$$

  where $\alpha_j < 1$ for all $j \in K$, $\sum_{j=1}^{k} \alpha_j = 1$, and $\alpha_j \neq \alpha_\ell$ for some distinct $j, \ell \in K$.

- Let $I_2 : [0,1]^k \to \mathbb{R}$ be a geometric mean multiplied by some number $c \neq 1$

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_2(\boldsymbol{y}) = c \prod_{j=1}^{k} y_j{}^{\frac{1}{k}}.$$

- Let $I_3 : [0,1]^k \to \mathbb{R}$ an index funtion such that

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_3(\boldsymbol{y}) = \prod_{j=1}^{k} (\boldsymbol{y}[j]^{\alpha_j}),$$

  where $\alpha_k < 1$, $\sum_{j=1}^{k} \alpha_j = 1$, and $\alpha_j \neq \alpha_\ell$ for some distinct $j, \ell \in K$.

- Let $I_4 : [0,1]^k \to \mathbb{R}$ be an index function such that

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_4(\boldsymbol{y}) = \exp\left(\prod_{j=1}^{k} \left(\log\left(y_j + 1\right)\right)^{\frac{1}{k}} \right) - 1.$$

- Let $I_5 : [0,1]^k \to \mathbb{R}$ be a power mean with exponent $\frac{1}{2}$

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_5(\boldsymbol{y}) = \left(\frac{1}{k} \sum_{j=1}^{k} y_j^{\frac{1}{2}} \right)^2.$$

- Let $I_6 : [0,1]^k \to \mathbb{R}$ be a power mean with exponent -1

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_6(\boldsymbol{y}) = \left(\frac{1}{k} \sum_{j=1}^{k} \frac{1}{y_j} \right)^{-1}.$$

- Let $I_7 : [0,1]^k \to \mathbb{R}$ be an asymmetrically weighted sum function such that

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_7(\boldsymbol{y}) = \frac{1}{k}\sum_{j=1}^{k} \alpha_j y_j,$$

  where $\sum_{j=1}^{k} \alpha_j = 1$ and $\alpha_1 = \alpha_2 \neq \alpha_3$.

- Let $I_8 : [0,1]^k \to \mathbb{R}$ be an arithmetic mean multiplied by some number $c \neq 1$

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_8(\boldsymbol{y}) = c\left(\frac{1}{k}\sum_{j=1}^{k} y_j\right).$$

- Let $I_9 : [0,1]^k \to \mathbb{R}$ be the minimum function

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_9(\boldsymbol{y}) = \min_{j\in K} y_j.$$

- Let $I_{10} : [0,1]^k \to \mathbb{R}$ be an index function such that

$$\forall \boldsymbol{y} \in [0,1]^k,\ I_{10}(\boldsymbol{y}) = \eta^{-1}\left(\frac{1}{k}\sum_{j=1}^{k} \eta(y_j)\right),$$

  where for each $y_j \in [0,1]$,

$$\eta(y_j) = \begin{cases} 2y_j^2 & \text{if } y_j \leq \frac{1}{2}, \\ \left(\frac{y_j}{2}\right)^{\frac{1}{2}} & \text{if } y_j > \frac{1}{2}. \end{cases}$$

	SYM	NORM	SEP	HMG	MLB	SENS
I_1	−	+	+	+	+	+
I_2	+	−	+	+	+	+
I_3	+	+	−	+	+	+
I_4	+	+	+	−	+	+
I_5	+	+	+	+	−	+
I_6	+	+	+	+	+	−

Table 1: Tightness of axioms in Theorem 1(i)

The satisfaction and the violation of axioms by these functions are summarized by Tables 1 and 2. It shows the independence of the axioms in our Propositions 1 and 2, and Theorem 1.

	SYM	NORM	SEP	HMG	LS
I_7	−	+	+	+	+
I_8	+	−	+	+	+
I_9	+	+	−	+	+
I_{10}	+	+	+	−	+
I_5	+	+	+	+	−

Table 2: Tightness of axioms in Theorem 1(ii)

Appendix C: Amendment of Zambrano's (2014) Theorem 1

To state our amendment of Zambrano(2014)'s Theorem 1 precisely, let us introduce Zambrano's model, which differs from that of Herrero, Martínez, and Villar (2010).

Zambrano considers the problem of designing an index function that aggregates partial indices for the three characteristics: health, education, and income. A *social state* is a vector (h, e, y), each component of which is the *aggregate level* of achievements of the members of a society in health, education, or income. Let $H \equiv [h^0, h^*] \subset \mathbb{R}$ be the space of the aggregate level of achievements in health. Similarly, let $E \equiv [e^0, e^*]$ and $Y \equiv [y^0, y^*]$.

Partial index functions in health, education, and income are continuous functions $C_h : H \to [0, 1]$, $C_e : E \to [0, 1]$, and $C_y : Y \to [0, 1]$, respectively. An *aggregator function* is a continuous function $I : [0, 1]^3 \to \mathbb{R}$ that aggregates partial indices yielded by these three functions. Note that an aggregator function herein corresponds to an index function in Herrero, Martínez, and Villar's model, where $n = 1$ and $k = 3$.

An *index function* $C : H \times E \times Y \to \mathbb{R}$ is defined by the functions C_h, C_e, C_y, and I as follows: for each $(h, e, y) \in H \times E \times Y$,

$$C(h, e, y) \equiv I\Big(C_h(h), C_e(e), C_y(y)\Big).$$

The new HDI is an index function $C^* : H \times E \times Y \to \mathbb{R}$ that is defined by the following functions C^*_h, C^*_e, C^*_y, and I^*. For each $(h, e, y) \in H \times E \times Y$,

$$C^*_h(h) = \frac{h - h^0}{h^* - h^0},$$
$$C^*_e(e) = \frac{e - e^0}{e^* - e^0},$$
$$C^*_y(y) = \frac{\log y - \log y^0}{\log y^* - \log y^0},$$
$$I^*\Big(C^*_h(h), C^*_e(e), C^*_y(y)\Big) = C^*_h(h)^{\frac{1}{3}} \cdot C^*_e(e)^{\frac{1}{3}} \cdot C^*_y(y)^{\frac{1}{3}}.$$

Zambrano proposes six axioms: *Monotonicity*, *aggregation symmetry*, *normalization*, *minimal lower boundedness*, *separability*, and *partial capability growth*. The first five axioms can be defined in a similar manner. *Partial capability growth* requires that different functionings improve capabilities in different ways. In particular, an improvement of health achievements changes health capabilities at a constant rate, and so does an improvement of education achievements; on the other hand, an improvement of income achievements changes income capabilities at a decreasing rate.

