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Chapter 10

Newtonian mechanics in measurement Theory

In the previous chapter, we propose the W^*-algebraic formulation of SMT:

$$\text{SMT}^{W^*} = \underset{[\text{Proclaim}^{W^*}1\ (9.9)]}{\text{statistical measurement}} + \underset{[\text{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} \quad \text{in } W^*\text{-algebra} . \tag{10.1}$$

As mentioned in Remark 1.1 (b), in this book, "Newtonian mechanics" in MT is called the "classical system theory (or dynamical system theory)". In this sense, we will study "Newtonian mechanics" in SMT^{W^*}. We first introduce "the W^*-algebraic generalization of Kolmogorov's extension theorem". This theorem is essential to MT just like Kolmogorov's extension theorem is so in his probability theory. Using this theorem, we can define "particle's trajectory" by "the sequence of measured values". And further we prove:

(i) the existence of "particle's trajectory" in Newtonian mechanics,

(ii) the existence of Brownian motion.

Thus, we can understand the difference between the concepts of "particle's trajectory" and "state's evolution" in both classical and quantum mechanics. Throughout this chapter, readers will see that, from the mathematical point of view, the W^*-algebraic formulation is more handy than the C^*-algebraic formulation.

10.1 Kolmogorov's extension theorem in W^*-algebra

In this section we study "Kolmogorov's extension theorem" in the (W^*-algebraic) Statistical MT. It is generally said that Kolmogorov's extension theorem is most fundamental in Kolmogorov's probability theory. That is because this theorem assures the existence of a probability space (i.e., sample space). On the other hand, our theorem (= Theorem 10.1, i.e., the W^*-algebraic generalization of Kolmogorov's extension theorem) assures the existence of a measurement (or, observable). Recall the our spirit $\Big($see Remark (in §2.3(I))$\Big)$:

(♯) there is no probability without measurements.

Thus, in measurement theory, the concept of "measurement" is more fundamental than that of "sample space". Therefore, this theorem (i.e., the W^*-algebraic generalization of Kolmogorov's extension theorem) is very important in MT. That is, this theorem (= Theorem 10.1) is essential to MT just like Kolmogorov's extension theorem is so in his probability theory. Using this theorem, we can define "particle's trajectory" by "the sequence of measured values". And further we prove:

(i) the existence of "particle's trajectory" in Newtonian mechanics,

(ii) the existence of Brownian motion.

Thus, we can understand the difference between the concepts of "particle's trajectory" and "state's evolution" in both classical and quantum mechanics.

Let $\widehat{\Lambda}$ be an index set. For each $\lambda \in \widehat{\Lambda}$, consider a set X_λ. For any subsets $\Lambda_1 \subseteq \Lambda_2 (\subseteq \widehat{\Lambda})$, $\pi_{\Lambda_1,\Lambda_2}$ is the natural projection such that:

$$\pi_{\Lambda_1,\Lambda_2} : \mathop{\times}_{\lambda \in \Lambda_2} X_\lambda \longrightarrow \mathop{\times}_{\lambda \in \Lambda_1} X_\lambda.$$

Especially, put $\pi_\Lambda = \pi_{\Lambda,\widehat{\Lambda}}$. For each $\lambda \in \widehat{\Lambda}$, consider a W^*-observable $(X_\lambda, \mathcal{F}_\lambda, F_\lambda)$ in W^*-algebra $\mathcal{N}$. Note that the quasi-product observable $\overline{\mathbf{O}} \equiv (\times_{\lambda\in\widehat{\Lambda}} X_\lambda, \ \times_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda, \ F_{\widehat{\Lambda}})$ of $\{ \ (X_\lambda, \mathcal{F}_\lambda, \ F_\lambda) \ | \ \lambda \in \widehat{\Lambda} \ \}$ is characterized as the observable such that:

$$F_{\widehat{\Lambda}}(\pi^{-1}_{\{\lambda\}}(\Xi_\lambda)) = F_\lambda(\Xi_\lambda) \qquad (\forall \Xi_\lambda \in \mathcal{F}_\lambda, \forall \lambda \in \widehat{\Lambda}), \tag{10.2}$$

though the existence and the uniqueness of a quasi-product observable are not guaranteed in general. The following theorem says something about the existence and uniqueness of the quasi-product observable.

Theorem 10.1. [W^*-algebraic generalization of Kolmogorov's extension theorem, *cf.* [43]]. *For each $\lambda \in \widehat{\Lambda}$, consider a Borel measurable space $(X_\lambda, \mathcal{F}_\lambda)$, where X_λ is a separable complete metric space. Define the set $\mathcal{P}_0(\widehat{\Lambda})$ such as $\mathcal{P}_0(\widehat{\Lambda}) \equiv \{\Lambda \subseteq \widehat{\Lambda} \mid \Lambda$ is finite $\}$. Assume that the family of the W^*-observables $\big\{ \ \overline{\mathbf{O}}_\Lambda \equiv (\ \times_{\lambda\in\Lambda} X_\lambda, \ \times_{\lambda\in\Lambda} \mathcal{F}_\lambda, \ F_\Lambda \ ) \ | \ \Lambda \in \mathcal{P}_0(\widehat{\Lambda}) \ \big\}$ in a W^*-algebra $\mathcal{N}$ satisfies the following "W^*-algebraic consistency condition":*

- *for any* $\Lambda_1,\ \Lambda_2 \in \mathcal{P}_0(\widehat{\Lambda})$ *such that* $\Lambda_1 \subseteq \Lambda_2$,

$$F_{\Lambda_2}\big(\pi^{-1}_{\Lambda_1,\Lambda_2}(\Xi_{\Lambda_1})\big) = F_{\Lambda_1}\big(\Xi_{\Lambda_1}\big) \quad (\forall \Xi_{\Lambda_1} \in \mathop{\times}_{\lambda\in\Lambda_1} \mathcal{F}_\lambda). \tag{10.3}$$

Then, there uniquely exists the W^*-observable $\widetilde{\mathbf{O}}_{\widehat{\Lambda}} \equiv \big(\times_{\lambda\in\widehat{\Lambda}} X_\lambda,\ \times_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda,\ \widetilde{F}_{\widehat{\Lambda}}\big)$ *in* $\mathcal{N}$ *such that:*

$$\widetilde{F}_{\widehat{\Lambda}}\big(\pi^{-1}_{\Lambda}(\Xi_\Lambda)\big) = F_\Lambda\big(\Xi_\Lambda\big) \quad (\forall \Xi_\Lambda \in \mathop{\times}_{\lambda\in\Lambda} \mathcal{F}_\lambda,\ \forall \Lambda \in \mathcal{P}_0(\widehat{\Lambda})). \tag{10.4}$$

