Chapter 11

Measurement error

Let $\overline{\mathbf{Q}} \equiv (\mathbf{R}, \mathcal{B}, G)$ and $\overline{\mathbf{O}} \equiv (\mathbf{R}, \mathcal{B}, F)$ be respectively a crisp W^*-observable (i.e., quantity) and a W^*-observable in a W^*-algebra $\mathcal{N}$ such that $\overline{\mathbf{Q}}$ and $\overline{\mathbf{O}}$ commute. Under the assumption that $\overline{\mathbf{O}}$ is regarded as the approximation of $\overline{\mathbf{Q}}$, we define the measurement error $\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}} \, (\overline{\mathbf{Q}} \times \overline{\mathbf{O}}, \overline{S}(\overline{\rho}))\Big)$ by

$$\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}, \overline{S}(\overline{\rho}))\Big) = \Big[\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \overline{\rho}\big((G \times F)(d\lambda_1 d\lambda_2)\big)\Big]^{1/2}. \tag{11.1}$$

This is also called the *distance between* $\overline{\mathbf{Q}}$ *and* $\overline{\mathbf{O}}$ *concerning* $\overline{\rho}$. The purpose of this chapter is to investigate the measurement error. Readers will see that the $\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}} \, (\overline{\mathbf{Q}} \times \overline{\mathbf{O}}, \overline{S}(\overline{\rho}))\Big)$ is superior to the "conventional definition" such as |"true value" −"measured value" |.

11.1 Approximate measurements for quantities

Let $\mathcal{N}$ be a W^*-algebra. Let $\overline{\mathbf{Q}} \equiv (\mathbf{R}, \mathcal{B}, G)$ be a crisp W^*-observable (i.e., quantity) in $\mathcal{N}$. Let $\overline{\mathbf{O}} \equiv (\mathbf{R}, \mathcal{B}, F)$ be a W^*-observable in $\mathcal{N}$ such that $\overline{\mathbf{Q}}$ and $\overline{\mathbf{O}}$ commute. Let $\overline{\mathbf{Q}} \times \overline{\mathbf{O}} \equiv (\mathbf{R}^2, \mathcal{B}^2, G \times F)$ be the product observable of $\overline{\mathbf{Q}}$ and $\overline{\mathbf{O}}$. Consider the simultaneous measurement $\overline{\mathbf{M}}_{\mathcal{N}} \, (\overline{\mathbf{Q}} {\times} \overline{\mathbf{O}}, \, \overline{S}(\overline{\rho}))$. According to Proclaim$^{W^*}$1 (9.9), the probability that the measured value (λ_1, λ_2) ($\in \mathbf{R}^2$) belong to $\Xi_1 \times \Xi_2$ ($\in \mathcal{B}^2$) is given by $\overline{\rho}((G \times F)(\Xi_1 \times \Xi_2))$. Thus, the variance of $|\lambda_1 - \lambda_2|$ is given by

$$\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \overline{\rho}\big((G \times F)(d\lambda_1 d\lambda_2)\big) \tag{11.2}$$

Here we have the following definition.

Definition 11.1. [Error (or precisely, Measurement error), *cf.* [44]]. *Assume the above notations. And assume the situation that we hope to approximate* $\overline{\mathbf{Q}}$ ($\equiv (\mathbf{R}, \mathcal{B}, G)$) *by*

$\overline{\mathbf{O}}$ $(\equiv (\mathbf{R}, \mathcal{B}, F))$, *that is,* $\overline{\mathbf{O}}$ *is the approximation of* $\overline{\mathbf{Q}}$. *Then the measurement error,* $\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}, \, \overline{S}(\overline{\rho}))\Big)$, *is defined by*

$$\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}, \overline{S}(\overline{\rho}))\Big) = \Big[\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \overline{\rho}\big((G \times F)(d\lambda_1 d\lambda_2)\big)\Big]^{1/2}. \tag{11.3}$$

This is also called the distance between $\overline{\mathbf{Q}}$ *and* $\overline{\mathbf{O}}$ *concerning* $\overline{\rho}$ (or, *the error of* $\overline{\mathbf{O}}$ *for* $\overline{\mathbf{Q}}$ *concerning* $\overline{\rho}$).

■

It should be noted that every measurement is *exact.* Thus the above definition is based on the following assumption:

($\sharp$) We want to take a measurement $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}}, \, \overline{S}(\overline{\rho}))$. But it is impossible for some reason. Thus, instead of the $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}}, \, \overline{S}(\overline{\rho}))$, we take a measurement $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}}, \, \overline{S}(\overline{\rho}))$. In this sense, we regard $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}}, \, \overline{S}(\overline{\rho}))$ as the approximation of $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}}, \, \overline{S}(\overline{\rho}))$.

The following examples will promote the understanding of Definition 11.1.

Example 11.2. [(i): Gaussian observables]. Consider the exact observable $\overline{\mathbf{O}}_{\text{EXA}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \chi_{(\cdot)})$ and Gaussian observable $\overline{\mathbf{O}}_G \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, G^\sigma)$ in $\mathcal{N} \equiv L^\infty(\mathbf{R}, d\mu)$ such that:

$$[G^\sigma(\Xi)](\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\Xi} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad (\forall \mu \in \mathbf{R} \ \forall \Xi \in \mathcal{B}_{\mathbf{R}}), \tag{11.4}$$

(where σ^2 is a variance). Then we see, for each density function $\overline{\rho}$ $(\in L^1_{+1}(\mathbf{R}, d\mu))$,

$$\begin{aligned} &\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}}_{\text{EXA}} \times \overline{\mathbf{O}}_G, \overline{S}(\overline{\rho}))\Big) = \Big[\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \overline{\rho}\big((G \times G^\sigma)(d\lambda_1 d\lambda_2)\big)\Big]^{1/2} \\ =&\Big[\int_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \Big(\int_{\mathbf{R}} \chi_{d\lambda_1}(\mu) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{d\lambda_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \times \overline{\rho}(\mu) d\mu\Big)\Big]^{1/2} \\ =&\sigma, \end{aligned} \tag{11.5}$$

which is independent of $\overline{\rho}$.

[(ii): Triangle observable, *cf.* Example 2.19]. Let ϵ be any positive number. Define the membership function (i.e., triangle function) $\mathcal{Z}_\epsilon : \mathbf{R} \to \mathbf{R}$ such that:

$$\mathcal{Z}_\epsilon(\omega) = \begin{cases} 1 - \frac{\omega}{\epsilon} & 0 \le \omega \le \epsilon \\ \frac{\omega}{\epsilon} + 1 & -\epsilon \le \omega \le 0 \\ 0 & \text{otherwise} . \end{cases} \tag{11.6}$$

Put $\mathbb{Z}_\epsilon \equiv \big\{\epsilon k \ : \ k \in \mathbb{Z} \equiv \{0, \pm 1, \pm 2, ...\}\big\}$. Define the W^*-observable $\overline{\mathbf{O}}_T \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, T^\epsilon_{(\cdot)})$ in the commutative W^*-algebra $L^\infty(\mathbf{R}, d\omega)$ such that $T^\epsilon_\Xi(\omega) = \sum_{x \in \Xi \cap \mathcal{Z}_\epsilon} \mathcal{Z}_\epsilon(\omega - x)$ $(\forall \Xi \in$

