Colorings of Graphs and Vertex Partitions of Graphs with Degree Conditions

by

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Preface

This thesis deals with two topics in graph theory; colorings of graphs and degree conditions of graphs.

After an introductory chapter and a chapter for basic terminologies, readers will see two parts, each of which is on colorings of graphs and on vertex partitions of graphs with degree conditions, respectively.

In the first part entitled colorings of graphs, we focus on three different variations of a coloring of graphs. The first is the online list coloring, which was introduced in 2009 as an extended concept of list coloring. In Chapter 3, we focus on online list coloring of complete bipartite graphs.

Chapters 4-7 are devoted to the odd coloring and the proper conflict-free coloring of graphs, which were introduced, respectively, in 2022 and in 2023. A coloring of a graph is called proper if every adjacent pair of vertices receives distinct colors. Both odd coloring and proper conflict-free coloring of a graph are proper colorings with constraints on colors appearing at the neighborhood of each vertex. One major problem on odd coloring of graphs is how many colors are necessary for planar graphs. In Chapter 5, we focus on odd coloring of outerplanar graphs. In Chapter 6, we investigate odd coloring of k-trees, which are related to maximal outerplanar graphs.

Brooks-type theorem for proper conflict-free coloring is one of the actively studied topics. Motivated by such results and a classical concept of degree-choosability of graphs, in Chapter 7, we introduce a concept of proper conflict-free (degree $+\, k$)-choosability of a graph. We investigate the concept and show some results on the cases $k = 1$, 2, and 3.

In the second part entitled degree conditions for some vertex partition problems, we mainly consider a new degree condition named "minimum degree sum on large independent sets", introduced recently by the authors. A 2-proper partition of a graph is a partition of the vertex set of the graph such that each part induces a 2-connected subgraph. The concept of a 2-proper partition was introduced in 2013, and some degree conditions have been known for a graph to have a 2-proper partition. In Chapter 9, we discuss conditions on minimum degree sum on large independent sets for a graph to have a 2-proper partition.

A 2-factor of a graph is a 2-regular spanning subgraph of the graph. Since a connected 2-factor is a hamilton cycle, sufficient conditions for a graph to have a 2-factor have been intensively studied. In Chapter 10, we discuss conditions on minimum degree sum on large independent sets for a graph to have a 2-factor.

Contributions

[1] M. Furuya, M. Kashima, and K. Ota, New Invariants for Partitioning a Graph into 2-connected Subgraphs, to appear in Journal of Graph Theory.

[2] M. Kashima, Paintability of complete bipartite graphs, Discrete Applied Mathematics 346 (2024), 279–289.

[3] M. Kashima, New degree conditions for a graph to have a 2-factor, preprint.

[4] M. Kashima and K. Ozeki, Odd colorings of k-trees, preprint.

[5] M. Kashima, R. Škrekovski, and R. Xu, Results on proper conflict-free list colorings, submitted.

[6] M. Kashima and X. Zhu, Odd 4-coloring of outerplanar graphs, Graphs and Combinatorics 40 (2024), Paper No. 108.

Contents

Chapter 1

Introduction

In this thesis, we deal with two topics in graph theory; colorings of graphs and vertex partitions of graphs with specified properties. These two topics look completely different, but, in fact, both are problems of finding a vertex partition. In the first part of this thesis, we discuss colorings of graphs, which are partitions of the vertex set into independent sets. In the second part, we consider problems of finding a partition into 2-connected subgraphs, and problems of finding a partition into hamiltonian subgraphs, equivalently, finding a 2-factor. For both parts, our interest is the minimum number of parts among all such partitions of a given graph. In this chapter, we outline this thesis.

1.1 Colorings of graphs

The origin of graph coloring problems is the map coloring problem, especially the *Four-Color Problem*, one of the most famous problems in graph theory. Map coloring problems ask how many colors are needed to color a given map so that no adjacent two areas receive the same color. Map coloring problems can be translated into vertex-coloring problems of graphs, which ask how many colors are needed to color vertices of a given graph so that no adjacent two vertices receive the same color. Such a coloring of a graph is called a *proper coloring* of a graph. For a given graph G, let $\chi(G)$ denote the minimum number of colors to give a proper coloring. The Four-Color Problem asks whether $\chi(G) \leq 4$ for every planar graph G. This problem attracted many researchers and was finally solved positively by Appel and Haken [5] in 1976. Thus, the Four-Color Problem turned into the *Four-Color Theorem*.

Even after that, since their proof uses computer assists, finding an alternative proof of the Four-Color Theorem is an important problem in graph theory. Furthermore, various related problems have been posed, and many techniques in graph coloring problems have been developed in the stream.

Another important topic in graph coloring problems is the relationship between colorings and degree of vertices. It is easy to see that every graph G satisfies $\chi(G) \leq \Delta(G) + 1$ by giving a greedy coloring, where $\Delta(G)$ is the maximum degree of G. This bound is tight since the complete graph K_n of order n satisfies $\chi(K_n) = n$ and $\Delta(K_n) = n - 1$. Brooks showed in [11] that complete graphs and odd cycles are the only graphs which attain the equality.

Theorem 1.1 ([11])**.** For every connected graph G, $\chi(G) \le \Delta(G)$ unless G is a complete graph or an odd cycle.

This is a fundamental theorem in graph coloring problems, and there are many proofs using different kinds of technique in graph coloring problems. (See Cranston and Rabern [21] for some of them.) Brooks' theorem was extended to a characterization of degree-choosable graphs, which are defined in the context of list coloring of graphs.

The concept of *list coloring* of graphs was introduced in Vizing [62] and in Erdős, Rubin, and Taylor [27] as a variation of graph coloring. In list coloring, each vertex has a list of available colors, and the goal is to obtain a coloring by choosing a color from the list for each vertex. For a list assignment $L : V(G) \to 2^{\mathbb{N}}$ of a graph G, an L-coloring of G is a proper coloring φ such that $\varphi(v) \in L(v)$ for every vertex v of G. The *choice number* of a graph G, denoted by $\mathrm{ch}(G)$, is the least integer k such that G admits an L-coloring for any list assignment L of G such that $|L(v)| \ge k$ for every vertex v of G. For each vertex v of G, let $d_G(v)$ denote the degree of v. If a list assignment L of G satisfies $|L(v)| \ge d_G(v) + 1$ for every vertex v of G, then again by greedy colorings, we obtain an L-coloring of G. This obviously implies $\chi(G) \le \mathrm{ch}(G) \le \Delta(G) + 1$, and hence a question is when we can remove "+1" in the list setting. A graph G is called *degree-choosable* if G admits an L-coloring for any list assignment L of G with $|L(v)| \ge d_G(v)$ for every vertex v of G. Borodin [9] and Erdős, Rubin, and Taylor [27] independently characterized the degree-choosable graphs as follows. A *block* of a connected graph G is a maximal subgraph that does not contain a cut vertex of G.

Theorem 1.2 ([9], [27])**.** A connected graph G is degree-choosable if and only if G has a block that is neither a complete graph nor an odd cycle.

If a list assignment L of a graph G satisfies $|L(v)| \ge d_G(v)$ for every vertex v and $|L(v_0)| \ge d_G(v_0) + 1$ for some vertex v_0, then G admits an L-coloring. Thus, every connected graph G with at least two blocks satisfies $\chi(G) \le \Delta(G)$, and thus Theorem 1.2 implies Theorem 1.1.

In the long history of graph coloring problems, many variations of coloring, including list coloring, have been introduced. There are mainly two directions of variations; the problems of finding a coloring from available colors for each vertex with constraints and the problems of finding a coloring with additional properties. In this thesis, we investigate the online list coloring of graphs, which belongs to the first direction, the odd coloring, and the proper conflict-free coloring of graphs, which belong to the second direction.

1.1.1 Online list coloring of complete bipartite graphs

In 2009, Schauz [57] and Zhu [67] independently introduced a new variation of coloring named *online list coloring*, as a generalization of list coloring. For a graph G and a mapping f from $V(G)$ to (possibly negative) integers, the f-paintability of G is defined recursively as follows:

- When G has no edge, G is f-paintable if and only if $f(v) \geq 1$ for every vertex v of G.
- When G has at least one edge, G is f-paintable if and only if for every non-empty set of vertices $U \subseteq V(G)$, there is an independent set X of G such that $X \subseteq U$ and $G-X$ is f'-paintable where $f'(v) = f(v)$ for each $v \in V(G) \setminus U$ and $f'(v) = f(v)-1$ for each $v \in U \setminus X$.

Note that we consider the case where $f(v)$ has a negative value only for well-definedness, and if there is a vertex $v \in V(G)$ with $f(v) \leq 0$, then it implies that G is not f-paintable. The *paint number* of a graph G, denoted by $\mathrm{pnt}(G)$, is the least positive integer k such that G is f-paintable for the constant map f from $V(G)$ to k. It follows from a little observation that $\chi(G) \leq \mathrm{ch}(G) \leq \mathrm{pnt}(G)$ for every graph G, and it is known that the gap $\mathrm{pnt}(G) - \mathrm{ch}(G)$ can be arbitrarily large. In this thesis, we investigate online list colorings of complete bipartite graphs. For positive integers s and t, let $K_{s,t}$ denote the complete bipartite graph with partite sets of order s and t. In Carraher, Loeb, Mahoney, Puleo, Thai, and West [14], they investigated the least integer $r_{\mathrm{pnt}} = r_{\mathrm{pnt}}(m)$ such that $\mathrm{pnt}(K_{m+1,r_{\mathrm{pnt}}}) \geq m+1$, and showed that there is a gap from the least integer $r_{\mathrm{ch}} = r_{\mathrm{ch}}(m)$ such that $\mathrm{ch}(K_{m+1,r_{\mathrm{ch}}}) \geq m+1$. We strengthen their result by showing the following.

Theorem 1.3 ([37])**.** Let $r_{\mathrm{pnt}} = r_{\mathrm{pnt}}(m)$ be the least integer such that $\mathrm{pnt}(K_{m+1,r_{\mathrm{pnt}}}) \geq m+1$. Then there are constants c and c' such that $0 < c < c'$ and $cm^{m-1} < r_{\mathrm{pnt}} < c'm^{m-1}$.

Since it was shown in Hoffman and Johnson Jr. [36] that $r_{\mathrm{ch}}(m) = m^m - (m-1)^m = \Theta(m^m)$, Theorem 1.3 implies that the ratio $r_{\mathrm{pnt}}(m)/r_{\mathrm{ch}}(m)$ tends to 0 as m goes to infinity. We give a proof of Theorem 1.3 in Chapter 3.

1.1.2 Odd coloring and proper conflict-free coloring of graphs

Next, we focus on colorings with constraints on neighborhoods. One of such colorings is the *square coloring* of graphs, which is also called the distance-2 coloring of graphs. For a graph G, a mapping φ from $V(G)$ to positive integers is called a square coloring of G if every pair of vertices with distance at most 2 maps to distinct colors by φ. This is equivalent to a proper coloring of G^2, where G^2 is the graph obtained from G by adding an edge between every pair of vertices with distance 2 in G. The *square chromatic number* of a graph G, denoted by $\chi_2(G)$, is the minimum number of colors to give a square coloring of G. The square coloring of graphs was first introduced in Wegner [65], where he investigated the square coloring of planar graphs. What we should point out is that we cannot expect the existence of a constant k such that $\chi_2(G) \leq k$ for every planar graph G. Indeed, since the closed neighborhood of each vertex of a graph G becomes a clique of G^2, we have a trivial lower bound $\Delta(G) + 1 \leq \chi_2(G)$, which cannot be bounded by any constant. This shows that the behaviors of $\chi(G)$ and that of $\chi_2(G)$ are absolutely different.

Recently, two other variations of colorings, the *odd coloring* and the *proper conflict-free coloring* of graphs were introduced respectively in Petruševski and Škrekovski [55] and in Fabrici, Lužar, Rindošová, and Soták [29]. For a graph G, a mapping φ from $V(G)$ to positive integers is called an odd coloring of G if φ is a proper coloring of G such that for every non-isolated vertex v of G, there is a color c such that $|\varphi^{-1}(c) \cap N_G(v)|$ is an odd integer,

where $N_G(v)$ denotes the neighborhood of v. For a graph G, a mapping φ from $V(G)$ to positive integers is called a proper conflict-free coloring of G if φ is a proper coloring of G such that

for every non-isolated vertex v of G, there is a color c such that $|\varphi^{-1}(c) \cap N_G(v)| = 1$.

The *odd chromatic number* of a graph G, denoted by $\chi_o(G)$, is the least integer k such that G admits an odd coloring from $V(G)$ to $\{1, 2, \ldots, k\}$, and the *proper conflict-free chromatic number* of a graph G, denoted by $\chi_{\mathrm{pcf}}(G)$, is the least integer k such that G admits a proper conflict-free coloring from $V(G)$ to $\{1, 2, \ldots, k\}$. By the definitions, a proper conflict-free coloring of a graph G is an odd coloring of G as well, and an odd coloring of a graph G is a proper coloring of G as well. Furthermore, if φ is a square coloring of G, then for every non-isolated vertex v of G, $N_G(v)$ is colored by distinct colors, which implies that φ is a proper conflict-free coloring of G. Thus, the following inequality holds.

Proposition 1.4. For every graph G, $\chi(G) \leq \chi_o(G) \leq \chi_{\mathrm{pcf}}(G) \leq \chi_2(G)$.

It is observed in Petruševski and Škrekovski [55] that $\chi_o(G)$ cannot be upper bounded by any function of $\chi(G)$ because $\chi(S(K_n)) = 2$ and $\chi_o(S(K_n)) = n$, where $S(K_n)$ is a graph obtained from the complete graph K_n by subdividing each edge once. More strongly, we prove the following.

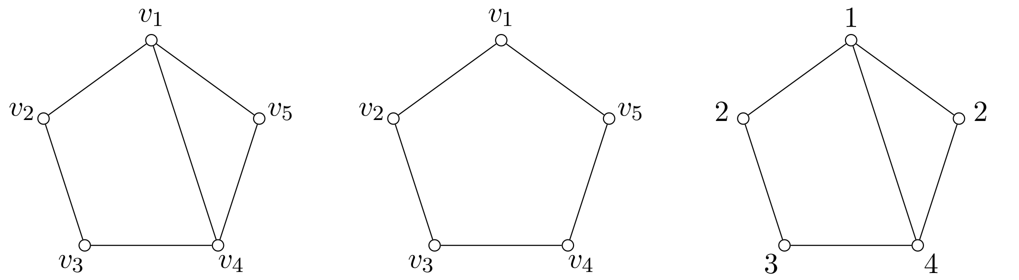
Theorem 1.5. Let k and l be any positive integers with $3 \leq k < l$.

(1) If k is odd, then there is a graph G such that $\chi(G) = k$ and $\chi_o(G) \geq l$.

(2) If k is even, then there is a graph G such that $\chi_o(G) = k$ and $\chi_{\mathrm{pcf}}(G) \geq l$.

It is easy to see that $\chi_{\mathrm{pcf}}(K_{m,m}) = 4$ and $\chi_2(K_{m,m}) = 2m$ for every positive integer m, where $K_{m,m}$ is the complete bipartite graph with two partite sets of m vertices. As a result, for any two invariants in the equation in Proposition 1.4, the right one cannot be upper bounded by any function of the left one. We give a proof of Theorem 1.5 in Chapter 4.

It is known that the odd chromatic number does not satisfy the monotonicity with respect to subgraph relations, that is, there are graphs G and H such that H is a subgraph of G and $\chi_o(G) < \chi_o(H)$. Indeed, let H be a graph isomorphic to the 5-cycle C_5 with vertices v_0, v_1, v_2, v_3, v_4 appearing in this order along the cycle, and let $G = H + v_1v_4$ (Figures 1.1a and 1.1b). Although $\chi_o(C_5) = 5$, G admits an odd 4-coloring as in Figure 1.1c. Thus, H is a subgraph of G, and we have $\chi_o(H) = 5 > 4 \geq \chi_o(G)$. By the same example, it follows that the proper conflict-free chromatic number does not satisfy the monotonicity with respect to subgraph relations either. Most of known variations of colorings including



(a) A graph G (b) A graph H (c) An odd 4-coloring of G

Figure 1.1: Graphs G and H such that H is a subgraph of G and $\chi_o(G) < \chi_o(H)$.

square colorings satisfy the monotonicity, and hence, this is a peculiar property of odd coloring and proper conflict-free coloring.

Based on this background, we investigate the following problems on odd coloring and proper conflict-free coloring; odd coloring of outerplanar graphs and related classes of graphs; and a Brooks-type theorem and degree-choosability for proper conflict-free coloring.

1.1.3 Odd coloring of outerplanar graphs and related classes of graphs

Studies on odd coloring and proper conflict-free coloring of planar graphs start with two conjectures. Petruševski and Škrekovski conjectured in [55] that every planar graph is odd 5-colorable, and Fabrici, Lužar, Rindošová, and Soták conjectured in [29] that every planar graph is proper conflict-free 6-colorable. Though the Four-Color Theorem states that 4 colors are enough to give a proper coloring of a planar graph, both bounds in the conjectures are best possible.

To our knowledge, the smallest known upper bounds of the odd chromatic numbers and the proper conflict-free chromatic numbers of planar graphs are both equal to 8 by a result in Fabrici, Lužar, Rindošová, and Soták [29].

When we restrict the class to outerplanar graphs, it was shown in Caro, Petruševski, and Škrekovski [12] that the upper bound of odd chromatic number can be reduced to 5, and thus the conjecture on odd coloring of planar graphs holds for outerplanar graphs. The bounds 5 for the odd chromatic numbers of outerplanar graphs are tight since the 5-cycle C_5 is outerplanar and $\chi_o(C_5) = 5$. On the other hand, when we consider maximal outerplanar graphs, the upper bound of odd chromatic numbers can be reduced to 4 by a result in Kashima, Maezawa, Osako, Ozeki, and Tsuchiya [39].

A natural question is to characterize odd 4-colorable outerplanar graphs. It was shown in Caro, Petruševski, and Škrekovski [12] that a graph whose every block is isomorphic to the 5-cycle is not odd 4-colorable. We prove that the other outerplanar graphs are odd 4-colorable.

Theorem 1.6 ([42])**.** An outerplanar graph G is odd 4-colorable if and only if G contains a block which is not isomorphic to C_5.

Thus, odd 4-colorable outerplanar graphs are completely characterized by Theorem 1.6. We give a proof of Theorem 1.6 in Chapter 5.

Another class of graphs related to maximal outerplanar graphs is the class of 2-trees. For a positive integer k, a graph G is a *k-tree* if there is a sequence $G_0, G_1, \dots, G_s$ of subgraphs of G such that

- G_0 is the complete graph of order $k+1$, and $G_s = G$,
- for every $i \in [s]$, there is a vertex $v_i \in V(G_i)$ such that $V(G_i) = V(G_{i-1}) \cup \{v_i\}$ and $N_{G_i}[v_i]$ is a clique of order $k+1$ of G_i.

It is easy to see that every maximal outerplanar graph is a 2-tree.

A theorem in Cranston, Lafferty, and Song [19] implies an upper bound $\chi_o(G) \leq 2k+1$ for every k-tree G. We improve this upper bound for general k as follows.

Theorem 1.7 ([40])**.** For every positive integer k, every k-tree is odd $(k + 2\lfloor \log_2 k \rfloor + 3)$-colorable.

Furthermore, for the cases $k = 2$ and $k = 3$, we prove the following tight upper bounds of odd chromatic numbers.

Theorem 1.8 ([40])**.** Every 2-tree is odd 4-colorable.

Theorem 1.9 ([40])**.** Every 3-tree is odd 5-colorable.

Note that Theorem 1.8 implies that every maximal outerplanar graph is odd 4-colorable. We give proofs of Theorems 1.7, 1.8 and 1.9 in Chapter 6.

1.1.4 Brooks-type theorems and degree-choosability for proper conflict-free coloring

As we mentioned before, for the proper coloring, there is a trivial bound $\chi(G) \leq \Delta(G)+1$, and Brooks' theorem characterizes graphs which attain the equality. The same bound does not work for proper conflict-free coloring in general due to cycles. However, as a Brooks-type statement, Caro, Petruševski, and Škrekovski conjecutured in [13] that $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 1$ holds for every connected graph G with maximum degree $\Delta(G) \geq 3$. The assumption $\Delta(G) \geq 3$ is necessary to exclude cycles. The conjecture is widely open except for the case $\Delta(G) = 3$, which was shown in Liu and Yu [48].

A characterization of degree-choosable graphs is a natural extension of Brooks' theorem. Motivated by this classical concept, we introduce a concept of *proper conflict-free* (degree$+k$)*-choosability* as an analogy of degree-choosability for proper conflict-free coloring. For a non-negative integer k, we say that a graph G is proper conflict-free (degree$+k$)-choosable if for any list assignment L of G with $|L(v)| \geq d_G(v) + k$ for every vertex v of G, G admits a proper conflict-free coloring φ such that $\varphi(v) \in L(v)$ for every vertex v. By the definition, if a graph G is proper conflict-free (degree $+\, k$)-choosable for some integer k, then we have $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + k$.

While it was conjectured in Caro, Petruševski, and Škrekovski [13] that a connected graph G except for cycles is proper conflict-free $(\Delta(G)+1)$-colorable, there are infinitely many graphs that are not proper conflict-free (degree + 1)-choosable. In fact, we prove the following.

Theorem 1.10 ([41])**.** For every connected graph H, there is a connected graph G such that H is an induced subgraph of G and G is not proper conflict-free (degree+1)-choosable.

As for proper conflict-free (degree+2)-choosability, C_5 is not (degree+2)-choosable, but we know no other graphs which are not proper conflict-free (degree + 2)-choosable. Thus, we believe that every graph other than C_5 is proper conflict-free (degree+2)-choosable. We show the following theorems on some classes of proper conflict-free (degree + 2)-choosable graphs. For a graph G, let $S(G)$ be a graph obtained from G by subdividing each edge once.

Theorem 1.11 ([41])**.** Every graph with maximum degree at most 3 other than C_5 is proper conflict-free (degree + 2)-choosable.

Theorem 1.12 ([41])**.** For every graph G, $S(G)$ is proper conflict-free (degree + 2)-choosable.

Theorem 1.13 ([41])**.** Every 2-connected outerplanar graph other than C_5 is proper conflict-free (degree + 2)-choosable.

Furthermore, for the class of graphs with maximum degree at most 4, we show proper conflict-free (degree + 3)-choosability without any exceptions.

Theorem 1.14 ([41])**.** Every graph with maximum degree at most 4 is proper conflict-free (degree + 3)-choosable.

We give proofs of Theorems 1.10, 1.11, 1.12, 1.13 and 1.14 in Chapter 7.

1.2 Degree conditions for some vertex partition problems

In the rest of this chapter, we outline the second part, which deals with some problems on vertex partitions into 2-connected subgraphs and 2-factors. In contrast to coloring problems, such partitions do not always exist, and roughly, such partitions are more likely to exist when a given graph has many edges.

The most basic assumption for graphs to have many edges is minimum degree conditions. Indeed, if $\delta(G) \geq a$, then it follows that G has at least $\frac{a}{2}|V(G)|$ edges, where $\delta(G)$ is the *minimum degree* of G. While there are many results on minimum degree conditions for the existence of certain structures or partitions, for some problems, the minimum degree condition is too strong in some sense. One such example is the relationship between two well-known results on hamilton cycles of graphs by Dirac [23] and Ore [52], where a hamilton cycle is a cycle that passes through all vertices of a graph. For a graph G, the *minimum degree sum* $\sigma_2(G)$ is defined as the minimum value of the sum of degrees for every non-adjacent pair of vertices. Ore proves that every graph G with $\sigma_2(G) \geq |V(G)|$ has a hamilton cycle, which implies a result by Dirac to show that every graph with $\delta(G) \geq \frac{1}{2}|V(G)|$ has a hamilton cycle. Motivated by these results, minimum degree sum conditions are studied in many problems in graph theory.

Recently, another degree condition named minimum degree product conditions was introduced in Furuya and Tsuchiya [32]. For a graph G, the *minimum degree product* $\pi_2(G)$ is defined as the minimum value of the product of degrees for every non-adjacent pair of vertices. They investigated a problem of finding a spanning tree without vertices of degree 2, and showed that a minimum degree product condition works well in the problem.

In this thesis, we introduced another new degree condition named *minimum degree sum on large independent sets*, and show that the condition works well in some vertex partition problems.

1.2.1 Minimum degree sum on large independent sets

For a graph G, let $\mathcal{I}(G)$ be the family of independent sets of G. For each independent set $I \in \mathcal{I}(G)$, let $\delta_G(I)$ be the minimum degree of vertices contained in I, and let $w_G(I)$ be the sum of degrees of vertices contained in I. For a graph G, $\sigma_0^*(G)$ is defined as the minimum value of $w_G(I)$ for every independent set I of G satisfying the condition $|I| \geq \delta_G(I)$. Similarly, $\sigma_1^*(G)$ is defined as the minimum value of $w_G(I)$ for every independent set I of G satisfying the condition $|I| \geq \delta_G(I)+1$. In particular, for $t = 0, 1$, we define $\sigma_t^*(G) = \infty$ when no independent set I of G satisfies $|I| \geq \delta_G(I) + t$.

When a graph G has the minimum degree at least a, then it follows that $\sigma_0^*(G) \geq a^2$ and $\sigma_1^*(G) \geq a(a+1)$ (possibly $\sigma_0^*(G) = \infty$ or $\sigma_1^*(G) = \infty$). Thus, these two new invariants can be regarded as some kinds of degree conditions. We see more relationship between σ_0^*, σ_1^* and other degree conditions in Chapter 8.

We see some fundamental properties of σ_0^* and σ_1^*. Although there is a polynomial time algorithm to calculate $\sigma_2(G)$ and $\pi_2(G)$ of a given graph G, it is NP-complete to decide whether $\sigma_t^*(G) < \infty$ or not for each $t = 0, 1$. Formally, for $t = 0, 1$, let Sigma Star-t be a decision problem that asks whether an input graph G satisfies $\sigma_t^*(G) < \infty$ or not. We prove the following.

Theorem 1.15 ([38])**.** Both Sigma Star-0 and Sigma Star-1 are NP-complete.

One classical result on extremal graph theory is Turán's theorem in [60], which determined the maximum number of edges of a graph with n vertices and without large cliques. By using Turán's theorem, we show the following almost tight lower bound of the number of edges of a graph G with $\sigma_1^*(G) = \infty$.

Theorem 1.16 ([38])**.** Let G be a graph of order n. If $\sigma_1^*(G) = \infty$, then G has at least $\frac{1}{2}n\left\lfloor\sqrt{n+1}\right\rfloor$ edges.

By the definitions of $\sigma_0^*(G)$ and $\sigma_1^*(G)$, if $\sigma_0^*(G) = \infty$, then we have $\sigma_1^*(G) = \infty$ and it follows from Theorem 1.16 that G has at least $\frac{1}{2}n\left\lfloor\sqrt{n+1}\right\rfloor$ edges. We give proofs of Theorems 1.15 and 1.16 in Chapter 8.

1.2.2 Vertex partitions into 2-connected subgraphs

For an integer $k \geq 2$ and a graph G, a *k-proper partition* of G is a partition $\mathcal{P}$ of the vertex set of G such that every part $P \in \mathcal{P}$ induces a k-connected subgraph of G. This concept was introduced in Ferrara, Magnant, and Wenger [30]. They showed that every graph G with minimum degree at least $2k\sqrt{|V(G)|}$ has a k-proper partition with a bounded number of parts. Later, in [10], Borozan, Ferrara, Fujita, Furuya, Manoussakis, Narayanan, and Stolee improved both the minimum degree condition and the upper bound of the number of parts. In particular, for the case $k = 2$, they showed a tight minimum degree condition for graphs to have a 2-proper partition. Recently, their result on 2-proper partition was extended to a minimum degree sum condition in Chen, Guo, and Yang [15].

