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Chapter 12

Heisenberg's uncertainty relation

Quantum mechanics is surely one of the most successful theories in all science. In fact, most of the Nobel prizes of physics and chemistry are due to quantum mechanics. Also, as recent topics (particularly, related to measurements), we see quantum computer [80], quantum cryptography [91], quantum teleportation [10], etc. Although these are quite interesting and promising, in this chapter, we devote ourselves to Heisenberg's uncertainty relation, which is the most fundamental in quantum mechanics.

Heisenberg's uncertainty relation (*cf.* [31]).

(i) *The particle position q and momentum p can be measured "simultaneously", if the "errors" $\Delta(q)$ and $\Delta(p)$ in determining the particle position and momentum are permitted to be non-zero.*

(ii) *Moreover, for any $\epsilon > 0$, we can take the above "approximate simultaneous" measurement of the position q and momentum p such that $\Delta(q) < \epsilon$ (or $\Delta(p) < \epsilon$).*

(iii) *However, the following Heisenberg's uncertainty relation holds:*

$$\Delta(q) \cdot \Delta(p) \geq \frac{\hbar}{2}, \tag{12.1}$$

for all "approximate simultaneous" measurements of the particle position and momentum.

However, it should be noted that some ambiguous terms (i.e., "approximate simultaneous", "error") are included in the above statement, Thus, we believe that it is not a scientific statement but a "catch phrase" that was used to promote the paradigm shift from classical mechanics to quantum mechanics. Thus, in this last chapter[1] we try to describe this uncertainty relation precisely in terms of mathematics and further to derive it in the framework of the W^*-algebraic formulation of MT. For this, we first give the mathematical definitions of "$\Delta(q)$" (or "$\Delta(p)$") and "approximate simultaneous measurement", etc. in terms of MT.

[1]Every result mentioned in this chapter was published in [36], which was the oldest result in our study of "measurement theory". That is, our research of "measurement theory" starts from the paper [36]. On the other hand, the philosophical assertion mentioned in Chapter 1 is the latest result in our study. In this sense, the progress of our research is symbolically summarized as

$$\underset{\text{(in Chapter 12)}}{\text{"quantum" (physics)}} \longrightarrow \underset{\text{(in Chapters 2}\sim\text{11)}}{\text{"classical" (engineering)}} \longrightarrow \underset{\text{(in Chapter 1)}}{\text{"philosophical" (epistemology)}}$$

.

12.1 Introduction

Although the uncertainty relation (discovered by Heisenberg in 1927) has a long history, the various discussions about its interpretations are even now continued. Mainly there are two interpretations of uncertainty relations. One is the statistical interpretation. By repeating the exact (i.e. the "error" $\Delta(q) = 0$) measurements of the position q of particles with same states, we can obtain its average value $\bar{q}$ and its variance $var(q)$. Also, by repeating the exact (i.e. the "error" $\Delta(p) = 0$) measurements of the momentum p of the same particles, we can similarly get its average value $\bar{p}$ and its variance $var(p)$. From the simple mathematical deduction, we can obtain the following uncertainty relation:

$$[var(q)]^{\frac{1}{2}} \cdot [var(p)]^{\frac{1}{2}} \geq \frac{\hbar}{2}, \tag{12.2}$$

where $\hbar =$"Plank's constant"$/2\pi$. This is the statistical aspect of the uncertainty relation. The mathematical derivation of the uncertainty relation (12.2) was proposed by Kennard in 1927 (or more generally, Robertson 1n 1929). Cf. [54, 73]. Thus, this inequality (12.2) is called Robertson's uncertainty relation.

On the other hand, Heisenberg's uncertainty relation is rather individualistic. Most physicists will agree that the content of Heisenberg's uncertainty relation is roughly as stated in the following proposition (though it includes some ambiguous sentences as well as some ambiguous words, i.e. "approximate simultaneous" and "error").

Proposition 12.1. [Heisenberg's uncertainty relation, *cf.* [31]].[2]

(i) *The particle position q and momentum p can be measured "approximately" and "simultaneously", if the "errors" $\Delta(q)$ and $\Delta(p)$ in determining the particle position and momentum are permitted to be non-zero.*

(ii) *Moreover, for any $\epsilon > 0$, we can take the "approximate simultaneous" measurement of the position q and momentum p such that $\Delta(q) < \epsilon$ (or $\Delta(p) < \epsilon$).*

[2] It may be usually considered that the (12.2) is the mathematical representation of the (12.3). However, it is not true. In fact, in [84], J. von Neumann pointed out the difference between Robertson's uncertainty relation (= (12.2)) and Heisenberg's uncertainty relation (= (12.3)).

(iii) *However, the following Heisenberg's uncertainty relation holds:*

$$\Delta(q) \cdot \Delta(p) \geq \frac{\hbar}{2}, \tag{12.3}$$

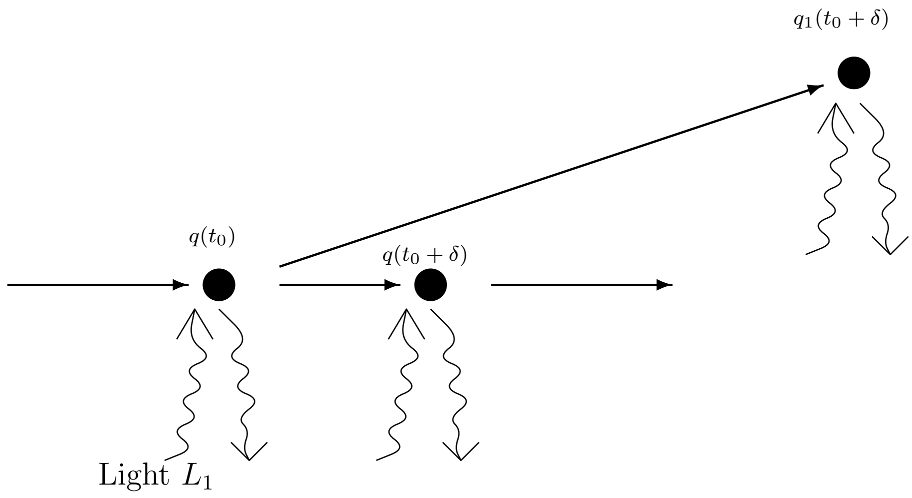
for all "approximate simultaneous" measurements of the particle position and momentum.

■

It should be noted that the above "proposition (= Heisenberg's assertion)" is ambiguous, that is, *it is not a scientific statement but a "catch phrase" that was used to promote the paradigm shift from classical mechanics to quantum mechanics.* In fact, the above "proposition" is powerless to solve the paradox (i.e., the paradox between EPR-experiment and Heisenberg's uncertainty relation), *cf.* §12.7.

Several authors have contributed to the problem to deduce Heisenberg's uncertainty relation. In [2] (Ali and Emach, 1974), [3] (Ali and Prugovečki, 1977), these were done by means of the concept of (generalized) observable which has been developed by E.B. Davies [17] (*cf.* Definition 9.3 for $B(V)$). Hence, a certain part of this problem has been already solved. In particular, the statements (i) and (ii) in the above Proposition 12.1 were deduced satisfactorily. However, concerning the statement (iii), there still seems to be some questions. The mathematical formulation and derivation of the Heisenberg's uncertainty relation (iii) (in the above Proposition 12.1) was proposed by M. Ozawa [67], S. Ishikawa [36] independently. We believe that this is the final version of Heisenberg's uncertainty relation concerning measurement errors. Thus, in this chapter we shall introduce this formulation and derivation of the above Proposition 12.1.

Remark 12.2. [(i): A classical understanding of Heisenberg's uncertainty relation]. Let us explain the classical understanding of Heisenberg's uncertainty relation (which is essentially equal to the thought experiment of γ-rays microscope (*cf.* [31])). In order to know the position $q(t_0)$ and momentum $p(t_0)$ of a particle A at time t_0, it suffices to measure the position $q(t_0)$ of a particle A at time t_0 (i.e., light L_1 is irradiated at the particle at time t_0), and continuously (i.e., after δ seconds), measure the position $q(t_0+\delta)$ at time $t_0+\delta$. That is because $(q(t_0), p(t_0)(\equiv \frac{mdq}{dt}(t_0)))$ is approximately calculated by $(q(t_0), \frac{m(q(t_0+\delta)-q(t_0))}{\delta})$.