Proof. Let $\overline{\rho}$ be any normal state, i.e., $\overline{\rho} \in \mathfrak{S}^n(\mathcal{N}_*)$. Then, the $\overline{\rho}(F_\Lambda(\cdot))$ is a probability measure on the product measurable space $(\times_{\lambda\in\Lambda} X_\lambda,\ \times_{\lambda\in\Lambda} \mathcal{F}_\lambda)$ for all $\Lambda \in \mathcal{P}_0(\widehat{\Lambda})$. $\Big($If $\mathcal{N} = L^\infty(\Omega,\mu)$, the existence is assured.$\Big)$ It is clear that the family $\{\ (\times_{\lambda\in\Lambda} X_\lambda,\ \times_{\lambda\in\Lambda} \mathcal{F}_\lambda,\ \overline{\rho}(F_\Lambda(\ \cdot\ )))\ \ |\ \ \Lambda \in \mathcal{P}_0(\widehat{\Lambda})\ \}$ satisfies the "usual consistency condition" in Kolmogorov's probability theory. Therefore, by Kolmogorov's extension theorem[56], there uniquely exists a probability measure $P^{\overline{\rho}}_{\widehat{\Lambda}}$ on the product measurable space $\big(\ \times_{\lambda\in\widehat{\Lambda}} X_\lambda,\ \times_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda\ \big)$ such that:

$$P^{\overline{\rho}}_{\widehat{\Lambda}}\big(\pi^{-1}_{\Lambda}(\Xi_\Lambda)\big) = \overline{\rho}(F_\Lambda(\Xi_\Lambda)) \quad (\forall \Xi_\Lambda \in \mathop{\times}_{\lambda\in\Lambda} \mathcal{F}_\lambda,\ \forall \Lambda \in \mathcal{P}_0(\widehat{\Lambda})). \tag{10.5}$$

Define the subfield $\times^\sharp_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda$ of $\times_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda$ such that:

$$\mathop{\times}^{\sharp}_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda = \{\pi^{-1}_{\Lambda}(\Xi_\Lambda) \mid \Xi_\Lambda \in \mathop{\times}_{\lambda\in\Lambda} \mathcal{F}_\lambda,\ \Lambda \in \mathcal{P}_0(\widehat{\Lambda})\}. \tag{10.6}$$

Then, we see, by (10.5), that there uniquely exists the countably additive function $F^\sharp_{\widehat{\Lambda}} : \times^\sharp_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda \to \mathcal{N}$ (in the sense of weak*-topology $\sigma(\mathcal{N},\mathcal{N}_*)$) such that:

$$P^{\overline{\rho}}_{\widehat{\Lambda}}\big(\Xi^\sharp_{\widehat{\Lambda}}\big) = \overline{\rho}(F^\sharp_{\widehat{\Lambda}}\big(\Xi^\sharp_{\widehat{\Lambda}}\big)) \quad (\forall \Xi^\sharp_{\widehat{\Lambda}} \in \mathop{\times}^{\sharp}_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda). \tag{10.7}$$

Define the map $\widetilde{F}_{\widehat{\Lambda}} : \times_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda \to \mathcal{N}$ such that:

$$\widetilde{F}_{\widehat{\Lambda}}\big(\Xi_{\widehat{\Lambda}}\big) = \inf_{\{\Xi^{\sharp,k}_{\widehat{\Lambda}}\}_{k=1}^\infty \in Q(\Xi_{\widehat{\Lambda}})} \sum_{k=1}^\infty F^\sharp_{\widehat{\Lambda}}\big(\Xi^{\sharp,k}_{\widehat{\Lambda}}\big), \tag{10.8}$$

where $Q(\Xi_{\widehat{\Lambda}}) \equiv \Big\{\{\Xi^{\sharp,k}_{\widehat{\Lambda}}\}_{k=1}^\infty \mid \Xi_{\widehat{\Lambda}} \subseteq \cup_{k=1}^\infty \Xi^{\sharp,k}_{\widehat{\Lambda}},\ \Xi^{\sharp,k}_{\widehat{\Lambda}} \in \times^\sharp_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda\Big\}$ $(\forall \Xi_{\widehat{\Lambda}} \in \times_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda)$. It clearly holds that

$$\widetilde{F}_{\widehat{\Lambda}}\big(\Gamma^1_{\widehat{\Lambda}} \cup \Gamma^2_{\widehat{\Lambda}}\big) \le \widetilde{F}_{\widehat{\Lambda}}\big(\Gamma^1_{\widehat{\Lambda}}\big) + \widetilde{F}_{\widehat{\Lambda}}\big(\Gamma^2_{\widehat{\Lambda}}\big) \qquad (\forall \Gamma^1_{\widehat{\Lambda}}, \Gamma^2_{\widehat{\Lambda}} \in \mathop{\times}_{\lambda\in\widehat{\Lambda}} \mathcal{F}_\lambda).$$

Also, we see that, for any $\Xi_{\widehat{\Lambda}}$ in $\times_{\lambda \in \widehat{\Lambda}} \mathcal{F}_\lambda$,

$$\begin{aligned} P^{\overline{\rho}}_{\widehat{\Lambda}}(\Xi_{\widehat{\Lambda}}) &= \inf_{\{\Xi^{\sharp,k}_{\widehat{\Lambda}}\}_{k=1}^\infty \in Q(\Xi_{\widehat{\Lambda}})} \sum_{k=1}^\infty P^{\overline{\rho}}_{\widehat{\Lambda}}(\Xi^{\sharp,k}_{\widehat{\Lambda}}) \qquad \text{(by Caratheodory theorem, } cf. \text{ [29])} \\ &= \inf_{\{\Xi^{\sharp,k}_{\widehat{\Lambda}}\}_{k=1}^\infty \in Q(\Xi_{\widehat{\Lambda}})} \sum_{k=1}^\infty \overline{\rho}(F^{\sharp}_{\widehat{\Lambda}}(\Xi^{\sharp,k}_{\widehat{\Lambda}})) \qquad \text{(by (10.7))} \\ &\ge \overline{\rho}\Big(\inf_{\{\Xi^{\sharp,k}_{\widehat{\Lambda}}\}_{k=1}^\infty \in Q(\Xi_{\widehat{\Lambda}})} \sum_{k=1}^\infty F^{\sharp}_{\widehat{\Lambda}}(\Xi^{\sharp,k}_{\widehat{\Lambda}}) \Big) \qquad \text{(by the property of } \mathcal{N}) \\ &= \overline{\rho}(\widetilde{F}_{\widehat{\Lambda}}(\Xi_{\widehat{\Lambda}})) \qquad \text{(by (10.8)).} \end{aligned}$$

