Chapter 7

Practical logic

It is certain that pure logic (*cf.* [89]) is merely a kind of rule in mathematics (or meta-mathematics). However, if it is so, the logic is not guaranteed to be applicable to our world. For instance, (pure) logic does not assure the following famous statement:

[♯] *Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.*

That is, we think that the problem: "Is the [♯] true or not?" should be answered. Thus, the purpose of this chapter is to prove the statement [♯], or more generally, to propose "practical logic", i.e., "logic with an interpretation", [1] which is formulated in the framework of the measurement theory:

$$\text{PMT} = \underset{\text{[Axiom 1 (2.37)]}}{\text{measurement}} + \underset{\text{[Axiom 2 (3.26)]}}{\text{the relation among systems}} \quad \text{in } C^*\text{-algebra} \quad . \tag{7.1}$$

(=(1.4))

Firstly, the symbol "$A \Rightarrow B$" (i.e., "implication") is defined in terms of measurements. And we prove the standard syllogism for classical systems:

$$\text{“}A \Rightarrow B,\ B \Rightarrow C\text{” implies “}A \Rightarrow C\text{”} \tag{7.2}$$ [2]

(This is not trivial, because the (7.2) does not necessarily hold in quantum systems.) We can assert, by "Declaration (1.11)" in §1.4, that this theorem (7.2) guarantees that the above (7.2) (or, the statement [♯]) is "theoretical true". Several variants may be interesting. For example, under the condition that "$A \Rightarrow B$, $B \Rightarrow C$", we can assert a kind of conclusion such as "$C \Rightarrow A$". For completeness, "pure logic" and "practical logic" must not be confused. The former is a basic rule on which mathematics is founded. On the other hand, the latter is a collection of theorems (whose forms are similar to that of "pure logic") in MT. All results in this chapter are due to [41]. Also, this chapter can be skipped if readers want to study statistics in the framework of SMT firstly (*cf.* Chapters 8).

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[1] We have no confidence for the naming "practical logic". We may choose the other namings: "empirical logic", "applied logic", "usual logic". etc.

[2] It is said that the syllogism is said to be, for the first time, introduced by Aristotle (B.C.384-B.C.322)

7.1 Measurement, inference, control and practical logic

The PMT has various aspects. For example, we believe that three concepts: "measurement", "inference", and "control" are different aspects of the same thing. Let us explain it as follows: Let $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O} \equiv (X, \mathcal{F}, F),\ S_{[\rho^p]}\big)$ be a measurement formulated in a C^*-algebra $\mathcal{A}$. Note that Axiom 1 can be regarded as the answer to the following problem:

(M) What kind of measured value is obtained by a measurement $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O},\ S_{[\rho^p]}\big)$?

As mentioned in Chapter 5, the measurement $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O},\ S_{[\rho^p]}\big)$ is often denoted by $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O},\ S_{[*]}\big)$, if we want to stress that we do not know the state ρ^p. Using this notation, we can respectively characterize "inference (I)" and "control (C)" as follows:

(I) Assume that we get a measured value $x(\in X)$ by a measurement $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O},\ S_{[*]}\big)$. Then, infer the state $[\,*\,]$,

and

(C) Assume that we want to get a measured value $x(\in X)$ by a measurement $\mathbf{M}_{\mathcal{A}}\big(\mathbf{O},\ S_{[*]}\big)$. Then, settle the state $[\,*\,]$.

Of course, Fisher's maximum likelihood method is one of the answers of the above problems (I) and (C).

Also, we think that

(L) "Practical logic" is characterized as "a qualitative theory concerning conditional probability (*cf.* §2.5 (IV)) in PMT".

Thus "practical logic" is also one of the aspects of Axiom 1. Also, since "(practical) logic" is a qualitative aspect of "inference", we can say that "(practical) logic" [resp. "inference"] is used in rough [resp. precise] arguments. For completeness, "pure logic" and "practical logic" must not be confused. The former is a basic rule on which mathematics is built. And thus it is not related to our world. On the other hand, the latter is a collection of theorems (whose forms are similar to that of "pure logic") in PMT. Since practical logic is regarded as a theorem in PMT, it automatically possesses the measurement theoretical interpretation. That is, we think that

"practical logic" = "theorems (whose forms are similar to (pure) logic) in MT".

Recall, throughout this book, that the *measured value set* (or, *label set*) X is assumed to be finite if we write $(X, 2^X, F)$ (or, $(X, \mathcal{P}(X), F)$ and not $(X, \mathcal{F}, F)$. In this chapter we always assume that X is finite.

7.2 Quasi-product observables with dependence

We begin with the following definition.

Definition 7.1. [Marginal observable, quasi-product observable, consistency. (*cf.* Definition 2.10.)]. *Let $\mathcal{A}$ be a C^*-algebras. Let $K = \{1, 2, ..., |K|\}$.*

(i). *Consider an observable $\mathbf{O} \equiv (\times_{k \in K} X_k,\ 2^{\times_{k\in K} X_k},\ F)$ (with a label set $\times_{k \in K} X_k$) in $\mathcal{A}$. Let D be $D \subseteq K$. An observable $\mathbf{O}_D \equiv (\times_{k \in D} X_k,\ 2^{\times_{k\in D} X_k},\ F_D)$ in $\mathcal{A}$ is called a D-marginal observable of $\mathbf{O}$ if it satisfies that*

$$F_D(\mathop{\times}_{k\in D} \Xi_k) = F\Big((\mathop{\times}_{k\in D} \Xi_k) \times (\mathop{\times}_{k\in K\setminus D} X_k)\Big),$$

for all $\Xi_k \in 2^{X_k}$, $k \in D$. Also this $\mathbf{O}_D$ is denoted by $\mathbf{O}\big|_D$. Here note that the marginal observable $\mathbf{O}\big|_D$ is equal to the image observable $\mathbf{O}_{[g_D]}$ where $\times_{k \in K} X_k \ni (x_k)_{k \in K} \overset{g_D}{\longmapsto} (x_k)_{k\in D} \in \times_{k\in D} X_k$.

(ii). *For each $k \in K$, consider an observable $\mathbf{O}_k \equiv (X_k, 2^{X_k}, F_k)$ in $\mathcal{A}$. If there exists an observable $\mathbf{O}_K \equiv (\times_{k \in K} X_k, 2^{\times_{k\in K} X_k}, F)$ in $\mathcal{A}$ such that $\mathbf{O}_K\big|_{\{k\}} = \mathbf{O}_k$ for all $k \in K$, then $[\mathbf{O}_k : k \in K]$ is called consistent. Also, this $\mathbf{O}_K$ is called a quasi-product observable of $[\mathbf{O}_k : k \in K]$, and is sometimes denoted by $(\times_{k \in K} X_k, 2^{\times_{k\in K} X_k}, \times^{\mathbf{O}_K}_{k\in K} F_k)$, or $\times^{\mathbf{O}_K}_{k\in K} \mathbf{O}_k$ $\Big($or, $(\times_{k \in K} X_k, 2^{\times_{k\in K} X_k}, \overset{\text{qp}}{\times}_{k\in K} F_k)$, or $\overset{\text{qp}}{\times}_{k\in K} \mathbf{O}_k\Big)$.*

■

Note that the consistency of observables $[(X_k, 2^{X_k}, F_k) : k \in K]$ in $\mathcal{A}$ is not guaranteed in general. If the commutativity condition:

$$F_{k_1}(\Xi_{k_1})F_{k_2}(\Xi_{k_2}) = F_{k_2}(\Xi_{k_2})F_{k_1}(\Xi_{k_1}) \quad (\forall \Xi_{k_1} \in 2^{X_{k_1}},\ \forall \Xi_{k_2} \in 2^{X_{k_2}},\ k_1 \neq k_2)$$

holds, then we can construct a quasi-product observable $\mathbf{O} \equiv (\times_{k \in K} X_k, 2^{\times_{k \in K} X_k}, F \equiv \times^{\mathbf{O}}_{k \in K} F_k)$ such that:

$$F(\Xi_1 \times \Xi_2 \times \cdots \times \Xi_{|K|}) = F_1(\Xi_1) F_2(\Xi_2) \cdots F_{|K|}(\Xi_{|K|}).$$

It is, of course, the case that the uniqueness is not guaranteed even under the above commutativity condition.

Remark 7.2. [Only one measurement is permitted (*cf.* §2.5. Remarks (II))]. If we want the data concerning both $\mathbf{O}_1$ and $\mathbf{O}_2$ for the system $S_{[\rho^p]}$, we must take a simultaneous measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{12} \equiv \mathbf{O}_1 \times^{\mathbf{O}_{12}} \mathbf{O}_2, S_{[\rho^p]})$. Therefore, if a quasi-product observable $\mathbf{O}_{12}$ does not exist (i.e., $[\mathbf{O}_1, \mathbf{O}_2]$ is not consistent), the concept of "the data concerning $\mathbf{O}_1$ and $\mathbf{O}_2$ for the system $S_{[\rho^p]}$" is nonsense, i.e., it has no reality. This is a prevalent notion in quantum theory as in the case that the concept "the momentum and position of a particle" or "the trajectory of a particle" is meaningless in quantum theory. (For the recent results, see [37, 40].) It should be emphasized that the importance of this spirit (i.e., "the consistency of $[\mathbf{O}_1, \mathbf{O}_2]$" $\Leftrightarrow$ "the reality of data") is essential.

■

As the classical PMT is rather easy, people tend to overlook important facts in classical systems. Since quantum theory is moderately difficult, it is rather handy compared to classical theory.

Let $X = \{x^1, x^2, ..., x^J\}$. Let $\mathbf{O} \equiv (X, 2^X, F)$ be an observable in a commutative C^*-algebra $\mathcal{A}$ (hence by Gelfand theorem, we can assume that $\mathcal{A} = C(\Omega)$). We can consider the following identification:

$$(X, 2^X, F) \longleftrightarrow \Big[\, [F(\{x^j\})](\omega) \; : \; j = 1, 2, ..., J \, \Big]$$

(where $F(\{x^j\}) \equiv [F(\{x^j\})] \in C(\Omega)$), and therefore denote

$$\mathrm{Rep}[\mathbf{O}] = \mathrm{Rep}[(X, 2^X, F)] = \Big[[F(\{x^j\})](\omega) \; : \; j = 1, 2, ..., J \, \Big].$$

It is clear that

$$0 \le [F(\{x^j\})](\omega) \le 1 \quad \text{and} \quad \sum_{j=1}^{J} [F(\{x^j\})](\omega) = 1 \qquad (\forall \omega \in \Omega),$$

that is, $\mathrm{Rep}[(X, 2^X, F)]$ is considered to be the resolution of the identity (*cf.* §2.3).