[a]. However, if we want to know the exact position $q(t_0)$ (i.e., if we want $\Delta q \approx 0$), the wavelength λ of the light L_1 must be short (i.e., the energy ($= \frac{\text{“Plank constant”} \times \text{“lightspeed”}}{\lambda}$) of the light L_1 must be large), and therefore, the particle A is strongly perturbed. Thus, the position of the particle A at time $t_0+\delta$ will be changed to $q_1(t_0+\delta)$. Thus we observe that the momentum of the particle A at time t_0 is equal to $\frac{m(q_1(t_0+\delta)-q(t_0))}{\delta}$, which is away from $p(t_0)(\equiv \frac{mdq}{dt}(t_0) \approx \frac{m(q(t_0+\delta)-q(t_0))}{\delta})$ (i.e., Δp is large).
[b]. Also, if we want to know the exact momentum $p(t_0)$ (i.e., if we want $\Delta p \approx 0$), the wavelength λ of the light L_1 must be long, and therefore, the particle A is weakly perturbed. Although the position of the particle A at time $t_0+\delta$ will be changed to $q_1(t_0+\delta)$, it is almost the same as $q(t_0+\delta)$. Thus we observe that the momentum of the particle A at time t_0 is equal to $\frac{m(q_1(t_0+\delta)-q(t_0))}{\delta}$, which is near $p(t_0)(\equiv \frac{mdq}{dt}(t_0) \approx \frac{m(q(t_0+\delta)-q(t_0))}{\delta})$ (i.e., Δp is small) if δ is large. However it should be noted that Δq is large since the wavelength λ of the light L_1 is long.
[c]. Therefore, $\Delta p \approx 0$ and $\Delta q \approx 0$ are not compatible, that is, the inequality “$\Delta p \cdot \Delta q >$ constant” always holds. Although this explanation is, of course, rough, there is something thought-provoking in the above argument.
[(ii): EPR-experiment [22]]. Let A and B be particles with the same masses m. Consider the situation described in the following figure:

where “the velocity of A” $= -$“the velocity of B”. The position q_A of the particle A can be measured, and moreover, the velocity of v_B of the particle B can be measured. Thus, we can conclude that the position and momentum of the particle A are respectively equal to q_A and $-mv_B$. Is this contradictory to Heisenberg's uncertainty relation? This question is significant though their (i.e. Einstein, Podolosky and Rosen) interest is concentrated on "the reality of physics".

■

12.2 Example due to Arthurs-Kelley

Here, we mainly consider the following identification:

$$\underset{(\|u\|_{L^2(\mathbf{R},dx)}=1,\ u\approx e^{i\theta}u)}{L^2(\mathbf{R}, dx) \ni u} \quad \underset{\text{identification}}{\longleftrightarrow} \quad |u\rangle\langle u| \in Tr^p_{+1}(L^2(\mathbf{R}, dx)).$$

We first introduce Robertson's uncertainty relation, which generally seems to be understood (or, misunderstood) as the mathematical representation of Heisenberg's uncertainty relation. By repeating the exact (i.e. the "error" $\Delta(q) = 0$) measurements of the position q of particles with same states, we can obtain its average value $\bar{q}$ and its variance $var(q)$. Also, by repeating the exact (i.e. the "error" $\Delta(p) = 0$) measurements of the momentum p of the same particles, we can similarly get its average value $\bar{p}$ and its variance $var(p)$. A simple calculation shows:

$$\bar{q} = \int_{\mathbf{R}} x\Big|u(x)\Big|^2 dx \quad \text{and} \quad \bar{p} = \int_{\mathbf{R}} \overline{u(x)}\Big[\frac{\hbar d}{idx}u(x)\Big]dx \quad \Big(= \int_{\mathbf{R}} p\Big|\tilde{u}(p)\Big|^2 dp\Big) \tag{12.4}$$

where $\tilde{u}$ is the Fourier transform of u, $\Big($that is, $\tilde{u}(p) = \sqrt{\frac{\hbar}{2\pi}}\int_{\mathbf{R}} u(x)e^{-i\hbar xp}dx\Big)$. And further, we see,

$$\begin{aligned} var(q) &= \int_{\mathbf{R}} |x-\bar{q}|^2\Big|u(x)\Big|^2 dx = \int_{\mathbf{R}} |x|^2\Big|u(x)\Big|^2 dx - \bar{q}^2, \\ var(p) &= \int_{\mathbf{R}} |p-\bar{p}|^2\Big|\tilde{u}(p)\Big|^2 dp = \int_{\mathbf{R}} |\frac{\hbar d}{idx}u(x)|^2 dx - \bar{p}^2. \end{aligned} \tag{12.5}$$

Immediately after Heisenberg's discovery (="Proposition 12.1", 1927), Kennard, by a simple calculation, showed the following uncertainty relation:

$$[var(q)]^{\frac{1}{2}} \cdot [var(p)]^{\frac{1}{2}} \geq \frac{\hbar}{2}. \tag{12.6}$$

(=(12.2))

(*cf.* Lemma 12.13 later). Of course, it is clear that there is a great gap between Heisenberg's uncertainty relation (12.3) and Kennard's uncertainty relation (12.6).

Next we shall introduce the nice idea by Arthurs-Kelly [7], that is, a certain approximate simultaneous measurement of the position q and the momentum p of a particle A in one dimensional real line $\mathbf{R}$, which has a state function $u(x)$ ($\in L^2(\mathbf{R}), \|u\|_{L^2(\mathbf{R})} = 1$).

Note that the position observable $Q(\equiv x)$ and the momentum observable $P(\equiv \frac{\hbar d}{idx})$ do not commute, that is,

$$QP - PQ = i\hbar\Big(\neq 0\Big). \tag{12.7}$$

Therefore, any simultaneous measurement of the position observable x and the momentum observable $\frac{\hbar d}{idx}$ for a particle "A" can not be realized. However, Arthurs-Kelly's idea is excellent as follows: We first prepare another particle "B" with the state $u_0(y)$ such that:

$$\int_{\mathbf{R}} y\Big|u_0(y)\Big|^2 dy = \int_{\mathbf{R}} \overline{u_0(y)}\Big[\frac{\hbar d}{idy}u_0(y)\Big]dy = 0 \tag{12.8}$$

for example, $u_0(y) = \frac{1}{(\pi\hbar)^{1/4}}\exp(- \frac{y^2}{2\hbar})$. Further we regard these two particles "A" and "B" as a "particle C" in two dimensional Euclidean space $\mathbf{R}^2$ with the state $u(x)u_0(y)$ ($\in L^2(\mathbf{R}^2), \|u \cdot u_0\|_{L^2(\mathbf{R}^2)} = 1$). Now consider the self-adjoint operators $(x - y)$ and $\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y}$ in $L^2(\mathbf{R}^2)$, which commute, that is, it holds that:

$$(\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y})(x-y) = (x-y)(\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y}) \tag{12.9}$$

That is because we can easily calculate:

$$\begin{aligned}&[(\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y})(x-y)]f(x,y)\\ =&\frac{\hbar}{i}f(x,y) + x\frac{\hbar\partial}{i\partial x}f(x,y) - y\frac{\hbar\partial}{i\partial x}f(x,y) + x\frac{\hbar\partial}{i\partial y}f(x,y) - \frac{\hbar}{i}f(x,y) - y\frac{\hbar\partial}{i\partial y}f(x,y)\\ =&[(x-y)(\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y})]f(x,y).\end{aligned}$$

Thus the simultaneous measurement of observables $(x - y)$ and $\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y}$ for a "particle C" (= "A" + "B") can be realized. Moreover, we can easily calculate these expectations as follows:

$$\iint_{\mathbf{R}^2} \overline{u(x)u_0(y)}\Big[(x-y)u(x)u_0(y)\Big]dxdy = \int_{\mathbf{R}} x\Big|u(x)\Big|^2 dx \tag{12.10}$$

and

$$\iint_{\mathbf{R}^2} \overline{u(x)u_0(y)}\Big[(\frac{\hbar\partial}{i\partial x} + \frac{\hbar\partial}{i\partial y})u(x)u_0(y)\Big]dxdy = \int_{\mathbf{R}} \overline{u(x)}\Big[\frac{\hbar d}{idx}u(x)\Big]dx. \tag{12.11}$$

By the reason that the equalities (12.10)$= \bar{q}$ and (12.11)$= \bar{p}$ hold, we may say that

(♯) An "approximate simultaneous measurement" of the position observable $Q(\equiv x)$ and the momentum observable $P(\equiv \frac{\hbar d}{idx})$ can be realized.

Here, the variances $var_{asm}(q)$ and $var_{asm}(p)$ in the *approximate simultaneous measurement* of the position q and the momentum p of a particle "C" are given respectively by:

$$
\begin{aligned}
var_{asm}(q) &= \iint_{\mathbf{R}^2} [(x-y)u(x)u_0(y)\Big]^2 dxdy - \Big| \iint_{\mathbf{R}^2} \overline{u(x)u_0(y)} \Big[(x-y)u(x)u_0(y)\Big] dxdy \Big|^2 \\
&= \int_{\mathbf{R}} \Big| xu(x) \Big|^2 dx - \Big| \int_{\mathbf{R}} x \Big| u(x) \Big|^2 dx \Big|^2 + \Big| \int_{\mathbf{R}} \Big| yu_0(y) \Big|^2 dy \Big|^2
\end{aligned}
\tag{12.12}
$$

and

$$
var_{asm}(p) = \int_{\mathbf{R}} \Big| \frac{\hbar d}{idx} u(x) \Big|^2 dx - \Big| \int_{\mathbf{R}} \overline{u(x)} \Big[\frac{\hbar d}{idx} u(x) \Big] dx \Big|^2 + \Big| \int_{\mathbf{R}} \overline{u(y)} \Big[\frac{\hbar d}{idy} u_0(y) \Big] dy. \tag{12.13}
$$

Hence, we can get, by the arithmetic-geometric inequality and the well-known uncertainty relation (Robertson uncertainty relation, *cf.* Lemma 12.13 later), the following simultaneous uncertainty relation;

$$
\begin{aligned}
&[var_{asm}(q)]^{1/2} \cdot [var_{asm}(p)]^{1/2} \\
=&2 \Big[\int_{\mathbf{R}} \Big| xu(x) \Big|^2 dx - \Big| \int_{\mathbf{R}} x \Big| u(x) \Big|^2 dx \Big|^2 \Big]^{1/4} \times \Big[\Big| \int_{\mathbf{R}} \Big| yu_0(y) \Big|^2 dy \Big| \Big]^{1/4} \\
&\times \Big[\int_{\mathbf{R}} \Big| \frac{\hbar d}{idx} u(x) \Big|^2 dx - \Big| \int_{\mathbf{R}} \overline{u(x)} \Big[\frac{\hbar d}{idx} u(x) \Big] dx \Big|^2 \Big]^{1/4} \times \Big[\Big| \int_{\mathbf{R}} \overline{u_0(y)} \Big[\frac{\hbar d}{idy} u_0(y) \Big] dy \Big|^2 \Big]^{1/4} \\
\geq & \hbar.
\end{aligned}
\tag{12.14}
$$

This is Arthurs-Kelly's idea. We believe that Arthurs-Kelly's discovery (12.14) is the first great step to the understanding of Heisenberg's uncertainty relation.