$\mathcal{B}_{\mathbf{R}}$, $\forall \omega \in \mathbf{R}$). This W^*-observable $\overline{\mathbf{O}}_T$ is called a *triangle observable* in $L^\infty(\mathbf{R}, d\omega)$. Consider the exact observable $\overline{\mathbf{O}}_{\text{EXA}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \chi_{(\cdot)})$ and the triangle observable $\overline{\mathbf{O}}_T \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, T^\epsilon_{(\cdot)})$ in $\mathcal{N} \equiv L^\infty(\mathbf{R}, d\omega)$. Then we see, for each density function $\overline{\rho}$ ($\in L^1_{+1}(\mathbf{R}, d\omega)$),

$$\Delta\Big(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}}_{\text{EXA}} \times \overline{\mathbf{O}}_T, \overline{S}(\overline{\rho}))\Big) = \epsilon\Big[\int_{\mathbf{R}} (\omega - [\omega]_G)(1 - [\omega]_G + \omega)\overline{\rho}(\omega) d\omega\Big]^{1/2} \leq \frac{\epsilon}{2}$$

where $[\omega]_G$ is the integer such that $[\omega]_G \leq \omega < [\omega]_G + 1$. ■

Example 11.3. [Self-adjoint operators]. Let A_1 and A_2 be commutative self-adjoint operators on a Hilbert space H. For each i ($= 1, 2$), consider the crisp observable $\overline{\mathbf{O}}_i \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E_{A_i})$ in $B(H)$ which is the spectral measure of A_i, i.e., $A_i = \int_{\mathbf{R}} \lambda E_{A_i}(d\lambda)$. Then, we see that

$$\begin{aligned}\Delta\Big(\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{O}}_1 \times \overline{\mathbf{O}}_2, \overline{S}(|u\rangle\langle u|))\Big) &= \Big[\int_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \Big\langle u, E_{A_1}(d\lambda_1) E_{A_2}(d\lambda_2) u\Big\rangle\Big]^{1/2} \\ &= \|(A_1 - A_2)u\|^2. \end{aligned} \tag{11.7}$$

■

11.2 The estimation under loss function in statistics

Let $\overline{\mathbf{Q}} \equiv (\mathbf{R}, \mathcal{B}, G)$ and $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ be a quantity (i.e., a crisp observable on $\mathbf{R}$) and a W^*-observable in a W^*-algebra $\mathcal{N}$ respectively. Consider the measurable map $h : X \to \mathbf{R}$, and the image observable $\overline{\mathbf{O}}_{[h]} \equiv (\mathbf{R}, \mathcal{B}, F(h^{-1}(\,\cdot\,))\,)$ in $\mathcal{N}$. This measurable map $h : X \to \mathbf{R}$ is called a *statistic.* Also assume that $\overline{\mathbf{Q}}$ and $\overline{\mathbf{O}}_{[h]}$ commute. Thus, the distance between $\overline{\mathbf{Q}}$ and $\overline{\mathbf{O}}_h$ (concerning $\overline{\rho} \in \mathfrak{S}^n(\mathcal{N}_*)$) is defined by $\Delta(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}_h, \overline{S}(\overline{\rho}))\,)$ as in the above definition.

Now we have the following problem:

Problem 11.4. [The estimation under loss function in statistics]. Assume the above notations. Then our present problem is as follows:

($\sharp$) how to choose a proper image observable $\overline{\mathbf{O}}_{[h]}$ (i.e., $\overline{\mathbf{O}}$ ($\equiv (X, \mathcal{F}, F)$) and $h : X \to \mathbf{R}$) as the approximation of a quantity $\overline{\mathbf{Q}}$ ($\equiv (\mathbf{R}, \mathcal{B}, G)$).

Our interest is concentrated on the problem ($\sharp$), which is regarded as a kind of "inference". Note that this ($\sharp$) is entirely different from Fisher's spirit in Chapter 5, that is, *how to infer the unknown state from the measured data obtained by a measurement.*

Of course, it is desirable that $\overline{\mathbf{O}}$ and h in the above ($\sharp$) satisfy the following (A_1) and (A_2).

(A_1) (unbias condition). There exists a dense set D ($\in \mathfrak{S}^n(\mathcal{N}_*)$) such that:

$$\int_{\mathbf{R}} \lambda \ {}_{\mathcal{N}_*}\langle \rho, G(d\lambda)\rangle_{\mathcal{N}} = \int_{\mathbf{R}} \lambda \ {}_{\mathcal{N}_*}\langle \rho, F(h^{-1}(d\lambda))\rangle_{\mathcal{N}} \qquad (\forall \rho \in D)$$

(A_2) $\Delta(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}_{[h]}, \overline{S}(\overline{\rho}))$) is small (where $\overline{\mathbf{Q}}$ and $\overline{\mathbf{O}}_{[h]}$ commute).

In what follows, we shall study Problem 11.4 in Example 11.5 and Problem 11.6.

Example 11.5. [Heisenberg's uncertainty relation, *cf.* [31], [36], Chapter 12]. Let A_1 and A_2 be a position quantity and a momentum quantity respectively (i.e. A_1 and A_2 are self-adjoint operators on a Hilbert space H satisfying that $A_1A_2 - A_2A_1 = i\hbar$, $\hbar$ is "Plank constant" $/(2\pi)$). As mentioned before, we identify A_i with the spectral measure $\overline{\mathbf{A}}_i \equiv (\mathbf{R}, \mathcal{B}, G_i)$ in $B(H)$, i.e., $A_i = \int_{\mathbf{R}} \lambda G_i(d\lambda)$. Since A_1 and A_2 do not commute, the product observable does not exist. Therefore, consider an observable $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ in $B(H)$ and measurable maps $h_i : X \to \mathbf{R}$, $(i = 1, 2)$, and define the image observables $\overline{\mathbf{O}}_{[h_i]} \equiv (\mathbf{R}, \mathcal{B}, F(h_i^{-1}(\,\cdot\,)) \equiv F_i(\,\cdot\,))$ in $B(H)$. And furthermore, assume the conditions:

(i) There exists a set D ($\subset H$) such that $\overline{D}$ ($\equiv$"closure on D)= $\{u \in H \mid \|u\| = 1\}$ and it holds that $\langle u, A_i u\rangle_H = \int_{\mathbf{R}} \lambda \langle u, F_i(d\lambda))u\rangle_H$ ($\forall u \in D$, $i = 1, 2$

(ii) $\overline{\mathbf{Q}}_i$ and $\overline{\mathbf{O}}_{[h_i]}$ commute $(i = 1, 2)$.

Then we get the following inequality:

$$\Delta\Big(\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{Q}}_1 \times \overline{\mathbf{O}}_{[h_1]}, \overline{S}(\overline{\rho}))\Big) \cdot \Delta\Big(\overline{\mathbf{M}}_{B(H)}(\overline{\mathbf{Q}}_2 \times \overline{\mathbf{O}}_{[h_2]}, \overline{S}(\overline{\rho}))\Big) \geq \hbar/2 \quad \text{for all } \overline{\rho} \in \mathrm{Tr}_{+1}(H). \tag{11.8}$$

This is just Heisenberg's uncertainty relation, of which non-mathematical representation was proposed by W. Heisenberg in the famous thought experiment of γ-rays microscope (*cf.* [31]). This will be discussed in Chapter 12.

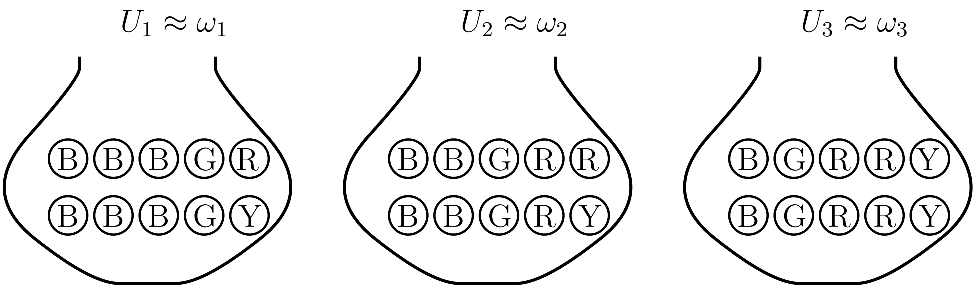
■

The following problem is a main part of this section. The reader should find "estimation under loss function in statistics" in the following problem.