Consider two observables $\mathbf{O}_1 \equiv (X_1, 2^{X_1}, F_1)$ and $\mathbf{O}_2 \equiv (X_2, 2^{X_2}, F_2)$ in $C(\Omega)$ such that:

$$X_1 = \{x_1^1, x_1^2, ..., x_1^{J_1}\} \quad \text{and} \quad X_2 = \{x_2^1, x_2^2, ..., x_2^{J_2}\}.$$

Let $\mathbf{O}_{12} \equiv (X_1 \times X_2, 2^{X_1 \times X_2}, F \equiv F_1 \overset{\mathbf{O}_{12}}{\times} F_2)$ be a quasi-product observable with the marginal observables $\mathbf{O}_1$ and $\mathbf{O}_2$. (The existence of $\mathbf{O}_{12}$ is guaranteed by Theorem 2.11 since $C(\Omega)$ is commutative.) Put

$$\text{Rep}[\mathbf{O}_{12}] = \begin{bmatrix} [F(\{(x_1^1, x_2^1)\})](\omega) & [F(\{(x_1^1, x_2^2)\})](\omega) & \dots & [F(\{(x_1^1, x_2^{J_2})\})](\omega) \\ [F(\{(x_1^2, x_2^1)\})](\omega) & [F(\{(x_1^2, x_2^2)\})](\omega) & \dots & [F(\{(x_1^2, x_2^{J_2})\})](\omega) \\ \vdots & \vdots & \ddots & \vdots \\ [F(\{(x_1^{J_1}, x_2^1)\})](\omega) & [F(\{(x_1^{J_1}, x_2^2)\})](\omega) & \dots & [F(\{(x_1^{J_1}, x_2^{J_2})\})](\omega) \end{bmatrix}.$$

Let $X = \{x^1, x^2, ..., x^J\}$. Let $\mathbf{O} \equiv (X, 2^X, F)$ be an observable in a C^*-algebra $\mathcal{A}$. Put $X = \Xi_y \bigcup \Xi_n$ (where $\Xi_y \bigcap \Xi_n = \emptyset$). Define the map $g : X \to X_{(2)} \equiv \{y, n\}$ such that $g(x) = y$ (if $x \in \Xi_y$), $= n$ (if $x \in \Xi_n$). Here we can define the two-valued observable $(X_{(2)} \equiv \{y, n\}, 2^{X_{(2)}}, F_{(2)})$ in $\mathcal{A}$ as the image observable $\mathbf{O}_{[g]}$. This two-valued observable is also called *yes-no observable* or $1-0$ *observable.* The following lemma says about the conditions that a quasi-product observable of yes-no observables should satisfy.

Lemma 7.3. [The existence condition of yes-no quasi-product observable]. *Consider yes-no observables* $\mathbf{O}_1 \equiv (X_1, 2^{X_1}, F_1)$ *and* $\mathbf{O}_2 \equiv (X_2,\ 2^{X_2},\ F_2)$ *in a commutative* C^**-algebra* $C(\Omega)$ *such that:*

$$X_1 = \{y_1, n_1\} \quad \textit{and} \quad X_2 = \{y_2, n_2\}.$$

Let $\mathbf{O}_{12} \equiv (X_1 \times X_2, 2^{X_1 \times X_2}, F \equiv F_1 \overset{\mathbf{O}_{12}}{\times} F_2)$ *be a quasi-product observable with the marginal observables* $\mathbf{O}_1$ *and* $\mathbf{O}_2$.
Put

$$\begin{aligned} \text{Rep}[\mathbf{O}_{12}] &= \begin{bmatrix} [F(\{(y_1, y_2)\})](\omega) & [F(\{(y_1, n_2)\})](\omega) \\ [F(\{(n_1, y_2)\})](\omega) & [F(\{(n_1, n_2)\})](\omega) \end{bmatrix} \\ &= \begin{bmatrix} \alpha(\omega) & [F_1(\{y_1\})](\omega) - \alpha(\omega) \\ [F_2(\{y_2\})](\omega) - \alpha(\omega) & 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega) \end{bmatrix}, \end{aligned} \tag{7.3}$$

where $\alpha \in C(\Omega)$. $\Big($*Note that* $[F(\{(y_1, y_2)\})](\omega) + [F(\{(y_1, n_2)\})](\omega) = [F_1(\{y_1\})](\omega)$ *and* $[F(\{(y_1, y_2)\})](\omega) + [F(\{(n_1, y_2)\})](\omega) = [F_2(\{y_2\})](\omega)\Big)$.
That is,

	$[F_2(\{y_2\})](\omega)$	$[F_2(\{n_2\})](\omega)$
$[F_1(\{y_1\})](\omega)$	$\alpha(\omega)$	$[F_1(\{y_1\})](\omega) - \alpha(\omega)$
$[F_1(\{n_1\})](\omega)$	$[F_2(\{y_2\})](\omega) - \alpha(\omega)$	$1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega)$

Then, it holds that

$$\max\{0, [F_1(\{y_1\})](\omega) + [F_2(\{y_2\})](\omega) - 1\} \le \alpha(\omega) \le \min\{[F_1(\{y_1\})](\omega),\ [F_2(\{y_2\})](\omega)\} \quad (\forall \omega \in \Omega). \tag{7.4}$$

Conversely, for any α $(\in C(\Omega))$ *that satisfies (7.4), the observable* $\mathbf{O}_{12}$ *defined by (7.3) is a quasi-product observable with the marginal observables* $\mathbf{O}_1$ *and* $\mathbf{O}_2$*. Also, note that*

$$[F(\{(y_1, n_2)\})](\omega) = 0 \ \Leftrightarrow\ \alpha(\omega) = [F_1(\{y_1\})](\omega) \ \Rightarrow\ [F_1(\{y_1\})](\omega) \le [F_2(\{y_2\})](\omega). \tag{7.5}$$

Proof. Though this lemma is easy, we add a brief proof for completeness. Since $0 \le [F(\{(x_1^1, x_2^2)\})]\ (\omega) \le 1$, $(\forall x^1, x^2 \in \{y, n\})$, we see, by (7.3), that

$$\begin{aligned} &0 \le \alpha(\omega) \le 1, \quad 0 \le [F_1(\{y_1\})](\omega) - \alpha(\omega) \le 1, \quad 0 \le [F_2(\{y_2\})](\omega) - \alpha(\omega) \le 1, \\ &0 \le 1 + \alpha(\omega) - [F_1(\{y_1\})](\omega) - [F_2(\{y_2\})](\omega) \le 1, \end{aligned} \tag{7.6}$$

which clearly implies (7.4). Conversely. if α satisfies (7.4), then we easily see (7.6). Also, (7.5) is obvious. This completes the proof. □

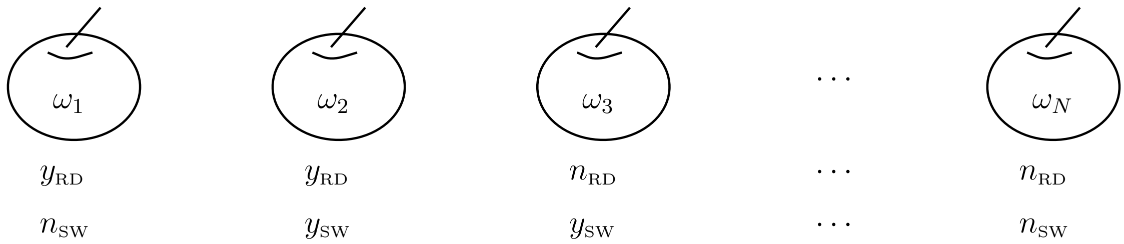
Next we provide several examples, which will promote a understanding of our theory.

Example 7.4. [Tomatoes' example]. Let $\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$ be a set of tomatoes, which is regarded as a compact Hausdorff space with the discrete topology. Consider yes-no observables $\mathbf{O}_{\text{RD}} \equiv (X_{\text{RD}}, 2^{X_{\text{RD}}}, F_{\text{RD}})$ and $\mathbf{O}_{\text{SW}} \equiv (X_{\text{SW}}, 2^{X_{\text{SW}}}, F_{\text{SW}})$ in $C(\Omega)$ such that:

$$X_{\text{RD}} = \{y_{\text{RD}}, n_{\text{RD}}\} \text{ and } X_{\text{SW}} = \{y_{\text{SW}}, n_{\text{SW}}\},$$

where we consider that "y_{RD}" and "n_{RD}" respectively mean "RED" and "NOT RED". Similarly, "y_{SW}" and "n_{SW}" respectively mean "SWEET" and "NOT SWEET".

For example, the ω_1 is red and not sweet, the ω_2 is red and sweet, etc. as follows.



We see that

(∗) the probability that x_{RD} ($\in X_{\mathrm{RD}} \equiv \{y_{\mathrm{RD}}, n_{\mathrm{RD}}\}$), the measured value by the measurement $\mathbf{M}_{C(\Omega)}(\mathbf{O}_{\mathrm{RD}}, S_{[\delta_{\omega_n}]})$, belongs to Ξ_{RD} ($\subseteq X_{\mathrm{RD}} \equiv \{y_{\mathrm{RD}}, n_{\mathrm{RD}}\}$) is given by

$$\delta_{\omega_n}(F_{\mathrm{RD}}(\Xi_{\mathrm{RD}})) \quad (= [F_{\mathrm{RD}}(\Xi_{\mathrm{RD}})](\omega_n)\ ).$$

That is, the probability that the tomato ω_n is observed as "RED" $\big[$ resp. "NOT RED"$\big]$ is given by $[F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})]\ (\omega_n)$ $\big[$ resp. $[F_{\mathrm{RD}}(\{n_{\mathrm{RD}}\})]\ (\omega_n)$ $\big]$. (*Continued to Example 7.5*).

