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Chapter 9

Statistical measurements in W^*-algebraic formulation

The Statistical MT (= SMT) has two kinds of formulations. One is SMT^{C^*} (i.e., the C^*-algebraic formulation of SMT), which was introduced in the previous chapter, that is,

$$\mathrm{SMT}^{C^*} = \underset{[\text{Proclaim 1 (8.10)}]}{\text{statistical measurement}} + \underset{[\text{Axiom 2 (3.26)}]}{\text{the relation among systems}} \quad \text{in } C^*\text{-algebra.} \tag{9.1}$$
$$(=(8.2))$$

In this chapter we introduce another formulation of SMT (i.e., SMT^{W^*}), that is,

$$\mathrm{SMT}^{W^*} = \underset{[\text{Proclaim}^{W^*}1\ (9.9)]}{\text{statistical measurement}} + \underset{[\text{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} \quad \text{in } W^*\text{-algebra,} \tag{9.2}$$

which is called the W^*-algebraic formulation of SMT. Of course, "SMT^{C^*}" and "SMT^{W^*}" are essentially the same. The difference between the two is that of the mathematical tools (i.e., C^*-algebra and W^*-algebra). Thus, "SMT^{W^*}" should be understood by an analogy of "SMT^{C^*}". Although the C^*-algebraic formulation is most fundamental, the W^*-algebraic formulation is rather handy from the mathematical point of view.

9.1 Statistical measurements (W^*-algebraic formulation)

The Statistical MT (= SMT) has two kinds of formulations. One is the C^*-algebraic formulation of SMT (= SMT^{C^*}), which was introduced in the previous chapter. In order to develop "Statistical MT", in this chapter we introduce the W^*-algebraic formulation of Statistical MT (= SMT^{W^*}).[1] Here, it should be noted that "SMT^{C^*}" and "SMT^{W^*}" are

[1] Of course, the (pure) measurement theory (= PMT) has also two kinds of formulations, i.e., PMT^{C^*} and PMT^{W^*}. However, the commutative PMT^{W^*} has a demerit such that a pure state can not be represented in the commutative PMT^{W^*} in general. (*cf.* the statement (9.3)). Thus, we usually focus on SMT^{W^*} and not PMT^{W^*}. However, it should be noted that as far as quantum mechanics, PMT^{W^*} is superior to PMT^{C^*}. Cf. §9.3.

essentially the same. The difference between the two is that of the mathematical tools (i.e., C^*-algebra and W^*-algebra).

The C^*-algebraic formulation stated in the previous chapter is, of course, most fundamental. However, from the mathematical (or technical) point of view, the topology of a C^*-algebra $\mathcal{A}$ is somewhat too strong. Note that any C^*-algebra $\mathcal{A}$ can be imbedded into $B(V)$, the algebra composed of all bounded linear operators on a Hilbert space V (*cf.* Theorem 2.4 (the GNS-construction in [50, 76, 82])). Thus, using the imbedding: $\mathcal{A} \subseteq B(V)$, we may start from the weak*-closure $\overline{\mathcal{A}}$ (of $\mathcal{A}$) in $B(V)$. This $\overline{\mathcal{A}}$ is called a W^*-*algebra.* This method (i.e., to formulate measurement theory in terms of W^*-algebras) is called the W^*-algebraic formulation. Though this method is somewhat methodological, it is rather handy from the mathematical point of view. $\Big($For example, this will be seen in Theorem 10.1 in Chapter 10.$\Big)$

Let $\mathcal{N}$ be a W^*-algebra, that is,

[$\sharp_1$] $\mathcal{N}$ is a weak* closed subalgebra of a certain $B(V)$.

It is well known (see, for example, [76]) that this is equivalent to

[$\sharp_2$] $\mathcal{N}$ is a C^*-algebra with the pre-dual Banach space $\mathcal{N}_*$ (i.e., $\mathcal{N} = (\mathcal{N}_*)^*$).

Also, it is well known that the uniqueness of the pre-dual Banach space $\mathcal{N}_*$ is assured. However, we may sometimes call the pair $(\mathcal{N}, \mathcal{N}_*)$ a W^*-algebra.

An element F in $\mathcal{N}$ is called *self-adjoint* if it holds that $F = F^*$. A self-adjoint element F in $\mathcal{N}$ is called *positive* (and denoted by $F \geq 0$) if there exists an element F_0 in $\mathcal{N}$ such that $F = F_0^* F_0$ where F_0^* is the adjoint element of F_0. Also, a positive element F is called a *projection* if $F = F^2$ holds.

Now we can define the *normal state-class* $\mathfrak{S}^n(\mathcal{N}_*)$ such as

$$\mathfrak{S}^n(\mathcal{N}_*) \equiv \{\rho^n \in \mathcal{N}_* \ : \ \|\rho^n\|_{\mathcal{N}_*} = 1 \text{ and } \rho^n \geq 0 \text{ (i.e., } \rho^n(T^*T) \geq 0 \text{ for all } T \in \mathcal{N})\}.$$

The element ρ^n (in $\mathfrak{S}^n(\mathcal{N}_*)$) is called a *normal state* (or, *density state*). The linear functional $\rho^n(T)$ is sometimes denoted by $\Big\langle \rho^n, T \Big\rangle$, or precisely, ${}_{\mathcal{N}_*}\Big\langle \rho^n, T \Big\rangle_{\mathcal{N}}$. Also, note that

- a W^*-algebra $\mathcal{N}$ has a lot of projections,

that is, the set of all finite linear combinations of projections is dense in $\mathcal{N}$ in the weak* topology $\sigma(\mathcal{N}; \mathcal{N}_*)$. Also, note that

- $\mathcal{N}$ has always the identity $I_{\mathcal{N}}$.

Example 9.1. [(i): Commutative W^*-algebras ; $L^\infty(\Omega,\mu)$)]. Let $(\Omega,\mathcal{B}_\Omega,\mu)$ be a measure space. For any $1 \le p \le \infty$, define $L^p(\Omega,\mu)$ $\Big(\equiv L^p(\Omega,\mathcal{B}_\Omega,\mu)\Big) = \{f : f$ is a complex valued measurable function such that $\|f\|_{L^p} \equiv [\int_\Omega |f(\omega)|^p \mu(d\omega)]^{1/p} < \infty$ $\}$. $\Big($Here, of course, $\|f\|_{L^\infty} = \text{ess.sup}\ \{|f(\omega)| \ : \ \omega \in \Omega\}.\Big)$ Then, the $\mathcal{N} \equiv L^\infty(\Omega,\mu)$ is a commutative W^*-algebra with the pre-dual Banach space $\mathcal{N}_* = L^1(\Omega,\mu)$. We see, of course, that $\mathfrak{S}^n(\mathcal{N}_*) = L^1_{+1}(\Omega,\mu) \equiv \{\rho^n \in L^1(\Omega,\mu) : \rho^n \ge 0, \int_\Omega \rho^n(\omega)\mu(d\omega) = 1$, i.e., ρ^n is a density function on Ω $\}$. Also, it is well known that any commutative W^*-algebra $\mathcal{N}$ is represented by some $L^\infty(\Omega,\mu)$. It should be noted that

- a "pure state" can not be generally represented in terms of the commutative W^*-algebra $L^\infty(\Omega,\mu)$, (9.3)

since we see[2] that δ_{ω_0} (i.e., a point measure at ω_0 ($\in \Omega$)) does not necessarily belong to $L^1(\Omega,\mu)$. Summing up (and recalling Example 2.2), we see,

	commutative C^*-algebra	commutative W^*-algebra
concrete form	$C(\Omega)$	$L^\infty(\Omega;\mu)$
dual space	$\mathcal{M}(\Omega)$ ($= C(\Omega)^*$)	not important
pre-dual space	nothing	$L^1(\Omega;\mu)$ ($= L^\infty(\Omega;\mu)_*$)
pure state	$\delta_{\omega_0} \in \mathcal{M}^p_{+1}(\Omega) \approx \Omega$	(no representation in general)
mixed (normal) state	$\nu \in \mathcal{M}^m_{+1}(\Omega)$	$\overline{\rho} \in L^1_{+1}(\Omega,\mu)$
characteristics[3]	topological approach	measure theoretical approach

[(ii): The case that Ω is countable or finite]. Of course, the above table is in the case that Ω is general. In the case that $\Omega \equiv \{\omega_1,\omega_2,...,\omega_n\}$ is finite, we can easily see that "commutative C^*-algebra" = "commutative W^*-algebra", that is, we see the following identifications:

$$C(\{\omega_1,\omega_2,...,\omega_n\}) \approx \mathbf{C}^n (cf.\ \text{the formula (2.15)}) \approx L^\infty(\{\omega_1,\omega_2,...,\omega_n\},\mu) \tag{9.4}$$

where μ is a measure such that $\mu(\{\omega_k\}) > 0$ $(\forall k = 1,2,...,n)$. Next consider the case that $\Omega \equiv \{\omega_1,\omega_2,...,\omega_k,...\}$ is countable infinite. The commutative W^*-algebra $\mathcal{N}$ is defined by

[2]In this sense the W^*-algebraic formulation is fit to SMT rather than PMT. However note our spirit (8.12) : *"There is no SMT without PMT".* Thus we think that PMT (i.e., the concept of "pure state") is not only hidden in the C^*-algebraic formulation of SMT but also in the W^*-algebraic formulation of SMT.

[3]The Ω in $C(\Omega)$ is a topological space. On the other hand, the Ω in $L^\infty(\Omega;\mu)$ is a measure space. Cf. Remark 9.14 later.

$L^\infty(\Omega, \mu)$, where $\mu(\{\omega_k\}) > 0$ $(\forall k = 1, 2, ...)$. In this case, a pure state ρ_{ω_k} $(k = 1, 2, ...)$, is defined by $\rho_{\omega_k}(\omega) = \frac{1}{\mu(\{\omega_k\})}$ (if $\omega = \omega_k$), $= 0$ (if $\omega \neq \omega_k$).

■

Example 9.2. [Non-commutative W^*-algebras; $B(V)$]. When $\mathcal{N} = B(V)$, we see that $\mathcal{N}_* = Tr(V)$ (*cf.* Example 2.3) and $\mathfrak{S}^n(\mathcal{N}_*) = Tr^m_{+1}(V) \equiv \{\rho^n \in Tr(V) : \rho^n \geq 0, \|\rho^n\|_{Tr(V)} = 1\}$. Also, note that ${}_{Tr(V)}\left\langle \rho^n, T \right\rangle_{B(V)} = tr[\rho^n \cdot T]_V$. Here, $tr[A]_V \equiv \sum_{\lambda\in\Lambda} \langle e_\lambda, Ae_\lambda\rangle_V$ where $\{e_\lambda | \lambda \in \Lambda\}$ is a complete orthonormal basis in V. Also, it is well known that the value $tr[A]_V$ is independent of the choice of a complete orthonormal basis $\{e_\lambda | \lambda \in \Lambda\}$ in V. Further, any ρ^n ($\in Tr^m_{+1}(V)$) is represented by $\rho^n = \sum_{\lambda\in\lambda} \alpha_\lambda |e_\lambda\rangle\langle e_\lambda|$ (in the trace norm $\|\cdot\|_{Tr(V)}$) for some complete orthonormal basis $\{e_\lambda | \lambda \in \Lambda\}$ in V and some sequence $\{a_\lambda\}_{\lambda\in\Lambda}$ of non-negative numbers such that $\sum_{\lambda\in\Lambda} a_\lambda = 1$. Also it should be noted that any $|v\rangle\langle v|$, $(\|v\| = 1)$, is just a pure state[4]. Summing up (and recalling Example 2.3), we see,

	non-commutative C^*-algebra	non-commutative W^*-algebra
concrete form	$\mathcal{C}(V)$	$B(V)$
dual space	$Tr(V)$ ($= \mathcal{C}(V)^*$)	not important
pre-dual space	nothing	$Tr(V)$ ($= B(V)_*$)
pure state	$\|v\rangle\langle v\| \in Tr^p_{+1}(V)$	$\|v\rangle\langle v\| \in Tr^p_{+1}(V)$
mixed (normal) state	mixed state: $\rho^m \in Tr^m_{+1}(V)$	normal state: $\rho^n \in Tr^m_{+1}(V)$

(9.5)

■

The following definition is the W^*-algebraic form of Definition 2.7 (C^*-observables).

Definition 9.3. [W^*-observables]. *Let $\mathcal{N}$ be a W^*-algebra. A W^*-observable (or in short, observable, fuzzy observable) $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ in $\mathcal{N}$ is defined such that it satisfies that*

(i) [*σ-field*]. *$(X, \mathcal{F})$ is a measurable space, that is, $\mathcal{F}$ ($\subseteq 2^X$) is a σ-field on X, i.e., it satisfies that*

$$\emptyset \in \mathcal{F}, \quad \Xi_k \in \mathcal{F}\ (k = 1, 2, ...) \Longrightarrow \cup_{k=1}^\infty \Xi_k \in \mathcal{F}, \quad \Xi \in \mathcal{F} \Longrightarrow \Xi^c \in \mathcal{F},$$

[4]This fact (i.e., a pure state can be represented in terms of W^*-algebra $B(V)$) is remarkable. Thus, The W^*-algebra $B(V)$ has a power to describe quantum PMT as well as quantum SMT. Cf. §9.4.

(ii) *for every* $\Xi \in \mathcal{F}$, $F(\Xi)$ *is a positive element in* $\mathcal{N}$ *(i.e.,* $0 \leq F(\Xi) \in \mathcal{N}$*) such that* $F(\emptyset) = 0$ *and* $F(X) = I_{\mathcal{N}}$, *where* 0 *is the* 0*-element and* $I_{\mathcal{N}}$ *is the identity element in* $\mathcal{N}$, *and,*

(iii) [*countably additivity*]. *For any countable decomposition* $\{\Xi_1, \Xi_2, ..., \Xi_j, ...\}$ *of* Ξ, $\Big($*i.e.,* $\Xi, \Xi_j \in \mathcal{F}, \cup_{j=1}^{\infty} \Xi_j = \Xi$, $\Xi_j \cap \Xi_i = \emptyset($ *if* $j \neq i)\Big)$, *it holds that*

$$F(\Xi) = \sum_{j=1}^{\infty} F(\Xi_j)$$

where the series is convergent in the sense of the weak**-topology* $\sigma(\mathcal{N}; \mathcal{N}_*)$ *in* $\mathcal{N}$.

If $F(\Xi)$ *is a projection for every* Ξ ($\in \mathcal{F}$), *a* W^**-observable* $(X, \mathcal{F}, F)$ *in* $\mathcal{N}$ *is called a crisp* W^**-observable in* $\mathcal{N}$. *Also, a crisp observable* $\overline{\mathbf{O}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F)$ *(or,* $(\mathbf{R}^n, \mathcal{B}_{\mathbf{R}^n}, F)$*) in* $\mathcal{N}$ *is called a quantity* (or, $\mathbf{R}^n$*-valued quantity*) in W^*-algebra $\mathcal{N}$.

■

Now we show several W^*-observables (in Example 9.4 $\sim$ 9.7).