Similarly we see that $P^{\overline{\rho}}_{\widehat{\Lambda}}(\Xi^c_{\widehat{\Lambda}}) \ge \overline{\rho}(\widetilde{F}_{\widehat{\Lambda}}(\Xi^c_{\widehat{\Lambda}}))$ where $\Xi^c_{\widehat{\Lambda}} = (\times_{\lambda \in \widehat{\Lambda}} X_\lambda) \setminus \Xi_{\widehat{\Lambda}}$. Thus we see, by (10.9), that

$$1 = P^{\overline{\rho}}_{\widehat{\Lambda}}(\Xi_{\widehat{\Lambda}}) + P^{\overline{\rho}}_{\widehat{\Lambda}}(\Xi^c_{\widehat{\Lambda}}) \ge \overline{\rho}(\widetilde{F}_{\widehat{\Lambda}}(\Xi_{\widehat{\Lambda}})) + \overline{\rho}(\widetilde{F}_{\widehat{\Lambda}}(\Xi^c_{\widehat{\Lambda}})) \ge \overline{\rho}(\widetilde{F}_{\widehat{\Lambda}}(\times_{\lambda \in \widehat{\Lambda}} X_\lambda)) = 1.$$

This implies that $P^{\overline{\rho}}_{\widehat{\Lambda}}(\Xi_{\widehat{\Lambda}}) = \overline{\rho}(\widetilde{F}_{\widehat{\Lambda}}(\Xi_{\widehat{\Lambda}}))$. Thus we see that

$$\overline{\rho}\Big(\widetilde{F}_{\widehat{\Lambda}}(\pi_\Lambda^{-1}(\Xi_\Lambda))\Big) = P^{\overline{\rho}}_{\widehat{\Lambda}}(\pi_\Lambda^{-1}(\Xi_\Lambda)) = \overline{\rho}\Big(F_\Lambda(\Xi_\Lambda)\Big) \quad (\forall \Xi_\Lambda \in \times_{\lambda \in \Lambda} \mathcal{F}_\lambda, \ \forall \Lambda \in \mathcal{P}_0(\widehat{\Lambda})),$$

which implies (10.4). This completes the proof. □

10.2 The definition of "trajectories"

Now we shall propose the definition of the "trajectories" in SMT$^{W^*}$. Let $\overline{\mathbf{S}}(\overline{\rho}_0) \equiv [\overline{S}(\overline{\rho}_0), \{\Psi_{t_1,t_2} : \mathcal{N} \to \mathcal{N}\}_{(t_1,t_2) \in \mathbf{R}^2_{\le}}]$ be a W^*-general system with an initial system $\overline{S}(\overline{\rho}_0)$. Let $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ be a crisp observable in $\mathcal{N}$. For each time $t \in \overline{\mathbf{R}}^+ \equiv \{t \in \mathbf{R} \mid t \ge 0\}$, consider a W^*-observable $\overline{\mathbf{O}}_t \equiv (X_t, \mathcal{F}_t, F_t)$ in $\mathcal{N}$ such that:

$$\bullet \ (X_t, \mathcal{F}_t, F_t) = (X, \mathcal{F}, F) \text{ for all } t \in \overline{\mathbf{R}}^+. \tag{10.9}$$

Let us represent the "measurement $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t \in \overline{\mathbf{R}}^+}, \overline{\mathbf{S}}(\overline{\rho}_0))$" in what follows. Let $\Lambda \in \mathcal{P}_0(\overline{\mathbf{R}}^+)$ $\Big(\equiv \{\Lambda_0 \in 2^{\overline{\mathbf{R}}^+} : \Lambda_0 \text{ is finite }\}\Big)$, that is, $\Lambda = \{t_1, t_2, ..., t_n\}$ where $0 \le t_1 < t_2 < \cdots < t_n$. Then, we can uniquely define the observable $\overline{\mathbf{O}}_\Lambda \equiv (X^\Lambda, \mathcal{F}^\Lambda, F_\Lambda)$ at time 0 such that:

$$F_\Lambda(\Xi_{t_1} \times \Xi_{t_2} \times \cdots \times \Xi_{t_n}) = \Psi_{0,t_1}\Big(F(\Xi_{t_1}) \cdots \Psi_{t_{n-2},t_{n-1}}\Big(F(\Xi_{t_{n-1}})(\Psi_{t_{n-1},t_n} F(\Xi_{t_n}))\Big) \cdots\Big), \tag{10.10}$$

though the existence of $\overline{\mathbf{O}}_\Lambda$ is not always guaranteed except for the classical cases. $\Big($For the uniqueness, recall Theorem 9.8. $\Big)$ Assume that the observable $\overline{\mathbf{O}}_\Lambda$ exists for any $\Lambda \in \mathcal{P}_0(\overline{\mathbf{R}}^+)$. It is clear that the family $\{ \ \overline{\mathbf{O}}_\Lambda \mid \Lambda \in \mathcal{P}_0(\overline{\mathbf{R}}^+) \ \}$ satisfies the consistency condition (10.3). Thus, by Theorem 10.1 we have the observable $\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+} \equiv (X^{\overline{\mathbf{R}}^+}, \mathcal{F}^{\overline{\mathbf{R}}^+}, \widetilde{F}_{\overline{\mathbf{R}}^+})$ in $\mathcal{N}$, which is called a *trajectory observable* (*concerning* $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$). Therefore, we get the Heisenberg picture representation $\overline{\mathbf{M}}_{\mathcal{N}}(\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+}, \overline{S}(\overline{\rho}_0))$ of $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t \in \overline{\mathbf{R}}^+}, \mathbf{S}(\overline{\rho}_0) \ )$.

Now we can propose the following definition, which is our main assertion in this chapter.

Definition 10.2. [Trajectory (= particle's trajectory)]. *Assume the above notations. The measured value obtained by the measurement* $\overline{\mathbf{M}}_{\mathcal{N}}(\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+}, \overline{S}(\overline{\rho}_0))$ *is called a trajectory* (*concerning* $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$) *of the* W^**-general system* $\overline{\mathbf{S}}(\overline{\rho}_0) \equiv [\overline{S}(\overline{\rho}_0), \{\Psi_{t_1,t_2} : \mathcal{N} \to \mathcal{N}\}_{(t_1,t_2) \in \mathbf{R}^2_{\leq}}]$.