■

Example 7.5. [Tomatoes' example; continued from Example 7.4]. Consider the quasi-product observable as follows:

$$\mathbf{O} = (X_{\mathrm{RD}} \times X_{\mathrm{SW}}, 2^{X_{\mathrm{RD}} \times X_{\mathrm{SW}}}, F \equiv F_{\mathrm{RD}} \overset{\mathbf{O}}{\times} F_{\mathrm{SW}}),$$

that is,

$$\mathrm{Rep}[\mathbf{O}] = \begin{bmatrix} [F(\{(y_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega) & [F(\{(y_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega) \\ [F(\{(n_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega) & [F(\{(n_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega) \end{bmatrix}$$
$$= \begin{bmatrix} \alpha(\omega) & [F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega) - \alpha(\omega) \\ [F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega) - \alpha(\omega) & 1 + \alpha(\omega) - [F_{\mathrm{RD}}(\{y_{\mathrm{RD}}\})](\omega) - [F_{\mathrm{SW}}(\{y_{\mathrm{SW}}\})](\omega) \end{bmatrix}$$

where $\alpha(\omega)$ satisfies (7.4). Hence by Axiom 1, when we observe that the tomato ω_n is "RED", we can see that the probability that the tomato ω_n is "SWEET" is given by

$$\frac{[F(\{(y_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega_n)}{[F(\{(y_{\mathrm{RD}}, y_{\mathrm{SW}})\})](\omega_n) + [F(\{(y_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega_n)}. \tag{7.7}$$

$\Big($For the conditional probability, see §2.5(IV).$\Big)$ Here note that (7.7) implies ;

$$\text{"}[F(\{(y_{\mathrm{RD}}, n_{\mathrm{SW}})\})](\omega_n) = 0\text{"} \qquad \text{if and only if} \qquad \text{"RED"} \Rightarrow \text{"SWEET"}\ , \tag{7.8}$$

which is also clearly equivalent to "NOT SWEET" $\Rightarrow$ "NOT RED".

■

Being motivated by the above (7.8), we introduce the following definition of "implication" as a general form which is applicable to classical and quantum systems.

Definition 7.6. [Implication]. *Let* $\mathbf{O}_1 \equiv (X_1, 2^{X_1}, F_1)$ *and* $\mathbf{O}_2 \equiv (X_2, 2^{X_2}, F_2)$ *be observables (not necessarily two-valued observables) in a* C^**-algebra* $\mathcal{A}$. *Let* $\mathbf{O}_{12} = (X_1 \times X_2, 2^{X_1} \times X_2, F_1 \overset{\mathbf{O}_{12}}{\times} F_2)$ *be a quasi-product observable of* $\mathbf{O}_1$ *and* $\mathbf{O}_2$. *Let* $\rho^p \in \mathfrak{S}^p(\mathcal{A}^*)$. *Let* $\Xi_1 \in \mathcal{P}(X_1)$ *and* $\Xi_2 \in \mathcal{P}(X_2)$. *Then, the condition*

$$\rho^p\Big((F_1 \overset{\mathbf{O}_{12}}{\times} F_2)(\Xi_1 \times (X_2 \setminus \Xi_2))\Big) = 0 \tag{7.9}$$

is denoted by

$$\mathbf{O}_1^{\Xi_1} \underset{\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{12}, S_{[\rho^p]})}{\Longrightarrow} \mathbf{O}_2^{\Xi_2}. \tag{7.10}$$

■

Remark 7.7. [Contraposition]. Assume that we get a measured value (x_1, x_2) ($\in X_1 \times X_2$) by the measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{12}, S_{[\rho^p]})$. And assume the condition (7.10). If we know that $x_1 \in \Xi_1$, then we can assure that $x_2 \in \Xi_2$. Also, (7.9) is of course also equal to $\mathbf{O}_1^{X_1 \setminus \Xi_1} \underset{\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{12}, S_{[\rho^p]})}{\Longleftarrow} \mathbf{O}_2^{X_2 \setminus \Xi_2}$ since $\mathbf{O}_{12} = \mathbf{O}_{\{1,2\}} = \mathbf{O}_{21}$ (i.e., $K = \{1, 2\}$ is not regarded as an ordered set). That is, "$\mathbf{O}_1^{X_1 \setminus \Xi_1} \underset{\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{12}, S_{[\rho^p]})}{\Longleftarrow} \mathbf{O}_2^{X_2 \setminus \Xi_2}$" is the contraposition of (7.10).

■

7.3 Consistency and syllogism

In this section we study the consistent condition for observables. We show several theorems of practical syllogisms (i.e., theorems concerning "implication" in Definition 7.6).

7.3.1 Consistent condition

Though we are not concerned with quantum theory in this chapter, our investigations for classical systems become clearer in comparison with quantum theory. Therefore, the following definitions (Definitions 7.8 and 7.9) are common in both classical and quantum theory.

Definition 7.8. [Covering family]. *Let $\mathcal{A}$ be a C^*-algebra. For each $k \in K \equiv \{1, 2, ..., |K|-1, |K|\}$, consider a label set X_k. Consider $\mathcal{D}$ ($\subseteq 2^K$) such that $\bigcup_{D \in \mathcal{D}} D = K$. Then, $\mathcal{G} \equiv [\ \mathbf{O}_D \equiv (\ \times_{k \in D} X_k,\ 2^{\times_{k \in D} X_k},\ F_D)\ \ : D \in \mathcal{D}\ ]$ is called a covering family of observables in $\mathcal{A}$, if it satisfies the following condition:*

$$\mathbf{O}_{D_1}\big|_{D_1 \cap D_2} = \mathbf{O}_{D_2}\big|_{D_1 \cap D_2} \qquad (\forall D_1, \forall D_2 \in \mathcal{D} \text{ such that } D_1 \cap D_2 \neq \emptyset).$$

Note that, if $\mathcal{G}$ is a covering family, it holds that $\mathbf{O}_{D_1}\big|_{\{k\}} = \mathbf{O}_{D_2}\big|_{\{k\}}$ for any $k \in K$ and any $D_1, D_2 \in \mathcal{D}$ such that $k \in D_1 \bigcap D_2$. Thus, a covering family of observables $\mathcal{G}$ determines a unique $\{k\}$-marginal observable $\mathbf{O}_k \equiv (X_k, 2^{X_k}, F_k)$ for each $k \in K$.

■

The following definition is a generalization of Definition 7.1 (i.e., the case that $\mathcal{D} = \{\{1\}, \{2\}, ..., \{|K|\}\}$).

Definition 7.9. [Consistent condition]. *Let $\mathcal{A}$ be a C^*-algebra. A covering family of observable $\mathcal{G} \equiv [\ \mathbf{O}_D \equiv (\ \times_{k \in D} X_k,\ 2^{\times_{k \in D} X_k},\ F_D)\ \ : D \in \mathcal{D}\ (\ \subseteq 2^{K)}\ ]$ in $\mathcal{A}$ is called consistent, if there exists an observable $\mathbf{O}_K \equiv (\ \times_{k \in K} X_k,\ 2^{\times_{k \in K} X_k},\ F)$ in $\mathcal{A}$ such that:*

$$\mathbf{O}_K\big|_D = \mathbf{O}_D \qquad (\forall D \in \mathcal{D}). \tag{7.11}$$

Also, the above relation (7.11) is denoted by

$$[\mathbf{O}_D : D \in \mathcal{D}] \sqsubset \mathbf{O}_K. \tag{7.12}$$

■

Remark 7.10. [Consistent condition]. Under the condition (7.12), the data concerning $\mathcal{G} \equiv [\mathbf{O}_D : D \in \mathcal{D}]$ for the system $S_{[\rho^p]}$ is obtained by the simultaneous measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_K, S_{[\rho^p]})$. Thus, a covering family $\mathcal{G}$ has no reality, if it is not consistent. Recall the arguments in Remark 7.2, which correspond to the above definition for the case that $\mathcal{D} = \{\{1\}, \{2\}\}$.

■

Lemma 7.11. [Consistent condition]. *Let $\mathcal{A}$ be a C^*-algebra. Let $\mathcal{G}_1 \equiv [\ \mathbf{O}^1_{D_1}\ : D_1 \in \mathcal{D}_1$ $(\ \subseteq 2^K)\ ]$ be a covering family of observables in $\mathcal{A}$. And let $\mathcal{G}_2 \equiv [\ \mathbf{O}^2_{D_2}\ : D_2 \in \mathcal{D}_2\ (\ \subseteq 2^{K)}$ $]$ be a consistent covering family of observables in $\mathcal{A}$. Assume that for any $D_1 \in \mathcal{D}_1$ there* *exists an* D_2 $(\in \mathcal{D}_2)$ *such that:*

$$D_1 \subseteq D_2 \quad \text{and} \quad \mathbf{O}^1_{D_1} = \mathbf{O}^2_{D_2}\big|_{D_1}. \tag{7.13}$$

Then, $\mathcal{G}_1$ *is consistent.*

Proof. Since a covering family $\mathcal{G}_2$ is consistent, there exists an observable $\mathbf{O}_K \equiv (\times_{k\in K} X_k,\ 2^{\times_{k\in K} X_k},\ F_K)$ in $\mathcal{A}$ such that $\mathbf{O}^2_{D_2} = \mathbf{O}_K\big|_{D_2}$ $(\forall D_2 \in \mathcal{D}_2)$. Let D_1 be any element in $\mathcal{D}_1$. Then, by choosing $D_2(\in \mathcal{D}_2)$ satisfying (7.13), we see that $\mathbf{O}^1_{D_1} = \mathbf{O}^2_{D_2}\big|_{D_1}$ $= (\mathbf{O}_K\big|_{D_2})\big|_{D_1} = \mathbf{O}_K\big|_{D_1}$. This completes the proof. □

Lemma 7.12. [Consistent condition and quasi-product observables]. *Let* $\mathcal{A}$ *be a commutative* C^**-algebra (i.e.,* $\mathcal{A} = C(\Omega)$*). Let* D_{12} *and* D_{23} *be subsets of* K*. Put* $D_{123} \equiv D_{12} \bigcup D_{23} \equiv (D_{12} \setminus D_{23}) \bigcap (D_{12} \cap D_{23}) \bigcap (D_{23} \setminus D_{12}) \equiv D_1 \bigcup D_2 \bigcup D_3$*. Consider the following observables in* $C(\Omega)$ *:*

$$\mathbf{O}_{D_{12}} \equiv (\mathop{\times}_{k\in D_{12}} X_k, \mathcal{P}(\mathop{\times}_{k\in D_{12}} X_k), F_{D_{12}}) \quad \text{and} \quad \mathbf{O}_{D_{23}} \equiv (\mathop{\times}_{k\in D_{23}} X_k, \mathcal{P}(\mathop{\times}_{k\in D_{23}} X_k), F_{D_{23}})$$

such that $\mathbf{O}_{D_{12}}\big|_{D_2} = \mathbf{O}_{D_{23}}\big|_{D_2}$*. Then, there exists an observable* $\mathbf{O}_{D_{123}} \equiv (\times_{k\in D_{123}} X_k,$ $\mathcal{P}(\times_{k\in D_{123}} X_k),\ F_{D_{123}})$ *such that* $\mathbf{O}_{D_{123}}\big|_{D_{12}} = \mathbf{O}_{D_{12}}$ *and* $\mathbf{O}_{D_{123}}\big|_{D_{23}} = \mathbf{O}_{D_{23}}$*.*