Example 9.4. [Crisp W^*-observables in $L^\infty(\Omega, \mu)$]. (i). As a typical crisp W^*-observable in $L^\infty(\Omega, \mu)$, the *exact observable* $\overline{\mathbf{O}}_{\text{EXA}} \equiv (\Omega, \mathcal{B}_\Omega, \chi_{(\cdot)})$ is frequently used where χ_Ξ is the characteristic function of Ξ ($\in \mathcal{B}_\Omega$) $\Big($i.e., $\chi_\Xi(\omega) = 1(\omega \in \Xi), = 0$ (otherwise)$\Big)$. This observable is finest in $L^\infty(\Omega, \mu)$, i.e., it includes all projections.
(ii). Consider the commutative W^*-algebra $L^\infty(\Omega, \mu)$. Let $a : \Omega \to \mathbf{R}$ be a measurable function. Then, we can define the crisp W^*-observable $\overline{\mathbf{O}}_a = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F)$ in $L^\infty(\Omega, \mu)$ such that $[F(\Xi)](\omega) = \chi_{a^{-1}(\Xi)}(\omega)$ $(\forall \Xi \in \mathcal{B}_{\mathbf{R}}, \forall \omega \in \Omega)$. Note that we can identify the real-valued measurable function $a(\,\cdot\,)$ with the $\overline{\mathbf{O}}_a$. That is, we see

$$\underset{\text{(real valued measurable function on } \Omega)}{a : \Omega \to \mathbf{R}} \quad \longleftrightarrow \quad \underset{\text{(crisp observable)}}{(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F)} \quad \text{in } L^\infty(\Omega, \mu).$$

That is because it holds that $[F((-\infty, \lambda))](\omega) = 0$ (if $\lambda < a(\omega)$), $= 1$ (if $\lambda \geq a(\omega)$), and therefore, the $a(\omega)$ is determined by the equality $a(\omega) = \int_{\mathbf{R}} \lambda \delta_{a(\omega)}(d\lambda) = \int_{\mathbf{R}} \lambda [F(d\lambda)](\omega)$ (a.e. μ). A real-valued measurable function on Ω is called a (*classical*) *quantity* in $L^\infty(\Omega, \mu)$ (though it is not always a bounded function).

■

Example 9.5. [Gaussian W^*-observable]. Define the W^*-observable $\overline{\mathbf{O}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F_{(\cdot)})$ in $\mathcal{N} \equiv L^\infty(\mathbf{R}, d\omega)$ such that:

$$F^{\sigma}_{\Xi}(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\Xi} e^{-\frac{(u-\omega)^2}{2\sigma^2}} du \qquad (\forall\omega \in \mathbf{R},\ \forall\Xi \in \mathcal{B}_{\mathbf{R}}). \quad (\sigma^2\text{: variance}).$$

This is, of course, the W^*-algebraic form of the Gaussian C^*-observable $\mathbf{O} \equiv (\mathbf{R}, \mathcal{B}^{\mathrm{bd}}_{\mathbf{R}}, F_{(\cdot)})$ (*cf.* Example 2.17). Note that the $\mathcal{B}_{\mathbf{R}}$ in $\overline{\mathbf{O}}$ is a σ-field, and the $\mathcal{B}^{\mathrm{bd}}_{\mathbf{R}}$ in $\mathbf{O}$ is a σ-ring.

■

Remark 9.6. [The vagueness of a crisp observable]. Let ν be a probability measure on an index set Θ. For each θ ($\in \Theta$), consider a crisp observable $\overline{\mathbf{O}}_\theta \equiv (X, \mathcal{F}, E_\theta)$ in W^*-algebra $\mathcal{N}$. Define the observable $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$ in W^*-algebra $\mathcal{N}$ such that:

$$F(\Xi) = \int_{\Theta} E_\theta(\Xi)\nu(d\theta) \qquad (\forall\Xi \in \mathcal{F})$$

which is not crisp but fuzzy in general. Thus we think that

(F) "fuzzy observable"$\Longleftrightarrow$ "To understand a dearth of information concerning a crisp observable by a fuzzy observable".

This is one of the aspects of "fuzzy observable". When we want to stress this statistical aspect, the "observable" is often called a *"fuzzy observable"* (or, *"random observable"*). This will be again discussed in §11.4.

■

Example 9.7. [(i): Crisp W^*-observables in quantum $B(V)$]. Here, consider the quantum version of the (ii) in Example 9.4. Let A be a self-adjoint operator (not necessarily bounded) on a Hilbert space V. Recall the spectral representation: $A = \int_{\mathbf{R}} \lambda E_A(d\lambda)$. Here, the spectral measure $\overline{\mathbf{O}}_A \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E_A)$ is of course the crisp W^*-observable in $B(V)$. Conversely, any crisp W^*-observable $(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F)$ in $B(V)$ determines a unique self-adjoint operator A_F on V such that $A_F = \int_{\mathbf{R}} \lambda F(d\lambda)$. Therefore, we have the identification:

$$\underset{\text{(self-adjoint operator on } V)}{A} \longleftrightarrow \underset{\text{(crisp observable)}}{\overline{\mathbf{O}}_A = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, F)} \text{ in } B(V) \qquad \left(\text{i.e., } A = \int_{\mathbf{R}} \lambda F(d\lambda)\right).$$

A self-adjoint operator A on a Hilbert space V is called a (*unbounded*) *quantity* in $B(V)$ (though A is not always a bounded linear operator).

[(ii): Position quantity, momentum quantity]. Put $V \equiv L^2(\mathbf{R}; dq))$, and define the (unbounded) self-adjoint operator Q [resp. P], which is called the *position quantity* [resp. *momentum quantity*], such that:

$$(Q\psi)(q) = q \cdot \psi(q), \qquad \left[\text{ resp. } \quad (P\psi)(q) = -i\frac{\hbar d\psi(q)}{dq}\right].$$

By the following spectral representations,

$$Q = \int_{\mathbf{R}} \lambda E_Q(d\lambda) \qquad \text{and} \qquad P = \int_{\mathbf{R}} \lambda E_P(d\lambda),$$

we see the following identifications:

$$\underset{\text{(self-adjoint operator on } V)}{Q} \longleftrightarrow \underset{\text{(crisp observable)}}{\overline{\mathbf{O}}_Q = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E_Q) \text{ in } B(V)}$$

and

$$\underset{\text{(self-adjoint operator on } V)}{P} \longleftrightarrow \underset{\text{(crisp observable)}}{\overline{\mathbf{O}}_P = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E_P) \text{ in } B(V)}.$$

Here note that

$$[E_Q(\Xi)\psi](q) = \chi_\Xi(q) \cdot \psi(q) \qquad (\forall \psi \in L^2(\mathbf{R}), \forall \Xi \in \mathcal{B}_{\mathbf{R}}, q \in \mathbf{R})$$

and

$$E_P(\Xi)\psi = \mathfrak{F}^*(\chi_\Xi \cdot (\mathfrak{F}\psi)) \qquad (\forall \psi \in L^2(\mathbf{R}),\ \forall \Xi \in \mathcal{B}_{\mathbf{R}}, q \in \mathbf{R}),$$

where the Fourier transform $\mathfrak{F} : L^2(\mathbf{R}, dx) \to L^2(\mathbf{R}, dy)$ is defined by

$$(\mathfrak{F}\psi)(y) = \sqrt{\frac{\hbar}{2\pi}} \int_{\mathbf{R}} \psi(x) e^{-i\hbar xy} dx.$$

Note that both the position observable and momentum observable, which are most important in quantum mechanics, can not be defined in the C^*-algebraic formulation.

[(iii): Glauber-Sudarshan representation]. Consider ψ_0 ($\in V \equiv L^2(\mathbf{R}; dq)$) such that $\|\psi_0\|_{L^2(\mathbf{R})} = 1$ and

$$\langle \psi_0, P\psi_0 \rangle_V = 0, \qquad \langle \psi_0, Q\psi_0 \rangle_V = 0.$$

If we define $\phi_{x,y}(q) = e^{ixy}\psi_0(q - x)$, then an elementary computation shows that

$$\langle P\phi_{x,y}, \phi_{x,y} \rangle_{L^2(\mathbf{R})} = y, \quad \langle Q\phi_{x,y}, \phi_{x,y} \rangle_{L^2(\mathbf{R})} = x. \tag{9.6}$$

Here we can define the W^*-observable $(\mathbf{R}^2, \mathcal{B}_{\mathbf{R}^2}, G)$ in $B(L^2(\mathbf{R}; dq))$ such that:

$$G(\Xi) = \iint_\Xi |\phi_{x,y}\rangle\langle\phi_{x,y}| dxdy \qquad (\forall \Xi \in \mathcal{B}_{\mathbf{R}^2}).$$

This observable is essential in semi-classical mechanics (*cf.* [34]).

■

The following theorem is the W^*-algebraic form of Theorem 2.13. Since W^*-algebra $\mathcal{N}$ has a lot of projections, it is much more useful than Theorem 2.13.

Theorem 9.8. [The W^*-algebraic form of Theorem 2.13, *cf.* [42]]. *Let $\mathcal{N}$ be a W^*-algebra. Let $\overline{\mathbf{O}}_1 \equiv (X_1, \mathcal{F}_1, F_1)$ and $\overline{\mathbf{O}}_2 \equiv (X_2, \mathcal{F}_2, F_2)$ be W^*-observables in $\mathcal{N}$ such that at least one of them is crisp. $\Big($So, without loss of generality, we assume that $\overline{\mathbf{O}}_2$ is crisp$\Big)$. Then, the following statements are equivalent:*

(i) *There exists a quasi-product observable $\overline{\mathbf{O}}_{12} \equiv (X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2, F_1 \overset{\text{qp}}{\times} F_2)$ with marginal observables $\overline{\mathbf{O}}_1$ and $\overline{\mathbf{O}}_2$.*

(ii) *$\overline{\mathbf{O}}_1$ and $\overline{\mathbf{O}}_2$ commute, that is, $F_1(\Xi_1)F_2(\Xi_2) = F_2(\Xi_2)F_1(\Xi_1)$ $(\forall \Xi_1 \in \mathcal{F}_1, \forall \Xi_2 \in \mathcal{F}_2)$.*

Furthermore, if the above statements (i) and (ii) hold, the uniqueness of $\overline{\mathbf{O}}_{12}$ is guaranteed. (So, we can write that $\overline{\mathbf{O}}_{12} = (X_1 \times X_2,\ \mathcal{F}_1 \times \mathcal{F}_2,\ F_1 \times F_2) = \overline{\mathbf{O}}_1 \times \overline{\mathbf{O}}_2$.)

Proof. The proof is essentially the same as that of Theorem 2.13. □

The purpose of this chapter is to propose the W^*-algebraic formulation of SMT, that is,

$$\mathrm{SMT}^{W^*} = \underset{[\mathrm{Proclaim}^{W^*}1\ (9.9)]}{\text{statistical measurement}} + \underset{[\mathrm{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} \qquad \text{in } W^*\text{-algebra}\ . \tag{9.7}$$
$$(=(9.2))$$

In order to do it, we must recall the C^*-algebraic formulation of SMT, that is,

$$\mathrm{SMT}^{C^*} = \underset{[\mathrm{Proclaim}\ 1\ (8.10)]}{\text{statistical measurement}} + \underset{[\mathrm{Axiom}\ 2\ (3.26)]}{\text{the relation among systems}} \qquad \text{in } C^*\text{-algebra}\ . \tag{9.8}$$
$$(=(9.1))$$

As mentioned before, we want to understand SMT^{W^*} by an analogy of SMT^{C^*}. Here, it should be recalled that

- [Proclaim 1 (8.10), (The probabilistic interpretation of mixed states)]. *Consider a statistical measurement $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O} \equiv (X, \mathcal{F}, F),\ S(\rho^m)\big)$ formulated in a C^*-algebra $\mathcal{A}$. Then, the probability that x ($\in X$), the measured value obtained by the statistical measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S(\rho^m)\ )$, belongs to a set Ξ ($\in \mathcal{F}$) is given by*

$$\rho^m(F(\Xi))\ \Big(\equiv {}_{\mathcal{A}^*}\Big\langle \rho^m, F(\Xi)\Big\rangle_{\mathcal{A}}\Big).$$

By an analogy of Proclaim 1, we can propose Proclaim$^{W^*}$1 as follows: Cf [44].

PROCLAIM$^{W^*}$ 1. [Statistical measurements in the W^*-algebraic formulation]. *Consider a statistical measurement* $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F), \overline{S}(\rho^n))$ *formulated in a* W^**-algebra* $\mathcal{N}$. *The probability that* x $(\in X)$, *the measured value obtained by the statistical measurement* $\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F), \overline{S}(\rho^n))$, *belongs to* Ξ $(\in \mathcal{F})$ *is given by*

$$\rho^n(F(\Xi)) \left(\equiv {}_{\mathcal{N}_*}\Big\langle \rho^n, F(\Xi) \Big\rangle_{\mathcal{N}} \right). \tag{9.9}$$

This will be easily read by the above [Proclaim 1] and the following [TABLE (Statistical measurement theory)].

Statistical measurement theory (9.10)

[C^*-algebraic formulation]	$\longleftrightarrow$	[W^*-algebraic formulation]
Proclaim 1 (8.10)	$\longleftrightarrow$	Proclaim$^{W^*}$1 (9.9)
$\mathfrak{S}^m(\mathcal{A}^*) \ni \rho^m$	$\longleftrightarrow$	$\rho^n \in \mathfrak{S}^n(\mathcal{N}_*)$
C^*-observable $\mathbf{O} \equiv (X, \mathcal{F}, F)$	$\longleftrightarrow$	W^*-observable $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$
$\mathbf{M}_{\mathcal{A}}(\mathbf{O} \equiv (X, \mathcal{F}, F), S(\rho^m))$	$\longleftrightarrow$	$\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F), \overline{S}(\rho^n))$

Remark 9.9. [The W^*-algebraic formulation of PMT]. Though the commutative PMT$^{W^*}$ has a demerit such that a pure state can not be represented in the commutative PMT$^{W^*}$ in general (*cf.* the statement (9.3)), a pure state can be represented in the non-commutative PMT$^{W^*}$ (i.e., in $B(V)$, *cf.* Example 9.2). Thus, it is worthwhile mentioning the following Axiom$^{W^*}$1 (9.11). If $\mathcal{N} = B(V)$ or $\mathcal{N} = L^\infty(\Omega, \mu)$ (where Ω is finite or countable infinite), the concept "pure state" is valid (*cf.* (9.4) and (9.5)). Thus, in this case, we can propose "Axiom$^{W^*}$1 (9.11)" (i.e., the W^*-algebraic formulation of Axiom 1) as follows:

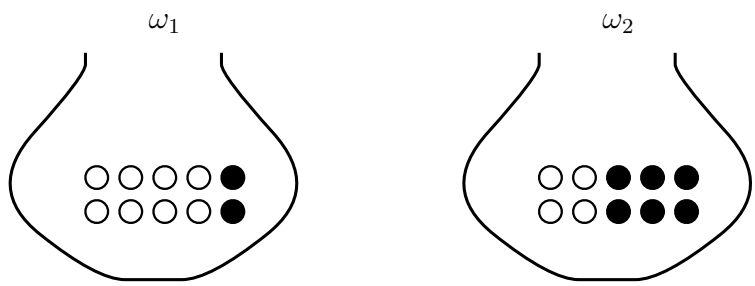
AXIOM$^{W^*}$1. [The W^*-algebraic formulation of Axiom 1]. *Consider a measurement* $\overline{\mathbf{M}}_{\mathcal{N}}\big(\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F), \overline{S}_{[\rho^p]}\big)$ *formulated in a W^*-algebra $\mathcal{N}$, where ρ^p is a pure state. Assume that the measured value x ($\in X$) is obtained by the measurement $\overline{\mathbf{M}}_{\mathcal{N}}\big(\overline{\mathbf{O}}, \overline{S}_{[\rho^p]}\big)$. Then, the probability that the x ($\in X$) belongs to a set Ξ ($\in \mathcal{F}$) is given by $\rho^p(F(\Xi))$ $\big($ $\equiv$ ${}_{\mathcal{N}_*}\Big\langle \rho^p, F(\Xi)\Big\rangle_{\mathcal{N}}$ $\big)$.* (9.11)

■

In the following example, we see that the C^*-algebraic formulation and the W^*-algebraic formulation are essentially the same.