■

The difference of "particle's trajectory" and "state's evolution" is clear in Definition 10.2. That is,

$$
\begin{cases}
\text{"state's evolution"} & \cdots (\Psi_{0,t})_* \overline{\rho}_0, \quad (0 \leq t < \infty), \\
\text{"particle's trajectory"} & \cdots \ \text{the measured value of } \overline{\mathbf{M}}_{\mathcal{N}}(\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+}, \overline{S}(\overline{\rho}_0)).
\end{cases}
\tag{10.11}
$$

Note that in quantum mechanics, the existence of $\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+}$ is not usually guaranteed, and thus, the concept of "particle's trajectory" is meaningless in general (*cf.* [37, 40]).

Recall DST(1.2a), that is,

$$
\boxed{\text{"dyn. syst. theor."}} =
\begin{cases}
\frac{dx(t)}{dt} = g(x(t), u_1(t), t), \ x(0) = \ x_0 \ \cdots \text{(state equation)} , \\
y(t) = f(x(t), u_2(t), t) \qquad \text{(measurement equation)}.
\end{cases}
\tag{10.12}
$$
(=(1.2a))

In order to compare (10.11) and (10.12), we add the following remark.

Remark 10.3. [(i): The case that $u_2 = 0$ in (10.12)] (The generalization of Definition 10.2). The condition (10.9) can be easily generalized as follows:

$$
\bullet \quad (X_t, \mathcal{F}_t, F_t) \text{ is crisp for all } t \in \overline{\mathbf{R}}^+. \tag{10.13}
$$

Under the condition, by a similar way of (10.10) we can easily define a *trajectory* (concerning $\{(X_t, \mathcal{F}_t, F_t) \mid t \in \overline{\mathbf{R}}^+\}$) of the W^*-general system $\overline{\mathbf{S}}(\overline{\rho}_0) \equiv [S(\overline{\rho}_0), \{\Psi_{t_1,t_2} : \mathcal{N} \to$

$\mathcal{N}\}_{(t_1,t_2)\in\mathbf{R}^2_{\le}}]$. Here, consider classical cases, i.e., $\mathcal{N} = L^\infty(\Omega,\mu)$. And, for each $t \in \overline{\mathbf{R}}^+$, consider a measurable function $f_t : \Omega \to \mathbf{R}^m$, which can be identified with a crisp observable $(\mathbf{R}^m, \mathcal{B}_{\mathbf{R}^m}, F_t)$, (*cf.* (ii) in Example 9.4). Thus, by Theorem 10.1 we have the observable $\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+} \equiv ((\mathbf{R}^m)^{\overline{\mathbf{R}}^+}, (\mathcal{B}_{\mathbf{R}^m})^{\overline{\mathbf{R}}^+}, \widetilde{F}_{\overline{\mathbf{R}}^+})$ in $\mathcal{N}$, which is called a *trajectory observable* (*concerning* $\{\ \overline{\mathbf{O}}_t \equiv (\mathbf{R}^m, \mathcal{B}_{\mathbf{R}^m}, F_t) \mid t \in \overline{\mathbf{R}}^+\}$. Thus we can also define a *trajectory* (concerning $\{f_t \mid t \in \overline{\mathbf{R}}^+\}$) of the W^*-dynamical system $\overline{\mathbf{S}}(\overline{\rho}_0)$ as the trajectory concerning $\{\ (\mathbf{R}^m, \mathcal{B}_{\mathbf{R}^m}, F_t) \mid t \in \overline{\mathbf{R}}^+\}$

[(ii): The case that $u_2 \neq 0$ in (10.12)] (The generalization of Definition 10.2). The condition (10.13) can be easily generalized as follows:

- $(X_t, \mathcal{F}_t, F_t)$ is not always crisp for all $t \in \overline{\mathbf{R}}^+$. (10.14)

By a similar way as in the above (i), we have the observable $\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+} \equiv (\times_{t\in\overline{\mathbf{R}}^+} X_t, \times_{t\in\overline{\mathbf{R}}^+} \mathcal{F}_t, \widetilde{F}_{\overline{\mathbf{R}}^+})$ in $\mathcal{N}$ ($= L^\infty(\Omega;\mu)$), which is called a *trajectory observable* (*concerning* $\{\ \overline{\mathbf{O}}_t \equiv (X_t, \mathcal{F}_t, F_t) \mid t \in \overline{\mathbf{R}}^+\}$).

■

10.3 Trajectories and Newtonian mechanics

In the previous section, we proposed Definition 10.2, in which the concept of "particle's trajectory" is characterized as a measured value of the measurement. Thus, our concern in this section is to show that the "particle's trajectory" is represented by the Newton equation. If it is true, we can completely understand "Newtonian mechanics" in measurement theory.

First we review Liouville's equation. Put $\mathcal{N} = L^\infty(\mathbf{R}^s_q \times \mathbf{R}^s_p, m^{2s})$ and $\mathcal{N}_* = L^1(\mathbf{R}^s_q \times \mathbf{R}^s_p, m^{2s})$, where $\mathbf{R}^s_q \times \mathbf{R}^s_p \equiv \{\ (q,p) = (q_1, q_2, \cdots, q_s, p_1, p_2, \cdots, p_s) \mid q_j, p_j \in \mathbf{R}, j = 1, 2, \cdots, s\ \}$ and $(\mathbf{R}^s_q \times \mathbf{R}^s_p, \mathcal{B}(\mathbf{R}^s_q \times \mathbf{R}^s_p), m^{2s})$ is the $2s$-dimensional Lebesgue measure space. Liouville's equation with an initial density function $\overline{\rho}_0$ is as follows:

$$\frac{\partial \overline{\rho}_t(q,p)}{\partial t} = \sum_{j=1}^{s} \Big(\frac{\partial \mathcal{H}(q,p,t)}{\partial q_j} \frac{\partial \overline{\rho}_t(q,p)}{\partial p_j} - \frac{\partial \mathcal{H}(q,p,t)}{\partial p_j} \frac{\partial \overline{\rho}_t(q,p)}{\partial q_j} \Big), \tag{10.15}$$

$$\overline{\rho}_0 \in \mathfrak{S}^n(\mathcal{N}_*) \equiv \{\overline{\rho}\ :\ \|\overline{\rho}\|_{L^1} = 1,\ \overline{\rho} \ge 0\}, \tag{10.16}$$

where $\mathcal{H} : \mathbf{R}_q^s \times \mathbf{R}_p^s \times \mathbf{R} \to \mathbf{R}$ is a Hamiltonian. By using the solution of (10.15), we can define the operator $[\Psi_{t_1,t_2}]_* : L^1(\mathbf{R}_q^s \times \mathbf{R}_p^s, m^{2s}) \to L^1(\mathbf{R}_q^s \times \mathbf{R}_p^s, m^{2s})$ such that:

$$\Big([\Psi_{t_1,t_2}]_* \overline{\rho}_{t_1}\Big)(q,p) = \overline{\rho}_{t_2}(q,p) \quad \forall (q,p) \in \mathbf{R}_q^s \times \mathbf{R}_p^s,\ \forall (t_1,t_2) \in \mathbf{R}_{\leq}^2. \tag{10.17}$$