Problem 11.6. [= Example 5.9 (Urn problem)]. Let U_j, $j = 1, 2, 3$, be urns that contain sufficiently many colored balls as follows:

	blue balls	green balls	red balls	yellow balls
urn U_1	60%	20%	10%	10%
urn U_2	40%	20%	30%	10%
urn U_3	20%	20%	40%	20%

Put $\mathbf{U} = \{U_1, U_2, U_3\}$. By the same argument in Example 5.9, we consider the state space Ω $\big(\equiv \{\omega_1, \omega_2, \omega_3\} \big)$ with the discrete topology, which is identified with $\mathbf{U}$, that is, $\mathbf{U} \ni U_j \leftrightarrow \omega_j \in \Omega \approx \mathcal{M}_{+1}^p(\Omega)$.



Let Q be a quantity in $C(\Omega)$, i.e., $Q : \Omega$ $(\approx \mathcal{M}_{+1}^p(\Omega)) \to \mathbf{R}$ is a real valued continuous function on Ω. For example we may consider in what follows. Assume that the weight of a blue ball is given by 10 (gram), and green 20, red 30 and yellow 10. $\Big($Thus, we can define the map $W : X \to \mathbf{R}$ such that $W(b) = 10$, $W(g) = 20$, $W(r) = 30$ and $W(y) = 10.\Big)$ Therefore, we can define the quantity $Q : \Omega \to [0, 50]$ such that the average weight $Q(\omega_1)$ of the balls in the urn U_1 is given by 14 $(= (10{\cdot}60{+}20{\cdot}20{+}30{\cdot}10{+}10{\cdot}10)/100)$, and similarly, $Q(\omega_2) = 18$ and $Q(\omega_3) = 20$. Define the observable $\mathbf{O} \equiv (X = \{b, g, r, y, \}, 2^X, F_{(\cdot)})$ in $C(\Omega)$ by the usual way. That is,

$$
\begin{array}{llll}
F_{\{b\}}(\omega_1) = 6/10 & F_{\{g\}}(\omega_1) = 2/10 & F_{\{r\}}(\omega_1) = 1/10 & F_{\{y\}}(\omega_1) = 1/10 \\
F_{\{b\}}(\omega_2) = 4/10 & F_{\{g\}}(\omega_2) = 2/10 & F_{\{r\}}(\omega_2) = 3/10 & F_{\{y\}}(\omega_2) = 1/10 \\
F_{\{b\}}(\omega_3) = 2/10 & F_{\{g\}}(\omega_3) = 2/10 & F_{\{r\}}(\omega_3) = 4/10 & F_{\{y\}}(\omega_3) = 2/10.
\end{array}
$$

Now consider the iterated measurement $\mathbf{M}_{C(\Omega)}(\times_{k=1}^2 \mathbf{O} \equiv (X^2, 2^{X^2}, \times_{k=1}^2 F), S_{[*]})$ where $(\times_{k=1}^2 F)_{\Xi_1 \times \Xi_2}(\omega) = F_{\Xi_1}(\omega) \cdot F_{\Xi_2}(\omega)$. Also, assume that

- the measured value (b, r) is obtained by the simultaneous measurement $\mathbf{M}_{C(\Omega)}(\times_{k=1}^2 \mathbf{O}, S_{[*]})$.

Now we have the following problem.

($\sharp$) *How do we infer $Q(*)$ from the measured value (b, r) obtained by the simultaneous measurement $\mathbf{M}_{C(\Omega)}(\times_{k=1}^{2} \mathbf{O}, S_{[*]})$?*

■

In what follows, we provide four answers to the above problem.

Answer 1. [Fisher's method, *cf.* [44]]. Recall "[II]" in Example 5.8, in which we infer, by Fisher's method, that the unknown urn is U_2. That is, applying Fisher's method (*cf.* Corollary 5.6), we get the conclusion as follows: Put $E(\omega) = F_{\{b\}}(\omega)F_{\{r\}}(\omega)$. Clearly it holds that $E(\omega_1) = 6 \cdot 1/10^2 = 0.06$, $E(\omega_2) = 4 \cdot 3/10^2 = 0.12$ and $E(\omega_3) = 4 \cdot 2/10^2 = 0.08$. Therefore, there is a very reason to think that $[\, * \,] = \delta_{\omega_2}$, that is, the unknown urn is U_2. Since we inferred that $[\, * \,] = \delta_{\omega_2}$ ($\leftrightarrow \omega_2$) in Example 5.8(II), we can immediately conclude that (or more precisely, Regression analysis II (6.48))

$$Q(\, * \,) = Q(\omega_2) = 18.$$

Answer 2. [Moment method] Recall "Remark" in Example 5.8, in which we infer, by the moment method, that the unknown urn is U_2. Thus, we conclude that $Q(*) = Q(U_2) = 18$.

Answer 3. [Bayes' method in SMT_{PEP}]. Next study the above problem ($\sharp$) in SMT_{PEP}-method (*cf.* §8.6.2, and Theorem 11.12 later). Thus, we assume that the $[\, * \,]$ is chosen by a fair rule (e.g., a fair coin-tossing, a fair dice-throwing, etc.). Consider a statistical measurement $\mathbf{M}_{C(\Omega)}(\times_{k=1}^{2} \mathbf{O},\ S_{[*]}(\rho_0^m)\ )$, where we assume that $\rho_0^m = \rho_{\text{uni}}^m$, i.e., $\rho_{\text{uni}}^m = \frac{1}{3}\sum_{j=1}^{3} \delta_{\omega_j}$ on Ω. When we get the measured value (b, r) by the measurement $\mathbf{M}_{C(\Omega)}(\times_{k=1}^{2} \mathbf{O},\ S_{[*]}(\rho_0^m)\ )$, we infer, by Bayes' method (for example, (B_1) in Remark 8.14, or more precisely, Theorem 8.13), that the new state ρ_{new}^m is

$$\begin{aligned}\rho_{\text{new}}^m &= \frac{1}{0.06 + 0.12 + 0.08}(0.06 \cdot \delta_{\omega_1} + 0.12 \cdot \delta_{\omega_2} + 0.08 \cdot \delta_{\omega_3}) \\ &= \frac{1}{6 + 12 + 8}(6 \cdot \delta_{\omega_1} + 12 \cdot \delta_{\omega_2} + 8 \cdot \delta_{\omega_3}).\end{aligned}$$

Thus there is a very reason to consider that

$$Q(\, * \,) \text{ is approximated by } \int_\Omega Q(\omega)\rho_{\text{new}}^m(d\omega) = \tfrac{14 \cdot 6 + 18 \cdot 12 + 20 \cdot 8}{6+12+8} = 17.69 \cdots.$$

Also, the variance σ^2 is given by

$$\sigma = \Big[\frac{(14 - 17.69)^2 \cdot 6 + (18 - 17.69)^2 \cdot 12 + (20 - 17.69)^2 \cdot 8}{6 + 12 + 8}\Big]^{1/2} = 2.19...$$

