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Chapter 3

The relation among systems (Axiom 2)

As mentioned in Chapter 1, (pure) measurement theory (PMT) is formulated as follows:

$$\text{PMT} = \underset{\text{[Axiom 1 (2.37)]}}{\text{measurement}} + \underset{\text{[Axiom 2 (3.26)]}}{\text{the relation among systems}} \quad \text{in } C^*\text{-algebra} \;. \tag*{(3.1) (=(1.4))}$$

In Chapter 2 we studied "measurement (= Axiom 1)". In this chapter we intend to explain "the relation among systems (= Axiom 2)".

3.1 Newton Equation and Schrödinger equation

In this section, we review the Newton equation and Schrödinger equation.

[I: Newtonian Mechanics]

Put $\mathcal{A} = C_0(\mathbf{R}_q^s \times \mathbf{R}_p^s)$ and $\mathcal{A}^* = \mathcal{M}(\mathbf{R}_q^s \times \mathbf{R}_p^s)$, where $\mathbf{R}_q^s \times \mathbf{R}_p^s \equiv \{(q,p) = (q_1, q_2, \cdots, q_s, p_1, p_2, \cdots, p_s) \mid q_j, p_j \in \mathbf{R}, j = 1, 2, \cdots, s\}$ and $(\mathbf{R}_q^s \times \mathbf{R}_p^s)$ is the $2s$-dimensional space (*cf.* Example 2.2). It is well known that the Newton equation is mathematically equivalent to the following Hamilton equation:

$$\frac{d}{dt} q_j(t) = \frac{\partial \mathcal{H}}{\partial p_j}(q(t), p(t), t), \quad \frac{d}{dt} p_j(t) = -\frac{\partial \mathcal{H}}{\partial q_j}(q(t), p(t), t), \quad j = 1, 2, \cdots, s \tag{3.2}$$

$$(q(0), p(0)) \in \mathbf{R}_q^s \times \mathbf{R}_p^s. \tag{3.3}$$

where $\mathcal{H} : \mathbf{R}_q^s \times \mathbf{R}_p^s \times \mathbf{R} \to \mathbf{R}$ is a Hamiltonian. Using the solution of Newton equation (i.e., Hamilton equation (3.2)), we define the continuous map $\psi_{t_1,t_2} : \mathbf{R}_q^s \times \mathbf{R}_p^s \to \mathbf{R}_q^s \times \mathbf{R}_p^s$,

$\forall t_1 \le \forall t_2$, such that:

$$\psi_{t_1,t_2}(q(t_1),p(t_1)) = (q(t_2),p(t_2)) \qquad (\forall (q(t_1),p(t_1)) \in \mathbf{R}^s_q \times \mathbf{R}^s_p), \tag{3.4}$$

which is equivalent to (3.2).

Put $\Omega = \mathbf{R}^s_q \times \mathbf{R}^s_p$. Also, put $\Omega_t = \Omega$ $(\forall t \in \mathbf{R})$, and $\omega_0^0 = (q(0),p(0))$ $(\in \Omega_0)$. Thus, the pair $[\omega_0^0, \{\psi_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2}\}_{t_1 \le t_2}]$ can be considered to be equivalent to "(3.3)+(3.2)".

Using the continuous map $\psi_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2}$ $(\forall t_1 \le \forall t_2)$, we define the continuous linear operator $\Phi_{t_1,t_2} : C_0(\Omega_{t_2}) \to C_0(\Omega_{t_1})$ such that:

$$[\Phi_{t_1,t_2}(f_{t_2})](\omega_{t_1}) = f_{t_2}(\phi_{t_1,t_2}(\omega_{t_1})) \qquad (\forall f_{t_2} \in C_0(\Omega_{t_2}), \forall \omega_{t_1} \in \Omega_{t_1}).$$

And therefore, we can consider the following identifications:

$$\text{"(3.3)+(3.2)"} \Leftrightarrow [\omega_0^0, \{\psi_{t_1,t_2} : \Omega_{t_1} \to \Omega_{t_2}\}_{t_1 \le t_2}] \Leftrightarrow [\delta_{\omega_0^0}, \{\Phi_{t_1,t_2} : C_0(\Omega_{t_2}) \to C_0(\Omega_{t_1})\}_{t_1 \le t_2}]$$

where $\delta_{\omega_0^0}$ is the point measure at ω_0^0. The pair $[\delta_{\omega_0^0}, \{\Phi_{t_1,t_2} : C_0(\Omega_{t_2}) \to C_0(\Omega_{t_1})\}_{t_1 \le t_2}]$ will be called "general system" (*cf.* Definition 3.1), and will play an important role in our theory, that is, it is a special case of "the relation among systems" in (3.1).

[II:Quantum Mechanics in $\mathcal{C}(L^2(\mathbf{R}_q, dq))$]

We begin with the classical mechanics. For simplicity, consider the one dimensional case, i.e., $\mathbf{R}_q = \{q \mid q \in \mathbf{R}\}$. Thus $q(t)$, $-\infty < t < \infty$, means the particle's position at time t, and thus, $p(t)$ $(\equiv m\frac{dq(t)}{dt})$ means the particle's momentum at time t. Let $\mathbf{R}^2_{q,p}$ $(\equiv \{(q,p) \mid q,p \in \mathbf{R}\}$ be a phase space. Define a Hamiltonian $\mathcal{H} : \mathbf{R}^2_{q,p} \to \mathbf{R}$ such that:

$$\mathcal{H}(q,p) = \frac{p^2}{2m}\Big(\text{=kinetic energy=}\ \tfrac{1}{2}m(\tfrac{dq(t)}{dt})^2\Big) \quad + \quad V(q)\Big(\text{=potential energy}\Big). \tag{3.5}$$

Thus we see

$$\underset{\text{(total energy)}}{E} = \mathcal{H}(q,p) = \underset{\text{(kinetic energy)}}{\frac{p^2}{2m}} + \underset{\text{(potential energy)}}{V(q)} . \tag{3.6}$$

Put $H = L^2(\mathbf{R}_q, dq)$, that is, the Hilbert space composed of all complex valued L^2-functions f on $\mathbf{R}_q$, i.e., $\|f\|_{L^2(\mathbf{R}_q,dq)} \equiv [\int_{-\infty}^{\infty} |f(q)|^2 dq]^{1/2} < \infty$. And put $\mathcal{A} = \mathcal{C}(H) = \mathcal{C}(L^2(\mathbf{R}_q, dq))$, (i.e., the algebra composed of all compact operators on H, *cf.* Example 2.3). Applying the quantumization:

$$E \mapsto i\hbar\frac{\partial}{\partial t}, \quad p \mapsto -i\hbar\frac{\partial}{\partial q}, \quad q \mapsto q \quad (\text{where } i = \sqrt{-1},\ \hbar = \text{"Plank constant" } /2\pi) \tag{3.7}$$

to the (3.6), we obtain the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t} = \mathcal{H}(q, -i\hbar\frac{\partial}{\partial q}) = -\frac{\hbar^2\partial^2}{2m\partial q^2} + V(q) \tag{3.8}$$

or precisely

$$i\hbar\frac{\partial}{\partial t}\psi(q,t) = -\frac{\hbar^2\partial^2}{2m\partial q^2}\psi(q,t) + V(q)\psi(q,t). \tag{3.9}$$

This solution is, formally, written by

$$\psi(q,t) = e^{-\frac{i}{\hbar}\mathcal{H}(q,-i\hbar\frac{\partial}{\partial q})t}\psi(q,0).$$

Put $U(t) = e^{-\frac{i}{\hbar}\mathcal{H}(q,-i\hbar\frac{\partial}{\partial q})t}$, and $\psi(\cdot,t) = \psi_t$. Then, we see,

$$\psi_t = U(t)\psi_0 \qquad (\|\psi_0\|_H = 1).$$

Thus, the time-evolution of the state $|\psi_t\rangle\langle\psi_t|$ $\Big(\equiv (\Psi_t^0)^*(|\psi_0\rangle\langle\psi_0|)\Big)$ is represented by

$$|\psi_t\rangle\langle\psi_t| = (\Psi_t^0)^*\Big(|\psi_0\rangle\langle\psi_0|\Big) = |U(t)\psi_0\rangle\langle U(t)\psi_0| \quad \Big(\in Tr_{+1}^p(H)\Big).$$

Let $\Psi_t^0 : \mathcal{C}(H) \to \mathcal{C}(H)$ be the pre-adjoint operator of $(\Psi_t^0)^*$. Let $\mathbf{O}_0 = (X, \mathcal{F}, F_0)$ be a C^*-observable in $\mathcal{C}(H)$. Then, the time-evolution of the observable $\mathbf{O}_t = (X, \mathcal{F}, F_t)$ is represented by

$$(X, \mathcal{F}, F_t) = (X, \mathcal{F}, U(t)F_0U(t)^*) = (X, \mathcal{F}, \Psi_t^0 F_0).$$

Putting $\Phi_{t_1,t_2} = \Psi^0_{t_2-t_1}$, we get the pair $[|\psi_0\rangle\langle\psi_0|, \{\Phi_{t_1,t_2} : \mathcal{C}(H) \to \mathcal{C}(H)\}_{t_1\le t_2}]$. Also, it should be note that the above F_t is the solution of the following Heisenberg kinetic equation:

$$i\hbar\frac{dF_t}{dt} = F_t\mathcal{H} - \mathcal{H}F_t \qquad \text{in } \mathcal{C}(H)\ , \tag{3.10}$$

which is equivalent to the Schrödinger equation (3.9). (Cf. [84].) The pair $\Big[|\psi_0\rangle\langle\psi_0|,$ $\{\Phi_{t_1,t_2} : \mathcal{C}(L^2(\mathbf{R}_q, dq)) \to \mathcal{C}(L^2(\mathbf{R}_q, dq))\}_{t_1\le t_2}\Big]$ will be called "general system" (*cf.* Definition 3.1), and will play an important role in our theory, that is, it is also a special case of "the relation among systems" in (3.1).