Partial capabilities growth. For each $(h, e, y), (h', e', y') \in H \times E \times Y$ and each $\Delta h, \Delta e, d_y$ such that $h + \Delta h, h' + \Delta h \in H$, $e + \Delta e, e' + \Delta e \in E$, and $y + y \cdot d_y, y' + y' \cdot d_y \in Y$,

- $C_h(h + \Delta h) - C_h(h) = C_h(h' + \Delta h) - C_h(h')$;
- $C_e(e + \Delta e) - C_e(e) = C_e(e' + \Delta e) - C_e(e')$; and
- $C_y(y + y \cdot d_y) - C_y(y) = C_y(y' + y' \cdot d_y) - C_y(y')$.

Claim 2 (Zambrano, 2010, Theorem 1)**.** *The new HDI $C^* : H \times E \times Y \to \mathbb{R}$ is the only index function satisfying monotonicity, aggregation symmetry, normalization, minimal lower boundedness, separability, and partial capabilities growth.*

However, the following modification of the index function in our Example 2 shows that the new HDI is not a unique index function satisfying the axioms.

Example 3. Let $C : H \times E \times Y \to \mathbb{R}$ be an index function such that for each $(h, e, y) \in H \times E \times Y$,

$$C(h, e, y) \equiv \log\left[\left(\exp\left(C_h^*(h)\right) - 1\right)^{\frac{1}{3}}\left(\exp\left(C_e^*(e)\right) - 1\right)^{\frac{1}{3}}\left(\exp\left(C_y^*(y)\right) - 1\right)^{\frac{1}{3}} + 1\right].^{*37}$$

Then C satisfies *monotonicity*, *aggregation symmetry*, *normalization*, *minimal lower boundedness*, *separability*, and *partial capability growth*, but it is not the new HDI.

Zambrano's Theorem 1 can be fixed by applying our Theorem 1(i). *Sensitivity to lowest-level characteristics* is similarly defined as an axiom on aggregator function $I : [0, 1]^3 \to \mathbb{R}$. To this end, we introduce *aggregation homogeneity*.

Aggregation homogeneity. For each $(h, e, y) \in H \times E \times Y$ and each $\lambda > 0$ with $\left(\lambda C_h(h), \lambda C_e(e), \lambda C_y(y)\right) \in [0, 1]^3$,

$$I\left(\lambda C_h(h), \lambda C_e(e), \lambda C_y(y)\right) = \lambda \cdot I\left(C_h(h), C_e(e), C_y(y)\right).$$

[*37] Note that the natural exponential function $\exp(\cdot)$ is used here instead of Napier's constant in order not to be confused with the aggregate level in health e.

Theorem 2. *The new HDI $C^*: H \times E \times Y \to \mathbb{R}$ is the only index function satisfying aggregation symmetry, normalization, minimal lower boundedness, separability, partial capabilities growth,* aggregate homogeneity*, and* sensitivity to lowest-level characteristics.

Proof. Slightly modify Zambrano's proof for the uniqueness as follows. Here note that Zambrano refers to *normalization*, *minimal lower boundedness*, *separability* as *scale*, *subsistence*, and *independence*, respectively and that *symmetry*, *normalization* and *separability* together imply *monotonicity*.

- (p.868, ℓ.14-15) "Since C satisfies Monotonicity, Subsistence and Independence it is a consequence of *Theorem 1 in Herrero et al. (2010a)* that..."
  $\longrightarrow$ a consequence of *our Theorem 1(i)*
- (p.868, ℓ.19) "By Scale and Aggregation Symmetry, ..."
  $\longrightarrow$ By Scale, Aggregation Symmetry, *Aggregation Homogeneity*, and *Sensitivity to lowest-level characteristics* ∎

For the tightness of the axioms in Theorem 2, that in our Theorem 1(i) is applicable. Examples I_1, I_2, I_3, I_4 and I_5 in Appendix 4.5 together with C_h^*, C_e^*, C_y^* generate index functions that satisfy all the axioms except *aggregation symmetry*, *normalization*, *separability*, *aggregation homogeneity*, *sensitivity to lowest-level characteristics*, and *minimal lower boundedness*, respectively. The following example borrowed from Zambrano (2014) satisfies all the axioms except *partial capabilities growth*:

$$C(h,e,y) \equiv \left(\frac{\log h - \log h^0}{\log h^* - \log h^0}\right)^{\frac{1}{3}} \cdot \left(\frac{\log e - \log e^0}{\log e^* - \log e^0}\right)^{\frac{1}{3}} \cdot \left(\frac{y - y^0}{y^* - y^0}\right)^{\frac{1}{3}}.$$

Chapter 5

A Characterization of the Esteban-Ray Polarization Measures

[*38]

5.1 Introduction

A seminal work by Esteban and Ray (1994; henceforth "ER") formalizes an idea of *polarization* and develop a theory for its measurement. In their main theorem (ER, Theorem 1), they claim that a class of allowable polarization measures, called the *Esteban-Ray measures*, is characterized by a set of axioms that capture an idea of polarization. However, we show that this claim does not hold by presenting a counterexample. We strengthen their Axiom 1 so that the characterization result can be reestablished.