Answer 4. [The estimation under loss function in statistics, *cf.* [44]]. Let $\mathbf{M}_{C(\Omega)}(\times_{k=1}^2 \mathbf{O}, S_{[*]}(\rho_0^m))$ and $Q : \Omega \to [0, 50]$ be as in Problem 11.6. Put $\mathbf{O} = (X = \{b, g, r, y\}, 2^X, F_{(\cdot)})$ in $C(\Omega)$ ($\equiv C(\{\omega_1, \omega_2, \omega_3\})$) and ρ_0^m is any mixed state $\in \mathcal{M}_{+1}^m(\Omega)$. Consider a measure ν on Ω, for example, $\nu(\{\omega_j\}) = 1$ $(j = 1, 2, 3)$. Define the W^*-observable $\overline{\mathbf{O}}$ in $L^\infty(\Omega, \nu)$ such that $\overline{\mathbf{O}} = \mathbf{O}$, and define the normal state $\overline{\rho}$ ($\in L_{+1}^1(\Omega, \nu)$) such that $\rho_0^m(B) = \int_B \overline{\rho}(\omega)\nu(d\omega)$ for all B ($\subseteq \Omega$). Then, we can identify $\mathbf{M}_{C(\Omega)}(\times_{k=1}^2 \mathbf{O}, S_{[*]}(\rho_0^m))$ with $\overline{\mathbf{M}}_{L^\infty(\Omega,\nu)}(\times_{k=1}^2 \overline{\mathbf{O}}, \overline{S}(\overline{\rho}))$. Note that Q is equivalent to the crisp observable $\overline{\mathbf{Q}} \equiv (\mathbf{R}, \mathcal{B}, G^Q)$ in $L^\infty(\Omega, \nu)$ such that $G_\Xi^Q(\omega) = \chi_{\{\omega' \in \Omega : Q(\omega') \in \Xi\}}(\omega)$ for all $\Xi \in \mathcal{B}$ and all $\omega \in \Omega$. Define the map $h : X^2 \to \mathbf{R}$ such that:

$$h(x_1, x_2) = \frac{1}{2}\Big(W(x_1) + W(x_2)\Big) \qquad (\forall (x_1, x_2) \in X^2 \equiv \{b, g, r, y\}^2) \tag{11.9}$$

where $W(b) = 10$, $W(g) = 20$, $W(r) = 30$ and $W(y) = 10$. Consider the image observable $(\times_{k=1}^2 \overline{\mathbf{O}})_h \equiv (\mathbf{R}, \mathcal{B}, \widehat{F} = (\times_{k=1}^2 F)_{h^{-1}(\cdot)})$. Then, $\Delta\Big(\overline{\mathbf{M}}_{L^\infty(\Omega,\nu)}(\overline{\mathbf{Q}} \times (\times_{k=1}^2 \overline{\mathbf{O}})_h, \overline{S}(\overline{\rho}))\Big)$, the distance between $\overline{\mathbf{Q}}$ and $(\times_{k=1}^2 \overline{\mathbf{O}})_h$ concerning $\overline{\rho}$, is calculated as

$$\begin{aligned} &\Delta\Big(\overline{\mathbf{M}}_{L^\infty(\Omega,\nu)}(\overline{\mathbf{Q}} \times (\times_{k=1}^2 \overline{\mathbf{O}})_h, \overline{S}(\overline{\rho}))\Big) = \Big[\iint_{\mathbf{R}^2} |\lambda_1 - \lambda_2|^2 \overline{\rho}((G^Q \times \widehat{F})(d\lambda_1 d\lambda_2))\Big]^{1/2} \\ =& \Big[\sum_{j=1}^3 \sum_{(x_1,x_2)\in X^2} \overline{\rho}(\omega_j)|Q(\omega_j) - h(x_1, x_2)|^2 F_{\{x_1\}}(\omega_j) F_{\{x_2\}}(\omega_j)\Big]^{1/2} \\ =& \Big[22\overline{\rho}(\omega_1) + 38\overline{\rho}(\omega_2) + 38\overline{\rho}(\omega_3)\Big]^{1/2}. \end{aligned} \tag{11.10}$$

Therefore, we see that (11.10) $\leq \sqrt{38} \approx 6.17$ for all $\overline{\rho} \in L_{+1}^1(\Omega, \nu)$. Now we can also answer the problem (♯) in Problem 11.6. That is, we see,

$$Q(\,*\,) = \tfrac{1}{2}(W(r) + W(b)) = (30 + 10)/2 = 20,$$

though it of course includes the error 6.17.

■

The map $h : X^n \to \mathbf{R}$, $(n = 2)$, in (11.9) may be chosen by the hint of "the law of large numbers". That is, if n is sufficiently large, the map $h : X^n \to \mathbf{R}$ (defined by $h(x_1, ..., x_n) = \frac{1}{n}\sum_{k=1}^n W(x_k)$) has a proper property, i.e., $\lim_{n\to\infty} \Delta\Big(\overline{\mathbf{M}}_{L^\infty(\Omega,\nu)}(\overline{\mathbf{Q}} \times (\times_{k=1}^n \overline{\mathbf{O}})_h, \overline{S}(\overline{\rho}))\Big) = 0$ for all $\overline{\rho} \in L_{+1}^1(\Omega, \nu)$. However, there are several ideas for the choice of h.

Definition 11.7. [Admissible]. *Let $\overline{\mathbf{Q}} \equiv (\mathbf{R}, \mathcal{B}, G)$ and $\overline{\mathbf{O}} \equiv (X, \overline{\mathcal{F}}, F)$ be a quantity and W^*-observable in a W^*-algebra $\mathcal{N}$ respectively. For each $i = 1, 2$, consider a measurable* *map* $h_i : X \to \mathbf{R}$, *and the image observable* $\overline{\mathbf{O}}_{h_i} \equiv (\mathbf{R}, \mathcal{B}, F(h_i^{-1}(\,\cdot\,))\,)$ *in* $\mathcal{N}$. *Also assume that* $\overline{\mathbf{Q}}$ *and* $\overline{\mathbf{O}}_{h_i}$ *commute.*

(i) *When it holds that*

$$\Delta(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}_{h_1}, \overline{S}(\overline{\rho}))\,) \le \Delta(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}_{h_2}, \overline{S}(\overline{\rho}))\,) \quad \forall \overline{\rho} \in \mathfrak{S}^n(\mathcal{N}_*), \tag{11.11}$$

we say that $\overline{\mathbf{O}}_{h_1}$ *is better* than $\overline{\mathbf{O}}_{h_2}$ as the approximation of $\overline{\mathbf{Q}}$.

(ii) *Also,* $\overline{\mathbf{O}}_{h_2}$ *is called admissible* as the approximation of $\overline{\mathbf{Q}}$, if there exists no h_1 that satisfies (11.11) and the following condition:

$$\Delta(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}_{h_1}, \overline{S}(\overline{\rho}_0))\,) < \Delta(\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{Q}} \times \overline{\mathbf{O}}_{h_2}, \overline{S}(\overline{\rho}_0))\,) \quad \text{for some } \overline{\rho}_0 \in \mathfrak{S}^n(\mathcal{N}_*). \tag{11.12}$$

■

As a well known result concerning "admissibility", we mention the following example.

Example 11.8. [Gaussian observable and admissibility]. Let $\overline{\mathbf{O}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, G^{\sigma})$ be the Gaussian observable in $\mathcal{N} \equiv L^\infty(\mathbf{R}, d\mu)$, that is,

$$G^{\sigma}_{\Xi}(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\Xi} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du \qquad (\forall \mu, \in \mathbf{R},\ \forall \Xi \in \mathcal{B}_{\mathbf{R}}). \tag{11.13}$$

Consider the quantity $Q : \mathbf{R} \to \mathbf{R}$ such that $Q(\mu) = \mu$ $(\forall \mu \in \mathbf{R})$, which is identified with the observable $\overline{\mathbf{Q}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F^{Q}_{(\cdot)})$ where $F^{Q}_{\Xi}(\mu) = \chi_{\Xi}(\mu)$. Consider the product observable $\times_{k=1}^{n} \overline{\mathbf{O}} \equiv (\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, \times_{k=1}^{n} G^{\sigma})$ in $L^\infty(\mathbf{R}, d\mu)$. Define the map $h : \mathbf{R}^n \to \mathbf{R}$ such that $\mathbf{R}^n \ni (\lambda_1, ..., \lambda_n) \overset{h}{\mapsto} \frac{\lambda_1 + \cdots + \lambda_n}{n} \in \mathbf{R}$. Then, it is well known (*cf.* [86]) that $(\times_{k=1}^{n} \overline{\mathbf{O}})_h$ is *admissible* as the approximation of $\overline{\mathbf{Q}}$.