Example 9.10. [(i): The review of Example 8.1] . There are two urns ω_1 and ω_2. The urn ω_1 [resp. ω_2] contains 8 white and 2 black balls [resp. 4 white and 6 black balls]. We regard Ω $\big($ $\equiv \{\omega_1, \omega_2\}\big)$ as the state space. And consider the observable $\mathbf{O}\big(\equiv (X \equiv \{w, b\}, 2^{\{w,b\}}, F)\big)$ in $C(\Omega)$ where

$$[F(\{w\})](\omega_1) = 0.8, \qquad [F(\{b\})](\omega_1) = 0.2,$$
$$[F(\{w\})](\omega_2) = 0.4, \qquad [F(\{b\})](\omega_2) = 0.6.$$



Here consider the following procedures (P_1) and (P_2).

(P_1) One of the two (i.e., ω_1 or ω_2) is chosen by an unfair tossed-coin ($C_{p,1-p}$), i.e.,

$$\text{Head } (100p\%) \to \omega_1, \text{ Tail } (100(1-p)\%) \to \omega_2 \qquad (0 \le p \le 1).$$

The chosen urn is denoted by $[*](\in \{\omega_1, \omega_2\})$. Note, for completeness, that we do not know whether $[*]$ is ω_1 or ω_2. Here define the mixed state $\nu_0(\in \mathcal{M}^m_{+1}(\Omega))$ such that $\nu_0(\{\omega_1\}) = p$, $\nu_0(\{\omega_2\}) = 1-p$, which is considered to be *"the distribution of* $[*]$*."*

(P_2) Take one ball, at random, out of the urn chosen by the procedure (P_1). $\Big($That is, we take the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]})$.$\Big)$

[(ii): Continued from the above (i): C^*-algebraic formulation]. As seen in Example 8.1,

- "$(P_1) + (P_2)$" is notated by $\mathbf{M}_{C(\Omega)}(\mathbf{O} \equiv (X, 2^X, F), S(\nu_0))$.

Of course, we see

- the probability that the measured value x ($\in \{w, b\}$) is obtained by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_0))$, is given by

$$
\begin{aligned}
&{}_{C(\Omega)^*}\Big\langle \nu_0, F(\{x\})\Big\rangle_{C(\Omega)} \Big(\equiv \int_\Omega {}_{C(\Omega)^*}\Big\langle \delta_\omega, F(\{x\})\Big\rangle_{C(\Omega)} \nu_0(d\omega)\Big) \\
&= \begin{cases} 0.8p + 0.4(1-p) & (\text{ if } x = w), \\ 0.2p + 0.6(1-p)) & (\text{ if } x = b). \end{cases}
\end{aligned} \tag{9.12}
$$

[(iii): Continued from the above (i): W^*-algebraic formulation]. Define the measure μ on Ω, for example, such that

$$
\mu(\{\omega_1\}) = \mu(\{\omega_2\}) = 1.
$$

Thus we have the commutative W^*-algebra $L^\infty(\Omega, \mu)$. And consider the observable $\overline{\mathbf{O}}\big(\equiv (X \equiv \{w, b\}, 2^{\{w,b\}}, F)\big)$ in $L^\infty(\Omega, \mu)$ where

$$
\begin{aligned}
&[F(\{w\})](\omega_1) = 0.8, && [F(\{b\})](\omega_1) = 0.2, \\
&[F(\{w\})](\omega_2) = 0.4, && [F(\{b\})](\omega_2) = 0.6.
\end{aligned}
$$

Also define the normal state ρ^n ($\in L^1_{+1}(\Omega, \mu)$) such that[5]

$$
\rho^n(\omega_1) = p, \qquad \rho^n(\omega_2) = 1 - p.
$$

Then, we have the W^*-measurement $\overline{\mathbf{M}}_{L^\infty(\Omega,\mu)}(\overline{\mathbf{O}}, \overline{S}(\rho^n))$ in $L^\infty(\Omega, \mu)$. Of course, we see,

- the probability that the measured value x ($\in \{w, b\}$) is obtained by the measurement $\overline{\mathbf{M}}_{L^\infty(\Omega,\mu)}(\overline{\mathbf{O}}, \overline{S}(\rho^n))$, is given by

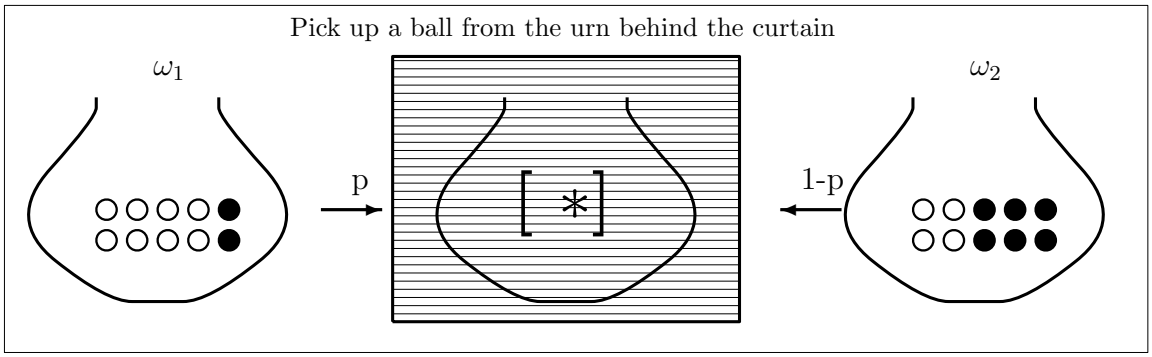
$$
\begin{aligned}
&{}_{L^1(\Omega,\mu)}\Big\langle \rho^n, F(\{x\})\Big\rangle_{L^\infty(\Omega,\mu)} \Big(\equiv \int_\Omega [F(\{x\})](\omega)\rho^n(\omega)\mu(d\omega)\Big) \\
&= \begin{cases} 0.8p + 0.4(1-p) & (\text{ if } x = w), \\ 0.2p + 0.6(1-p)) & (\text{ if } x = b). \end{cases}
\end{aligned} \tag{9.13}
$$

[5]Note that μ is arbitrary (*cf.* the formula (9.4)). If $\mu(\{\omega_1\}) = 1/3$ and $\mu(\{\omega_2\}) = 2$, it suffices to define that $\rho^n(\omega_1) = 3p$ and $\rho^n(\omega_1) = (1-p)/2$.

Thus we see that $\mathbf{M}_{C(\Omega)}(\mathbf{O}, S_{[*]}(\nu_0))$ and $\overline{\mathbf{M}}_{L^\infty(\Omega,\mu)}(\overline{\mathbf{O}}, \overline{S}(\rho^n))$ are essentially the same (*cf.* (9.12) and (9.13)).

Also, we see:

The illustration of $\overline{\mathbf{M}}_{L^\infty(\Omega,\mu)}(\overline{\mathbf{O}}, \overline{S}(\rho^n))$



■

9.2 The relation among systems (Proclaim$^{W^*}$2 in SMT$^{W^*}$)

We mentioned "statistical measurement" [Proclaim$^{W^*}$1 (9.9)] in the previous section. Thus in this section, we devote ourselves to the "relation among systems (i.e., Proclaim$^{W^*}$2)" in the W^*-algebraic formulation of SMT [6]. That is, we want to propose

$$\mathrm{SMT}^{W^*} = \underset{[\mathrm{Proclaim}^{W^*}1\ (9.9)]}{\text{statistical measurement}} + \underset{[\mathrm{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} \qquad \text{in } W^*\text{-algebra } \mathcal{N}\ . \tag{9.14}$$
$$(=(9.2))$$

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ with weak*-topologies $\sigma(\mathcal{N}_1, (\mathcal{N}_1)_*)$ and $\sigma(\mathcal{N}_2, (\mathcal{N}_2)_*)$ respectively. A continuous linear operator $\Psi_{1,2} : \mathcal{N}_2 \to \mathcal{N}_1$ is called a *Markov operator,* if it satisfies that

(i) $\Psi_{1,2}(F_2) \geq 0$ for any positive element F_2 in $\mathcal{N}_2$,

(ii) $\Psi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{N}_k$ $(k = 1, 2)$.

Here note that, for any observable $(X, \mathcal{F}, F_2)$ in $\mathcal{N}_2$, the $(X, \mathcal{F}, \Psi_{1,2}F_2)$ is an observable in $\mathcal{N}_1$, which is denoted by $\Psi_{12}\mathbf{O}_2$. For example, it is easy to see that, for any countable decomposition $\{\Xi_j\}_{j=1}^\infty$ of Ξ, $(\Xi_j,\ \Xi \in \mathcal{F})$,

[6] If $\mathcal{N} = B(V)$ or $\mathcal{N} = L^\infty(\Omega, \mu)$ (where Ω is finite), the concept "pure state" is valid (*cf.* (9.4) and (9.5)). Thus, in the case, we can propose "PMT$^{W^*}$" (i.e., the W^*-algebraic formulation of PMT) as follows:

$$PMT^{W^*} = \underset{[\mathrm{Axiom}^{W^*}1\ (9.11)]}{\text{statistical measurement}} + \underset{[\mathrm{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} \qquad \text{in } W^*\text{-algebra } \mathcal{N}\ .$$

$$[\Psi_{1,2}F_2](\Xi) = (w^*)\text{-}\lim_{n\to\infty} \Psi_{1,2}(F_2(\cup_{j=1}^n \Xi_j)) = (w^*)\text{-}\lim_{n\to\infty} \Psi_{1,2}(\sum_{j=1}^n F_2(\Xi_j))$$

$$=(w^*)\text{-}\lim_{n\to\infty} \sum_{j=1}^n [\Psi_{1,2}(F_2)](\Xi_j).$$

Also, a Markov operator $\Psi_{1,2} : \mathcal{N}_2 \to \mathcal{N}_1$ is called a *homomorphism* (or precisely, W^*-*homomorphism*), if it satisfies that

(i) $\Psi_{1,2}(F_2)\Psi_{1,2}(G_2) = \Psi_{1,2}(F_2G_2)$ for any F_2 and G_2 in $\mathcal{N}_2$,

(ii) $(\Psi_{1,2}(F_2))^* = \Psi_{1,2}(F_2^*)$ for any F_2 in $\mathcal{N}_2$.

Then the following mathematical result is well known.

(a) $(\Psi_{1,2})_*(\mathfrak{S}^n((\mathcal{N}_1)_*)) \subseteq \mathfrak{S}^n((\mathcal{N}_2)_*)$.

Let $(\Psi_{1,2})_* : (\mathcal{N}_1)_* \to (\mathcal{N}_2)_*$ be the pre-dual operator[7] of a Markov operator $\Psi_{1,2} : \mathcal{N}_2 \to \mathcal{N}_1$, that is, it holds that

$$_{(\mathcal{N}_1)_*}\left\langle \rho_1^n, \Psi_{1.2}F_2 \right\rangle_{\mathcal{N}_1} = {}_{(\mathcal{N}_2)_*}\left\langle (\Psi_{1.2})_*\rho_1^n, F_2 \right\rangle_{\mathcal{N}_2} \quad (\forall \rho_1^n \in (\mathcal{N}_1)_*, \forall F_2 \in \mathcal{N}_2). \tag{9.15}$$

Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are commutative W^*-algebras, i.e., $\mathcal{N}_1 = L^\infty(\Omega_1, \mu_1)$ and $\mathcal{N}_2 = L^\infty(\Omega_2, \mu_2)$. Then, under the identification that $\mathfrak{S}^n(\mathcal{N}_1^*) = L^1_{+1}(\Omega_1, \mu_1)$ and $\mathfrak{S}^n((\mathcal{N}_2)_*) = L^1_{+1}(\Omega_2, \mu_2)$ (*cf.* Example 9.2), the above (a) implies that the pre-dual operator $(\Psi_{1,2})_*$ of a Markov operator Ψ_{12} can be identified with a *transition probability rule* $M(\omega_1, B_2)$, $(\omega_1 \in \Omega_1,\ B_2 \in \mathcal{B}_{\Omega_2})$, such that:

$$\int_{B_2} [(\Phi_{12})_*(\rho_1^n)](\omega_2)\mu_2(d\omega_2) = \int_{\Omega_1} M(\omega_1, B_2)\rho_1^n(\omega_1)\mu_1(d\omega_1) \quad (\forall \rho_1^n \in L^1_{+1}(\Omega_1, \mu_1), \forall B_2 \in \mathcal{B}_{\Omega_2}).$$

Also, it is well known that, a Markov operator $\Psi_{1,2} : L^\infty(\Omega_2, \mu_2) \to L^\infty(\Omega_1, \mu_1)$ is homomorphic, if and only if there exists a measurable map $\psi_{1,2}$ from Ω_1 into Ω_2 such that:

$$(\Psi_{1,2}f_2)(\omega_1) = f_2(\psi_{1,2}(\omega_1)) \qquad (\text{almost all } \mu_1) \tag{9.16}$$

for all $f_2 \in L^\infty(\Omega_2, \mu_2)$.

[7]The symbol $*$ is used in the three following ways (1) $\sim$ (v) in this book. (i) involution operator (e.g., F^*), (ii) dual operator (e.g., Ψ^*), (iii) dual space (e.g., $\mathcal{A}^*$), (iv) pre-dual operator (e.g., Ψ_*), (v) pre-dual space (e.g., $\mathcal{N}_*$).

Let $(T, \leq)$ be a tree-like partial ordered set, i.e., a partial ordered set such that "$t_1 \leq t_3$ and $t_2 \leq t_3$" implies "$t_1 \leq t_2$ or $t_2 \leq t_1$". Put $T^2_{\leq} = \{(t_1, t_2) \in T^2 : t_1 \leq t_2\}$. An element $t_0 \in T$ is called a *root* if $t_0 \leq t$ $(\forall t \in T)$ holds. Note that the sub-tree $T_{t_0} \equiv \{t \in T \mid t \geq t_0\}$ has the root t_0. Thus we always assume that the tree-like ordered set $(T, \leq)$ has a root. We assume that T is not always finite. $\Big($In the next Chapter 10, T is always assumed to be infinite.$\Big)$

Definition 9.11. [General systems]. *The pair* $\overline{\mathbf{S}}(\rho^n_{t_0}) \equiv [\overline{S}(\rho^n_{t_0}), \{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$ *is called a general system with an initial state* $\overline{S}(\rho^n_{t_0})$ *if it satisfies the following conditions* (i)$\sim$(iii).

(i) *With each* t $(\in T)$, *a* W^**-algebra* $\mathcal{N}_t$ *is associated.*

(ii) *Let* t_0 $(\in T)$ *be the root of* T. *And, assume that a system* S *has the normal state* $\rho^n_{t_0}$ $(\in \mathfrak{S}^n((\mathcal{N}_{t_0})_*))$ *at* t_0, *that is, the initial state is equal to* $\rho^n_{t_0}$.

(iii) *For every* $(t_1, t_2) \in T^2_{\leq}$, *Markov operator* $\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}$ *is defined such that* $\Phi_{t_1,t_2}\Phi_{t_2,t_3} = \Phi_{t_1,t_3}$ *holds for all* (t_1, t_2), $(t_2, t_3) \in T^2_{\leq}$.

The family $\{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}$ *is also called a "Markov relation among systems". Let an observable* $\overline{\mathbf{O}}_t \equiv (X_t, 2^{X_t}, F_t)$ *in a* W^**-algebra* $\mathcal{N}_t$ *be given for each* $t \in T$. *The pair* $[\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$ *is called a "sequential observable", which is denoted by* $[\overline{\mathbf{O}}_T]$, *i.e.,* $[\overline{\mathbf{O}}_T] \equiv [\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$.