12.3 Approximate simultaneous measurement

Since our main purpose in this chapter is to describe Proposition 12.1 in terms of mathematics and further to prove it, we must clarify the ambiguous words (i.e., "approximate simultaneous", "error") in Proposition 12.1. For this, we prepare several definitions in this section.

According to the well-known spectral representation theorem (*cf.* [92]), there is a bijective correspondence of a crisp observable $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, E)$ in $B(H)$ to an n-tuple (A_1, ..., A_n) of commutative (unbounded) self-adjoint operators on H such that $A_i = \int_{\mathbf{R}^n} \lambda_i$ $E(d\lambda_1 ... d\lambda_n)$. That is,

$$\underset{\text{(commutative self-adjoint operators on } H)}{(A_1, A_2, ..., A_n)} \quad \underset{A_i = \int_{\mathbf{R}^n} \lambda_i E(d\lambda_1 ... d\lambda_n)}{\longleftrightarrow} \quad \underset{\text{(crisp observable in } B(V))}{(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, E)} \tag{12.15}$$

In particular, we frequently identify a crisp observable $(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E)$ in $B(H)$ with a (unbounded) self-adjoint operator A $\Big(= \int_{\mathbf{R}} \lambda \ E(d\lambda)\Big)$ on H.

Note that Proclaim$^{W^*}$1 (9.9) $\Big($or, Axiom$^{W^*}$1 (9.11)$\Big)$ says as follows:

[♯] *Let* $\overline{\mathbf{O}} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ *be an observable in* $B(H)$. *And consider a measurement* $\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F), \overline{S}_{[\rho_u]})$, *where* $\rho_u = |u\rangle\langle u|$. *When we take a measurement* $\overline{\mathbf{M}}_{B(H)}(\mathbf{O} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F), \overline{S}_{[\rho_u]})$, *the probability that the measured value* $\lambda(\in \mathbf{R}^n)$ *belongs to a set* Ξ $(\in \mathcal{B}_{\mathbf{R}^n})$ *is given by*

$$\langle u, F(\Xi) u \rangle_H \ \Big(= tr[\rho_u F(\Xi)]\Big). \tag{12.16}$$

Therefore, the expectation $\mathbf{E}\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \overline{S}_{[\rho_u]})\Big] \ \left(\equiv \Big(\mathbf{E}^{(i)}\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \overline{S}_{[\rho_u]})\Big]\Big)_{i=1}^n \right)$ of the measured value obtained by the measurement $\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F), \overline{S}_{[\rho_u]})$ is given by

$$\begin{aligned} &\mathbf{E}^{(i)}\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \overline{S}_{[\rho_u]})\Big] \\ = &\int_{\mathbf{R}^n} \lambda_i \langle u, F(d\lambda_1 \cdots d\lambda_n) u \rangle_H \qquad i = 1, 2, ..., n, \end{aligned} \tag{12.17}$$

where $\rho_u = |u\rangle\langle u|$. Further, its variance $var\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \ \overline{S}_{[\rho_u]})\Big] \ \left(\equiv \Big(var^{(i)}\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \overline{S}_{[\rho_u]})\Big]\Big)_{i=1}^n \right)$ is given by

$$\begin{aligned} &var^{(i)}\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \overline{S}_{[\rho_u]})\Big] \\ = &\int_{\mathbf{R}^n} \Big|\lambda_i - \mathbf{E}^{(i)}\Big[\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}, \overline{S}_{[\rho_u]})\Big]\Big|^2 \langle u, F(d\lambda_1 \cdots d\lambda_n) u \rangle_H && (12.18) \\ = &\int_{\mathbf{R}^n} |\lambda_i|^2 \langle u, F(d\lambda_1 \cdots d\lambda_n) u \rangle_H - \Big| \int_{\mathbf{R}^n} \lambda_i \langle u, F(d\lambda_1 \cdots d\lambda_n) u \rangle_H \Big|^2 && (12.19) \end{aligned}$$

$(i = 1, 2, ..., n)$.

We begin with the following definition.

Definition 12.3. *Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$.*

(1). *A triplet $\widehat{\mathbf{O}}^{tnsr}_{H \otimes K} = (K, s, (X, \mathcal{F}, \widehat{F}))$ is called a "tensor observable" (or precisely, "tensor represented observable") in $B(H \otimes K)$, if it satisfies the following conditions* (i) and (ii):

(i) *K is a Hilbert space and s is an element in K such that $\|s\| = 1$,*

(ii) *$(X, \mathcal{F}, \widehat{F})$ is a crisp observable in $B(H \otimes K)$, where $H \otimes K$ is a tensor Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{H \otimes K}$.*

(2). *Let $(X, \mathcal{F}, F)$ be any observable in $B(H)$. A tensor observable $\widehat{\mathbf{O}}^{tnsr}_{H \otimes K} = (K, s, (X, \mathcal{F}, \widehat{F}))$ is called a realization of the observable $(X, \mathcal{F}, F)$ in tensor Hilbert space $H \otimes K$, if it holds that*

$$\langle u \otimes s, \widehat{F}(\Xi)(u \otimes s) \rangle_{H \otimes K} = \langle u, F(\Xi) u \rangle_H \quad (\forall u \in H, \forall \Xi \in \mathcal{F}). \tag{12.20}$$

■

The following proposition is essential to our argument.

Proposition 12.4. [Holevo [34]]. *Let $(X, \mathcal{F}, F)$ be an observable in $B(H)$. Then, there exists a tensor observable $\widehat{\mathbf{O}}^{tnsr}_{H \otimes K} = (K, s, (X, \mathcal{F}, \widehat{F}))$ that is the realization of $(X, \mathcal{F}, F)$, that is, it holds that*

$$\langle u \otimes s, \widehat{F}(\Xi)(u \otimes s) \rangle_{H \otimes K} = \langle u, F(\Xi) u \rangle_H \quad (u \in H, \Xi \in \mathcal{F}). \tag{12.21}$$

Conversely any crisp observable $(X, \mathcal{F}, \widehat{F})$ in $B(H \otimes K)$ and any $s (\in K, \|s\|_K = 1)$ give rise to the unique observable $(X, \mathcal{F}, F)$ in $B(H)$ satisfying (12.21).

■

We shall use the following notations.

Notation 12.5. [Domain]. *Let $A \left(= \int_{\mathbf{R}} \lambda \; E_A(d\lambda)\right.$, the spectral representation of $\left.A\right)$ be a (unbounded) self-adjoint operator on H. Then, we define the* Dom(A), *the domain of A, by*

$$\mathrm{Dom}(A) := \{u \in H : \int_{\mathbf{R}} |\lambda|^2 \langle u, E_A(d\lambda) u \rangle < \infty \}.$$

Let $\overline{\mathbf{O}} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ *and* $\widehat{\mathbf{O}}^{tnsr}_{H\otimes K} = (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F}))$ *be an observable and a tensor observable in* $B(H)$ *and in* $B(H \otimes K)$ *respectively. Then, we define that*

$$[\overline{\mathbf{O}}]^{mar}_{(k)} := (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, [F]^{mar}_{(k)}) \qquad \text{(it will be called the } k\text{th marginal observable of } \overline{\mathbf{O}}) ,$$

where

$$[F]^{mar}_{(k)}(\Xi) := F(\underbrace{\mathbf{R}\times\cdots\times\mathbf{R}}_{k-1 \text{ times}} \times \Xi \times \underbrace{\mathbf{R}\times\cdots\times\mathbf{R}}_{n-k \text{ times}}) \qquad (\forall \Xi \in \mathcal{B}_{\mathbf{R}})$$

Further, define that

$$\mathrm{Dom}([\overline{\mathbf{O}}]^{mar}_{(k)})\Big(\equiv \mathrm{Dom}([F]^{mar}_{(k)})\Big) := \{u \in H : \int_{\mathbf{R}^n} |\lambda_k|^2 \langle u, F(d\lambda_1 ... d\lambda_n) u\rangle < \infty\},$$

$$\mathrm{Dom}([\widehat{\mathbf{O}}]^{mar}_{(k)})\Big(\equiv \mathrm{Dom}([\widehat{F}]^{mar}_{(k)})\Big) := \{\hat{v} \in H\otimes K : \int_{\mathbf{R}^n} |\lambda_k|^2 \langle \hat{v}, \widehat{F}(d\lambda_1 ... d\lambda_n) \hat{v}\rangle_{H\otimes K} < \infty\},$$

$$\mathrm{Dom}_{\otimes s}([\widehat{\mathbf{O}}^{tnsr}_{H\otimes K}]^{mar}_{(k)})\Big(\equiv \mathrm{Dom}_{\otimes s}([\widehat{F}]^{mar}_{(k)})\Big)$$
$$:= \{u \in H : \int_{\mathbf{R}^n} |\lambda_k|^2 \langle u\otimes s, \widehat{F}(d\lambda_1 ... d\lambda_n)(u\otimes s)\rangle_{H\otimes K} < \infty\}, \tag{12.22}$$

where $\mathrm{Dom}([\overline{\mathbf{O}}]^{mar}_{(k)})$ *(or* $\mathrm{Dom}([\widehat{\mathbf{O}}]^{mar}_{(k)})$*) is called the k-th domain of* $\overline{\mathbf{O}}$ *(or* $\widehat{\mathbf{O}}$*).*

■

Now we have the following main definition.