In this thesis, we show the following two theorems which give conditions on the minimum degree sum on large independent sets for a graph to have a 2-proper partition. Let $\alpha^*(G)$ be the maximum order of an independent set I of G such that $w_G(I) < |V(G)|$. The definitions of $\{K_2, F_5\} \cup \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{H}_n$ in Theorem 1.18 are given in Chapter 9.

Theorem 1.17 ([31])**.** *Let G be a graph of order n. If $\sigma_0^*(G) \geq n$, then G contains a 2-proper partition with at most $\alpha^*(G)$ parts.*

Theorem 1.18 ([31])**.** *Let G be a graph of order n. If $\sigma_1^*(G) \geq n$, then either G contains a 2-proper partition with at most $\alpha^*(G)$ parts, or G is isomorphic to a graph in $\{K_2, F_5\} \cup \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{H}_n$.*

The lower bound of the minimum degree and the upper bound of the number of parts are tight in both Theorems 1.17 and 1.18. We can show that Theorem 1.18 implies two known results on 2-proper partition in Borozan et al. [10] and in Chen, Guo, and Yang [15]. We give proofs of Theorems 1.17 and 1.18 in Chapter 9.

1.2.3 2-factors

A 2-factor of a graph G is a 2-regular spanning subgraph of G. Obviously, a 2-factor is a vertex-disjoint union of cycles that covers all vertices, and a hamilton cycle is a 2-factor with exactly one component. Motivated by problems on the hamiltonicity of graphs, sufficient conditions for a graph to have a 2-factor have been studied actively. For general graphs, it follows from Dirac's theorem in [23] and Ore's theorem in [52] respectively that every graph G with $\delta(G) \geq |V(G)|/2$ has a 2-factor and that every graph G with $\sigma_2(G) \geq |V(G)|$ has a 2-factor. While these assumptions ensure the existence of a hamilton cycle, the bounds cannot be improved even for the existence of a 2-factor. Indeed, the complete bipartite graph $K_{n,n+1}$ satisfies $\delta(K_{n,n+1}) = n = (|V(K_{n,n+1})|-1)/2$, $\sigma_2(K_{n,n+1}) = 2n = |V(K_{n,n+1})|-1$, and $K_{n,n+1}$ does not have a 2-factor. Thus, in a sense, minimum degree conditions and minimum degree sum conditions cannot describe the gap between hamilton cycles and 2-factors in general.

Another aspect of a 2-factor is a vertex partition into hamiltonian subgraphs. As every hamiltonian graph is 2-connected, it follows immediately that a 2-factor of a graph induces a 2-proper partition of the graph. Considering this relationship and Theorems 1.17

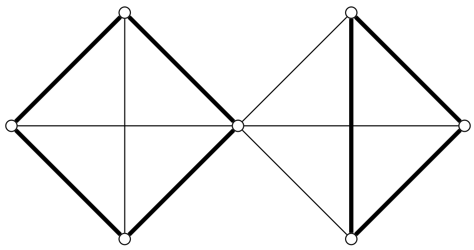


Figure 1.2: An example of non-hamiltonian graph G with $\sigma_0^*(G) = \infty$. Thick edges represent a 2-factor of G.

and 1.18, we prove the following two theorems which give conditions on the minimum degree sum on large independent sets for a graph to have a 2-factor. The definition of $\mathcal{G}$ in Theorem 1.20 is given in Chapter 10.

Theorem 1.19 ([38])**.** Let G be a graph. If $\sigma_0^*(G) = \infty$, then G has a 2-factor.

Theorem 1.20 ([38])**.** Let G be a graph. If $\sigma_1^*(G) = \infty$, then either G has a 2-factor, or G is isomorphic to a graph in $\mathcal{G}$.

Note that there are infinitely many non-hamiltonian graphs which satisfy the assumptions of Theorems 1.19 and 1.20 (see Figure 1.2).

As we mentioned in the beginning of this chapter, our interest is the minimum number of parts in a partition, that is, the minimum number of components of a 2-factor. As an analogy of Theorems 1.17 and 1.18, we believe that every graph G with $\sigma_0^*(G) = \infty$ has a 2-factor with at most $\alpha^*(G)$ components. For example, while the graph G in Figure 1.2 is not hamiltonian, we have $\alpha^*(G) = 2$ and G has a 2-factor with 2 components. To support the statement, we prove the following. For a graph G, $\sigma_k(G)$ is the minimum value of the degree sum on independent sets of order k of G. For a graph G, let $\omega(G)$ denote the number of components of G.

Theorem 1.21 ([38])**.** Let $k \geq 2$ be an integer and let G be a graph of order n. Suppose that $\sigma_0^*(G) = \infty$ and $\sigma_k(G) \geq n$. If there is a cut set $S \subseteq V(G)$ of G such that $\omega(G-S) \geq |S| + k - 2$, then G has a 2-factor with exactly $k-1$ components.

This together with a result in Bauer, Veldman, Morgana, and Schmeichel [6] implies the following, which is a partial result of the above statement.

Corollary 1.22 ([38])**.** Let G be a graph. If $\sigma_0^*(G) = \infty$ and $\alpha^*(G) \leq 2$, then G has a 2-factor with at most $\alpha^*(G)$ components.

We give proofs of Theorems 1.19, 1.20, 1.21 and Corollary 1.22 in Chapter 10.

Chapter 2

Fundamentals

In this chapter, we define some basic notations and terminology in graph theory which are used in the following chapters. Let $\mathbb{N}$ be the set of positive integers and for each $k \in \mathbb{N}$, let $[k]$ be the set of integers $\{1, 2, \ldots, k\}$ and let $[0, k]$ be the set of integers $\{0, 1, \ldots, k\}$.

2.1 Graphs

A *graph* G is a pair of a vertex set $V(G)$ and an edge set $E(G)$, where each element of $E(G)$ maps to a unordered pair of two vertices (possibly the same vertex). An element of $V(G)$ is called a *vertex* of G, and an element of $E(G)$ is called an *edge* of G. For vertices u and v of G and an edge $e \in E(G)$, if e maps to a pair of u and v, then we often write $e = uv$ or $e = vu$ and say e *joins* u and v. An edge $e = uv$ with $u = v$ is called a *loop*, and two edges e and e' which map to the same pair of vertices are called *multiple edges*. A graph is *simple* if there are no loops and no multiple edges. If both $V(G)$ and $E(G)$ are finite sets, then we say that the graph G is *finite*. In this thesis, we only consider simple and finite graphs. Figure 2.1 shows an example of a simple and finite graph.

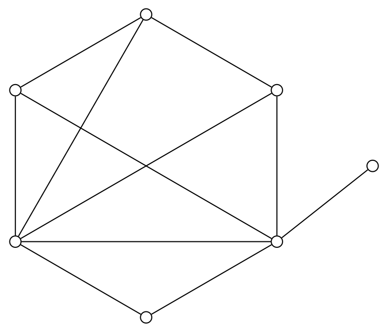


Figure 2.1: A simple and finite graph.

2.2 Subgraphs and graph isomorphism

Let G be a graph. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and we write $H \subseteq G$. In particular, when H is a subgraph of G and is not G itself, we say H is a *proper subgraph* of G. A subgraph H of G with $V(H) = V(G)$ is called a *spanning subgraph* of G. For a set of vertices $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G defined by $V(G[S]) = S$ and $E(G[S]) = E(G) \cap \{uv \mid u, v \in S\}$. We call $G[S]$ the subgraph of G *induced by* S. We define $G - S = G[V(G) \setminus S]$. For a vertex u of G we write $G - u$ instead of $G - \{u\}$. For a set of edges $F \subseteq E(G)$, the subgraph induced by F is a subgraph H of G defined by $V(H) = \{v \in V(G) \mid v$ is incident with an edge in $F\}$ and $E(H) = F$. For an edge e of G, let $G - e$ denote the subgraph of G defined by $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

For two graphs G and H, G is *isomorphic to* H if there is a bijection $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ if and only if $uv \in E(G)$. If a graph G is isomorphic to a graph H, then we write $G \simeq H$.

2.3 Union and join of graphs

Let G and H be graphs. The *union* of G and H, denoted by $G \cup H$ is a graph defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. In particular, we say that $G \cup H$ is the *disjoint union* of G and H when $V(G) \cap V(H) = \emptyset$. Similarly, for graphs $G_1, G_2, \ldots, G_s$ which is pairwise vertex-disjoint, the union of $G_1, G_2, \ldots, G_s$ is a graph G' defined by $V(G') = \bigcup_{i=1}^{s} V(G_i)$ and $E(G') = \bigcup_{i=1}^{s} E(G_i)$. When $V(G) \cap V(H) = \emptyset$, the *join* of G and H, denoted by $G + H$, is a graph defined by $V(G+H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

2.4 Neighborhoods, degrees and independent set

Let G be a graph. For vertices u and v, if there is an edge $e = uv \in E(G)$, then we say that u is *adjacent to* v, and vice versa. We also say that the vertices u and v are adjacent. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of all vertices that are adjacent to v. Each vertex in $N_G(v)$ is a *neighbor* of v. More generally, for a set of vertices $S \subseteq V(G)$, let $N_G(S)$ denote the set of vertices in $V(G) \setminus S$ that have a neighbor in S. We define $N_G[v] := N_G(v) \cup \{v\}$. For a subgraph H of G and a vertex v of G (not necessarily $v \in V(H)$), let $N_H(v) := N_G(v) \cap V(H)$. For two disjoint sets of vertices $S \subseteq V(G)$ and $T \subseteq V(G)$, let $E(S,T) = \{uv \in E(G) \mid u \in S, v \in T\}$, and let $e(S,T) = |E(S,T)|$. In particular, for a vertex v of G, we often write $E(v,T)$ and $e(v,T)$, respectively, for $E(\{v\},T)$ and $e(\{v\},T)$.

The degree of a vertex v, denoted by $d_G(v)$, is the number of vertices that are adjacent to v, and hence $d_G(v) = |N_G(v)|$. Let $\delta(G) := \min\{d_G(v) \mid v \in V(G)\}$ be the *minimum* *degree* of G, let $\Delta(G) := \max\{d_G(v) \mid v \in V(G)\}$ be the *maximum degree* of G, and let

$$d(G) := \frac{1}{|V(G)|} \sum_{v \in V(G)} d_G(v)$$

be the *average degree* of G. Note that $\delta(G) \leq d(G) \leq \Delta(G)$ for every graph G according to the definitions. It follows immediately from the Handshaking Lemma that $d(G) = \frac{2|E(G)|}{|V(G)|}$. For a positive integer r, G is called *r-regular* if $d_G(v) = r$ for every vertex v of G.

A set of vertices $I \subseteq V(G)$ is an *independent set* of G if no pair of vertices of I are adjacent. In particular, a set of a single vertex is always an independent set of G. The maximum cardinality among all independent sets of G is called the *independence number* of G, denoted by $\alpha(G)$. If $\alpha(G) \geq k$ for some positive integer k, then let

$$\sigma_k(G) := \min \left\{ \sum_{v \in I} d_G(v) \;\middle|\; I \text{ is an independent set of } G, |I| = k \right\},$$

and let $\sigma_k(G) = \infty$ otherwise. Note that $\sigma_1(G) = \delta(G)$ and $\sigma_k(G) \geq k\delta(G)$ for every positive integer k. Similarly, we let

$$\pi_2(G) := \min \{d_G(u) \cdot d_G(v) \mid \{u, v\} \text{ is an independent set of } G\}$$

if $\alpha(G) \geq 2$, and let $\pi_2(G) = \infty$ otherwise. By the equation of geometric and arithmetic means, the following is true.

Proposition 2.1. Let G be a graph with $\alpha(G) \geq 2$, and let a be a positive real. If $\delta(G) \geq a$, then $\pi_2(G) \geq a^2$. If $\pi_2(G) \geq a^2$, then $\sigma_2(G) \geq 2a$.

Proof. The former statement is trivial from the definitions. Suppose that $\pi_2(G) \geq a^2$. Let u and v be two non-adjacent vertices of G. As $d_G(u) \cdot d_G(v) \geq \pi_2(G) \geq a^2$, by the equation of geometric and arithmetic means, we have

$$d_G(u) + d_G(v) \geq 2\sqrt{d_G(u) \cdot d_G(v)} \geq 2\sqrt{a^2} = 2a.$$

Hence $\sigma_2(G) \geq 2a$. $\square$

2.5 Colorings and list colorings

Let G be a graph. A mapping φ from $V(G)$ to $\mathbb{N}$ is a *proper coloring* of G if $\varphi(u) \neq \varphi(v)$ for every edge uv of G. In this thesis, we always assume a coloring is proper, and thus we just say that such φ is a coloring of G. A coloring φ of G that uses only k colors is called a *k-coloring* of G. Without loss of generality, in a k-coloring of G, we may assume that each vertex maps to an integer in $[k]$. The *chromatic number* of G, denoted by $\chi(G)$, is the least integer k such that G admits a k-coloring.

The concept of list coloring was introduced independently by Vizing [62] and by Erdős, Rubin, and Taylor [27]. A *list assignment* of G is a mapping L from $V(G)$ to the power

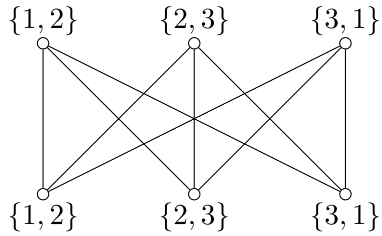


Figure 2.2: A 2-colorable graph G and its list assignment L such that G is not L-colorable.

set of $\mathbb{N}$. For a list assignment L of G, an *L-coloring* of G is a coloring φ of G such that $\varphi(v) \in L(v)$ for every vertex v of G. For a mapping f from $V(G)$ to positive integers, G is *f-choosable* if G is L-colorable for any list assignment L of G such that $|L(v)| \geq f(v)$ for every vertex v of G. In particular, if G is f-choosable for the constant map f to a positive integer k, then we say that G is *k-choosable*. The *choice number* of G, denoted by $\mathrm{ch}(G)$, is the least integer k such that G is k-choosable. Since a k-coloring of G is exactly an L-coloring of G, where L is a list assignment of G defined by $L(v) = [k]$ for every vertex v of G, it follows that $\chi(G) \leq \mathrm{ch}(G)$ for every graph G. On the other hand, there is a 2-colorable graph which is not 2-choosable (Figure 2.2).

For an integer k, we say that G is *k-degenerate* if every subgraph of G contains a vertex of degree at most k. The following relation between the degeneracy and colorings is well known.

Proposition 2.2. If a graph G is k-degenerate for some positive integer k, then G is $(k+1)$-choosable. In particular, G is $(k+1)$-colorable.

Proof. Suppose that G is a k-degenerate graph for some positive integer k. Let L be a list assignment of G such that $|L(v)| \geq k+1$ for every vertex v of G. We shall prove that every subgraph H of G is L-colorable by induction on the order of H. When $|V(H)| = 1$, then obviously H is L-colorable. Suppose that $|V(H)| \geq 2$ and that every subgraph of G of smaller order is L-colorable. Since G is k-degenerate, there is a vertex v of H such that $d_H(v) \leq k$. By the induction hypothesis, $H - v$ admits a proper L-coloring φ. By letting $\varphi(v) \in L(v) \setminus \{\varphi(u) \mid u \in N_H(v)\}$, we can extend φ to a proper L-coloring of H. $\square$

By the definition, we know that every graph G is $\Delta(G)$-degenerate, and hence the following proposition immediately follows from Proposition 2.2.

Proposition 2.3. For every graph G, $\chi(G) \leq \mathrm{ch}(G) \leq \Delta(G) + 1$.

2.6 Paths and cycles

A *path* is a graph P defined by $V(P) = \{v_0, v_1, \ldots, v_l\}$ and $E(P) = \{v_{i-1}v_i \mid i \in [l]\}$. We often write a path as a sequence of vertices such as $P = v_0v_1 \cdots v_l$. The vertices v_0 and v_l are called the *end vertices* of a path $P = v_0v_1 \cdots v_l$. The number of edges contained in a path is the *length* of the path. A path which is a subgraph of a graph G is called a path of G. For two distinct vertices u and v of a graph G, a path of G with end vertices u and v is called a *uv-path* (a *vu-path*) of G.

A *cycle* is a graph C defined by $V(C) = \{v_0, v_1, \dots, v_{l-1}\}$ and $E(C) = \{v_{i-1}v_i \mid i \in [l]\}$ where the indices are on modulo l. We often write a cycle as a sequence of vertices such as $C = v_0v_1 \cdots v_{l-1}v_0$. The number of edges contained in a cycle is the *length* of the cycle. A cycle which is a subgraph of a graph G is called a cycle of G. For a cycle C of G, a *chord* of C is an edge $uv \in E(G)$ such that $u, v \in V(C)$ and $uv \notin E(C)$.

2.7 Connectivity

Let G be a graph. We say G is *connected* if for every pair of distinct vertices u and v of G, there is a uv-path of G. For a connected graph G, the *distance* between two vertices u and v is the minimum length of all uv-paths of G. Every maximal connected subgraph of G is called a *component* of G, and let $\omega(G)$ denote the number of components of G. For a connected graph G, a set of vertices $S \subseteq V(G)$ is a *cut set* of G if $G - S$ is not connected. In particular, if $\{v\} \subseteq V(G)$ is a cut set of a connected graph G, then the vertex v is called a *cut vertex* of G. For a positive integer k, we say that G is *k-connected* if $|V(G)| \geq k+1$ and $G - S$ is connected for every set of vertices $S \subseteq V(G)$ with $|S| < k$. The *connectivity* of G, denoted by $\kappa(G)$, is the least integer k such that G is k-connected.

Every maximal subgraph of G without cut vertices is a *block* of G. By the definition, if G does not have an isolated vertex, then every block of G is either 2-connected, or isomorphic to a path of length 1.

2.8 Planar graphs and outerplanar graphs

Let G be a graph. A drawing of a graph G in the plane is an injection f from $V(G)$ to $\mathbb{R}^2$ together with continuous maps from edges to $\mathbb{R}^2$ such that each edge $e = uv \in E(G)$ maps to a curve joining two ends $f(u)$ and $f(v)$. A graph G is called a *planar graph* if there is a drawing of G in the plane without any crossing of edges. In particular, a planar graph that can be drawn in the plane so that all vertices lie in the boundary of the outer region is called an *outerplanar graph*. A graph G is called a *maximal outerplanar graph* if G is an outerplanar graph and $G \cup uv$ is not an outerplanar graph for any non-adjacent pair of vertices u and v of G.

2.9 Some special classes of graphs

A graph G is *complete* if $uv \in E(G)$ for any two distinct vertices $u, v \in V(G)$. The complete graph of order n is denoted by K_n. For an integer $n \geq 3$, the n-cycle C_n is the cycle of length n. For a positive integer r, a graph G is an *r-partite graph* if there is a partition of the vertices $V(G) = \bigcup_{i=1}^{r} V_i$ such that $E(G) \subseteq \bigcup_{1 \leq i < j \leq r} \{uv \mid u \in V_i, v \in V_j\}$. In particular, a 2-partite graph is called a *bipartite graph*. Each of $V_1, V_2, \dots, V_r$ is called a *partite set* of G. An r-partite graph G is *complete* if every pair of vertices in distinct

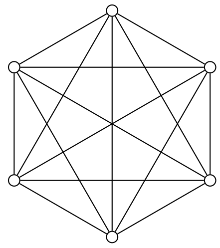


Figure 2.3: K_6

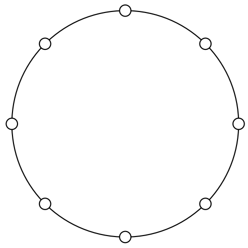


Figure 2.4: C_8

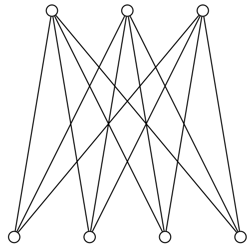


Figure 2.5: $K_{3,4}$

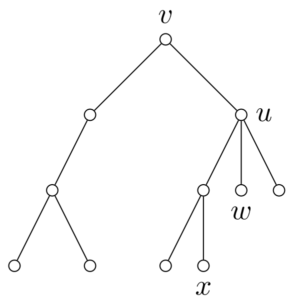


Figure 2.6: A rooted tree (T, v). The vertex w is a child of the vertex u, and the vertex x is a descendant of u. The height of (T, v) is equal to 3.

partite sets are joined by an edge of G. For two positive integers m and n, let $K_{m,n}$ denote the complete bipartite graph with two partite sets of order m and n. See Figures 2.3, 2.4, and 2.5 for examples of a complete graph, a cycle, and a complete bipartite graph.

A connected graph without cycles is called a *tree*. A vertex of a tree T with degree exactly 1 is called a *leaf* of T. It is easy to see that every tree of order at least two contains at least two leaves. It follows that a connected graph T is a tree if and only if T is 1-degenerate. A graph without cycles is called a *forest*. Thus, every component of a forest is a tree.

For a tree T and a vertex v of T, we call a pair (T, v) a *rooted tree* with the *root* v. For a rooted tree (T, v) and a vertex $u \in V(T) \setminus \{v\}$, there is a unique uv-path of T. The *parent* of u is the unique neighbor of u on the uv-path. For a vertex u of a rooted tree (T, v) (possibly $u = v$), a vertex w of T is a *descendant* of u if the wv-path of T contains u and $w \neq u$. In particular, a descendant of a vertex u which is adjacent to u is called a *child* of u. The *height* of a rooted tree (T, v) is the maximum length of a path with an end vertex v.

Part I

Colorings of graphs

Chapter 3

Online list coloring of complete bipartite graphs

The online list coloring was introduced independently by Schauz [57] and Zhu [67] as a generalized concept of the list coloring. While the choice number and the paint number (also called online list chromatic number) of a graph can be far apart, many techniques developed in studies of list coloring are useful even for online list coloring. In particular, for sparse graphs like planar graphs, many results on the list coloring can be reconsidered as results on the online list coloring by a few modifications in proofs. For dense graphs, the results in [14, 45] show that the paint number behaves differently from the choice number. In this chapter, we focus on online list coloring of complete bipartite graphs, which are the most basic examples of dense graphs. In Section 3.1, we introduce the online list coloring and show some basic properties of the online list coloring. We discuss online list colorings of complete bipartite graphs in Section 3.2, and in Sections 3.4 and 3.5, we give a proof of Theorem 3.8, which shows the difference between list coloring and online list coloring on complete bipartite graphs. All the new results in this chapter are based on [37].

We introduce some notation used in this chapter. For a graph G and a set of vertices $U \subseteq V(G)$, let $\mathbf{1}_U$ be a mapping from $V(G)$ to $\{0,1\}$ such that every vertex in U maps to 1 and every other vertex maps to 0. For an integer k and a mapping f from $V(G)$ to positive integers, let $f-k\mathbf{1}_U$ be a mapping from $V(G)$ to integers such that $(f-k\mathbf{1}_U)(v) = f(v)-k$ if $v \in U$ and $(f-k\mathbf{1}_U)(v) = f(v)$ otherwise. In particular, we write $f-\mathbf{1}_U$ instead of $f-1\mathbf{1}_U$. For two mappings f and f' from $V(G)$ to positive integers, we write $f \leq f'$ when $f(v) \leq f'(v)$ for every vertex v of G.

3.1 Introduction to online list coloring

For a graph G and a mapping f from $V(G)$ to (not necessarily positive) integers, the f-paintability of G is defined recursively as follows.

- When G has no edge, G is f-paintable if and only if $f(v) \geq 1$ for every vertex v of G.

- When G has at least one edge, G is f-paintable if and only if for every non-empty set of vertices $U \subseteq V(G)$, there is an independent set X of G such that $X \neq \emptyset$, $X \subseteq U$, and $G - X$ is f'-paintable where f' is the restriction of $f - \mathbf{1}_U$ to $V(G) \setminus X$.

It follows easily from the definition that if $f(v) \leq 0$ for some vertex v of G, then G is not f-paintable. If a graph G is f-paintable for the constant map f from $V(G)$ to a positive integer k, then we say that G is *k-paintable*. The *paint number* of a graph G, denoted by $\mathrm{pnt}(G)$, is the least integer k such that G is k-paintable. For a subgraph H of a graph G and a mapping f from $V(G)$ to positive integers, we say that H is f-paintable if H is f_H-paintable where f_H is the restriction of f to $V(H)$. Then the following relation between f-choosability and f-paintability holds.

Proposition 3.1. Let G be a graph, and let f be a mapping from $V(G)$ to positive integers. If G is f-paintable, then G is f-choosable. In particular, for every graph G, $\mathrm{ch}(G) \leq \mathrm{pnt}(G)$.

Proof. The proof goes by induction on the order of G. Suppose that G is f-paintable for some mapping f from $V(G)$ to positive integers. If G consists of a single vertex v, then obviously G is f-choosable since $f(v) \geq 1$. Thus, we assume that $|V(G)| = n \geq 2$ and the statement holds for every graph of order less than n and for every mapping on the graph. Let L be an arbitrary list assignment of G such that $|L(v)| \geq f(v)$ for every vertex v of G. Without loss of generality, we may assume that $1 \in \bigcup_{v \in V(G)} L(v)$, and let $U = \{v \in V(G) \mid 1 \in L(v)\}$. Since G is f-paintable, there is an independent set X of G such that $X \subseteq U$ and $G - X$ is $(f - \mathbf{1}_U)$-paintable. Let L' be a list assignment of $G - X$ defined by $L'(v) = L(v)$ for each $v \in V(G) \setminus U$ and $L'(v) = L(v) \setminus \{1\}$ for each $v \in U \setminus X$. Note that $1 \notin \bigcup_{v \in V(G-X)} L'(v)$. Since $|L'(v)| \geq (f - \mathbf{1}_U)(v)$ for every vertex v of $G - X$, by the induction hypothesis, there is an L'-coloring φ' of $G - X$. A coloring φ of G defined by $\varphi(v) = \varphi'(v)$ for each $v \in V(G) \setminus X$ and $\varphi(v) = 1$ for each $v \in X$ is an L-coloring of G. Thus, G is f-choosable. The latter statement follows immediately from the former statement. $\square$

By the definition, the coloring of the online list satisfies the following monotonicity with respect to subgraph relations and the number of colors, which was observed in Zhu [67].

Proposition 3.2. Let G be a graph and f be a mapping from $V(G)$ to positive integers. If G is f-paintable, then for any subgraph H of G and any mapping f' such that $f \leq f'$, H is f'-paintable.

The following basic property of online list coloring was first shown in Zhu [67].

Proposition 3.3. Let G be a connected graph of order at least 2 and f be a mapping from $V(G)$ to positive integers. Suppose that $f(v) \geq d_G(v) + 1$ for a vertex v of G. Then G is f-paintable if and only if $G - v$ is f-paintable.

Proof. The "only if" part is trivial by Proposition 3.2. We only show "if" part. The proof goes by induction on the number of vertices. Suppose first that G has only two vertices

u and v, and a mapping f satisfies $f(u) \geq 1$ and $f(v) \geq d_G(v) + 1 = 2$. Obviously $G - v$ is f-paintable, and hence it suffices to show that G is f-paintable. Let U be a subset of $\{u, v\}$. If $u \in U$, then let $X = \{u\}$. As $(f - \mathbf{1}_U)(v) \geq f(v) - 1 \geq 1$, $G - X$ is $(f - \mathbf{1}_U)$-paintable. Otherwise, we have $U = \{v\}$, so let $X = \{v\}$. Then $(f - \mathbf{1}_U)(u) = f(u) \geq 1$, and thus $G - X$ is $(f - \mathbf{1}_U)$-paintable. Combining these two cases, we conclude that G is f-paintable.

Hence we assume that G has $n \geq 3$ vertices, f is a mapping from $V(G)$ to positive integers such that $f(v) \geq d_G(v) + 1$ for a vertex v of G, and suppose that the statement holds for any graph of order less than n. Suppose that $G - v$ is f-paintable. We show that G is f-paintable. Let U be an arbitrarily non-empty set of vertices of G, and let $U' = U \setminus \{v\}$. Since $G - v$ is f-paintable, there is an independent set X' of $G - v$ such that $X' \subseteq U'$, $X' \neq \emptyset$, and $G - (\{v\} \cup X')$ is $(f - \mathbf{1}_{U'})$-paintable. We consider the following three cases.