The rest of this chapter is organized as follows. Section 2 introduces definitions and axioms. Section 3 presents our results. Proofs are relegated to Appendix.

5.2 Model and Axioms

Our model follows that of ER. Let $\mathbb{R}$ be the set of *attributes* (a basic perceptual variable is the natural logarithm of income). We consider population distributions on $\mathbb{R}$ with finite supports. That is, a distribution is denoted by a pair of n-dimensional vectors $(\boldsymbol{\pi}, \boldsymbol{y}) = ((\pi_1, \ldots, \pi_n), (y_1, \ldots, y_n)) \in \mathbb{R}^n_{++} \times \mathbb{R}^n$ for some $n \in \mathbb{N}$, where π_i is the population of individuals with attribute y_i and $y_i \neq y_j$ for distinct $i, j \in \{1, \ldots, n\}$. Let

$$\mathscr{D} \equiv \bigcup_{n=1}^{\infty} \mathbb{R}^n_{++} \times \left\{ \boldsymbol{y} \in \mathbb{R}^n : \; y_i \neq y_j \text{ for all distinct } i, j \in \{1, \ldots, n\} \right\}$$

be the set of distributions. A *polarization measure* is a function $P : \mathscr{D} \to \mathbb{R}_+$ that maps each distribution $(\boldsymbol{\pi}, \boldsymbol{y}) \in \mathscr{D}$ to a non-negative real number $P(\boldsymbol{\pi}, \boldsymbol{y}) \in \mathbb{R}_+$. ER's analysis focuses on

[*38]This chapter is co-authored with Yoko Kawada and Keita Sunada, and based on Kawada, Nakamura, and Sunada (2018).

polarization measures that take the following functional form:

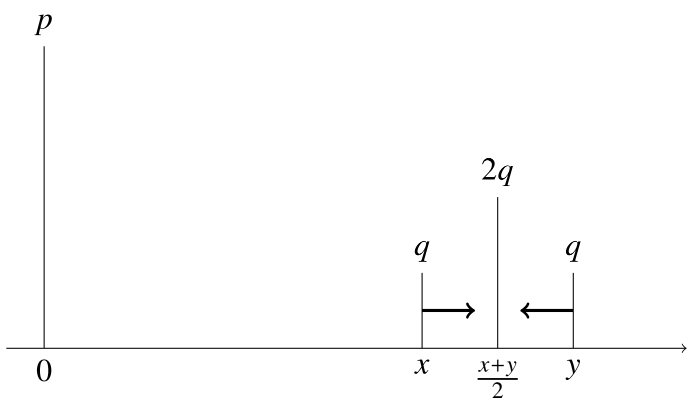
$$P((\boldsymbol{\pi}, \mathbf{y})) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \pi_j \theta\left(\pi_i, \left|y_i - y_j\right|\right), \tag{5.100}$$

where θ is a function $\mathbb{R}_+^2 \to \mathbb{R}$ such that $\theta(\cdot, \cdot)$ is strictly increasing in the second argument (distance), continuous in each argument, $\theta(0, \cdot) = 0$, $\theta(\cdot, 0) = 0$, and $\theta(\pi_i, \delta) > 0$ for all $\pi_i > 0$ and $\delta > 0$.[*39] An interpretation and background of this form is discussed in ER.

ER propose three axioms that capture an idea of polarization. Axiom 1 says that when a large mass exists at attribute 0, unifying two close masses increases polarization.

Axiom 1. For any $p > 0$ and any $x > 0$, there exist $\varepsilon > 0$ and $\mu > 0$ such that, for any $y > x$ and any $q < p$ with $y - x < \varepsilon$ and $0 < q < \mu p$,

$$P(((p, q, q), (0, x, y))) < P\left(\left((p, 2q), \left(0, \frac{x+y}{2}\right)\right)\right).$$

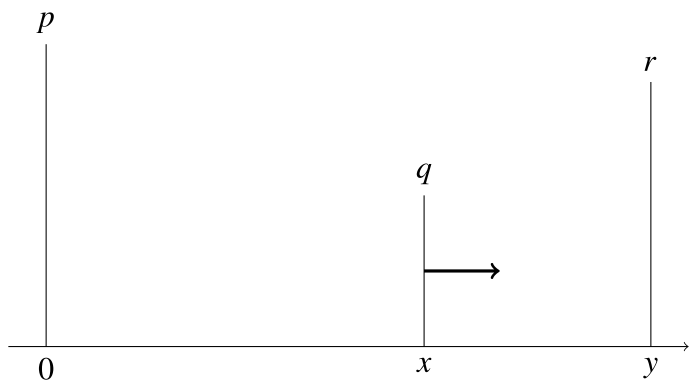


Axiom 2 requires that when an intermediate mass gets closer to the right extreme mass, polarization go up.

Axiom 2. For any $p, q, r > 0$ with $p > r$, any $x, y > 0$ with $|y - x| < x < y$, and any $\Delta \in (0, y - x)$,

$$P((p, q, r), (0, x, y)) < P((p, q, r), (0, x + \Delta, y)).$$

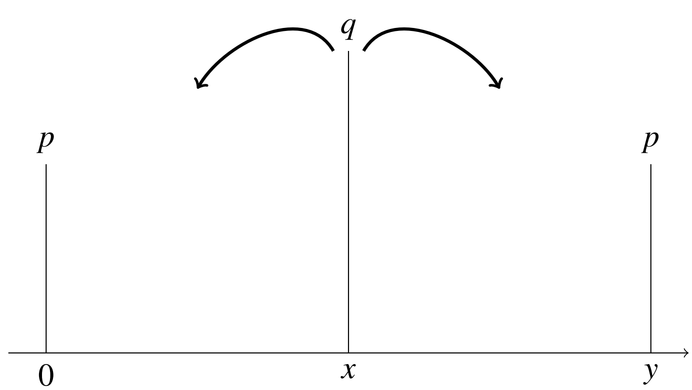
[*39] The assumption $\theta(0, \cdot) = 0$ is not imposed by ER. However, this assumption is necessary to deduce equation (6) in their proof of Theorem 1.