Definition 12.6. [Approximate simultaneous observable]. *Let* $A_1, ..., A_n$ *be (unbounded) self-adjoint operators in* H*. An observable* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ *in* $B(H)$ *is called the approximate simultaneous observable* of $A_1, ..., A_n$, if it satisfies the following conditions

(i) *(domain condition) for each* i $(= 1, 2, ..., n)$, $\mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \cap \mathrm{Dom}(A_i)$ *is dense in* H

(ii) *(unbias condition) for each* i $(= 1, 2, ..., n)$,

$$\langle u, A_i u\rangle = \int_{\mathbf{R}} \lambda \langle u, [F]^{mar}_{(i)}(d\lambda) u\rangle, \qquad (u \in \mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \cap \mathrm{Dom}(A_i)). \tag{12.23}$$

■

Remark 12.7. [1]. As seen later (*cf.* Lemma 12.14(iii)), it holds that $\mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \subseteq \mathrm{Dom}(A_i)$ holds. Thus, $\mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \cap \mathrm{Dom}(A_i) = \mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)})$

[2]. There is a very reason to assume the following condition (iii) or (iv) instead of the above (i). $\Big($(iii) and (iv) are stronger than (i), more precisely, (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).$\Big)$

(iii) (*self-adjointness*) *for each* i $(= 1, 2, ..., n)$, A_i *is essentially self-adjoint on* $\text{Dom}([\overline{\mathbf{O}}{}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \cap \text{Dom}(A_i)$,

or

(iv) (*commutative condition*) *for each* i $(= 1, 2, ..., n)$, $A_i \Big(= \int_{\mathbf{R}} \lambda \, E_i(d\lambda)\Big)$ *and* $[\overline{\mathbf{O}}{}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}$ *commute.*

Although each of (i), (iii) and (iv) has merit and demerit respectively, the physical meaning of the (iv) is the clearest. (Continued on Remark 12.12.)

[3]. Also, see the condition (i) in Example 11.5. This condition is equivalent to

- the formula (12.23) holds on a dense set $\cap_{i=1}^n \Big(\text{Dom}([\overline{\mathbf{O}}{}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)} \cap \text{Dom}(A_i)\Big)$.

■

Definition 12.8. [Approximate simultaneous tensor observable]. *Let* $A_1, ..., A_n$ *be (unbounded) self-adjoint operators in* H. *A tensor observable* $\widehat{\mathbf{O}}{}^{ASTO}_{[A_l]_{l=1}^n} = (K,\, s,\, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F}))$ *is called an approximate simultaneous tensor observable of* $A_1, ..., A_n$, *if* $\widehat{\mathbf{O}}{}^{ASTO}_{[A_l]_{l=1}^n} = (K,\, s,$ $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F}))$ *satisfies the following conditions:*

(i) (*domain condition*) *for each* i $(= 1, 2, ..., n)$, $\text{Dom}_{\otimes s}([\widehat{\mathbf{O}}{}^{ASTO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \cap \text{Dom}(A_i)$ *is dense in* H

(ii) (*unbias condition*) *for each* i $(= 1, 2, ..., n)$,

$$\langle u, A_i u\rangle = \int_{\mathbf{R}^n} \lambda_i \langle u \otimes s, \widehat{F}(d\lambda_1 \cdots d\lambda_n)(u \otimes s)\rangle \tag{12.24}$$
$$(u \in \text{Dom}_{\otimes s}([\widehat{\mathbf{O}}{}^{ASTO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \cap \text{Dom}(A_i),).$$

■

The relation between $\overline{\mathbf{O}}{}^{ASO}_{[A_l]_{l=1}^n}$ and $\widehat{\mathbf{O}}{}^{ASTO}_{[A_l]_{l=1}^n}$ is characterized by the following proposition.

Proposition 12.9. *Let* $A_1, ..., A_n$ *be (unbounded) self-adjoint operators in* H.

(i) *Let* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n} \equiv (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F})$ *be an approximate simultaneous tensor observable of* $A_1, ..., A_n$ *in* H. *Then, there exists an approximate simultaneous observable* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ *such as* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}$ *is a realization of* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}$.

(ii) *Let* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ *be an approximate simultaneous observable of* $A_1, ..., A_n$ *in* H. *Then, there exists a approximate simultaneous tensor observable* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}$ $\equiv (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F})$ *such as it is a realization of* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}$.

(iii) *Let* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ *be an approximate simultaneous observable of* $A_1, ..., A_n$ *in* H. *Let* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n} \equiv (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F})$ *be an approximate simultaneous tensor observable of* $A_1, ..., A_n$ *in* H. *And assume that* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}$ *is a realization of* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}$. *Then, for each* i $(= 1, 2, ..., n)$,

$$\mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) = \mathrm{Dom}_{\otimes s}([\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \subseteq \mathrm{Dom}(A_i). \tag{12.25}$$

Proof. The statement (i) is trivial. Also the statement (ii) and the equality "=" in (12.25) immediately follow from Proposition 12.4. Also, the inclusion "⊆" in (12.25) is proved in Lemma 12.14(iii) later. □

Definition 12.10. [Uncertainty] *Let* $A_1, ..., A_n$ *be (unbounded) self-adjoint operators on a Hilbert space* H.

[I]. *Let* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} = (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$) *be an approximate simultaneous observable of* $A_1, \ldots, A_n$.

(i). Then, the uncertainty $\Big(\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u)\Big)_{i=1}^n$ *of* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}$ *for a state* u $(\|u\|_H = 1)$ *is defined by*

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u) = \int_{\mathbf{R}^n} \lambda_i^2 \langle u, F(d\lambda_1 \cdots d\lambda_n)u\rangle - \int_{\mathbf{R}} \lambda^2 \langle u, A_i(d\lambda)u\rangle \tag{12.26}$$
$$(u \in H \text{ such that } \|u\| = 1 \ ),$$

where (12.26) *should be interpreted that* $\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u) = \infty$ *for* $u \notin \mathrm{Dom}([F]^{mar}_{(i)})$ *(cf.* $\mathrm{Dom}([F]^{mar}_{(i)}) \subseteq \mathrm{Dom}(A_i)$ *in (12.25)).* $\Big($"$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u) \geq 0$" *will be shown in Theorem 12.15 later.*$\Big)$

(ii). Also the i*-th variance* $var_{(i)}[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}, u]$ *is defined by*

$$var_{(i)}[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}, u] = \int_{\mathbf{R}^n} |\lambda_i - \langle u, A_i u\rangle|^2 \langle u, F(d\lambda_1 \cdots d\lambda_n)u\rangle_H \tag{12.27}$$

$$(i = 1, 2, ..., n),$$

[II] Let $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n} = (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F})\ )$ be an approximate simultaneous tensor observable of $A_1, \ldots, A_n$.

(i). Then, the uncertainty $\Big(\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u)\Big)_{i=1}^n$ of $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}$ for a state u ($\|u\|_H = 1$) is defined by

$$\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u) = \int_{\mathbf{R}^n} \lambda_i^2 \langle u \otimes s, \widehat{F}(d\lambda_1 \cdots d\lambda_n)(u \otimes s)\rangle - \int_{\mathbf{R}} \lambda^2 \langle u, A_i(d\lambda) u\rangle \tag{12.28}$$
$$(u \in H \text{ such that } \|u\| = 1\ ),$$

where (12.28) *should be interpreted that* $\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u) = \infty$ *for* $u \notin \mathrm{Dom}_{\otimes s}([\widehat{F}]^{mar}_{(i)})$ *(cf.* $\mathrm{Dom}_{\otimes s}([\widehat{F}]^{mar}_{(i)}) \subseteq \mathrm{Dom}(A_i)$ *in (12.25)).* $\Big($*"*$\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u) \geq 0$*" will be shown in Theorem 12.15 later.*$\Big)$

(ii). *Also the i-th variance* $var_{(i)}[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}, u]$ *is defined by*

$$var_{(i)}[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}, u] = \int_{\mathbf{R}^n} |\lambda_i - \langle u, A_i u\rangle|^2 \langle u \otimes s, \widehat{F}(d\lambda_1 \cdots d\lambda_n)(u \otimes s)\rangle_{H \otimes K} \tag{12.29}$$
$$(i = 1, 2, ..., n).$$

■

Proposition 12.11. *Let* $A_1, ..., A_n$ *be (unbounded) self-adjoint operators on a Hilbert space* H. *Assume that* $\widehat{\mathbf{O}}^{ASTO}_{[A_i]_{i=1}^n} = (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F}))$ *is a realization of* $\overline{\mathbf{O}}^{ASO}_{[A_i]_{i=1}^n} = (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$. *Let* $u \in H$ ($\|u\|_H = 1$). *Then it holds that*

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u) = \Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u) \tag{12.30}$$

and

$$var_{(i)}[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}, u] = var_{(i)}[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}, u]. \tag{12.31}$$

Proof. This immediately follows from Definition 12.10. □

Remark 12.12. [Continued from Remark 12.7]. Again note that, if the commutative condition (iv) in Remark 12.7 is assumed in the Definition 12.10, we can define $\Delta\Big(\overline{\mathbf{M}}_{B(H)}(A_i \times [\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}, \overline{S}(\rho_u))\Big)$, the distance between A_i and $[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}$, *cf.* Definition 11.1. And further we see that

$$\underset{\text{("error" defined in Definition 11.1)}}{\Delta\Big(\overline{\mathbf{M}}_{B(H)}(A_i \times [\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}, \overline{S}(\rho_u))\Big)} = \underset{\text{("uncertainty" defined in Definition 12.10)}}{\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u)} \tag{12.32}$$

Thus, in this case, the physical meaning of "uncertainty" is clear.