■

11.3 Random observable

Recall the probabilistic measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}([\delta_{\omega_1}; p] \oplus [\delta_{\omega_2}; 1-p]))$ in Example 8.1 (8.8). Here, the symbol $[\delta_{\omega_1}; p] \oplus [\delta_{\omega_2}; 1-p]$ is called a "probabilistic state". The concept of "probabilistic state" urges us to propose the "random observable" as follows:

For simplicity, in this section we devote ourselves to the classical case (i.e., $C(\Omega)$ and $L^\infty(\Omega, \mu)$).

Let $\mathbf{O}_1 \equiv (X, \mathcal{F}, F_1)$, $\mathbf{O}_2 \equiv (X, \mathcal{F}, F_2)$, $\cdots$, $\mathbf{O}_N \equiv (X, \mathcal{F}, F_N)$ be observables in $C(\Omega)$. In a similar way in the procedures (P_1) and (P_2) of Example 8.1, define the "random observable" $\oplus_{n=1}^N[\mathbf{O}_n;p_n]$, where $\sum_{n=1}^N p_n = 1$ $(0 \le p_n \le 1$ $(n=1,2,...,N))$. That is, we assume that:

- To take a measurement $\mathbf{M}_{C(\Omega)}(\oplus_{n=1}^N[\mathbf{O}_n;p_n],\ S_{[\delta_\omega]})$. $\Big($This measurement is called a *"random measurement".*$\Big)$

$\Longleftrightarrow$

- To take one of $\{\mathbf{M}_{C(\Omega)}(\mathbf{O}_n, S_{[\delta_\omega]}) \mid n = 1,2,...,N\}$ according to the probabilistic rule $(p_1, p_2, ..., p_N)$. That is, to take the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_n, S_{[\delta_\omega]})$ with probability p_n.

Here, it should be noted that

- the statistical property of $\mathbf{M}_{C(\Omega)}(\oplus_{n=1}^N[\mathbf{O}_n;p_n],\ S_{[\delta_\omega]})$ is equal to that of $\mathbf{M}_{C(\Omega)}(\widehat{\mathbf{O}},\ S_{[\delta_\omega]})$, where $\widehat{\mathbf{O}} \equiv (X.\mathcal{F}, \widehat{F})$ is defined by $\widehat{F}(\Xi) = \sum_{n=1}^N p_n F_n(\Xi)$. That is, for each $\Xi(\in \mathcal{F})$ and ω $(\in \Omega)$,

$$
\begin{aligned}
&\text{"the probability that a measured value obtained by } \mathbf{M}_{C(\Omega)}(\oplus_{n=1}^N[\mathbf{O}_n;p_n],\ S_{[\delta_\omega]}) \\
&\quad \text{belongs to } \Xi\text{"} \\
&= \sum_{n=1}^N p_n[F(\Xi)](\omega) \qquad (11.14) \\
&= \text{"the probability that a measured value obtained by } \mathbf{M}_{C(\Omega)}(\widehat{\mathbf{O}},\ S_{[\delta_\omega]}) \text{ belongs} \\
&\quad \text{to } \Xi\text{ "},
\end{aligned}
$$

which is easily seen by a similar argument such as stated in Example 8.1.

Again note that

(1) to take a random measurement $\mathbf{M}_{C(\Omega)}(\oplus_{n=1}^N[\mathbf{O}_n;p_n],\ S_{[\delta_\omega]})$ (11.15)

$\Longleftrightarrow$

to take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_n, S_{[\delta_\omega]})$ with probability p_n $(n = 1,2,...,N)$.

(2) to take a probabilistic measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O},\ S_{[*]}(\oplus_{n=1}^N [\delta_{\omega_n};p_n]))$ (11.16)

$\Longleftrightarrow$

to take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_n}]})$ with probability p_n $(n = 1,2,...,N)$.

In the case that $N = \infty$. it suffices to prepare a probability space $(\Lambda, \mathcal{F}(\Lambda), \nu)$. And, for each $\lambda (\in \Lambda)$, consider an observable $\mathbf{O}_\lambda$ $(\equiv (X, \mathcal{F}, F_\lambda))$ in $C(\Omega)$. Then, the random observable $\oplus_{n=1}^N [\mathbf{O}_n; p_n]$ is generalized as $\oint_\Lambda \mathbf{O}_\lambda \nu(d\lambda)$ $\Big(\equiv (X, \mathcal{F}, \oint_\Lambda F_\lambda \nu(d\lambda))\Big)$.

The following example is typical (though the description is due to the W^*-algebraic formulation).

Example 11.9. [Gaussian observable as a random observable]. For each $\lambda(\in \mathbf{R}(\equiv \Lambda))$, consider an observable $\mathbf{O}_\lambda$ $(\equiv (\mathbf{R}(\equiv X), \mathcal{B}_{\mathbf{R}}, E_\lambda))$ in $L^\infty(\mathbf{R}(\equiv \Omega), d\omega)$ such that

$$[F_\lambda(\Xi)](\omega) = \chi_\Xi(\omega - \lambda) \quad (\forall \Xi \in \mathcal{B}_{\mathbf{R}}(\subseteq 2^X), \forall \omega \in \mathbf{R}(\equiv \Omega), \forall \lambda \in \mathbf{R}(\equiv \Lambda)).$$

Define the probability space $(\mathbf{R}(\equiv \Lambda), \mathcal{B}_{\mathbf{R}}, \nu)$ such that:

$$\nu(S) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_S e^{-\frac{\lambda^2}{2\sigma^2}} d\lambda \qquad (\forall S \in \mathcal{B}_{\mathbf{R}}). \tag{11.17}$$

Thus, we have the random observable:

$$\oint_\Lambda \mathbf{O}_\lambda \nu(d\lambda) \Big(\equiv (\mathbf{R}(\equiv X), \mathcal{B}_{\mathbf{R}}(\equiv \mathcal{F}), \oint_\Lambda F_\lambda \nu(d\lambda))\Big) \tag{11.18}$$

which the probabilistic form of the Gaussian observable $(\mathbf{R}(\equiv X), \mathcal{B}_{\mathbf{R}}(\equiv \mathcal{F}), G^\sigma)$ in $L^\infty(\mathbf{R}(\equiv \Omega), d\omega)$ such that:

$$\begin{aligned}[G^\sigma(\Xi)](\omega) &= \int_\Lambda [F_\lambda(\Xi)](\omega)\nu(d\lambda) = \int_\Lambda \chi_\Xi(\omega - \lambda) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\lambda^2}{2\sigma^2}} d\lambda \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_\Xi e^{-\frac{(x-\omega)^2}{2\sigma^2}} dx \qquad (\forall \omega \in \mathbf{R}(\equiv \Omega)\ \forall \Xi \in \mathcal{B}_{\mathbf{R}}(\subseteq 2^X)),\end{aligned} \tag{11.19}$$

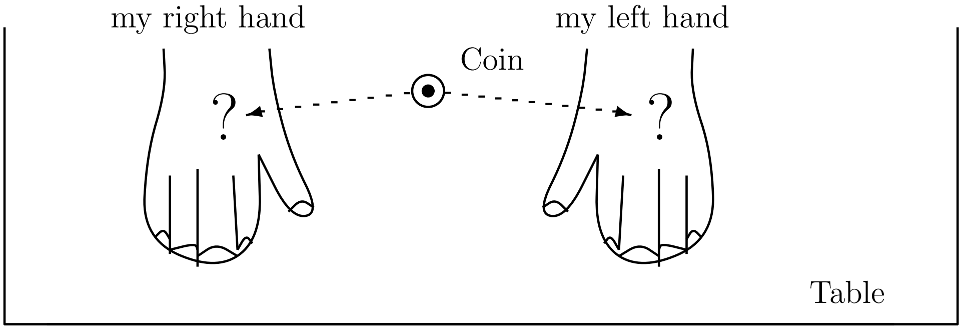
(Cf. Example 11.8.)