3.2 The relation among systems (Definition)

By the hint of the arguments in the previous section, we shall devote ourselves to "the relation among systems (i.e., Axiom 2)" in PMT (3.1) (=(1.4)).

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras. A continuous linear operator $\Psi_{1,2} : \mathcal{A}_2 \to \mathcal{A}_1$ is called a *Markov operator,* if it satisfies that

(i) $\Psi_{1,2}(F_2) \geq 0$ for any positive element F_2 in $\mathcal{A}_2$,

(ii) $\Psi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{A}_k$ $(k = 1, 2)$.

Here note that, for any observable $(X, \mathcal{F}, F_2)$ in $\mathcal{A}_2$, the $(X, \mathcal{F}, \Psi_{1,2}F_2)$ is an observable in $\mathcal{A}_1$, which is denoted by $\Psi_{12}\mathbf{O}_2$. For example, it is easy to see that

$$\begin{aligned}&[\Psi_{1,2}F_2](\Xi \cup \Xi') = \Psi_{1,2}(F_2(\Xi \cup \Xi')) = \Psi_{1,2}(F_2(\Xi) + F_2(\Xi'))\\ =&[\Psi_{1,2}(F_2)](\Xi) + [\Psi_{1,2}(F_2)](\Xi') \quad (\text{for all } \Xi, \Xi'(\in \mathcal{F}) \text{ such that } \Xi \cap \Xi' = \emptyset).\end{aligned} \tag{3.11}$$

Also, a Markov operator $\Psi_{1,2} : \mathcal{A}_2 \to \mathcal{A}_1$ is called a *homomorphism* (or precisely, *C^*-homomorphism),* if it satisfies that

(i) $\Psi_{1,2}(F_2)\Psi_{1,2}(G_2) = \Psi_{1,2}(F_2G_2)$ for any F_2 and G_2 in $\mathcal{A}_2$,

(ii) $(\Psi_{1,2}(F_2))^* = \Psi_{1,2}(F_2^*)$ for any F_2 in $\mathcal{A}_2$.

Let $\Psi_{1,2}^* : \mathcal{A}_1^* \to \mathcal{A}_2^*$ be the dual operator[1] of a Markov operator $\Psi_{1,2} : \mathcal{A}_2 \to \mathcal{A}_1$, that is, it holds that

$${}_{\mathcal{A}_1^*}\Big\langle \rho_1, \Psi_{1.2}F_2 \Big\rangle_{\mathcal{A}_1} = {}_{\mathcal{A}_2^*}\Big\langle \Psi_{1.2}^*\rho_1, F_2 \Big\rangle_{\mathcal{A}_2} \quad (\forall \rho_1 \in \mathcal{A}_1^*, \forall F_2 \in \mathcal{A}_2). \tag{3.12}$$

Then the following mathematical results are well known (*cf.* [50, 76, 82]).

(a) $\Psi_{1,2}^*(\mathfrak{S}^m(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$, (3.13)

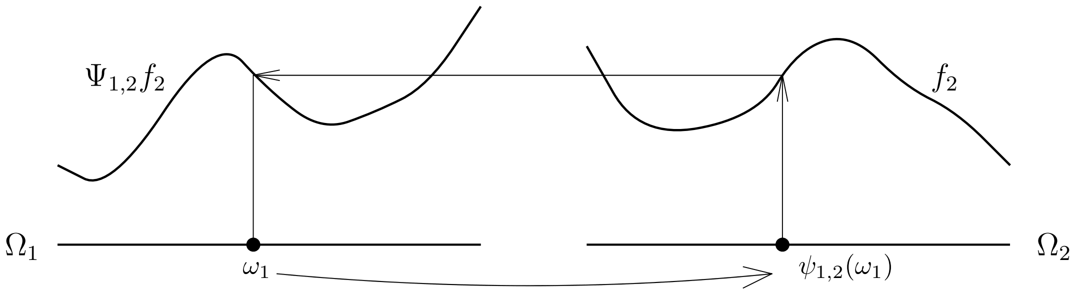
(b) $\Psi_{1,2}^*(\mathfrak{S}^p(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^p(\mathcal{A}_2^*)$ if $\Psi_{1,2} : \mathcal{A}_2 \to \mathcal{A}_1$ is homomorphic.

Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are commutative unital C^*-algebras, i.e., $\mathcal{A}_1 = C(\Omega_1)$ and $\mathcal{A}_2 = C(\Omega_2)$. Then, under the identification that $\mathfrak{S}^p(\mathcal{A}_1^*) = \mathcal{M}_{+1}^p(\Omega_1) = \Omega_1$ and $\mathfrak{S}^m(\mathcal{A}_2^*) = \mathcal{M}_{+1}^m(\Omega_2)$ (*cf.* §2.1), the above (a) implies that the dual operator $\Psi_{1,2}^*$ of a Markov operator

[1] The symbol $*$ is used in the three following ways (i) $\sim$ (iii) in this book. (i) involution operator (e.g., F^*), (ii) dual operator (e.g., Ψ^*), (iii) dual space (e.g., $\mathcal{A}^*$).

Ψ_{12} can be identified with a *transition probability rule* $M(\omega_1, B_2)$, $(\omega_1 \in \Omega_1,\ B_2 \in \mathcal{B}_{\Omega_2})$, such that $M(\omega_1, B_2) = (\Psi^*_{1,2}(\delta_{\omega_1}))(B_2)$. Also, under the identification that $\mathcal{M}^p_{+1}(\Omega_1) = \Omega_1$ and $\mathcal{M}^p_{+1}(\Omega_2) = \Omega_2$, the above (b) implies that the dual operator $\Psi^*_{1,2}$ of a homomorphism $\Psi_{1,2}$ can be identified with a continuous map $\psi_{1,2}$ from Ω_1 into Ω_2 such that:

$$(\Psi_{1,2} f_2)(\omega_1) = f_2(\psi_{1,2}(\omega_1)) \qquad (\forall \omega_1 \in \Omega_1, \forall f_2 \in C(\Omega_2)). \tag{3.14}$$



Let $(T, \leq)$ be a tree-like partial ordered set, i.e., a partial ordered set such that "$t_1 \leq t_3$ and $t_2 \leq t_3$" implies "$t_1 \leq t_2$ or $t_2 \leq t_1$". Put $T^2_{\leq} = \{(t_1, t_2) \in T^2 : t_1 \leq t_2\}$. An element $t_0 \in T$ is called a *root* if $t_0 \leq t$ $(\forall t \in T)$ holds. Since we usually consider the subtree T_{t_0} $(\subseteq T)$ with the root t_0, we assume that the tree-like ordered set has a root. In this chapter, assume, for simplicity, that T is finite (though it is sometimes infinite in applications).

Definition 3.1. [Markov relation among systems, General systems, Sequential observable]. *The pair* $\mathbf{S}_{[\rho^p_{t_0}]} \equiv [S_{[\rho^p_{t_0}]}, \{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2) \in T^2_{\leq}}]$ *is called a general system with an initial state* $\rho^p_{t_0}$ *if it satisfies the following conditions* (i)$\sim$(iii).

(i) *With each* t $(\in T)$, *a* C^**-algebra* $\mathcal{A}_t$ *is associated.*

(ii) *Let* t_0 $(\in T)$ *be the root of* T. *And, assume that a system* S *has the state* $\rho^p_{t_0}$ $(\in \mathfrak{S}^p(\mathcal{A}^*_{t_0}))$ *at* t_0, *that is, the initial state is equal to* $\rho^p_{t_0}$.

(iii) *For every* $(t_1, t_2) \in T^2_{\leq}$, *a Markov operator* $\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}$ *is defined such that* $\Phi_{t_1,t_2}\Phi_{t_2,t_3} = \Phi_{t_1,t_3}$ *holds for all* (t_1, t_2), $(t_2, t_3) \in T^2_{\leq}$.

The family $\{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2) \in T^2_{\leq}}$ *is also called a "Markov relation among systems". Let an observable* $\mathbf{O}_t \equiv (X_t, 2^{X_t}, F_t)$ *in a* C^**-algebra* $\mathcal{A}_t$ *be given for each* $t \in T$. *The pair* $[\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2) \in T^2_{\leq}}]$ *is called a "sequential observable", which is* *denoted by* $[\mathbf{O}_T]$, *i.e.*, $[\mathbf{O}_T] = [\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2) \in T^2_{\leq}}\ ]$.