■

12.4 Lemmas

In this section, we shall prepare some Lemmas.

Lemma 12.13. [Robertson's uncertainty relation]. *Let A_1 and A_2 be any symmetric operators on a Hilbert space H. Then, it holds that*

$$\Big[\|A_1u\|^2 - |\langle u, A_1u\rangle|^2\Big]^{1/2} \cdot \Big[\|A_2u\|^2 - |\langle u, A_2u\rangle|^2\Big]^{1/2} \geq \frac{1}{2}\Big|\Big\langle A_1u, A_2u\Big\rangle - \Big\langle A_2u, A_1u\Big\rangle\Big| \tag{12.33}$$

for all $u \in \mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2)$.

Proof. Using Schwartz inequality, we see

$$\begin{aligned} & |\langle A_1u, A_2u\rangle - \langle A_2u, A_1u\rangle| \\ =& \Big|\Big\langle A_1u - \langle u, A_1u\rangle u, A_2u - \langle u, A_2u\rangle u\Big\rangle - \Big\langle A_2u - \langle u, A_2u\rangle u, A_1u - \langle u, A_1u\rangle u\Big\rangle\Big| \\ \leq& 2\Big[\|A_1u\|^2 - |\langle u, A_1u\rangle|^2\Big]^{1/2} \cdot \Big[\|A_2u\|^2 - |\langle u, A_2u\rangle|^2\Big]^{1/2}. \end{aligned} \tag{12.34}$$

□

Lemma 12.14. *Let $A_1, \cdots, A_n$ be any (unbounded) self-adjoint operators in a Hilbert space H. Let $(K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F})$ be an approximate simultaneous tensor observable for $A_1, \cdots, A_n$. Put $\widehat{A}_k = \int_{\mathbf{R}^n} \lambda_k \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)$ $\Big(\equiv \int_{\mathbf{R}} \lambda[\widehat{F}] = (i)^{mar}(d\lambda)\Big)$ $(k = 1, 2, ..., n)$. Then, the following equalities* (i) ∼ (iii) *hold*

(i)

$$\langle v, A_ku\rangle = \langle v \otimes s, \widehat{A}_i(u \otimes s)\rangle = \int_{\mathbf{R}^2} \lambda_k \langle v \otimes s, \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle \tag{12.35}$$

for all $u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_k)$ and all $v \in H$ $(k = 1, 2, ..., n)$,

(ii)

$$\begin{aligned} & \int_{\mathbf{R}^n} \lambda_i\lambda_j \langle u \otimes s, \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle \\ & = \langle \widehat{A}_i(u \otimes s), \widehat{A}_j(u \otimes s)\rangle \end{aligned}$$

$$= \langle A_i u, A_j u\rangle + \langle (\widehat{A}_i - A_i \otimes I)(u \otimes s), (\widehat{A}_j - A_j \otimes I)(u \otimes s)\rangle \tag{12.36}$$

for all $i \neq j$ *and all* $u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_i) \cap \mathrm{Dom}_{\otimes s}(\widehat{A}_j)$,

(iii)

$$\begin{aligned}&\int_{\mathbf{R}^2} |\lambda_k|^2 \langle u \otimes s, \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle \\ &= \|\widehat{A}_k(u \otimes s)\|^2 = \|A_k u\|^2 + \|(\widehat{A}_k - A_k \otimes I)(u \otimes s)\|^2 \geq \|A_k u\|^2\end{aligned} \tag{12.37}$$

for all $u \in \mathrm{Dom}_s(\widehat{A}_k)$ $(k = 1, 2, ..., n)$. *Thus, it holds that, for each* i $(= 1, 2, ..., n)$,

$$\mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) = \mathrm{Dom}_{\otimes s}([\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \subseteq \mathrm{Dom}(A_i). \tag{12.38}$$

Proof. First we prove (i). Fix $k \in \{1, 2\}$. We can see that, for any $v, u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_k)$.

$$\begin{aligned}&\langle v, A_k u\rangle \\ &= \frac{1}{4}\{\langle (v+u), A_k(v+u)\rangle - \langle (v-u), A_k(v-u)\rangle \\ &\quad - i\langle (v+iu), A_k(v+iu)\rangle + i\langle (v-iu), A_k(v-iu)\rangle\} \\ &= \frac{1}{4}\{\langle (v+u)\otimes s, \widehat{A}_k((v+u)\otimes s)\rangle - \langle (v-u)\otimes s, \widehat{A}_k((v-u)\otimes s)\rangle \\ &\quad - i\langle (v+iu)\otimes s, \widehat{A}_k((v+iu)\otimes s)\rangle + i\langle (v-iu)\otimes s, \widehat{A}_k((v-iu)\otimes s)\rangle\} \\ &= \langle v \otimes s, \widehat{A}_k(u \otimes s)\rangle \\ &= \langle v \otimes s, \int_{\mathbf{R}^n} \lambda_k \widehat{F}_k(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle = \int_{\mathbf{R}^n} \lambda_k \langle v \otimes s, \widehat{F}_k(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle.\end{aligned} \tag{12.39}$$

Since $\mathrm{Dom}_{\otimes s}(\widehat{A}_k)$ is dense in H, we see that

$$\langle v, A_k u\rangle = \langle v \otimes s, \widehat{A}_k(u \otimes s)\rangle = \int_{\mathbf{R}^n} \lambda_k \langle v \otimes s, \widehat{A}_k(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle \tag{12.40}$$

for all $u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_k)$ and all $v \in H$. This completes the proof of (i).

Next, we prove (ii). Without loss of generality, we put $i = 1$ and $j = 2$. Let u be any element in $\mathrm{Dom}_{\otimes s}(\widehat{A}_1) \cap \mathrm{Dom}_{\otimes s}(\widehat{A}_2)$. Then, we see, by the above (i), that

$$\begin{aligned}&\int_{\mathbf{R}^n} \lambda_1 \lambda_2 \langle u \otimes s, \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle \\ &= \langle \int_{\mathbf{R}^n} \lambda_1 \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s), \int_{\mathbf{R}^n} \lambda_2 \widehat{F}(d\lambda_1 d\lambda_2 \cdots d\lambda_n)(u \otimes s)\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \widehat{A}_1(u\otimes s), \widehat{A}_2(u\otimes s)\rangle \\
&= \langle (\widehat{A}_1 - A_1\otimes I)(u\otimes s) + (A_1u\otimes s), (\widehat{A}_2 - A_2\otimes I)(u\otimes s) + (A_2u\otimes s)\rangle \\
&= \langle (\widehat{A}_1 - A_1\otimes I)(u\otimes s), (\widehat{A}_2 - A_2\otimes I)(u\otimes s)\rangle \\
&\qquad + \langle (\widehat{A}_1 - A_1\otimes I)(u\otimes s), A_2u\otimes s\rangle \\
&\qquad + \langle A_1u\otimes s, (\widehat{A}_2 - A_2\otimes I)(u\otimes s)\rangle + \langle A_1u\otimes s, A_2u\otimes s\rangle \\
&= \langle (\widehat{A}_1 - A_1\otimes I)(u\otimes s), (\widehat{A}_2 - A_2\otimes I)(u\otimes s)\rangle \\
&\qquad + \langle \widehat{A}_1(u\otimes s), A_2u\otimes s\rangle - \langle A_1u, A_2u\rangle \\
&\qquad + \langle A_1u\otimes s, \widehat{A}_2(u\otimes s)\rangle - \langle A_1u, A_2u\rangle + \langle A_1u, A_u\rangle \\
&= \langle (\widehat{A}_1 - A_1\otimes I)(u\otimes s), (\widehat{A}_2 - A_2\otimes I)(u\otimes s)\rangle - \langle A_1u, A_2u\rangle \\
&\qquad + \int_{\mathbf{R}^n} \lambda_2 \langle A_1u\otimes s, \widehat{F}(d\lambda_1 d\lambda_2)(u\otimes s)\rangle + \int_{\mathbf{R}^n} \lambda_1 \langle \widehat{F}(d\lambda_1 d\lambda_2)(u\otimes s), A_2u\otimes s\rangle \\
&= \langle A_1u, A_2u\rangle + \langle (\widehat{A}_1 - A_1\otimes I)(u\otimes s), (\widehat{A}_2 - A_2\otimes I)(u\otimes s)\rangle. \qquad (12.41)
\end{aligned}
$$

Hence, the proof of (ii) is completed. Also, the proof of (12.37) is carried out just in a similar way. Lastly, we can easily see that (12.37) implies (12.38) since we see that $\mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) = \mathrm{Dom}_{\otimes s}([\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}]^{mar}_{(i)})$ in (12.25). □

Now we have the following theorem, which is one of our main results.