That is, the "state's evolution" is represented by the Schrödinger picture $\{[\Psi_{t_1,t_2}]_* \mid (t_1,t_2) \in \mathbf{R}_{\leq}^2\}$, which is induced by Liouville's equation (10.15) for states. And furthermore, putting $\Psi_{t_1,t_2} = ([\Psi_{t_1,t_2}]_*)^*$, we get the Heisenberg picture $\{\Psi_{t_1,t_2} \mid (t_1,t_2) \in \mathbf{R}_{\leq}^2\}$, which is also induced by Liouville's adjoint equation (i.e., Liouville's equation for observables). Thus, we get the W^*-dynamical system $\overline{\mathbf{S}}(\overline{\rho}_0) \equiv [S(\overline{\rho}_0), \{\Psi_{t_1,t_2} : \mathcal{N} \to \mathcal{N}\}_{(t_1,t_2)\in\mathbf{R}_{\leq}^2}]$. Also, it should be noted that the dynamical system $\overline{\mathbf{S}}(\overline{\rho}_0)$ is deterministic, i.e., each $\Psi_{t_1,t_2} : \mathcal{N} \to \mathcal{N}$ is (bijective) homomorphic.

It is well known that Liouville's equation is mathematically equivalent to the following Newton equation:

$$\frac{d}{dt}q_j(t) = \frac{\partial \mathcal{H}}{\partial p_j}(q(t),p(t),t), \quad \frac{d}{dt}p_j(t) = -\frac{\partial \mathcal{H}}{\partial q_j}(q(t),p(t),t), \quad j = 1,2,\cdots,s \tag{10.18}$$

$$(q(0),p(0)) \in \mathbf{R}_q^s \times \mathbf{R}_p^s. \tag{10.19}$$

Using the solution of the Newton equation (10.18), we define the continuous map $\psi_{t_1,t_2} : \mathbf{R}_q^s \times \mathbf{R}_p^s \to \mathbf{R}_q^s \times \mathbf{R}_p^s$ such that:

$$\psi_{t_1,t_2}(q(t_1),p(t_1)) = (q(t_2),p(t_2)) \qquad (\forall (q(t_1),p(t_1)) \in \mathbf{R}_q^s \times \mathbf{R}_p^s). \tag{10.20}$$

Thus we can get the (bijective) homomorphism $\Psi_{t_1,t_2} : L^\infty(\mathbf{R}_q^s \times \mathbf{R}_p^s, m^{2s}) \to L^\infty(\mathbf{R}_q^s \times \mathbf{R}_p^s, m^{2s})$ such that:

$$(\Psi_{t_1,t_2}F)(q,p) = F(\psi_{t_1,t_2}(q,p)) \quad (\forall (q,p) \in \mathbf{R}_q^s \times \mathbf{R}_p^s, \forall F \in L^\infty(\mathbf{R}_q^s \times \mathbf{R}_p^s), \forall (t_1,t_2) \in \mathbf{R}_{\leq}^2). \tag{10.21}$$

Of course, this Ψ_{t_1,t_2} is the same as the Ψ_{t_1,t_2} derived from Liouville's equation. Since Liouville's equation and Newton equation are mathematically equivalent, there is a reason to say that the time evolution is also represented by Newton equation. However, note that the term "Newton equation" [resp. "Liouville's equation"] is, in this book, defined to be the equation that represents "particle's trajectory" [resp. "time evolution of states or observables"].

For simplicity, we put $(\Omega, \mathcal{B}, d\omega) = (\mathbf{R}_q^s \times \mathbf{R}_p^s, \mathcal{B}(\mathbf{R}_q^s \times \mathbf{R}_p^s), m^{2s})$. And, put $(\mathcal{N}, \mathcal{N}_*)$ $= (L^\infty(\Omega), L^1(\Omega))$. Consider the deterministic W^*-dynamical system $\overline{\mathbf{S}}(\overline{\rho}_0) \equiv [S(\overline{\rho}_0)$, $\{\Psi_{t_1,t_2} : \mathcal{N} \to \mathcal{N}\}_{(t_1,t_2)\in \mathbf{R}^2_{\le}}]$, which is induced by Liouville's equation (10.15) and (10.16).

Define the *state space observable* (or, *exact observable*) $\overline{\mathbf{O}} \equiv (\Omega, \mathcal{B}, F)$ in $\mathcal{N}$ $\big(\equiv L^\infty(\Omega)\big)$ such that:

$$F(\Xi) = \chi_{\Xi} \quad \forall \Xi \in \mathcal{B}, \tag{10.22}$$

which is, of course, crisp. Thus, by the same arguments appearing above Definition 10.2, we can get the trajectory observable $\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+} \equiv (\Omega^{\overline{\mathbf{R}}^+}, \mathcal{B}^{\overline{\mathbf{R}}^+}, \widetilde{F}_{\overline{\mathbf{R}}^+})$ concerning the state space observable $\overline{\mathbf{O}} \equiv (\Omega, \mathcal{B}, F)$. And therefore, we get the measurement $\overline{\mathbf{M}}_{L^\infty(\Omega)}(\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+}, S(\overline{\rho}_0))$ (*cf.* Remark 10.3). Assume that

- a measured value $\widehat{\omega}$ $(= (\omega_t)_{t\in\overline{\mathbf{R}}^+} \in \Omega^{\overline{\mathbf{R}}^+})$ is obtained by $\overline{\mathbf{M}}_{L^\infty(\Omega)}(\widetilde{\mathbf{O}}_{\overline{\mathbf{R}}^+}, S(\overline{\rho}_0))$.

Note that the measured value $\widehat{\omega}$ is precisely the "particle's trajectory" in Definition 10.2.