■

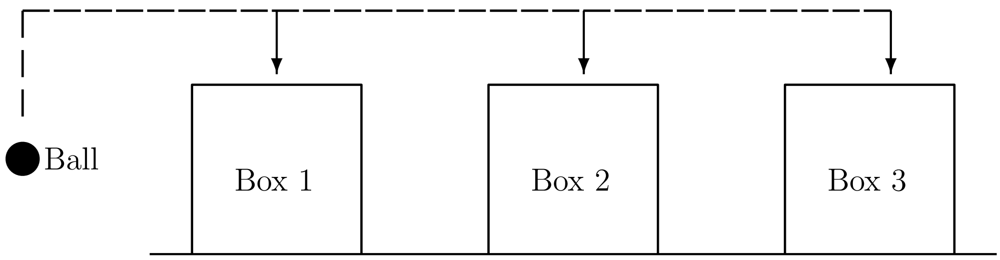
Although the following problem is easy, its measurement theoretical answer is quite important.

Problem 11.10. [Which hand is the coin under?]. The following problems (P_1) and (P_2) are essentially the same.

(P_1) A coin is, intentionally or unintentionally, put under my right hand or my left hand. Suppose that you do not know which hand the coin is under, and you choose one of my hands which you guess that the coin is under. Is it reasonable to believe that the probability that the ball is under the hand you choose is equal to 1/2. How do you think about it?



(P_2) There are three boxes (i.e., Box 1, Box 2 and Box 3) and a ball. A ball is, intentionally or unintentionally, put in one box (i.e., Box 1 or Box 2 or Box 3). Suppose that you do not know which box contains the ball, and you choose one of three boxes which you guess the ball is in. In this case, it is often believed that the probability that the ball is in Box 1 [resp. in Box 2; in Box 3] is 1/3 [resp. 1/3; 1/3]. How do you think about it?



•[The experimental answer to Problem (P_1)]. We can easily say "Yes", that is,

(A_1) the probability that the ball is under the hand you choose is equal to 1/2.

In fact, it can be easily tested experimentally. For example, it suffices to ask to 1000 persons "Which hand is the coin under?". About 500 persons will say "Right hand", and the other persons will say "Left hand". In either case, about 500 persons' guess is hit. Thus the above (A_1) is true. Although this (P_1) is the easiest problem throughout this book, what I want to say is the measurement theoretical answer mentioned in what follows.

•[The measurement theoretical answer to Problem (P_2)]. Since the two (P_1) and (P_2) are essentially the same, it suffices to answer Problem (P_2) from the measurement theoretical point of view. When the conclusion is said first, we can say that:

(A_2) the probability that the ball is in your chosen box is equal to 1/3.

In what follows we shall explain it. Put $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where ω_1 [resp. ω_2, ω_3] means the state that the ball is in Box 1 [resp. Box 2; Box 3]. First we consider the case ω_1, that is, the ball is in Box 1.

[(i): The case ω_1, that is, the ball is in Box 1]. Define three observables $\mathbf{O}_1^e$ ($= (\{0,1\}, 2^{\{0,1\}}, F_1^e)$), $\mathbf{O}_2^e$ ($= (\{0,1\}, 2^{\{0,1\}}, F_2^e)$), $\mathbf{O}_2^e$ ($= (\{0,1\}, 2^{\{0,1\}}, F_3^e)$) such that:

$$
\begin{aligned}
&[F_1^e(\{0\})](\omega_1) = 0, \quad [F_1^e(\{0\})](\omega_2) = 1, \quad [F_1^e(\{0\})](\omega_3) = 1, \\
&[F_1^e(\{1\})](\omega_1) = 1, \quad [F_1^e(\{1\})](\omega_2) = 0, \quad [F_1^e(\{1\})](\omega_3) = 0,
\end{aligned} \tag{11.20}
$$

$$
\begin{aligned}
&[F_2^e(\{0\})](\omega_1) = 1, \quad [F_2^e(\{0\})](\omega_2) = 0, \quad [F_2^e(\{0\})](\omega_3) = 1, \\
&[F_2^e(\{1\})](\omega_1) = 0, \quad [F_2^e(\{1\})](\omega_2) = 1, \quad [F_2^e(\{1\})](\omega_3) = 0,
\end{aligned} \tag{11.21}
$$

$$
\begin{aligned}
&[F_3^e(\{0\})](\omega_1) = 1, \quad [F_3^e(\{0\})](\omega_2) = 1, \quad [F_3^e(\{0\})](\omega_3) = 0, \\
&[F_3^e(\{1\})](\omega_1) = 0, \quad [F_3^e(\{1\})](\omega_2) = 0, \quad [F_3^e(\{1\})](\omega_3) = 1.
\end{aligned} \tag{11.22}
$$

Note that we identify the following (S_1^1) and (S_2^1):

(S_1^1) We take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_1^e, S_{[\delta_{\omega_1}]})$. And we obtain a measured value 1. (Or, we obtain a measured value 0.) (11.23)

(S_2^1) We open Box 1. And we find the ball. (Or, we do not find the ball.) (11.24)

Similarly, we see the following identification:

(S_1^{23}) We take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_2^e, S_{[\delta_{\omega_1}]})$ [resp. $\mathbf{M}_{C(\Omega)}(\mathbf{O}_3^e, S_{[\delta_{\omega_1}]})$]. And we obtain a measured value 1. (Or, we obtain a measured value 0.)

(S_2^{23}) We open Box 2. [resp. Box 3.]. And we find the ball. (Or, we do not find the ball.)

Since "the state ω_1" = "the case that the ball is in Box 1", we can assume that

- the measured value obtained by $\mathbf{M}_{C(\Omega)}(\mathbf{O}_1^e, S_{[\delta_{\omega_1}]})$ [resp. $\mathbf{M}_{C(\Omega)}(\mathbf{O}_2^e, S_{[\delta_{\omega_1}]})$; $\mathbf{M}_{C(\Omega)}(\mathbf{O}_3^e, S_{[\delta_{\omega_1}]})$] is 1 [resp. 0; 0].

Since you have no information about the [*], your choice is the same as the choice by a fair coin-tossing. That is, we assume that

$$
\text{"decision without having information"} \iff \text{"decision by a fair coin-tossing"}, \tag{11.25}
$$

which is the fundamental spirit of "the principle of equal probability" in the following section. Thus, it is reasonable to consider that

$$\begin{aligned}&\text{the probability that Box 1 is opened} = \text{the probability that Box 2 is opened}\\ =&\text{the probability that Box 3 is opened} = 1/3. \end{aligned} \tag{11.26}$$

Therefore, we see that

(a) the probability that the measured value obtained by $\mathbf{M}_{C(\Omega)}(\oplus_{k=1}^3 [\mathbf{O}_k^e; 1/3], S_{[\delta_{\omega_1}]})$ is 1 [resp. 0] is given by 1/3 [resp. 2/3].

[(ii): The case ω_2, that is, the ball is in Box 2]. Similarly we see that

(b) the probability that the measured value obtained by the "measurement" $\mathbf{M}_{C(\Omega)}(\oplus_{k=1}^3 [\mathbf{O}_k^e; 1/3], S_{[\delta_{\omega_2}]})$ is 1 [resp. 0] is given by 1/3. [resp. 2/3].