Axiom 3 requires that if the population of an intermediate mass decreases, and if the population of left and right extreme masses increase equally, then polarization go up.

Axiom 3. For any $p, q > 0$, any $x, y > 0$ with $x = y - x$, and any $\Delta \in (0, q/2)$,

$$P\left((p, q, p), (0, x, y)\right) < P\left((p + \Delta, q - 2\Delta, p + \Delta), (0, x, y)\right).$$



Condition H is a homotheticity property requiring any bilateral comparison be invariant to the scale of population.

Condition H. For any $(\boldsymbol{\pi}, \boldsymbol{y}), (\boldsymbol{\pi}', \boldsymbol{y}') \in \mathscr{D}$ and $\lambda > 0$, if $P(\boldsymbol{\pi}, \boldsymbol{y}) \geq P(\boldsymbol{\pi}', \boldsymbol{y}')$, then $P(\lambda\boldsymbol{\pi}, \boldsymbol{y}) \geq P(\lambda\boldsymbol{\pi}', \boldsymbol{y}')$.

5.3 Main Results

5.3.1 Counterexamples

ER claim that the class of the Esteban-Ray measures is characterized by Axioms 1–3 and Condition H.

Claim 1 (ER, Theorem 1)**.** A polarization measure P^* of the family defined in (5.100) satisfies Axioms 1, 2, and 3, and Condition H if and only if it is of the form

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \left| y_i - y_j \right| \tag{5.101}$$

for some constants $K > 0$ and $\alpha \in (0, \alpha^*]$ where $\alpha^* \simeq 1.6$.[*40]

We show that Claim 1 does not hold because Axiom 1 is too weak to characterize the class of the Esteban-Ray measures. In their proof of Claim 1, ER show that Axiom 1 and the continuity of $\theta(\cdot, \cdot)$ in equation (5.100) imply that $\theta(\pi_i, \cdot)$ must be *locally concave* with respect to the distance; that is, for each $x > 0$, there exists $\varepsilon > 0$ such that $\theta(\pi_i, \cdot)$ is concave on a half-open interval $[x, x + \varepsilon)$. Then, they claim that this *local concavity* of θ implies that $\theta(\pi_i, \cdot)$ must be *concave* on $\mathbb{R}_+$. However, this claim is not correct. To see this, fix any $c \in \mathbb{R}_{++}$ and let $\hat{f} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that for each $\delta \in \mathbb{R}_+$,

$$\hat{f}(\delta) = \begin{cases} K\delta & \text{if } \delta < c, \\ K'\delta - (K' - K)c & \text{if } \delta \geq c, \end{cases}$$

where $0 < K < K'$. Then, a convex piecewise linear function $\theta(\pi_i, \delta) = \pi_i^{\alpha} \hat{f}(\delta)$ is not *concave* on $\mathbb{R}_+$, but satisfies the *local concavity*.[*41] Therefore, Axiom 1 cannot exclude the convex piecewise linear function. In fact, a polarization measure with the convex piecewise linear function $\theta(\pi_i, \delta) = \pi_i^{\alpha} \hat{f}(\delta)$ satisfies Axioms 1–3 and Condition H, and hence Claim 1 does not hold.

Proposition 1 (Counterexample to Claim 1)**.** *Let $\hat{P} : \mathscr{D} \to \mathbb{R}_+$ be such that*

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \hat{f}(|y_i - y_j|), \tag{5.102}$$

where $\alpha \in (0, \alpha^*]$. Then, $\hat{P}$ satisfies Axioms 1, 2, *and* 3, *and Condition H, but does not take the form of* (5.101).

Proof. See Appendix A. ■

[*40]For the definition of α^*, see ER's equation (2) and subsequent arguments on page 833.

[*41]To confirm this, consider any $x > 0$. If $x \geq c$, then for any $\varepsilon > 0$, $\theta(\pi_i, \cdot)$ is concave on the half-open interval $[x, x + \varepsilon)$. Conversely, if $x < c$, then by letting $\varepsilon = \frac{1}{2}(c - x) > 0$, $\theta(\pi_i, \cdot)$ becomes concave on the half-open interval $[x, x + \varepsilon)$.

We have two remarks on this proposition; (i) our counterexample function $\hat{P}$ and an Esteban-Ray measure generate different orderings; (ii) a set of counterexamples is *dense* in a set of polarization measures with standard properties. These discussions are relegated to Supplementary Appendix B.

5.3.2 Modification

We provide a modified axiom that excludes the convex piecewise linear functions, and amend a characterization of the Esteban-Ray measures. For any $x > 0$ and $\varepsilon > 0$, let $B(x, \varepsilon) \equiv \{z \in \mathbb{R}_+ : |x - z| < \varepsilon\}$.

Axiom 1′. For any $p > 0$ and any $x > 0$, there exist $\varepsilon > 0$ and $\mu > 0$ such that for any $a, b \in B(x, \varepsilon)$ and $q < p$ with $0 < q < \mu p$,

$$P\left(\left((p, q, q), (0, a, b)\right)\right) < P\left(\left((p, 2q), \left(0, \frac{a+b}{2}\right)\right)\right).$$

In contrast to Axiom 1, Axiom 1′ implies that $\theta(\pi_i, \cdot)$ must satisfy the following stronger version of *local concavity* with respect to the second argument: for each $x > 0$, there exists $\varepsilon > 0$ such that $\theta(p, \cdot)$ is concave on an open interval $(x - \varepsilon, x + \varepsilon)$. Since the convex piecewise linear function $\theta(\pi_i, \delta) = \pi_i^\alpha \hat{f}(\delta)$ with kink point c is not concave on any open interval $(c - \varepsilon, c + \varepsilon)$ with $\varepsilon > 0$, $\hat{\theta}$ does not satisfy this *strong local concavity*. This is why Axiom 1′ can exclude the convex piecewise linear functions.