Now we shall investigate the properties of the measured value $\widehat{\omega}$ $(= (\omega_t)_{t\in\overline{\mathbf{R}}^+} \in \Omega^{\overline{\mathbf{R}}^+})$, that is, we shall show that the trajectory $\widehat{\omega}$ is represented by the Newton equation (10.18) and (10.19). Let $D = \{t_0, t_1, t_2, \cdots, t_n\}$ be a finite subset of $\overline{\mathbf{R}}^+$, where $t_0 = 0 < t_1 < t_2 < \cdots < t_n$. Put $\widehat{\Xi} = \times^D_{t\in\overline{\mathbf{R}}^+}\Xi_t$ $\big(\in \mathcal{B}^{\overline{\mathbf{R}}^+}\big)$ where $\Xi_t = \Omega$ $(\forall t \notin D)$. Then, we see that

- the probability that $\widehat{\omega}(= (\omega_t)_{t\in\overline{\mathbf{R}}^+})$ belongs to the set $\widehat{\Xi} = \times^D_{t\in\overline{\mathbf{R}}^+}\Xi_t$ is given by

$$\begin{aligned}
\overline{\rho}_0(\widetilde{F}_{\overline{\mathbf{R}}^+}(\widehat{\Xi})) =& \overline{\rho}_0\Big(F(\Xi_0)\Psi_{0,t_1}\Big(F(\Xi_{t_1})\cdots\Psi_{t_{n-2},t_{n-1}}\Big(F(\Xi_{t_{n-1}})(\Psi_{t_{n-1},t_n}F(\Xi_{t_n}))\Big)\cdots\Big) \\
=& \overline{\rho}_0\Big(\Pi_{k=0}^n\big(\Psi_{0,t_k}F(\Xi_{t_k})\big)\Big) \qquad \text{(because each } \Psi_{t_{k-1},t_k} \text{ is homomorphic)} \\
=& \overline{\rho}_0\Big(\Pi_{k=0}^n F(\psi^{-1}_{0,t_k}(\Xi_{t_k}))\Big) \\
=& \int_\Omega \Big(\Pi_{k=0}^n \chi_{\psi^{-1}_{0,t_k}(\Xi_{t_k})}(\omega)\Big)\overline{\rho}_0(\omega)d\omega.
\end{aligned} \tag{10.23}$$

Let Ξ_0 be any element in $\mathcal{B}$ such that $\int_{\Xi_0}\overline{\rho}_0(\omega)d\omega \neq 0$. Thus, under the hypothesis that we know that $\omega_0 \in \Xi_0$, i.e., $\widehat{\omega}(= (\omega_t)_{t\in\overline{\mathbf{R}}^+}) \in \Xi_0 \times \Omega^{\mathbf{R}^+}$ (where $\mathbf{R}^+ = (0, \infty)$), we can calculate the following conditional probability:

$$\frac{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\times^D_{t\in\overline{\mathbf{R}}^+}\Xi_t)\big)}{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\Xi_0 \times \Omega^{\mathbf{R}^+})\big)} = \frac{\int_{\Xi_0}\Big(\Pi_{k=1}^n \chi_{\psi^{-1}_{0,t_k}(\Xi_{t_k})}(\omega)\Big)\overline{\rho}_0(\omega)d\omega}{\int_{\Xi_0}\overline{\rho}_0(\omega)d\omega}. \tag{10.24}$$

Thus, we see that

$$\lim_{\Xi_0 \to \{\omega_0\}} \frac{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\times_{t\in\overline{\mathbf{R}}^+}^{D} \Xi_t)\big)}{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\Xi_0 \times \Omega^{\mathbf{R}^+})\big)} = \begin{cases} 1 & \text{if } \omega_0 \in \cap_{k=1}^{n} \psi_{0,t_k}^{-1}(\Xi_{t_k}) \\ 0 & \text{otherwise.} \end{cases} \tag{10.25}$$

(Though the above argument is somewhat rough from the mathematical point of view, we can easily check it in mathematics.) This implies that

$$\omega_t = \psi_{0,t}(\omega_0) \qquad (\forall t \in \overline{\mathbf{R}}^+). \tag{10.26}$$

That is, the measured value $\widehat{\omega}$ $(= (\omega_t)_{t\in\overline{\mathbf{R}}^+} \in \Omega^{\overline{\mathbf{R}}^+})$ is the solution of the Newton equation. Also, note that the (10.25) is independent of the choice of the initial normal state $\overline{\rho}_0$.

Summing up, we see,

- In Newtonian mechanics, the state's evolution is represented by Liouville equation, and the existence of the trajectory (concerning the state space observable) is always guaranteed. That is, it can be represented by the Newton equation. Also, in quantum mechanics, the state's evolution is represented by Schrödinger equation. However, the existence of the trajectory is not always guaranteed.

That is,

	state's evolution	particle's trajectory
Newtonian mechanics	Liouville equation	Newton equation
quantum mechanics	Schrödinger equation	(meaningless)[1]

(10.27)

10.4 Brownian motions

As emphasized throughout this chapter, the concepts of "state's evolution" and "particle's trajectory" are completely different. This is, of course, a matter of common knowledge in quantum mechanics. And moreover, we can point out that the difference is clear in diffusion processes for classical systems. Therefore, in this section we examine diffusion processes in SMT^{W^*}. The examination will promote a better understanding of our theory.

[1] For the measurement theoretical model of Wilson chamber and its numerical analysis, see [37, 40].

Put $\mathcal{N} = L^\infty(\mathbf{R}_q, m)$ and $\mathcal{N}_* = L^1(\mathbf{R}_q, m)$, where $(\mathbf{R}_q, \mathcal{B}(\mathbf{R}_q), m)$ is the 1-dimensional Lebesgue measure space. The diffusion equation with an initial density function $\overline{\rho}_0$ at the time $t = 0$ is as follows:

$$\frac{\partial \overline{\rho}_t(q)}{\partial t} = \frac{\partial^2 \overline{\rho}_t(q)}{\partial q^2}, \tag{10.28}$$

$$\overline{\rho}_0 \in \{\overline{\rho} \in L^1(\mathbf{R}_q, m) \ : \ \|\overline{\rho}\|_{L^1} = 1, \ \overline{\rho} \geq 0\}. \tag{10.29}$$

By using the solution of (12.28), we can define the operator $[\Psi_{t_1,t_2}]_* : L^1(\mathbf{R}_q, m) \to L^1(\mathbf{R}_q, m)$ such that:

$$\Big([\Psi_{t_1,t_2}]_*(\overline{\rho}_{t_1})\Big)(q) = \overline{\rho}_{t_2}(q) = \int_{-\infty}^{\infty} \overline{\rho}_{t_1}(y) G_{t_2 - t_1}(q-y) m(dy), \quad (\forall (t_1, t_2) \in \mathbf{R}^2_{\leq}) \tag{10.30}$$

where $G_t(q)$ is the Gaussian function, that is, $G_t(q) = \frac{1}{\sqrt{2\pi t}} \exp\Big[- \frac{q^2}{2t} \Big]$. The "state's evolution" is, of course, represented by the Schrödinger picture $\{[\Psi_{t_1,t_2}]_* \mid (t_1, t_2) \in \mathbf{R}^2_{\leq}\}$.