Case 1. $v \notin U$, and thus $U' = U$.

Let $X_1 = X'$ and $G_1 = G - X_1$. Since $G_1 - v = G - (\{v\} \cup X')$ is $(f - \mathbf{1}_U)$-paintable and $(f - \mathbf{1}_U)(v) = f(v) \geq d_G(v) + 1 \geq d_{G_1}(v) + 1$, by the induction hypothesis, G_1 is $(f - \mathbf{1}_U)$-paintable. Thus, X_1 is a desired independent set of G.

Case 2. $v \in U$ and $X' \cap N_G(v) = \emptyset$.

Let $X_2 = X \cup \{v\}$ and $G_2 = G - X_2$. Since $G_2 = G - (\{v\} \cup X')$ is $(f - \mathbf{1}_U)$-paintable and X_2 is an independent set of G such that $X_2 \subseteq U$, X_2 is a desired independent set of G.

Case 3. $v \in U$ and $X' \cap N_G(v) \neq \emptyset$.

Let $X_3 = X'$ and $G_3 = G - X_3$. Since $G_3 - v = G - (\{v\} \cup X')$ is $(f - \mathbf{1}_U)$-paintable and $(f - \mathbf{1}_U)(v) = f(v) - 1 \geq d_G(v) + 1 - 1 \geq d_{G_3}(v) + 1$, by the induction hypothesis, G_3 is $(f - \mathbf{1}_U)$-paintable. Thus, X_3 is a desired independent set of G.

Hence G is f-paintable, as desired. □

It follows from Proposition 3.3 that every k-degenerate graph is $(k+1)$-paintable, and thus $\mathrm{pnt}(G) \leq \Delta(G) + 1$ for every graph G. Furthermore, it has been shown in literature [43, 45, 58] that many theorems on list coloring hold for online list coloring.

3.2 Complete bipartite graphs

Although complete bipartite graphs are highly symmetric, the problems on the list coloring and the online list coloring of complete bipartite graphs have many open problems. For every pair of integers m and n with $1 \leq m \leq n$, since the complete bipartite graph $K_{m,n}$ is m-degenerate, we have $\mathrm{ch}(K_{m,n}) \leq \mathrm{pnt}(K_{m,n}) \leq m + 1$ by Proposition 2.2. This bound is tight since $\mathrm{ch}(K_{m,m^m}) = m + 1$, which was first shown in Erdős, Rubin, and Taylor [27]. On the other hand, when n becomes closer to m, then both $\mathrm{ch}(K_{m,n})$ and $\mathrm{pnt}(K_{m,n})$ are strictly less than $m + 1$. Indeed, we have $\log_2 m - (2 + o(1)) \log_2 \log_2 m \leq \mathrm{ch}(K_{m,m}) \leq \log_2 m - \left(\frac{1}{2} - o(1)\right) \log_2 \log_2 m$ from the results in Erdős and Hajnal [26] and in Radhakrishnan and Srinivasan[56], and we have $\mathrm{pnt}(K_{m,m}) = \log_2 m + O(1)$ from a result in Duraj, Gutowski, and Kozik [25].

One possible question is when $\mathrm{ch}(K_{m,n})$ and $\mathrm{pnt}(K_{m,n})$ drop to m as n becomes closer to m. The following classical result on list coloring is known.

Theorem 3.4 (Erdős, Rubin, and Taylor [27])**.** *Let m and n be integers such that $2 \leq m \leq n$. The complete bipartite graph $K_{m,n}$ is m-choosable if and only if $n < m^m$.*

Proof. We first show that K_{m,m^m} is not m-choosable. Let $G \simeq K_{m,m^m}$, and let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_{m^m}\}$ be partite sets of G. Let $S_1, S_2, \ldots, S_m$ be pairwise disjoint sets of positive integers of order m, and let $\mathcal{S}$ be the family of sets S such that $|S \cap S_i| = 1$ for every $i \in [m]$. Note that the order of $\mathcal{S}$ is equal to m^m. Let f be an arbitrary bijection from $[m^m]$ to $\mathcal{S}$, and define a list assignment L of G by $L(a_i) = S_i$ for each $i \in [m]$ and $L(b_j) = f(j)$ for each $j \in [m^m]$. If there is an L-coloring φ of G, there is an integer $j \in [m^m]$ such that $L(b_j) = f(j) = \{\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_m)\}$, a contradiction. Thus, G is not L-colorable.

Next, we assume that $G \simeq K_{m,n}$ for some integers m and n with $2 \leq m \leq n < m^m$, and show that G is m-choosable. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be partite sets of G, and let L be a list assignment of G such that $|L(v)| = m$ for every vertex v of G. Suppose that $L(a_i) \cap L(a_j) \neq \emptyset$ for some i and j with $1 \leq i < j \leq m$. We let $\varphi(a_i) = \varphi(a_j)$, and choose $\varphi(a) \in L(a)$ arbitrarily for each $a \in A \setminus \{a_i, a_j\}$. Since at most $m - 1$ colors appear in A, at least one color is still available for each vertex $b \in B$, thus G is L-colorable. Hence, we assume that $L(a_i) \cap L(a_j) = \emptyset$ for any i and j with $1 \leq i < j \leq m$. Let $\mathcal{S}_L$ be the family of a set S such that $|S \cap L(a_i)| = 1$ for every $i \in [m]$. Then we have $|\mathcal{S}_L| = m^m > |B|$, and thus there is a tuple $(c_1, c_2, \ldots, c_m)$ such that $\{c_1, c_2, \ldots, c_m\} \in \mathcal{S}_L \setminus \{L(b_j) \mid 1 \leq j \leq n\}$. Let $\varphi(a_i) = c_i$ for each $i \in [m]$. Then at least one color is still available for each vertex $b \in B$, and thus G is L-colorable. □

Analogously to Theorem 3.4, Carraher, Loeb, Mahoney, Puleo, Thai, and West showed the following in [14].

Theorem 3.5 (Carraher et al. [14])**.** *Let m and n be integers such that $2 \leq m \leq n$. Let G be a complete bipartite graph with partite sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, and let f be a mapping from $V(G)$ to positive integers such that $f(b_j) = m$ for every index $j \in [n]$. Then G is f-paintable if and only if $n < \prod_{i=1}^{m} f(a_i)$. In particular, the complete graph $K_{m,n}$ is m-paintable if and only if $n < m^m$.*

As a result, both $\mathrm{ch}(K_{m,n})$ and $\mathrm{pnt}(K_{m,n})$ drop to m from $m+1$ when n is smaller than m^m. The next step, that is, the threshold of n such that $\mathrm{ch}(K_{m,n})$ and $\mathrm{pnt}(K_{m,n})$ drop to $m - 1$ have been studied as well. Hoffman and Johnson Jr. determined the maximum value of n such that $\mathrm{ch}(K_{m+1,n}) \leq m$ in [36].

Theorem 3.6 (Hoffman and Johnson Jr. [36])**.** *Let m and n be integers such that $2 \leq m \leq n$. Then the complete graph $K_{m+1,n}$ is m-choosable if and only if $n < m^m - (m-1)^m$.*

In [14], Carraher et al. showed the following lower bound of the integer n such that the complete graph $K_{m+1,n}$ is not m-paintable.

Theorem 3.7 (Carraher et al. [14])**.** *Let m and n be integers such that $2 \leq m \leq n$. If $n > \frac{1}{2}(m^m + 3m^{m-1} - m - 1)$, then the complete bipartite graph $K_{m+1,n}$ is not m-paintable.*

To clarify the relationship of results, we define two functions r_{ch} and r_{pnt} on integers at $m \geq 2$. Let $r_{\text{ch}} = r_{\text{ch}}(m)$ be the function on m such that $r_{\text{ch}}(m)$ is the least integer n such that $\text{ch}(K_{m+1,n}) \geq m+1$. Similarly, let $r_{\text{ch}} = r_{\text{ch}}(m)$ be the function on m such that $r_{\text{pnt}}(m)$ is the least integer n such that $\text{pnt}(K_{m+1,n}) \geq m+1$. By the relationship between list coloring and online list coloring, it follows that $r_{\text{pnt}}(m) \leq r_{\text{ch}}(m)$ for every integer $m \geq 2$. Using these notation, Theorem 3.6 can be restated as $r_{\text{ch}}(m) = m^m - (m-1)^m$, and Theorem 3.7 can be restated as $r_{\text{pnt}}(m) \leq \frac{1}{2}(m^m + 3m^{m-1} - m - 1)$. Since $m^m - (m-1)^m \approx m^m(1 - e^{-1}) > \frac{1}{2}m^m$, Theorems 3.6 and 3.7 derive that there is a gap between $r_{\text{pnt}}(m)$ and $r_{\text{ch}}(m)$ for sufficiently large m. In this thesis, we give tighter lower and upper bounds of $r_{\text{pnt}}(m)$ as follows.

Theorem 3.8 ([37])**.** *Let $r_{\text{pnt}} = r_{\text{pnt}}(m)$ be the least integer such that $\text{pnt}(K_{m+1,r_{\text{pnt}}}) \geq m+1$. Then there are constants c and c' such that $0 < c < c'$ and $cm^{m-1} < r_{\text{pnt}} < c'm^{m-1}$.*

As a result, we have $r_{\text{pnt}}(m) = \Theta(m^{m-1})$, and thus the ratio $\frac{r_{\text{pnt}}(m)}{r_{\text{ch}}(m)}$ tends to 0 as m goes to infinity.

3.3 Lemmas

In this section, we show some lemmas used in our proof of Theorem 3.8. The following lemma was first shown in Zhu [67].

Lemma 3.9. *Let G be a graph, and let f be a mapping from $V(G)$ to positive integers. If $f(v) = 1$ for a vertex v of G, then G is f-paintable if and only if $G - v$ is $(f - \mathbf{1}_{N_G[v]})$-paintable.*

When a given graph is a complete bipartite graph, we can relax the definition of f-paintability as follows.

Lemma 3.10 ([37])**.** *Let G be a complete bipartite graph with non-empty partite sets A and B, and let f be a mapping from $V(G)$ to positive integers. Then G is f-paintable if and only if for every nonempty set of vertices $U \subseteq V(G)$ such that both $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$, either $G - (U \cap A)$ or $G - (U \cap B)$ is $(f - \mathbf{1}_U)$-paintable.*

Proof. First, we assume that G is f-paintable. Let $U \subseteq V(G)$ be a set of vertices such that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$. As G is f-paintable, there is an independent set X of G such that $G - X$ is $(f - \mathbf{1}_U)$-paintable. We know that either $X \subseteq U \cap A$ or $X \subseteq U \cap B$ holds. Without loss of generality, we assume that $X \subseteq U \cap A$. Since $G - (U \cap A)$ is a subgraph of $G - X$, by Proposition 3.2, $G - (U \cap A)$ is $(f - \mathbf{1}_U)$-paintable, as desired.

Next we suppose that for every nonempty set of vertices $U \subseteq V(G)$ such that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$, either $G - (U \cap A)$ or $G - (U \cap B)$ is $(f - \mathbf{1}_U)$-paintable. Assume to the contrary that G is not f-paintable. Let H be a minimum subgraph of G which is not f-paintable. Then H has at least 2 vertices and thus there is a set of vertices $U \subseteq V(H)$

such that neither $H-(U\cap A)$ nor $H-(U\cap B)$ is $(f-\mathbf{1}_U)$-paintable. If both $U\cap A$ and $U\cap B$ are non-empty, then by the assumption, either $G-(U\cap A)$ or $G-(U\cap B)$ is $(f-\mathbf{1}_U)$-paintable. It follows that either $H-(U\cap A)$ or $H-(U\cap B)$ is $(f-\mathbf{1}_U)$-paintable, a contradiction. Hence, without loss of generality, we may assume that $U\cap B=\emptyset$. Then $H-U$ is not $(f-\mathbf{1}_U)$-paintable, equivalently, $H-U$ is not f-paintable, a contradiction by the minimality of H. Thus, G is f-paintable, as desired. □

3.4 Lower bound in Theorem 3.8

Before we move to a proof of Theorem 3.8, we introduce some notation used in Sections 3.4 and 3.5. Let $\mathcal{G}$ be the set of pairs (G,f) of a graph and a mapping that satisfies the following conditions. The graph G is a complete bipartite graph with partite sets A and B, and suppose that B is divided into B_- and B_0. (Either B_- or B_0 can be empty.) The mapping f gives any positive integer for each $v\in A$, $|A|-1$ for each $v\in B_-$ and $|A|$ for each $v\in B_0$. Let $A=\{a_1,a_2,\dots,a_n\}$ and we regard f on A as an n-tuple of positive integers $\boldsymbol{t}=(t_1,t_2,\dots,t_n)$ by $t_i=f(a_i)$. For an n-tuple $\boldsymbol{t}$, let

$$P(\boldsymbol{t})=\prod_{i=1}^{n}t_i \qquad \text{and} \qquad S(\boldsymbol{t})=\sum_{i=1}^{n}(t_i-1).$$

We introduce two other tuples defined from U, which is a set of vertices of G. Let $\boldsymbol{t}_{\bar{U}}$ denote the $|A\setminus U|$-tuple obtained by restricting $\boldsymbol{t}$ to $A\setminus U$, and let $\boldsymbol{t}_U$ denote the n-tuple obtained from $\boldsymbol{t}$ by reducing by 1 at the vertices in $U\cap A$.

In this section, we prove the following.

Theorem 3.11 ([37])**.** Let G be a complete bipartite graph with partite sets A and B, and let $A=\{a_1,a_2,\dots,a_n\}$ where $n\ge 2$. We divide B into B_- and B_0 and let k be the order of B_-. Suppose that f is a mapping from $V(G)$ to positive integers such that

$$f(v)=\begin{cases} t_i(>0) & v=a_i\in A,\\ n-1 & v\in B_-,\\ n & v\in B_0,\end{cases}$$

and let $\boldsymbol{t}=(t_1,t_2,\dots,t_n)$. If G satisfies the following condition

$$|B_0|\le P(\boldsymbol{t})-kS(\boldsymbol{t})-2, \tag{3.4.1}$$

then G is f-paintable.

Theorem 3.11 immediately implies the following lower bound of $r_{\mathrm{pnt}}(m)$, which implies the lower bound in Theorem 3.8.

Corollary 3.12 ([37])**.** For every integer $m\ge 2$, $K_{m+1,r}$ is m-paintable where $r=\left\lfloor\frac{m^{m+1}-2}{m^2-1}\right\rfloor$. In particular, $r_{\mathrm{pnt}}(m)>\left\lfloor\frac{m^{m+1}-2}{m^2-1}\right\rfloor$ for every integer $m\ge 2$.

Proof. We fix an integer $m \geq 2$. Let $r = \left\lfloor \frac{m^{m+1}-2}{m^2-1} \right\rfloor$, and let G be a complete bipartite graph with partite sets $A = \{a_1, a_2, \ldots, a_{m+1}\}$ and $B = \{b_1, b_2, \ldots, b_r\}$. Let f be a constant map from $V(G)$ to m, and let $\boldsymbol{t}$ be an $(m+1)$-tuple defined from f on A. Then we have $P(\boldsymbol{t}) = m^{m+1}$ and $S(\boldsymbol{t}) = (m+1)(m-1) = m^2 - 1$, and thus

$$P(\boldsymbol{t}) - rS(\boldsymbol{t}) - 2 = m^{m+1} - \left\lfloor \frac{m^{m+1}-2}{m^2-1} \right\rfloor (m^2-1) - 2 \geq 0.$$

By using Theorem 3.11 with $B_- = B$ and $B_0 = \emptyset$, we conclude that G is f-paintable. $\square$

3.4.1 Proof of Theorem 3.11

The proof of Theorem 3.11 goes by induction on the number of vertices. Assume $(G, f) \in \mathcal{G}$ and (G, f) satisfies (3.4.1). Since $|B_0| \geq 0$, we have

$$P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2 \geq 0. \tag{3.4.2}$$

If $|A| = 2$, then by using Lemma 3.9 to each vertex of B_-, G is f-paintable if and only if $G - B_-$ is $(f - k\mathbf{1}_A)$-paintable. By (3.4.2), we have $t_1t_2 - k(t_1 + t_2 - 2) - 2 \geq 0$, and thus $k < \min\{t_1, t_2\}$. Then, (3.4.1) implies

$$\begin{aligned} |B_0| \leq t_1t_2 - k(t_1 + t_2 - 2) - 2 &= (t_1 - k)(t_2 - k) - 1 - (k-1)^2 \\ &\leq (f - \mathbf{1}_A)(a_1)(f - \mathbf{1}_A)(a_2) - 1, \end{aligned}$$

and hence $G - B_-$ is $(f - \mathbf{1}_A)$-paintable by Theorem 3.5. Thus, G is f-paintable.

When $t_i = 1$ for some $i \in [n]$, Lemma 3.9 implies that G is f-paintable if and only if $G - a_i$ is $(f - \mathbf{1}_B)$-paintable. In addition, (3.4.1) for (G, f) implies (3.4.1) for $(G - a_i, f - \mathbf{1}_B)$. By the induction hypothesis, $G - a_i$ is $(f - \mathbf{1}_B)$-paintable, and thus G is f-paintable.

Suppose that $|A| \geq 3$ and $t_i \geq 2$ for all $i \in [n]$. Let U be a set of vertices such that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$, and let $f' = f - \mathbf{1}_U$. For $(G - X, f')$ where $X \in \{U \cap A, U \cap B\}$, we define $A' = A \setminus X$, $B' = B \setminus X$, and $n' = |A'|$. Let $\boldsymbol{t}'$ be an n'-tuple of positive integers defined from f' on A'. Then we have $f'(v) \geq n' - 1$ for every vertex v of B'; if $X = U \cap A$, then $f'(v) \geq f(v) - 1 \geq (n-1) - 1 \geq n' - 1$ for every vertex $v \in B'$; otherwise, $f'(v) = f(v) \geq n - 1 = n' - 1$ for every vertex $v \in B'$. We define a partition $B'_- \cup B'_0 \cup B'_+$ of B' as

$$\begin{aligned} B'_- &= \{v \in B' \mid f'(v) = n' - 1\}, \\ B'_0 &= \{v \in B' \mid f'(v) = n'\}, \text{and} \\ B'_+ &= \{v \in B' \mid f'(v) > n'\}. \end{aligned}$$

Then, by using Proposition 3.3 for each vertex of B'_+, it follows that $G - X$ is f'-paintable if and only if $G - (X \cup B'_+)$ is f'-paintable.

In the rest of the proof, we shall show that either $G - (U \cap A)$ or $G - (U \cap B)$ is f'-paintable.

Case 1. $|U \cap A| \geq 3$.

Let $X = U \cap A$ and we show that $G - X$ is f'-paintable. We have $d_{G-X}(v) = n' \leq n - 3 \leq f(v) - 2 < f'(v)$ for each $v \in B'$, which implies $B'_+ = B'$. Since $G - (X \cup B'_+)$ has no edge, $G - (X \cup B'_+)$ is f'-paintable, and so is $G - X$.

Case 2. $|U \cap A| = 2$.

Without loss of generality, we may assume that $U \cap A = \{a_1, a_2\}$. Note that $P(\boldsymbol{t}_{\bar{U}}) = \prod_{i=3}^{n} t_i$, $S(\boldsymbol{t}_{\bar{U}}) = \sum_{i=3}^{n}(t_i - 1)$, $P(\boldsymbol{t}_U) = (t_1 - 1)(t_2 - 1)\prod_{i=3}^{n} t_i$ and $S(\boldsymbol{t}_U) = \sum_{i=1}^{n}(t_i - 1) - 2$ for this case. We consider the following two subcases.

Subcase 2.1. $|B_- \cap U| < P(\boldsymbol{t}_{\bar{U}})$.

Let $X = U \cap A$ and we show that $G - X$ is f'-paintable. We have $n' = n - 2$, $\boldsymbol{t}' = \boldsymbol{t}_{\bar{U}}$, $B'_- = \emptyset$, and $B'_0 = B_- \cap U$. Since $|B'_0| = |B_- \cap U| < P(\boldsymbol{t}_{\bar{U}}) = P(\boldsymbol{t}')$, $G - (X \cup B'_+)$ is f'-paintable by Theorem 3.5, and so is $G - X$.

Subcase 2.2. $|B_- \cap U| \geq P(\boldsymbol{t}_{\bar{U}})$.

Let $X = U \cap B$ and we show that $G - X$ is f'-paintable. We have $n' = n$, $\boldsymbol{t}' = \boldsymbol{t}_U$, $B'_- = B_- \setminus U$, and $B'_0 = B_0 \setminus U$. By the assumption of the subcase, $|B'_-| = |B_- \setminus U| \leq k - P(\boldsymbol{t}_{\bar{U}})$ and $|B'_0| \leq |B_0| \leq P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2$. As $P(\boldsymbol{t}_U) = (t_1 - 1)(t_2 - 1)P(\boldsymbol{t}_{\bar{U}})$ and $P(\boldsymbol{t}) = t_1 t_2 P(\boldsymbol{t}_{\bar{U}})$,

$$\begin{aligned}
P(\boldsymbol{t}') - |B'_-|S(\boldsymbol{t}') - 2 &\geq (t_1 - 1)(t_2 - 1)P(\boldsymbol{t}_{\bar{U}}) - (k - P(\boldsymbol{t}_{\bar{U}}))\,(S(\boldsymbol{t}) - 2) - 2 \\
&= t_1 t_2 P(\boldsymbol{t}_{\bar{U}}) - kS(\boldsymbol{t}) - 2 + (1 - t_1 - t_2 + S(\boldsymbol{t}) - 2)\,P(\boldsymbol{t}_{\bar{U}}) + 2k \\
&\geq P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2 + (1 - t_1 - t_2 + S(\boldsymbol{t}) - 2)\,P(\boldsymbol{t}_{\bar{U}}) + 2P(\boldsymbol{t}_{\bar{U}}) \\
&= P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2 + (S(\boldsymbol{t}_{\bar{U}}) - 1)\,P(\boldsymbol{t}_{\bar{U}}) \\
&\geq P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2 \geq |B'_0|.
\end{aligned}$$

By the induction hypothesis, $G - (X \cup B'_+)$ is f'-paintable, and so is $G - X$.

Case 3. $|U \cap A| = 1$.

Without loss of generality, we may assume that $U \cap A = \{a_1\}$. Note that $P(\boldsymbol{t}_{\bar{U}}) = \prod_{i=2}^{n} t_i$, $S(\boldsymbol{t}_{\bar{U}}) = \sum_{i=2}^{n}(t_i - 1)$, $P(\boldsymbol{t}_U) = (t_1 - 1)\prod_{i=2}^{n} t_i$ and $S(\boldsymbol{t}_U) = \sum_{i=1}^{n}(t_i - 1) - 1$ for this case. Let $s = |B_- \cap U|$ and we consider the following four subcases.

Subcase 3.1. $s = 0$ and $|B \cap U| < P(\boldsymbol{t}_{\bar{U}}) - k$.

Let $X = U \cap A$ and we show that $G - X$ is f'-paintable. We have $n' = n - 1$, $\boldsymbol{t}' = \boldsymbol{t}_{\bar{U}}$, $B'_- = \emptyset$, and $B'_0 = B_- \cup (B_0 \cap U)$. Since $|B'_0| = k + |B_0 \cap U| \leq P(\boldsymbol{t}_{\bar{U}}) = P(\boldsymbol{t}')$, $G - (X \cup B'_+)$ is f'-paintable by Theorem 3.5, and so is $G - X$.

Subcase 3.2. $s = 0$ and $|B \cap U| \geq P(\boldsymbol{t}_{\bar{U}}) - k$.

Let $X = U \cap B$ and we show that $G - X$ is f'-paintable. We have $n' = n$, $\boldsymbol{t}' = \boldsymbol{t}_U$, $B'_- = B_-$, and $B'_0 = B_0 \setminus U$. Since $P(\boldsymbol{t}_U) = P(\boldsymbol{t}) - P(\boldsymbol{t}_{\bar{U}})$ and $S(\boldsymbol{t}_U) = S(\boldsymbol{t}) - 1$,

$$\begin{aligned} |B'_0| = |B_0| - |B_0 \cap U| &\leq (P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2) - (P(\boldsymbol{t}_{\bar{U}}) - k) \\ &= P(\boldsymbol{t}_U) - kS(\boldsymbol{t}_U) - 2 = P(\boldsymbol{t}') - kS(\boldsymbol{t}') - 2. \end{aligned}$$

By the induction hypothesis, $G - (X \cup B'_+)$ is f'-paintable, and so is $G - X$.

Subcase 3.3. $s \geq 1$ and $|B_0 \cap U| < P(\boldsymbol{t}_{\bar{U}}) - sS(\boldsymbol{t}_{\bar{U}}) + s - k - 1$.

Let $X = U \cap A$, and we show that $G - X$ is f'-paintable. We have $n' = n - 1$, $\boldsymbol{t}' = \boldsymbol{t}_{\bar{U}}$, $B'_- = B_- \cap U$, and $B'_0 = (B_- \setminus U) \cup (B_0 \cap U)$. By the assumption of this case, $|B'_-| = s$ and $|B_0 \cap U| \leq P(\boldsymbol{t}_{\bar{U}}) - sS(\boldsymbol{t}_{\bar{U}}) + s - k - 2$. Therefore,

$$\begin{aligned} |B'_0| = |(B_- \setminus U) \cup (B_0 \cap U)| &\leq (k - s) + (P(\boldsymbol{t}_{\bar{U}}) - sS(\boldsymbol{t}_{\bar{U}}) + s - k - 2) \\ &= P(\boldsymbol{t}_{\bar{U}}) - sS(\boldsymbol{t}_{\bar{U}}) - 2 \\ &= P(\boldsymbol{t}') - |B'_-|S(\boldsymbol{t}') - 2. \end{aligned}$$

By the induction hypothesis, $G - (X \cup B'_+)$ is f'-paintable, and so is $G - X$.

Subcase 3.4. $s \geq 1$ and $|B_0 \cap U| \geq P(\boldsymbol{t}_{\bar{U}}) - sS(\boldsymbol{t}_{\bar{U}}) + s - k - 1$.

Let $X = U \cap B$ and we show that $G - X$ is f'-paintable. We have $n' = n$, $\boldsymbol{t}' = \boldsymbol{t}_U$, $B'_- = B_- \setminus U$, and $B'_0 = B_0 \setminus U$. Since $P(\boldsymbol{t}) = P(\boldsymbol{t}_{\bar{U}}) + P(\boldsymbol{t}_U)$ and $S(\boldsymbol{t}_U) = S(\boldsymbol{t}) - 1 = S(\boldsymbol{t}_{\bar{U}}) + t_1 - 2$, we have

$$\begin{aligned} |B'_0| = |B_0| - |B_0 \cap U| &\leq (P(\boldsymbol{t}) - kS(\boldsymbol{t}) - 2) - (P(\boldsymbol{t}_{\bar{U}}) - sS(\boldsymbol{t}_{\bar{U}}) + s - k - 1) \\ &= (P(\boldsymbol{t}) - P(\boldsymbol{t}_{\bar{U}})) - k\,(S(\boldsymbol{t}_U) + 1) + s\,(S(\boldsymbol{t}_U) - t_1 + 2) - s + k - 1 \\ &= P(\boldsymbol{t}_U) - (k - s)S(\boldsymbol{t}_U) - 2 - (st_1 - s - 1) \\ &\leq P(\boldsymbol{t}_U) - (k - s)S(\boldsymbol{t}_U) - 2 = P(\boldsymbol{t}') - |B'_-|S(\boldsymbol{t}') - 2. \end{aligned}$$

By the induction hypothesis, $G - (X \cup B'_+)$ is f'-paintable, and so is $G - X$. This completes Case 3.

Combining these cases, either $G - (U \cap A)$ or $G - (U \cap B)$ is f'-paintable for any set of vertices U such that $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$. Thus, by Lemma 3.10, G is f-paintable. This completes the proof of Theorem 3.11.