Proof. Assume that $D_{12} \bigcap D_{23} \neq \emptyset$. (If $D_{12} \bigcap D_{23} = \emptyset$, this lemma is trivial. Put $Y_m = \times_{k\in D_m} X_k = \{y^1_m, y^2_m, ..., y^{j_m}_m, ..., y^{M_m}_m\}$, $m = 1, 2, 3$. (So, $M_m = \prod_{k\in D_m} |X_k|$.) Thus, we can put, by $Y_1 \times Y_2 = \times_{k\in D_{12}} X_k$ and $Y_2 \times Y_3 = \times_{k\in D_{23}} X_k$, that

$$\mathbf{O}_{D_{12}} = (Y_1 \times Y_2, \mathcal{P}(Y_1 \times Y_2), F_{12} \equiv F_{D_{12}})$$

and

$$\mathbf{O}_{D_{23}} = (Y_2 \times Y_3, \mathcal{P}(Y_2 \times Y_3), F_{23} \equiv F_{D_{23}}).$$

Define the observable $\mathbf{O}_{D_{123}} \equiv (\times^3_{m=1} Y_m, \mathcal{P}(\times^3_{m=1} Y_m), F_{123})$ in $C(\Omega)$ such that:

$$\begin{aligned} &[F_{123}(\{(y^{j_1}_1, y^{j_2}_2, y^{j_3}_3)\})](\omega) \\ &= \begin{cases} \frac{[F_{12}(\{(y^{j_1}_1, y^{j_2}_2)\})](\omega)\cdot[F_{23}(\{(y^{j_2}_2, y^{j_3}_3)\})](\omega)}{[F_2(\{y^{j_2}_2\})](\omega)} & \text{if } [F_2(\{y^{j_2}_2\})](\omega) \neq 0 \\ 0 & \text{if } [F_2(\{y^{j_2}_2\})](\omega) = 0 \end{cases} \end{aligned}$$

for $1 \leq \forall j_1 \leq M_1$, $1 \leq \forall j_2 \leq M_2$, $1 \leq \forall j_3 \leq M_3$. Therefore, it is clear that this lemma holds. For example, $\mathbf{O}_{D_{123}}\big|_{D_{23}} = \mathbf{O}_{D_{23}}$ is easily seen as follows:

$$
\begin{aligned}
&[F_{123}(Y_1 \times \{(y_2^{j_2}, y_3^{j_3})\})](\omega) = \sum_{y_1^{j_1} \in Y_1} [F_{123}(\{(y_1^{j_1}, y_2^{j_2}, y_3^{j_3})\})](\omega) \\
&= \sum_{y_1^{j_1} \in Y_1} \frac{[F_{12}(\{(y_1^{j_1}, y_2^{j_2})\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)} \\
&= \frac{[F_{12}(Y_1 \times \{y_2^{j_2}\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)} = \frac{[F_2(\{y_2^{j_2}\})](\omega) \cdot [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega)}{[F_2(\{y_2^{j_2}\})](\omega)} \\
&= [F_{23}(\{(y_2^{j_2}, y_3^{j_3})\})](\omega) \qquad (\forall \omega \in \Omega,\ 1 \le \forall j_2 \le M_2,\ 1 \le \forall j_3 \le M_3).
\end{aligned}
$$

This completes the proof. □

The following theorem is a kind of generalization of Theorem 2.11 $\Big($which essentially corresponds to the result for $\mathcal{D} = \{\{1\},\ \{2\}, ..., \{|K|\}\}$ in the following theorem$\Big)$. Here note that a covering family $[\mathbf{O}_D : D \in \mathcal{D}]$ is equivalent to $[\mathbf{O}_{D'} : D' \in \{D' : D' \subseteq D$ for some $D \in \mathcal{D}\}]$ where $\mathbf{O}_{D'} = \mathbf{O}_D\big|_{D'}$ for any D' such that $D' \subseteq D$.

Theorem 7.13. [Consistent condition and quasi-product observables]. *Let* $\mathcal{D} = \{\{1, 2\}, \{2, 3\}, ..., \{|K| - 1, |K|\}\}$ $(\subseteq 2^K)$. *Let* $\mathcal{G} = [\mathbf{O}_D = (\times_{k \in D} X_k, 2^{\times_{k \in D} X_k}, F_D) : D \in \mathcal{D}]$ *be a covering family of observables in a commutative* C^**-algebra* $C(\Omega)$. $\Big($*Here we can put* $\mathcal{G}$ $= [\ \mathbf{O}_{k,k+1} \equiv (X_k \times X_{k+1}, \mathcal{P}(X_k \times X_{k+1}), F_{k,k+1} \equiv F_k \overset{\mathbf{O}_{k,k+1}}{\times} F_{k+1}) : k = 1, 2, ..., |K| - 1$ $].\Big)$ *Then,* $\mathcal{G} = [\ \mathbf{O}_{k,k+1} : k = 1, 2, ..., |K| - 1\ ]$ *is consistent.*

Proof. Put $D_{12} = \{1, 2\}$ and $D_{23} = \{2, 3\}$. By Lemma 7.12, we get $\mathbf{O}_{123}$ $(= \mathbf{O}_{D_{123}})$ such that $\mathcal{G}_3 = [\mathbf{O}_{123}, \mathbf{O}_{34}, \mathbf{O}_{45}, ..., \mathbf{O}_{|K|-1,|K|}]$ is a covering family in $C(\Omega)$ where $\mathbf{O}_{12} = \mathbf{O}_{123}\big|_{\{1,2\}}$ and $\mathbf{O}_{23} = \mathbf{O}_{123}\big|_{\{2,3\}}$. Iteratively, we get $\mathcal{G}_{|K|-1} = [\mathbf{O}_{123\cdots|K|-1}, \mathbf{O}_{|K|-1,|K|}]$ and $\mathcal{G}_{|K|} = [\mathbf{O}_{123\cdots|K|-1,|K|}] \equiv [\mathbf{O}_K]$, which is clearly consistent. So, by Lemma 7.11, we see that $\mathcal{G}_{|K|-1} \sqsubset \mathbf{O}_K$. Therefore, we iteratively get $\mathcal{G} \sqsubset \mathbf{O}_K$. This completes the proof. □

Remark 7.14. [Quantum PMT]. This theorem is due to the commutativity of a C^*-algebra $C(\Omega)$. In general (particularly in quantum systems, i.e., $\mathcal{A} = \mathcal{C}(V)$), there exists no $\mathbf{O}_{123}$ such that $[\mathbf{O}_{12}, \mathbf{O}_{23}] \sqsubset \mathbf{O}_{123}$ (i.e., $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ is not consistent in general). Thus, we have no simultaneous measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{123}, S_{[\rho^p]})$. Therefore, in general, we can not get information (i.e., data) concerning the covering family $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ for the quantum system $S_{[\rho^p]}$. That is, in general, the covering family $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ has no reality in quantum mechanics.

■

The following notation is the preparation for Theorems 7.19 and 7.23.

Notation 7.15. [Preparation for Theorems 7.19 and 7.23]. *Let* $\mathcal{G} = [\mathbf{O}_{12}, \mathbf{O}_{23}, ...\mathbf{O}_{|K|-1,|K|}]$ $\equiv [\ (X_k \times X_{k+1},\ \mathcal{P}(X_k \times X_{k+1}),\ F_{k,k+1} \equiv F_k \overset{\mathbf{O}_{k,k+1}}{\times} F_{k+1}) : k = 1, 2, ..., |K| - 1\ ]$ *be a covering family of observables in a commutative* C^**-algebra* $C(\Omega)$*. (So,* $\mathcal{G}$ *is consistent as in Theorem 7.13). Suppose that* $X_k = \{y_k, n_k\}$ *for each* $k \in K$*. As in Definition 7.8, put*

$$\mathrm{Rep}[\mathbf{O}_\mathrm{k}] = \mathrm{Rep}[(\mathrm{X_k}, 2^{\mathrm{X_k}}, \mathrm{F_k})] = \Big[[\mathrm{F_k}(\{\mathrm{y_k}\})](\omega), [\mathrm{F_k}(\{\mathrm{n_k}\})](\omega)\Big] \equiv \Big[\mathrm{p_k^1}(\omega), \mathrm{p_k^0}(\omega)\Big]$$

for all $k = 1, 2, 3, ..., |K|$*. And put*

$$\begin{aligned}
&\mathrm{Rep}[\mathbf{O}_{\mathrm{k,k+1}}] = \mathrm{Rep}[(\mathrm{X_k} \times \mathrm{X_{k+1}}, 2^{\mathrm{X_k} \times \mathrm{X_{k+1}}}, \mathrm{F_{k,k+1}})] \\
&= \begin{bmatrix} [F_{k,k+1}(\{y_k\} \times \{y_{k+1}\})](\omega) & [F_{k,k+1}(\{y_k\} \times \{n_{k+1}\})](\omega) \\ [F_{k,k+1}(\{n_k\} \times \{y_{k+1}\})](\omega) & [F_{k,k+1}(\{n_k\} \times \{n_{k+1}\})](\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p_{k,k+1}^{11}(\omega) & p_{k,k+1}^{10}(\omega) \\ p_{k,k+1}^{01}(\omega) & p_{k,k+1}^{00}(\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p_{k,k+1}^{11}(\omega) & p_k^1(\omega) - p_{k,k+1}^{11}(\omega) \\ p_{k+1}^1(\omega) - p_{k,k+1}^{11}(\omega) & 1 + p_{k,k+1}^{11}(\omega) - p_k^1(\omega) - p_{k+1}^1(\omega) \end{bmatrix}
\end{aligned} \tag{7.14}$$

for all $k = 1, 2, ..., |K| - 1$*, where* $p_{k,k+1}^{11}(\omega)$ *satisfies (7.4). Let* $\mathbf{O}_K \equiv (\times_{k\in K} X_k$, $\mathcal{P}(\times_{k\in K} X_k)$, $F_K)$ *be any observable in* $C(\Omega)$ *such that:*

$$[\mathbf{O}_{12}, \mathbf{O}_{23}, ...\mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K. \tag{7.15}$$