[(iii): The case ω_3, that is, the ball is in Box 3]. Similarly we see that

(c) the probability that the measured value obtained by the "measurement" $\mathbf{M}_{C(\Omega)}(\oplus_{k=1}^3 [\mathbf{O}_k^e; 1/3], S_{[\delta_{\omega_3}]})$ is 1 [resp. 0] is given by 1/3. [resp. 2/3].

[(iv): The case that we do not know which box contains the ball]. By the above (a), (b) and (c), we see that

- the probability that the measured value obtained by the "measurement" $\mathbf{M}_{C(\Omega)}(\oplus_{k=1}^3 [\mathbf{O}_k^e; 1/3], S_{[*]})$ is 1 [resp. 0] is given by 1/3 [resp. 2/3].

Note that "measured value 1 is obtained" $\Leftrightarrow$ "open the box that contains the ball". Thus, we can believe that the probability that the ball is in Box 1 [resp. in Box 2; in Box 3] is 1/3 [resp. 1/3 ; 1/3].

[Remark]. Recall BMT (in §8.6). Then, the system in Problem (P_2) is clearly represented by $S_{[*]}((\nu_u))_{bw}$, *cf.* §8.6.1. Here, $\nu_u(\{\omega_k\}) = 1/3$ $(k = 1, 2, 3)$. However, in the above argument, we conclude that the "probability" that the ball is in Box 1 [resp. in Box 2; in Box 3] is 1/3 [resp. 1/3; 1/3]. Therefore, we have the following question:

- Is the system represented by $S_{[*]}(\nu_u)$ (as well as $S_{[*]}((\nu_u))_{bw}$)?

This will be discussed in the following section. ■

11.4 The principle of equal probability

Consider a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O} \equiv (X, \mathcal{F}, F),\ S_{[*]})$, where Ω is finite, i.e., $\Omega \equiv \{\omega_1, \omega_2, ..., \omega_N\}$. There may be several definitions of "Having no information about the [∗]". As mentioned in §8.6, in this book we introduce three definitions of "Having no information about the [∗]" such as:

$$\begin{cases} \text{(a). iterative likelihood function method in §5.6,} \\ \text{(b). SMT}_{\text{PEP}} \text{ in SMT in this section and §11.4,} \\ \text{(c). BMT in §8.6.} \end{cases}$$

We want to change $S_{[*]}(\!(\nu_u)\!)_{bw}$ (belief weight) to $S_{[*]}(\nu_u)$ (statistical state). This will be done according to the spirit (11.25), that is,

"decision without having information" $\Longleftrightarrow$ "decision by a fair coin-tossing",

which assures that the principle of equal probability holds. This is the purpose of this section.

Let Ω be a finite set, i.e., $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$. A map $\phi : \Omega \to \Omega$ is said to be ergodic, if it is a bijection and if it holds that $\Omega = \{\phi^n(\omega) \mid n = 0, 1, ..., N-1\}$ for any ω ($\in \Omega$). Also, a homomorphism $\Phi : C(\Omega) \to C(\Omega)$ is said to be ergodic, if there exists an ergodic bijection $\phi : \Omega \to \Omega$ such that

$$(\Phi f)(\omega) = f(\phi(\omega)) \quad (\forall f \in C(\Omega), \forall \omega \in \Omega). \tag{11.27}$$

Theorem 11.12. [The principle of equal probability (="PEP"), SMT$_{\text{PEP}}$ method].
Consider a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O} \equiv (X, \mathcal{F}, F), S_{[*]})$, *where* Ω *is finite, i.e.,* $\Omega \equiv \{\omega_1, \omega_2, ..., \omega_N\}$. *And consider the measurement* $\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{N-1} [\Phi^n \mathbf{O}; 1/N], S_{[*]})$ *(where* $\Phi : C(\Omega) \to C(\Omega)$ *is ergodic), which is called an unintentional random measurement.*[1] *Then we see*

$$\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{N-1} [\Phi^n \mathbf{O}; 1/N], S_{[*]}) \underset{identification}{\Longleftrightarrow} \mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\oplus_{n=1}^{N} [\delta_{\omega_n}; 1/N])) \tag{11.28}$$

and

$$\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\oplus_{n=1}^{N} [\delta_{\omega_n}; 1/N])) \underset{\text{statistical form}}{\overset{\text{probabilistic form}}{\rightleftarrows}} \mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_u)) \tag{11.29}$$
(=(8.9))

[1] Also, it is called a "completely random measurement", "coin-tossing measurement", "no information measurement".

where $\nu_u = \frac{1}{N}\sum_{n=1}^{N}\delta_{\omega_n}$*. That is, we can assert that:*

$$\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{N-1} [\Phi^n \mathbf{O}; 1/N], S_{[*]}) \underset{identification}{\Longleftrightarrow} \mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_u)). \tag{11.30}$$

Proof. Let $\omega \in \Omega$. Then we see that:

to take an unintentional random measurement $\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{N-1} [\Phi^n \mathbf{O}; 1/N], S_{[\delta_\omega]})$

$\Longleftrightarrow$

to take a measurement $\mathbf{M}_{C(\Omega)}(\Phi^n \mathbf{O}, S_{[\delta_\omega]})$
with probability $1/N$, $(n = 1, 2, ..., N)$

$\Longleftrightarrow$

to take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\phi^n(\omega)}]})$ with probability $1/N$
$(n = 0, 1, 2, ..., N-1)$

$\Longleftrightarrow$ (Note that $\Omega = \{\phi^n(\omega) \mid n = 0, 1, ..., N-1\}$.)

to take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[\delta_{\omega_n}]})$ with probability $1/N$, $(n = 1, 2, ..., N)$

$\Longleftrightarrow$

to take a probabilistic measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\oplus_{n=0}^{N-1} [\delta_{\omega_n}; 1/N]))$

$\Longleftrightarrow$

to take a measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_u))$.

Thus we see that:

$$\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{N-1} [\Phi^n \mathbf{O}; 1/N], S_{[*]}) \underset{\text{identification}}{\Longleftrightarrow} \mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_u)). \tag{11.31}$$

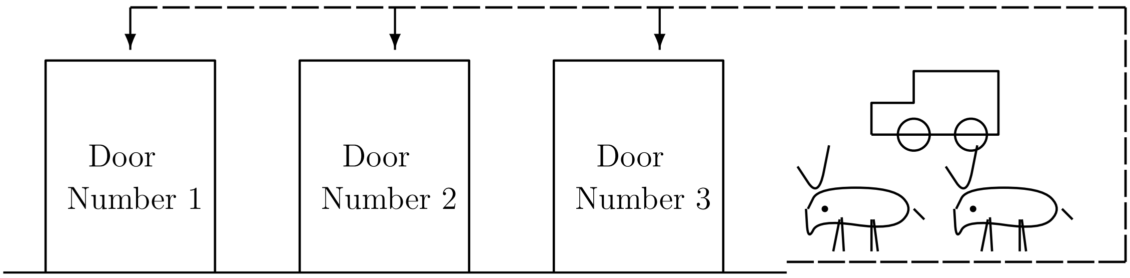
□

Problem 11.13. [Monty Hall problem, *cf.*[33]].
The Monty Hall problem is as follows (*cf.* Problem 5.12, Remark 5.13, Problem 8.8) :

(P) Suppose you are on a game show, and you are given the choice of three doors (i.e., "number 1", "number 2", "number 3"). Behind one door is a car, behind the others, goats.

(C) The host knows the fact that the probability that the car was set behind the k-th door (i.e., "number k") is given by p_k $(k = 1, 2, 3)$, for example, $p_1 = 3/7$, $p_2 = 1/7$, $p_3 = 3/7$. But you do not know this fact.