■

Before we explain Proclaim$^{W^*}$ 2, we prepare some notations. For simplicity, assume that T is finite, or a finite subtree of a whole tree. Let T $(= \{0, 1, ..., N\})$ be a tree with the root 0. Define the *parent map* $\pi : T \setminus \{0\} \to T$ such that $\pi(t) = \max\{s \in T : s < t\}$. It is clear that the tree $(T \equiv \{0, 1, ..., N\}, \leq)$ can be identified with the pair $(T \equiv \{0, 1, ..., N\}, \pi : T \setminus \{0\} \to T)$. Also, note that, for any $t \in T \setminus \{0\}$, there uniquely exists a natural number $h(t)$ (called the height of t) such that $\pi^{h(t)}(t) = 0$. Here, $\pi^2(t) = \pi(\pi(t))$, $\pi^3(t) = \pi(\pi^2(t))$, etc. Thus, the general system $\overline{\mathbf{S}}(\rho^n_0) \equiv [\overline{S}(\rho^n_0), \{\Psi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in\{0,1,...,N\}^2_{\leq}}]$ is sometimes represented by $[\overline{S}(\rho^n_0), \mathcal{N}_t \overset{\Psi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}$ ($t \in \{0, 1, ..., N\} \setminus \{0\})]$. Also, we define the $\Phi_{0,\tau} : \mathcal{N}_\tau \to \mathcal{N}_0$ such that $\Phi_{0,\tau} = \Psi_{0,\tau}$, that is,

$$\Phi_{0,\tau} = \Psi_{0,\pi^{h(\tau)-1}(\tau)}\Psi_{\pi^{h(\tau)-1}(\tau)),\pi^{h(\tau)-2}(\tau)} \cdots \Psi_{\pi^2(\tau),\pi(\tau)}\Psi_{\pi(\tau),\tau}. \tag{9.17}$$

Let $\overline{\mathbf{O}}_t \equiv (X_t, \mathcal{F}_t, F_t)$ be an observable in $\mathcal{N}_t$ ($\forall t \in T$). The "measurement" of $\{\overline{\mathbf{O}}_t : t \in T\}$ for the $\overline{\mathbf{S}}(\rho^n_{t_0})$ is symbolically described by $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T},\ \overline{\mathbf{S}}(\rho^n_{t_0}))$. The Markov relation $\{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}$ is also denoted by $\{\mathcal{N}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}$

Example 9.12. [A simple general system. Compared to Examples 3.4 and 8.12]. Suppose that a tree $(T \equiv \{0, 1, ..., 6, 7\}, \pi)$ has an ordered structure such that $\pi(1) = \pi(6) = \pi(7) = 0$, $\pi(2) = \pi(5) = 1$, $\pi(3) = \pi(4) = 2$. $\Big($See the figure (9.18).$\Big)$ Consider a general system $\overline{\mathbf{S}}(\rho^n_0) \equiv [\overline{S}(\rho^n_0),\ \{\mathcal{N}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}]$ with the initial system $\overline{S}(\rho^n_0)$.

$$\begin{tikzcd}
 & & \mathcal{N}_3 \arrow[dl, "\Phi_{2,3}"'] & \\
 & \mathcal{N}_1 \arrow[dl, "\Phi_{0,1}"'] & \mathcal{N}_2 \arrow[dl, "\Phi_{1,2}"'] & \mathcal{N}_4 \arrow[ul, "\Phi_{2,4}"] \\
\mathcal{N}_0 & \mathcal{N}_6 \arrow[l, "\Phi_{0,6}"] & \mathcal{N}_5 \arrow[ul, "\Phi_{1,5}"] & \\
 & \mathcal{N}_7 \arrow[ul, "\Phi_{0,7}"] & &
\end{tikzcd} \tag{9.18}$$

Also, for each $t \in \{0, 1, ..., 6, 7\}$, consider an observable $\overline{\mathbf{O}}_t \equiv (X_t, 2^{X_t}, F_t)$ in a W^*-algebra $\mathcal{N}_t$. Thus, we have a sequential observable $[\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{N}_t \to \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}\ ]$. Now we want to consider the following "measurement",

(♯) for a system $\overline{S}((\rho^n_0))$ where $\rho^n_0 \in \mathfrak{S}^n((\mathcal{N}_0)_*)$, take a measurement of *"a sequential observable* $[\overline{\mathbf{O}}_T] \equiv [\{\overline{\mathbf{O}}_t\}_{t\in T},\ \{\mathcal{N}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}]$*"*, i.e., take a measurement of an observable $\overline{\mathbf{O}}_0$ at $0(\in T)$, and next, take a measurement of an observable $\overline{\mathbf{O}}_1$ at $1(\in T)$, $\cdots\cdots$, and finally take a measurement of an observable $\overline{\mathbf{O}}_7$ at $7(\in T)$,

which is symbolized by $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T},\ \overline{\mathbf{S}}(\rho^n_0))$. Note that the $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T},\ \overline{\mathbf{S}}(\rho^n_0))$ is merely a symbol since only one measurement is permitted (*cf.* §2.5 Remark (II)). In what follows let us describe the above (♯) $(= \overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T},\ \overline{\mathbf{S}}(\rho^n_0)))$ precisely. Put

$$\widetilde{\mathbf{O}}_t = \overline{\mathbf{O}}_t \quad \text{and thus} \quad \widetilde{F}_t = F_t \quad (t = 3, 4, 5, 6, 7).$$

First we construct the quasi-product observable $\widetilde{\mathbf{O}}_2$ in $\mathcal{N}_2$ such as

$$\widetilde{\mathbf{O}}_2 = (X_2 \times X_3 \times X_4, 2^{X_2\times X_3\times X_4}, \widetilde{F}_2) \quad \text{where } \widetilde{F}_2 = F_2 \overset{\text{qp}}{\times} (\overset{\text{qp}}{\times}_{t=3,4}\, \Phi_{2,t}\widetilde{F}_t), \tag{9.19}$$

if it exists. Iteratively, we construct the following:

$$\begin{array}{ccccc}
\mathcal{N}_0 & \xleftarrow{\Phi_{0,1}} & \mathcal{N}_1 & \xleftarrow{\Phi_{1,2}} & \mathcal{N}_2 \\
F_0 \overset{\mathrm{qp}}{\times} \Phi_{0,6}\widetilde{F}_6 \overset{\mathrm{qp}}{\times} \Phi_{0,7}\widetilde{F}_7 & & F_1 \overset{\mathrm{qp}}{\times} \Phi_{1,5}\widetilde{F}_5 & & \\
\downarrow & & \downarrow & & \\
\underset{(F_0 \overset{\mathrm{qp}}{\times} \Phi_{0,6}\widetilde{F}_6 \overset{\mathrm{qp}}{\times} \Phi_{0,7}\widetilde{F}_7 \overset{\mathrm{qp}}{\times} \Phi_{0,1}\widetilde{F}_1)}{\widetilde{F}_0} & \xleftarrow{\Phi_{0,1}} & \underset{(F_1 \overset{\mathrm{qp}}{\times} \Phi_{1,5}\widetilde{F}_5 \overset{\mathrm{qp}}{\times} \Phi_{1,2}\widetilde{F}_2)}{\widetilde{F}_1} & \xleftarrow{\Phi_{1,2}} & \underset{(F_2 \overset{\mathrm{qp}}{\times} \Phi_{2,3}\widetilde{F}_3 \overset{\mathrm{qp}}{\times} \Phi_{2,4}\widetilde{F}_4)}{\widetilde{F}_2}
\end{array} . \tag{9.20}$$

That is, we get the quasi-product observable $\widetilde{\mathbf{O}}_1 \equiv (\prod_{t=1}^5 X_t, 2^{\Pi_{t=1}^5 X_t}, \widetilde{F}_1)$ of $\overline{\mathbf{O}}_1$, $\Phi_{1,2}\widetilde{\mathbf{O}}_2$ and $\Phi_{1,5}\widetilde{\mathbf{O}}_5$, and finally, the quasi-product observable $\widetilde{\mathbf{O}}_0 \equiv (\prod_{t=0}^7 X_t, 2^{\Pi_{t=0}^7 X_t}, \widetilde{F}_0)$ of $\overline{\mathbf{O}}_0$, $\Phi_{0,1}\widetilde{\mathbf{O}}_1$, $\Phi_{0,6}\widetilde{\mathbf{O}}_6$ and $\Phi_{0,7}\widetilde{\mathbf{O}}_7$, if it exists. Here, $\widetilde{\mathbf{O}}_0$ is called *the realization* (*or, the Heisenberg picture representation*) of *a sequential observable* $[\overline{\mathbf{O}}_T] \equiv [\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\mathcal{N}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}]$. Then, we have the measurement

$$\overline{\mathbf{M}}_{\mathcal{N}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, 2^{\Pi_{t\in T} X_t}, \widetilde{F}_0), \overline{S}(\rho_0^n)), \tag{9.21}$$

which is called *the realization* (*or, the Heisenberg picture representation*) *of the symbol* $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T}, \overline{\mathbf{S}}(\rho_0^n))$.

■

Examining Example 9.12, we have the following arguments. Let $(T \equiv \{0, 1, ..., N\}, \pi : T\setminus\{0\} \to T)$ be a tree with root 0 and let $\overline{\mathbf{S}}(\rho_0^n) \equiv [\overline{S}(\rho_0^n), \mathcal{N}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}\ (t \in T\setminus\{0\})]$ be a general system with the initial system $\overline{S}(\rho_0^n)$. And, let an observable $\overline{\mathbf{O}}_t \equiv (X_t, \mathcal{F}_t, F_t)$ in a W^*-algebra $\mathcal{N}_t$ be given for each $t \in T$. For each s ($\in T$), define the observable $\widetilde{\mathbf{O}}_s \equiv (\prod_{t\in T_s} X_t, \prod_{t\in T_s} \mathcal{F}_t, \widetilde{F}_s)$ in $\mathcal{N}_s$ such that:

$$\widetilde{\mathbf{O}}_s = \begin{cases} \overline{\mathbf{O}}_s & (\text{if } s \in T\setminus\pi(T)) \\ \overline{\mathbf{O}}_s \overset{\mathrm{qp}}{\times} (\overset{\mathrm{qp}}{\times}_{t\in\pi^{-1}(\{s\})} \Phi_{\pi(t),t}\widetilde{\mathbf{O}}_t) & (\text{if } s \in \pi(T)) \end{cases} \tag{9.22}$$

if possible. Then, if an observable $\widetilde{\mathbf{O}}_0$ (i.e., the Heisenberg picture representation of the sequential observable $[\overline{\mathbf{O}}_T] \equiv [\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{N}_t \to \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}\ ]$) in $\mathcal{N}_0$ exists (such as in Example 9.12), we have the measurement

$$\overline{\mathbf{M}}_{\mathcal{N}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0), \overline{S}(\rho_0^n)),$$

which is called the Heisenberg picture representation of the symbol $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T}, \overline{\mathbf{S}}(\rho_0^n))$.

Summing up the essential part of the above argument, we can propose the following axiom, which corresponds to "the rule of the relation among systems" in SMT^{W^*}.

PROCLAIM$^{W^*}$2. [The Markov relation among systems, the Heisenberg picture] *The relation among systems is represented by a Markov relation* $\{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}$. *Let* $\overline{\mathbf{O}}_t$ ($\equiv (X_t, \mathcal{F}_t, F)$) *be an observable in* $\mathcal{N}_t$ *for each* t ($\in T$). *If the procedure (9.22) is possible, a sequential observable* $[\overline{\mathbf{O}}_T] \equiv [\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$ *can be realized as the observable* $\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0)$ *in* $\mathcal{N}_0$. (9.23)

Also, we must add the following statement:

- *Let* $\overline{\mathbf{S}}(\rho^n_{t_0}) \equiv [S(\rho^n_{t_0}), \{\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$ *be a general system with an initial state* $\rho^n_{t_0}$ ($\in \mathfrak{S}^n((\mathcal{N}_{t_0})_*)$). *And then, a measurement represented by the symbol* $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T}, \overline{\mathbf{S}}(\rho^n_{t_0}))$ *can be realized by* $\overline{\mathfrak{M}}_{\mathcal{N}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0), \overline{S}(\rho^n_0))$, *if* $\widetilde{\mathbf{O}}_0$ *exists.*

which explains the relation between Proclaim$^{W^*}$1 and Proclaim$^{W^*}$2.

Remark 9.13. [How to read Proclaim$^{W^*}$2]. For completeness, we mention how to read Proclaim$^{W^*}$2 as follows: Recall Axiom 2 (3.26), that is,

- [Axiom 2. (The Markov relation among systems, the Heisenberg picture)] *The relation among systems is represented by a Markov relation* $\{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}$. *Let* $\mathbf{O}_t$ ($\equiv (X_t, \mathcal{F}_t, F)$) *be an observable in* $\mathcal{A}_t$ *for each* t ($\in T$). *If the procedure (3.25) is possible, a sequential observable* $[\mathbf{O}_T] \equiv [\{\mathbf{O}_t\}_{t\in T}, \{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$ *can be realized as the observable* $\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0)$ *in* $\mathcal{A}_0$.

Using this and the following correspondence, we can easily read the above Proclaim$^{W^*}$2.

$$\text{Statistical measurement theory} \tag{9.24}$$

$[\mathrm{SMT}^{C^*}(C^*\text{-algebraic formulation})]$	$\longleftrightarrow$	$[\mathrm{SMT}^{W^*}(W^*\text{-algebraic formulation})]$
Proclaim 1 (8.10)	$\longleftrightarrow$	$\mathrm{Proclaim}^{W^*}1$ (9.11)
$\mathfrak{S}^m(\mathcal{A}^*) \ni \rho^m$	$\longleftrightarrow$	$\rho^n \in \mathfrak{S}^n(\mathcal{N}_*)$
C^*-observable $\mathbf{O} \equiv (X, \mathcal{F}, F)$	$\longleftrightarrow$	W^*-observable $\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F)$
$\mathbf{M}_{\mathcal{A}}(\mathbf{O} \equiv (X, \mathcal{F}, F), S(\rho^m))$	$\longleftrightarrow$	$\overline{\mathbf{M}}_{\mathcal{N}}(\overline{\mathbf{O}} \equiv (X, \mathcal{F}, F), \overline{S}(\rho^n))$
Axiom 2 (3.26)	$\longleftrightarrow$	$\mathrm{Proclaim}^{W^*}2$ (9.23)
general system$\mathbf{S}(\rho^m)$ $\left(=[S(\rho^m), \{\Psi_{t_1,t_2}:\mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]\right)$	$\longleftrightarrow$	general system$\overline{\mathbf{S}}(\rho^n)$ $\left(=[\overline{S}(\rho^n), \{\Psi_{t_1,t_2}:\mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]\right)$
sequential observable$[\mathbf{O}_T]$ $\left(=[\{\mathbf{O}_t\}_{t\in T}, \{\Psi_{t_1,t_2}:\mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]\right)$	$\longleftrightarrow$	sequential observable$[\overline{\mathbf{O}}_T]$ $\left(=[\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Psi_{t_1,t_2}:\mathcal{N}_{t_2} \to \mathcal{N}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]\right)$
$\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}(\rho^m))$	$\longleftrightarrow$	$\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T}, \overline{\mathbf{S}}(\rho^n))$

Remark 9.14. [The C^*-algebraic and the W^*-algebraic formulations]. Now we have two formulations of SMT, i.e., the C^*-algebraic formulation and the W^*-algebraic formulation. Recall that any commutative C^*-algebra [resp. commutative W^*-algebra] is represented by some $C(\Omega)$ [resp. $L^\infty(\Omega, \mu)$]. Thus, we can say that the C^*-algebraic formulation and the W^*-algebraic formulation are respectively topological and measure theoretical. Therefore, from the mathematical point of view, the W^*-algebraic formulation is handy for us to deal with "limit" or "convergence". For example, this will be seen in Theorem 10.1 (the W^*-algebraic generalization of Kolmogorov's extension theorem).[8]

■

The following theorem is essentially the same as Theorem 3.7.