(The existence of $\mathbf{O}_K$ *is guaranteed by Theorem 7.13.) Put*

$$\Big[p_{1,2,...,|K|}^{j_1,j_2,...,j_{|K|}}(\omega)\ :\ j_1, j_2, ..., j_{|K|} = 1, 0\Big] \equiv \Big[[F_K(\overset{|K|}{\underset{k=1}{\times}} \{x_k^{j_k}\})](\omega)\ :\ j_1, j_2, ..., j_{|K|} = 1, 0\Big], \tag{7.16}$$

where $x_k^{j_k} = y_k$ *(if* $j_k = 1$ *) and* $x_k^{j_k} = n_k$ *(if* $j_k = 0$ *). Define* $\mathbf{O}_{1,|K|} \equiv (X_1 \times X_{|K|}$, $\mathcal{P}(X_1 \times X_{|K|})$, $F_{1,|K|})$ *such that* $\mathbf{O}_{1,|K|} = \mathbf{O}_K\big|_{\{1,|K|\}}$*. Put*

$$\begin{aligned}
&\mathrm{Rep}[\mathbf{O}_{1,|\mathrm{K}|}] = \mathrm{Rep}[(\mathrm{X_1} \times \mathrm{X_{|K|}}, 2^{\mathrm{X_1} \times \mathrm{X_{|K|}}}, \mathrm{F_{1,|K|}})] \\
&= \begin{bmatrix} [F_{1,|K|}(\{y_1\} \times \{y_{|K|}\})](\omega) & [F_{1,|K|}(\{y_1\} \times \{n_{|K|}\})](\omega) \\ [F_{1,|K|}(\{n_1\} \times \{y_{|K|}\})](\omega) & [F_{1,|K|}(\{n_1\} \times \{n_{|K|}\})](\omega) \end{bmatrix} \\
&\equiv \begin{bmatrix} p_{1,|K|}^{11}(\omega) & p_{1,|K|}^{10}(\omega) \\ p_{1,|K|}^{01}(\omega) & p_{1,|K|}^{00}(\omega) \end{bmatrix} \equiv \begin{bmatrix} p_{1,|K|}^{11}(\omega) & p_1^1(\omega) - p_{1,|K|}^{11}(\omega) \\ p_{|K|}^1(\omega) - p_{1,|K|}^{11}(\omega) & 1 + p_{1,|K|}^{11}(\omega) - p_1^1(\omega) - p_{|K|}^1(\omega) \end{bmatrix}.
\end{aligned} \tag{7.17}$$

(*Continued to Lemmas 7.16 and 7.17 and Theorem 7.19 for* $K = \{1, 2, 3\}$, *and to Theorem 7.23 for general case*).

■

Lemma 7.16. [Continued from Notation 7.15]. *Under Notation 7.15 for* $K = \{1, 2, 3\}$, *we see, (putting* $p_{123}^{j_1 j_2 j_3} = p_{123}^{j_1 j_2 j_3}(\omega)$ *in (7.16),* $p_{123}^{111} = A$ *and* $p_{123}^{101} = B$*),*

$$
\begin{aligned}
&p_{123}^{111} = A(\omega), && p_{123}^{011} = p_{23}^{11} - A(\omega),\\
&p_{123}^{110} = p_{12}^{11} - A(\omega), && p_{123}^{010} = p_2^1 - p_{12}^{11} - p_{23}^{11} + A(\omega),\\
&p_{123}^{101} = B(\omega), && p_{123}^{001} = p_3^1 - p_{23}^{11} - B(\omega),\\
&p_{123}^{100} = p_1^1 - p_{12}^{11} - B(\omega), && p_{123}^{000} = 1 - p_1^1 - p_2^1 - p_3^1 + p_{12}^{11} + p_{23}^{11} + B(\omega),
\end{aligned} \tag{7.18}
$$

where

$$
\max\{0, -p_2^1(\omega) + p_{12}^{11}(\omega) + p_{23}^{11}(\omega)\} \leq A(\omega) \leq \min\{p_{12}^{11}(\omega), p_{23}^{11}(\omega)\} \tag{7.19}
$$

and

$$
\begin{aligned}
&\max\{0, p_1^1(\omega) + p_2^1(\omega) + p_3^1(\omega) - p_{12}^{11}(\omega) - p_{23}^{11}(\omega) - 1\}\\
&\leq B(\omega) \leq \min\{p_1^1(\omega) - p_{12}^{11}(\omega), p_3^1(\omega) - p_{23}^{11}(\omega)\}.
\end{aligned} \tag{7.20}
$$

Proof. From (7.16), (7.15) and (7.14) for $K = \{1, 2, 3\}$, we see

$$
\begin{aligned}
&p_{123}^{111} + p_{123}^{110} = p_{12}^{11}, && p_{123}^{101} + p_{123}^{100} = p_{12}^{10} = p_1^1 - p_{12}^{11},\\
&p_{123}^{011} + p_{123}^{010} = p_{12}^{01} = p_2^1 - p_{12}^{11}, && p_{123}^{001} + p_{123}^{000} = p_{12}^{00} = 1 + p_{12}^{11} - p_1^1 - p_2^1,\\
&p_{123}^{111} + p_{123}^{011} = p_{23}^{11}, && p_{123}^{110} + p_{123}^{010} = p_{23}^{10} = p_2^1 - p_{23}^{11},\\
&p_{123}^{101} + p_{123}^{001} = p_{23}^{01} = p_3^1 - p_{23}^{11}, && p_{123}^{100} + p_{123}^{000} = p_{23}^{00} = 1 - p_{23}^{11} - p_2^1 - p_3^1.
\end{aligned}
$$

After a small computation, we get (7.18). Since $0 \leq p_{123}^{j_1 j_2 j_3}(\omega) \leq 1$, we see, from (7.18), that

$$
\begin{aligned}
&0 \leq A \leq 1, \quad p_{23}^{11} - 1 \leq A \leq p_{23}^{11}, \quad p_{12}^{11} - 1 \leq A \leq p_{12}^{11},\\
&-p_2^1 + p_{12}^{11} + p_{23}^{11} \leq A \leq 1 - p_2^1 + p_{12}^{11} + p_{23}^{11},\\
&0 \leq B \leq 1, \quad p_3^1 - p_{23}^{11} - 1 \leq B \leq p_3^1 - p_{23}^{11}, \quad p_1^1 - p_{12}^{11} - 1 \leq B \leq p_1^1 - p_{12}^{11},\\
&p_1^1 + p_2^1 + p_3^1 - p_{12}^{11} - p_{23}^{11} - 1 \leq B \leq p_1^1 + p_2^1 + p_3^1 - p_{12}^{11} - p_{23}^{11}.
\end{aligned}
$$

This implies (7.19) and (7.20). This completes the proof. □

Lemma 7.17. [Continued from Notation 7.15]. *Under Notation 7.15 for* $K = \{1,2,3\}$, *we see*

$$\begin{aligned} &\max\{0, -p_2^1(\omega) + p_{12}^{11}(\omega) + p_{23}^{11}(\omega)\} \\ &\qquad\qquad + \max\{0, p_1^1(\omega) + p_2^1(\omega) + p_3^1(\omega) - p_{12}^{11}(\omega) - p_{23}^{11}(\omega) - 1\} \\ &\le p_{13}^{11}(\omega) \qquad (7.21) \\ &\le \min\{p_{12}^{11}(\omega), p_{23}^{11}(\omega)\} + \min\{p_1^1(\omega) - p_{12}^{11}(\omega), p_3^1(\omega) - p_{23}^{11}(\omega)\}. \qquad (7.22) \end{aligned}$$

Proof. Since $p_{13}^{11}(\omega) = p_{123}^{111}(\omega) + p_{123}^{101}(\omega) = A(\omega) + B(\omega)$ in Lemma 7.16, by (7.19) and (7.20) we can easily get (7.21) and (7.22). This completes the proof. □

Remark 7.18. [Comparison]. Let us compare the result in Lemma 7.17 with the result (7.4) in Lemma 7.3 (i.e., the result without consistent condition). Note that (7.4) implies

$$C_1 \equiv \max\{0, p_1^1(\omega) + p_3^1(\omega) - 1\} \le p_{13}^{11}(\omega) \le \min\{p_1^1(\omega), p_3^1(\omega)\} \equiv C_2.$$

Here we can easily see that $C_1 \le$ (7.21) and (7.22) $\le C_2$ from the following trivial inequalities:

$$\max\{0, \alpha_1 + \alpha_2\} \le \max\{0, \max\{0, \alpha_1\} + \max\{0, \alpha_2\}\} = \max\{0, \alpha_1\} + \max\{0, \alpha_2\}$$

and

$$\begin{aligned} &\min\{\alpha_1, \alpha_2\} + \min\{\alpha_3, \alpha_4\} = \min\{\ \alpha_1 + \alpha_3,\ \alpha_1 + \alpha_4,\ \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_4\ \} \\ &\le \min\{\ \alpha_1 + \alpha_3,\ \alpha_2 + \alpha_4\ \}\ . \end{aligned}$$

Therefore, we see in Lemma 7.17 that the value $p_{13}^{11}(\omega)$ is restricted under the consistent condition of $[\mathbf{O}_{12}, \mathbf{O}_{23}]$.

■

7.3.2 Practical syllogism

Now we show several theorems of practical syllogisms (i.e., theorems concerning "implication" in Definition 7.6) as the consequences of our arguments.

Theorem 7.19. [Practical syllogism, [41]]. *Assume Notation 7.15 for* $K = \{1,2,3\}$. *That is,* $[\mathbf{O}_{12}, \mathbf{O}_{23}]$ *is a covering family of observables in a commutative* C^**-algebra* $C(\Omega)$.