Theorem 12.15. *Let* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n} = (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F})\ )$ *be a realization of an approximate simultaneous tensor observable* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} = (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F}\ )$ *of* $A_1, \ldots, A_n$. *Put* $\widehat{A}_i = \int_{\mathbf{R}} \lambda [\widehat{F}]^{mar}_{(i)}(d\lambda)$. *Then, we see that*

$$
\begin{aligned}
\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u) = &\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u) = \int_{\mathbf{R}} \lambda^2 \langle u, [F]^{mar}_{(i)}(d\lambda)u\rangle - \int_{\mathbf{R}} \lambda^2 \langle u, A_i(d\lambda)u\rangle \\
= &\|\widehat{A}_i(u\otimes s)\|^2 - \|A_iu\|^2 \qquad (12.42) \\
= &\|(\widehat{A}_i - A_i\otimes I)(u\otimes s)\|^2 \qquad (\forall u \in H \text{ such that } \|u\| = 1 \qquad (12.43)
\end{aligned}
$$

Proof. It immediately follows from Lemma 12.14. □

12.5 Existence theorem

Now we shall mention the following theorem, which assures the existence of an approximate simultaneous tensor observable of arbitrary observables $A_1, ..., A_n$. For two observables A_1 and A_2, the similar theorem was proved by P. Busch, *et al.* [15, 14].

Theorem 12.16. [Cf.[36]] *Let $A_1, ..., A_n$ be (unbounded) self-adjoint operators on a Hilbert space H. Let $a_1, ..., a_n$ be any positive numbers such that $\sum_{i=1}^n (1 + {a_i}^2)^{-1} = 1$. Then, we see,*

(i) there exists an approximate simultaneous tensor observable $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n} \equiv (K, s, (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{F}))$ of $A_1, ..., A_n$ such that:

$$\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u) = a_i \|A_i u\| \quad (u \in \mathrm{Dom}_{\otimes s}([\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \quad i = 1, 2, ..., n). \tag{12.44}$$

and equivalently,

(ii) there exists an approximate simultaneous observable $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$ of $A_1, ..., A_n$ such that:

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}}(A_i, u) = a_i \|A_i u\| \quad (u \in \mathrm{Dom}([\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^n}]^{mar}_{(i)}) \quad i = 1, 2, ..., n). \tag{12.45}$$

Proof. By Proposition 12.11, it suffices to prove (i). Put $K = \mathbf{C}^n = \{z = (z_1, ..., z_n) : z_i \in \mathbf{C} \ (i = 1, 2, ..., n)\}$, i.e., the n-dimensional Hilbert space with the norm $\|z\|_n = [\Sigma_{i=1}^n |z_i|^2]^{1/2}$. Put $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$,...., $e_n = (0, 0, ..., 1) \in \mathbf{C}^n$. Put $s = e_1$. And put $P_i : \mathbf{C}^n \to \mathbf{C}^n$, $(i = 1, 2, ..., n)$, a projection such that $P_i e_i = e_i, P_i e_k = 0 (k \neq i)$, that is, $P_i = |e_i\rangle\langle e_i|$. Put $b_i = (1 + {a_i}^2)^{1/2}$ and $B_i = {b_i}^2 A_i$ $(i = 1, 2, ..., n)$. Consider the spectral representations

$$A_i = \int_{\mathbf{R}} \lambda E_{A_i}(d\lambda), \quad B_i = \int_{\mathbf{R}} \lambda E_{B_i}(d\lambda), \quad 0 = \int_{\mathbf{R}} \lambda E_0(d\lambda) \quad \text{in } H$$

and

$$P_i = \int_{\mathbf{R}} \lambda E^{\mathbf{C}^n}_{P_i}(d\lambda), \quad I = \int_{\mathbf{R}} \lambda E^{\mathbf{C}^n}_I(d\lambda) \quad \text{in } \mathbf{C}^n.$$

Note that $E_{A_i}(d(\lambda/{b_i}^2)) = E_{B_i}(d\lambda)$. Define the unitary operator $\widehat{U} : H \otimes \mathbf{C}^n \to H \otimes \mathbf{C}^n$ by $\widehat{U} = I \otimes U$ where a unitary operator U on $\mathbf{C}^n$ satisfies that $U e_1 = \Sigma_{i=1}^n e_i / b_i$. And define the crisp observable $(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \widehat{E}_{\widehat{A}_i})$ in $B(H \otimes \mathbf{C}^n)$ by

$$\widehat{E}_{\widehat{A}_i}(d\xi) = \widehat{U}^* [E_{B_i}(d\xi) \otimes P_i + E_0(d\xi) \otimes (I - P_i)] \widehat{U} \quad (i = 1, 2, ..., n). \tag{12.46}$$

Since $\widehat{E}_{\widehat{A}_1}, ..., \widehat{E}_{\widehat{A}_n}$ commute, we can define a crisp observable $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \widehat{E}_{\widehat{A}})$ in $B(H \otimes \mathbf{C}^n)$ such that:

$$\widehat{E}_{\widehat{A}}(d\xi_1 d\xi_2 ... d\xi_n) = \prod_{i=1}^n \widehat{E}_{\widehat{A}_i}(d\xi_i). \tag{12.47}$$

Now, we shall show that the tensor observable $\widehat{\mathbf{O}}^{tnsr}_{H\otimes K} = (\mathbf{C}^n, e_1, (\mathbf{R}^n, B_n, \widehat{E}_{\widehat{A}}))$ is an approximate simultaneous tensor observable of $A_1, ..., A_n$. Put $\widehat{A}_i = \int_{\mathbf{R}^n} \xi_i \, \widehat{E}_{\widehat{A}} \, (d\xi_1 d\xi_2 ... d\xi_n)$ $(i = 1, ..., n)$. Then we see that,

$$
\begin{aligned}
&\int_{\mathbf{R}^n} |\xi_i|^2 \langle u \otimes e_1, \widehat{E}_{\widehat{A}}(d\xi_1 d\xi_2 ... d\xi_n)(u \otimes e_1)\rangle \\
=& \int_{\mathbf{R}} |\xi_i|^2 \langle u \otimes e_1, \widehat{E}_{\widehat{A}_i}(d\xi_i)(u \otimes e_1)\rangle \\
=& \int_{\mathbf{R}} |\xi_i|^2 \langle u \otimes e_1, [(I \otimes U^*)\Big(E_{B_i}(d\xi_i) \otimes P_i + E_0(d\xi_i) \otimes (I - P_i)\Big)(I \otimes U)](u \otimes e_1)\rangle \\
=& \int_{\mathbf{R}} |\xi|^2 \langle u, E_{B_i}(d\xi) u\rangle \cdot \langle e_1, U^* P_i U e_1\rangle \\
=& \int_{\mathbf{R}} |\xi|^2 \langle u, E_{B_i}(d\xi) u\rangle \cdot \langle \Sigma_{j=1}^n \frac{e_j}{b_j}, P_i \Sigma_{k=1}^n \frac{e_k}{b_k}, \rangle \\
=& |b_i|^{-2} \int_{\mathbf{R}} |\lambda|^2 \langle u, E_{B_i}((d\lambda) u\rangle = |b_i|^2 \int_{\mathbf{R}} |\lambda|^2 \langle u, E_{A_i}(d\lambda) u\rangle.
\end{aligned} \tag{12.48}
$$

Hence, $\mathrm{Dom}_{\otimes s}(\widehat{A}_i) = \mathrm{Dom}(A_i)$ (where $s = e_1$). Similarly we see

$$
\begin{aligned}
&\int_{\mathbf{R}^n} \xi_i \langle u \otimes e_1, \widehat{E}_{\widehat{A}}(d\xi_1 d\xi_2 ... d\xi_n)(u \otimes e_1)\rangle \\
=& |b_i|^{-2} \int_{\mathbf{R}} \lambda \langle u, E_{B_i}((d\lambda) u\rangle = \int_{\mathbf{R}} \lambda \langle u, E_{A_i}(d\lambda) u\rangle.
\end{aligned} \tag{12.49}
$$

Thus, $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}$ satisfies the condition (ii) in Definition 12.6. Also, noting that $I(d\lambda) = I(1 \in d\lambda), = 0(1 \notin d\lambda)$, we also see that, for each i $(i = 1, 2, ..., n)$ and $\Xi_k \in \mathcal{B}$,

$$
\begin{aligned}
&\widehat{E}_{\widehat{A}_i}(\Xi_1) \cdot (E_{A_i}(\Xi_2) \otimes I) \\
=& (I \otimes U^*)\Big(E_{B_i}(\Xi_1) \otimes P_i + E_0(\Xi_1) \otimes (I - P_i)\Big)(I \otimes U)(E_{A_i}(\Xi_2) \otimes I) \\
=& (E_{A_i}(\Xi_2) \otimes I)(I \otimes U^*)\Big(E_{B_i}(\Xi_1) \otimes P_i + E_0(\Xi_1) \otimes (I - P_i)\Big)(I \otimes U) \\
=& (E_{A_i}(\Xi_2) \otimes I) \cdot \widehat{E}_{\widehat{A}_i}(\Xi_1).
\end{aligned} \tag{12.50}
$$

So, $\widehat{A}_i$ and $A_i \otimes I$ commute since $\widehat{A}_i = \int_{\mathbf{R}} \xi \; E_{\widehat{A}_i}(d\xi)$ and $A_i \otimes I = \int_{\mathbf{R}} \xi \; (E_{A_i}(d\xi) \otimes I)$. Hence, $\widehat{A}_i - A_i \otimes I$ on $\mathrm{Dom}(\widehat{A}_i) \cap \mathrm{Dom}(A_i \otimes I)$ has the unique self-adjoint extension $[\widehat{A}_i - A_i \otimes I]$, which has the spectral representation

$$
[\widehat{A}_i - A_i \otimes I] = \int_{\mathbf{R}^2} (\xi_1 - \xi_2) \widehat{E}_{\widehat{A}_i}(d\xi_1)(E_{A_i}(d\xi_2) \otimes I). \tag{12.51}
$$