Though Axiom 1′ and Axiom 1 are mathematically quite different, Axiom 1’ has almost the same interpretation as the original compelling axiom. In this sense, this modification does not change the spirit of the original axiom.[*42] Now we restore an axiomatic foundation of the Esteban-Ray polarization measures.

Proposition 2. *A polarization measure P^* of the family defined in* (5.100) *satisfies Axioms* 1′, 2*, and* 3*, and Condition H if and only if it is of the form*

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \left| y_i - y_j \right|$$

for some constants $K > 0$ and $\alpha \in (0, \alpha^*]$ where $\alpha^* \simeq 1.6$.

Proof. See Appendix C. ∎

[*42] Assuming differentiability of measures is one way to exclude convex piecewise linear functions. However, differentiability is irrelevant to the original Axiom 1. Moreover, we cannot find a normative reason for adopting differentiability as an axiom in this context. For these reasons, we do not assume differentiability to characterize the Esteban-Ray measures.

Appendix A: Proof of Proposition 1.

Let us show that $\hat{P}$ satisfies Condition H and Axioms 1, 2, and 3.

Condition H. Take any $(\boldsymbol{\pi}, \boldsymbol{y}), (\boldsymbol{\pi}', \boldsymbol{y}') \in \mathscr{D}$ and any $\lambda > 0$. Suppose that

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) \geq \hat{P}(\boldsymbol{\pi}', \boldsymbol{y}').$$

Then, by the definition of $\hat{P}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \hat{f}(|y_i - y_j|) \geq \sum_{i=1}^{n'} \sum_{j=1}^{n'} \pi_i'^{1+\alpha} \pi_j' \hat{f}(|y_i' - y_j'|).$$

Since $\lambda > 0$, this equation implies that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda\pi_i)^{1+\alpha} (\lambda\pi_j) \hat{f}(|y_i - y_j|) \geq \sum_{i=1}^{n'} \sum_{j=1}^{n'} (\lambda\pi_i')^{1+\alpha} (\lambda\pi_j') \hat{f}(|y_i' - y_j'|).$$

Therefore,

$$\hat{P}(\lambda\boldsymbol{\pi}, \boldsymbol{y}) \geq \hat{P}(\lambda\boldsymbol{\pi}', \boldsymbol{y}').$$

Axiom 1. Take any $p > 0$ and $x > 0$. Let us show that there exist $\varepsilon > 0$ and $\mu > 0$ such that for any $y > x$ with $y - x < \varepsilon$ and any $q > 0$ with $q < \mu p$,

$$\hat{P}\left((p, q, q), (0, x, y)\right) < \hat{P}\left((p, 2q), \left(0, \frac{x+y}{2}\right)\right).$$

Case 1 ($\boldsymbol{x < c}$). Let $\varepsilon > 0$ and $\mu > 0$ be such that

$$\varepsilon = \min\left\{\frac{1}{2}(c - x), (2^\alpha - 1)x\right\},$$
$$\mu = \frac{1}{2}.$$

Take any $y > x$ with $y - x < \varepsilon$ and any $q > 0$ with $q < \mu p$. Note that $x, y, \frac{x+y}{2}, |y - x| < c$. Therefore,

$$\hat{P}\left((p, q, q), (0, x, y)\right) = K\left(\left(p^{1+\alpha} q + q^{1+\alpha} p\right)(x + y) + 2q^{2+\alpha}|y - x|\right),$$

and

$$\hat{P}\left((p, 2q), \left(0, \frac{x+y}{2}\right)\right) = K\left(2p^{1+\alpha} q + 2^{1+\alpha} q^{1+\alpha} p\right)\left(\frac{x+y}{2}\right).$$

Moreover, since $|y-x|<\varepsilon$, $2q<p$ and $\varepsilon \leq (2^\alpha-1)x$, it follows that

$$2q|y-x| < \varepsilon p \leq (2^\alpha-1)xp < (2^\alpha-1)(x+y)p.$$

Then, by the same argument as ER (line 14 on page 837),

$$\hat{P}\left((p,q,q),(0,x,y)\right) < \hat{P}\left((p,2q),\left(0,\frac{x+y}{2}\right)\right).$$

Case 2 ($x \geq c$). Let $\varepsilon>0$ and $\mu>0$ be such that

$$\varepsilon = \min\left\{\frac{1}{2}c, (2^\alpha-1)(K'x-(K'-K)c)\right\},$$
$$\mu = \frac{1}{2K}.$$

Take any $y>x$ with $y-x<\varepsilon$ and any $q>0$ with $q<\mu p$. Note that $x, y, \frac{x+y}{2} \geq c$ and $|y-x|<c$. Therefore,

$$\hat{P}\left((p,q,q),(0,x,y)\right) = \left(p^{1+\alpha}q + q^{1+\alpha}p\right)\left(K'x+K'y-2(K'-K)c\right) + 2q^{2+\alpha}K|y-x|,$$

and

$$\hat{P}\left((p,2q),\left(0,\frac{x+y}{2}\right)\right) = \left(2p^{1+\alpha}q + 2^{1+\alpha}q^{1+\alpha}p\right)\left(\frac{x+y}{2}K' - (K'-K)c\right)$$
$$= \left(p^{1+\alpha}q + q^{1+\alpha}p\right)\left(K'x+K'y-2(K'-K)c\right) + (2^\alpha-1)q^{1+\alpha}p(K'x+K'y-2(K'-K)c).$$

Then, $\hat{P}\left((p,q,q),(0,x,y)\right) < \hat{P}\left((p,2q),\left(0,\frac{x+y}{2}\right)\right)$ if and only if

$$2qK|y-x| < (2^\alpha-1)p\left(K'x+K'y-2(K'-K)c\right).$$

Since $|y-x|<\varepsilon$, $2Kq<p$ and $\varepsilon \leq (2^\alpha-1)(K'x-(K'-K)c)$, we have

$$2qK|y-x| < \varepsilon p \leq (2^\alpha-1)p\left(K'x-(K'-K)c\right) < (2^\alpha-1)p\left(K'x+K'y-2(K'-K)c\right).$$

Therefore,

$$\hat{P}\left((p,q,q),(0,x,y)\right) < \hat{P}\left((p,2q),\left(0,\frac{x+y}{2}\right)\right).$$

Hence $\hat{P}$ satisfies Axiom 1.