For simplicity, we put $(\Omega, \mathcal{B}, d\omega) = (\mathbf{R}_q, \mathcal{B}(\mathbf{R}_q), m)$. And therefore, put $(\mathcal{N}, \mathcal{N}_*) = (L^\infty(\Omega), L^1(\Omega))$. Putting $\Psi_{t_1,t_2} = ([\Psi_{t_1,t_2}]_*)^*$, we get the Heisenberg picture $\{\Psi_{t_1,t_2} \mid (t_1, t_2) \in \mathbf{R}^2_{\leq}\}$, and consequently, the W^*-dynamical system $\overline{\mathbf{S}}(\overline{\rho}_0) \equiv [S(\overline{\rho}_0), \{\Psi_{t_1,t_2} : \mathcal{N} \to \mathcal{N}\}_{(t_1,t_2) \in \mathbf{R}^2_{\leq}}]$. Consider the state space observable $\overline{\mathbf{O}} \equiv (\Omega, \mathcal{B}, F)$ in $\mathcal{N}$ $\big(\equiv L^\infty(\Omega)\big)$ such as in Example 9.4.(i). Thus, by a similar way in the previous section, we get the measurement $\overline{\mathbf{M}}_{L^\infty(\Omega)}(\overline{\mathbf{O}}_{\overline{\mathbf{R}}^+}, S(\overline{\rho}_0))$. Assume that

- a measured value $\widehat{\omega}$ $(= (\omega_t)_{t \in \overline{\mathbf{R}}^+} \in \Omega^{\overline{\mathbf{R}}^+})$ is obtained by $\overline{\mathbf{M}}_{L^\infty(\Omega)}(\overline{\mathbf{O}}_{\overline{\mathbf{R}}^+}, S(\overline{\rho}_0))$.

Note that the measured value $\widehat{\omega}$ is precisely the "particle's trajectory" in Definition 10.2. Also, it may be usually called a "path".

By a similar way in the previous section, we shall investigate the properties of the measured value $\widehat{\omega}$ $(= (\omega_t)_{t \in \overline{\mathbf{R}}^+} \in \Omega^{\overline{\mathbf{R}}^+})$. Let $D = \{t_0, t_1, t_2, \cdots, t_n\}$ be a finite subset of $\overline{\mathbf{R}}^+$, where $t_0 = 0 < t_1 < t_2 < \cdots < t_n$. Put $\widehat{\Xi} = \times^D_{t \in \overline{\mathbf{R}}^+} \Xi_t$ $\big(\in \mathcal{B}^{\overline{\mathbf{R}}^+}\big)$ where $\Xi_t = \Omega$ $(\forall t \notin D)$. Then, by Proclaim$^{W^*}$2, we see

- the probability that $\widehat{\omega}(= (\omega_t)_{t \in \overline{\mathbf{R}}^+})$ belongs to the set $\widehat{\Xi} \equiv \times^D_{t \in \overline{\mathbf{R}}^+} \Xi_t$ is given by

$$\begin{aligned} &\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\widehat{\Xi})\big) = \overline{\rho}_0\Big(F(\Xi_0)\Psi_{0,t_1}\Big(F(\Xi_{t_1}) \cdots \Psi_{t_{n-2},t_{n-1}}\Big(F(\Xi_{t_{n-1}})\big(\Psi_{t_{n-1},t_n} F(\Xi_{t_n})\big)\Big) \cdots\Big) \\ = \ &\int_{\Xi_0} \overline{\rho}_0(\omega_0)\Big(\int_{\Xi_1}\Big(\cdots\big(\int_{\Xi_{t_{n-1}}}\big(\int_{\Xi_{t_n}} \prod_{k=1}^n G_{t_k - t_{k-1}}(\omega_k - \omega_{k-1}) d\omega_n\big) d\omega_{n-1}\big) \cdots\Big) d\omega_1\Big) d\omega_0. \end{aligned} \tag{10.31}$$

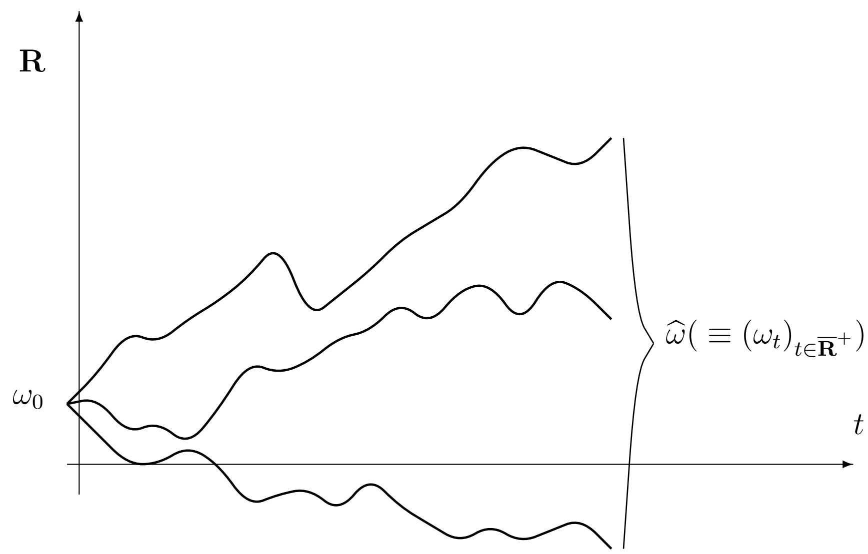
Let Ξ_0 be any element in $\mathcal{B}$ such that $\int_{\Xi_0} \overline{\rho}_0(\omega)d\omega \neq 0$. Suppose that we know that $\omega_0 \in \Xi_0$. i.e., $\widehat{\omega}(\equiv (\omega_t)_{t \in \overline{\mathbf{R}}^+}) \in \Xi_0 \times \Omega^{\mathbf{R}^+}$. Under the hypothesis, we can calculate the following conditional probability:

$$\frac{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\times^D_{t \in \overline{\mathbf{R}}^+}\Xi_t)\big)}{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\Xi_0 \times \Omega^{\mathbf{R}^+})\big)} = \frac{\int_{\Xi_0} \overline{\rho}_0(\omega_0) \Big(\int_{\Xi_{t_1}} \cdots \int_{\Xi_{t_n}} \prod_{k=1}^n G_{t_k - t_{k-1}}(\omega_k - \omega_{k-1}) d\omega_n \cdots d\omega_1\Big) d\omega_0}{\int_{\Xi_0} \overline{\rho}_0(\omega_0) d\omega_0}. \tag{10.32}$$