Theorem 9.15. [The measurability of a general system; Compared to Theorem 3.7]. *Let* $(T \equiv \{0, 1, ..., N\}, \pi : T \setminus \{0\} \to T)$ *be a tree with root* 0 *and let* $\overline{\mathbf{S}}(\rho_0^n) \equiv [\overline{S}(\rho_0^n), \mathcal{N}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{N}_{\pi(t)}\ (t \in T \setminus \{0\})]$ *be a general system with the initial system* $\overline{S}(\rho_0^n)$. *And, let an observable* $\overline{\mathbf{O}}_t \equiv (X_t, \mathcal{F}_t, F_t)$ *in a* C^**-algebra* $\mathcal{N}_t$ *be given for each* $t \in T$. *For each* s

[8] If readers have some knowledge of Riemann integral (defined in terms of topology) and Lebesgue integral (defined in terms of measure, *cf.* [29]), they can easily understand the mathematical handiness of "measure theoretical approach".

($\in T$), define the observable $\widetilde{\mathbf{O}}_s \equiv (\prod_{t\in T_s} X_t, \prod_{t\in T_s} \mathcal{F}_t, \widetilde{F}_s)$ in $\mathcal{N}_s$ such that:

$$\widetilde{\mathbf{O}}_s = \begin{cases} \overline{\mathbf{O}}_s & (\text{if } s \in T \setminus \pi(T)) \\ \overline{\mathbf{O}}_s \overset{\text{qp}}{\times} (\overset{\text{qp}}{\times}_{t\in\pi^{-1}(\{s\})} \Phi_{\pi(t),t}\widetilde{\mathbf{O}}_t) & (\text{if } s \in \pi(T)) \end{cases}$$

if possible. Then, if an observable $\widetilde{\mathbf{O}}_0$ (i.e., the Heisenberg picture representation of the sequential observable $[\{\overline{\mathbf{O}}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{N}_t \to \mathcal{N}_{\pi(t)}\}_{t\in T\setminus\{0\}}]$) in $\mathcal{N}_0$ exists, we have the measurement

$$\overline{\mathbf{M}}_{\mathcal{N}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0), \overline{S}(\rho_0^n)), \tag{9.25}$$

which is called the Heisenberg picture representation of the symbol $\overline{\mathfrak{M}}(\{\overline{\mathbf{O}}_t\}_{t\in T}, \overline{\mathbf{S}}\rho_{t_0}^n))$. If the system is classical, i.e., $\mathcal{N}_t \equiv L^\infty(\Omega, \mu)$ ($\forall t \in T$), then the measurement always exists, while the uniqueness is not always guaranteed. Also, it should be noted that, for each s($\in T$), it holds that $\Phi_{\pi(s),s}\widetilde{F}_s(\prod_{t\in T_s} \Xi_t) = \widetilde{F}_{\pi(s)}((\Pi_{t\in T_{\pi(s)}\setminus T_s} X_t) \times (\prod_{t\in T_s} \Xi_t))$ ($\forall \Xi_t \in \mathcal{F}_t$ ($\forall t \in T$)).

Proof. The proof is the same as that of Theorem 3.7. □

Remark 9.16. [Summing up]. In Chapters 2 ∼ 8, we studied the C^*-algebraic formulation such that

$$\mathrm{MT}^{C^*} \begin{cases} \mathrm{PMT}^{C^*} = \underset{[\text{Axiom 1 (2.37)}]}{\text{measurement}} + \underset{[\text{Axiom 2 (3.26)}]}{\text{the relation among systems}} & (\text{In Chap. } 2\sim 7) \\ \mathrm{SMT}^{C^*} = \underset{[\text{Proclaim 1 (8.10)}]}{\text{statistical measurement}} + \underset{[\text{Axiom 2 (3.26)}]}{\text{the relation among systems}} & (\text{In Chap. } 8) \end{cases}$$

In this chapter, we study the W^*-algebraic formulation as follows:

$$\mathrm{MT}^{W^*} \begin{cases} \mathrm{PMT}^{W^*} = \underset{[\text{Axiom}^{W^*}1\ (9.11)]}{\text{measurement}} + \underset{[\text{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} & (\text{in } \mathcal{N}) \\ \mathrm{SMT}^{W^*} = \underset{[\text{Proclaim}^{W^*}1\ (9.9)]}{\text{statistical measurement}} + \underset{[\text{Proclaim}^{W^*}2\ (9.23)]}{\text{the relation among systems}} & (\text{in } \mathcal{N}) \end{cases} \tag{9.26}$$

Here we add the remarks as follows:

(i) MT$^{C^*}$ is fundamental,

(ii) MT$^{W^*}$ should be understood by an analogy of MT$^{C^*}$. Cf. Table (9.24).

(iii) From the mathematical point of view, SMT$^{W^*}$ is more handy than SMT$^{C^*}$. (Cf. Remark 9.14).

(iv) When $\mathcal{N} = B(V)$ or $\mathcal{N} = L^\infty(\Omega, \mu)$ (where Ω is finite or countable infinite), PMT$^{W^*}$ is meaningful (*cf.* Example 9.1).

(v) Most results in MT$^{C^*}$ hold in MT$^{W^*}$. However, we omit "Fisher's maximum likelihood method" and "Generalized Bayes theorem", etc. in MT$^{W^*}$ since the proofs are the same.

■

9.3 Quantum mechanics in $B(L^2(\mathbf{R}))$

9.3.1 Schrödinger equation and Heisenberg kinetic equation

Recall the C^*-algebraic formulation (in $\mathcal{C}(L^2(\mathbf{R}))$) of quantum mechanics (*cf.* §3.1). However, as far as quantum mechanics, the W^*-algebraic formulation (in $B(L^2(\mathbf{R}))$) is more handy than the C^*-algebraic formulation (in $\mathcal{C}(L^2(\mathbf{R}))$). (Cf. [71].) Thus, in this section, we explain the W^*-algebraic formulation of quantum mechanics (*cf.* §3.1). though it is similar to the C^*-algebraic formulation of quantum mechanics,

We begin with the classical mechanics. For simplicity, consider the one dimensional case, i.e., $\mathbf{R}_q = \{q \mid q \in \mathbf{R}\}$. Thus $q(t)$, $-\infty < t < \infty$, means the particle's position at time t, and thus, $p(t)$ ($\equiv m\frac{dq(t)}{dt}$) means the particle's momentum at time t. Let $\mathbf{R}^2_{q,p}$ ($\equiv \{(q,p) \mid q,p \in \mathbf{R}\}$ be a phase space. Define a Hamiltonian $\mathcal{H} : \mathbf{R}^2_{q,p} \to \mathbf{R}$ such that:

$$\mathcal{H}(q,p) = \frac{p^2}{2m}(\text{=kinetic energy}) \quad + \quad V(q)(\text{=potential energy}). \tag{9.27}$$

Thus we see

$$\underset{\text{(total energy)}}{E} = \mathcal{H}(q,p) = \underset{\text{(kinetic energy)}}{\frac{p^2}{2m}} + \underset{\text{(potential energy)}}{V(q)}. \tag{9.28}$$

Put $H = L^2(\mathbf{R}_q, dq)$, i.e., the Hilbert space composed of all L^2-functions on $\mathbf{R}_q$. And put $\mathcal{N} = B(L^2(\mathbf{R}_q, dq))$. Applying the quantumization:

$$E \mapsto i\hbar\frac{\partial}{\partial t}, \quad p \mapsto -i\hbar\frac{\partial}{\partial q}, \quad q \mapsto q \tag{9.29}$$

to the (9.27), we obtain the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t} = \mathcal{H}(q, -i\hbar\frac{\partial}{\partial q}) = -\frac{\hbar^2\partial^2}{2m\partial q^2} + V(q) \tag{9.30}$$

or, precisely

$$i\hbar\frac{\partial}{\partial t}\psi(q,t) = -\frac{\hbar^2\partial^2}{2m\partial q^2}\psi(q,t) + V(q)\psi(q,t). \tag{9.31}$$

This solution is formally written by

$$\psi(q,t) = e^{-\frac{i}{\hbar}\mathcal{H}(q,-i\hbar\frac{\partial}{\partial q})t}\psi(q,0). \tag{9.32}$$

Put $U(t) = e^{-\frac{i}{\hbar}\mathcal{H}(q,-i\hbar\frac{\partial}{\partial q})t}$, and $\psi(\cdot,t) = \psi_t$. Then, we see,

$$\psi_t = U(t)\psi_0. \tag{9.33}$$

Thus, the time-evolution of the state $|\psi_t\rangle\langle\psi_t|$ is represented by

$$|\psi_t\rangle\langle\psi_t| = (\Phi_t^0)_*\Big(|\psi_0\rangle\langle\psi_0|\Big) = |U(t)\psi_0\rangle\langle U(t)\psi_0|$$

Let $\overline{\mathbf{O}}_0 = (X,\mathcal{F},F_0)$ be a W^*-observable in $B(H)$. Then, the time-evolution of the observable $\overline{\mathbf{O}}_t = (X,\mathcal{F},F_t)$ is represented by

$$(X,\mathcal{F},F_t) = (X,\mathcal{F},U(t)F_0U(t)^*) = (X,\mathcal{F},\Phi_t^0F_0). \tag{9.34}$$

Also, it should be note that it holds that

$$\frac{dF_t}{dt} = F_t\mathcal{H} - \mathcal{H}F_t, \tag{9.35}$$

which is the Heisenberg kinetic equation. Put $\Psi_{t_1,t_2} = \Phi^0_{t_2-t_1}$, And let ρ be any element in $Tr^m_{+1}(H)$, i.e, a normal state. Then, we get the general statistical system $[\overline{S}(\rho), \{\Psi_{t_1,t_2} : B(H) \to B(H)\}_{t_1\le t_2}]$. Also, let ρ_u be any element in $Tr^p_{+1}(H)$, i.e, $\rho_u = |u\rangle\langle u|$, a pure state. Then, we get the general system $[\overline{S}_{[\rho_u]}, \{\Psi_{t_1,t_2} : B(H) \to B(H)\}_{t_1\le t_2}]$.

Although the two formulations (i.e., the W^*-algebraic formulation (in $B(L^2(\mathbf{R}))$) and the C^*-algebraic formulation (in $\mathcal{C}(L^2(\mathbf{R}))$) are similar, it should be noted that the position observable and the momentum observable can not be represented in the C^*-algebraic formulation but the W^*-algebraic formulation (*cf.* Example 9.7).

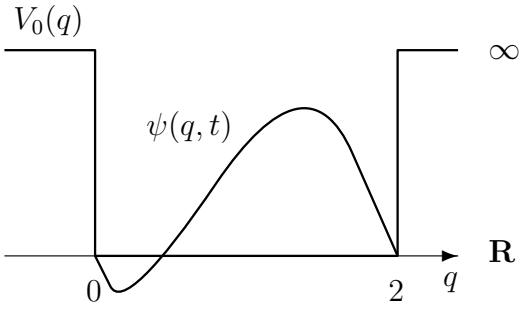
9.3.2 A simplest example of Schrödinger equation

Consider a particle with the mass m in the box (i.e., the closed interval $[0,2]$) in the one dimensional space $\mathbf{R}$. The motion of this particle (i.e., the wave function of the particle) is represented by the following Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(q,t) = -\frac{\hbar^2\partial^2}{2m\partial q^2}\psi(q,t) + V_0(q)\psi(q,t).$$

where

$$V_0(q) = \begin{cases} 0 & (0 \le q \le 2) \\ \infty & (\text{ otherwise }) \end{cases}$$



Put

$$\phi(q,t) = T(t)X(q) \qquad (0 \le q \le 2).$$

And consider the following equation:

$$i\hbar\frac{\partial}{\partial t}\phi(q,t) = -\frac{\hbar^2\partial^2}{2m\partial q^2}\phi(q,t).$$

Then, we see

$$\frac{iT'(t)}{T(t)} = -\frac{X''(q)}{2mX(q)} = K(= \text{ constant }).$$

Then,

$$\phi(q,t) = T(t)X(q) = C_3\exp(iKt)\Big(C_1\exp(i\sqrt{2mK/\hbar}\,q) + C_2\exp(-i\sqrt{2mK/\hbar}\,q).\Big)$$

Since $X(0) = X(2) = 0$ (perfectly elastic collision), putting $K = \frac{n^2\pi^2\hbar}{8m}$, we see

$$\phi(q,t) = T(t)X(q) = C_3\exp(\frac{in^2\pi^2\hbar t}{8m})\sin(n\pi q/2) \qquad (n = 1, 2, ...).$$

Assume the initial condition:

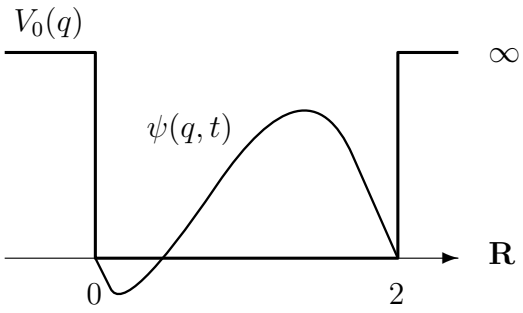
$$\psi(q,0) = c_1\sin(\pi q/2) + c_2\sin(2\pi q/2) + c_3\sin(3\pi q/2) + \cdots .$$

where $\int_{\mathbf{R}}|\psi(q,0)|^2dq = 1$. Then we see

$$\begin{aligned}&\psi(q,t)\\ =&c_1\exp(\frac{i\pi^2\hbar t}{8m})\sin(\pi q/2) + c_2\exp(\frac{i4\pi^2\hbar t}{8m})\sin(2\pi q/2) + c_3\exp(\frac{i9\pi^2\hbar t}{8m})\sin(3\pi q/2) + \cdots .\end{aligned}$$

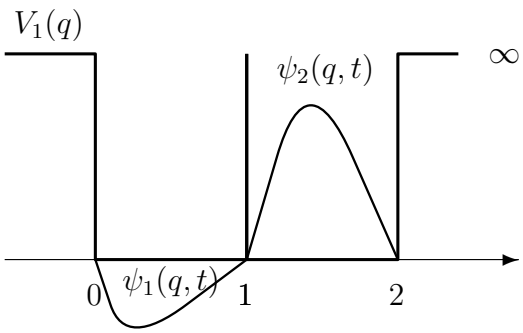
9.3.3 The de Broglie paradox

Consider the same situation in §9.3.2, i.e., a particle with the mass m in the box (i.e., the closed interval $[0, 2]$) in the one dimensional space $\mathbf{R}$.