3.5 Upper bound in Theorem 3.8

In order to give an upper bound of $r_{\text{pnt}}(m)$, we prove the following two theorems. We say that a tuple $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ of positive integers is *semi-uniform* if $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - 1$.

Theorem 3.13. Let G be a complete bipartite graph with partite sets A and B, and let $A = \{a_1, a_2, \ldots, a_n\}$ where $n \geq 2$. We divide B into B_- and B_0, and let k be the order of B_-. Suppose that f is a mapping from $V(G)$ to positive integers such that

$$f(v) = \begin{cases} t_i(>0) & v = a_i \in A, \\ n-1 & v \in B_-, \\ n & v \in B_0, \end{cases}$$

and let $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$. Then, there exists a constant $c_1 \geq 0$ such that if (G, f) satisfies the following conditions (1-a), (1-b), and (1-c), then G is not f-paintable.

(1-a) $\boldsymbol{t}$ is semi-uniform,

(1-b) $|B_0| \geq P(\boldsymbol{t}) - \frac{k}{2}(S(\boldsymbol{t}) - c_1)$, and

(1-c) if $S(\boldsymbol{t}) > c_1$, then $k \leq \frac{2P(\boldsymbol{t})}{S(\boldsymbol{t})+t_1-c_1-1}$.

Theorem 3.14. Let G be a complete bipartite graph with partite sets A and B, and let $A = \{a_1, a_2, \ldots, a_n\}$ where $n \geq 2$. Suppose that f is a mapping from $V(G)$ to positive integers such that

$$f(v) = \begin{cases} t_i(>0) & v = a_i \in A, \\ n-1 & v \in B, \end{cases}$$

and let $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$. Then, there exist constants $c_1, c_2 \geq 0$ such that if (G, f) satisfies the following conditions (2-a), (2-b) and (2-c), then G is not f-paintable.

(2-a) $\boldsymbol{t}$ is semi-uniform,

(2-b) $|B| \geq c_2$ and

(2-c) if $S(\boldsymbol{t}) > c_1$, then $|B| \geq \frac{2P(\boldsymbol{t})}{S(\boldsymbol{t})-c_1} + c_2$.

Theorem 3.13 is used in a proof of Theorem 3.14. Before going to the proofs of Theorems 3.13 and 3.14, we give an upper bound of $r_{\text{pnt}}(m)$, which implies the upper bound in Theorem 3.8.

Corollary 3.15. There are constants $c_1 > 0$ and $c_2 > 0$ such that for every integer $m \geq 2$, the complete bipartite graph $K_{m+1,r}$ is not m-paintable where $r = \max\left\{\frac{2m^{m+1}}{m^2-1-c_1}, 0\right\} + c_2$. In particular, $r_{\text{pnt}}(m) \leq \max\left\{\frac{2m^{m+1}}{m^2-1-c_1}, 0\right\} + c_2$ for every integer $m \geq 2$.

Proof. We fix an integer $m \geq 2$. Let c_1 and c_2 be constants in Theorem 3.14, and let $r = \max\left\{\frac{2m^{m+1}}{m^2-1-c_1}, 0\right\} + c_2$. Let G be a complete bipartite graph with partite sets $A = \{a_1, a_2, \ldots, a_{m+1}\}$ and $B = \{b_1, b_2, \ldots, b_r\}$. Let f be the constant map from $V(G)$ to m, and let $\boldsymbol{t}$ be an $(m+1)$-tuple defined from f on A. Then $\boldsymbol{t}$ is semi-uniform, $P(\boldsymbol{t}) = m^{m+1}$, and $S(\boldsymbol{t}) = (m+1)(m-1) = m^2 - 1$, and thus

$$|B| = r = \max\left\{\frac{2m^{m+1}}{m^2-1-c_1}, 0\right\} + c_2 \geq \frac{2m^{m+1}}{m^2-1-c_1} + c_2 = \frac{2S(\boldsymbol{t})}{S(\boldsymbol{t})-c_1} + c_2.$$

Since $|B| = r = \max\left\{\frac{2m^{m+1}}{m^2-1-c_1}, 0\right\} + c_2 \geq c_2$, by Theorem 3.14, G is not f-paintable. $\square$

3.5.1 Proof of Theorem 3.13

For a tuple $\boldsymbol{x} = (x_1, x_2, \ldots, x_h)$, let F_1 be a function defined by

$$F_1(\boldsymbol{x}) = 2(x_1 - 1)P(\boldsymbol{x}) - x_1(S(\boldsymbol{x}) + x_1 - 2)(S(\boldsymbol{x}) + x_1 - 1).$$

Then, it is easy to see that there are only finite semi-uniform tuples $\boldsymbol{x} = (x_1, x_2, \ldots, x_h)$ such that $h \geq 3$, $x_h \geq 2$ and $F_1(\boldsymbol{x}) < 0$, and let T_1 be the set of such tuples. A constant c_1 is defined by $c_1 = \max\{S(\boldsymbol{x}) \mid \boldsymbol{x} \in T_1\}$. Let $\tilde{S}(\boldsymbol{x}) = S(\boldsymbol{x}) - c_1$.

Let $(G, f) \in \mathcal{G}$ be a pair of a graph and a mapping satisfying the conditions of Theorem 3.13. We define A, B, B_-, B_0 and $\boldsymbol{t}$ as in the statement.

Claim 1. If $\tilde{S}(\boldsymbol{t}) \leq 0$, then G is not f-paintable. In particular, if $\boldsymbol{t} \in T_1$ then G is not f-paintable.

Proof. When $\tilde{S}(\boldsymbol{t}) \leq 0$, the condition (1-b) implies $|B_0| \geq P(\boldsymbol{t})$. Hence, $G - B_-$ is not f-paintable by Theorem 3.5 and G is not f-paintable either. If $\boldsymbol{t} \in T_1$, by the definition of T_1, it follows that $\tilde{S}(\boldsymbol{t}) \leq 0$, and thus G is not f-paintable. □

Claim 2. If $n = 2$, then G is not f-paintable.

Proof. When $n = 2$, $f(v) = 1$ for every $v \in B_-$. If $t_2 \leq k$, we remove the vertices in B_- by using Lemma 3.9 and eventually we have $(f - t_2\mathbf{1}_A)(a_2) = 0$, and thus G is not f-paintable. If $t_2 \geq k+1$, then again by Lemma 3.9, G is f-paintable if and only if $G - B_-$ is $(f - k\mathbf{1}_A)$-paintable. The condition (1-b) implies that

$$\begin{aligned} |B_0| \geq t_1t_2 - \frac{k}{2}(t_1 + t_2 - 2 - c_1) = (t_1 - k)(t_2 - k) + \frac{k}{2}(t_1 + t_2 + 2 + c_1) - k^2 \\ \geq (t_1 - k)(t_2 - k), \end{aligned}$$

and thus $G - B_-$ is not $(f - k\mathbf{1}_A)$-paintable by Theorem 3.5. □

By Claims 1 and 2, we assume that $\tilde{S}(\boldsymbol{t}) > 0$ and $n \geq 3$. We also assume that Theorem 3.13 holds for every $(G', f') \in \mathcal{G}$ with $|V(G')| < |V(G)|$.

Claim 3. If $t_n = 1$, then G is not f-paintable.

Proof. When $t_n = 1$, G is f-paintable if and only if $G - a_n$ is $(f - \mathbf{1}_B)$-paintable by Lemma 3.9. Since $P(\boldsymbol{t}') = P(\boldsymbol{t})$ and $S(\boldsymbol{t}') = S(\boldsymbol{t})$ for $\boldsymbol{t}' = (t_1, t_2, \ldots, t_{n-1})$, $(G - a_n, f - \mathbf{1}_B)$ satisfies the condition (1-c). By the induction hypothesis, $G - a_n$ is not $(f - \mathbf{1}_B)$-paintable, and thus G is not f-paintable. □

By Claims 1, 2, and 3, we assume $n \geq 3$, $t_n \geq 2$, and $\boldsymbol{t} \notin T_1$, which imply $F_1(\boldsymbol{t}) \geq 0$. For the tuple $\boldsymbol{t}$, we define $\boldsymbol{t}_A = (t_2, t_3, \ldots, t_n)$ and $\boldsymbol{t}_B = (t_2, t_3, \ldots, t_n, t_1 - 1)$. Let $l = \left\lceil P(\boldsymbol{t}) - \frac{k}{2}\tilde{S}(\boldsymbol{t}) \right\rceil$.

Claim 4. Let $s = \left\lfloor \frac{k-1}{t_1} \right\rfloor \in [0, k]$. Then,

$$s \leq \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t}_A) + t_2 - 1} \quad \text{and} \tag{3.5.1}$$

$$k - s \leq \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B) + t_2 - 1}. \tag{3.5.2}$$

Proof. We define a function $\tilde{F}_1$ by

$$\tilde{F}_1(\boldsymbol{x}) = 2(x_1 - 1)P(\boldsymbol{x}) - x_1 \left(\tilde{S}(\boldsymbol{x}) + x_1 - 2\right) \left(\tilde{S}(\boldsymbol{x}) + x_1 - 1\right).$$

Since $S(\boldsymbol{t}) \geq \tilde{S}(\boldsymbol{t}) > 0$, we know that $\tilde{F}_1(\boldsymbol{t}) \geq F_1(\boldsymbol{t}) \geq 0$. By the condition (1-c) and the assumption $\tilde{S}(\boldsymbol{t}) \geq 0$, we have $k \leq \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t}) + t_1 - 1}$. Since $\tilde{S}(\boldsymbol{t}_A) < \tilde{S}(\boldsymbol{t})$ and $t_2 \leq t_1$, we have

$$s < \frac{k}{t_1} \leq \frac{2P(\boldsymbol{t})/t_1}{\tilde{S}(\boldsymbol{t}) + t_1 - 1} = \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t}) + t_1 - 1} < \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t}_A) + t_2 - 1},$$

and thus (3.5.1) holds. Since

$$k - s = k - \left\lfloor \frac{k-1}{t_1} \right\rfloor \leq \frac{t_1 - 1}{t_1} k + 1 \leq \frac{2(t_1 - 1)P(\boldsymbol{t})/t_1}{\tilde{S}(\boldsymbol{t}) + t_1 - 1} + 1 = \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}) + t_1 - 1} + 1,$$

(3.5.2) holds by

$$\begin{aligned}
\frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B) + t_2 - 1} - (k - s) &\geq \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B) + t_2 - 1} - \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}) + t_1 - 1} - 1 \\
&\geq \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B) + t_1 - 1} - \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}) + t_1 - 1} - 1 \\
&= \frac{2(t_1 - 1)P(\boldsymbol{t}) - t_1 \left(\tilde{S}(\boldsymbol{t}) + t_1 - 2\right) \left(\tilde{S}(\boldsymbol{t}) + t_1 - 1\right)}{t_1 \left(\tilde{S}(\boldsymbol{t}) + t_1 - 2\right) \left(\tilde{S}(\boldsymbol{t}) + t_1 - 1\right)} \\
&= \frac{\tilde{F}_1(\boldsymbol{t})}{t_1 \left(\tilde{S}(\boldsymbol{t}) + t_1 - 2\right) \left(\tilde{S}(\boldsymbol{t}) + t_1 - 1\right)} \geq 0.
\end{aligned}$$

□

Claim 5. There exists an integer $r \in [0, l]$ such that

$$k - s + r \geq P(\boldsymbol{t}_A) - \frac{s}{2}\tilde{S}(\boldsymbol{t}_A) \quad \text{and} \tag{3.5.3}$$

$$l - r \geq P(\boldsymbol{t}_B) - \frac{k - s}{2}\tilde{S}(\boldsymbol{t}_B). \tag{3.5.4}$$

Proof. Let $Q_1 = P(\boldsymbol{t}_A) - \frac{s}{2}\tilde{S}(\boldsymbol{t}_A) + s - k$ and $Q_2 = l - P(\boldsymbol{t}_B) + \frac{k-s}{2}\tilde{S}(\boldsymbol{t}_B)$. Then, (3.5.3) and (3.5.4) are equivalent to $r \geq Q_1$ and $r \leq Q_2$. For the existence of an integer $r \in [0, l]$ with $r \geq Q_1$ and $r \leq Q_2$, it suffices to show that $Q_1 \leq Q_2 - \frac{1}{2}$, $Q_1 \leq l - \frac{1}{2}$ and $Q_2 \geq 0$. By the definition of s, we have $st_1 \leq k - 1$. Since $P(\boldsymbol{t}) = P(\boldsymbol{t}_A) + P(\boldsymbol{t}_B)$ and $\tilde{S}(\boldsymbol{t}) = \tilde{S}(\boldsymbol{t}_B) + 1 = \tilde{S}(\boldsymbol{t}_A) + t_1 - 1$,

$$\begin{aligned} Q_2 - \frac{1}{2} - Q_1 &= l - P(\boldsymbol{t}_B) + \frac{k-s}{2}\tilde{S}(\boldsymbol{t}_B) - \frac{1}{2} - P(\boldsymbol{t}_A) + \frac{s}{2}\tilde{S}(\boldsymbol{t}_A) - s + k \\ &\geq P(\boldsymbol{t}) - \frac{k}{2}\tilde{S}(\boldsymbol{t}) - P(\boldsymbol{t}_B) + \frac{k-s}{2}\tilde{S}(\boldsymbol{t}_B) - \frac{1}{2} - P(\boldsymbol{t}_A) + \frac{s}{2}\tilde{S}(\boldsymbol{t}_A) - s + k \\ &= \frac{k - st_1 - 1}{2} \geq 0. \end{aligned}$$

By (3.5.2) and the inequality $Q_2 - \frac{1}{2} - Q_1 \geq 0$,

$$\begin{aligned} l - \frac{1}{2} - Q_1 = (l - Q_2) + \left(Q_2 - \frac{1}{2} - Q_1\right) \geq l - Q_2 &= P(\boldsymbol{t}_B) - \frac{k-s}{2}\tilde{S}(\boldsymbol{t}_B) \\ &\geq P(\boldsymbol{t}_B) - \frac{P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B) + t_2 - 1}\tilde{S}(\boldsymbol{t}_B) > 0. \end{aligned}$$

Since $\tilde{S}(\boldsymbol{t}) > 0$ and k satisfies the condition (1-c),

$$\begin{aligned} Q_2 &\geq P(\boldsymbol{t}) - \frac{k}{2}\tilde{S}(\boldsymbol{t}) - P(\boldsymbol{t}_B) + \frac{k-s}{2}\tilde{S}(\boldsymbol{t}_B) = P(\boldsymbol{t}_A) - \frac{k}{2} - \frac{s}{2}\tilde{S}(\boldsymbol{t}_B) \\ &\geq P(\boldsymbol{t}_A) - \frac{k}{2} - \frac{k}{2t_1}\tilde{S}(\boldsymbol{t}_B) = P(\boldsymbol{t}_A) - \frac{k}{2t_1}\left(\tilde{S}(\boldsymbol{t}) + t_1 - 1\right) \geq 0. \end{aligned}$$

Thus, there exists an integer $r \in [0, l]$ that satisfies (3.5.3) and (3.5.4). □

Let $s \in [0, k]$ and $r \in [0, l]$ be integers as in Claims 4 and 5. Let $U \subseteq V(G)$ be a set of integers such that $U \cap A = \{a_1\}$, $|U \cap B_-| = s$, and $|U \cap B_0| = r$, and let $f' = f - \mathbf{1}_U$. We shall show that neither $G - (U \cap A)$ nor $G - (U \cap B)$ is f'-paintable. For each case $X = U \cap A$ and $X = U \cap B$, we define A', B', B'_-, B'_0, B'_+, n', $\boldsymbol{t}'$ as in the proof of Theorem 3.11. It suffices to show that $G - (X \cup B'_+)$ is not f'-paintable for each case $X = U \cap A$ and $X = U \cap B$.

When $X = U \cap A$, we have $n' = n - 1$, $\boldsymbol{t}' = \boldsymbol{t}_A$, $B'_- = B_- \cap U$ and $B'_0 = (B_- \setminus U) \cup (B_0 \cap U)$. The condition (1-a) obviously holds, and (3.5.1) and (3.5.3) respectively imply the conditions (1-b) and (1-c) for $(G - (X \cup B'_+), f')$. By the induction hypothesis, $G - (X \cup B'_+)$ is not f'-paintable, and thus $G - X$ is not f'-paintable.

When $X = U \cap B$, we have $n' = n$, $\boldsymbol{t}' = \boldsymbol{t}_B$, $B'_- = B_- \setminus U$ and $B'_0 = B_0 \setminus U$. The condition (1-a) obviously holds, and (3.5.2) and (3.5.4) respectively imply the conditions (1-b) and (1-c) for $(G - (X \cup B'_+), f')$. By the induction hypothesis, $G - (X \cup B'_+)$ is not f'-paintable, and thus $G - X$ is not f'-paintable.

Hence G is not f-paintable, which completes the proof of Theorem 3.13.

3.5.2 Proof of Theorem 3.14

Let c_1 be the constant in Theorem 3.13, and let $\tilde{S}(\boldsymbol{t}) = S(\boldsymbol{t}) - c_1$. For a tuple $\boldsymbol{x} = (x_1, x_2, \dots, x_h)$, let F_2 be a function defined by

$$F_2(\boldsymbol{x}) = 2(x_1 - 1)P(\boldsymbol{x}) - \tilde{S}(\boldsymbol{x})\left(\tilde{S}(\boldsymbol{x}) - 1\right)^2.$$

Let T_2 be the set of semi-uniform tuples $\boldsymbol{x} = (x_1, x_2, \dots, x_h)$ such that $h \geq 3$, $x_h \geq 2$, $\tilde{S}(\boldsymbol{x}) > 0$, and $F_2(\boldsymbol{x}) < 0$, and let T_2' be the set of semi-uniform tuples $\boldsymbol{x} = (x_1, x_2, \dots, x_h)$ such that $h \geq 2$, $x_h \geq 1$ and $\tilde{S}(\boldsymbol{x}) \leq x_1$. Then, both T_2 and T_2' are finite sets, and we define a constant c_2 by $c_2 = \max\{P(\boldsymbol{x}) \mid \boldsymbol{x} \in T_2 \cup T_2'\}$.

Let (G, f) be a pair of a graph and a mapping satisfying the conditions of Theorem 3.14. We define A, B, and $\boldsymbol{t}$ as in the statement.

Claim 1. If $\boldsymbol{t} \in T_2 \cup T_2'$, then G is not f-paintable.

Proof. When $\boldsymbol{t} \in T_2 \cup T_2'$, we have $|B| \geq c_2 \geq P(\boldsymbol{t})$ by the conditions (2-b) and (2-c). By Theorem 3.5 and Proposition 3.2, G is not $(f + \mathbf{1}_B)$-paintable, and thus G is not f-paintable. □

Claim 2. If $\tilde{S}(\boldsymbol{t}) \in \{1, 2\}$, then G is not f-paintable.

Proof. When $\tilde{S}(\boldsymbol{t}) \in \{1, 2\}$, the condition (2-c) implies that $|B| \geq \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} + c_2 \geq P(\boldsymbol{t}) + c_2 \geq P(\boldsymbol{t})$. Thus, again by Theorem 3.5 and Proposition 3.2, G is not f-paintable. □

Claim 3. If $n = 2$, then G is not f-paintable.

Proof. By Claim 1, we may assume that $\tilde{S}(\boldsymbol{t}) > t_1 > 0$. When $n = 2$, we have $f(v) = 1$ for every $v \in B$, and the condition (2-c) implies

$$|B| \geq \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} + c_2 > \frac{2P(\boldsymbol{t})}{S(\boldsymbol{t}) + 2} = \frac{2t_1t_2}{t_1 + t_2} \geq t_2.$$

By using Lemma 3.9 to delete vertices in B, we eventually have $(f - t_2\mathbf{1}_A)(a_2) = 0$, and thus G is not f-paintable. □

By Claims 1, 2 and 3, we assume $\boldsymbol{t} \notin T_2 \cup T_2'$, $\tilde{S}(\boldsymbol{t}) > \max\{t_1, 2\}$ and $n \geq 3$. We also assume that the theorem holds for every (G', f') with $|V(G')| < |V(G)|$. Let $k = \left\lceil \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} + c_2 \right\rceil$. By Proposition 3.2, we may assume that $|B| = k$. Furthermore, by a similar argument to the proof of Claim 3, we may assume that $t_n \geq 2$. Thus, we have $F_2(\boldsymbol{t}) \geq 0$.

For the tuple $\boldsymbol{t}$, we define $\boldsymbol{t}_A = (t_2, t_3, \dots, t_n)$ and $\boldsymbol{t}_B = (t_2, t_3, \dots, t_n, t_1 - 1)$. Note that $\tilde{S}(\boldsymbol{t}_B) = \tilde{S}(\boldsymbol{t}) - 1 \geq 2$ by $\tilde{S}(\boldsymbol{t}) > 2$.

Claim 4. There exists an integer $s \in [0, k]$ such that

$$s \leq \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t}_A) + t_2 - 1}, \tag{3.5.5}$$

$$k - s \geq P(\boldsymbol{t}_A) - \frac{s}{2}\tilde{S}(\boldsymbol{t}_A), \quad \text{and} \tag{3.5.6}$$

$$k - s \geq \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B)} + c_2. \tag{3.5.7}$$

Proof. Let $k' = \left\lceil \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B)} + c_2 \right\rceil$. Since

$$\frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} \geq \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} \cdot \frac{(t_1 - 1)\tilde{S}(\boldsymbol{t})}{t_1(\tilde{S}(\boldsymbol{t}) - 1)} = \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B)}$$

where the inequality holds by $\tilde{S}(\boldsymbol{t}) > t_1$, we have $k' \leq k$. We define $s \in [0, k]$ by $s = k - k'$. By the definition of s, $k - s = k' \geq \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B)} + c_2$ and thus (3.5.7) holds. By the definitions of $\boldsymbol{t}_A$ and $\boldsymbol{t}_B$, we have $P(\boldsymbol{t}) = P(\boldsymbol{t}_A) + P(\boldsymbol{t}_B)$ and $\tilde{S}(\boldsymbol{t}_B) = \tilde{S}(\boldsymbol{t}) - 1$ and thus

$$\begin{aligned}
s &= \left\lceil \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} + c_2 \right\rceil - \left\lceil \frac{2P(\boldsymbol{t}_B)}{S(\boldsymbol{t}_B)} + c_2 \right\rceil \leq \frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} + \frac{\tilde{S}(\boldsymbol{t}) - 1}{\tilde{S}(\boldsymbol{t})} - \frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B)} \\
&= \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t})} - \frac{2P(\boldsymbol{t}_B) - \left(\tilde{S}(\boldsymbol{t}) - 1\right)^2}{\tilde{S}(\boldsymbol{t})\left(\tilde{S}(\boldsymbol{t}) - 1\right)} \\
&\leq \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t}_A) + t_2 - 1} - \frac{2P(\boldsymbol{t}_B) - \left(\tilde{S}(\boldsymbol{t}) - 1\right)^2}{\tilde{S}(\boldsymbol{t})\left(\tilde{S}(\boldsymbol{t}) - 1\right)} \\
&\leq \frac{2P(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t}_A) + t_2 - 1},
\end{aligned}$$

where the last inequality holds by

$$2P(\boldsymbol{t}_B) - \left(\tilde{S}(\boldsymbol{t}) - 1\right)^2 = \frac{2(t_1 - 1)P(\boldsymbol{t}) - t_1\left(\tilde{S}(\boldsymbol{t}) - 1\right)^2}{t_1} \geq \frac{F_2(\boldsymbol{t})}{t_1} \geq 0.$$

Thus, (3.5.5) is fulfilled. By the definitions of $\boldsymbol{t}_A$ and $\boldsymbol{t}_B$, we have

$$P(\boldsymbol{t}_A) + (t_1 + 1)P(\boldsymbol{t}_B) = \frac{1}{t_1}P(\boldsymbol{t}) + \frac{(t_1 + 1)(t_1 - 1)}{t_1}P(\boldsymbol{t}) = t_1 P(\boldsymbol{t}).$$

Since $\tilde{S}(\boldsymbol{t}_A) = \tilde{S}(\boldsymbol{t}) - t_1 + 1$ and $\tilde{S}(\boldsymbol{t}_B) = \tilde{S}(\boldsymbol{t}) - 1$,

$$
\begin{aligned}
k' - P(\boldsymbol{t}_A) + \frac{k-k'}{2}\tilde{S}(\boldsymbol{t}_A) &= \frac{k}{2}\tilde{S}(\boldsymbol{t}_A) - \frac{k'}{2}\left(\tilde{S}(\boldsymbol{t}_A) - 2\right) - P(\boldsymbol{t}_A) \\
&\geq \left(\frac{2P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})} + c_2\right)\frac{\tilde{S}(\boldsymbol{t}_A)}{2} - \left(\frac{2P(\boldsymbol{t}_B)}{\tilde{S}(\boldsymbol{t}_B)} + c_2 + 1\right)\frac{\tilde{S}(\boldsymbol{t}_A) - 2}{2} - P(\boldsymbol{t}_A) \\
&= \frac{P(\boldsymbol{t})\tilde{S}(\boldsymbol{t}_A)}{\tilde{S}(\boldsymbol{t})} - \frac{P(\boldsymbol{t}_B)(\tilde{S}(\boldsymbol{t}_A) - 2)}{\tilde{S}(\boldsymbol{t}_B)} - P(\boldsymbol{t}_A) - \frac{\tilde{S}(\boldsymbol{t}_A) - 2}{2} + c_2 \\
&= \frac{1}{\tilde{S}(\boldsymbol{t})(\tilde{S}(\boldsymbol{t}) - 1)}\Big(P(\boldsymbol{t})(\tilde{S}(\boldsymbol{t}) - t_1 + 1)(\tilde{S}(\boldsymbol{t}) - 1) \\
&\quad - P(\boldsymbol{t}_B)(\tilde{S}(\boldsymbol{t}) - t_1 - 1)\tilde{S}(\boldsymbol{t}) - P(\boldsymbol{t}_A)\tilde{S}(\boldsymbol{t})(\tilde{S}(\boldsymbol{t}) - 1)\Big) - \frac{\tilde{S}(\boldsymbol{t}_A) - 2}{2} + c_2 \\
&= \frac{1}{\tilde{S}(\boldsymbol{t})(\tilde{S}(\boldsymbol{t}) - 1)}\Big((P(\boldsymbol{t}) - P(\boldsymbol{t}_B) - P(\boldsymbol{t}_A))\tilde{S}(\boldsymbol{t})^2 \\
&\quad - (t_1P(\boldsymbol{t}) - (t_1 + 1)P(\boldsymbol{t}_B) - P(\boldsymbol{t}_B))\tilde{S}(\boldsymbol{t}) + (t_1 - 1)P(\boldsymbol{t})\Big) - \frac{\tilde{S}(\boldsymbol{t}_A) - 2}{2} + c_2 \\
&= \frac{(t_1 - 1)P(\boldsymbol{t})}{\tilde{S}(\boldsymbol{t})\tilde{S}(\boldsymbol{t}_B)} - \frac{\tilde{S}(\boldsymbol{t}_A) - 2}{2} + c_2 \\
&= \frac{2(t_1 - 1)P(\boldsymbol{t}) - \tilde{S}(\boldsymbol{t})\left(\tilde{S}(\boldsymbol{t}) - 1\right)\left(\tilde{S}(\boldsymbol{t}) - t_1 - 1\right)}{2\tilde{S}(\boldsymbol{t})\left(\tilde{S}(\boldsymbol{t}) - 1\right)} + c_2 \\
&> \frac{F_2(\boldsymbol{t})}{2\tilde{S}(\boldsymbol{t})\left(\tilde{S}(\boldsymbol{t}) - 1\right)} + c_2 \geq 0,
\end{aligned}
$$

and (3.5.6) holds. □

Let $s \in [0, k]$ be an integer as in Claim 4. Let $U \subseteq V(G)$ be a set of vertices such that $U \cap A = \{a_1\}$ and $|U \cap B| = s$, and let $f' = f - \mathbf{1}_U$. We shall show that neither $G - (U \cap A)$ nor $G - (U \cap B)$ is f'-paintable. For each case $X = U \cap A$ and $X = U \cap B$, we define A', B', B'_-, B'_0, B'_+, n', and $\boldsymbol{t}'$ as in the proof of Theorem 3.11. It suffices to show that $G - (X \cup B'_+)$ is not f'-paintable for each case $X = U \cap A$ and $X = U \cap B$.