And therefore, we see that

$$\lim_{\Xi_0 \to \{\omega_0\}} \frac{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\times^D_{t \in \overline{\mathbf{R}}^+}\Xi_t)\big)}{\overline{\rho}_0\big(\widetilde{F}_{\overline{\mathbf{R}}^+}(\Xi_0 \times \Omega^{\mathbf{R}^+})\big)} = \int_{\Xi_{t_1}} \cdots \int_{\Xi_{t_n}} \prod_{k=1}^n G_{t_k - t_{k-1}}(\omega_k - \omega_{k-1}) d\omega_n \cdots d\omega_1. \tag{10.33}$$

Thus, under the hypothesis that we know that $\widehat{\omega}(\equiv (\omega_t)_{t \in \overline{\mathbf{R}}^+}) \in \{\omega_0\} \times \Omega^{\mathbf{R}^+}$, the measured value $\widehat{\omega}(\equiv (\omega_t)_{t \in \overline{\mathbf{R}}^+})$ has the property like Brownian motion with the initial value ω_0. Also note that the (10.33) is independent of $\overline{\rho}_0$.



Remark 10.4. [Complex system theory]. Here I shall mention my opinion for the relation between Brownian motions and "complex system theory" (or, "chaotic system theory") as follows:

[(i): Chaotic system theory]. It is a matter of course that Brownian motion is used to analyze stochastic phenomena (*cf.* [32]). It should be noted that Brownian motion is, from the computational point of view, generated by "pseudo-random number". And moreover, it should be noted that random number generator is regarded as a kind of chaotic equation (*cf.* [19]). In this sense, we consider, from the computational point of view, that Brownian motion analysis is regarded as a kind of chaotic equation. However, chaotic theory (or complex system theory, cf [87]) should not be overestimated as "the third physics (i.e., relativity theory, quantum mechanics, complex system theory)" [2]. Chaotic theory is not such a theory. This is easily seen if chaotic theory is investigated in the framework of MT (in which "probability" (related to Axiom 1) is never born from "equations" (related to Axiom 2), *cf.* Chapter 4 ("staying time interpretation" and not "probabilistic interpretation") and Remark 8.4 (Bertrand's paradox)).

[(ii): Information compression]. Newtonian mechanics may be regarded as a kind of "information compression". In fact, if we want to know the motion of particles, it suffices to solve the Newtonian kinetic differential equation. Also, it should be noted that the differential equation is, numerically, solved by iteration method (= "loop (in computer programming)"). Thus, there is a reason to think that an iteration (= "loop"), which is mainly related to Axiom 2, is regarded as a kind of information compression method of "analytic function", "pseudo-random number", "self-similar figure (Julia and Mandelbrot set)", etc. In other words, *any figure (or graph) treated in mathematical science is always generated by iteration.* Thus, we assert that MT is also a kind of information compression method. That is, mathematical science always has the aspect such as *"mathematical method of information compression."*

[(iii): Butterfly effect]. "Butterfly effect" is mentioned as follows:

(♯) *The flutter of a butterfly's wings in China could, in fact, actually effect weather patterns in New York City, thousands of miles away.*

It is impossible to test the above (♯). In this sense, we do not tell whether the (♯) is true or not. However, recall the spirit of the mechanical world view (1.12), i.e., *"at any rate, study every problem in the framework of MT".* Thus, if a certain differential equation

[2]This overestimation is like the proverb *"It's always darkest just beneath the lighthouse".* I have an opinion that Einstein's relativity theory, quantum mechanics and dynamical system theory (=DST(1.2)) are the most influential mathematical scientific theories in the 20th century, though DST is too familiar to us. The dropping of two atomic bombs (Einstein's relativity theory) is obviously one of the most tragic events in World War II. Also, Kalman filter (DST) and IC technology (quantum mechanics) lead the Apollo plan to success. This feat promoted the end of Cold War. And further, I think that this opinion is improved in this book (i.e., "quantum theory" + "DST" $\Longrightarrow$ "MT") and it is realized in Table (1.7), in which we may assert that "relativity theory (or, TOE)" $\leftrightarrow$ "the first physics", and "MT" $\leftrightarrow$"the second physics".

suggests the above fact (♯), we have to agree that there is a possibility that the above (♯) is true.

■

10.5 Conclusions

Summing up, we conclude (*cf.* [43]),

	state's evolution (≈Axiom 2)	particle's trajectory(sample space)
Newtonian mechanics	Liouville equation	Newton equation
quantum mechanics	Schrödinger equation	(meaningless)
diffusion process	diffusion equation	stochastic differential equation[3]

(10.34)

Thus there is a reason to say that the *state equation* in DST(1.2) should be called *"trajectory equation"*, though DST(1.2) is sometimes called *"state space method."* Therefore, in this book we say that DST(1.2) is the *"sample space method"*, in which the theory of differential equations and Kolmogorov's probability theory play essential roles.[4] Thus we can symbolically say:

$$\text{“MT”} \xleftarrow{\text{(our proposal)}} \underset{\text{(sample space method)}}{\text{“DST”} + \text{“statistics”}} \tag{10.35}$$

Here we have the following problem:

- Can we propose another mathematical scientific theory for data analysis? (*cf.* the third theory in Table (1.7))

I think that it is impossible to propose "the third theory" in mathematical science but computer science. Cf. Remark 1.5.

Also, recall we are not concerned with "Newtonian mechanics" in physics (which is represented in terms of differential geometry) but "Newtonian mechanics" in MT (which is represented in terms of operator algebra). Thus, it should be noted that our viewpoint (proposed in this book) is, of course, one-sided.

[3]Recall (1.2). It should be noted that the stochastic state equation (= stochastic differential equation) in (1.2) is not "state equation" but "trajectory equation (i.e., the equation that represents particle's trajectory)".

[4]I believe that "Kolmogorov's probability space" is essentially the same as "the sample space in MT".