Axiom 2. Fix any $p,q,r>0$ with $p>r$, and any $x<y$ with $x>y-x$. Let us show that for any $\Delta \in (0, y-x)$,

$$\hat{P}\left((p,q,r),(0,x,y)\right) < \hat{P}\left((p,q,r),(0,x+\Delta,y)\right).$$

Take any $\Delta \in (0, y-x)$. For simplicity, we write

$$\hat{P} \equiv \hat{P}\left((p,q,r),(0,x,y)\right),$$
$$\hat{P}_\Delta \equiv \hat{P}\left((p,q,r),(0,x+\Delta,y)\right).$$

Then,

$$\hat{P} = \hat{f}(x)\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + \hat{f}(y-x)\big(q^{1+\alpha}r + r^{1+\alpha}q\big) + \hat{f}(y)\big(p^{1+\alpha}r + r^{1+\alpha}p\big),$$
$$\hat{P}_\Delta = \hat{f}(x+\Delta)\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + \hat{f}(y-x-\Delta)\big(q^{1+\alpha}r + r^{1+\alpha}q\big) + \hat{f}(y)\big(p^{1+\alpha}r + r^{1+\alpha}p\big).$$

We shall show $\hat{P}_\Delta - \hat{P} > 0$.

Since $\hat{f}$ is convex and $x > y-x$, the slope of $\hat{f}$ at x is larger than that at $y-x$; that is,

$$\hat{f}(x+\Delta) - \hat{f}(x) \geq \hat{f}(y-x) - \hat{f}(y-x-\Delta).$$

Therefore,

$$\begin{aligned}&\hat{P}_\Delta - \hat{P}\\ &\geq \left(\hat{f}(x+\Delta) - \hat{f}(x)\right)\left(p^{1+\alpha}q + q^{1+\alpha}p\right) - \left(\hat{f}(y-x) - \hat{f}(y-x-\Delta)\right)\left(q^{1+\alpha}r + r^{1+\alpha}q\right)\\ &\geq \left(\hat{f}(y-x) - \hat{f}(y-x-\Delta)\right)\left(\left(p^{1+\alpha}q + q^{1+\alpha}p\right) - \left(q^{1+\alpha}r + r^{1+\alpha}q\right)\right).\end{aligned}$$

Hence, $\hat{P}_\Delta - \hat{P}$ is positive whenever $p > r$ since $\hat{f}(y-x) - \hat{f}(y-x-\Delta) > 0$.

Axiom 3. Fix any $p, q > 0$, and any $x, y > 0$ with $x = y - x$. Let us show that for any $\Delta \in (0, q/2)$,

$$\hat{P}\left((p,q,p),(0,x,y)\right) < \hat{P}\left((p+\Delta, q-2\Delta, p+\Delta),(0,x,y)\right).$$

Take any $\Delta \in (0, q/2)$. For simplicity, we write

$$\hat{P} \equiv \hat{P}\left((p,q,p),(0,x,y)\right),$$
$$\hat{P}_\Delta \equiv \hat{P}\left((p+\Delta, q-2\Delta, p+\Delta),(0,x,y)\right).$$

Then,

$$\hat{P} = 2\hat{f}(d)\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 2\hat{f}(2d)\big(p^{2+\alpha}\big),$$
$$\hat{P}_\Delta = 2\hat{f}(d)\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 2\hat{f}(2d)\big((p+\Delta)^{2+\alpha}\big).$$

We shall show $\hat{P}_\Delta - \hat{P} > 0$.

Case 1 ($2d < c$). In this case,

$$\hat{P} = K\Big(2d\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 4d\big(p^{2+\alpha}\big)\Big),$$
$$\hat{P}_\Delta = K\Big(2d\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 4d\big((p+\Delta)^{2+\alpha}\big)\Big).$$

Then, by the same argument as ER (paragraph of verifying axiom 3 on page 837), it follows that $\hat{P}_\Delta - \hat{P} > 0$.

Case 2 ($d < c$ and $c < 2d$). By definition of $\hat{f}$, it follows that $\hat{f}(2d) \geq 2Kd$. Then, since $(p+\Delta)^{2+\alpha} \geq p^{2+\alpha}$,

$$2\hat{f}(2d)(p+\Delta)^{2+\alpha} - 2\hat{f}(2d)p^{2+\alpha} \geq 4Kd(p+\Delta)^{2+\alpha} - 4Kdp^{2+\alpha}.$$

Therefore,

$$P_\Delta - P \geq K\Big(2d\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 4d\big((p+\Delta)^{2+\alpha}\big)\Big) - K\Big(2d\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 4d\big(p^{2+\alpha}\big)\Big). \quad (5.103)$$

Then, by the same argument as Case 1, the right hand side of (5.103) is positive. Hence $\hat{P}_\Delta - \hat{P} > 0$.