You pick a door (strictly speaking, you pick a door at random), say number 1, and the host, who knows what's behind the doors, opens another door, say "number 3", which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?



[*Answer*]. Put $\Omega = \{\omega_1, \omega_2, \omega_3\}$, where ω_1 [resp. ω_2, ω_3] means the state that the car is behind the door number 1 [resp. the door number 2, the door number 3]. Define the observable $\mathbf{O} \equiv (\{1,2,3\}, 2^{\{1,2,3\}}, F)$ in $C(\Omega)$ such that

$$
\begin{aligned}
&[F(\{1\})](\omega_1) = 0.0, \quad &[F(\{2\})](\omega_1) = 0.5, \quad &[F(\{3\})](\omega_1) = 0.5,^{2} \\
&[F(\{1\})](\omega_2) = 0.0, \quad &[F(\{2\})](\omega_2) = 0.0, \quad &[F(\{3\})](\omega_2) = 1.0, \\
&[F(\{1\})](\omega_3) = 0.0, \quad &[F(\{2\})](\omega_3) = 1.0, \quad &[F(\{3\})](\omega_3) = 0.0, \qquad (11.32)
\end{aligned}
$$

Thus, we have the unintentional random measurement $\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{2}[\Phi^n\mathbf{O}; 1/3], S_{[*]})$ (where $\Phi : C(\Omega) \to C(\Omega)$ is ergodic). Theorem 11.12 says that

$$
\mathbf{M}_{C(\Omega)}(\oplus_{n=0}^{2} [\Phi^n\mathbf{O}; 1/3], S_{[*]}) \Longleftrightarrow \mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_u)) \qquad (11.33)
$$

where $\nu_u(\{\omega_1\}) = \nu_u(\{\omega_2\}) = \nu_u(\{\omega_3\}) = 1/3$. Thus, it suffices to consider the statistical measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S(\nu_u))$. Here, note that

- By the statistical measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_u))$, you obtain a measured value 3,

which corresponds to the fact that the host said "Door (number 3) has a goat". Then, the posttest state ν_{post} ($\in \mathcal{M}_{+1}^m(\Omega)$) is given by

$$
\nu_{\text{post}} = \frac{F(\{3\}) \times \nu_u}{\big\langle \nu_u, F(\{3\}) \big\rangle}. \qquad (11.34)
$$

[2]Strictly speaking, $F(\{1\})(\omega_1) = 0.5$ and $F(\{2\})(\omega_1) = 0.5$ should be assumed in the problem (P)

That is,

$$\nu_{\rm post}(\{\omega_1\}) = 1/3, \quad \nu_{\rm post}(\{\omega_2\}) = 2/3, \quad \nu_{\rm post}(\{\omega_3\}) = 0, \tag{11.35}$$

and thus, *you should pick door number 2.*

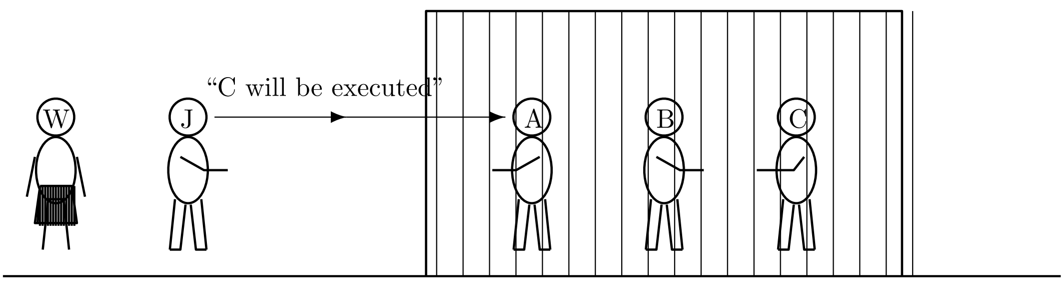
■

Remark 11.14. [Four answers to Monty Hall problem]. In this book four answers to the Monty Hall problem are presented in Problem 5.12, Remark 5.13, Problem 8.8, Problem 11.13. However, I believe that the Monty Hall problem in Problem 11.13 is the most natural.

■

Problem 11.15. [The problem of three prisoners, *cf.* Problem 8.10 and Remark 8.11]. Consider the following problem:

(P) Three men, A, B, and C were in jail. A knew that one of them was to be set free and the other two were to be executed. But he did not know who was the one to be spared. To the jailer who did know, A said, "Since two out of the three will bee executed, it is certain that either B or C will be, at least. You will give me no information about my own chances if you give me the name of one man, B or C, who is going to be executed." Accepting this argument after some thinking, the jailer said, "C will be executed." Thereupon A felt happier because now either he or C would go free, so his chance had increased from 1/3 to 1/2. This prisoner's happiness may or may not be reasonable. What do you think?



(Q) (Continued from the above (P)). There is a woman, who was proposed to by the three prisoners A, B and C. She listened to the conversation between A and the jailer. Thus, assume that she has the same information as A has. Then, we have the following problem:

($\sharp$) Whose proposal should she accept?

[*Answer to* (P)]. Let Ω ($\equiv \{\omega_a, \omega_b, \omega_c\}$) and $\mathbf{O} \equiv (X \equiv \{x_A, x_B, x_C\}, 2^{\{x_A, x_B, x_C\}}, F)$ be as in Problem 8.10. Since A has no information, the unintentional random measurement $\mathbf{M}_{C(\Omega)}(\oplus_{k=0}^{2}[\Phi^k \mathbf{O}; 1/3], S_{[*]}(\nu_0))$ (where $\Phi : C(\Omega) \to C(\Omega)$ is ergodic) is considered. Theorem 11.12 asserts the following identification:

$$\mathbf{M}_{C(\Omega)}(\oplus_{k=0}^{2}[\Phi^k \mathbf{O}; 1/3], S_{[*]}) \underset{\text{identification}}{\Longleftrightarrow} \mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_0)) \tag{11.36}$$

where ν_0 ($\in \mathcal{M}_{+1}^m(\Omega)$) is defined by

$$\nu_0(\{\omega_a\}) = 1/3, \quad \nu_0(\{\omega_b\}) = 1/3, \quad \nu_0(\{\omega_c\}) = 1/3. \tag{11.37}$$

Thus, we can assume that the (P) in the above is the same as the (P) in Problem 8.10. Therefore, we get that

$$\nu_{\text{post}}(\{\omega_a\}) = \frac{\frac{\nu_0(\{\omega_a\})}{2}}{\frac{\nu_0(\{\omega_a\})}{2} + \nu_0(\{\omega_b\})} = 1/3, \quad \nu_{\text{post}}(\{\omega_b\}) = \frac{\nu_0(\{\omega_b\})}{\frac{\nu_0(\{\omega_a\})}{2} + \nu_0(\{\omega_b\})} = 2/3,$$
$$\nu_{\text{post}}(\{\omega_c\}) = 0. \tag{11.38}$$

Therefore, we conclude that

- *the prisoner's happiness is not reasonable.* That is because $\nu_0(\{\omega_a\}) = 1/3 = \nu_{\text{post}}(\{\omega_a\})$.

[*Answer to* (Q)]. In the above (11.38), we see that

$$\nu_{\text{post}}(\{\omega_a\}) = 1/3, \quad \nu_{\text{post}}(\{\omega_b\}) = 2/3, \quad \nu_{\text{post}}(\{\omega_c\}) = 0. \tag{11.39}$$

Thus, we conclude that

- *she should choose the prisoner B.* That is because

$$\nu_{\text{post}}(\{\omega_c\}) = 0 < \nu_{\text{post}}(\{\omega_a\}) = 1/3 < \nu_{\text{post}}(\{\omega_b\}) = 2/3. \tag{11.40}$$

■