■

3.3 Examples (Several tree structures)

Before we propose Axiom 2 (3.26), we prepare some notations and examples. For simplicity, assume that T is finite, or a finite subtree of a whole tree. Let T ($= \{0, 1, ..., N\}$) be a tree with the root 0. Define the *parent map* $\pi : T \setminus \{0\} \to T$ such that $\pi(t) = \max\{s \in T : s < t\}$. It is clear that the tree $(T \equiv \{0, 1, ..., N\}, \leq\ )$ can be identified with the pair $(T \equiv \{0, 1, ..., N\}, \pi : T \setminus \{0\} \to T)$. Also, note that, for any $t \in T \setminus \{0\}$, there uniquely exists a natural number $h(t)$ (called the *height* of t) such that $\pi^{h(t)}(t) = 0$. Here, $\pi^2(t) = \pi(\pi(t))$, $\pi^3(t) = \pi(\pi^2(t))$, etc. Also, put $\{0, 1, ..., N\}^2_{\leq} = \{(m, n) \mid 0 \leq m \leq n \leq N\}$. Thus, the general system $\mathbf{S}_{[\rho_0^p]} \equiv [S^0_{[\rho_0^p]}$, $\{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2) \in \{0,1,...,N\}^2_{\leq}}]$ is sometimes represented by $[S^0_{[\rho_0^p]}$, $\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}$ ($t \in \{0, 1, ..., N\} \setminus \{0\})]$. Let $\mathbf{O}_t \equiv (X_t, \mathcal{F}_t, F_t)$ be an observable in $\mathcal{A}_t$ ($\forall t \in T$). The "measurement" of $\{\mathbf{O}_t : t \in T\}$ for the $\mathbf{S}_{[\rho^p_{t_0}]}$ is symbolically described by $\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T}, \mathbf{S}_{[\rho^p_{t_0}]})$. The Markov relation $\{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2) \in T^2_{\leq}}$ is also denoted by $\{\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\}_{t \in T \setminus \{0\}}$

The following Examples 3.2, 3.3 and 3.4 will promote the understanding of Axiom 2 later.

Example 3.2. [Series structures[2]]. Suppose that a tree $(T \equiv \{0, 1, ..., N\}, \pi)$ has a "series" structure, i.e., $\pi(t) = t - 1$ ($\forall t \in T \setminus \{0\}$). Consider a general system $\mathbf{S}_{[\rho_0^p]} \equiv [S^0_{[\rho_0^p]}$, $\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}$ ($t \in T \setminus \{0\})]$ with the initial system $S^0_{[\rho_0^p]}$, that is,

$$\mathcal{A}_0 \overset{\Phi_{0,1}}{\longleftarrow} \mathcal{A}_1 \overset{\Phi_{1,2}}{\longleftarrow} \mathcal{A}_2 \overset{\Phi_{2,3}}{\longleftarrow} \cdots\cdots\cdots \overset{\Phi_{N-2,N-1}}{\longleftarrow} \mathcal{A}_{N-1} \overset{\Phi_{N-1,N}}{\longleftarrow} \mathcal{A}_N. \tag{3.15}$$

For each $t \in T$, consider an observable $\mathbf{O}_t \equiv (X_t, \mathcal{F}_t, F_t)$ in a C^*-algebra $\mathcal{A}_t$. Thus, we have a sequential observable $[\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t \in T \setminus \{0\}}\ ]$. Put $\widetilde{\mathbf{O}}_N$ ($\equiv (X_N, \mathcal{F}_N, \widetilde{F}_N)) = \mathbf{O}_N$ ($\equiv (X_N, \mathcal{F}_N, F_N))$. According to the Heisenberg picture (*cf.* §3.5), the observable $\mathbf{O}_N$ in $\mathcal{A}_N$ can be identified with the observable $\Phi_{N-1,N}\widetilde{\mathbf{O}}_N$ in $\mathcal{A}_{N-1}$. Thus, we can consider the quasi-product observable $\widetilde{\mathbf{O}}_{N-1} \equiv \mathbf{O}_{N-1} \overset{\text{qp}}{\times} \Phi_{N-1,N}\mathbf{O}_N \equiv (X_{N-1} \times$

[2]Most problems in dynamical system theory are formulated as the general systems with series trees (i.e., T="time") Cf. Kalman filter in §8.4.

$X_N, \mathcal{F}_{N-1} \times \mathcal{F}_n, \widetilde{F}_{N-1})$ in $\mathcal{A}_{N-1}$, that is,

$$\widetilde{F}_{N-1}(\Xi_{N-1} \times \Xi_N) = \big(F_{N-1} \overset{\text{qp}}{\times} (\Phi_{N-1,N} F_N)\big)(\Xi_{N-1} \times \Xi_N), \tag{3.16}$$

(though the existence and the uniqueness are not guaranteed in general). By a similar way, we can define the quasi-product observable $\widetilde{\mathbf{O}}_{N-2} \equiv \mathbf{O}_{N-2} \overset{\text{qp}}{\times} \Phi_{N-2,N-1} \widetilde{\mathbf{O}}_{N-1} \equiv (X_{N-2} \times X_{N-1} \times X_N,\ \mathcal{F}_{N-2} \times \mathcal{F}_{N-1} \times \mathcal{F}_n,\ \widetilde{F}_{N-2})$ in $\mathcal{A}_{N-2}$, that is,

$$\widetilde{F}_{N-2}(\Xi_{N-2} \times \Xi_{N-1} \times \Xi_N) = \big(F_{N-2} \overset{\text{qp}}{\times} (\Phi_{N-2,N-1} \widetilde{F}_{N-1})\big)(\Xi_{N-2} \times \Xi_{N-1} \times \Xi_N). \tag{3.17}$$

Iteratively we get as follows:

$$\begin{array}{ccccccccccc}
[\mathcal{A}_0] & \xleftarrow{\Phi} & [\mathcal{A}_1] & \xleftarrow{\Phi} \cdots \xleftarrow{\Phi} & [\mathcal{A}_{N-2}] & \xleftarrow{\Phi} & [\mathcal{A}_{N-1}] & \xleftarrow{\Phi} & [\mathcal{A}_N] \\
F_0 & & F_1 & \cdots & F_{N-2} & & F_{N-1} & & F_N \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\begin{matrix}(F_0 \overset{\text{qp}}{\times} \Phi \widetilde{F}_1) \\ = \widetilde{F}_0\end{matrix} & \xleftarrow{\Phi} & \begin{matrix}(F_1 \overset{\text{qp}}{\times} \Phi \widetilde{F}_2) \\ = \widetilde{F}_1\end{matrix} & \xleftarrow{\Phi} \cdots \xleftarrow{\Phi} & \begin{matrix}(F_{N-2} \overset{\text{qp}}{\times} \Phi \widetilde{F}_{N-1}) \\ = \widetilde{F}_{N-2}\end{matrix} & \xleftarrow{\Phi} & \begin{matrix}(F_{N-1} \overset{\text{qp}}{\times} \Phi \widetilde{F}_N) \\ = \widetilde{F}_{N-1}\end{matrix} & \xleftarrow{\Phi} & \begin{matrix}(F_N) \\ = \widetilde{F}_N\end{matrix}
\end{array}$$

And finally, we get the quasi-product observable $\widetilde{\mathbf{O}}_0 \equiv \mathbf{O}_0 \overset{\text{qp}}{\times} \Phi_{0,1} \widetilde{\mathbf{O}}_1 \equiv (\times_{t=0}^N X_t,\ \times_{t=0}^N \mathcal{F}_t,\ \widetilde{F}_0)$ in $\mathcal{A}_0$, that is,

$$\widetilde{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N) = \big(F_0 \overset{\text{qp}}{\times} (\Phi_{0,1} \widetilde{F}_1)\big)(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N). \tag{3.18}$$

Here $\widetilde{\mathbf{O}}_0$ is a realization of the sequential observable $[\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t \in T \setminus \{0\}}$]. Then, we have the "measurement" $\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T},\ \mathbf{S}_{[\rho_0^p]})$ such as

$$\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T}, \mathbf{S}_{[\rho_0^p]}) = \mathbf{M}_{\mathcal{A}_0}(\widetilde{\mathbf{O}}_0 \equiv (\mathop{\times}_{t \in T} X_t, \mathop{\times}_{t \in T} \mathcal{F}_t, \widetilde{F}_0), S^0_{[\rho_0^p]}). \tag{3.19}$$

Also, note that the above arguments can be executed under the hypothesis that quasi-product observables (i.e., $\widetilde{\mathbf{O}}_n$, $n = 0, 1., , , .N$) exist. In other words, the existence of the "measurement" $\mathfrak{M}(\{\mathbf{O}\}_{t \in T},\ \mathbf{S}_{[\rho_0^p]})$ is equivalent to that of the observable $\widetilde{\mathbf{O}}_0$.

■

Example 3.3. [Parallel structures[3]]. Suppose that a tree $(T \equiv \{0, 1, ..., N\}, \pi)$ has a "parallel" structure, i.e., $\pi(t) = 0$ $(\forall t \in T \setminus \{0\})$. Consider a general system $\mathbf{S}_{[\rho_0^p]} \equiv [S^0_{[\rho_0^p]}$, $\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}$ $(\ t \in T \setminus \{0\})]$ with the initial system $S^0_{[\rho_0^p]}$, that is,

[3] Most problems in statistics are formulated as the general systems with parallel trees. Cf. Figure (6.12) in regression analysis.

$$
\begin{tikzcd}[row sep=small]
 & \mathcal{A}_1 \arrow[dl, "\Phi_{0,1}"'] \\
\mathcal{A}_0 & \mathcal{A}_2 \arrow[l, "\Phi_{0,2}"'] \\
 & \cdots\cdots \\
 & \cdots\cdots \\
 & \mathcal{A}_N \arrow[uuul, "\Phi_{0,N}"]
\end{tikzcd}
\tag{3.20}
$$

For each $t \in T$, consider an observable $\mathbf{O}_t \equiv (X_t, \mathcal{F}_t, F_t)$ in a C^*-algebra $\mathcal{A}_t$. Thus, we have a sequential observable $[\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t \in T\setminus\{0\}}\ ]$. Then, we get the quasi-product observable $\widetilde{\mathbf{O}}_0 \equiv (\times_{t=0}^N X_t,\ \times_{t=0}^N \mathcal{F}_t,\ \widetilde{F}_0)$ in $\mathcal{A}_0$ such that:

$$
\widetilde{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N) = \big(\overset{\mathrm{qp}}{\underset{t \in T}{\times}} \Phi_{0,t} F_t)\big)(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N). \tag{3.21}
$$

Here $\widetilde{\mathbf{O}}_0$ is a realization of the sequential observable $[\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t \in T\setminus\{0\}}$]. Then, we have the "measurement" $\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T},\ \mathbf{S}_{[\rho_0^p]})$ such as

$$
\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T}, \mathbf{S}_{[\rho_0^p]}) = \mathbf{M}_{\mathcal{A}_0}(\widetilde{\mathbf{O}}_0 \equiv (\underset{t \in T}{\times} X_t, \underset{t \in T}{\times} \mathcal{F}_t, \widetilde{F}_0), S^0_{[\rho_0^p]}). \tag{3.22}
$$

Also, note that the above arguments can be executed under the hypothesis that quasi-product observables exist. In other words, the existence of the "measurement" $\mathfrak{M}(\{\mathbf{O}\}_{t \in T}$, $\mathbf{S}_{[\rho_0^p]})$ is equivalent to that of the observable $\widetilde{\mathbf{O}}_0$.