When $X = U \cap A$, we have $n' = n - 1$, $\boldsymbol{t}' = \boldsymbol{t}_A$, $B'_- = U \cap B$ and $B'_0 = B \setminus U$. The condition (1-a) obviously holds, and (3.5.5) and (3.5.6) respectively imply the conditions (1-b) and (1-c) for $(G - (X \cup B'_+), f')$. By Theorem 3.13, $G - (X \cup B'_+)$ is not f'-paintable, and thus $G - X$ is not f'-paintable.

When $X = U \cap B$, we have $n' = n$, $\boldsymbol{t}' = \boldsymbol{t}_B$, $B'_- = B \setminus U$ and $B'_0 = \emptyset$. The condition (2-a) obviously holds, and (3.5.7) implies the condition (2-c) for $(G - (X \cup B'_+), f')$. The condition (2-b) is implied by the condition (2-c) since $\tilde{S}(\boldsymbol{t}_B) > 0$. By the induction hypothesis, $G - (X \cup B'_+)$ is not f'-paintable, and thus $G - X$ is not f'-paintable.

Hence G is not f-paintable, which completes the proof of Theorem 3.14.

Chapter 4

Odd coloring and Proper conflict-free coloring

The odd coloring and the proper conflict-free coloring are relatively new concepts introduced in Petruševski and Škrekovski [55] and in Fabrici, Lušar, Rindošová, and Soták [29], respectively. Each of the colorings is a proper coloring with some additional constraints on the colors appearing in the neighborhood of each vertex. In this chapter, as preparation for the following chapters of Part I, we introduce odd coloring and proper conflict-free coloring. We focus on the relationship between odd coloring, proper conflict-free coloring and square coloring in Section 4.2. In Section 4.3, we give a proof of Theorem 4.2, which shows that the gap between the chromatic numbers can be arbitrarily large.

4.1 Introduction to odd coloring and proper conflict-free coloring of graphs

Let G be a graph. A mapping φ from $V(G)$ to positive integers is called an *odd coloring* of G if φ is a proper coloring of G such that

> for every non-isolated vertex v of G, there is a color c such that $|\varphi^{-1}(c) \cap N_G(v)|$ is an odd integer.

For a positive integer k, an *odd k-coloring* of G is an odd coloring of G such that every vertex of G maps to an integer in $[k]$. The *odd chromatic number* of G, denoted by $\chi_o(G)$, is the least integer k such that G admits an odd k-coloring. For a proper (partial) coloring φ of G, we say a vertex v of G *satisfies the odd condition with respect to* φ if there is a color that appears at odd number of neighbors of v. Therefore, an odd coloring of G is a proper coloring φ of G such that every non-isolated vertex v of G satisfies the odd condition. It follows from the definitions that for every vertex v and any k-coloring φ of G, $\sum_{c\in[k]} |\varphi^{-1}(c) \cap N_G(v)| = d_G(v)$. Thus, if $d_G(v)$ is odd, then for any proper k-coloring φ of G, there is a color $c \in [k]$ such that $|\varphi^{-1}(c) \cap N_G(v)|$ is an odd integer, that is, v satisfies the odd condition with respect to φ. For a list assignment $L : V(G) \to 2^{\mathbb{N}}$ of G, an *odd L-coloring* of G is an odd coloring φ of G such that $\varphi(v) \in L(v)$ for every vertex v of G.

A mapping φ from $V(G)$ to positive integers is called a *proper conflict-free coloring* of G if φ is a proper coloring of G such that

> for every non-isolated vertex v of G, there is a color c such that $|\varphi^{-1}(c) \cap N_G(v)| = 1$.

For a positive integer k, a *proper conflict-free k-coloring* of G is a proper conflict-free coloring of G such that every vertex of G maps to an integer in $[k]$. The *proper conflict-free chromatic number* of G, denoted by $\chi_{\text{pcf}}(G)$, is the least integer k such that G admits a proper conflict-free k-coloring. The word "conflict-free" comes from the conflict-free coloring of hypergraphs, which was introduced in Even, Lokter, Ron, and Smorodinsky [28]. For a proper (partial) coloring φ of G, we say that a vertex v of G *satisfies the conflict-free condition with respect to* φ if there is a color that appears at exactly one neighbor of v. Therefore, a proper conflict-free coloring of G is a proper coloring φ of G such that every vertex v of G satisfies the conflict-free condition. For a list assignment $L : V(G) \to 2^{\mathbb{N}}$ of G, a *proper conflict-free L-coloring* of G is a proper conflict-free coloring φ of G such that $\varphi(v) \in L(v)$ for every vertex v of G.

By the definitions, it follows that an odd coloring of G is also a proper coloring of G, and a proper conflict-free coloring of G is an odd coloring of G. It was shown in Ahn, Im, and Oum [2] that to determine whether a given bipartite graph is odd k-colorable (resp. proper conflict-free k-colorable) is NP-complete.

4.2 Relationship between odd coloring, proper conflict-free coloring, and square coloring

One strong motivation for studies on odd coloring and proper conflict-free coloring is the relationship to square coloring. For a positive integer k, a mapping φ from $V(G)$ to $[k]$ is called a *square k-coloring* of G if $\varphi(u) \neq \varphi(v)$ for every pair of vertices of G with distance at most 2. The *square chromatic number* of G, denoted by $\chi_2(G)$, is the least integer k such that G admits a square k-coloring. If φ is a square coloring of G, then for every vertex v of G, $N_G[v]$ is colored by $d_G(v) + 1$ distinct colors, each of which appears exactly once in $N_G[v]$. Thus, every vertex of G satisfies the conflict-free condition with respect to φ. This together with the relationship between coloring, odd coloring, and proper conflict-free coloring, we have the following proposition.

Proposition 4.1. For every graph G, $\chi(G) \leq \chi_o(G) \leq \chi_{\text{pcf}}(G) \leq \chi_2(G)$.

There are many graphs known as examples of graph G with $\chi(G) < \chi_o(G)$. The cycle C_n satisfies $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ otherwise. Caro, Petruševski, and

Škrekovski observed the following in [12].

$$\chi_o(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \mod 3 \\ 4 & \text{if } n \not\equiv 0 \mod 3 \text{ and } n \neq 5 \\ 5 & \text{if } n = 5 \end{cases} \tag{4.2.1}$$

They also showed that a tree T is odd 2-colorable if and only if every vertex of T has an odd degree, while every tree is 2-colorable. Furthermore, they observed that the ratio $\frac{\chi_o(G)}{\chi(G)}$ can be arbitrarily large. Indeed, for a graph G, let $S(G)$ be a graph obtained from G by subdividing each edge once. Obviously, $S(G)$ is bipartite and thus $\chi(S(G)) = 2$. On the other hand, since every vertex of degree 2 satisfies the odd condition, the restriction of an odd coloring of $S(G)$ to $V(G)$ must be a proper coloring of G. Hence $\chi_o(S(G)) \geq \chi(G)$, and therefore the ratio $\frac{\chi_o(S(G))}{\chi(S(G))} \geq \frac{\chi(G)}{2}$ can be arbitrarily large.

Considering the complete bipartite graph $K_{m,m}$, we see that $\chi_2(G)$ cannot be bounded from above by any function of $\chi_{\rm pcf}(G)$. Since every two distinct vertices of $K_{m,m}$ have distance at most 2, $K_{m,m}^2$ is the complete graph of order $2m$, and thus $\chi_2(K_{m,m}) = |V(K_{m,m})| = 2m$. On the other hand, we have $\chi_{\rm pcf}(K_{m,m}) = 4$. Indeed, let G be a complete bipartite graph with partite sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Then a coloring φ of G defined by $\varphi(a_1) = 1$, $\varphi(b_1) = 2$, and $\varphi(a_i) = 3$, $\varphi(b_i) = 4$ for every $i \in \{2, 3, \ldots, m\}$ is a proper conflict-free coloring of G. Thus, $\chi_2(K_{m,m})$ become arbitrarily large as m increases, while $\chi_{\rm pcf}(K_{m,m}) = 4$ for every m.

In this thesis, we construct a family of graphs that shows the relationship between $\chi(G)$, $\chi_o(G)$ and $\chi_{\rm pcf}(G)$.

Theorem 4.2. Let k and l be any positive integers with $3 \leq k < l$.

(1) If k is odd, then there is a graph G such that $\chi(G) = k$ and $\chi_o(G) \geq l$.

(2) If k is even, then there is a graph G such that $\chi_o(G) = k$ and $\chi_{\rm pcf}(G) \geq l$.

By Theorems 4.2 (2), $\chi_{\rm pcf}(G)$ cannot be bounded from above by any function of $\chi_o(G)$.

4.3 Proof of Theorem 4.2

For positive integers $s \geq 2$ and $t \geq 2$, we define a graph $G_{s,t}$ as follows. Let H be a disjoint union of t copies of the complete graph K_{s-1}, and let $\binom{V(H)}{s-1} = \left\{X \mid X \in 2^{V(H)}, |X| = s-1\right\}$. Let $U = \left\{u_X \mid X \in \binom{V(H)}{s-1}\right\}$ be a set of vertices disjoint from $V(H)$. We define a graph $G_{s,t}$ by $V(G_{s,t}) = V(H) \cup U$ and $E(G_{s,t}) = E(H) \cup \left\{vu_X \mid X \in \binom{V(H)}{s-1}, v \in X\right\}$. See Figure 4.1 for an example of $G_{s,t}$. It is easy to see that $G_{s,t}$ is $(s-1)$-degenerate and $G_{s,t}$ contains a clique of order s, and thus $\chi(G_{s,t}) = s$. We show the following on the odd chromatic number of $G_{s,t}$.

Proposition 4.3. Let $s \geq 3$ and $t \geq 2$ be integers. If s is even, then $\chi_o(G_{s,t}) = s$. If s is odd, then $\chi_o(G_{s,t}) > (s-1)(t-1)$.

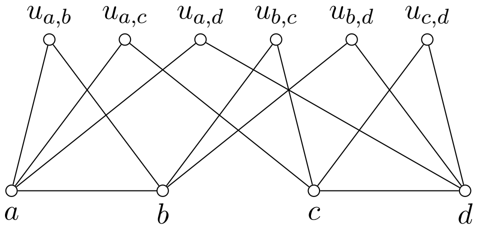


Figure 4.1: The graph $G_{3,2}$. The top vertices represent U and the bottom vertices represent $V(H) = \{a, b, c, d\}$.

Proof. We first consider the case where s is even. Since $\chi_o(G_{s,t}) \geq \chi(G_{s,t}) = s$, it suffices to show that $G_{s,t}$ admits an odd s-coloring. We color the vertices of H properly with the colors $1, 2, \ldots, s-1$, and give a color s to each vertex in U. Then every vertex of H satisfies the odd condition since a color in $\{1, 2, \ldots, k-1\}$ appears exactly once in the neighborhood of the vertex. Furthermore, every vertex of U satisfies the odd condition since the degree of vertices in U is $k-1$, which is an odd integer. Hence $G_{s,t}$ is odd s-colorable.

Next, we consider the case where s is odd. Suppose that $\chi_o(G_{s,t}) = r \leq (s-1)(t-1)$ and let φ be an odd r-coloring of $G_{s,t}$. For each $i \in [r]$, let $V_i = \varphi^{-1}(i) \cap V(H)$, and let $V_i' \subseteq V_i$ be the set of arbitrarily chosen $2\left\lfloor \frac{|V_i|}{2} \right\rfloor$ vertices of V_i. Since $t \geq 2$ and $r \leq (s-1)(t-1)$, we have

$$\sum_{i\in[r]} |V_i'| \geq \sum_{i\in[r]} (|V_i| - 1) = t(s-1) - r \geq t(s-1) - (s-1)(t-1) = s-1,$$

which implies that there is a set $V_i'' \subseteq V_i'$ for each $i \in [r]$ such that $\sum_{i\in[r]} |V_i''| = s-1$ and $|V_i''|$ is even for every $i \in [r]$. Let $X = \bigcup_{i\in[r]} V_i''$. Then the vertex u_X has a neighborhood X, and no color appears at an odd number of neighbors of u_X, a contradiction. Hence, we have $\chi_o(G_{s,t}) > (s-1)(t-1)$ for any odd integer s. $\square$

The following proposition is on the proper conflict-free chromatic number of $G_{s,t}$.

Proposition 4.4. Let $s \geq 3$ and $t \geq 2$ be integers. Then $\chi_{\mathrm{pcf}}(G_{s,t}) \geq \frac{t(s-1)}{2}$.

Proof. Suppose that $\chi_{\mathrm{pcf}}(G) = r < \frac{t(s-1)}{2}$, and let φ be a proper conflict-free r-coloring of $G_{s,t}$. For each $i \in [r]$, let $V_i = \varphi^{-1}(i) \cap V(H)$. Since $r < \frac{t(s-1)}{2}$ and $\sum_{i\in[r]} |V_i| = t(s-1)$, there is a color $c \in [r]$ such that V_c contains at least 3 vertices. Without loss of generality, we may assume that $|V_1| \geq 3$. Let $A = \{i \in [r] \mid |V_i| \geq 2\}$. Since

$$t(s-1) = \sum_{i\in[r]} |V_i| = \sum_{i\in A} |V_i| + (r - |A|) \leq \sum_{i\in A} |V_i| + r < \sum_{i\in A} |V_i| + \frac{t(s-1)}{2},$$

we have $\sum_{i\in A} |V_i| > \frac{t(s-1)}{2} \geq s-1$. Let A' be a minimal subset of A such that $1 \in A'$ and $\sum_{a\in A'} |V_a| \geq s-1$. If $A' = \{1\}$, then there is a subset $X \subseteq V_1$ of order $s-1$, and therefore the vertex u_X does not satisfy the conflict-free condition, a contradiction. Hence, without loss of generality, we may assume that $2 \in A'$. By the minimality of A', we have $s-1 = \sum_{i \in A'} |V_i| - p$ for some integer $p \leq |V_2| - 1$. If $p \leq |V_2| - 2$, then let V_2' be a set of arbitrarily chosen $|V_2| - p$ vertices of V_2, and let $X' = V_2' \cup \bigcup_{i \in A' \setminus \{2\}} |V_i|$. Then we have $|X'| = s-1$, and thus $u_{X'}$ does not satisfy the conflict-free condition, a contradiction. Otherwise, let V_1'' be a set of arbitrarily chosen $|V_1| - 1$ vertices of V_1, let V_2'' be a set of arbitrarily chosen 2 colors of V_2, and let $X'' = V_1'' \cup V_2'' \cup \bigcup_{i \in A' \setminus \{1,2\}} V_i$. Then we have $|X'| = s-1$, $|V_1''| \geq 3-1 = 2$ and $|V_2''| = 2$, which implies that $u_{X''}$ does not satisfy the conflict-free condition, a contradiction. Hence, we have $\chi_{\text{pcf}}(G_{s,t}) \geq \frac{t(s-1)}{2}$ for every $s \geq 3$ and $t \geq 2$. $\square$

Note that if $s \geq 3$ is odd, then the lower bound in Proposition 4.4 can be improved since $\chi_{\text{pcf}}(G_{s,t}) \geq \chi_o(G_{s,t}) > (s-1)(t-1)$ by Proposition 4.3. When $s \geq 3$ is even, it is not hard to show that $\chi_{\text{pcf}}(G_{s,t}) \leq \left\lceil \frac{t(s-1)}{2} \right\rceil$.

Now we prove Theorem 4.2. By Propositions 4.3 and 4.4, for any integers k and l such that $3 \leq k < l$ and k is odd, we have $\chi(G_{k,t}) = k$ and $\chi_o(G_{k,t}) \geq l$ where $t = \left\lceil \frac{2l}{k-1} \right\rceil$. Similarly, for any integers k and l such that $3 \leq k \leq l$ and k is even, we have $\chi_o(G_{k,t}) = k$ and $\chi_{\text{pcf}}(G_{k,t}) \geq l$ where $t = \left\lceil \frac{2l}{k-1} \right\rceil$.

Chapter 5

Odd coloring of planar graphs and outerplanar graphs

Motivated by the Four-Color Theorem and related studies, odd coloring of planar graphs are actively studied. It was conjectured that every planar graph is odd 5-colorable, and the conjecture is still open. For outerplanar graphs, it is known that every outerplanar graph is odd 5-colorable, hence the above conjecture holds for outerplanar graphs. In this chapter, we focus on the odd coloring of planar graphs and outerplanar graphs. We see studies on the odd coloring of planar graphs and related classes of graphs in Section 5.1. In Section 5.4, we give a proof of Theorem 5.10, which gives a complete characterization of odd 4-colorable outerplanar graphs. Sections 5.2 and 5.3 are for the lemmas used in a proof of Theorem 5.10. All the new results in this chapter are based on [42].

5.1 Studies on odd coloring of planar graphs and outerplanar graphs

Petruševski and Škrekovski posed the following conjecture on the odd coloring of planar graphs in [55].

Conjecture 5.1 (Petruševski and Škrekovski [55]). Every planar graph is odd 5-colorable.

Since there are planar graphs with the odd chromatic number 5 such as the 5-cycle and the octahedron (Figure 5.1), the bound 5 in Conjecture 5.1 is best possible. The first result on the odd chromatic number of planar graphs is the following, which was proved in Petruševski and Škrekovski [55].

Theorem 5.2 (Petruševski and Škrekovski [55]). Every planar graph is odd 9-colorable.

Their proof of Theorem 5.2 uses a discharging method, which is one of the basic tools in the problem of graph coloring. Soon after that, Petr and Portier improved the bound to 8 in [54].

Theorem 5.3 (Petr and Portier [54]). Every planar graph is odd 8-colorable.

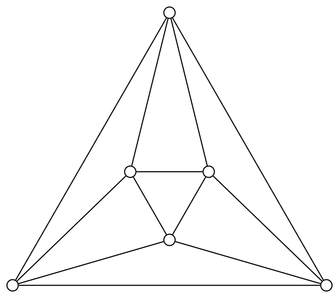


Figure 5.1: Octahedron.

Their proof is based on a result of partitioning a given planar graph into trees with specified properties, which was shown in Aashtab, Akbari, and Ghanbari [1]. Note that the result of Aashtab, Akbari, and Ghanbari [1] was proved by using the Four-Color Theorem. It was shown in Fabrici, Lužar, Rindošová, and Soták [29] that every planar graph is proper conflict-free 8-colorable, which implies Theorem 5.3, but their proof also uses the Four-Color Theorem.

Another important conjecture on odd coloring of planar graphs is the following, posed in Caro, Petruševski, and Škrekovski [12]. For a graph G with at least one cycle, the *girth* of G is the length of a shortest cycle of G.

Conjecture 5.4 (Caro, Petruševski, and Škrekovski [12]). Every planar graph with girth at least 6 is odd 4-colorable.

If Conjecture 5.4 is true, then it implies the Four-Color Theorem. Indeed, let G be a planar graph, and we consider the graph $S(G)$, which is a planar graph of girth at least 6. If Conjecture 5.4 is true, then $S(G)$ is odd 4-colorable. By the relation between $\chi(G)$ and $\chi_o(S(G))$ that we mentioned in Chapter 4, we have $\chi(G) \leq \chi_o(S(G)) \leq 4$, and thus every planar graph is 4-colorable. Towards Conjecture 5.4, the odd coloring of planar graphs with large girth has been studied in literature [4, 16, 51, 64].

Theorem 5.5 (Cho, Choi, Kwon, and Park [16], Anderson, Chau, Cho, Crawford, Hartke, Heath, Hendershedt, Kwon, and Zhang [4]). The followings are true.

- Every planar graph with girth at least 5 is odd 6-colorable. [16]
- Every planar graph with girth at least 10 is odd 4-colorable. [4]
- For every integer $c \geq 5$, every planar graph with girth at least $\left\lceil \frac{4c}{c-2} \right\rceil$ is odd c-colorable. [16]

Note that Miao, Sun, Tu, and Yu showed in [51] that every triangle-free planar graph is odd 7-colorable, but Wang [63] pointed out that their proof is wrong. Odd coloring of other related classes such as 1-planar graphs in [19, 53], toroidal graphs in [59] and K_t-minor free graphs in [35] have been studied.

In this thesis, we focus on odd coloring of outerplanar graphs. Caro, Petruševski, and Škrekovski showed the following in [12].

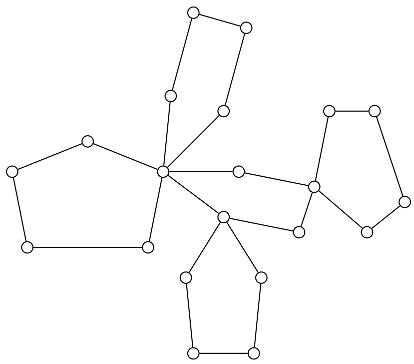


Figure 5.2: A graph in $\mathcal{G}_{C_5}$.

Theorem 5.6 (Caro, Petruševski, and Škrekovski [12])**.** Every outerplanar graph is odd 5-colorable.

They observed that there are infinitely many outerplanar graphs which are not odd 4-colorable. Indeed, let $\mathcal{G}_{C_5}$ be the family of graphs whose every block is isomorphic to the cycle C_5 (Figure 5.2). Then it is easy to show that every graph in $\mathcal{G}_{C_5}$ is an outerplanar graph, and the following proposition holds.

Proposition 5.7 (Caro, Petruševski, and Škrekovski [12])**.** Every graph $G \in \mathcal{G}_{C_5}$ is not odd 4-colorable.

Proof. Let G be a graph in $\mathcal{G}_{C_5}$. If G is disconnected, then we apply the following argument to a component of G. We may assume that G is connected. The proof goes by induction on the number of blocks of G. When G has only one block, G is isomorphic to C_5 and is not odd 4-colorable. Suppose that G has $s \geq 2$ blocks and that every graph in $\mathcal{G}_{C_5}$ with $s-1$ blocks is not odd 4-colorable. Assume to the contrary that G admits an odd 4-coloring φ. Let $B = v_1v_2v_3v_4v_5v_1$ be a leaf block of G such that v_1 is a cut vertex of G, and let $G' = G - \{v_2, v_3, v_4, v_5\}$. Note that $G' \in \mathcal{G}_{C_5}$ and G' has $s-1$ blocks. Without loss of generality, we may assume that $\varphi(v_1) = 1$. Then, we have $1 \notin \{\varphi(v_2), \varphi(v_3), \varphi(v_4), \varphi(v_5)\}$ and $|\{\varphi(v_2), \varphi(v_3), \varphi(v_4)\}| = |\{\varphi(v_3), \varphi(v_4), \varphi(v_5)\}| = 3$, which imply that $\varphi(v_2) = \varphi(v_5)$. Hence, the restriction of φ to $V(G) \setminus \{v_2, v_3, v_4, v_5\}$ is an odd 4-coloring of G', a contradiction. $\square$

On the other hand, when we consider maximal outerplanar graphs, the bound is reduced to 4 by the following result in Kashima, Maezawa, Osako, Ozeki, and Tsuchiya [39].

Theorem 5.8 (Kashima, Maezawa, Osako, Ozeki, and Tsuchiya [39])**.** For every maximal outerplanar graph G and every list assignment L of G, if $|L(v)| \geq 4$ for every vertex v of G, then G admits an odd L-coloring. In particular, every maximal outerplanar graph is odd 4-colorable.

As we see in Chapter 1, a deletion of an edge can increase the odd chromatic number of a graph. Motivated by Theorems 5.6 and 5.8, in this thesis, we completely characterize odd 4-colorable outerplanar graphs as follows.

Theorem 5.9. A connected outerplanar graph G is odd 4-colorable if and only if G has a block that is not isomorphic to C_5.

As every graph in $\mathcal{G}_{C_5}$ is odd 5-colorable, Theorem 5.9 immediately implies Theorem 5.6. Furthermore, no graph in $\mathcal{G}_{C_5}$ is a maximal outerplanar graph, and hence Theorem 5.9 partially implies Theorem 5.8 in terms of the odd chromatic number.

For the purpose of using an induction, we prove a slightly stronger statement. For a graph G and a proper k-coloring φ of G, we say that a vertex $v \in V(G)$ *satisfies the even condition with respect to* φ if there is a color $c \in [k] \setminus \{\varphi(v)\}$ such that $|\varphi^{-1}(c) \cap N_G(v)|$ is an even integer. For a graph G and a vertex v of G, we define an odd k-coloring of (G, v) as an odd k-coloring φ of G such that v satisfies the even condition with respect to φ. We show the following.

Theorem 5.10. For every outerplanar graph $G \notin \mathcal{G}_{C_5}$ and every vertex v of G, (G, v) is odd 4-colorable.

Theorem 5.9 is directly derived from Theorem 5.10 and Proposition 5.11. Although the graphs in $\mathcal{G}_{C_5}$ are not odd 4-colorable, they are almost odd 4-colorable in a sense. The following proposition is used in our proof of Theorem 5.10.

Proposition 5.11. For every graph $G \in \mathcal{G}_{C_5}$ and every vertex v of G, there is a proper 4-coloring φ of G such that every vertex other than v satisfies the odd condition and v satisfies the even condition with respect to φ.

Proof. The proof goes by induction on the number of blocks of G. If G has only one block, then let G be a cycle $v_1v_2v_3v_4v_5v_1$ of length 5. By symmetry, we may assume that $v = v_1$. Then a 4-coloring φ of G defined by $\varphi(v_1) = 1$, $\varphi(v_2) = \varphi(v_5) = 2$, $\varphi(v_3) = 3$ and $\varphi(v_4) = 4$ is a desired coloring of G.

Suppose that G has at least two blocks. Let v be a vertex of G. As G has at least two leaf blocks, there is one, denoted by $B = v_1v_2v_3v_4v_5v_1$, with a cut vertex v_1 of G such that $v \neq v_i$ for $i \in \{2, 3, 4, 5\}$. Let $G' = G - \{v_2, v_3, v_4, v_5\}$. By the induction hypothesis, there is a proper 4-coloring φ of G' such that any vertex other than v satisfies the odd condition and v satisfies the even condition. Without loss of generality, we may assume that $\varphi(v_1) = 1$. We extend φ to G by letting $\varphi(v_i) = i$ for $i \in \{2, 3, 4\}$ and $\varphi(v_5) = 2$. It is easy to check that every vertex other than v satisfies the odd condition and that the vertex v satisfies the even condition. □

Our proof of Theorem 5.10 is based on two ideas; exchanges of colors and an unavoidable set. We prove a lemma on color exchanges in Section 5.2, introduce an unavoidable set in Section 5.3, and finally give a proof of Theorem 5.10 in Section 5.4.

5.2 Color exchanging lemma

Lemma 5.12. Let G be an outerplane graph that is not a tree, and let v be a vertex of G. Let xy be an edge on the boundary of the outer face that is not a cut edge of G. If

G admits an odd 4-coloring such that v satisfies the even condition, then there exists an odd 4-coloring φ of G such that the vertex v and at least one of $\{x, y\}$ satisfy the even condition with respect to φ.

Proof. Suppose that φ_0 is an odd 4-coloring of G such that v satisfies the even condition. If $v \in \{x, y\}$, then there is nothing to prove. We assume that $v \notin \{x, y\}$. Without loss of generality, we may assume that $\varphi_0(x) = 1$ and $\varphi_0(y) = 2$. If x or y satisfies the even condition with respect to φ_0, then we are done. We assume that neither x nor y satisfies the even condition with respect to φ_0. Let z be a cut vertex of $G - xy$ that separates x and y. Let G_x be a subgraph induced by the union of $\{z\}$ and the vertices of the component of $(G - xy) - z$ containing x, and let $G_y = G - (V(G_x) \setminus \{z\})$. Depending on the color of z, we consider the following two cases.