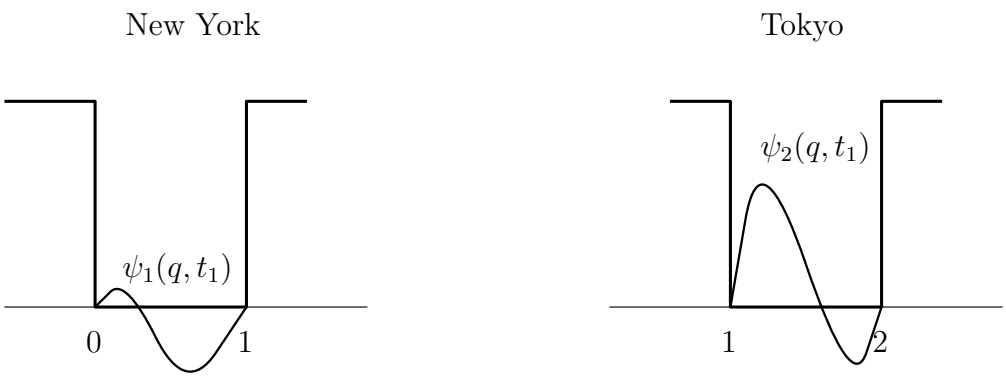


Now let us partition the box $[0, 2]]$ into $[0, 1]]$ and $[1, 2]$. That is, we change $V_0(q)$ to $V_1(q)$, where

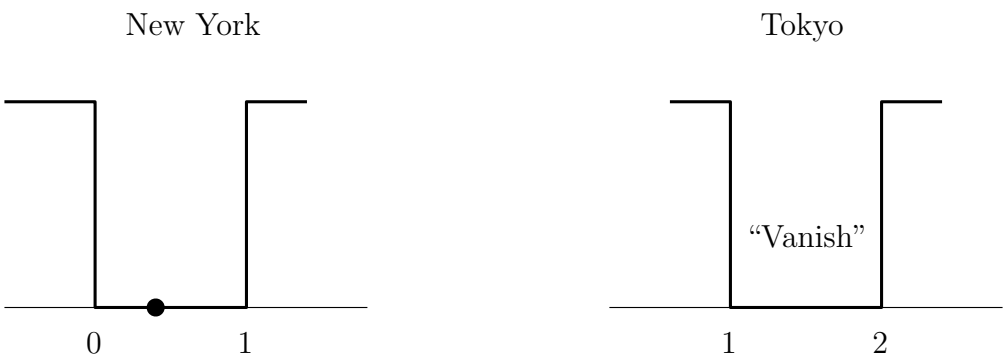
$$V_1(q) = \begin{cases} 0 & (0 \leq q < 1) \\ \infty & (q = 1) \\ 0 & (1 < q \leq 2) \\ \infty & (\text{ otherwise }) \end{cases}$$



Next, we carry the box $[0, 1]$ $\big[$resp. the box $[1, 2]\big]$ to New York (or, the earth) $\big[$resp. Tokyo (or, the polar star)$\big]$.



Note that the probability that we find the particle in the box $[0,1]$ $\big[$resp. the box $[1,2]\big]$ is given by $\int_{\mathbf{R}} |\psi_1(q,t_1)|^2 dq$ $\big[$resp. $\int_{\mathbf{R}} |\psi_2(q,t_1)|^2 dq\big]$. Here, we open the box $[0,1]$ at New York. And assume that we find the particle in the box $[0,1]$. Then, quantum mechanics says that at the moment the wave function ψ_2 vanishes.



Note that New York $\big[$resp. Tokyo$\big]$ may be the earth $\big[$resp. the polar star$\big]$. Thus, the above argument implies that there is something faster than light. This is called "the de Broglie paradox" (*cf.* §2.9.1, [78]).

9.4 The method of moments

9.4.1 The moment method

In this book we mainly devoted ourselves to Fisher's maximum likelihood method (*cf.* Corollary 5.6) in (pure) measurements, and Bayes' method (Cf. Theorem 6.6 and Theorem 8.13) in statistical measurements. In this section we study *"the method of moments"* (or, *the moment method*) in measurements theory (particularly, repeated measurements, cf.

Definition 2.27).

In what follows, we shall review "the method of moments" (*cf.* Definition 2.27). Let $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O} \equiv (X, \mathcal{F}, F),\ S_{[\rho_0^p]}\big)$ be a (pure) measurement, which may be constructed as in (8.13) of Remark 8.3. Assume the ρ_0^p (in $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O},\ S_{[\rho_0^p]}\big)$) is unknown. And further, we get the sample space $(X, \mathcal{F}, \nu_0)$ from the measured value $\widehat{x}$ $(= (x_1, x_2, ..., x_T) \in X^T)$ obtained by the repeated measurement $\otimes_{t=1}^T \mathbf{M}_{\mathcal{A}}\big(\mathbf{O} \equiv (X, \mathcal{F}, F),\ S_{[\rho_0^p]}\big)$ $\Big(= \mathbf{M}_{\otimes\mathcal{A}}\big(\bigotimes_{t=1}^T \mathbf{O}, S_{[\otimes_{t=1}^T \rho_0^p]}\big)\Big)$. That is, $\nu_0 = \frac{1}{T}\sum_{t=1}^T \delta_{x_t}$ $\Big($i.e., $\nu_0(\Xi) = \frac{\sharp[\{k : x_t \in \Xi\}]}{T}\Big)$. Theorem 2.25 says that that $\rho^p(F(\Xi)) \approx \nu_0(\Xi)$ $(\forall \Xi \in \mathcal{F})$ if T is sufficiently large. Therefore,

- [Generalized moment method]; there is a very reason to infer the unknown ρ_0^p $(\in \mathfrak{S}^p(\mathcal{A}^*))$ such that:

$$\Delta(\nu_0, \rho_0^p(F(\,\cdot\,))\,) = \min_{\rho^p \in \mathfrak{S}^p(\mathcal{A}^*)} \Delta(\nu_0, \rho^p(F(\,\cdot\,))\,), \tag{9.36}$$

  where Δ is a certain semi-distance on $\mathcal{M}_{+1}^m(X)$.

This method is called "generalized moment method" or "moment method".

Note that the "semi-distance Δ on $\mathcal{M}_{+1}^m(X)$" is not always unique. In this sense, the moment method is somewhat artificial. If X is a finite set, it is usual to define the distance Δ on $\mathcal{M}_{+1}^m(X)$ such that:

$$\Delta(\nu_1, \nu_2) = \sum_{x \in X} |\nu_1(\{x\}) - \nu_2(\{x\})| \quad (\forall \nu_1, \nu_2 \in \mathcal{M}_{+1}^m(X)). \tag{9.37}$$

More generally, assume that X is an infinite set (and moreover, a metric space). Let $f_l : X \to \mathbf{R}$, $l = 1, 2, ..., L$, be a continuous function on X. Then, the semi-distance $\Delta_{\{f_l\}_{l=1}^L}$ on $\mathcal{M}_{+1}^m(X)$ is defined by

$$\Delta_{\{f_l\}_{l=1}^L}(\nu_1, \nu_2) = \sum_{l=1}^L \Big| \int_X f_l(x)(\nu_1(dx) - \nu_2(dx)) \Big| \quad (\forall \nu_1, \nu_2 \in \mathcal{M}_{+1}^m(X)). \tag{9.38}$$

The above argument is quite general. We usually use the following moment method.

Remark 9.17. [The simple case of (9.36)]. The minimization problem (9.36) may be somewhat troublesome. Thus, we often want to solve the equation $\Delta(\nu_0, \rho_0^p(F(\,\cdot\,))\,) = 0$ (i.e., the case of "$\min_{\rho^p \in \mathfrak{S}^p(\mathcal{A}^*)} \Delta(\nu_0, \rho^p(F(\,\cdot\,))\,) = 0$"). That is, our concern is to solve the following equation:

$$\sum_{l=1}^L \Big| \int_X f_l(x)\nu_0(dx) - \int_X f_l(x)\rho_0^p(F(dx)) \Big| = 0.$$

Or, equivalently,

$$\begin{cases} \int_X f_1(x)\nu_0(dx) = \int_X f_1(x)\rho_0^p(F(dx)) \\ \int_X f_2(x)\nu_0(dx) = \int_X f_2(x)\rho_0^p(F(dx)) \\ \qquad\qquad \cdots\cdots \\ \int_X f_L(x)\nu_0(dx) = \int_X f_L(x)\rho_0^p(F(dx)). \end{cases} \tag{9.39}$$

This is usually called the method of moments.

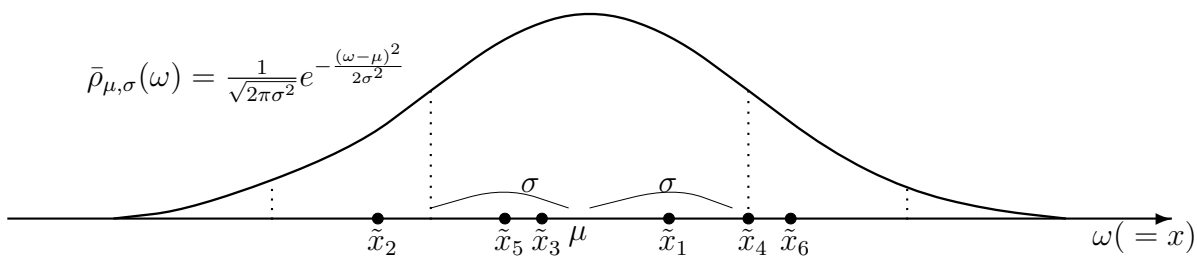
■

9.4.2 Example 1 [Normal distribution (= Gaussian distribution)]

Let $\overline{\rho}_{\mu,\sigma}$ be the Gaussian state in the commutative W^*-algebra $L^\infty(\mathbf{R}, d\omega)$ such that:

$$\rho_{\mu,\sigma}(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{(\omega-\mu)^2}{2\sigma^2}] \quad (\forall \omega \in \mathbf{R}),$$

where the average μ and the variance σ^2 are assumed to be unknown. Let $\overline{\mathbf{O}}_{\text{EXA}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \chi_{(\cdot)})$ be the *exact observable* in $L^\infty(\mathbf{R}, d\omega)$ (*cf.* Example 9.4 (i)).



Consider the statistical measurement $\overline{\mathbf{M}}_{L^\infty(\mathbf{R},d\omega)}$ $(\overline{\mathbf{O}}_{\text{EXA}}, \overline{S}(\overline{\rho}_{\mu,\sigma}))$, which may be identified with the (pure) measurement $\mathbf{M}_{C_0(\mathbf{R}\times\mathbf{R}^+)}$ $(\mathbf{O}_{\text{G}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, G),\ S_{[\delta_{(\mu,\sigma)}]})$ in $C_0(\mathbf{R}\times\mathbf{R}^+)$ (*cf.* Remark 8.3 (hybrid measurements)), where $\mathbf{O}_{\text{G}}$ is defined by i.e.,

$$[G(\Xi)](\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_\Xi \exp[-\frac{(x-\mu)^2}{2\sigma^2}]dx \quad (\forall \Xi \in \mathcal{B}_{\mathbf{R}}, \forall(\mu,\sigma) \in \mathbf{R}\times\mathbf{R}^+). \tag{9.40}$$

Assume that we take the measurement $\overline{\mathbf{M}}_{L^\infty(\mathbf{R},d\omega)}$ $(\overline{\mathbf{O}}_{\text{EXA}}, \overline{S}(\overline{\rho}_{\mu,\sigma}))$ T times, that is, we take the measurement $\overline{\mathbf{M}}_{L^\infty(\mathbf{R}^T,\otimes_{t=1}^T d\omega)}$ $(\otimes_{t=1}^T \overline{\mathbf{O}}_{\text{EXA}}, \overline{S}(\otimes_{t=1}^T \overline{\rho}_{\mu,\sigma}))$, which may be identified with

the (pure) measurement $\otimes_{t=1}^{T}\mathbf{M}_{C_0((\mathbf{R}\times\mathbf{R}^+))}$ $(\mathbf{O}_{\mathrm{G}} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, G),\ S_{[\delta_{(\mu,\sigma)}]})$ $\Big($i.e., $\mathbf{M}_{C_0((\mathbf{R}\times\mathbf{R}^+)^T)}$ $(\otimes_{t=1}^{T}\mathbf{O}_{\mathrm{G}} \equiv (\mathbf{R}^T, \mathcal{B}_{\mathbf{R}^T}, \otimes_{t=1}^{T}G),\ S_{[\otimes_{t=1}^{T}\delta_{(\mu,\sigma)}]})$ in $C_0((\mathbf{R}\times\mathbf{R}^+)^T)$ (*cf.* Remark 8.3)$\Big)$. Again note that the average μ and variance σ^2 are assumed to be unknown. Here, we have the following problem:

(P) *Under the assumption that the measured value* $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_T)$ $(\in \mathbf{R}^T)$ *is obtained by the measurement* $\otimes_{t=1}^{T}\mathbf{M}_{C_0((\mathbf{R}\times\mathbf{R}^+))}$ $(\mathbf{O}_G \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, G),\ S_{[\delta_{(\mu,\sigma)}]})$, *infer the unknown average* μ *and variance* σ^2. (9.41)

[(i): Answer (Moment method)]. The problem (P) says that we have the sample space $(\mathbf{R}, \mathcal{B}_{\mathbf{R}}, \nu_0)$ such that:

$$\nu_0 = \frac{1}{T}\sum_{t\in T}\delta_{\tilde{x}_t}\Big(\in \mathcal{M}_{+1}^{m}(\mathbf{R})\Big). \tag{9.42}$$

Thus, it suffices to solve the following equation:

$$\Delta_{\{f_1,f_2\}}(\nu_0, [G(\,\cdot\,)](\mu_0,\sigma_0)\ ) = 0, \tag{9.43}$$

where $f_k : \mathbf{R} \to \mathbf{R}$ is usually defined by $f_1(x) = x$ and $f_2(x) = x^2$. That is, seeing (9.39), we have to solve

$$\begin{cases} (1). \quad \int_{\mathbf{R}} x\nu_0(dx) = \int_{\mathbf{R}} x[G(dx)](\mu_0,\sigma_0) \\ (2). \quad \int_{\mathbf{R}} x^2\nu_0(dx) = \int_{\mathbf{R}} x^2[G(dx)](\mu_0,\sigma_0). \end{cases} \tag{9.44}$$

The above (1) clearly implies that

$$\mu_0 = \frac{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_T}{T}\Big(\equiv A_T \quad \text{say,}\Big) \tag{9.45}$$

Also, calculating (2)- (1)×(1), we get that

$$\sigma_0 = \sqrt{\frac{(\tilde{x}_1 - A_T)^2 + (\tilde{x}_2 - A_T)^2 + \cdots + (\tilde{x}_T - A_T)^2}{T}}. \tag{9.46}$$

This is the answer by the moment method.

[(ii): Answer (Fisher's maximum likelihood method)].