Let $\delta_{\omega_0} \in \mathcal{M}^p_{+1}(\Omega)$ for any fixed $\omega_0 \in \Omega$. Let $\mathbf{O}_{123}$ $(= \mathbf{O}_K)$ be any observable such that $[\mathbf{O}_{12}, \mathbf{O}_{23}] \sqsubset \mathbf{O}_{123}$ and let $\mathbf{O}_{13} = \mathbf{O}_{123}\big|_{\{1,3\}}$. (The existence of $\mathbf{O}_{123}$ is guaranteed by Theorem 7.13.) Then we have the following statements [1] $\sim$ [3]:

[1]. *Assume that*

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{12}, S_{[\delta_{[\omega_0]}]})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{23}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{7.23}$$

Then, we see that

$$\begin{bmatrix} p_{13}^{11}(\omega_0) & p_{13}^{10}(\omega_0) \\ p_{13}^{01}(\omega_0) & p_{13}^{00}(\omega_0) \end{bmatrix} = \begin{bmatrix} p_1^1(\omega_0) & 0 \\ p_3^1(\omega_0) - p_1^1(\omega_0) & 1 - p_3^1(\omega_0) \end{bmatrix}, \tag{7.24}$$

hence, we see that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{13}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{7.25}$$

[2]. *Assume that*

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{12}, S_{[\delta_{\omega_0}]})}{\Longleftarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{23}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{7.26}$$

Then, we see that

$$\begin{bmatrix} p_{13}^{11}(\omega_0) & p_{13}^{10}(\omega_0) \\ p_{13}^{01}(\omega_0) & p_{13}^{00}(\omega_0) \end{bmatrix} = \begin{bmatrix} \alpha(\omega_0) & p_1^1(\omega_0) - \alpha(\omega_0) \\ p_3^1(\omega_0) - \alpha(\omega_0) & 1 + \alpha(\omega_0) - p_1^1(\omega_0) - p_3^1(\omega_0) \end{bmatrix}$$

where

$$\max\{p_2^1(\omega_0), p_1^1(\omega_0) + p_3^1(\omega_0) - 1\} \le \alpha(\omega_0) \le \min\{p_1^1(\omega_0), p_3^1(\omega_0)\}. \tag{7.27}$$

Also (7.26) is equivalent to

$$\mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{123}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_{13}^{\{(y_1, y_3)\}}. \tag{7.28}$$

[3]. *Assume that*

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{12}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{23}, S_{[\delta_{\omega_0}]})}{\Longleftarrow} \mathbf{O}_3^{\{y_3\}}. \tag{7.29}$$

Then, we see that

$$\begin{bmatrix} p_{13}^{11}(\omega_0) & p_{13}^{10}(\omega_0) \\ p_{13}^{01}(\omega_0) & p_{13}^{00}(\omega_0) \end{bmatrix} = \begin{bmatrix} \alpha(\omega_0) & p_1^1(\omega_0) - \alpha(\omega_0) \\ p_3^1(\omega_0) - \alpha(\omega_0) & 1 + \alpha(\omega_0) - p_1^1(\omega_0) - p_3^1(\omega_0) \end{bmatrix}$$

where

$$\max\{0, p_1^1(\omega_0) + p_3^1(\omega_0) - p_2^1(\omega_0)\} \leq \alpha(\omega_0) \leq \min\{p_1^1(\omega_0), p_3^1(\omega_0)\}. \tag{7.30}$$

Also (7.29) is equivalent to

$$\mathbf{O}_{13}^{\{(y_1,y_3),(y_1,n_3),(n_1,y_3)\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{123}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}. \tag{7.31}$$

Proof. [1]. By (7.23) and (7.5), we see that $p_{12}^{10} = p_{23}^{10} = 0$, so, $p_{12}^{11} = p_1^1 \leq p_2^1 = p_{23}^{11} \leq p_3^1$. Therefore, we see that (7.21) $= p_{12}^{11} + \max\{0, p_3^1 - 1\} = p_1^1$. And (7.22) $= p_1^1 + 0 = p_1^1$. This implies that $p_{13}^{11} = p_1^1$, i.e., (7.24). Also, (7.25) follows from $p_{13}^{10} = 0$.
[2]. By (7.26) and (7.5), we see that $p_{12}^{01} = p_{23}^{10} = 0$, so, $p_{12}^{11} = p_2^1 \leq p_1^1$ and $p_{23}^{11} = p_2^1 \leq p_3^1$. Therefore, we see that (7.21) $= p_{23}^{11} + \max\{0, p_1^1 - p_2^1 + p_3^1 - 1\} = \max\{p_2^1, p_1^1 + p_3^1 - 1\}$. And (7.22) $= \min\{p_2^1, p_2^1\} + \min\{p_1^1 - p_2^1, p_3^1 - p_2^1\} = \min\{p_1^1, p_3^1\}$. This implies (7.27). Also, we see that (7.26) $\Leftrightarrow p_{12}^{01} = p_{23}^{10} = 0 \Leftrightarrow p_{123}^{010} = p_{123}^{011} = p_{123}^{110} = 0 \Leftrightarrow$ (7.28).
[3]. By (7.29) and (7.5), we see that $p_{12}^{10} = p_{23}^{01} = 0$, so, $p_{12}^{11} = p_1^1 \leq p_2^1$ and $p_{23}^{11} = p_3^1 \leq p_2^1$. Therefore, we see that (7.21) $= \max\{0, p_1^1 - p_2^1 + p_3^1\} + \max\{0, p_2^1 - 1\} = \max\{0, p_1^1 - p_2^1 + p_3^1\}$. And (7.22) $= \min\{p_1^1, p_3^1\}$. This implies (7.30). Also, (7.29) $\Leftrightarrow p_{12}^{10} = p_{23}^{01} = 0 \Leftrightarrow p_{123}^{101} = p_{123}^{100} = p_{123}^{001} = 0 \Leftrightarrow$ (7.31). This completes the proof. □

Remark 7.20. [Practical logic and pure logic]. The reader must not confuse the result (for example, (7.23)⇒(7.25)) in Theorem 7.19 with pure logic (i.e., mathematical logic). Theorem 7.19 is a consequence of Axiom 1. Note that Theorem 7.19 is due to Theorem 7.13, i.e., the commutativity of C^*-algebra $C(\Omega)$. That means the results in Theorem 7.19 can not be expected in quantum systems. In comparison with quantum theory, Theorem 7.19 becomes clearer. For example, in general, the syllogism is meaningless in quantum systems. This is easily shown as follows. Put $V = \mathbf{C}^5$, and $\mathcal{A} = B(\mathbf{C}^5)$. And

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \quad \vec{e}_4 = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \quad \vec{e}_5 = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix},$$

and put $\vec{f}_4 = \frac{\vec{e}_4}{\sqrt{2}} + \frac{\vec{e}_5}{\sqrt{2}}$, $\vec{f}_5 = \frac{\vec{e}_4}{\sqrt{2}} - \frac{\vec{e}_5}{\sqrt{2}}$. Define the three observables $\mathbf{O}_1 \equiv (X_1 \equiv \{a_1, b_1, c_1\}, 2^{X_1}, F_1)$, $\mathbf{O}_2 \equiv (X_2 \equiv \{a_2, b_2, c_2\}, 2^{X_2}, F_2)$ and $\mathbf{O}_3 \equiv (X_3 \equiv \{a_3, b_3, c_3\}, 2^{X_3}, F_3)$ such that

$$F_1(\{a_1\}) = |\vec{e}_1\rangle\langle\vec{e}_1|, \quad F_1(\{b_1\}) = |\vec{e}_2\rangle\langle\vec{e}_2| + |\vec{e}_3\rangle\langle\vec{e}_3| + |\vec{e}_4\rangle\langle\vec{e}_4|, \quad F_1(\{c_1\}) = |\vec{e}_5\rangle\langle\vec{e}_5|,$$

$$F_2(\{a_2\}) = |\vec{e}_1\rangle\langle\vec{e}_1| + |\vec{e}_2\rangle\langle\vec{e}_2|, \quad F_2(\{b_2\}) = |\vec{e}_3\rangle\langle\vec{e}_3|, \quad F_2(\{c_2\}) = |\vec{e}_4\rangle\langle\vec{e}_4| + |\vec{e}_5\rangle\langle\vec{e}_5|,$$

$$F_3(\{a_3\}) = |\vec{e}_1\rangle\langle\vec{e}_1| + |\vec{e}_2\rangle\langle\vec{e}_2| + |\vec{e}_3\rangle\langle\vec{e}_3|, \quad F_3(\{b_3\}) = |\vec{f}_4\rangle\langle\vec{f}_4|, \quad F_3(\{c_3\}) = |\vec{f}_5\rangle\langle\vec{f}_5|.$$

Note that $\mathbf{O}_1$ and $\mathbf{O}_2$ [resp. $\mathbf{O}_2$ and $\mathbf{O}_3$] commute. Let $\mathbf{O}_{12} = (X_1 \times X_2, 2^{X_1 \times X_2}, F_1 \times F_2)$ be the product observable of $\mathbf{O}_1$ and $\mathbf{O}_2$. And let $\mathbf{O}_{23} = (X_2 \times X_3, 2^{X_2 \times X_3}, F_2 \times F_3)$ be the product observable of $\mathbf{O}_2$ and $\mathbf{O}_3$. Let ρ^p be any pure state (i.e., $\rho^p \in \mathfrak{S}^p(B(\mathbf{C}^5)^*)$). Then, we have

$$\mathbf{O}_1^{\{a_1\}} \underset{\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{12}, S_{[\rho^p]})}{\Longrightarrow} \mathbf{O}_2^{\{a_2\}}, \quad \mathbf{O}_2^{\{a_2\}} \underset{\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{23}, S_{[\rho^p]})}{\Longrightarrow} \mathbf{O}_3^{\{a_3\}}.$$

since we see

$$\rho^p\Big((F_1 \times F_2)(\{a_1\} \times (\{b_2, c_2\}))\Big) = 0, \quad \rho^p\Big((F_1 \times F_2)(\{a_2\} \times (\{b_3, c_3\}))\Big) = 0.$$

However, it should be noted that we have no product observable of $\mathbf{O}_1$, $\mathbf{O}_2$ and $\mathbf{O}_3$. Thus, the implication:

$$\mathbf{O}_1^{\{a_1\}} \underset{\mathbf{M}_{\mathcal{A}}(\mathbf{O}_{13}, S_{[\rho^p]})}{\Longrightarrow} \mathbf{O}_3^{\{a_3\}}$$

is nonsense since $\mathbf{O}_{13}$ can not be defined.