■

Example 3.4. [A simple general system, Heisenberg picture]. Suppose that a tree $(T \equiv \{0, 1, ..., 6, 7\}, \pi)$ has an ordered structure such that $\pi(1) = \pi(6) = \pi(7) = 0$, $\pi(2) = \pi(5) = 1$, $\pi(3) = \pi(4) = 2$. $\Big($See the figure (3.23).$\Big)$ Consider a general system $\mathbf{S}_{[\rho_0^p]} \equiv [S_{[\rho_0^p]},\ \{\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\}_{t \in T\setminus\{0\}}]$ with the initial system $S_{[\rho_0^p]}$.

$$
\begin{tikzcd}[row sep=small]
 & & & \mathcal{A}_3 \arrow[dl, "\Phi_{2,3}"'] \\
 & & \mathcal{A}_2 \arrow[dl, "\Phi_{1,2}"'] & \\
 & & & \mathcal{A}_4 \arrow[ul, "\Phi_{2,4}"] \\
 & \mathcal{A}_1 \arrow[dl, "\Phi_{0,1}"'] & & \\
 & & \mathcal{A}_5 \arrow[uul, "\Phi_{1,5}"'] & \\
\mathcal{A}_0 & \mathcal{A}_6 \arrow[l, "\Phi_{0,6}"] & & \\
 & \mathcal{A}_7 \arrow[uul, "\Phi_{0,7}"] & &
\end{tikzcd}
\tag{3.23}
$$

Also, for each $t \in \{0, 1, ..., 6, 7\}$, consider an observable $\mathbf{O}_t \equiv (X_t, 2^{X_t}, F_t)$ in a C^*-algebra $\mathcal{A}_t$. Thus, we have a sequential observable $[\{\mathbf{O}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t\in T\setminus\{0\}}\ ]$. Now we want to consider the following “measurement”,

($\sharp$) for a system $S_{[\rho_0^p]}$, take a measurement of *“a sequential observable* $[\{\mathbf{O}_t\}_{t\in T}, \{\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\}_{t\in T\setminus\{0\}}]$*”* i.e., take a measurement of an observable $\mathbf{O}_0$ at $0(\in T)$, and next, take a measurement of an observable $\mathbf{O}_1$ at $1(\in T)$, $\cdots\cdots$, and finally take a measurement of an observable $\mathbf{O}_7$ at $7(\in T)$,

which is symbolized by $\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho_0^p]})$. Note that the $\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho_0^p]})$ is merely a symbol since only one measurement is permitted (*cf.* §2.5 Remark(II)). In what follows let us describe the above ($\sharp$) ($= \mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho_0^p]})$) precisely. Put

$$\widetilde{\mathbf{O}}_t = \mathbf{O}_t \quad \text{and thus} \quad \widetilde{F}_t = F_t \quad (t = 3, 4, 5, 6, 7).$$

First we construct the quasi-product observable $\widetilde{\mathbf{O}}_2$ in $\mathcal{A}_2$ such as

$$\widetilde{\mathbf{O}}_2 = (X_2 \times X_3 \times X_4, 2^{X_2\times X_3\times X_4}, \widetilde{F}_2) \quad \text{where } \widetilde{F}_2 = F_2 \overset{\text{qp}}{\times} (\overset{\text{qp}}{\times}_{t=3,4} \Phi_{2,t}\widetilde{F}_t),$$

if it exists. Iteratively, we construct the following:

$$\begin{array}{ccccc}
\mathcal{A}_0 & \xleftarrow{\Phi_{0,1}} & \mathcal{A}_1 & \xleftarrow{\Phi_{1,2}} & \mathcal{A}_2 \\
F_0 \overset{\text{qp}}{\times} \Phi_{0,6}\widetilde{F}_6 \overset{\text{qp}}{\times} \Phi_{0,7}\widetilde{F}_7 & & F_1 \overset{\text{qp}}{\times} \Phi_{1,5}\widetilde{F}_5 & & \\
\downarrow & & \downarrow & & \\
\widetilde{F}_0 & \xleftarrow{\Phi_{0,1}} & \widetilde{F}_1 & \xleftarrow{\Phi_{1,2}} & \widetilde{F}_2 \\
(F_0 \overset{\text{qp}}{\times} \Phi_{0,6}\widetilde{F}_6 \overset{\text{qp}}{\times} \Phi_{0,7}\widetilde{F}_7 \overset{\text{qp}}{\times} \Phi_{0,1}\widetilde{F}_1) & & (F_1 \overset{\text{qp}}{\times} \Phi_{1,5}\widetilde{F}_5 \overset{\text{qp}}{\times} \Phi_{1,2}\widetilde{F}_2) & & (F_2 \overset{\text{qp}}{\times} \Phi_{2,3}\widetilde{F}_3 \overset{\text{qp}}{\times} \Phi_{2,4}\widetilde{F}_4)
\end{array}. \tag{3.24}$$

That is, we get the quasi-product observable $\widetilde{\mathbf{O}}_1 \equiv (\prod_{t=1}^5 X_t, 2^{\prod_{t=1}^5 X_t}, \widetilde{F}_1)$ of $\mathbf{O}_1$, $\Phi_{1,2}\widetilde{\mathbf{O}}_2$ and $\Phi_{1,5}\widetilde{\mathbf{O}}_5$, and finally, the quasi-product observable $\widetilde{\mathbf{O}}_0 \equiv (\prod_{t=0}^7 X_t, 2^{\prod_{t=0}^7 X_t}, \widetilde{F}_0)$ of $\mathbf{O}_0$, $\Phi_{0,1}\widetilde{\mathbf{O}}_1$, $\Phi_{0,6}\widetilde{\mathbf{O}}_6$ and $\Phi_{0,7}\widetilde{\mathbf{O}}_7$, if it exists. Here, $\widetilde{\mathbf{O}}_0$ is called *the realization* (or, *the Heisenberg picture representation*) of *a sequential observable* $[\{\mathbf{O}_t\}_{t\in T}, \{\mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\}_{t\in T\setminus\{0\}}]$. Then, we have the measurement

$$\mathbf{M}_{\mathcal{A}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, 2^{\prod_{t\in T} X_t}, \widetilde{F}_0), S_{[\rho_0^p]}),$$

which is called *the realization* (or, *the Heisenberg picture representation) of the symbol* $\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho^p_{t_0}]})$.

■

Remark 3.5. Let $(T \equiv \{0,1,...,N\}, \pi : T\setminus\{0\} \to T)$ be any tree with the root 0. Let τ be any element of T. Consider a series structure $\widetilde{T}_\tau$ such that $\widetilde{T}_\tau = \{\pi^k(\tau) \mid k = 0,1,2,...,h(\tau)\}$ ($\subseteq T$), where $h(\tau)$ is the height of τ, i.e., $\pi^{h(\tau)}(\tau) = 0$. Note that Example 3.4 (i.e, diagram (3.24)) means that any general system (with a tree structure T) can be regarded as a general system with a series structure $\widetilde{T}_\tau$.

■

3.4 The relation among systems (Axiom 2)

Examining Example 3.4, we see as follows: Let $(T \equiv \{0,1,...,N\}, \pi : T\setminus\{0\} \to T)$ be a tree with root 0 and let $\mathbf{S}_{[\rho^p_0]} \equiv [S_{[\rho^p_0]}, \mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\ (t \in T\setminus\{0\})]$ be a general system with the initial system $S_{[\rho^p_0]}$. And, let an observable $\mathbf{O}_t \equiv (X_t, \mathcal{F}_t, F_t)$ in a C^*-algebra $\mathcal{A}_t$ be given for each $t \in T$. Thus, we have a sequential observable $[\{\mathbf{O}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t\in T\setminus\{0\}}\ ]$. For each s ($\in T$), define the observable $\widetilde{\mathbf{O}}_s \equiv (\prod_{t\in T_s} X_t, \prod_{t\in T_s}\mathcal{F}_t, \widetilde{F}_s)$ in $\mathcal{A}_s$ such that:

$$\widetilde{\mathbf{O}}_s = \begin{cases} \mathbf{O}_s & (\text{if } s \in T\setminus\pi(T)) \\ \mathbf{O}_s \overset{\text{qp}}{\times} (\overset{\text{qp}}{\times}_{t\in\pi^{-1}(\{s\})} \Phi_{\pi(t),t}\widetilde{\mathbf{O}}_t) & (\text{if } s \in \pi(T)) \end{cases} \tag{3.25}$$

if possible. Then, if an observable $\widetilde{\mathbf{O}}_0$ (i.e., the Heisenberg picture representation of the sequential observable $[\{\mathbf{O}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t\in T\setminus\{0\}}\ ])$ in $\mathcal{A}_0$ exists (such as in Example 3.4), we have the measurement

$$\mathbf{M}_{\mathcal{A}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T}\mathcal{F}_t, \widetilde{F}_0), S_{[\rho^p_0]}),$$

which is called the Heisenberg picture representation of the symbol $\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho^p_{t_0}]})$.