Next, we present the answer by Fisher's likelihood method. Note that the observable $\otimes_{t=1}^{T}\mathbf{O}_G = (\mathbf{R}^T, \mathcal{B}_{\mathbf{R}^T}^{\mathrm{bd}}, \otimes_{t=1}^{T}G \equiv \widehat{G})$ in $C_0((\mathbf{R}\times\mathbf{R}^+)^T)$ is represented by

$$[\widehat{G}(\Xi_1 \times \cdots \times \Xi_T)](\mu_1,\sigma_1,\mu_2,\sigma_2,\cdots,\mu_T,\sigma_T) = \Pi_{t=1}^{T}[G(\Xi_t)](\mu_t,\sigma_t).$$

Assume the condition in the above (P), and further add that

$$\Xi_t^\epsilon = [\tilde{x}_t - \epsilon, \tilde{x}_t + \epsilon], \quad \text{(for sufficiently small positive } \epsilon).$$

Since we take the (pure) measurement $\mathbf{M}_{C_0((\mathbf{R}\times\mathbf{R}^+)^T)}$ $(\otimes_{t=1}^T \mathbf{O}_{\mathrm{G}} \equiv (\mathbf{R}^T, \mathcal{B}_{\mathbf{R}^T}, \otimes_{t=1}^T G)$, $S_{[\otimes_{t=1}^T \delta_{(\mu,\sigma)}]})$ in $C_0((\mathbf{R}\times\mathbf{R}^+)^T)$, we see

$$\begin{aligned}
&\text{“maximum problem”} : \max_{(\mu,\sigma)\in\mathbf{R}\times\mathbf{R}^+} [\widehat{G}(\Xi_1^\epsilon \times \cdots \times \Xi_T^\epsilon)](\mu,\sigma,\mu,\sigma,\cdots,\mu,\sigma) \\
\Longleftrightarrow &\text{“maximum problem”} : \max_{(\mu,\sigma)\in\mathbf{R}\times\mathbf{R}^+} \frac{1}{\sigma^T} \exp\Big[-\sum_{t=1}^T \frac{(\tilde{x}_t-\mu)^2}{2\sigma^2}\Big] \quad \text{(since } \epsilon \text{ is small)}
\end{aligned} \tag{9.47}$$

$$\Longleftrightarrow \begin{cases} \text{(i) } \mu = \frac{\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_T}{T} & (\leftarrow \ \frac{\partial}{\partial\mu}(9.47) = 0) \\ \text{(ii) } \sigma^2 = \frac{(\tilde{x}_1-\mu)^2+(\tilde{x}_2-\mu)^2+\cdots+(\tilde{x}_T-\mu)^2}{T} & (\leftarrow \ \frac{\partial}{\partial\sigma}(9.47) = 0) \\ \text{(where } \mu \text{ is defined by in the above (i))} . & \end{cases} \tag{9.48}$$

Thus, Fisher's maximum likelihood method says that there is a reason to infer that

$$\mu = \frac{\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_T}{T} \equiv A_T, \qquad \sigma = \sqrt{\frac{(\tilde{x}_1-A_T)^2+(\tilde{x}_2-A_T)^2+\cdots+(\tilde{x}_T-A_T)^2}{T}}. \tag{9.49}$$

This is the answer by Fisher's likelihood method

9.4.3 Example 2 (measurement error model in SMT)

Put $\Omega_0 = \Omega_1 = \mathbf{R}, \Theta = \mathbf{R}^2$ and define the map $\psi^{(\theta_0,\theta_1)} : \Omega_0(\equiv \mathbf{R}) \to \Omega_1(\equiv \mathbf{R})$ such that:

$$\psi^{(\theta_0,\theta_1)}(\omega) = \theta_1\omega + \theta_0 \quad (\forall\omega \in \Omega_0(\equiv \mathbf{R}), \forall(\theta_0,\theta_1) \in \Theta \equiv \mathbf{R}^2). \tag{9.50}$$

Also, put $(X, \mathcal{F}, F) = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}^{\mathrm{bd}}, G^{\sigma_1})$ in $C_0(\Omega_0)$ and $(Y, \mathcal{G}, G) = (\mathbf{R}, \mathcal{B}_{\mathbf{R}}^{\mathrm{bd}}, G^{\sigma_2})$ in $C_0(\Omega_1)$ (*cf.* Example 2.17 (Gaussian observable)), that is,

$$[G^{\sigma_i}(\Xi)](\omega) = \frac{1}{\sqrt{2\pi}\sigma_i}\int_\Xi \exp[-\frac{(x-\omega)^2}{2\sigma_i^2}]dx \qquad (\forall\Xi \in \mathcal{B}_{\mathbf{R}}^{\mathrm{bd}}, \quad \forall\omega \in \mathbf{R}, \quad i = 1,2).$$

Define the product observable $\widetilde{\mathbf{O}}_{(\sigma_1,\sigma_2)}^{(\theta_0,\theta_1)} = (X\times Y, \mathcal{F}\times\mathcal{G}, H_{(\sigma_1,\sigma_2)}^{(\theta_0,\theta_1)} \equiv G^{\sigma_1} \times \Psi^{(\theta_0,\theta_1)} G^{\sigma_2})$ such that:

$$\begin{aligned}
&[H_{(\sigma_1,\sigma_2)}^{(\theta_0,\theta_1)}(\Xi\times\Gamma)](\omega) \\
=&\frac{1}{(2\pi)^{2/2}\sigma_1\sigma_2}\int_{\Omega_0}\int_{\Xi\times\Gamma} \exp[-\frac{(x-\omega)^2}{2\sigma_1^2} - \frac{(y-(\theta_1\omega+\theta_0))^2}{2\sigma_2^2}]dxdyd\omega
\end{aligned} \tag{9.51}$$

$$(\forall \Xi, \forall \Gamma \in \mathcal{B}^{\rm bd}_{\mathbf{R}}, \quad \forall \omega \in \Omega_0 \equiv \mathbf{R}),$$

where θ_0, θ_1 and σ_2 are assumed to be unknown, but σ_1 is known.

Let ν_{μ,σ_3} be the Gaussian state in $\mathcal{M}^m_{+1}(\Omega_0)$ such that:

$$\nu_{\mu,\sigma_3}(D) = \frac{1}{\sqrt{2\pi}\sigma_3}\int_D \exp[-\frac{(\omega-\mu)^2}{2\sigma_3^2}]d\omega \quad (\forall D \in \mathcal{B}_{\Omega_0}), \tag{9.52}$$

where the average μ and the variance $(\sigma_3)^2$ are assumed to be unknown.

Here we have the measurement $\mathbf{M}_{C_0(\Omega_0)}$ $(\widetilde{\mathbf{O}}^{(\theta_0,\theta_1)}_{(\sigma_1,\sigma_2)}, S(\overline{\rho}_{\mu,\sigma_3}))$. Define the observable $\widehat{\mathbf{O}}$ $= (X \times Y, \mathcal{F} \times \mathcal{G}, \widehat{H})$ in $C_0(\Theta \times ((\mathbf{R}^+)^3 \times \mathbf{R})^9$ such that:

$$\begin{aligned}&[\widehat{H}(\Xi \times \Gamma)](\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu) = {}_{\mathcal{M}(\Omega_0)}\Big\langle \nu_{\mu,\sigma_3}, H^{(\theta_0,\theta_1)}_{(\sigma_1,\sigma_2)}(\Xi \times \Gamma)\Big\rangle_{C_0(\Omega_0)}\\ =&\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\int_{\Omega_0}\int_{\Xi\times\Gamma}\exp[-\frac{(x-\omega)^2}{2\sigma_1^2}-\frac{(y-(\theta_1\omega+\theta_0))^2}{2\sigma_2^2}-\frac{(\omega-\mu)^2}{2\sigma_3^2}]dxdyd\omega\end{aligned} \tag{9.53}$$

$$(\forall \Xi, \forall \Gamma \in \mathcal{B}^{\rm bd}_{\mathbf{R}}, \quad \forall \omega \in \Omega_0 \equiv \mathbf{R}).$$

Thus we have the identification:

$$\mathbf{M}_{C_0(\Omega_0)}(\widetilde{\mathbf{O}}^{(\theta_0,\theta_1)}_{(\sigma_1,\sigma_2)}, S(\nu_{\mu,\sigma_3})) \longleftrightarrow \mathbf{M}_{C_0(\Theta\times((\mathbf{R}^+)^3\times\mathbf{R})}(\widehat{\mathbf{O}}, S_{[\delta_{(\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu)}]}).$$

Thus, we have the sample space $(\mathbf{R}^2, \mathcal{B}^{\rm bd}_{\mathbf{R}^2}, \nu^{(\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu)})$ such that:

$$\nu^{(\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu)}(\Xi \times \Gamma) = [\widehat{H}(\Xi \times \Gamma)](\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu) \quad (\forall \Xi, \forall \Gamma \in \mathcal{B}_{\mathbf{R}}). \tag{9.54}$$

Here, we have the following problem:

(P) *Assume that we take the measurement* $\mathbf{M}_{C_0(\Theta\times((\mathbf{R}^+)^3\times\mathbf{R})}(\widehat{\mathbf{O}},\ S_{[\delta_{(\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu)}]})$ *T-times, and get the measured value* $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ ..., \tilde{x}_T, \tilde{y}_T)$ $(\in \mathbf{R}^{2T})$. *Here it is assumed that* $\theta_0, \theta_1, \sigma_2, \sigma_3$ *and* μ *are unknown (but* σ_1 *is known). Then, infer* θ_0 *and* θ_1 *(and moreover* σ_2, σ_3 *and* μ*) from the measured value* $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ ..., \tilde{x}_T, \tilde{y}_T)$ $(\in \mathbf{R}^{2T})$ *and the known* σ_1. (9.55)

[(i): Answer (Moment method)].

[9] If $\Theta \times (\mathbf{R}^+)^3 \times \mathbf{R}$ is required to be compact, it suffices to consider $[-L, L]^2 \times [(1/L), L]^3 \times [-L, L]$ (for sufficiently large L) instead of $\Theta \times (\mathbf{R}^+)^3 \times \mathbf{R}$.

Under the notation in the problem (P), put

$$A_T^{\tilde{X}} = \frac{\widetilde{x}_1 + \widetilde{x}_2 + \cdots + \widetilde{x}_T}{T}, \qquad A_T^{\tilde{Y}} = \frac{\widetilde{y}_1 + \widetilde{y}_2 + \cdots + \widetilde{y}_T}{T}, \tag{9.56}$$

$$\overline{V}_T^{\tilde{X}\tilde{X}} = \frac{(\widetilde{x}_1 - A_T^{\tilde{X}})^2 + (\widetilde{x}_1 - A_T^{\tilde{X}})^2 + \cdots + (\widetilde{x}_T - A_T^{\tilde{X}})^2}{T}, \tag{9.57}$$

$$\overline{V}_T^{\tilde{Y}\tilde{Y}} = \frac{(\widetilde{y}_1 - A_T^{\tilde{X}})^2 + (\widetilde{y}_1 - A_T^{\tilde{X}})^2 + \cdots + (\widetilde{y}_T - A_T^{\tilde{X}})^2}{T}, \tag{9.58}$$

$$\overline{V}_T^{\tilde{X}\tilde{Y}} = \frac{(\widetilde{x}_1 - A_T^{\tilde{X}})(\widetilde{y}_1 - A_T^{\tilde{X}}) + (\widetilde{x}_2 - A_T^{\tilde{X}})(\widetilde{y}_2 - A_T^{\tilde{X}}) + \cdots + (\widetilde{x}_T - A_T^{\tilde{X}})(\widetilde{y}_T - A_T^{\tilde{X}})}{T}. \tag{9.59}$$

Recall (9.54), and put

$$\mu_X = \mu, \quad \sigma_{uu} = \sigma_1, \quad \sigma_{ee} = \sigma_2, \quad \sigma_{XX} = \sigma_3, \quad \mu_Y = \int_{\mathbf{R}} y[\widehat{H}(\mathbf{R} \times dy)]. \tag{9.60}$$

Then we see that

$$A_T^{\tilde{Y}} = \int_{\mathbf{R}} y[\widehat{H}(\mathbf{R} \times dy)](\equiv \mu_Y) = \theta_0 + \theta_1 \mu_X, \qquad A_T^{\tilde{X}} = \int_{\mathbf{R}} x[\widehat{H}(dx \times \mathbf{R})] = \mu_X, \tag{9.61}$$

and

$$\overline{V}_T^{\tilde{Y}\tilde{Y}} = \int_{\mathbf{R}} (y - \mu_Y)^2 [\widehat{H}(\mathbf{R} \times dy)] = \theta_1^2 \sigma_{XX}^2 + \sigma_{ee}^2, \tag{9.62}$$

$$\overline{V}_T^{\tilde{X}\tilde{Y}} = \int_{\mathbf{R}} (x - \mu_X)^2 [\widehat{H}(dx \times \mathbf{R})] = \theta_1 \sigma_{XX}^2, \tag{9.63}$$

$$\overline{V}_T^{\tilde{X}\tilde{X}} = \int_{\mathbf{R}^2} (x - \mu_X)(y - \mu_Y) [\widehat{H}(dx \times dy)] = \sigma_{XX}^2 + \sigma_{uu}^2, \tag{9.64}$$

which is easily solved. Thus, the moment method says that there is a reason to infer that

$$\theta_1 = (\overline{V}_T^{\tilde{X}\tilde{X}} - \sigma_1^2)^{-1} \overline{V}_T^{\tilde{X}\tilde{Y}}, \qquad \theta_0 = A_T^{\tilde{Y}} - (\overline{V}_T^{\tilde{X}\tilde{X}} - \sigma_1^2)^{-1} A_T^{\tilde{X}} \overline{V}_T^{\tilde{X}\tilde{Y}}. \tag{9.65}$$

[(ii): Answer (Fisher's maximum likelihood method)].

Next, we shall answer the problem (P) by Fisher's likelihood method. Put, for sufficiently small positive ϵ,

$$\Xi_t^\epsilon = [\tilde{x}_t - \epsilon, \tilde{x}_t + \epsilon], \quad \Gamma_t^\epsilon = [\tilde{y}_t - \epsilon, \tilde{y}_t + \epsilon] \quad (t = 1, 2, \cdots, T). \tag{9.66}$$

The probability that the measured value $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ ..., \tilde{x}_T, \tilde{y}_T)$ ($\in \mathbf{R}^{2T}$) belongs to $\Pi_{t=1}^T (\Xi_t^\epsilon \times \Gamma_t^\epsilon)$ is given by

$$\Pi_{t=1}^T \Big[[\widehat{H}(\Xi_t^\epsilon \times \Gamma_t^\epsilon)](\theta_0, \theta_1, \sigma_1, \sigma_2, \sigma_3, \mu) \Big]. \tag{9.67}$$

Since ϵ is sufficiently small, we see, for some fixed σ_1, that

$$\max_{(\theta_0,\theta_1,\sigma_2,\sigma_3,\mu)\in\Theta\times(\mathbf{R}^+)^2\times\mathbf{R}} \Pi_{t=1}^T[\widehat{H}(\Xi_t^\epsilon \times \Gamma_t^\epsilon)](\theta_0,\theta_1,\sigma_1,\sigma_2,\sigma_3,\mu)$$
$$\Longleftrightarrow \max_{(\theta_0,\theta_1,\sigma_2,\sigma_3,\mu)\in\Theta\times(\mathbf{R}^+)^2\times\mathbf{R}} \Pi_{t=1}^T[\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\int_{\Omega_0(\equiv\mathbf{R})} e^{[-\frac{(\tilde{x}_t-\omega)^2}{2\sigma_1^2}-\frac{(\tilde{y}_t-(\theta_1\omega+\theta_0))^2}{2\sigma_2^2}-\frac{(\omega-\mu)^2}{2\sigma_3^2}]}d\omega]. \tag{9.68}$$

Thus, Fisher's maximum likelihood method says that it suffices to find the $(\theta_0,\theta_1,\sigma_2,\sigma_3,\mu)$ such that:

$$\Pi_{t=1}^T[\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\int_{\mathbf{R}} e^{[-\frac{(\tilde{x}_t-\omega)^2}{2\sigma_1^2}-\frac{(\tilde{y}_t-(\theta_1\omega+\theta_0))^2}{2\sigma_2^2}-\frac{(\omega-\mu)^2}{2\sigma_3^2}]}d\omega]$$
$$= \max_{(\theta_0,\theta_1,\sigma_2,\sigma_3,\mu)\in\Theta\times(\mathbf{R}^+)^2\times\mathbf{R}} \Pi_{t=1}^T[\frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\int_{\mathbf{R}} e^{[-\frac{(\tilde{x}_t-\omega)^2}{2\sigma_1^2}-\frac{(\tilde{y}_t-(\theta_1\omega+\theta_0))^2}{2\sigma_2^2}-\frac{(\omega-\mu)^2}{2\sigma_3^2}]}d\omega]. \tag{9.69}$$

However, it may be difficult to solve it analytically. Thus, the numerical computation may be recommended.