■

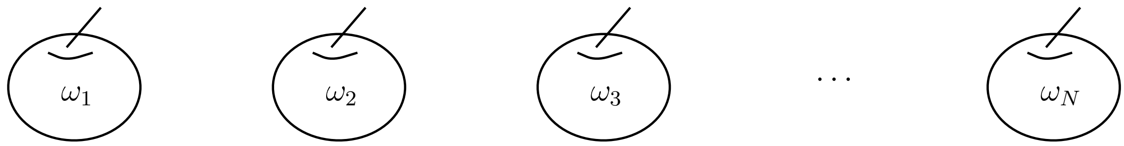
Example 7.21. [Continued from Example 7.4, [41]]. Let Ω, $C(\Omega)$, $\mathbf{O}_1 \equiv \mathbf{O}_{\text{SW}} \equiv (X_{\text{SW}}, 2^{X_{\text{SW}}}, F_{\text{SW}})$ and $\mathbf{O}_3 \equiv \mathbf{O}_{\text{RD}} \equiv (X_{\text{RD}}, 2^{X_{\text{RD}}}, F_{\text{RD}})$ be as in Example 7.4. Let $\mathbf{O}_2 \equiv \mathbf{O}_{\text{RP}} \equiv (X_{\text{RP}}, 2^{X_{\text{RP}}}, F_{\text{RP}})$ be an observable in $C(\Omega)$ such that:

$$X_{\text{RP}} = \{y_{\text{RP}}, n_{\text{RP}}\},$$

where "y_{RP}" and "n_{RP}" respectively mean "RIPE" and "NOT RIPE". Put

$$\begin{aligned}
\text{Rep}[\mathbf{O}_1] &= \Big[[F_{\text{SW}}(\{y_{\text{SW}}\})](\omega), [F_{\text{SW}}(\{n_{\text{SW}}\})](\omega)\Big], \\
\text{Rep}[\mathbf{O}_2] &= \Big[[F_{\text{RP}}(\{y_{\text{RP}}\})](\omega), [F_{\text{RP}}(\{n_{\text{RP}}\})](\omega)\Big], \\
\text{Rep}[\mathbf{O}_3] &= \Big[[F_{\text{RD}}(\{y_{\text{RD}}\})](\omega), [F_{\text{RD}}(\{n_{\text{RD}}\})](\omega)\Big].
\end{aligned}$$

For example,



$n_{\rm SW}$	$y_{\rm SW}$	$y_{\rm SW}$	$\cdots$	$n_{\rm SW}$
$y_{\rm RP}$	$n_{\rm RP}$	$y_{\rm RP}$	$\cdots$	$n_{\rm RP}$
$y_{\rm RD}$	$y_{\rm RD}$	$n_{\rm RD}$	$\cdots$	$n_{\rm RD}$

Consider the following quasi-product observables:

$$\mathbf{O}_{12} = (X_{\rm SW} \times X_{\rm RP}, 2^{X_{\rm SW} \times X_{\rm RP}}, F_{12} \equiv F_{\rm SW} \overset{\mathbf{O}_{12}}{\times} F_{\rm RP})$$

and

$$\mathbf{O}_{23} = (X_{\rm RP} \times X_{\rm RD}, 2^{X_{\rm RP} \times X_{\rm RD}}, F_{23} \equiv F_{\rm RP} \overset{\mathbf{O}_{23}}{\times} F_{\rm RD}).$$

Let $\delta_{\omega_n} \in \mathcal{M}_{+1}^p(\Omega)$ for any fixed $\omega_n \in \Omega$. Assume that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{12}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{23}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{7.32}$$

Then, we see, by Theorem 7.19 [1], that

$$\begin{aligned} {\rm Rep}[\mathbf{O}_{13}] &= \begin{bmatrix} [F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n) & [F_{13}(\{y_{\rm SW}\} \times \{n_{\rm RD}\})](\omega_n) \\ [F_{13}(\{n_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n) & [F_{13}(\{n_{\rm SW}\} \times \{n_{\rm RD}\})](\omega_n) \end{bmatrix} \\ &= \begin{bmatrix} [F_{\rm SW}(\{y_{\rm SW}\})](\omega_n) & 0 \\ [F_{\rm RD}(\{y_{\rm RD}\})](\omega_n) - [F_{\rm SW}(\{y_{\rm SW}\})](\omega_n) & 1 - [F_{\rm RD}(\{y_{\rm RD}\})](\omega_n) \end{bmatrix}. \end{aligned} \tag{7.33}$$

So, when we observe that the tomato ω_n is "RED", we can infer, by the fuzzy inference $\mathbf{M}_{C(\Omega)}(\mathbf{O}_{13}, S_{[\delta_{\omega_n}]})$ (equivalently, $\mathbf{M}_{C(\Omega)}(\mathbf{O}_{31}, S_{[\delta_{\omega_n}]})$), the probability that the tomato ω_n is "SWEET" is given by

$$\frac{[F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n)}{[F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n) + [F_{13}(\{n_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n)} = \frac{[F_{\rm SW}(\{y_{\rm SW}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}. \tag{7.34}$$

Also, when we observe that the tomato ω_n is "SWEET", we can infer, by the fuzzy inference $\mathbf{M}_{C(\Omega)}(\mathbf{O}_{13}, S_{[\delta_{\omega_n}]})$, the probability that the tomato ω_n is "RED" is given by

$$\frac{[F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n)}{[F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n) + [F_{13}(\{y_{\rm SW}\} \times \{n_{\rm RD}\})](\omega_n)} = \frac{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)} = 1. \tag{7.35}$$

Note that (7.32) implies (and is implied by)

$$\text{"SWEET"} \Longrightarrow \text{"RIPE"} \qquad \text{and} \qquad \text{"RIPE"} \Longrightarrow \text{"RED"} . \tag{7.36}$$

(Recall (7.8)). So, it is "reasonable" to reach the conclusion:

$$\text{"SWEET"} \Longrightarrow \text{"RED"} , \tag{7.37}$$

which is implied by the above (7.35). $\Big($Here we are afraid that the most important fact may be overlooked. For completeness, note that the conclusion "(7.36) ⇒ (7.37)" is a consequence of Theorem 7.19 (and therefore, our axiom).$\Big)$ However, the result (7.34) is due to the peculiarity of fuzzy inferences. That is, in spite of the fact (7.36), we get the conclusion (7.34) that is somewhat like

$$\text{"RED"} \Longrightarrow \text{"SWEET"} . \tag{7.38}$$

Note that the conclusion (7.37) is not valuable in the market. What we want in the market is the conclusion such as (7.38) (or (7.34)).

■

Example 7.22. [Continued from Example 7.21, [41]]. Instead of (7.32), assume that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{12}, S_{[\delta_{\omega_n}]})}{\Longleftarrow} \mathbf{O}_2^{\{y_2\}}, \qquad \mathbf{O}_2^{\{y_2\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{23}, S_{[\delta_{\omega_n}]})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}. \tag{7.39}$$

Assume the notation (7.33). When we observe that the tomato ω_n is "RED", we can infer, by the fuzzy inference $\mathbf{M}_{C(\Omega)}(\mathbf{O}_{13}, S_{[\delta_{\omega_n}]})$, the probability that the tomato ω_n is "SWEET" is given by

$$Q = \frac{[F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n)}{[F_{13}(\{y_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n) + [F_{13}(\{n_{\rm SW}\} \times \{y_{\rm RD}\})](\omega_n)} \tag{7.40}$$

which is, by (7.27), estimated as follows:

$$\begin{aligned} &\max\left\{ \frac{[F_{\rm RP}(\{y_{\rm RP}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}, \frac{[F_{\rm SW}(\{y_{\rm SW}\})] + [F_{\rm RD}(\{y_{\rm RD}\})] - 1}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)} \right\} \\ &\le Q \le \min\{ \frac{[F_{\rm SW}(\{y_{\rm SW}\})](\omega_n)}{[F_{\rm RD}(\{y_{\rm RD}\})](\omega_n)}, \ 1 \}. \end{aligned} \tag{7.41}$$

Note that (7.39) implies (and is implied by)

$$\text{"RIPE"} \Longrightarrow \text{"SWEET"} \qquad \text{and} \qquad \text{"RIPE"} \Longrightarrow \text{"RED"} . \tag{7.42}$$

And note that the conclusion (7.41) is somewhat like

$$\text{"RED"} \Longrightarrow \text{"SWEET"} . \tag{7.43}$$

Therefore, this conclusion is peculiar to "fuzziness".

■

The following theorem is a generalization of the first part of Theorem 7.19.

Theorem 7.23. [Standard syllogism, *cf.* [41]]. *Assume Notation 7.15. Let* $\delta_{\omega_0} \in \mathcal{M}_{+1}^p(\Omega)$. *Assume that*

$$\mathbf{O}_k^{\{y_k\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{k,k+1}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_{k+1}^{\{y_{k+1}\}} \qquad (\forall k = 1, 2, ..., |K| - 1), \tag{7.44}$$

Let $\mathbf{O}_K$ *be any observable as in Notation 7.15, i.e.,* $\mathcal{G} = [\mathbf{O}_{12}, \mathbf{O}_{23}, \mathbf{O}_{34}, ..., \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K$. *Put* $\mathbf{O}_{1,|K|} = \mathbf{O}_K\big|_{\{1,|K|\}}$. *Then, we see that*

$$\mathrm{Rep}[\mathbf{O}_{1,|\mathrm{K}|}]at\,\omega_0 = \begin{bmatrix} p^{11}_{1,|K|}(\omega_0) & p^{10}_{1,|K|}(\omega_0) \\ p^{01}_{1,|K|}(\omega_0) & p^{00}_{1,|K|}(\omega_0) \end{bmatrix} = \begin{bmatrix} p^1_1(\omega_0) & 0 \\ p^1_{|K|}(\omega_0) - p^1_1(\omega_0) & 1 - p^1_{|K|}(\omega_0) \end{bmatrix}, \tag{7.45}$$

hence, we see that

$$\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_{1,|K|}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_{|K|}^{\{y_{|K|}\}}. \tag{7.46}$$

Proof. Let $\mathbf{O}_K$ be any observable such that $\mathcal{G} = [\mathbf{O}_{12}, \mathbf{O}_{23}, \mathbf{O}_{34}, ..., \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K$. Thus, we see that $[\mathbf{O}_K\big|_{\{1,3\}}, \mathbf{O}_{34}, ..., \mathbf{O}_{|K|-1,|K|}] \sqsubset \mathbf{O}_K\big|_{K\setminus\{2\}}$. Note that $(\mathbf{O}_K\big|_{\{1,3\}})\big|_{\{m\}} = \mathbf{O}_m$, $m = 1, 3$. Also note, by (7.24), that

$$\mathrm{Rep}[\mathbf{O}_K\big|_{\{1,3\}}]\mathrm{at}\;\omega_0 = \begin{bmatrix} p^1_1(\omega_0) & 0 \\ p^1_3(\omega_0) - p^1_1(\omega_0) & 1 - p^1_3(\omega_0) \end{bmatrix},$$

and therefore $\mathbf{O}_1^{\{y_1\}} \underset{\mathbf{M}_{C(\Omega)}(\mathbf{O}_K|_{\{1,3\}}, S_{[\delta_{\omega_0}]})}{\Longrightarrow} \mathbf{O}_3^{\{y_3\}}$. Hence, by induction, we see that $\mathrm{Rep}[\mathbf{O}_{1,|K|}] \equiv \mathrm{Rep}[\mathbf{O}_K\big|_{\{1,|K|\}}] =$ (7.45) at $\omega = \omega_0$. This completes the proof. □

7.4 Conclusion

It is certain that (pure) logic is merely a kind of rule in mathematics. However, if it is so, the logic is not guaranteed to be applicable to our world. For instance, (pure) logic does not assure the truth of the following famous statement:

[♯] *Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.*

That is, we think that the problem: "Is this [♯] (theoretical) true or not?" is unsolved. Thus, the purpose of this chapter was to prove the [♯], or more generally, to propose "practical logic", i.e., a collection of theorems (whose forms are similar to that of "pure logic") in PMT.