Summing up the essential part of the above argument, we can propose the following axiom, which corresponds to "the rule of the relation among systems" in PMT (1.4). Cf. [43, 44, 46].

AXIOM 2. [The Markov relation among systems, the Heisenberg picture] *The relation among systems is represented by a Markov relation* $\{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}$. *Let* $\mathbf{O}_t$ $(\equiv (X_t, \mathcal{F}_t, F_t))$ *be an observable in* $\mathcal{A}_t$ *for each* t $(\in T)$. *If the procedure (3.25) is possible, a sequential observable* $[\mathbf{O}_T] \equiv [\{\mathbf{O}_t\}_{t\in T}, \{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}\ ]$ *can be realized as the observable* $\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0)$ *in* $\mathcal{A}_0$. (3.26)

It is quite important to note that Axiom 2 is stated in terms of $\mathcal{A}$ (and not in terms of $\mathcal{A}^*$).[4] Also, we must add the following statement:

- Let $\mathbf{S}_{[\rho^p_{t_0}]} \equiv [S_{[\rho^p_{t_0}]}, \{\Phi_{t_1,t_2} : \mathcal{A}_{t_2} \to \mathcal{A}_{t_1}\}_{(t_1,t_2)\in T^2_{\leq}}]$ be a general system with an initial state $\rho^p_{t_0}$ $(\in \mathfrak{S}^p(\mathcal{A}^*_{t_0}))$. Then, a measurement represented by the symbol $\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho^p_{t_0}]})$ can be realized by $\mathbf{M}_{\mathcal{A}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T} \mathcal{F}_t, \widetilde{F}_0), S_{[\rho^p_0]})$, if $\widetilde{\mathbf{O}}_0$ exists.

which explains the relation between Axiom 1 and Axiom 2.

Now we get the PMT (1.4). We have the following classification in PMT:

$$
\left\{
\begin{array}{l}
\text{deterministic PMT} = \underset{\text{[Axiom 1 (2.37)]}}{\text{“measurement”}} + \underset{\text{[each } \Phi_{t_1,t_2} \text{ is homomorphic in Axiom 2 (3.26)]}}{\text{“the deterministic relation among systems”.}} \\[3ex]
\text{stochastic PMT} = \underset{\text{[Axiom 1 (2.37)]}}{\text{“measurement”}} + \underset{\text{[Axiom 2 (3.26)]}}{\text{“the Markov relation among systems”.}}
\end{array}
\right.
\tag{3.27}
$$

Remark 3.6. (i). Roughly speaking, Axiom 2 asserts $\Phi_{0,1}\mathbf{O}_1$ is more fundamental than $\mathbf{O}_1$ in the following identification

$$\Phi_{0,1}\mathbf{O}_1 \text{ (in } \mathcal{A}_0) \longleftrightarrow \mathbf{O}_1 \text{ (in } \mathcal{A}_1)$$

where $\mathbf{O}_1$ is an observable in $\mathcal{A}_1$ and $\Phi_{0,1} : \mathcal{A}_1 \to \mathcal{A}_0$ is a Markov operator.
(ii). Also, it should be noted that Axiom 2 says that the time evolution of a system satisfies the Markov property. Thus, automata theory and circuit theory are characterized as special cases of measurement theory (especially, Axiom 2).
(iii). Axiom 2 has a great descriptive power. Note that “hysteresis” and “multiple Markov properties” can be described in the framework of Axiom 2.

■

[4] This fact makes us apply Axiom 2 to “statistical measurement theory” (in Chapter 8) as well as “PMT” (in this chapter).

3.5 Heisenberg picture and Schrödinger picture

Now let us mention something about the relation between Heisenberg picture and Schrödinger picture.

Suppose that a simplest tree ($T \equiv \{0,1\}, \pi$) has a "series" structure, i.e., $\pi(1) = 0$. Consider a general system $\mathbf{S}_{[\rho_0^p]} \equiv [S_{[\rho_0^p]}, \mathcal{A}_1 \overset{\Phi_{0,1}}{\to} \mathcal{A}_0]$ with the initial system $S_{[\rho_0^p]}$, that is,

$$\mathcal{A}_0 \overset{\Phi_{0,1}}{\longleftarrow} \mathcal{A}_1 \tag{3.28}$$

Let $\mathbf{O}_1 = (X_1, \mathcal{F}_1, F_1)$ be an observable in $\mathcal{A}_1$. Now we consider

(M) the measurement of the observable $\mathbf{O}_1 = (X_1, \mathcal{F}_1, F_1)$ for the general system $\mathbf{S}_{[\rho_0^p]} \equiv [S_{[\rho_0^p]}, \mathcal{A}_1 \overset{\Phi_{0,1}}{\to} \mathcal{A}_0]$

Under the following identification:

$$\boxed{\Phi_{0,1}\mathbf{O}_1 \text{ in } \mathcal{A}_0} \longleftrightarrow \boxed{\mathbf{O}_1 \text{ in } \mathcal{A}_1} \tag{3.29}$$

we think that

$$(M) = \mathbf{M}_{\mathcal{A}_0}(\Phi_{0,1}\mathbf{O}_1, S_{[\rho_0^p]}). \tag{3.30}$$

This viewpoint is standard, and it is called the *Heisenberg picture representation* of (M). Axiom 1 says that

- the probability that the measured value of the measurement (M) (i.e.,$\mathbf{M}_{\mathcal{A}_0}(\Phi_{0,1}\mathbf{O}_1, S^0_{[\rho_0^p]})$) belongs to Ξ_1 ($\in \mathcal{F}_1$) is given by

$$\rho_0^p(\Phi_{0,1}F(\Xi_1))\big(\equiv {}_{\mathcal{A}_0^*}\Big\langle \rho_0^p, \Phi_{0,1}F(\Xi_1)\Big\rangle_{\mathcal{A}_0}\big). \tag{3.31}$$

On the other hand, under the following identification:

$$\boxed{\rho_0^p \text{ in } \mathfrak{S}(\mathcal{A}_0^*)} \longleftrightarrow \boxed{\Phi_{0,1}^*\rho_0^p \text{ in } \mathfrak{S}(\mathcal{A}_1^*)},$$

we also consider that

$$(M) = \mathbf{M}_{\mathcal{A}_1}(\mathbf{O}_1, S_{[\Phi_{0,1}^*\rho_0^p]}) \tag{3.32}$$

(though $\Phi_{0,1}^*\rho_0^p$ is not in $\mathfrak{S}^p(\mathcal{A}^*)$ but in $\mathfrak{S}^m(\mathcal{A}^*)$ if $\Phi_{0,1}$ is not homomorphic. Cf. Chapter 8 (statistical measurement theory),) This viewpoint is called *the Schrödinger picture representation* of (M). We of course think that

- the probability that the measured value of the measurement (M) (i.e.,$\mathbf{M}_{\mathcal{A}_1}(\mathbf{O}_1, S_{[\Phi^*_{0,1}\rho^p_0]})$) belongs to Ξ_1 is given by

$$\rho^p_0(\Phi_{0,1}F(\Xi_1))\big(\equiv {}_{\mathcal{A}_1^*}\Big\langle \Phi^*_{0,1}\rho^p_0, F(\Xi_1)\Big\rangle_{\mathcal{A}_1}\big). \tag{3.33}$$

It should be noted that (3.31) = (3.33) holds. Thus it is usually and roughly said that

- Heisenberg picture (i.e., observable moves) and Schrödinger picture (i.e., state moves) are equivalent,

though the Heisenberg picture is fundamental (and the Schrödinger picture representation should be regarded as a kind of prescription). For the further arguments, see §6.2.

3.6 Measurability theorem

The following theorem is the most fundamental in classical PMT.