Case 3 ($c < d$). Let

$$A \equiv 2\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 4\big(p^{2+\alpha}\big),$$
$$A_\Delta \equiv 2\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 4\big((p+\Delta)^{2+\alpha}\big),$$
$$B \equiv 2\big(p^{1+\alpha}q + q^{1+\alpha}p\big) + 2\big(p^{2+\alpha}\big),$$
$$B_\Delta \equiv 2\big((p+\Delta)^{1+\alpha}(q-2\Delta) + (q-2\Delta)^{1+\alpha}(p+\Delta)\big) + 2\big((p+\Delta)^{2+\alpha}\big).$$

Then, we can compute that

$$\hat{P} = K'dA - (K'-K)cB,$$
$$\hat{P}_\Delta = K'dA_\Delta - (K'-K)cB_\Delta.$$

Moreover, by the same argument as Case 1, it follows that $A_\Delta \geq A$. Therefore,

$$K'd\,(A_\Delta - A) \geq K'c\,(A_\Delta - A),$$

and hence

$$\hat{P}_\Delta - \hat{P} \geq \Big(K'cA_\Delta - (K'-K)cB_\Delta\Big) - \Big(K'cA - (K'-K)cB\Big).$$

Therefore, it suffices to show that the derivative of the function

$$\hat{P}(\Delta) \equiv K' c A_{\Delta} - (K' - K) c B_{\Delta},$$

evaluated at $\Delta = 0$, is non-negative and positive for all but at most one ratio $z \equiv p/q$. By a simple computation, this derivative is given by

$$\hat{P}'(\Delta) = q^{1+\alpha} \Big(2c(K' - K)(2 + \alpha) z^{1+\alpha} - 4cK\varphi(z, \alpha) \Big), \tag{5.104}$$

where the function φ is defined by

$$\varphi(z, \alpha) = (1 + \alpha) \left(z - \frac{z^{\alpha}}{2} - z^{1+\alpha} \right) - \frac{1}{2}.$$

Then, since $\alpha \in (1, \alpha^*]$, (5.104) is non-negative and is positive for all but at most one ratio z (see ER; equation (2) and subsequent arguments on page 833). Therefore, $\hat{P}$ satisfies Axiom 3. □

Appendix B: Discussions on Proposition 1.

(i) Ordinal difference between a counterexample and ER's measure

Our counterexample and an Esteban-Ray measure generate different orderings. For example, let $(\boldsymbol{\pi}, \boldsymbol{y}) = ((1, 1, 1), (0, 4, 8))$ and $(\boldsymbol{\pi}', \boldsymbol{y}') = ((1, 1, 1), (0, 1, 7))$. Specify each parameter of $\hat{P}$ as $K = 1$, $K' = 10$ and $c = 4$; that is, let

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j |y_i - y_j|,$$

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \hat{f}(|y_i - y_j|),$$

where

$$\hat{f}\left(|y_i - y_j|\right) = \begin{cases} |y_i - y_j| & \text{if } |y_i - y_j| < 4, \\ 10|y_i - y_j| - 36 & \text{if } |y_i - y_j| \geq 4. \end{cases}$$

Then, for any $\alpha \in (1, \alpha^*]$,

$$P^*(\boldsymbol{\pi}, \boldsymbol{y}) = 32 > 28 = P^*(\boldsymbol{\pi}', \boldsymbol{y}'),$$

but

$$\hat{P}(\boldsymbol{\pi}, \boldsymbol{y}) = 104 < 118 = \hat{P}(\boldsymbol{\pi}', \boldsymbol{y}').$$

Hence, P^* and $\hat{P}$ yield different orderings.

(ii) Denseness of counterexamples

Many functions of the form of (5.100) satisfy Axioms 1–3 and Condition H, but do not take the form of (5.101). Indeed, let

$$F = \{f : \mathbb{R}_+ \to \mathbb{R}_+ \mid f \text{ is convex, strictly increasing, non-linear,} \\ \text{piecewise linear with discrete kink points, and } f(0) = 0.\}.$$

Then, for any $f \in F$, a function of the form (5.100)

$$P(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j f(|y_i - y_j|)$$

satisfies Axioms 1–3 and Condition H but does not take the form of (5.101).

Moreover, let

$$G = \{g : \mathbb{R}_+ \to \mathbb{R}_+ \mid g \text{ is convex, strictly increasing, and } g(0) = 0\},$$

then F is dense in G with a standard metric ρ, defined as $\rho(f, g) = \sup_{x \in \mathbb{R}_+} |f(x) - g(x)|$. That is, for all $g \in G$ and any $\varepsilon > 0$, there exists $f \in F$, such that $\rho(f, g) < \varepsilon$. Denseness of F suggests that Axiom 1 does not work well to characterize the Esteban-Ray measures.

Proposition. *For any $f \in F$, a function of the form* (5.100)

$$P(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j f(|y_i - y_j|)$$

where $\alpha \in (0, \alpha^*]$ satisfies Axioms 1, 2 and 3, and Condition H but does not take the form of (5.101)*. Moreover, F is dense in G.*

The first statement can be proven by the same way as Proposition 1. Here we show that F is dense in G. Consider any $g \in G$. Take any $\varepsilon > 0$.

Case 1 (g is non-linear). We will construct a piecewise linear function that uniformly approximates g. Consider the partition of $[0, \infty)$, $[0, \varepsilon), [\varepsilon, 2\varepsilon), [2\varepsilon, 3\varepsilon), \ldots$. Since g is strictly increasing and continuous[*43], for each $k \in \{1, 2, \ldots\}$, there exists a unique real number $d_k \in \mathbb{R}_+$ with $d_k = g^{-1}(k\varepsilon)$. For example, $g(d_1) = \varepsilon$ and $g(d_2) = 2\varepsilon$. Define $d_0 = 0$. Note

[*43] Since g convex, strict increasing, and $g(0) = 0$, it is continuous on $[0, \infty)$.

that $\mathbb{R}_+$ is partitioned by a family $[d_0, d_1), [d_1, d_2), [d_2, d_3), \ldots$. Let f be a piecewise linear function such that

$$f(x) = \begin{cases} \dfrac{\varepsilon}{d_1} x \ \text{ if } \ x \in [0, d_1), \\ \dfrac{\varepsilon}{d_2 - d_1}(x - d_1) + \varepsilon \ \text{ if } \ x \in [d_1, d_2), \\ \dfrac{\varepsilon}{d_3 - d_2}(x - d_2) + 2\varepsilon \ \text{ if } \ x \in [d_2, d_3), \\ \vdots \\ \dfrac{\varepsilon}{d_k - d_{k-1}}(x - d_{k-1}) + (k-1)\varepsilon \ \text{ if } \ x \in [d_{k-1}, d_k), \\ \vdots \end{cases}$$