Case 1. $\varphi_0(z) \in \{1, 2\}$.

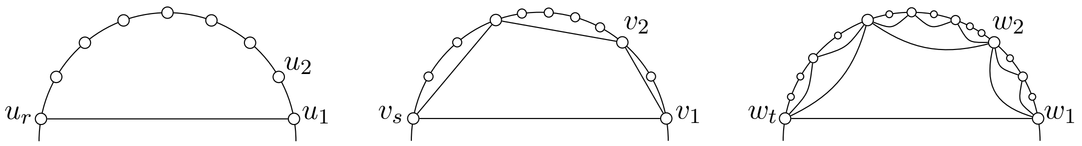
Without loss of generality, we may assume that $\varphi_0(z) = 1$. Let φ_1 be a coloring of G obtained from φ_0 by exchanging colors 2 and 3 in G_x; and φ_2 be a coloring of G obtained from φ_0 by exchanging colors 2 and 4 in G_x. Then both φ_1 and φ_2 are proper colorings of G, and the odd condition and the even condition at every vertex other than x and z are preserved in both colorings. We now consider the case $z = v$. When $d_G(z)$ is even, then z satisfies the even condition with respect to both φ_1 and φ_2. Indeed, if z does not satisfy the even condition with respect to φ_i, then $|\varphi_i^{-1}(2) \cap N_G(z)|$, $|\varphi_i^{-1}(3) \cap N_G(z)|$ and $|\varphi_i^{-1}(4) \cap N_G(z)|$ are all odd and thus $d_G(z)$ must be odd. Suppose z does not satisfy the odd condition with respect to φ_1. Then both $|\varphi_0^{-1}(2) \cap N_G(z)|$ and $|\varphi_0^{-1}(3) \cap N_G(z)|$ are odd, and they turn into even after the exchange of colors 2 and 3 in G_x. Hence, $|\varphi_2^{-1}(3) \cap N_G(z)| = |\varphi_0^{-1}(3) \cap N_G(z)|$ is odd, and thus z satisfies the odd condition with respect to φ_2. Similarly, when $d_G(z)$ is odd, z satisfies the odd condition with respect to both φ_1 and φ_2, and satisfies the even condition with respect to at least one of φ_1 and φ_2. In any case, z satisfies both the odd condition and the even condition with respect to at least one of φ_1 and φ_2. We consider the case where z satisfies the odd condition with respect to φ_1. (The other case can be easily proven with a similar argument.)

Since x does not satisfy the even condition with respect to φ_0, we conclude that $|\varphi_1^{-1}(3) \cap N_G(x)| = |\varphi_0^{-1}(2) \cap N_G(x)| - 1$ is even and $|\varphi_1^{-1}(4) \cap N_G(x)| = |\varphi_0^{-1}(4) \cap N_G(x)|$ is odd. Hence x satisfies both the odd condition and the even condition with respect to φ_1. Thus, φ_1 is a desired odd 4-coloring of G. The case $z \neq v$ can easily be proved with a similar argument.

Case 2. $\varphi_0(z) \in \{3, 4\}$.

Assume $\varphi_0(z) = 3$. Let φ_3 be a coloring of G obtained from φ_0 by exchanging colors 2 and 4 in G_x; and φ_4 be a coloring of G obtained from φ_0 by exchanging colors 1 and 4 in G_y. Similarly to Case 1, both φ_3 and φ_4 are proper colorings of G, and at least one of φ_3 and φ_4 is an odd 4-coloring of G such that the vertex v satisfies the even condition.

Since x does not satisfy the even condition with respect to φ_0, $|\varphi_3^{-1}(2) \cap N_G(x)| = |\varphi_0^{-1}(4) \cap N_G(x)| + 1$ is even and $|\varphi_3^{-1}(3) \cap N_G(x)| = |\varphi_0^{-1}(3) \cap N_G(x)|$ is odd. Hence x satisfies both the odd condition and the even condition with respect to φ_3. Similarly,



(a) An ear. (b) An ear chain. (c) An ear double chain.

Figure 5.3: An unavoidable set of 2-connected outerplanar graphs.

the vertex y satisfies both the odd condition and the even condition with respect to φ_4. Therefore, at least one of φ_3 and φ_4 is a desired odd 4-coloring of G. This completes the proof of Lemma 5.12. $\square$

5.3 An unavoidable set for 2-connected outerplanar graphs

In this section, we define an unavoidable set of 2-connected outerplanar graphs which is needed for our proof of Theorem 5.10. Let G be a 2-connected outerplanar graph which is not a cycle, and let v be a vertex of G.

- An *ear* H of G is a cycle $u_1u_2\cdots u_ru_1$ such that $d_G(u_i) = 2$ for $i \in \{2, 3, \dots, r-1\}$ (Figure 5.3a). The edge u_1u_r is the *root edge* of H. We say H is *good for* v if there exists $i \in \{1, r\}$ such that $d_G(u_i) = 3$ and $v \notin V(H) \setminus \{u_{r+1-i}\}$.
- An *ear chain* H is a subgraph induced by $\bigcup_{i=1}^{s-1} V(H_i)$ $(s \geq 3)$ where each H_i is an ear of G such that the root edges of the ears form a cycle $v_1v_2\cdots v_sv_1$ (the root edge of H_i is v_iv_{i+1}) and $d_G(v_i) = 4$ for $i \in \{2, 3, \dots, s-1\}$ (Figure 5.3b). The edge v_1v_s is the *root edge* of H. We say H is *good for* v if there exists $i \in \{1, s\}$ such that $d_G(v_i) \leq 5$ and $v \notin V(H) \setminus \{v_{s+1-i}\}$.
- An *ear double chain* H is a subgraph induced by $\bigcup_{i=1}^{t-1} V(H_i)$ $(t \geq 3)$ where each H_i is an ear chain of G such that the root edges of the ear chains form a cycle $w_1w_2\cdots w_tw_1$ (the root edge of H_i is w_iw_{i+1}) and $d_G(w_i) = 6$ for $i \in \{2, 3, \dots, t-1\}$ (Figure 5.3c). The edge w_1w_t is the *root edge* of H. We say H is *good for* v if $v \notin V(H) \setminus \{w_1, w_t\}$.

Lemma 5.13. Let G be a 2-connected outerplanar graph with at least 4 vertices. Let v be a vertex of G. If G is not a cycle, then G contains an ear, or an ear chain, or an ear double chain that is good for v.

Proof. Let F be the outercycle of G. We consider the following three cases.

Case 1. Every chord of F is the root edge of some ear.

Let T be the subgraph of G induced by the chords of F. By the assumption of this case, T is a cycle or a disjoint union of paths. Suppose first that T is a cycle $v_1v_2\cdots v_sv_1$, and let H_i be an ear whose root edge is v_iv_{i+1} for each $i \in [s]$ $(v_{s+1} = v_1)$. Without loss of generality, we may assume that $v \in V(H_s) \setminus \{v_1\}$. Then $\bigcup_{i=1}^{s-1} V(H_i)$ induces an ear chain with the root edge v_1v_s. Since $d_G(v_1) = 4$ and $v_1 \neq v$, the ear chain is good for v. Hence, we may assume that T is a disjoint union of paths. Since there are at least two leaves of T, there is an edge v_1v_2 of T such that $d_T(v_2) = 1$ and the corresponding ear H' of G satisfies $v \notin V(H') \setminus \{v_1\}$. Then $d_G(v_2) = d_T(v_2) + 2 = 3$, and thus H' is a good ear for v.

Case 2. Every chord of F is the root edge of some ear or some ear chain.

Note that if a chord xy of F is the root of an ear H, and also the root of an ear chain H', then $G = H \cup H'$, and every chord of F is the root of an ear. This falls into Case 1. Thus, we assume that each chord of F is either the root of an ear or the root of an ear chain, but not both. By Case 1, we may assume that there is a chord of F that is the root of an ear chain.

Let T (resp. T') be the subgraph of G induced by chords that are the root edges of ears (resp. ear chains). By our assumption, T' is a cycle or a disjoint union of paths. Suppose first that T' is a cycle $w_1w_2 \cdots w_tw_1$, and let H_i be an ear chain whose root edge is w_iw_{i+1} for each $i \in [t]$ ($w_{t+1} = w_1$). Without loss of generality, we may assume that $v \in V(H_t) \setminus \{w_1\}$. Then $\bigcup_{i=1}^{t-1} V(H_i)$ induces an ear double chain with the root edge w_1w_t, which is good for v.

Assume that T' is a disjoint union of paths. There is an edge w_1w_2 of T' such that $d_{T'}(w_2) = 1$ and the corresponding ear chain H' satisfies $v \notin V(H') \setminus \{w_1\}$. As $d_F(w_2) = 2$ and $d_T(w_2) \leq 2$, we have $d_G(w_2) = d_F(w_2) + d_T(w_2) + d_{T'}(w_2) \leq 5$. Hence H' is a good ear chain for v.

Case 3. There is a chord of F which is neither the root edge of an ear nor the root edge of an ear chain.

For each chord xy of F which is neither the root edge of an ear nor the root edge of an ear chain, let F' be a cycle in $F + xy$ such that $v \notin V(F') \setminus \{x, y\}$. We choose such a chord xy for which the length of F' is minimum. Let H be the subgraph of G induced by the vertices of F'. Note that H is a 2-connected outerplanar graph with the outercycle F'. By the minimality of F', for each edge $e \in E(H) \setminus E(F')$, e is either the root edge of some ear of G or the root edge of some ear chain of G, which is contained in H.

Let T_H (resp. T'_H) be the subgraph of H induced by edges which is the root edge of some ear (resp. ear chain) of H. Note that an ear or an ear chain H' of H is an ear or an ear chain of G, unless xy is an edge of H'. By definitions, each of T'_H and T_H is a cycle or a disjoint union of paths.

Assume $E(T'_H) \neq \emptyset$. If T'_H is a cycle, then there is an ear chain H' of H that contains xy. Let $x'y'$ be the root edge of H'. Then $H - (V(H') \setminus \{x', y'\})$ is an ear double chain of G that is good for v. If T'_H is an xy-path, then H is an ear double chain good for v with the root edge xy. Assume T'_H is neither a cycle nor an xy-path. Then there is an edge w_1w_2 of T'_H such that $d_{T'_H}(w_1) = 1, w_1 \notin \{x, y\}$, and w_1w_2 is the root edge of an ear chain H' that does not contain xy. As $d_{T_H}(w_1) \leq 2$ and $d_F(w_1) = 2$, we know that $d_G(w_1) \leq 5$. Hence H' is a good ear chain for v.

Assume $E(T'_H) = \emptyset$. If T_H is a cycle, then there is an ear H' of H that contains the edge xy. Let $x'y'$ be the root edge of H'. Then $H - (V(H') \setminus \{x', y'\})$ is an ear chain H'' of H with the root edge $x'y'$, contrary to $E(T'_H) = \emptyset$. Assume T_H is not a cycle. Since xy is not the root edge of an ear chain of G, T_H is not an xy-path of H. Hence, there is an edge v_1v_2 of T_H such that $d_{T_H}(v_1) = 1$, $v_1 \notin \{x, y\}$, and v_1v_2 is the root edge of an ear H' of H which does not contain xy. Then H' is an ear of G that is good for v. This completes the proof of Lemma 5.13. $\square$

5.4 Proof of Theorem 5.10

Assume that Theorem 5.10 is not true and (G, v) is a counterexample with minimum number of vertices. It is obvious that G has at least 5 vertices.

First, we consider the case where G is not 2-connected. Assume G is not 2-connected and x is a cut vertex of G. Let G_1 and G_2 be connected subgraphs of G such that $V(G_1) \cap V(G_2) = \{x\}$, $E(G_1) \cap E(G_2) = \emptyset$, and $E(G_1) \cup E(G_2) = E(G)$. Since $G \notin \mathcal{G}_{C_5}$, either G_1 or G_2 does not belong to $\mathcal{G}_{C_5}$. Without loss of generality, we may assume that $G_1 \notin \mathcal{G}_{C_5}$.

We first suppose that $v = x$. By the minimality of G, there is an odd 4-coloring φ_1 of G_1 such that v satisfies the even condition. Without loss of generality, we may assume that $\varphi_1(x) = 1$, $|\varphi_1^{-1}(2) \cap N_{G_1}(x)|$ is odd, and $|\varphi_1^{-1}(3) \cap N_{G_1}(x)|$ is even. By the minimality of G and Proposition 5.11, there is a proper 4-coloring φ_2 of G_2 such that every vertex other than x satisfies the odd condition. Without loss of generality, we may assume that $\varphi_2(x) = 1$, and $|\varphi_1^{-1}(2) \cap N_{G_2}(x)|$ and $|\varphi_1^{-1}(3) \cap N_{G_2}(x)|$ have the same parity, where the latter statement follows from the Pigeon-Hole principle. Let φ be a coloring of G such that if $u \in V(G_1)$ then $\varphi(u) = \varphi_1(u)$, otherwise $\varphi(u) = \varphi_2(u)$. Since one of $|\varphi^{-1}(2) \cap N_G(x)|$ and $|\varphi^{-1}(3) \cap N_G(x)|$ is odd and the other is even, φ is an odd 4-coloring of (G, v), a contradiction.

Suppose that $v \in V(G_1) \setminus \{x\}$. By the minimality of G, there is an odd 4-coloring φ_1 of G_1 such that v satisfies the even condition. Without loss of generality, we may assume that $\varphi_1(x) = 1$ and $|\varphi_1^{-1}(2) \cap N_{G_1}(x)|$ is odd. By the minimality of G and Proposition 5.11, there is a proper 4-coloring φ_2 of G_2 such that every vertex other than x satisfies the odd condition and the vertex x satisfies the even condition with respect to φ_2. Without loss of generality, we may assume that $\varphi_2(x) = 1$ and $|\varphi_2^{-1}(2) \cap N_{G_2}(x)|$ is even. Let φ be a coloring of G such that if $u \in V(G_1)$ then $\varphi(u) = \varphi_1(u)$, otherwise $\varphi(u) = \varphi_2(u)$. It is easy to check that φ is an odd 4-coloring of (G, v), a contradiction.

Suppose that $v \in V(G_2) \setminus \{x\}$. If $G_2 \notin \mathcal{G}_{C_5}$, then we are done by the symmetry of G_1 and G_2. We suppose that $G_2 \in \mathcal{G}_{C_5}$. As every vertex of $G_2 - x$ has an even degree, $d_G(v)$ is even, and thus v satisfies the even condition. By the minimality of G, there is an odd 4-coloring φ_1 of G_1. Without loss of generality, we may assume that $\varphi_1(x) = 1$ and $|\varphi_1^{-1}(2) \cap N_{G_1}(x)|$ is odd. By Proposition 5.11, there is a proper 4-coloring φ_2 of G_2 such that every vertex other than x satisfies the odd condition and x satisfies the even condition. Without loss of generality, we may assume that $\varphi_2(x) = 1$ and $|\varphi_2^{-1}(2) \cap N_{G_2}(x)|$ is even. Let φ be a coloring of G such that if $u \in V(G_1)$ then $\varphi(u) = \varphi_1(u)$, otherwise

$\varphi(u) = \varphi_2(u)$. It is easy to check that φ is an odd 4-coloring of (G, v), a contradiction. Therefore, G is 2-connected.

Let F be the outercycle of G. If G is a cycle, then it follows from (4.2.1) that a cycle is odd 4-colorable unless its length is equal to 5, and each vertex has one color missing at its neighbors and hence satisfies the even condition, a contradiction. Thus, F has at least one chord. By Lemma 5.13, G contains an ear, or an ear chain, or an ear double chain that is good for v. For a vertex $x \in V(G)$, we say that x satisfies the *parity condition* if x satisfies the odd condition, and in case $x = v$, then x satisfies the even condition as well. Note that one of the odd condition and the even condition for v is automatically satisfied depending on the parity of $d_G(v)$.

Lemma 5.14. Suppose that G has an ear H with the root edge u_1u_r such that $v \notin V(H) \setminus \{u_1\}$, and φ is a proper 4-coloring of a subgraph of $G - (V(H) \setminus \{u_1, u_r\})$. Then φ can be extended to a proper 4-coloring of H so that every vertex of $V(H) \setminus \{u_r\}$ satisfies the parity condition.

Proof. Let $H = u_1u_2 \cdots u_ru_1$ be an ear of G with the root edge u_1u_r. We sequentially color the vertices $u_2, u_3, \ldots, u_{r-1}$ in this order. For each $i \in \{2, 3, \ldots, r-3\}$, let $\varphi(u_i) \in [4] \setminus \{\varphi(u_{i-1})\}$ so that u_{i-1} satisfies the parity condition with respect to φ. For each $i \in \{r-2, r-1\}$, we choose $\varphi(u_i) \in [4] \setminus \{\varphi(u_{i-1}), \varphi(u_r)\}$ so that u_{i-1} satisfies the parity condition with respect to φ. Then φ is a desired coloring. □

Lemma 5.15. Suppose that G has an ear chain H with the root edge v_1v_s such that $v \notin V(H) \setminus \{v_1\}$, and φ is a proper 4-coloring of a subgraph of $G - (V(H) \setminus \{v_1, v_s\})$. Then φ can be extended to a proper 4-coloring of H so that every vertex of $V(H) \setminus \{v_s\}$ satisfies the parity condition.

Proof. Let H be an ear chain of G consisting of the sequence of ears $H_1, H_2, \ldots, H_{s-1}$, and let v_iv_{i+1} be the root edge of H_i for each $i \in [s-1]$. We first color $\{v_i \mid 2 \le i \le s-1\}$ properly as φ. By using Lemma 5.14 sequentially to ears $H_1, H_2, \ldots, H_{s-1}$ in this order, we obtain a coloring φ of H such that every vertex of $V(H) \setminus \{v_s\}$ satisfies the parity condition, as desired. □

Now we show that all of the unavoidable structures are reducible with respect to odd 4-coloring.

Case 1. G contains an ear H with the root edge u_1u_r such that $v \notin V(H) \setminus \{u_1\}$ and $d_G(u_r)$ is an odd integer. In particular, G contains an ear good for v.

Let H be an ear of G with the root edge u_1u_r such that $v \in V(H) \setminus \{u_1\}$ and $d_G(u_r)$ is an odd integer. We define a proper 4-coloring φ' of G' as follows: If G' is isomorphic to C_5, then let φ' be a proper 4-coloring of G' such that every vertex other than u_1 satisfies the odd condition. Otherwise, let φ' be an odd 4-coloring of G' such that the vertex v satisfies the even condition. Note that if G' is isomorphic to C_5, then every vertex of G' satisfies the even condition with respect to φ'. In particular, v satisfies the even condition. Let $\varphi(x) = \varphi'(x)$ for every $x \in V(G')$. By Lemma 5.14, we extend φ to a proper 4-coloring of

G such that every vertex of $V(H) \setminus \{u_r\}$ satisfies the parity condition. Since $d_G(u_r)$ is an odd integer and $u_r \neq v$, u_r satisfies the odd condition, and thus φ is an odd 4-coloring of (G, v), a contradiction.

Before we go to the cases where G contains either an ear chain or an ear double chain good for v, we consider the case where G contains an ear with more than 5 vertices.

Case 2. *G contains an ear H with the root edge u_1u_r such that $v \notin V(H) \setminus \{u_1, u_r\}$ and $|V(H)| \geq 6$.*

Let $H = u_1u_2 \cdots u_ru_1$ for some $r \geq 6$, where $d_G(u_i) = 2$ for each $i \in \{2, 3, \ldots, r-1\}$. Let $G' = G - \{u_2, u_3, \ldots, u_{r-1}\}$. If G' is isomorphic to C_5, then we have $d_G(u_r) = 3$, and thus this falls into Case 1. Hence G' is not isomorphic to C_5. By the minimality of G, there is an odd 4-coloring φ' of G' such that v satisfies the even condition. Without loss of generality, we may assume that $\varphi'(u_r) = 1$ and $|\varphi'^{-1}(2) \cap N_{G'}(u_r)|$ is odd. Let $\varphi(x) = \varphi'(x)$ for every $x \in V(G')$, and let $\varphi(u_{r-2}) = 2$. We sequentially color the vertices $u_2, u_3, \ldots, u_{r-3}, u_{r-1}$ in this order. If $r \geq 7$, then for each $i \in \{2, 3, \ldots, r-5\}$, we choose a color in $[4] \setminus \{\varphi(u_{i-1})\}$ as $\varphi(u_i)$ so that u_{i-1} satisfies the parity condition. For each $i \in \{r-4, r-3\}$, we choose a color in $[4] \setminus \{2, \varphi(u_{i-1})\}$ as $\varphi(u_i)$ so that u_{i-1} satisfies the parity condition. Finally, we let $\varphi(u_{r-1}) \in \{3, 4\}$ so that u_{r-2} satisfies the parity condition. By the choice of colors, φ is a proper 4-coloring of G, and every vertex other than u_r satisfies the parity condition with respect to φ. Furthermore, since $|\varphi^{-1}(2) \cap N_G(u_r)| = |\varphi'^{-1}(2) \cap N_{G'}(r_r)|$ is odd and $u_r \neq v$, u_r satisfies the odd condition. Thus, φ is an odd 4-coloring of (G, v), a contradiction.

By Case 2, we suppose that every ear of G without v in its internal vertices contains at most 5 vertices.

Case 3. *G contains a good ear chain H for v with the root edge v_1v_s such that $d_G(v_s) \in \{4, 5\}$ and $v \notin V(H) \setminus \{v_1\}$.*

Let $H_1, H_2, \ldots, H_{s-1}$ be the ears contained in H where the root edge of H_i is v_iv_{i+1} for each $i \in [s-1]$. Since the ear H_{s-1} satisfies $v \notin V(H_{s-1})$, the case $d_G(v_s) = 5$ falls into Case 1, a contradiction. Hence, we may assume that $d_G(v_s) = 4$. Let $H_{s-1} = v_{s-1}u_2u_3 \cdots u_{r-1}v_sv_{s-1}$. By Case 2, we know that $r \leq 5$.

Let $G' = G - (V(H) \setminus \{v_1, v_s\})$ and we define a proper 4-coloring φ' of G' as follows: If G' is isomorphic to C_5, then let φ' be a proper 4-coloring of G' such that every vertex other than v_1 satisfies the odd condition. Otherwise, let φ' be an odd 4-coloring of G' such that v satisfies the even condition. Without loss of generality, we may assume that $\varphi'(v_1) = 1$ and $\varphi'(v_s) = 2$. Let $\varphi(x) = \varphi'(x)$ for every $x \in V(G')$. We choose colors for $\{v_i \mid 2 \leq i \leq s-1\} \cup V(H_{s-1})$ as follows.

(a) If $r = 3$ and $s = 3$, then let $\varphi(u_2) = 3$ and $\varphi(v_2) = 4$.

(b) If $r = 3$ and $s \geq 4$, then let $\varphi(v_{s-1}) = 1$, $\{\varphi(u_2), \varphi(v_{s-2})\} = \{3, 4\}$ so that v_s satisfies the odd condition with respect to φ, and choose colors for $\{v_i \mid 2 \leq i \leq s-3\}$ properly.

(c) If $r = 4$ and $s = 3$, then let $\varphi(u_3) = 1$ and $\{\varphi(u_2), \varphi(v_2)\} = \{3, 4\}$ so that v_3 satisfies the odd condition with respect to φ.

(d) If $r = 4$ and $s \geq 4$, then let $\varphi(v_{s-1}) = 1$, $(\varphi(u_2), \varphi(u_3), \varphi(v_{s-2})) \in \{(4, 3, 3), (3, 4, 4)\}$ so that v_s satisfies the odd condition with respect to φ, and choose colors for $\{v_i \mid 2 \leq i \leq s-3\}$ properly.

(e) If $r = 5$ and $s = 3$, then let $\varphi(u_4) = \varphi(v_2) = 3$, $\varphi(u_2) = 4$, and $\varphi(u_3) = 1$.

(f) If $r = 5$ and $s \geq 4$, then let $\varphi(u_4) = \varphi(v_{s-1}) = 3$, $\varphi(u_2) = 1$, $\varphi(u_3) = \varphi(v_{s-2}) = 4$, and choose colors for $\{v_i \mid 2 \leq i \leq s-3\}$ properly.

In any case, every vertex in $V(H_{s-1}) \setminus \{v_{s-1}, v_s\}$ satisfies the odd condition with respect to φ. Furthermore, since $d_G(v_{s-1}) = 4$ and three colors appear in the neighborhood of v_{s-1}, v_{s-1} satisfies the odd condition with respect to φ no matter what color appears at the neighbor of v_{s-1} in $V(H_{s-2}) \setminus \{v_{s-2}\}$. In (b), (c) and (d), v_s satisfies the odd condition with respect to φ by the choice of colors. In (a), v_s satisfies the odd condition with respect to φ since $d_G(v_s) = 4$ and there are three distinct colors in the neighborhood of v_s. In (e) and (f), v_s satisfies the odd condition with respect to φ since for every color $j \neq \varphi(v_s)$, $|\varphi^{-1}(j) \cap N_G(v_s)|$ and $|\varphi^{-1}(j) \cap N_{G'}(v_s)|$ have the same parity. Applying a coloring in Lemma 5.14 sequentially to the ears $H_1, H_2, \ldots, H_{s-2}$, we extend φ to an odd 4-coloring of (G, v), a contradiction.

Note that the assumption that $d_G(v_s) = 4$ is required only in (a), which implies the following statement.

> $(*)$ Suppose that G contains an ear chain H with the root edge v_1v_s such that $v \notin V(H) \setminus \{v_1, v_s\}$. If either H contains at least 3 ears, or H contains an ear consisting of at least 4 vertices, then (G, v) is odd 4-colorable.

Case 4. G contains an ear double chain H good for v with the root edge w_1w_t such that $v \notin V(H) \setminus \{w_1, w_t\}$.

Let $H_1, H_2, \ldots, H_{t-1}$ be the ear chains contained in H where the root edge of H_i is w_iw_{i+1} for each $i \in [t-1]$. Let $G' = G - (V(H) \setminus \{w_1, w_t\})$. If G' is isomorphic to C_5, then we have $d_G(w_1) = d_G(w_t) = 5$. Thus, one of H_1 and H_{t-1} is a good ear chain for v of G, and hence we are done by Case 3. Hence, we may assume that G' is not isomorphic to C_5. By $(*)$, we may assume that $V(H_i) = \{w_i, u_i, v_i, u'_i, w_{i+1}\}$ and $E(H_i) = \{w_iw_{i+1}, w_iv_i, v_iw_{i+1}, w_iu_i, u_iv_i, v_iu'_i, u'_iw_{i+1}\}$ for every $i \in [t-1]$ (Figure 5.4).

By the minimality of G and Lemma 5.12, there is an odd 4-coloring φ' of G' such that v and at least one of w_1 and w_t satisfy the even condition with respect to φ'. By the symmetry of w_1 and w_t, we may assume that w_t satisfies the even condition. Without loss of generality, we may assume that $\varphi'(w_1) = 1$, $\varphi'(w_t) = 2$, and $|\varphi'^{-1}(j) \cap N_{G'}(w_t)|$ and $|\varphi'^{-1}(k) \cap N_{G'}(w_t)|$ have different parities for some $j, k \in \{1, 3, 4\}$. Let $\varphi(x) = \varphi'(x)$ for every $x \in V(G')$. Let $\varphi(u'_{t-1}) = 1$, $\varphi(v_{t-1}) = 3$, $\varphi(w_{t-1}) = 4$, and choose colors for $\{w_i \mid 2 \leq i \leq t-2\}$ properly. Then w_t satisfies both the odd condition and the

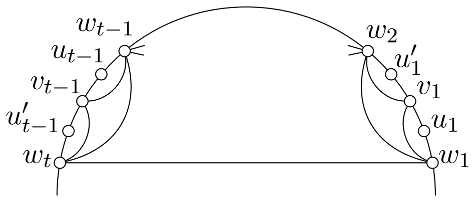


Figure 5.4: An ear double chain H in Case 4.

even condition with respect to φ since $|\varphi^{-1}(j) \cap N_G(w_t)| = |\varphi'^{-1}(j) \cap N_{G'}(w_t)| + 1$ and $|\varphi^{-1}(k) \cap N_G(w_t)| = |\varphi'^{-1}(k) \cap N_{G'}(w_t)| + 1$ have different parities. Furthermore, since $d_G(v_{t-1}) = 4$ and three colors appear in the neighborhood of v_{t-1}, v_{t-1} satisfies the odd condition with respect to φ no matter what color appears at u_{t-1}. We can extend φ to an odd 4-coloring of (G, v) by applying a coloring in Lemma 5.15 sequentially to the ear chains $H_1, H_2, \ldots, H_{t-2}$ in this order, and finally choosing $\varphi(u_{t-1}) \in \{1, 2\}$ so that w_{t-1} satisfies the parity condition with respect to φ, a contradiction. This completes the proof of Case 4 and the proof of Theorem 5.10.