Theorem 3.7. [The measurability theorem of a general system, *cf.* [43]]. *Let* $(T \equiv \{0,1,...,N\}, \pi : T\setminus\{0\} \to T)$ *be a tree with root* 0 *and let* $\mathbf{S}_{[\rho^p_0]} \equiv [S_{[\rho^p_0]}, \mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\ (t\in T\setminus\{0\})]$ *be a general system with the initial system* $S_{[\rho^p_0]}$*. And, let an observable* $\mathbf{O}_t \equiv (X_t, \mathcal{F}_t, F_t)$ *in a* C^**-algebra* $\mathcal{A}_t$ *be given for each* $t\in T$*. For each* s $(\in T)$*, define the observable* $\widetilde{\mathbf{O}}_s \equiv (\prod_{t\in T_s} X_t, \prod_{t\in T_s}\mathcal{F}_t, \widetilde{F}_s)$ *in* $\mathcal{A}_s$ *such that:*

$$\widetilde{\mathbf{O}}_s = \begin{cases} \mathbf{O}_s & (\text{if } s\in T\setminus\pi(T)) \\ \mathbf{O}_s \overset{\text{qp}}{\times} (\overset{\text{qp}}{\times}_{t\in\pi^{-1}(\{s\})} \Phi_{\pi(t),t}\widetilde{\mathbf{O}}_t) & (\text{if } s\in\pi(T)) \end{cases}$$

if possible. Then, if an observable $\widetilde{\mathbf{O}}_0$ *(i.e., the Heisenberg picture representation of the sequential observable* $[\{\mathbf{O}_t\}_{t\in T}, \{\Phi_{t,\pi(t)} : \mathcal{A}_t \to \mathcal{A}_{\pi(t)}\}_{t\in T\setminus\{0\}}\ ]$*) in* $\mathcal{A}_0$ *exists, we have the measurement*

$$\mathbf{M}_{\mathcal{A}_0}(\widetilde{\mathbf{O}}_0 \equiv (\prod_{t\in T} X_t, \prod_{t\in T}\mathcal{F}_t, \widetilde{F}_0), S_{[\rho^p_0]}), \tag{3.34}$$

$\Big(\bigotimes_{t\in T}\mathcal{F}_t$ *is sometimes denoted by* $\prod_{t\in T}\mathcal{F}_t$*, cf. Definition 2.10*$\Big)$*, which is called the Heisenberg picture representation of the symbol* $\mathfrak{M}(\{\mathbf{O}_t\}_{t\in T}, \mathbf{S}_{[\rho^p_{t_0}]})$*. If the system is classical, i.e.,* $\mathcal{A}_t \equiv C(\Omega_t)$ $(\forall t\in T)$*, then the measurement always exists, while the uniqueness is not always guaranteed. Also, it should be noted that, for each* $s(\in T)$*, it holds that* $\Phi_{\pi(s),s}\widetilde{F}_s(\prod_{t\in T_s}\Xi_t) = \widetilde{F}_{\pi(s)}((\Pi_{t\in T_{\pi(s)}\setminus T_s} X_t)\times(\prod_{t\in T_s}\Xi_t))$ $(\forall \Xi_t \in \mathcal{F}_t\ (\forall t\in T))$.

Proof. It suffices to prove it in classical measurements. However it is clear since, in classical measurements, the product observable of any observables always exists. Therefore the construction mentioned in Example 3.4 is always possible in classical systems.

□

Example 3.8. [Random walk]. Suppose that a tree $(T \equiv \{0, 1, ..., N\}, \pi)$ has a "series" structure, i.e., $\pi(t) = t - 1$ $(\forall t \in T \setminus \{0\})$. Consider a general system $\mathbf{S}_{[\delta_0]} \equiv [S_{[\delta_0]}, \mathcal{A}_t \overset{\Phi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}$ $(t \in T \setminus \{0\})]$ with the initial system $S_{[\delta_0]}$, that is,

$$\mathcal{A}_0 \overset{\Phi_{0,1}}{\longleftarrow} \mathcal{A}_1 \overset{\Phi_{1,2}}{\longleftarrow} \mathcal{A}_2 \overset{\Phi_{2,3}}{\longleftarrow} \cdots\cdots\cdots \overset{\Phi_{N-2,N-1}}{\longleftarrow} \mathcal{A}_{N-1} \overset{\Phi_{N-1,N}}{\longleftarrow} \mathcal{A}_N. \tag{3.35}$$

Let $\mathbb{Z}$ be the set of all integers, i.e., $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$. Consider a commutative C^*-algebra $C_0(\mathbb{Z})$. Here, put

$$\mathcal{A}_t = C_0(\mathbb{Z}) \quad (\forall t \in \{0, 1, ..., N\})$$

and define a Markov operator $\Phi_{t-1,t}(\equiv \Phi) : \mathcal{A}_t(\equiv C_0(\mathbb{Z})) \to \mathcal{A}_{t-1}(\equiv C_0(\mathbb{Z}))$ such that:

$$(\Phi f)(n) = (\Phi_{t-1,t} f)(n) = \frac{f(n+1) + f(n-1)}{2} \qquad (\forall f \in \mathcal{A}_t(\equiv C_0(\mathbb{Z})), \forall n \in \mathbb{Z}).$$

Also, for each $t = 0, 1, 2, ..., N$, consider the exact observable $\mathbf{O}_t \equiv (X_t, \mathcal{R}_t, E) \equiv (\mathbb{Z}, \mathcal{P}_0(\mathbb{Z}), E)$ in $\mathcal{A}_t(= C_0(\mathbb{Z}))$ such that, (*cf.* Example 2.20),

$$[E(\Xi)](n) = \begin{cases} 1 & n \in \Xi(\in \mathcal{P}_0(\mathbb{Z})) \\ \\ 0 & n \notin \Xi(\in \mathcal{P}_0(\mathbb{Z})). \end{cases} \tag{3.36}$$

Thus, we get the product observable $\widetilde{\mathbf{O}}_0 \equiv (\times_{t=0}^N X_t,\ \times_{t=0}^N \mathcal{F}_t,\ \widetilde{F}_0) \equiv (\mathbb{Z}^{N+1}, \mathcal{P}_0(\mathbb{Z}^{N+1}), \widetilde{F}_0)$ in $\mathcal{A}_0$ $(\equiv C_0(\mathbb{Z}))$, that is,

$$\widetilde{F}_0(\Xi_0 \times \Xi_1 \times \Xi_2 \times \cdots \times \Xi_N) = E(\Xi_0) \times \Phi(E(\Xi_1) \times \Phi(\cdots\cdots \Phi(E(\Xi_{N-1}) \times \Phi E(\Xi_N)) \cdots)).$$

Then, we have the "measurement" $\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T},\ \mathbf{S}_{[\delta_0]})$ such as

$$\mathfrak{M}(\{\mathbf{O}_t\}_{t \in T}, \mathbf{S}_{[\delta_0]}) = \mathbf{M}_{C(\mathbb{Z})}(\widetilde{\mathbf{O}}_0 \equiv (\mathbb{Z}^{N+1}, \mathcal{P}_0(\mathbb{Z}^{N+1}), \widetilde{F}_0), S_{[\delta_0]}).$$

where δ_0 is the point measure at 0 $(\in \mathbb{Z})$. The sample space $\Big(\mathbb{Z}^{N+1}, \mathcal{P}_0(\mathbb{Z}^{N+1}), [\widetilde{F}_0(\cdot)](0)\Big)$ is usually called a *random walk.*

■

For the further arguments, see §10.4 (Brown motion).

3.7 Appendix (Bell's inequality)

(Continued from §2.9 (Bell's Thought Experiment))[5]

3.7.1 Deterministic evolution or Stochastic evolution?

Recall the following classification (3.27) in PMT:

$$\begin{cases} \text{deterministic PMT} = \underset{\text{[Axiom 1 (2.37)]}}{\text{“measurement”}} + \underset{\text{[each } \Phi_{t_1,t_2} \text{ is homomorphic in Axiom 2 (3.26)]}}{\text{“the deterministic relation among systems”}}. \\ \\ \text{stochastic PMT} = \underset{\text{[Axiom 1 (2.37)]}}{\text{“measurement”}} + \underset{\text{[Axiom 2 (3.26)]}}{\text{“the Markov relation among systems”}}. \end{cases}$$

However, we know that in classical (or quantum) mechanics, the general system $\mathbf{S}_{[\rho^p]} \equiv [S_{[\rho^p]}, \mathcal{A}_t \overset{\Psi_{\pi(t),t}}{\to} \mathcal{A}_{\pi(t)}\ (t \in T \setminus \{0\})]$ is always deterministic, that is, $\Psi_{\pi(t),t}$ is always homomorphic. (*cf.* "Newtonian mechanics and quantum mechanics" in §3.1.)

Recall (2.76), i.e., the de Broglie paradox (*cf.* [20]. Also see §9.3.3). That is,

- if we admit quantum mechanics $\Big($= "Axiom 1 + Axiom 2 (homomorphic time evolution)"$\Big)$, we must admit the fact that there is something faster than light. (*cf.* [18, 78]). (3.37) (=(2.76))

Of course we admit quantum mechanics, and therefore, we believe that there is something faster than light. However, most people may hope that quantum mechanics is not true rather than admit the fact that there is something faster than light. That is,

($\sharp$) Using the Schrödinger picture representation, they may assert that the singlet state ρ_s is not fixed, but the Markov time evolution $\Big($i.e., "the Markov relation among systems (Axiom 2)" and not "the homomorphic relation among systems Axiom 2)"$\Big)$:

$$\rho_s \overset{\Phi^*}{\rightsquigarrow} \rho_0^m \tag{3.38}$$

should be considered.

[5] Although Bell's inequality is generally said to be one of the most profound discoveries in 20-th century science, I could not understand the arguments (in [9, 18, 78, 8]), particularly, I had the question: "In what framework is Bell's inequality discussed (in [9, 18, 78])?". I wonder if these arguments are confusing physics with mathematics. Thus, I add this section, in which all arguments are discussed in the framework of PMT (Axioms 1 and 2).

The purpose of the following section (i.e., §3.7.2) is to show that we must admit that there is something faster than light, even under the above assumption (♯). That is, if we assert that PMT $\Big($= "Axiom 1 + Axiom 2 (Markov time evolution)", i.e., quantum mechanics with Markov (and not homomorphic) time evolution$\Big)$ is true, we must admit the fact that there is something faster than light.

3.7.2 Generalized Bell's inequality in mathematics

First we prepare some mathematical inequalities. Of course, what is most important is how to interpret these theorems in physics. This will be discussed in the next section. In order to avoid to confuse physical results and mathematical ones, in this §3.7.2, we devote ourselves to mathematical arguments.