Firstly, the symbol "$A \Rightarrow B$" (i.e., "implication") is defined in terms of measurements (*cf.* Definition 7.6). And we prove the standard syllogism for classical systems:

$$\text{“}A \Rightarrow B,\ B \Rightarrow C\text{” implies “}A \Rightarrow C\text{”,} \tag{7.47}$$

which is the same as the above (♯). (This (7.47) is not trivial since it does not necessarily hold in quantum systems.) We can assert, by "Declaration (1.11)" in §1.4, that PMT guarantees that the above statement [♯] is true.

Several variants may be interesting. For example, under the condition that "$A \Rightarrow B$, $B \Rightarrow C$", we can assert a kind of conclusion such as "$C \Rightarrow A$". That is,

$$\text{“}A \Rightarrow B,\ B \Rightarrow C\text{” implies “}C \Rightarrow A\text{”} \qquad \text{in some sense.} \tag{7.48}$$

For completeness, "pure logic" and "practical logic" must not be confused. The former is a basic rule on which mathematics is founded. On the other hand, the latter is a collection of theorems (whose forms are similar to that of "pure logic") in PMT.

7.5 Appendix (Zadeh's fuzzy sets theory)

7.5.1 What is Zadeh's fuzzy sets theory?

As mentioned in Chapter 1 (i.e., the footnote below Problem 1.2), one of motivations of our research is motivated by Zadeh's fuzzy sets theory. In 1965, L.A. Zadeh proposed a certain system theory, in which *a membership function* $f : \Omega \to [0, 1]$, which is asserted to represent "fuzziness", plays an important role. The membership function is considered as a kind of generalization of a characteristic function. Here, the characteristic function χ_D of D ($\subseteq \Omega$) is defined by $\chi_D : \Omega \to \{0, 1\}$ such that:

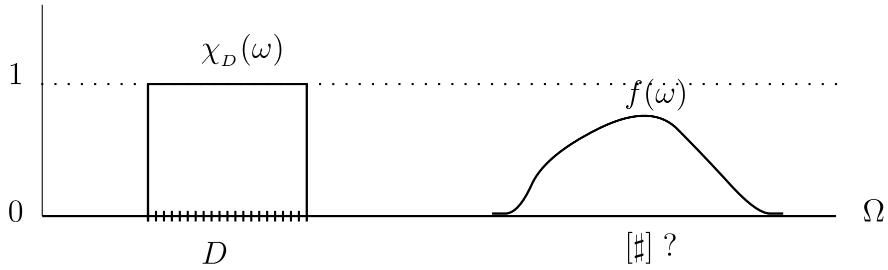
$$\chi_D(\omega) = \begin{cases} 1 & (\omega \in D) \\ 0 & (\omega \notin D). \end{cases}$$

Consider the identification:

$$\text{“characteristic function } \chi_D\text{”} \quad \longleftrightarrow \quad \text{“set } D\text{”},$$

which gives us the question “What is the following [♯]?”

$$\text{“membership function } f\text{”} \quad \longleftrightarrow [\sharp].$$



The [♯] is called a *fuzzy set* by Zadeh. Thus we think that Zadeh's fuzzy sets theory has two aspects $[A_1]$ and $[A_2]$ as follows:

$$\text{Zadeh's fuzzy sets theory} \begin{cases} [A_1] : \text{membership functions (analytic aspect)}, \\ \\ [A_2] : \text{fuzzy sets (logical aspect)}. \end{cases} \tag{7.49}$$

Zadeh's fuzzy sets theory acquired a lot of believers. In fact, his paper [93] is one of the most cited papers in all fields of 20th century science. However, his theory seems “fuzzy” rather than “difficult”. Thus, it is natural that the following problem arises:

$[\sharp_1]$ Is Zadeh's fuzzy sets theory true or not?

When we examine the problem, we are immediately confronted with the following problem:

$[\sharp_2]$ What is “true or not”? Or, if we want to assert “Zadeh's fuzzy sets theory is true [or not]”, what do we say?

And when we study the problems $[\sharp_1]$ and $[\sharp_2]$, we immediately notice the fact that we have not yet the clear answer to even the question: “Is Fisher's statistics true or not?”. [3] As mentioned in Chapter 1, our research starts from the above questions $[\sharp_1]$ and $[\sharp_2]$. And we conclude “Declaration (1.11)” in §1.4 as follows:

- MT is entitled to check all theories in theoretical informatics. In other words, we can, by using MT, introduce the criterion: “true or not” into theoretical informatics. That is, MT can be regarded as “the Construction of theoretical informatics”.

[3] In Chapters 5 and 6, it is proved that Fisher's statistics is theoretically true.

Now, consider an observable $(X, 2^X, F)$ in $C(\Omega)$. Note that, for any Ξ ($\subseteq X$), $F(\Xi)$ is a membership function on Ω. Since $F(\Xi) \in C(\Omega)$, the $F(\Xi)$, of course, has various analytic aspects. Also, in this chapter we see that the membership function $F(\Xi)$ has various logical aspects. Thus, someone may conclude that Zadeh's fuzzy sets theory (i.e., the analytic aspect $[A_1]$ and the logical aspect $[A_2]$ in (7.49)) is understood in the framework of measurement theory, that is, Zadeh's fuzzy sets theory is true (*cf.* "Declaration (1.11)" in §1.4). We may agree with this opinion. In fact, these kinds of aspects $[A_1]$ and $[A_2]$ can not be found in the conventional formulation of system theory (*cf.* (1.2)) such as

$$\boxed{\text{"dyn. syst. theor."}} = \begin{cases} \frac{dx(t)}{dt} = f(x(t), u_1(t), t), \; x(0) = x_0 & \cdots \text{(state equation)}, \\ y(t) = g(x(t), u_2(t), t) & \text{(measurement equation)}. \end{cases} \tag{7.50}$$
(=(1.2))

That is because the conventional formulation (7.50) does not possess the concept of "observable in the sense of Definition 2.7".

The believers of Zadeh's fuzzy sets theory say too much (*cf.* [64]). And thus, we have no firm answer to the question: "What is the essence of Zadeh's theory?". If we can assume that:

($\sharp$) Zadeh wanted to assert that *DST* (7.50) *and "logic" are closely connected* $\Big($*or precisely, "logic" is one of the aspects of DST* (7.50)$\Big)$ *though the two are, in appearance, independent,*

then we can understand his assertion. That is because in this section we study "logic" in measurement theory, which is a kind of generalization of the system theory (7.50). This is our opinion for Zadeh's theory. Of course, there may be another opinion, that is, someone may assert that Zadeh said something much more than the ($\sharp$). If it is so, we may not understand his theory in the framework of measurement theory.

Recall the arguments in Chapter 1 (particularly, "Declaration (1.11)" in §1.4, tables (1.7) and (1.8)). Now, we have only two options, i.e.,

(i) Zadeh's fuzzy sets theory is characterized as the theory concerning membership functions in measurement theory.

(ii) Zadeh's fuzzy sets theory is not characterized in measurement theory. Thus another fundamental theory (*cf.* $\boxed{\text{The third mathematical scientific theory}}$ in (1.7)) should be proposed.

Although there is a possibility that (ii) is reasonable, that is, Zadeh's fuzzy sets theory may be understood in another fundamental theory (*cf.* $\boxed{\text{The third mathematical scientific theory}}$ in (1.7)), we should note that the proposal of another fundamental theory is much more remarkable than the justification of Zadeh's fuzzy sets theory. Thus we choose the (i) even if the essential part of Zadeh's assertion (e.g., the scientific part asserted in [64]) can not be characterized in MT. Thus we conclude that Zadeh's assertion can not be completely understood in measurement theory, i.e.,

- Zadeh's assertion is not completely "theoretical true" (*cf.* Declaration 1.11), though practical logic somewhat has the property like "fuzzy set".

This is our present opinion.

7.5.2 Why is Zadeh's paper cited frequently?

Although we believe that the above argument in §7.5.1 is proper, it does not explain the reason why Zadeh's paper is cited frequently. As mentioned before, Zadeh's paper [93] is one of the most cited papers of all scientific papers. This is an established fact. This fact may imply that there is something interesting behind Zadeh's assertion. Thus, we think that the question "Why is Zadeh's paper cited frequently?" is more important than the question "What is Zadeh's fuzzy sets theory?". Thus we shall consider the question:

- Why does the term "fuzzy" look attractive?

We think that the reason is that Zadeh's spirit is regarded as *the antithesis of the myth: "Science must be exact, clear, strict, etc".* This myth seems to be due to Newtonian mechanics (and moreover, theoretical physics), which has been located in the center of all science. That is, we think that

- many people want another science, which is fuzzy, rough, vague, etc.

If it is so, we should recall Table 1.8 (in Chapter 1), which asserts mathematical science is classified as follows:

$$\begin{cases} \text{theoretical physics ('TOE')} & \text{..... exact mathematical science} \\ \\ \text{theoretical informatics (measurement theory)} & \text{..... fuzzy mathematical science.} \end{cases} \tag{7.51}$$

If it is true, we can understand the reason why the term "fuzzy" was accepted widely. Thus we do not deny the following opinion:

(♯) "measurement theory" = "fuzzy theory". (Cf. [42].) Or, the attractive parts of Zadeh's assertions are mostly included in measurement theory.

That is because we believe that

(♭) Measurement theory is the very theory that represents the anti-spirit against the myth: "Science must be exact, clear, strict, etc".

In fact, the terms

- *fuzzy statement (cf. the footnote below Example 2.16), ready-made, useful or not, subjective, popularity, likes or dislikes, (in "Theoretical informatics of Table (1.8)"*)

seem to belong to the category of "fuzziness". On the other hand, the terms

- *precise statement (cf. the footnote below Example 2.16), made to order, empirical true or not, objective, truth, (in "Theoretical informatics of Table (1.8)"*)

obviously belong to the category of "exactness".