That is, f is the piecewise linear function such that $f(d_k) = k\varepsilon = g(d_k)$ for each $k \in \{0, 1, 2, \ldots\}$. Obviously, $f \in F$.[*44]

We show that g can be uniformly approximated by f. Consider any $x \in \mathbb{R}_+$. Since the family $[d_0, d_1), [d_1, d_2), [d_2, d_3), \ldots$ is a partition of $\mathbb{R}_+$, there exists $k \in \{1, 2, \ldots\}$ such that $x \in [d_{k-1}, d_k)$, and so

$$f(x) \in [(k-1)\varepsilon, k\varepsilon) \ \text{ and } \ g(x) \in [(k-1)\varepsilon, k\varepsilon).$$

Therefore, $|f(x) - g(x)| < \varepsilon$.

Case 2 (g is linear). Since g is linear, there exists $k > 0$ such that $g(x) = kx$ for any $x \in \mathbb{R}_+$. Fix any $c \in \mathbb{R}_{++}$. Let $\tilde{f}$ be a convex piecewise linear function such that

$$\tilde{f}(x) = \begin{cases} \frac{1}{c}\left(kc - \frac{1}{2}\varepsilon\right) x \ \text{ if } \ x < c, \\ kx - \frac{1}{2}\varepsilon \ \text{ if } \ x \geq c. \end{cases}$$

Obviously, $\tilde{f} \in F$. Then, $\tilde{f}(x) \in (kx - \varepsilon, kx)$ for any $x \in \mathbb{R}_+$. Therefore, $|\tilde{f}(x) - g(x)| < \varepsilon$. Hence, F is dense in G.

Appendix C: Proof of Proposition 2.

(Sufficiency.) We can show that P^* satisfies Axiom 1′ as the same way in ER, so we omit the proof of this part.

[*44]Since g is convex, $d_k - d_{k-1} > d_{k+1} - d_k$ for each $k \in \{1, 2, \ldots\}$. Therefore, $\frac{\varepsilon}{d_k - d_{k-1}}$ is increasing in k, that is, f is a convex piecewise linear function.

(Necessity.) We show that Axiom $1'$ implies concavity of θ. since the proof of Proposition 2 is the same as that of Theorem 1 except for this point. Consider the distribution depicted in Axiom 1'. Initially, polarization is given by

$$P^1 \equiv pq[\theta(p,a)+\theta(p,b)] + pq[\theta(q,a)+\theta(q,b)] + 2q^2\theta(q,|b-a|),$$

and polarization after the distribution shifting is

$$P^2 \equiv 2pq\left[\theta\left(p, \frac{a+b}{2}\right)\right] + 2pq\left[\theta\left(2q, \frac{a+b}{2}\right)\right].$$

Axiom 1' implies that

$$\begin{aligned} 2p\left[\theta\left(p, \frac{a+b}{2}\right) + \theta\left(2q, \frac{a+b}{2}\right)\right] > & \ p[\theta(p,a)+\theta(p,b)] \\ & + p[\theta(q,a)+\theta(q,b)] \\ & + 2q\theta(q,|b-a|). \end{aligned}$$

Take the limit as $q \to 0$. Then, for any $x > 0$, there exists $\varepsilon > 0$ such that for any $a, b \in B(x, \varepsilon)$,

$$\theta\left(p, \frac{a+b}{2}\right) \geq \frac{\theta(p,a)+\theta(p,b)}{2}.$$

This means local mid-point concavity, but we show that it is sufficient for global and any convex-combination concavity; for any $a, b > 0$ and any $t \in [0,1]$,

$$\theta\left(p, ta+(1-t)b\right) \geq t\theta(p,a) + (1-t)\theta(p,b).$$

Suppose, by contradiction, that there exist $a, b > 0$ and $t \in [0,1]$ such that

$$\theta\left(p, ta+(1-t)b\right) < t\theta(p,a) + (1-t)\theta(p,b). \tag{5.105}$$

For each $s \in [0,1]$, define

$$h(s) = \theta\left(p, sa+(1-s)b\right) - s\theta(p,a) - (1-s)\theta(p,b).$$

Then h is locally mid-point concave. Indeed, since $\theta(p,\cdot)$ is locally mid-point concave at any point, $\theta(p,\cdot)$ is mid-point concave on $B(sa+(1-s)b, \varepsilon)$ for any $s \in [0,1]$ and for some $\varepsilon > 0$. Moreover, $-s\theta(p,a) - (1-s)\theta(p,b) = (\theta(p,b) - \theta(p,a))s - \theta(p,b)$ is a straight line, which is globally concave. Since the sum of two concave functions is also concave, h is locally mid-point concave. Note that $h(1) = h(0) = 0$ and $h(t) < 0$ by (5.105).

Since h is continuous, by the extreme value theorem, h has the minimum value on $[0,1]$. Let

$$m = \min\{h(s) : s \in [0,1]\},$$
$$u = \max\{s \in [0,1] : h(s) = m\}.$$

Then $h(s) > h(u)$ for every $s \in (u,1]$ and $h(s') \geq h(u)$ for every $s' \in [0,u]$. Thus h is not locally concave on any epsilon ball with center u. Indeed, take any $\varepsilon' > 0$. Let $a' \in B(u,\varepsilon')$ with $a' > u$. Let $b' = u - |a' - u|$. Note that $a', b' \in B(u,\varepsilon')$, $\frac{1}{2}(a' + b') = u$, $h(a') > h(u)$, and $h(b') \geq h(u)$. This implies that

$$h\left(\frac{1}{2}(a' + b')\right) = h(u) < \frac{h(a') + h(b')}{2}.$$

This contradicts local mid-point concavity of h, as desired. Therefore, we have restored a claim: "$\theta(p,\cdot)$ must be concave" (line 11, page 835 of ER).

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