Theorem 3.9. [Bell's inequality, *cf.* [9, 78]]. *Let $(Y, \mathcal{G}, m)$ be a probability space. Let g_1^1, g_1^2, g_2^1, g_2^2 be $\{-1,1\}$-valued measurable functions on Y. Define the correlation function $P'(g_1^i, g_2^j)$ such that:*

$$P'(g_1^i, g_2^j) = \int_Y g_1^i(y) g_2^j(y) \, m(dy). \tag{3.39}$$

Then, it holds that

$$|P'(g_1^1, g_2^1) - P'(g_1^1, g_2^2)| + |P'(g_1^2, g_2^1) + P'(g_1^2, g_2^2)| \le 2. \tag{3.40}$$

Proof. For completeness, we add the proof in what follows.

$$\begin{aligned}
&|P'(g_1^1, g_2^1) - P'(g_1^1, g_2^2)| + |P'(g_1^2, g_2^1) + P'(g_1^2, g_2^2)| \\
\le &\int_{X^4} |g_1^1(y)| \cdot |g_2^1(y) - g_2^2(y)| m(dy) + \int_Y |g_1^2(y)| \cdot |g_2^1(y) + g_2^2(y)| m(dy) \\
\le &\int_{X^4} |g_2^1(y) - g_2^2(y)| + |g_2^1(y) + g_2^2(y)| m(dy) = 2.
\end{aligned}$$

This completes the proof. □

Corollary 3.10. [Bell's inequality]. *Let $(Y, \mathcal{G}, m)$ be a probability space. Let g_1^{11}, g_1^{12}, g_1^{21}, g_1^{22}, g_2^{11}, g_2^{12}, g_2^{21} and g_2^{22} be $\{-1,1\}$-valued measurable functions on Y. Define the correlation function $P(g_1^{ij}, g_2^{ij})$ such that*

$$P(g_1^{ij}, g_2^{ij}) = \int_Y g_1^{ij}(y) g_2^{ij}(y) \, m(dy). \tag{3.41}$$

Further, assume that

$$g_1^{11} = g_1^{12}, \quad g_1^{21} = g_1^{22}, \quad g_2^{11} = g_2^{21}, \quad g_2^{12} = g_2^{22} \quad (\text{a.e. } m) \tag{3.42}$$

i.e., $m\big(\{y \in Y \ : \ g_1^{11}(y) = g_1^{12}(y)\}\big) = 1$, *etc. Then, it holds that*

$$|P(g_1^{11}, g_2^{11}) - P(g_1^{12}, g_2^{12})| + |P(g_1^{21}, g_2^{21}) + P(g_1^{22}, g_2^{22})| \leq 2. \tag{3.43}$$

Proof. It immediately follows from Theorem 3.9. □

Next we present the following theorem, which can be regarded as a generalization of the above corollary (*cf.* Remark 3.12 later).

Theorem 3.11. [Generalized Bell's inequality]. *Let* $(Y, \mathcal{G}, m)$ *be a probability space. Let* g_1^{11}, g_1^{12}, g_1^{21}, g_1^{22}, g_2^{11}, g_2^{12}, g_2^{21} *and* g_2^{22} *be* $\{-1, 1\}$*-valued measurable functions on* Y*. Assume that these satisfy*

$$m[(g_1^{ij}, g_2^{ij})^{-1}(B_1 \times B_2)] = \sum_{\ell \in L} \alpha_\ell \ \mu_{1,\ell}^i(B_1) \ \mu_{2,\ell}^j(B_2) \quad (\forall B_1, B_2 \subseteq \{-1, 1\}, \ \forall i, j = 1, 2) \tag{3.44}$$

for some probability measures $\mu_{k,\ell}^i$, $(k, i = 1, 2, \ell \in L)$, *on* $\{-1, 1\}$ *and some nonnegative sequence* $\{\alpha_\ell\}_{\ell \in L}$ *such that* $\sum_{\ell \in L} \alpha_\ell = 1$. *Then, it holds that*

$$|P(g_1^{11}, g_2^{11}) - P(g_1^{12}, g_2^{12})| + |P(g_1^{21}, g_2^{21}) + P(g_1^{22}, g_2^{22})| \leq 2, \tag{3.45}$$

where the correlation functions $P(g_1^{ij}, g_2^{ij})$ *are defined by (3.41).*

Proof. A simple calculation shows that

$$\begin{aligned} P(g_1^{ij}, g_2^{ij}) &= \sum_{\ell \in L} \alpha_\ell \ [\sum_{(x_1, x_2) \in \{-1,1\}^2} x_1 x_2 \ \mu_{1,\ell}^i(\{x_1\}) \mu_{2,\ell}^j(\{x_2\})] \\ &= \sum_{\ell \in L} \alpha_\ell (4\mu_{1,\ell}^i \mu_{2,\ell}^j + 1 - 2\mu_{1,\ell}^i - 2\mu_{2,\ell}^j), \end{aligned}$$

where $\mu^i_{k,\ell} = \mu^i_{k,\ell}(\{1\})$. Thus, we see that

$$
\begin{aligned}
&|P(g_1^{11}, g_2^{11}) - P(g_1^{12}, g_2^{12})| + |P(g_1^{21}, g_2^{21}) + P(g_1^{22}, g_2^{22})| \\
=&|\sum_{\ell\in L} \alpha_\ell(4\mu^1_{1,\ell}\mu^1_{2,\ell} + 1 - 2\mu^1_{1,\ell} - 2\mu^1_{2,\ell}) - \sum_{\ell\in L} \alpha_\ell(4\mu^1_{1,\ell}\mu^2_{2,\ell} + 1 - 2\mu^1_{1,\ell} - 2\mu^2_{2,\ell})| \\
&\qquad + |\sum_{\ell\in L} \alpha_\ell(4\mu^2_{1,\ell}\mu^1_{2,\ell} + 1 - 2\mu^2_{1,\ell} - 2\mu^1_{2,\ell}) + \sum_{\ell\in L} \alpha_\ell(4\mu^2_{1,\ell}\mu^2_{2,\ell} + 1 - 2\mu^2_{1,\ell} - 2\mu^2_{2,\ell})| \\
=&|\sum_{\ell\in L} \alpha_\ell(4\mu^1_{1,\ell}\mu^1_{2,\ell} - 2\mu^1_{2,\ell} - 4\mu^1_{1,\ell}\mu^2_{2,\ell} + 2\mu^2_{2,\ell})| \\
&\qquad + |\sum_{\ell\in L} \alpha_\ell(4\mu^2_{1,\ell}\mu^1_{2,\ell} + 2 - 4\mu^2_{1,\ell} - 2\mu^1_{2,\ell} + 4\mu^2_{1,\ell}\mu^2_{2,\ell} - 2\mu^2_{2,\ell})| \equiv |A| + |B|,
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
&= \begin{cases} |\sum_{\ell\in L} \alpha_\ell[2 - 4(\mu^2_{1,\ell} + \mu^1_{2,\ell} + \mu^1_{1,\ell}\mu^2_{2,\ell} - \mu^1_{1,\ell}\mu^1_{2,\ell} - \mu^2_{1,\ell}\mu^1_{2,\ell} - \mu^2_{1,\ell}\mu^2_{2,\ell})]| & (\text{if } A\cdot B \geq 0) \\ |\sum_{\ell\in L} \alpha_\ell[2 - 4(\mu^2_{1,\ell} + \mu^2_{2,\ell} + \mu^1_{1,\ell}\mu^1_{2,\ell} - \mu^1_{1,\ell}\mu^2_{2,\ell} - \mu^2_{1,\ell}\mu^2_{2,\ell} - \mu^2_{1,\ell}\mu^1_{2,\ell})]| & (\text{if } A\cdot B \leq 0) \end{cases} \\
&\leq \begin{cases} \sum_{\ell\in L} \alpha_\ell|2 - 4(\mu^2_{1,\ell} + \mu^1_{2,\ell} + \mu^1_{1,\ell}\mu^2_{2,\ell} - \mu^1_{1,\ell}\mu^1_{2,\ell} - \mu^2_{1,\ell}\mu^1_{2,\ell} - \mu^2_{1,\ell}\mu^2_{2,\ell})| & (\text{if } A\cdot B \geq 0) \\ \sum_{\ell\in L} \alpha_\ell|2 - 4(\mu^2_{1,\ell} + \mu^2_{2,\ell} + \mu^1_{1,\ell}\mu^1_{2,\ell} - \mu^1_{1,\ell}\mu^2_{2,\ell} - \mu^2_{1,\ell}\mu^2_{2,\ell} - \mu^2_{1,\ell}\mu^1_{2,\ell})| & (\text{if } A\cdot B \leq 0). \end{cases}
\end{aligned}
$$

Hence, it suffices to prove that $0 \leq C(x,y,z,w) \leq 1$ $(\forall(x,y,z,w) \in [0,1]^4)$, where $C(x,y,z,w) = y + z + xw - xz - yz - yw$. This is shown as follows:
[Case 1; $w - z \geq 0$].

$$
\begin{aligned}
0 &\leq y(1-w) + z(1-y) + x(w-z) \equiv C \leq C + (w-z)(1-x) \\
&= y(1-w) + w - yz \leq 1 - yz \leq 1.
\end{aligned}
$$

[Case 2; $w - z \leq 0$].

$$
\begin{aligned}
0 &\leq y(1-z) + w(1-y) = y + z + (w-z) - yz - yw \\
&\leq y + z + x(w-z) - yz - yw \equiv C \leq y + z - yz - yw \leq y(1-z) + z \leq 1.
\end{aligned}
$$