Chapter 6

Odd coloring of k-trees

For a positive integer k, a k-tree is a graph obtained from the complete graph of order $k+1$ by repeatedly adding a vertex adjacent to a clique of order k. It is easy to see that every k-tree has the chromatic number exactly $k+1$ for every positive integer k, but the bound does not work for odd coloring of k-trees. In this chapter, we focus on odd coloring of k-trees. We show some examples and a simple strategy for obtaining an odd coloring of a k-tree in Section 6.1. In Section 6.2, we give a proof of Theorem 6.3, which shows an upper bound of the odd chromatic number of k-trees. In Sections 6.3 and 6.4, we give proofs of Theorems 6.4 and 6.5, which give tight upper bounds of the odd chromatic number of 2-trees and 3-trees, respectively. All the new results in this chapter are based on [40].

6.1 Introduction to odd coloring of k-trees

The notion of a k-tree was originally introduced in Beineke and Pippert [7] as an analogy of the tree to a simplicial complex with higher dimensions. For a positive integer k, a graph G is a *k-tree* if there is a sequence $G_0, G_1, \dots, G_s$ of subgraphs of G such that

- G_0 is the complete graph of order $k+1$, and $G_s = G$,
- for every $i \in [s]$, there is a vertex $v_i \in V(G_i) \setminus V(G_{i-1})$ such that $V(G_i) = V(G_{i-1}) \cup \{v_i\}$ and $N_{G_i}[v_i]$ is a clique of order $k+1$ of G_i.

By the definition, every tree of order at least 2 is a 1-tree. It is known that every maximal outerplanar graph is obtained from K_3 by repeatedly adding a triangle consisting of two end vertices of an edge of the previous graph and one new vertex; hence, every maximal outerplanar graph is a 2-tree. Note that every k-tree G of order n has an ordering of vertices $(v_1, v_2, \dots, v_n)$ such that for every $i \in \{k+1, k+2, \dots, n\}$, $N_G(v_i) \cap \{v_j \mid 1 \leq j \leq i-1\}$ is a clique of order k of G.

It is easy to see that every k-tree is k-degenerate. Indeed, for a positive integer k, let G be a k-tree of order n and let H be a subgraph of G. By the definition of a k-tree, there is a sequence $G_0, G_1, \dots, G_s = G$ of subgraphs of G with the conditions above, and

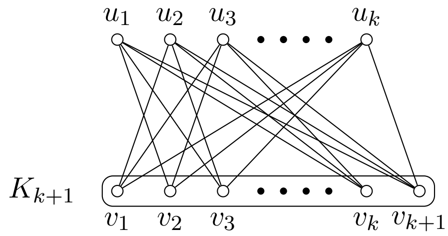


Figure 6.1: A k-tree G that is not odd $(k+1)$-colorable.

let i be the minimum index such that $H \subseteq G_i$. The vertex $v_i \in V(G_i) \setminus V(G_{i-1})$ satisfies $v_i \in V(H)$ and $d_{G_i}(v_i) = k$, which imply that $\delta(H) \leq d_H(v_i) \leq d_{G_i}(v_i) \leq k$. Thus, by Proposition 2.2 and the fact that every k-tree contains a clique of order $k+1$, we know that every k-tree G satisfies $\chi(G) = k+1$.

On the other hand, when it comes to odd coloring, $k+1$ colors for each vertex are not enough. For a positive integer k, let G be a k-tree defined by

- $V(G) = \{v_i \mid i \in [k+1]\} \cup \{u_j \mid j \in [k]\}$ and
- $E(G) = \{v_i v_{i'} \mid 1 \leq i < i' \leq k+1\} \cup \bigcup_{j=1}^{k} \{u_j v_i \mid i \in [k+1] \setminus \{j\}\}$ (Figure 6.1).

Let φ be a proper $(k+1)$-coloring of G. Without loss of generality, we may assume that $\varphi(v_i) = i$ for every $i \in [k+1]$. For each $j \in [k]$, since all colors in $[k+1] \setminus \{j\}$ appear in the neighborhood of u_j, we have $\varphi(u_j) = j$. Then, the vertex v_{k+1} has exactly two neighbors v_i and u_i with the color i for each $i \in [k]$, and hence v_{k+1} does not satisfy the odd condition with respect to φ.

An upper bound of the odd chromatic numbers of k-trees given by a simple strategy is as follows.

Proposition 6.1. For every positive integer k, every k-tree is odd $(2k+1)$-colorable.

Proof. Let k be a positive integer and let G be a k-tree of order $n \geq k+1$. The proof goes by induction on n. If $n \leq 2k+1$, then obviously G is odd $(2k+1)$-colorable. We assume $n \geq 2k+2$ and that every k-tree of order $n-1$ is odd $(2k+1)$-colorable. By the definition of a k-tree, there is a vertex v_0 of G such that $N_G[v_0]$ is a clique of order $k+1$ of G and $G' = G - v_0$ is a k-tree. Let $N_G(v_0) = \{v_1, v_2, \ldots, v_k\}$. By the induction hypothesis, there is an odd $(2k+1)$-coloring φ' of G'. Without loss of generality, we may assume that $\varphi'(v_i) = i$ for every $i \in [k]$. For each $i \in [k]$, let c_i be a color that appears at an odd number of neighbors of v_i with respect to φ'. Let $\varphi(v) = \varphi'(v)$ for every vertex $v \in V(G')$. We choose a color in $[2k+1] \setminus \bigcup_{i=1}^{k} \{\varphi(v_i), c_i\}$ as $\varphi(v_0)$ to obtain a $(2k+1)$-coloring of G. By the choice of colors, φ is a proper coloring and every vertex $v \in V(G')$ satisfies the odd condition with respect to φ. The vertex v_0 also satisfies the odd condition, and thus φ is an odd $(2k+1)$-coloring of G. □

In fact, Proposition 6.1 is an easy corollary of the following theorem in Cranston, Lafferty, and Song [19]. For two graphs G and H, we say that H is a *minor* of G if H

is obtained from G by a repetition of a deletion of a vertex, a deletion of an edge, and a contraction of an edge. A class $\mathcal{F}$ of graphs is called *minor-closed* if for every graph $G \in \mathcal{F}$, every minor of G belongs to $\mathcal{F}$ as well.

Theorem 6.2 (Cranston, Lafferty, and Song [19])**.** Let $\mathcal{F}$ be a minor-closed class of graphs such that every graph in $\mathcal{F}$ is d-degenerate for some positive integer d. Then, every graph in $\mathcal{F}$ is odd $(2d+1)$-colorable.

For a positive integer k, let $\mathcal{F}_k$ be the family consisting of all graphs with tree-width at most k. Then it is known that $\mathcal{F}_k$ is minor-closed and every graph in $\mathcal{F}_k$ is k-degenerate. As every k-tree belongs to $\mathcal{F}_k$, Theorem 6.2 implies that every k-tree is odd $(2k+1)$-colorable. We improve the bound in Proposition 6.1 as follows.

Theorem 6.3. For every positive integer k, every k-tree is odd $(k + 2\lfloor \log_2 k \rfloor + 3)$-colorable.

For a positive integer k, let $f(k)$ be the least positive integer such that every k-tree is odd $(k + f(k))$-colorable. Then, Proposition 6.1 implies $f(k) \le k+1$ and Theorem 6.3 implies $f(k) \le 2\lfloor \log_2 k \rfloor + 3$. Thus, Theorem 6.3 improves an upper bound of $f(k)$ in terms of the order of k. Since there is a k-tree G that is not odd $(k+1)$-colorable, $f(k)$ is bounded as $2 \le f(k) \le 2\lfloor \log_2 k \rfloor + 3$ for every positive integer k. For the cases $k = 2$ and $k = 3$, we show that the left inequality is attained by equalities.

Theorem 6.4. Every 2-tree is odd 4-colorable.

Theorem 6.5. Every 3-tree is odd 5-colorable.

Considering these two results, we pose the following conjecture.

Conjecture 6.6. For every positive integer k, every k-tree is odd $(k+2)$-colorable.

Before we go to proofs of theorems, we introduce some notation used in the following sections. Let k be a positive integer and let G be a k-tree of order at least $k+2$. Let V be a clique of G of order k, and let $u \in V(G)$ be a vertex such that $V \subseteq N_G(u)$. We define the *branch* $B(V,u)$ of G by $B(V,u) = G[V \cup V(G_u)]$ where G_u is the component of $G - V$ that contains u. The clique V is called the *root* of the branch $B(V,u)$ of G. Note that for any branch $B(V,u)$ of G, the graph $G - (V(B(V,u)) \setminus V)$ is either the complete graph of order k or a k-tree. The following property of a branch holds directly from the ordering of vertices of a k-tree.

Proposition 6.7. Let G be a k-tree of order at least $k+2$ for some positive integer k, and let $B = B(V,u)$ be a branch of G. Then, there is an ordering $(u_0, u_1, \ldots, u_s)$ of vertices of $V(B) \setminus V$ such that

- $u = u_0$,
- $|N_G(u_i) \cap (\{u_j \mid 0 \le j \le i-1\} \cup V)| \le k$ for each $i \in [s]$, and
- $\left|V \cap \bigcap_{j=0}^{i} N_G(u_j)\right| \ge \left|V \cap \bigcap_{j=0}^{i-1} N_G(u_j)\right| - 1$ for each $i \in [s]$.

6.2 k-trees

In this section, we give a proof of Theorem 6.3.

We fix a positive integer k and let $r = \lfloor \log_2 k \rfloor + 1$. If $k \leq 6$, then we have $k + 2\lfloor \log_2 k \rfloor + 3 \geq 2k+1$, and hence the statement follows from Proposition 6.1. Thus, we assume that $k \geq 7$, which implies that $r < k$. Let G be a k-tree of order n. The proof goes by induction on n. If $n \leq k+2r+1$, then it is obvious that G is odd $(k+2r+1)$-colorable. We assume that $n > k+2r+1$, and that every k-tree of order less than n is odd $(k+2r+1)$-colorable.

Let $B = B(V, u_0)$ be a minimal branch of G satisfying $|V(B)| \geq k+r+1$. Let $V = \{v_1, v_2, \ldots, v_k\}$. By Proposition 6.7, there is an ordering $(u_0, u_1, \ldots, u_s)$ of $V(B) \setminus V$ such that $|N_G(u_i) \cap (\{u_j \mid 0 \leq j \leq i-1\} \cup V)| \leq k$ and $\left|V \cap \bigcap_{j=0}^{i} N_G(u_j)\right| \geq \left|V \cap \bigcap_{j=0}^{i-1} N_G(u_j)\right| - 1$ for each $i \in [s]$. Note that $s \geq r$. By the minimality of B, we have $d_G(u_i) \leq k+r < 2k$ for each $i \in [s]$, which implies that every vertex of $V(B) \setminus (V \cup \{u_0\})$ satisfies the odd condition with respect to any proper coloring of G. Since $G' = G - (V(B) \setminus V)$ is a k-tree of order less than n, by the induction hypothesis, G' admits an odd $(k+2r+1)$-coloring φ'. Without loss of generality, we may assume that $\varphi'(v_i) = i$ for each $i \in [k]$. Let $\varphi(v) = \varphi'(v)$ for every vertex $v \in V(G')$. Let $U = \{u_i \mid i \in [r]\}$, and let $G'' = G[V(G') \cup U]$. We set $V_0 = V \cap \bigcap_{i=1}^{r} N_G(u_i)$, and let $\bar{V} = V \setminus V_0$. Without loss of generality, we may assume that $\bar{V} = \{v_1, v_2, \ldots, v_l\}$ for some integer $l \geq 1$.

Claim 1. There exists an injection $\sigma : \bar{V} \to U$ such that $v \notin N_G(\sigma(v))$ for every $v \in \bar{V}$. In particular, $l \leq r$.

Proof. For each $v \in \bar{V}$, let $i_v \in [r]$ be the minimum index such that $v \notin N_G(u_{i_v})$, and define $\sigma(v) = u_{i_v}$. Since $\left|V \cap \bigcap_{j=0}^{i} N_G(u_j)\right| \geq \left|V \cap \bigcap_{j=0}^{i-1} N_G(u_j)\right| - 1$ for each $i \in [s]$, σ is an injection from $\bar{V}$ to U, as desired. $\square$

Let σ be an injection in Claim 1, and let $A = \{i \in [r] \mid \sigma(v_j) = u_i \text{ for some } j \in [l]\}$. For each $i \in [r]$, we define a color c_i by $c_i = j$ if $u_i = \sigma(v_j)$ for some $j \in [l]$, and $c_i = k+i$ otherwise. Note that $c_i \neq c_{i'}$ for each pair of distinct $i, i' \in [r]$, and

$$C := \{c_i \mid i \in [r]\} = [l] \cup \{k+i \mid i \in [r] \setminus A\}.$$

For $i = 1, 2, \ldots, r$, we sequentially define a color $\varphi(u_i)$ and a set $V_i \subseteq V_0$ as follows. Suppose that V_{i-1} has been defined, and let

$$\begin{aligned} V_{i-1}^o &= \{v \in V_{i-1} \mid |\varphi^{-1}(c_i) \cap N_{G'}(v)| \text{ is odd}\} \text{ and} \\ V_{i-1}^e &= \{v \in V_{i-1} \mid |\varphi^{-1}(c_i) \cap N_{G'}(v)| \text{ is even}\}. \end{aligned}$$

We define V_i and $\varphi(u_i)$ as $V_i = V_{i-1}^o$ and $\varphi(u_i) = c_i$ if $|V_{i-1}^o| \leq |V_{i-1}^e|$, and $V_i = V_{i-1}^e$ and $\varphi(u_i) = k+r+i$ otherwise. By the definition, it is easy to verify that the resulting coloring of G'' is proper.

Claim 2. For every vertex $v \in V_0$, there is a color $c(v) \in C$ such that $|\varphi^{-1}(c(v)) \cap N_{G''}(v)|$ is odd.

Proof. By the definition of V_i, we have $|V_i| \leq \left\lfloor \frac{|V_{i-1}|}{2} \right\rfloor$ for each $i \in [r]$. Since $|V_0| \leq |V| = k$ and $r = \lfloor \log_2 k \rfloor + 1 > \log_2 k$, it follows that $|V_r| = 0$. Thus, for every vertex $v \in V_0$, there is an index $i(v) \in [r]$ such that $v \in V_{i(v)-1} \setminus V_{i(v)}$. Then, by the definition of $\varphi(u_{i(v)})$, $|\varphi^{-1}(c_{i(v)}) \cap N_{G''}(v)|$ is odd, as desired. $\square$

Now we color the remaining vertices. Since $\{\varphi(x) \mid x \in N_G(u_0) \cap V(G'')\} \cup C \subseteq [k+2r] \setminus \{k+i \mid i \in A\}$, we have

$$\begin{aligned}&|[k+2r+1] \setminus (\{\varphi(x) \mid x \in N_G(u_0) \cap V(G'')\} \cup C)| \\ &\geq (k+2r+1) - (k+2r-|A|) = |A| + 1 = l + 1 > |\bar{V}|.\end{aligned}$$

Thus, we choose a color in $[k+2r+1] \setminus (\{\varphi(x) \mid x \in N_G(u_0) \cap V(G'')\} \cup C)$ as $\varphi(u_0)$ so that every vertex of $\bar{V}$ satisfies the odd condition. We sequentially choose colors for $u_{r+1}, u_{r+2}, \ldots, u_s$ as follows. For each $i \in \{r+1, r+2, \ldots, s\}$, let $X_i = N_G(u_i) \cap (\{u_j \mid 0 \leq j \leq i-1\} \cup V)$. Since $|X_i| \leq k$, we have

$$|[k+2r+1] \setminus (\{\varphi(x) \mid x \in X_i\} \cup C)| \geq (k+2r+1) - k - r = r + 1 > |\bar{V}|.$$

Furthermore, if $N_G(u_i) \cap \bar{V} \neq \emptyset$, then we have $\{\varphi(x) \mid x \in X_i\} \cap C \neq \emptyset$ and thus

$$|[k+2r+1] \setminus (\{\varphi(x) \mid x \in X_i\} \cup C)| \geq (k+2r+1) - k - r + 1 = r + 2 > |\bar{V}| + 1.$$

Hence, we can choose a color $\varphi(u_i) \in [k+2r+1] \setminus (\{\varphi(x) \mid x \in X_i\} \cup C)$ so that every vertex in $N_G(u_i) \cap (\bar{V} \cup \{u_0\})$ satisfies the odd condition. It is easy to see that φ is a proper $(k+2r+1)$-coloring of G, and every vertex in $V(G')$ satisfies the odd condition with respect to φ. By the choice of colors for $\{u_0, u_{r+1}, u_{r+2}, \ldots, u_s\}$, each vertex of $\bar{V} \cup \{u_0\}$ satisfies the odd condition with respect to φ. Furthermore, for each $v \in V_0$, since $|\varphi^{-1}(c(v)) \cap N_{G''}(v)|$ is odd for some color $c(v) \in C$ by Claim 2 and no vertex of $V(G) \setminus V(G'')$ receives a color in C, v satisfies the odd condition with respect to φ. Thus, φ is an odd $(k+2r+1)$-coloring of G, which completes the proof of Theorem 6.3.

6.3 2-trees

In this section, we give a proof of Theorem 6.4.

6.3.1 Some special branches

We define some special branches used in our proof. Let G be a 2-tree.

- An *ear* of G is a branch $B(\{v_1, v_2\}, u_0)$ of G with 3 vertices $\{v_1, v_2, u_0\}$ such that $N_G(u_0) = \{v_1, v_2\}$.

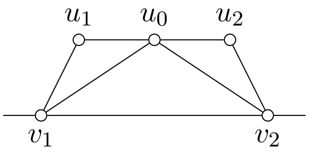


Figure 6.2: A hat of a 2-tree.

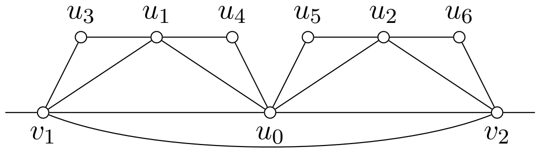


Figure 6.3: A double hat of a 2-tree.

- A *hat* of G is a branch $B(\{v_1, v_2\}, u_0)$ of G with 5 vertices $\{v_1, v_2, u_0, u_1, u_2\}$ such that $N_G(u_0) = \{v_1, v_2, u_1, u_2\}$, $N_G(u_1) = \{v_1, u_0\}$ and $N_G(u_2) = \{v_2, u_0\}$ (Figure 6.2). A hat with the root $\{v_1, v_2\}$ is *good* if $d_G(v_1)$ is equal to 4 or an odd integer.

- A *double hat* of G is a branch $B(\{v_1, v_2\}, u_0)$ of G with 7 vertices $\{v_1, v_2\} \cup \{u_i \mid i \in [0, 6]\}$ such that $N_G(u_0) \cap N_G(v_1) = \{v_2, u_1\}$, $N_G(u_0) \cap N_G(v_2) = \{v_1, u_2\}$, $N_G(u_1) = \{v_1, v_2, u_2, u_3, u_5, u_6\}$, $N_G(u_2) = \{v_1, u_1, u_4, u_5\}$, $N_G(u_3) = \{v_2, u_1, u_6, u_7\}$, $N_G(u_4) = \{v_1, u_2\}$, $N_G(u_5) = \{u_1, u_2\}$, $N_G(u_6) = \{u_1, u_3\}$ and $N_G(u_7) = \{v_2, u_3\}$ (Figure 6.3).

For a 2-tree G, we define two sets $\mathcal{H}^{(2)}(G)$ and $\mathcal{T}^{(2)}(G)$ of subgraphs as follows. In the following definition, we use the notation v_3 as v_1. For a tuple (a, b, c) of non-negative integers with $1 \leq a + b + c \leq 2$, $\mathcal{H}^{(2)}_{a,b,c}(G)$ is the set of branches $B = B(\{v_1, v_2\}, u_0)$ such that

- for each $i \in \{1, 2\}$, $|N_G(u_0) \cap N_G(v_i)| \leq 2$, and if the equality holds, then there is a vertex u_i such that $N_G(u_0) \cap N_G(v_i) = \{v_{i+1}, u_i\}$ and $B(\{v_i, u_0\}, u_i)$ is either an ear, a hat or a double hat of G where $v_3 = v_1$, and

- $\{B(\{v_1, u_0\}, u_1), B(\{v_2, u_0\}, u_2)\}$ consists of a ears, b hats and c double hats.

For a tuple (a, b, c) of non-negative integers with $a + b + c = 2$, $\mathcal{T}^{(2)}_{a,b,c}(G)$ is the set of subgraphs T induced by $V(B(\{v_1, v_2\}, u_0)) \cup V(B(\{v_1, v_2\}, w_0))$ such that

- each of $B(\{v_1, v_2\}, u_0)$ and $B(\{v_1, v_2\}, w_0)$ is either an ear, a hat or a double hat of G, and

- $\{B(\{v_1, v_2\}, u_0), B(\{v_1, v_2\}, w_0)\}$ consists of a ears, b hats and c double hats.

For a graph $T \in \mathcal{T}^{(2)}_{a,b,c}(G)$ induced by $V(B(\{v_1, v_2\}, u_0)) \cup V(B(\{v_1, v_2\}, w_0))$, the set of vertices $\{v_1, v_2\}$ is called the *root* of T. Let $\mathcal{H}^{(2)}(G) = \bigcup_{1 \leq a+b+c \leq 2} \mathcal{H}^{(2)}_{a,b,c}(G)$ and let $\mathcal{T}^{(2)}(G) = \bigcup_{a+b+c=2} \mathcal{T}^{(2)}_{a,b,c}(G)$. Note that a hat of G belongs to $\mathcal{H}^{(2)}_{2,0,0}(G)$, and a double hat of G belongs to $\mathcal{H}^{(2)}_{0,2,0}(G)$.

The following lemma holds.

Lemma 6.8. Let G be a 2-tree with at least 4 vertices. Then there is a subgraph $H \in \mathcal{H}^{(2)}(G) \cup \mathcal{T}^{(2)}(G)$ that is neither a hat nor a double hat of G.

(a) The case where $V(G) \setminus (V_0 \cup V_1) = \emptyset$. It follows that $B(\{u,v\},w) \in \mathcal{H}^{(2)}_{0,1,0}(G)$.

(b) The case where $V(G)\setminus(V_0 \cup V_1 \cup V_2) = \emptyset$. It follows that $B(\{u',v'\},w') \in \mathcal{H}^{(2)}_{1,0,1}(G)$.

Figure 6.4: Special graphs in the proof of Lemma 6.8.

Proof. Let V_0 be the set of vertices of degree 2 of G. For $i = 1,2,3$, we define a set of vertices V_i as follows: If $|V(G) \setminus \bigcup_{j=0}^{i-1} V_j| \geq 3$, then let V_i be the set of vertices of $G - \bigcup_{j=0}^{i-1} V_j$ whose degrees in $G - \bigcup_{j=0}^{i-1} V_j$ are equal to 2. If $|V(G) \setminus \bigcup_{j=0}^{i-1} V_j| = 2$, then let $V_i = V(G) \setminus \bigcup_{j=0}^{i-1} V_j$. Otherwise, let $V_i = \emptyset$. By the definition, V_0, V_1, V_2, and V_3 are pairwise disjoint.

The assumption $|V(G)| \geq 4$ forces $V_1 \neq \emptyset$. If $|V(G) \setminus V_0| = 2$, then the two vertices in $V(G) \setminus V_0$ form a root of a subgraph in $\mathcal{T}^{(2)}_{2,0,0}(G)$. Thus, we assume that $|V(G) \setminus V_0| \geq 3$. Let z_1 be a vertex in V_1 and let $\{x_1, y_1\} = N_G(z_1) \setminus V_0$ such that $|V_0 \cap N_G(z_1) \cap N_G(x_1)| \geq |V_0 \cap N_G(z_1) \cap N_G(y_1)|$. If there are distinct vertices $u, w \in V_0$ with $N_G(u) = N_G(w) = \{z_1, x_1\}$, then $B(\{z_1,x_1\},u) \cup B(\{z_1,x_1\},w) \in \mathcal{T}^{(2)}_{2,0,0}(G)$. Thus, we may assume that $|V_0 \cap N_G(z_1) \cap N_G(x_1)| = 1$, which implies that $B(\{x_1,y_1\},z_1) \in \mathcal{H}^{(2)}_{1,0,0}(G) \cup \mathcal{H}^{(2)}_{2,0,0}(G)$. If $B(\{x_1,y_1\},z_1)$ is not a hat of G, then we are done. Thus, since the choice of $z_1 \in V_1$ is arbitrary, we may assume that every vertex of V_1 is contained in a hat of G.

Now we consider V_2. If $V(G) \setminus (V_0 \cup V_1) = \emptyset$, then G is isomorphic to a graph in Figure 6.4a, and hence $\mathcal{H}^{(2)}_{0,1,0}(G) \neq \emptyset$. Suppose that $V_2 \neq \emptyset$. If $|V(G)\setminus(V_0 \cup V_1)| = 2$, then the two vertices in $V(G)\setminus(V_0 \cup V_1)$ form a root of a subgraph in $\mathcal{T}^{(2)}_{0,2,0}(G)$. Thus, we assume that $|V(G) \setminus (V_0 \cup V_1)| \geq 3$. Let z_2 be a vertex of V_2, and let $\{x_2, y_2\} = N_G(z_2) \setminus (V_0 \cup V_1)$ such that $|(V_0 \cup V_1) \cap N_G(z_2) \cap N_G(x_2)| \geq |(V_0 \cup V_1) \cap N_G(z_2) \cap N_G(x_2)|$. If there are distinct vertices $u, w \in V_0 \cup V_1$ such that $N_G(u) \cap (V(G) \setminus (V_0 \cup V_1)) = N_G(w) \cap (V(G) \setminus (V_0 \cup V_1)) = \{z_2, x_2\}$, then we are done since $B(\{z_2,x_2\},u) \cup B(\{z_2,x_2\},w) \in \mathcal{T}^{(2)}(G)$. Thus, we may assume that $|(V_0 \cup V_1) \cap N_G(z_2) \cap N_G(x_2)| = 1$, which implies that $B(\{x_2,y_2\},z_2) \in \mathcal{H}^{(2)}_{0,1,0}(G) \cup \mathcal{H}^{(2)}_{0,2,0}(G) \cup \mathcal{H}^{(2)}_{1,1,0}(G)$. If $B(\{x_1,y_1\},z_1)$ is not a double hat of G, then we are done. Since the choice of $z_2 \in V_2$ is arbitrary, we may assume that every vertex of V_2 is contained in a double hat of G.