Then, we see that

$$
\|[\widehat{A}_i - A_i \otimes I](u \otimes e_1)\|^2 \tag{12.52}
$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^2} |\xi_1 - \xi_2|^2 \langle u \otimes e_1, E_{\widehat{A}_i}(d\xi_1)(E_{A_i}(d\xi_2) \otimes I)(u \otimes e_1)\rangle \\
&= \int_{\mathbf{R}} |\xi|^2 \langle u \otimes e_1, E_{\widehat{A}_i}(d\xi_1)(u \otimes e_1)\rangle \\
&\quad - 2\int_{\mathbf{R}^2} \xi_1\xi_2 \langle u \otimes e_1, E_{\widehat{A}_i}(d\xi_1)(E_{A_i}(d\xi_2) \otimes I)(u \otimes e_1)\rangle \\
&\quad + \int_{\mathbf{R}} |\xi_2|^2 \langle u \otimes e_1, (E_{A_i}(d\xi_2) \otimes I)(u \otimes e_1)\rangle \\
&= (|b_i|^2 - 2 + 1) \int_{\mathbf{R}} |\xi|^2 \langle u, E_{A_i}(d\xi)u\rangle \\
&= |a_i|^2 \|A_i u\|^2, \qquad (12.53)
\end{aligned}
$$

which implies that $\mathrm{Dom}_{\otimes s}([\widehat{A}_i - A_i \otimes I]) = \mathrm{Dom}(A_i)$ (where $s = e_1$) and $\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^n}}(A_i, u)$ $= a_i\|A_i u\|$. Therefore, the proof of theorem is completed. □

Remark 12.17. In the above proof, the following statements were also proved:

(i) $\widehat{A}_i$ and $A_i \otimes I$ commute, so $\widehat{A}_i - A_i \otimes I$ on $\mathrm{Dom}(\widehat{A}_i) \cap \mathrm{Dom}(A_i \otimes I)$ has a unique self-adjoint extension $[\widehat{A}_i - A_i \otimes I]$ $(i = 1, 2)$,

(ii) $\mathrm{Dom}_{\otimes s}(\widehat{A}_i) = \mathrm{Dom}_{\otimes s}([\widehat{A}_i - A_i \otimes I]) = \mathrm{Dom}(A_i)$ $(i = 1, 2)$.

Thus the commutative condition (iv) in Remark 12.7 is satisfied.

■

12.6 Uncertainty relations

Now we propose the following theorem, which is our main result in this chapter. We believe that this theorem is the final version of Heisenberg's uncertainty relation concerning measurement errors.

Theorem 12.18. [Heisenberg's uncertainty relation, *cf.* [36, 67]]. *Let A_1 and A_2 be any (unbounded) self-adjoint operators on a Hilbert space H. Then, we see,*

(i) for any approximate simultaneous tensor observable $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2} \equiv (K, s, (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, \widehat{F}))$ of A_1 and A_2, the following inequality holds:

$$\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}}(A_1, u) \cdot \Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}}(A_2, u) \geq \frac{1}{2}|\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \qquad (12.54)$$

for all $u \in H$ *such that* $\|u\| = 1$*, where the left hand side of* (12.54) *is defined by* ∞ *if* $\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}}(A_i, u) = \infty$ *for some* i*,*

and equivalently,

(ii) for any approximate simultaneous observable $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2} \equiv (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, F)$ *of* A_1 *and* A_2*, the following inequality holds:*

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_1, u) \cdot \Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_2, u) \geq \frac{1}{2}|\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \tag{12.55}$$

for all $u \in H$ *such that* $\|u\| = 1$*, where the left hand side of* (12.55) *is defined by* ∞ *if* $\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_i, u) = \infty$ *for some* i*.*

Proof. By Proposition 12.11, it suffices to prove (i). Put $\widehat{A}_i = \int_{\mathbf{R}^2} \lambda_i \widetilde{F}(d\lambda_1 d\lambda_2)$ $(i = 1, 2)$. Let $u \in D(A_1) \cap D(A_2)$. If $u \notin \mathrm{Dom}_{\otimes s}(\widehat{A}_i)$ for some i, we see, by the definition of the uncertainty, that $\Delta_{\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}}(A_i, u) = \infty$, so (12.55) clearly holds. Hence, it is sufficient to prove (12.55) for $u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_1) \cap \mathrm{Dom}_{\otimes s}(\widehat{A}_2)$. Let u be any element in $u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_1) \cap \mathrm{Dom}_{\otimes s}(\widehat{A}_2)$. We see, by the part (ii) of Lemma 12.14, that

$$\begin{aligned}
&\langle A_1 u, A_2 u\rangle + \langle (\widehat{A}_1 - A_1 \otimes I)(u \otimes s), (\widehat{A}_2 - A_2 \otimes I)(u \otimes s)\rangle \\
&= \int_{\mathbf{R}^2} \lambda_1 \lambda_2 \langle u \otimes s, \widetilde{F}(d\lambda_1 d\lambda_2)(u \otimes s)\rangle \\
&= \langle A_2 u, A_1 u\rangle + \langle (\widehat{A}_2 - A_2 \otimes I)(u \otimes s), (\widehat{A}_1 - A_1 \otimes I)(u \otimes s)\rangle
\end{aligned} \tag{12.56}$$

from which, we get, by Schwarz inequality, that

$$\begin{aligned}
&\frac{1}{2}|\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \\
&= \frac{1}{2}|\langle (\widehat{A}_1 - A_1 \otimes I)(u \otimes s), (\widehat{A}_2 - A_2 \otimes I)(u \otimes s)\rangle \\
&\qquad - \langle (\widehat{A}_2 - A_2 \otimes I)(u \otimes s), (\widehat{A}_1 - A_1 \otimes I)(u \otimes s)\rangle| \\
&\leq \|(\widehat{A}_1 - A_1 \otimes I)(u \otimes s)\| \cdot \|(\widehat{A}_2 - A_2 \otimes I)(u \otimes s)\|.
\end{aligned} \tag{12.57}$$

Hence (by Theorem 12.15), the proof is completed. □

The following theorem was first discovered by Arthurs and Goodman [6]. However we did not know their discovery in the preparation of [36].

Theorem 12.19. [Approximate simultaneous uncertainty relation, cf [6]]. *Let* A_1 *and* A_2 *be any (unbounded) self-adjoint operators on a Hilbert space* H*. Then, we see,*

(i) *for any approximate simultaneous tensor observable* $\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2} = (K, s, (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, \widehat{F}),$ *) of* (A_1, A_2)*, the following inequality holds:*

$$(var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]_1)^{1/2} \cdot (var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]_2)^{1/2} \geq |\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \quad (12.58)$$

for all $u \in H$ *such that* $\|u\| = 1$*, where the left hand side of* (12.58) *is defined by* ∞ *if* $var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]^{mar}_{(i)} = \infty$ *for some* i*, also the right hand side of* (12.58) *is defined by* ∞ *if* $u \notin \mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2)$*,*

and equivalently

(ii) *for any approximate simultaneous observable* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2} = (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, \widehat{F}\ )$ *of* (A_1, A_2)*, the following inequality holds:*

$$(var[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, u]_1)^{1/2} \cdot (var[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, u]_2)^{1/2} \geq |\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle| \quad (12.59)$$

for all $u \in H$ *such that* $\|u\| = 1$*, where the left hand side of* (12.59) *is defined by* ∞ *if* $var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]^{mar}_{(i)} = \infty$ *for some* i*, also the right hand side of* (12.59) *is defined by* ∞ *if* $u \notin \mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2)$*.*

Proof. By Proposition 12.11, it suffices to prove (i). Put $\widehat{A}_i = \int_{\mathbf{R}^2} \lambda_i \widehat{F}(d\lambda_1 d\lambda_2)$ $(i = 1, 2)$. If $u \notin \mathrm{Dom}_{\otimes s}(\widehat{A}_i)$ for some i, we see, by the definition of the variance, that $var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]^{mar}_{(i)} = \infty$, so, (12.58) clearly holds. Hence, it is sufficient to prove (12.58) in the case that $u \in \mathrm{Dom}_{\otimes s}(\widehat{A}_1) \cap \mathrm{Dom}_{\otimes s}(\widehat{A}_2)$. Let u be any element in $\mathrm{Dom}_{\otimes s}(\widehat{A}_1) \cap \mathrm{Dom}_{\otimes s}(\widehat{A}_2)$. Then, we see, by (iii) in Lemma 12.14, that

$$\begin{aligned} &var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]^{mar}_{(i)} = \|\widehat{A}_i(u \otimes s)\|^2 - |\langle u \otimes s, \widehat{A}_i(u \otimes s)\rangle|^2 && (12.60)\\ =&\|A_i u\|^2 + \|(\widehat{A}_i - A_i \otimes I)(u \otimes s)\|^2 - |\langle u, A_i u\rangle|^2 \\ \leq& 2(\|A_i u\|^2 - |\langle u, A_i u\rangle|^2)^{1/2} \cdot \|(\widehat{A}_i - A_i \otimes I)(u \otimes s)\| \quad (i = 1, 2), && (12.61)\end{aligned}$$

therefore, by Lemma 12.13 and Theorem 12.18 we get,

$$\begin{aligned} &var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]_1 \cdot var[\widehat{\mathbf{O}}^{ASTO}_{[A_l]_{l=1}^2}, u]_2 \\ \geq& 4(\|A_1 u\|^2 - |\langle u, A_1 u\rangle|^2)^{1/2} \cdot (\|A_2 u\|^2 - |\langle u, A_2 u\rangle|^2)^{1/2} \\ &\quad \cdot \|(\widehat{A}_1 - A_1 \otimes I)(u \otimes s)\| \cdot \|(\widehat{A}_2 - A_2 \otimes I)(u \otimes s)\| \\ \geq& |\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle|^2. && (12.62)\end{aligned}$$

Hence, the proof is completed. □

Now we have the following corollary.[3]

Corollary 12.20. [Uncertainty relations concerning a pair of conjugate observables]. *Let A_1 and A_2 be a pair of conjugate observables in a Hilbert space H.*

(i: *cf.* [7]) *There exists an approximate simultaneous observable* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2} = (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}.F$) *of A_1 and A_2. Thus, we can take an approximate simultaneous measurement* $\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, \overline{S}_{[|u\rangle\langle u|]})$.