This completes the proof. □

Remark 3.12. It is interesting to see that Corollary 3.10 can be regarded as a particular case of Theorem 3.11. This can be easily shown as follows: Let $(Y, \mathcal{G}, m)$ and g_k^{ij} be as in Corollary 3.10. Thus, we assume that the condition (3.42) holds. Put $L = \{-1,1\}^4$. For each ℓ ($\equiv (\ell_1^1, \ell_1^2, \ell_2^1, \ell_2^2) \in L$), define the α_ℓ ($\in [0,1]$) such that $\alpha_{(\ell_1^1,\ell_1^2,\ell_2^1,\ell_2^2)} = m\big((g_1^{11}, g_1^{22}, g_2^{11}, g_2^{22})^{-1}\ (\{(\ell_1^1, \ell_1^2,\ \ell_2^1, \ell_2^2)\})\big)$. Clearly it holds that $\sum_{\ell\in L} \alpha_\ell = 1$. Define the probability measures $\widehat{\mu}_1$ and $\widehat{\mu}_{-1}$ on $\{-1,1\}$ such that $\widehat{\mu}_1(\{-1\}) = 0$, $\widehat{\mu}_1(\{1\}) = 1$

and $\widehat{\mu}_{-1} = 1-\widehat{\mu}_1$. It is easy to see that $m\big((g_1^{11}, g_1^{22}, g_2^{11}, g_2^{22})^{-1}(\{(x_1^1, x_1^2, x_2^1, x_2^2)\})\big) = \sum_{\ell\in L}$ $\alpha_\ell\ \widehat{\mu}_{\ell_1^1}(\{x_1^1\})\widehat{\mu}_{\ell_1^2}(\{x_1^2\})\widehat{\mu}_{\ell_2^1}(\{x_2^1\})\ \widehat{\mu}_{\ell_2^2}(\{x_2^2\})$ $(\forall (x_1^1, x_1^2, x_2^1, x_2^2) \in \{-1,1\}^4)$. Thus, putting $\mu^i_{k,(\ell_1^1,\ell_1^2,\ell_2^1,\ell_2^2)} = \widehat{\mu}_{\ell_k^i}$, we can immediately see that the $\{\alpha_\ell\}_{\ell\in L}$ and the $\{\mu^i_{k,\ell} : i,k = 1,2, \ell \in L\}$ satisfy the condition (3.44).

■

3.7.3 Generalized Bell's inequality in Measurements

Put $X = \{-1,1\}$. Consider a measurement $\mathbf{M}_{\mathcal{A}}(\ \mathbf{O} \equiv (X^8, \mathcal{P}(X^8), G),\ S_{[\rho_0]})$ formulated in arbitrary C^*-algebra $\mathcal{A}$. Putting $\nu^3_{\text{BI}}(\ \cdot\ ) = \rho_0(G(\ \cdot\ ))$, we have the sample space $(X^8, \mathcal{P}(X^8), \nu^3_{\text{BI}})$, which is induced by the measurement $\mathbf{M}_{\mathcal{A}}(\ \mathbf{O},\ S_{[\rho_0]})$. Consider the $\{-1,1\}$-valued functions g_k^{ij} on X^8, $(i,j,k = 1,2)$. And define the correlation functions $P(g_1^{ij}, g_2^{ij})$ $(i,j = 1,2)$ by (3.41). Assume the condition (3.44) in Theorem 3.11. Then, we see, by Theorem 3.11, that the following inequality holds:

$$|P(g_1^{11}, g_2^{11}) - P(g_1^{12}, g_2^{12})| + |P(g_1^{21}, g_2^{21}) + P(g_1^{22}, g_2^{22})| \le 2. \tag{3.46}$$

Therefore, it may be viable to compare the measurement $\mathbf{M}_{\mathcal{A}}(\ \mathbf{O},\ S_{[\rho_0]})$ with the measurement $\bigotimes_{i,j=1,2} \mathbf{M}_{B(\mathbf{C}^2\otimes\mathbf{C}^2)}\ (\mathbf{O}_{a^ib^j}, S_{[\rho_s]})$ in Bell's thought experiment, though it is also sure that these are not connected with each other. For example, some may, by some reason, consider that the singlet state ρ_s in Bell's thought experiment (*cf.* the formula (2.75)) is reduced to a certain state ρ_0 ($\in \mathfrak{S}^p(B(\mathbf{C}^2 \otimes \mathbf{C}^2)^*)$) such as

$$\rho_s \rightsquigarrow \rho_0 = |\vec{e} \otimes \vec{f}\rangle\langle \vec{e} \otimes \vec{f}| \tag{3.47}$$

for some $\vec{e}\otimes\vec{f}$ ($\in \mathbf{C}^2\otimes\mathbf{C}^2$) such that $\|\vec{e}\|_{\mathbf{C}^2} = \|\vec{f}\|_{\mathbf{C}^2} = 1$. If so, instead of the measurement $\bigotimes_{i,j=1,2} \mathbf{M}_{B(\mathbf{C}^2\otimes\mathbf{C}^2)}(\mathbf{O}_{a^ib^j}, S_{[\rho_s]})$, we must consider the measurement $\bigotimes_{i,j=1,2} \mathbf{M}_{B(\mathbf{C}^2\otimes\mathbf{C}^2)}$ $(\mathbf{O}_{a^ib^j}, S_{[\rho_0]})$, which has the sample space $(X^8, \mathcal{P}(X^8), \nu)$ such that:

$$\nu(\{(x_1^{11}, x_2^{11}, x_1^{12}, x_2^{12}, x_1^{21}, x_2^{21}, x_1^{22}, x_2^{22})\}) = \prod_{i,j=1,2} \rho_0\big((F_{a^i} \otimes F_{b^j})(\{(x_1^{ij}, x_2^{ij})\})\big)$$

$$= \prod_{i,j=1,2} \big[\langle \vec{e}, F_{a^i}(\{x_1^{ij}\})\vec{e}\rangle \langle \vec{f}, F_{b^j}(\{x_2^{ij}\})\vec{f}\rangle\big].$$

Or more generally (or, in the sense of "ensemble"), using the adjoint operator Φ^* of a Markov operator $\Phi : B(\mathbf{C}^2 \otimes \mathbf{C}^2) \to B(\mathbf{C}^2 \otimes \mathbf{C}^2)$, we may consider the following Markov evolution:

$$\rho_s \overset{\Phi^*}{\rightsquigarrow} \rho_0^m = \sum_{n=1}^{2}\sum_{m=1}^{2} \alpha_{mn} |\vec{e}_m \otimes \vec{f}_n\rangle\langle \vec{e}_m \otimes \vec{f}_n|, \tag{3.48}$$

where $\{\vec{e}_m\}_{m=1}^2$ and $\{\vec{f}_m\}_{i=1}^2$ are respectively the complete orthonormal basis in $\mathbf{C}^2$, and $0 \le \alpha_{mn} \le 1$ such that $\sum_{n=1}^2 \sum_{m=1}^2 \alpha_{mn} = 1$. Thus we have the (statistical) measurement $\bigotimes_{i,j=1,2} \mathbf{M}_{B(\mathbf{C}^2 \otimes \mathbf{C}^2)} \ (\Phi \mathbf{O}_{a^i b^j}, S_{[\rho_s]})$. Thus, we may have the sample space $(X^8, \mathcal{P}(X^8), \nu)$ such that:

$$
\begin{aligned}
&\nu(\{(x_1^{11}, x_2^{11}, x_1^{12}, x_2^{12}, x_1^{21}, x_2^{21}, x_1^{22}, x_2^{22})\}) = \prod_{i,j=1,2} \rho_s\big((\Phi F_{a^i} \otimes F_{b^j})(\{(x_1^{ij}, x_2^{ij})\})\big) \\
= &\prod_{i,j=1,2} (\Phi^* \rho_s)\big((F_{a^i} \otimes F_{b^j})(\{(x_1^{ij}, x_2^{ij})\})\big) = \prod_{i,j=1,2} \rho_0^m\big((F_{a^i} \otimes F_{b^j})(\{(x_1^{ij}, x_2^{ij})\})\big) \\
= &\prod_{i,j=1,2} \big[\sum_{m=1}^2 \sum_{n=1}^2 \alpha_{mn} \langle \vec{e}_m, F_{a^i}(\{x_1^{ij}\})\vec{e}_m \rangle \langle \vec{f}_n, F_{b^j}(\{x_2^{ij}\})\vec{f}_n \rangle\big].
\end{aligned}
$$

Note that the probability space $(X^8, \mathcal{P}(X^8),\ \nu)$ and the g_k^{ij} defined by (2.77) satisfy the condition (3.44) in Theorem 3.11. That is because it suffices to put $L = \{-1, 1\}^2$ and

$$
\begin{aligned}
\mu_{1,(m,n)}^1(\,\cdot\,) &= \langle \vec{e}_m, F_{a^1}(\,\cdot\,)\vec{e}_m \rangle, & \mu_{1,(m,n)}^2(\,\cdot\,) &= \langle \vec{e}_m, F_{a^2}(\,\cdot\,)\vec{e}_m \rangle, \\
\mu_{2,(m,n)}^1(\,\cdot\,) &= \langle \vec{f}_n, F_{b^1}(\,\cdot\,)\vec{f}_n \rangle, & \mu_{2,(m,n)}^2(\,\cdot\,) &= \langle \vec{f}_n, F_{b^2}(\,\cdot\,)\vec{f}_n \rangle,
\end{aligned}
$$

for each (m, n) $(\in L \equiv \{-1, 1\}^2)$. Thus, Theorem 3.11 says that such Markov evolution as the above (3.47) or (3.48) does not occur in Bell's thought experiment. Therefore we can conclude that

- if we admit PMT (= "Axiom 1 + Axiom 2 (Markov relation)"), we must also admit the fact that there is something faster than light. (3.49)

Of course we admit PMT, and therefore, we believe that there is something faster than light.