Remark 9.18. Comparing (9.65) and (9.69), readers may consider that the moment method is simple and powerful. However, it should be noted that the moment method is somewhat artificial since the semi-distance is not unique. Summing up, we see,

	Inference	Example
(pure) measurement	Fisher's likelihood method (Theorem 5.3, Corollary 5.6)	Examples 5.8 and 5.9 Regression analysis (6.7), (6.48)
statistical measurement	Bayes' method (Theorem 8.13, Remark 8.14)	Example 8.6 Generalized Bayes theorem (Theorem 8.13)
repeated measurement (product sample space)	moment method (Definition 2.27) See §9.4.1	Normal distribution (§9.4.2) measurement error model (§9.4.3)

(9.70)

■

9.5 Principal components analysis in MT

Our present purpose is to study "principal components analysis" in the framework of MT.

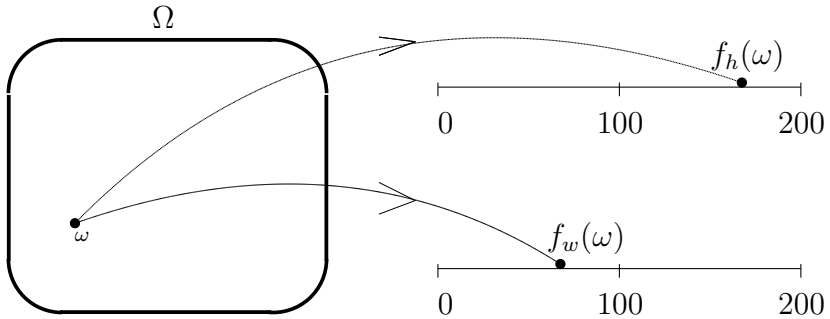
Consider the following two cases [I] and [II]:

[I: Homomorphic type]. Let Ω be a compact space. For each k ($= 1, 2, ..., K$), consider a continuous map $f_k : \Omega \to \mathbf{R}$. For example, we may consider that

(♯) the Ω ($= \{\omega_1, \omega_2, ..., \omega_N\}$ represents the class of students in some high school. And further, assume that

(a) $f_h(\omega_n) \cdots$ the student ω_n's height

(b) $f_w(\omega_n) \cdots$ the student ω_n's weight

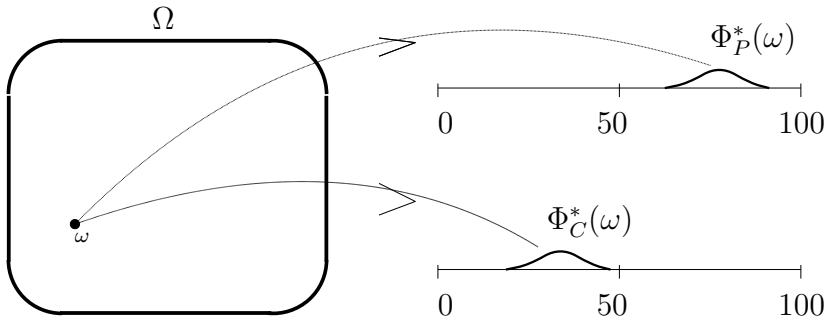


[II: Markov type]. Let Ω be a compact space. For each k ($= 1, 2, ..., K$), consider a map $\Phi_k^* : \Omega \to \mathcal{M}_{+1}^m(\mathbf{R})$ in the C^*-algebraic formulation $\Big($or, $\Phi_k^* : \Omega \to L_{+1}^1(\mathbf{R}; dm)$ in the W^*-algebraic formulation$\Big)$. For example, we may consider that

($\sharp$) the Ω ($\equiv \{\omega_1, \omega_2, ..., \omega_N\}$) represents the set of students in some high school. And further, assume that

(a) $\Phi_P^*(\omega_n) \cdots$ the student ω_n's scholastic ability of physics (or, the distribution of the student ω_n's marks (e.g., deviation values) in physics)

(b) $\Phi_C^*(\omega_n) \cdots$ the student ω_n's scholastic ability of chemistry (or, the distribution of the student ω_n's marks (e.g., deviation values) in chemistry)



Here consider the following problem:

(P) What kind of relation among the height and weight in [I] (or, the scores of physics and chemistry in [II]) of the students of the high school can we find?

This problem (P) is usually studied by "principal components analysis". Thus, in what follows, we shall study it in the framework of PMT^{W^*} (though it can be also studied

in PMT^{C^*} since a cyclic measurement is also formulated in PMT^{C^*}). Clearly the homomorphic type [I] is the special case of the Markov type [II]. Thus, from here, we devote ourselves to the Markov type [II].

Let Ω be a finite set, i.e., $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$, which is assumed to have the counting measure ν_c, that is, $\nu_c(A) = \sharp[A]$ $(\forall A \subseteq \Omega)$. For each k $(= 1, 2, ..., K)$, consider a Markov operator $\Phi_k : L^\infty(\mathbf{R}, m) \to L^\infty(\Omega, \nu_c)$, where m is the Lebesgue measure on $\mathbf{R}$. Let $\mathbf{O} \equiv (\mathbf{R}, \mathcal{B}_{\mathbf{R}}, E_{\mathrm{EXA}})$ be the exact observable in $L^\infty(\mathbf{R}, m)$. Define the observable $\widehat{\mathbf{O}} \equiv (\mathbf{R}^K, \mathcal{B}_{\mathbf{R}^K}, \times_{k=1}^K \Phi_k E_{\mathrm{EXA}})$ in $L^\infty(\Omega, \nu_c)$ such that

$$[\mathop{\times}_{k=1}^{K} \Phi_k E_{\mathrm{EXA}}](\Xi_1 \times \Xi_2 \times \cdots \times \Xi_K) = \mathop{\times}_{k=1}^{K} \Phi_k E_{\mathrm{EXA}}(\Xi_k) \quad (\Xi_1 \times \Xi_2 \times \cdots \times \Xi_K \in \mathcal{B}_{\mathbf{R}^K})$$

which is realization of the sequential observable $[\{\mathbf{O}\}_{k=1}^K, \{\Phi_k : L^\infty(\mathbf{R}, m) \to L^\infty(\Omega, \nu_c)\}_{k=1}^K]$. Thus we have the cyclic measurement $\otimes_{j=1}^{NL} \mathbf{M}_{L^\infty(\Omega,\nu_c)}(\widehat{\mathbf{O}}, \overline{S}(\bar{\rho}_{\omega_{\mathrm{mod}_N[j]}}))$, where $\bar{\rho}_{\omega_s} \in L^1_{+1}(\Omega, \nu_c)$, $(s = 1, 2, ..., N)$, is defined by $\bar{\rho}_{\omega_s}(\omega) = 1$ (if $\omega = \omega_s$), $= 0$ (if $\omega \neq \omega_s$).

Assume that, by the cyclic measurement $\otimes_{j=1}^{NL} \mathbf{M}_{L^\infty(\Omega,\nu_c)}(\widehat{\mathbf{O}}, \overline{S}(\bar{\rho}_{\omega_{\mathrm{mod}_N[j]}}))$ $\Big($or, the repeated measurement $\otimes_{j=1}^{LN} \mathbf{M}_{L^\infty(\Omega,\nu_c)}(\widehat{\mathbf{O}}, \overline{S}(1/N))$, *cf.* Example 8.7 (ii) $\Big)$, we get a measured value $(x_1, x_2, ..., x_{NL})$, where

$$\left.\begin{array}{l}
x_1 = (x_1^1, x_1^2, ..., x_1^K), \\
x_2 = (x_2^1, x_2^2, ..., x_2^K), \\
\quad\vdots \\
x_N = (x_N^1, x_N^2, ..., x_N^K) \\
x_{N+1} = (x_{N+1}^1, x_{N+1}^2, ..., x_{N+1}^K) \\
\quad\vdots \\
x_{2N} = (x_{2N}^1, x_{2N}^2, ..., x_{2N}^K) \\
\quad\vdots \\
x_{3N} = (x_{3N}^1, x_{3N}^2, ..., x_{3N}^K) \\
\quad\vdots \\
x_{LN} = (x_{NL}^1, x_{NL}^2, ..., x_{NL}^K)
\end{array}\right\} \tag{9.71}$$

Here, note that it holds:

$$\begin{aligned}
&\lim_{L\to\infty} \frac{\sharp[\{j \in \{1, 2, ..., NL\} \ : \ x_j \in \Xi_1 \times \Xi_2 \times \cdots \times \Xi_K\}]}{NL} \\
&= \int_\Omega \frac{[\times_{k=1}^K \Phi_k E_{\mathrm{EXA}}(\Xi_k)](\omega)}{N} \nu_c(d\omega) \qquad (\forall \Xi_1 \times \Xi_2 \times \cdots \times \Xi_K \in \mathcal{B}_{\mathbf{R}^K})
\end{aligned}$$

Put

$$(\mu_1, \mu_2, ..., \mu_K) = \Big(\frac{\sum_{j=1}^{NL} x_j^1}{NL}, \frac{\sum_{j=1}^{NL} x_j^2}{NL}, \cdots, \frac{\sum_{j=1}^{NL} x_j^K}{NL}\Big) \tag{9.72}$$

and put

$$C_{pq} = \frac{\sum_{j=1}^{NL}(x_p^j - \mu_p)(x_q^j - \mu_q)}{NL-1} \tag{9.73}$$

$\Big($For simplicity, here we are not concerned with the normalization, though it is reasonable.$\Big)$ Then, we have the correlation matrix C such that:

$$C = \Big[C_{pq}\Big]_{1\leq p,q\leq K} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1K} \\ C_{21} & C_{22} & \dots & C_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ C_{K1} & C_{K2} & \dots & C_{KK} \end{bmatrix}, \tag{9.74}$$

which is represented by

$$C = P\Lambda P^*$$

where Λ is a diagonal matrix such that:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_K \end{bmatrix} \quad (\lambda_1 \geq \lambda_2 \geq ... \geq 0)$$

and P is the orthonormal matrix such that:

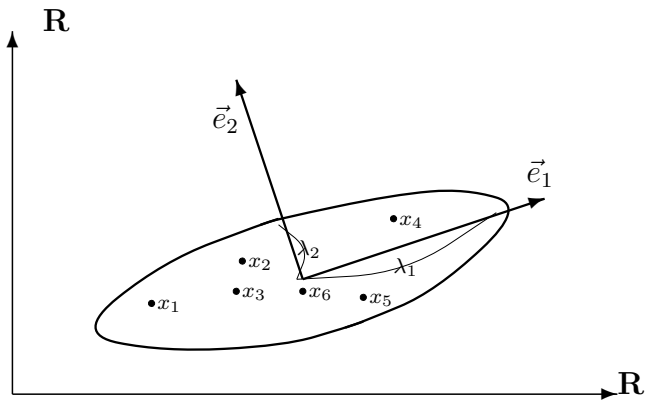
$$P = \Big[\vec{e}_1, \vec{e}_2, ..., \vec{e}_K\Big] = \begin{bmatrix} e_{11} & e_{12} & \dots & e_{1K} \\ e_{21} & e_{22} & \dots & e_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ e_{K1} & e_{K2} & \dots & e_{KK} \end{bmatrix}, \quad \vec{e}_k = \begin{bmatrix} e_{1k} \\ e_{1k} \\ \vdots \\ e_{1k} \end{bmatrix},$$

where

$$\langle \vec{e}_k, \vec{e}_{k'} \rangle_{\mathbf{R}^K} = \begin{cases} 1 & (\text{ if } k = k') \\ 0 & (\text{ if } k \neq k') \end{cases}.$$

Here, $\vec{e}_k$ is called the k-th principal component. Also, The k-contribution ratio is defined by $\frac{\lambda_k}{\sum_{i=1}^K \lambda_i}$.

Remark 9.19. [(i): Several interpretations of principal components analysis]. Principal components analysis (i.e., $\{(\vec{e}_k, \lambda_k)\}_{k=1}^K$) has several interpretations, which are important. For example, the following figure is frequently stated in usual books of statistics.



However, we are not concerned with it, because what we want to say here is the following (ii).

[(ii): Markov type and homomorphic type]. Note that the data (9.71) is obtained by the exact measurement. Thus the $\sqrt{C_{pp}}$ is not the error. In the case of Markov type, the following calculation is wrong. However, if $\Phi_k : L^\infty(\mathbf{R}, m) \to L^\infty(\Omega, \nu_c)$ is homomorphic, and if the observable $\widehat{\mathbf{O}}$ has the form such as $(\mathbf{R}^K.\mathcal{B}_{\mathbf{R}^K}, \times_{k=1}^K \Phi_k G^{\sigma_k})$ in $L^\infty(\Omega, \nu_c)$ where

$$G_\Xi^{\sigma_k}(\mu) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \int_\Xi e^{-\frac{(u-\mu)^2}{2\sigma_k^2}} du \qquad (\forall \mu \in \mathbf{R} \equiv \mathbf{R},\ \forall \Xi \in \mathcal{B}_{\mathbf{R}}). \quad (\sigma_k^2\text{: variance}),$$

(*cf.* Example 9.5 and Example 2.17), then the following calculation should be recommended: Put

$$\begin{aligned}
\bar{x}_1 &= \Big(\frac{\sum_{l=0}^{L-1} x_{1+lN}^1}{L}, \frac{\sum_{l=0}^{L-1} x_{1+lN}^2}{L}, \cdots, \frac{\sum_{l=0}^{L-1} x_{1+lN}^K}{L}\Big),\\
\bar{x}_2 &= \Big(\frac{\sum_{l=0}^{L-1} x_{2+lN}^1}{L}, \frac{\sum_{l=0}^{L-1} x_{2+lN}^2}{L}, \cdots, \frac{\sum_{l=0}^{L-1} x_{2+lN}^K}{L}\Big),\\
&\vdots\\
\bar{x}_N &= \Big(\frac{\sum_{l=0}^{L-1} x_{N+lN}^1}{L}, \frac{\sum_{l=0}^{L-1} x_{N+lN}^2}{L}, \cdots, \frac{\sum_{l=0}^{L-1} x_{N+lN}^K}{L}\Big).
\end{aligned}$$

Put

$$(\bar{\mu}_1, \bar{\mu}_2, ..., \bar{\mu}_K) = \Big(\frac{\sum_{j=1}^N \bar{x}_j^1}{N}, \frac{\sum_{j=1}^N \bar{x}_j^2}{N}, \cdots, \frac{\sum_{j=1}^N \bar{x}_j^K}{N}\Big), \tag{9.75}$$

and put

$$\bar{C}_{pq} = \frac{\sum_{j=1}^N (\bar{x}_p^j - \bar{\mu}_p)(\bar{x}_q^j - \bar{\mu}_q)}{N}. \tag{9.76}$$

Then, we have the correlation matrix $\bar{C}$ such that:

$$\bar{C} = \left[\bar{C}_{pq}\right]_{1\leq p,q\leq K} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \dots & \bar{C}_{1K} \\ \bar{C}_{21} & \bar{C}_{22} & \dots & \bar{C}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}_{K1} & \bar{C}_{K2} & \dots & \bar{C}_{KK} \end{bmatrix}.$$

Thus, by a similar way, we can get the k-th principal component and the k-contribution ratio, etc.

Note that it holds:

$$(9.74) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1K} \\ C_{21} & C_{22} & \dots & C_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ C_{K1} & C_{K2} & \dots & C_{KK} \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} + (\sigma_1)^2 & \bar{C}_{12} & \dots & \bar{C}_{1K} \\ \bar{C}_{21} & \bar{C}_{22} + (\sigma_2)^2 & \dots & \bar{C}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}_{K1} & \bar{C}_{K2} & \dots & \bar{C}_{KK} + (\sigma_K)^2 \end{bmatrix}$$

$$=(9.77) + \begin{bmatrix} (\sigma_1)^2 & 0 & \dots & 0 \\ 0 & (\sigma_2)^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\sigma_K)^2 \end{bmatrix}$$

though the situations are different.

■