(ii: *cf.* [36]) *For any positive number ϵ and any $k(=1,2)$, there exists an approximate simultaneous observable* $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2} = (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}.F$) *of A_1 and A_2 such that:*

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_k, u) \leq \epsilon \|A_k u\|_H \qquad (\forall u \in H \text{ such that } \|u\| = 1),$$

(iii: *cf.* [36, 67]) (*Heisenberg's uncertainty relation*) *However the following inequality holds*

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_1, u) \cdot \Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_2, u) \geq \hbar/2 \tag{12.63}$$

for all $u \in H$ $\quad (\|u\|_H = 1)$,

(iv: *cf.* [6]) *The following inequalities hold:* (*approximate simultaneous uncertainty relation*)

$$(var[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, u]_1)^{1/2} \cdot (var[\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, u]_2)^{1/2} \geq \hbar \tag{12.64}$$

for all $u \in H$ $\quad (\|u\|_H = 1)$.

Proof. Note that $\langle A_1 u, A_2 u\rangle - \langle A_2 u, A_1 u\rangle = i\hbar$ $(u \in \mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2), \|u\|_H = 1)$. Then, the above assertions (i) and (ii) are consequences of Theorem 12.16. Also, the above assertions (iii) and (iv) are respectively consequences of Theorem 12.18 and Theorem 12.19. □

[3] There are other uncertainty relations, For the recent variants, see [68].

12.7 EPR-experiment and Heisenberg's uncertainty relation

Now we have the complete form of Heisenberg's uncertainty relation as Corollary 12.20. To be compared with Corollary 12.20, we should note that the conventional Heisenberg's uncertainty relation (= Proposition 12.1) is ambiguous. Wrong conclusions are sometimes derived from the ambiguous statement (= Proposition 12.1). For example, in some books of physics, it is concluded that EPR-experiment (Einstein, Podolosky and Rosen [22]) contradicts with Heisenberg's uncertainty relation. That is,

(I) Heisenberg's uncertainty relation says that the position and the momentum of a particle can not be measured simultaneously and exactly.

On the other hand,

(II) EPR-experiment says that the position and the momentum of a certain "particle" can be measured simultaneously and exactly.

Thus someone may conclude that the above (i) and (ii) includes a paradox, and therefore, EPR-experiment contradicts with Heisenberg's uncertainty relation. Of course, this is a misunderstanding. This "paradox" was solved in [36]. Now we shall explain the solution of the paradox.

[Concerning the above (I)] Put $H = L^2(\mathbf{R}_q)$. Consider two-particles system in $H \otimes H = L^2(\mathbf{R}^2_{(q_1,q_2)})$. In the EPR problem, we, for example, consider the state u_s $(\in H \otimes H = L^2(\mathbf{R}^2_{(q_1,q_2)}))$ $\Big($or precisely, $|u_s\rangle\langle u_s|\Big)$ such that:

$$u_s(q_1, q_2) = \sqrt{\frac{1}{2\pi\epsilon\sigma}} e^{-\frac{1}{8\sigma^2}(q_1-q_2-a)^2 - \frac{1}{8\epsilon^2}(q_1+q_2-b)^2} \cdot e^{i\phi(q_1,q_2)} \tag{12.65}$$

where ϵ is assumed to be a sufficiently small positive number and $\phi(q_1, q_2)$ is a real-valued function. This is the quantum form of EPR-experiment in Remark 12.2(ii). Let $A_1 : L^2(\mathbf{R}^2_{(q_1,q_2)}) \to L^2(\mathbf{R}^2_{(q_1,q_2)})$ and $A_2 : L^2(\mathbf{R}^2_{(q_1,q_2)}) \to L^2(\mathbf{R}^2_{(q_1,q_2)})$ be self-adjoint operators such that

$$A_1 = q_1, \qquad A_2 = \frac{\hbar\partial}{i\partial q_1}. \tag{12.66}$$

Then, Corollary 12.20 (i) says that there exists an approximate simultaneous observable $\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2} = (\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}.F\ )$ of A_1 and A_2. Thus we can take an approximate simultaneous measurement $\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, \overline{S}_{[|u_s\rangle\langle u_s|]})$. And thus, the following Heisenberg's uncertainty relation (= Corollary 12.20 (iii)) holds,

$$\Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_1, u_s) \cdot \Delta_{\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}}(A_2, u_s) \geq \hbar/2 \tag{12.67}$$

[Concerning the above (II)] However, it should be noted that, in the above situation we assume that the state u_s is known before the measurement. In such a case, we may take another measurement as follows: Define the self-adjoint operators $\widehat{A}_1 : L^2(\mathbf{R}^2_{(q_1,q_2)}) \to L^2(\mathbf{R}^2_{(q_1,q_2)})$ and $\widehat{A}_2 : L^2(\mathbf{R}^2_{(q_1,q_2)}) \to L^2(\mathbf{R}^2_{(q_1,q_2)})$ such that

$$\widehat{A}_1 = b - q_2, \qquad \widehat{A}_2 = A_2 = \frac{\hbar\partial}{i\partial q_1} \tag{12.68}$$

Note that these operators commute. Therefore,

($\sharp$) we can take an exact simultaneous measurement of $\widehat{A}_1$ and $\widehat{A}_2$ (for the state u_s).

And moreover, we can easily calculate as follows (*cf.* Definition 11.1 and Remark 12.12).

$$\begin{aligned}
&\Delta\Big(\overline{\mathbf{M}}_{B(H)}(A_1 \times \widehat{A}_1, \overline{S}(\rho_{u_s}))\Big) = \|\widehat{A}_1 u_s - A_1 u_s\| \\
=&\Big[\iint_{\mathbf{R}^2} \Big|((b-q_2)-q_1)\sqrt{\frac{1}{2\pi\epsilon\sigma}} e^{-\frac{1}{8\sigma^2}(q_1-q_2-a)^2 - \frac{1}{8\epsilon^2}(q_1+q_2-b)^2} \cdot e^{i\phi(q_1,q_2)}\Big|^2 dq_1 dq_2\Big]^{1/2} \\
=&\Big[\iint_{\mathbf{R}^2} \Big|((b-q_2)-q_1)\sqrt{\frac{1}{2\pi\epsilon\sigma}} e^{-\frac{1}{8\sigma^2}(q_1-q_2-a)^2 - \frac{1}{8\epsilon^2}(q_1+q_2-b)^2}\Big|^2 dq_1 dq_2\Big]^{1/2} \\
=&\sqrt{2}\epsilon,
\end{aligned} \tag{12.69}$$

and

$$\Delta\Big(\overline{\mathbf{M}}_{B(H)}(A_2 \times \widehat{A}_2, \overline{S}(\rho_{u_s}))\Big) = \|\widehat{A}_2 u_s - A_2 u_s\| = 0. \tag{12.70}$$

Thus we see

$$\Delta\Big(\overline{\mathbf{M}}_{B(H)}(A_1 \times \widehat{A}_1, \overline{S}(\rho_{u_s}))\Big) \cdot \Delta\Big(\overline{\mathbf{M}}_{B(H)}(A_2 \times \widehat{A}_2, \overline{S}(\rho_{u_s}))\Big) = 0. \tag{12.71}$$

Since ϵ (> 0) can be taken sufficiently small, the above measurement ($\sharp$) is superior to the approximate simultaneous measurement $\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}^{ASO}_{[A_l]_{l=1}^2}, \overline{S}_{[|u_s\rangle\langle u_s|]})$. (Here, $\overline{S}_{[|u_s\rangle\langle u_s|]}$ is identified with $\overline{S}(|u_s\rangle\langle u_s|)$ since $|u_s\rangle\langle u_s|$ is a pure state.) However it should be again noted that, the measurement ($\sharp$) is made from the knowledge of the state u_s.

[(I) and (II) are consistent, *cf.* [36]] The above conclusion (12.71) does not contradicts with Heisenberg's uncertainty relation (12.67), since the measurement ($\sharp$) is not an approximate simultaneous measurement of A_1 and A_2.

■

In the above arguments, note that Theorem 12.19 (approximate simultaneous uncertainty relation) is powerless to solve the paradox (i.e., the paradox between EPR-experiment and Heisenberg's uncertainty relation). That is because the concept "error" (or "uncertainty") is not explicit in Theorem 12.19.