Finally, we consider V_3. If $V(G) \setminus (V_0 \cup V_1 \cup V_2) = \emptyset$, then G is isomorphic to a graph in Figure 6.4b, and hence $\mathcal{H}^{(2)}_{1,0,1}(G) \neq \emptyset$. Suppose that $V_3 \neq \emptyset$. If $|V(G) \setminus (V_0 \cup V_1 \cup V_2)| = 2$, then the two vertices in $V(G) \setminus (V_0 \cup V_1 \cup V_2)$ form a root of a subgraph in $\mathcal{T}^{(2)}_{0,0,2}(G)$. Thus, we assume that $|V(G) \setminus (V_0 \cup V_1 \cup V_2)| \geq 3$. Let z_3 be a vertex of V_3, and let

$\{x_3, y_3\} = N_G(z_3) \setminus (V_0 \cup V_1 \cup V_2)$ such that $|(V_0 \cup V_1 \cup V_2) \cap N_G(z_3) \cap N_G(x_3)| \geq |(V_0 \cup V_1 \cup V_2) \cap N_G(z_3) \cap N_G(x_3)|$. If there are distinct vertices $u, w \in V_0 \cup V_1 \cup V_2$ such that $N_G(u) \cap (V(G) \setminus (V_0 \cup V_1 \cup V_2)) = N_G(w) \cap (V(G) \setminus (V_0 \cup V_1 \cup V_2)) = \{z_3, x_3\}$, then we are done since $B(\{z_3, x_3\}, u) \cup B(\{z_3, x_3\}, w) \in \mathcal{T}^{(2)}(G)$. Thus, we may assume that $|(V_0 \cup V_1 \cup V_2) \cap N_G(z_3) \cap N_G(x_3)| = 1$, which implies that $B(\{x_3, y_3\}, z_3) \in \mathcal{H}^{(2)}_{0,0,1}(G) \cup \mathcal{H}^{(2)}_{0,0,2}(G) \cup \mathcal{H}^{(2)}_{0,1,1}(G) \cup \mathcal{H}^{(2)}_{1,0,1}(G)$. This completes the proof of Lemma 6.8. $\square$

The following lemma is used repeatedly in our proof of Theorem 6.4.

Lemma 6.9. Let G be a 2-tree and suppose that G has a branch $B = B(\{v_1, v_2\}, u_0)$ which is either an ear, a hat, or a double hat. Let G' be a subgraph of $G - (V(B) \setminus \{v_1, v_2\})$ such that $v_1, v_2 \in V(G')$, and assume that G' admits a proper 4-coloring φ'. Then φ' can be extended to a proper 4-coloring of $G' \cup B$ such that every vertex of $V(B) \setminus \{v_1\}$ satisfies the odd condition.

Proof. Let φ' be an odd 4-coloring of G'. Without loss of generality, we may assume that $\varphi'(v_1) = 1$ and $\varphi'(v_2) = 2$. For each vertex $v \in V(G')$, let $\varphi(v) = \varphi'(v)$. If B is an ear, then we choose $\varphi(u_0) \in \{3, 4\}$ so that v_2 satisfies the odd condition. If B is a hat, then let $V(B) = \{v_1, v_2, u_0, u_1, u_2\}$ such that $N_G(u_0) = \{v_1, v_2, u_1, u_2\}$, $N_G(u_1) = \{v_1, u_0\}$ and $N_G(u_2) = \{v_2, u_0\}$. We let $\varphi(u_0) = 3$, choose $\varphi(u_2) \in \{1, 4\}$ so that v_2 satisfies the odd condition, and choose $\varphi(u_1) \in \{2, 4\}$ so that u_0 satisfies the odd condition. If B is a double hat, then let $V(B) = \{v_1, v_2\} \cup \{u_i \mid i \in [0, 6]\}$ such that $N_G(u_0) = \{v_1, v_2, u_1, u_2, u_4, u_5\}$, $N_G(u_1) = \{v_1, u_0, u_3, u_4\}$, $N_G(u_2) = \{v_2, u_0, u_5, u_6\}$, $N_G(u_3) = \{v_1, u_1\}$, $N_G(u_4) = \{u_0, u_1\}$, $N_G(u_5) = \{u_0, u_2\}$, and $N_G(v_2, u_2)$. We first let $\varphi(u_0) = 3$ and $\varphi(u_1) = \varphi(u_2) = 4$. Then we choose $\varphi(u_6) \in \{1, 3\}$ so that v_2 satisfies the odd condition, choose $\varphi(u_4) \in \{1, 2\}$ so that u_2 satisfies the odd condition, choose $\varphi(u_3)$ so that $\varphi(u_0)$ satisfies the odd condition, and choose $\varphi(u_3) \in \{2, 3\}$ so that u_1 satisfies the odd condition. For each case, it is easy to verify that φ is a desired coloring of $G' \cup B$. $\square$

6.3.2 Proof of Theorem 6.4

Let G be a 2-tree of order n. The proof goes by induction on n. If $n \leq 4$, then trivially G is odd 4-colorable. Hence, we assume that $n \geq 5$ and every 2-tree of order less than n is odd 4-colorable. Note that every vertex of G of degree 2 satisfies the odd condition with respect to any proper coloring of G since the two neighbors are adjacent.

By Lemma 6.8, there is a subgraph $H \in \mathcal{H}^{(2)}(G) \cup \mathcal{T}^{(2)}(G)$ which is neither a hat nor a double hat of G. We show that H is a reducible structure with respect to odd 4-coloring.

Case 1. G has a good hat H'.

Let H' be a hat with 5 vertices $\{v_1, v_2, u_0, u_1, u_2\}$ such that $\{v_1, v_2\}$ is the root of H', $N_G(u_0) = \{v_1, v_2, u_1, u_2\}$, $N_G(u_1) = \{v_1, u_0\}$, $N_G(u_2) = \{v_2, u_0\}$, and $d_G(v_1)$ is equal to 4 or an odd integer. Let $G' = G - \{u_0, u_1, u_2\}$. By the induction hypothesis, G' admits an odd 4-coloring φ. Without loss of generality, we may assume that $\varphi(v_1) = 1$ and $\varphi(v_2) = 2$. If $d_G(v_1)$ is odd, then we may extend φ to a proper 4-coloring of G such that

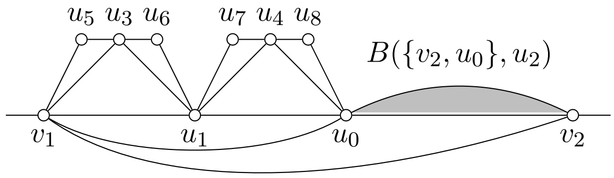


Figure 6.5: Case 3

every vertex other than v_1 satisfies the odd condition by Lemma 6.9, and v_1 satisfies the odd condition as well. Hence, we assume that $d_G(v_1) = 4$. We let $\varphi(u_0) = 3$, $\varphi(u_1) = 4$, and choose a color in $\{1, 4\}$ as $\varphi(u_2)$ so that v_2 satisfies the odd condition. By the choice of colors, every vertex other than $\{v_1, u_0\}$ satisfies the odd condition. Since $d_G(v_1) = 4$ and the three vertices $v_2, u_0, u_1 \in N_G(v_1)$ receive three distinct colors, v_1 satisfies the odd condition with respect to φ. Similarly, since $d_G(u_0) = 4$ and $v_1, v_2, u_1 \in N_G(u_0)$, u_0 satisfies the odd condition with respect to φ. Thus, φ is an odd 4-coloring of G, as desired.

If $H \in \mathcal{H}^{(2)}_{0,1,0}(G) \cup \mathcal{H}^{(2)}_{1,1,0}(G) \cup \mathcal{H}^{(2)}_{0,0,1}(G) \cup \mathcal{H}^{(2)}_{0,1,1}(G)$, then H contains a good hat of G. Thus, by Case 1, we may assume that $H \notin \mathcal{H}^{(2)}_{0,1,0}(G) \cup \mathcal{H}^{(2)}_{1,1,0}(G) \cup \mathcal{H}^{(2)}_{0,0,1}(G) \cup \mathcal{H}^{(2)}_{0,1,1}(G)$.

In each of the following cases, let $G' = G - (V(H) \setminus \{v_1, v_2\})$, where $\{v_1, v_2\}$ is the root of H. By the induction hypothesis, G' admits an odd 4-coloring φ'. Without loss of generality, we may assume that $\varphi'(v_1) = 1$ and $\varphi'(v_2) = 2$. Let $\varphi(x) = \varphi'(x)$ for every $x \in V(G')$, and we extend φ to G as in the following five cases.

Case 2. $H \in \mathcal{H}^{(2)}_{1,0,0}(G)$.

Let $H = G[\{v_1, v_2, u_0, u_1\}]$ such that $N_G(u_0) = \{v_1, v_2, u_1\}$ and $N_G(u_1) = \{v_1, u_0\}$. We choose $\varphi(u_0) \in \{3, 4\}$ so that v_2 satisfies the odd condition, and then choose $\varphi(u_1) \in \{2, 3, 4\} \setminus \{\varphi(u_0)\}$ so that v_1 satisfies the odd condition. It is easy to verify that φ is an odd 4-coloring of G.

Case 3. $H \in \mathcal{H}^{(2)}_{1,0,1}(G) \cup \mathcal{H}^{(2)}_{0,0,2}(G)$.

Let $H = B(\{v_1, v_2\}, u_0)$ such that $B(\{v_1, u_0\}, u_1)$ is a double hat and $B(\{v_2, u_0\}, u_2)$ is either an ear or a double hat (Figure 6.5). Let $V(B(\{v_1, u_0\}, u_1)) = \{v_1, u_0, u_1, u_3, u_4, u_5, u_6, u_7, u_8\}$ such that $N_G(u_1) = \{v_1, u_0, u_3, u_4, u_6, u_7\}$, $N_G(u_3) = \{v_1, u_1, u_5, u_6\}$, and $N_G(u_4) = \{u_0, u_1, u_7, u_8\}$. Let $G' = G - V(H) \setminus \{v_1, v_2\}$. We first let $\varphi(u_1) = 2$, $\varphi(u_3) = 3$, $\varphi(u_5) = 4$, and choose $\varphi(u_0) \in \{3, 4\}$ so that v_1 satisfies the odd condition. Then we sequentially use Lemma 6.9 to $B(\{u_0, v_2\}, u_2)$, $B(\{u_1, u_0\}, u_4)$, and $B(\{u_3, u_1\}, u_6)$ to obtain a proper 4-coloring of G such that every vertex other than u_3 satisfies the odd condition. Since $d_G(u_3) = 4$ and three colors appear in the neighborhood of u_3, u_3 also satisfies the odd condition. Thus, φ is an odd 4-coloring of G.

Case 4. $H \in \mathcal{T}^{(2)}_{2,0,0}(G)$.

Let $V(H) = \{v_1, v_2, u_0, w_0\}$ such that $N_G(u_0) = N_G(w_0) = \{v_1, v_2\}$. By setting $\varphi(u_0) = \varphi(w_0) = 3$, the parity of the number of neighbors with each color around v_1 and v_2 does not change. Thus, φ is an odd 4-coloring of G.

Case 5. $H \in \mathcal{T}^{(2)}_{a,b,c}(G)$ for some tuple (a, b, c) with $a + b + c = 2$ and $b \geq 1$.

Let $B_1 = B(\{v_1, v_2\}, u_0)$ and $B_2 = B(\{v_1, v_2\}, w_0)$ be branches such that $V(H) = V(B_1) \cup V(B_2)$ and B_1 is a hat of G. Let $V(B_1) = \{v_1, v_2, u_0, u_1, u_2\}$ such that $N_G(u_0) = \{v_1, v_2, u_1, u_2\}$, $N_G(u_1) = \{v_1, u_0\}$, and $N_G(u_2) = \{v_2, u_0\}$. We first let $\varphi(u_0) = 3$ and $\varphi(u_1) = 4$. By using Lemma 6.9, we choose colors for $V(B_2) \setminus \{v_1, v_2\}$ so that every vertex of $V(B_2) \setminus \{v_2\}$ satisfies the odd condition, and choose $\varphi(u_2) \in \{1, 4\}$ so that v_2 satisfies the odd condition. By the choice of colors, every vertex other than u_0 satisfies the odd condition. Since $d_G(u_0) = 4$ and three colors appear in the neighborhood of u_0, u_0 satisfies the odd condition and thus φ is an odd 4-coloring of G.

Case 6. $H \in \mathcal{T}^{(2)}_{a,b,c}(G)$ for some tuple (a, b, c) with $a + b + c = 2$ and $c \geq 1$.

Let $B_1 = B(\{v_1, v_2\}, u_0)$ and $B_2 = B(\{v_1, v_2\}, w_0)$ be branches such that $V(H) = V(B_1) \cup V(B_2)$ and B_1 is a double hat of G. Let $V(B_1) = \{v_1, v_2\} \cup \{u_i \mid i \in [0, 6]\}$ such that $N_G(u_0) = \{v_1, v_2, u_1, u_2, u_4, u_5\}$, $N_G(u_1) = \{v_1, u_0, u_3, u_4\}$, $N_G(u_2) = \{v_2, u_0, u_5, u_6\}$. We first let $\varphi(u_0) = 3$, $\varphi(u_1) = 4$, and $\varphi(u_3) = 2$. Using Lemma 6.9 to B_2 and $B(\{u_0, v_2\}, u_2)$, we can color $V(H) \setminus \{v_1, v_2, u_4\}$ so that every vertex other than u_0 and u_1 satisfies the odd condition, and finally we choose $\varphi(u_4) \in \{1, 2\}$ so that u_0 satisfies the odd condition. Since $d_G(u_1) = 4$ and three colors appear in the neighborhood of u_1, u_1 satisfies the odd condition. Thus, φ is an odd 4-coloring of G. This completes the proof of Theorem 6.4.

6.4 3-trees

In this section, we give a proof of Theorem 6.5.

6.4.1 Some special branches

We define some special branches of a 3-tree. Let G be a 3-tree.

- An *ear* of G is a branch $B(\{v_1, v_2, v_3\}, u_0)$ of G with 4 vertices $\{v_1, v_2, v_3, u_0\}$ such that $N_G(u_0) = \{v_1, v_2, v_3\}$.
- A *one-hat* of G is a branch $B(\{v_1, v_2, v_3\}, u_0)$ of G with 5 vertices $\{v_1, v_2, v_3, u_0, u_1\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1\}$ and $N_G(u_1) = \{v_1, v_2, u_0\}$ (Figure 6.6).
- A *one-hat plus* of G is a branch $B(\{v_1, v_2, v_3\}, u_0)$ of G with 7 vertices $\{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, 3]\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, $N_G(u_1) = \{v_1, v_2, u_0, u_3\}$, $N_G(u_2) = \{v_2, v_3, u_0\}$ and $N_G(u_3) = \{v_1, u_0, u_1\}$ (Figure 6.7).

For a 3-tree G, we define two sets $\mathcal{H}^{(3)}(G)$ and $\mathcal{T}^{(3)}(G)$ of subgraphs as follows.

Let (a, b, c) be a tuple of non-negative integers with $1 \leq a + b + c \leq 3$. For a clique $\{v_1, v_2, v_3, u_0\}$ of G, the branch $B(\{v_1, v_2, v_3\}, u_0)$ belongs to $\mathcal{H}^{(3)}_{a,b,c}(G)$ if

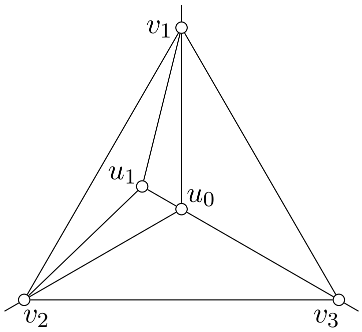


Figure 6.6: A one-hat of a 3-tree.

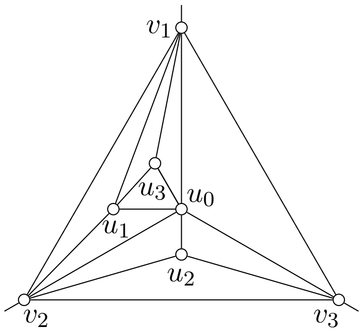


Figure 6.7: A one-hat plus of a 3-tree.

- $|N_G(u_0) \cap N_G(v_1) \cap N_G(v_2)| \leq 2$, and if the equality holds, then there is a vertex u_1 such that $N_G(u_0) \cap N_G(v_1) \cap N_G(u_2) = \{v_3, u_1\}$ and $B(\{v_1, v_2, u_0\}, u_1)$ is either an ear, a one-hat or a one-hat plus of G,

- $|N_G(u_0) \cap N_G(v_2) \cap N_G(v_3)| \leq 2$, and if the equality holds, then there is a vertex u_1 such that $N_G(u_0) \cap N_G(v_2) \cap N_G(u_3) = \{v_1, u_2\}$ and $B(\{v_2, v_3, u_0\}, u_2)$ is either an ear, a one-hat or a one-hat plus of G,

- $|N_G(u_0) \cap N_G(v_3) \cap N_G(v_1)| \leq 2$, and if the equality holds, then there is a vertex u_3 such that $N_G(u_0) \cap N_G(v_3) \cap N_G(u_1) = \{v_2, u_3\}$ and $B(\{v_3, v_1, u_0\}, u_3)$ is either an ear, a one-hat or a one-hat plus of G, and

- $\{B(\{v_1, v_2, u_0\}, u_1), B(\{v_2, v_3, u_0\}, u_2), B(\{v_3, v_1, u_0\}, u_3)\}$ consists of a ears, b one-hats and c one-hat pluses.

Let (a, b, c) be a tuple of non-negative integers with $a + b + c = 2$. For two cliques $\{v_1, v_2, v_3, u_0\}$ and $\{v_1, v_2, v_3, w_0\}$ of G sharing $\{v_1, v_2, v_3\}$, the subgraph of G induced by $V(B(\{v_1, v_2, v_3\}, u_0)) \cup V(B(\{v_1, v_2, v_3\}, w_0))$ belongs to $\mathcal{T}^{(3)}_{a,b,c}(G)$ if

- each of $B(\{v_1, v_2, v_3\}, u_0)$ and $B(\{v_1, v_2, v_3\}, w_0)$ is either an ear, a one-hat, or a one-hat plus of G, and

- $\{B(\{v_1, v_2, v_3\}, u_0), B(\{v_1, v_2, v_3\}, w_0)\}$ consists of a ears, b one-hats and c one-hat pluses.

For a graph $T \in \mathcal{T}^{(3)}_{a,b,c}(G)$ induced by $V(B(\{v_1, v_2, v_3\}, u_0)) \cup V(B(\{v_1, v_2, v_3\}, w_0))$, the set of vertices $\{v_1, v_2, v_3\}$ is called the *root* of T.

Let $\mathcal{H}^{(3)}(G) = \bigcup_{1 \leq a+b+c \leq 3} \mathcal{H}^{(3)}_{a,b,c}(G)$ and let $\mathcal{T}^{(3)}(G) = \bigcup_{a+b+c=2} \mathcal{T}^{(3)}_{a,b,c}(G)$. Note that a one-hat of G belongs to $\mathcal{H}^{(3)}_{1,0,0}(G)$ and a one-hat plus of G belongs to $\mathcal{H}^{(3)}_{1,1,0}(G)$.

The following lemma holds.

Lemma 6.10. Let G be a 3-tree with at least 5 vertices. Then there is a subgraph $H \in \mathcal{H}^{(3)}(G) \cup \mathcal{T}^{(3)}(G)$ which is neither a one-hat nor a one-hat plus of G.

Proof. Let V_0 be the set of vertices of degree 3 of G. For $i = 1, 2, 3$, we define a set of vertices V_i as follows: If $|V(G) \setminus \bigcup_{j=0}^{i-1} V_j| \geq 4$, then let V_i be the set of vertices of $G - \bigcup_{j=0}^{i-1} V_j$ whose degrees in $G - \bigcup_{j=0}^{i-1} V_j$ are equal to 3. If $|V(G) \setminus \bigcup_{j=0}^{i-1} V_j| = 3$, then let $V_i = V(G) \setminus \bigcup_{j=0}^{i-1} V_j$. Otherwise, let $V_i = \emptyset$. By the definition, V_0, V_1, V_2, and V_3 are pairwise disjoint.

The assumption $|V(G)| \geq 5$ forces $V_1 \neq \emptyset$. If $|V(G) \setminus V_0| = 3$, then the three vertices in $V(G) \setminus V_0$ form a root of a subgraph in $\mathcal{T}^{(3)}_{2,0,0}(G)$. Thus, we assume that $|V(G) \setminus V_0| \geq 4$. Let v_1 be a vertex in V_1 and let $N_G(v_1) \setminus V_0 = \{x_1, y_1, z_1\}$ such that $|N_G(x_1) \cap N_G(y_1) \cap N_G(v_1) \cap V_0| \geq |N_G(y_1) \cap N_G(z_1) \cap N_G(v_1) \cap V_0| \geq |N_G(z_1) \cap N_G(x_1) \cap N_G(v_1) \cap V_0|$. If there are two distinct vertices $u, w \in V_0$ such that $N_G(u) = N_G(w) = \{v_1, x_1, y_1\}$, then $B(\{v_1, x_1, y_1\}, u) \cup B(\{v_1, x_1, y_1\}, w) \in \mathcal{T}^{(3)}_{2,0,0}(G)$. Thus, we may assume that $|N_G(x_1) \cap N_G(y_1) \cap N_G(v_1) \cap V_0| = 1$, which implies that $B(\{x_1, y_1, z_1\}, v_1) \in \mathcal{H}^{(3)}_{a,0,0}(G)$ for some $a \in \{1, 2, 3\}$. If $B(\{x_1, y_1, z_1\}, v_1)$ is not a one-hat of G, then we are done. Thus, since the choice of $v_1 \in V_1$ is arbitrary, we may assume that every vertex of V_1 is contained in a one-hat of G.

If $V(G) = V_0 \cup V_1$, then since every vertex of V_1 belongs to a one-hat of G, G is isomorphic to a graph obtained from K_4 by adding one vertex of degree 3 to a triangle, and hence $\mathcal{T}^{(3)}_{2,0,0}(G) \neq \emptyset$. Suppose that $V_2 \neq \emptyset$. If $|V(G) \setminus (V_0 \cup V_1)| = 3$, then the three vertices in $V(G) \setminus (V_0 \cup V_1)$ form a root of a subgraph in $\mathcal{T}^{(3)}_{0,2,0}(G)$. Thus, we assume that $|V(G) \setminus (V_0 \cup V_1)| \geq 4$. Let v_2 be a vertex of V_2 and let $N_G(v_2) \setminus (V_0 \cup V_1) = \{x_2, y_2, z_2\}$ such that $|N_G(x_2) \cap N_G(y_2) \cap N_G(v_2) \cap (V_0 \cup V_1)| \geq |N_G(y_2) \cap N_G(z_2) \cap N_G(v_2) \cap (V_0 \cup V_1)| \geq |N_G(z_2) \cap N_G(x_2) \cap N_G(v_2) \cap (V_0 \cup V_1)|$. If there are two distinct vertices $u, w \in V_0 \cup V_1$ such that $N_G(u) \setminus (V_0 \cup V_1) = N_G(w) \setminus (V_0 \cup V_1) = \{v_2, x_2, y_2\}$, then $B(\{v_2, x_2, y_2\}, u) \cup B(\{v_2, x_2,_2\}, w) \in \mathcal{T}^{(3)}(G)$. Thus, we may assume that $|N_G(x_2) \cap N_G(y_2) \cap N_G(v_2) \cap (V_0 \cup V_1)| = 1$, which implies that $B(\{x_2, y_2, z_2\}, v_2) \in \mathcal{H}^{(3)}_{a,b,0}(G)$ for some $a \geq 0$ and $b \geq 1$ with $a + b \leq 3$. If $B(\{x_2, y_2, z_2\}, v_2)$ is not a one-hat plus of G, then we are done. Thus, since the choice of $v_2 \in V_2$ is arbitrary, we may assume that $V_2 \neq \emptyset$ and that every vertex of V_2 is contained in a one-hat plus of G. Then it is easy to verify that $V_3 \neq \emptyset$.

If $|V(G) \setminus (V_0 \cup V_1 \cup V_2)| = 3$, then the three vertices in $V(G) \setminus (V_0 \cup V_1 \cup V_2)$ form a root of a subgraph in $\mathcal{T}^{(3)}_{0,0,2}(G)$. Thus, we assume that $|V(G) \setminus (V_0 \cup V_1)| \geq 4$. Let $v_3 \in V_3$ and let $N_G(v_3) \setminus (V_0 \cup V_1 \cup V_2) = \{x_3, y_3, z_3\}$ such that $|N_G(x_3) \cap N_G(y_3) \cap N_G(v_3) \cap (V_0 \cup V_1 \cup V_2)| \geq |N_G(y_3) \cap N_G(z_3) \cap N_G(v_3) \cap (V_0 \cup V_1 \cup V_2)| \geq |N_G(z_3) \cap N_G(x_3) \cap N_G(v_3) \cap (V_0 \cup V_1 \cup V_2)|$. If there are two distinct vertices $u, w \in V_0 \cup V_1 \cup V_2$ such that $N_G(u) \setminus (V_0 \cup V_1 \cup V_2) = N_G(w) \setminus (V_0 \cup V_1 \cup V_2) = \{v_3, x_3, y_3\}$, then $B(\{v_3, x_3, y_3\}, u) \cup B(\{v_3, x_3,_3\}, w) \in \mathcal{T}^{(3)}(G)$. Thus, we may assume that $|N_G(x_3) \cap N_G(y_3) \cap N_G(v_3) \cap (V_0 \cup V_1 \cup V_2)| = 1$, which implies that $B(\{x_3, y_3, z_3\}, v_3) \in \mathcal{H}^{(3)}_{a,b,c}(G)$ for some $a \geq 0$, $b \geq 0$ and $c \geq 1$ with $a + b + c \leq 3$. This completes the proof of Lemma 6.10. $\square$

The following lemma is used repeatedly in our proof of Theorem 6.5.

Lemma 6.11. Let G be a 3-tree, and suppose that G has a branch $B = B(\{v_1, v_2, v_3\}, u_0)$ which is either an ear, a one-hat, or a one-hat plus of G. Suppose that $G - (V(B) \setminus$

$\{v_1, v_2, v_3\}$) admits an odd 5-coloring φ'. For any index $i \in \{1,2,3\}$, φ' can be extended to a proper 5-coloring of $G' \cup B$ such that every vertex of $V(B) \setminus \{v_i, v_{i+1}\}$ satisfies the odd condition, where $v_4 = v_1$. In particular, when B is either a one-hat or a one-hat plus, then φ' can be extended to a proper 5-coloring of $G' \cup B$ such that every vertex of $V(B) \setminus \{v_i\}$ satisfies the odd condition.

Proof. Let $G' = G - (V(B) \setminus \{v_1, v_2, v_3\})$ and let φ' be an odd 5-coloring of G'. Without loss of generality, we may assume that $\varphi'(v_i) = i$ for each $i \in \{1,2,3\}$. Let $\varphi(v) = \varphi'(v)$ for every vertex $v \in V(G')$, and we define $\varphi(u)$ for each $u \in V(B) \setminus \{v_1, v_2, v_3\}$ as follows.

We first consider the case where B is an ear of G. By the symmetry of B, it suffices to show that φ' can be extended to a proper 5-coloring φ of G such that every vertex of $V(G) \setminus \{v_2, v_3\}$ satisfies the odd condition. We simply choose $\varphi(u_0) \in \{4,5\}$ so that v_1 satisfies the odd condition, and then it is easy to verify that every vertex of $V(G) \setminus \{v_2, v_3\}$ satisfies the odd condition.

Next, we consider the case where B is a one-hat of G. Let $V(B) = \{v_1, v_2, v_3, u_0, u_1\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1\}$ and $N_G(u_1) = \{v_1, v_2, u_0\}$. By the symmetry of v_1 and v_2, it suffices to show that the statement holds for the cases $i = 1$ or $i = 3$. We extend φ' to a coloring φ_1 of G by choosing $\varphi_1(u_0) \in \{4,5\}$ so that v_3 satisfies the odd condition and choosing $\varphi_1(u_1) \in \{3,4,5\} \setminus \{\varphi_1(u_0)\}$ so that v_2 satisfies the odd condition. We extend φ' to a coloring φ_3 of G by letting $(\varphi_3(u_0), \varphi_3(u_1)) \in \{(4,1),(4,5),(5,1)\}$ so that both v_1 and v_2 satisfy the odd condition. In each case, it is easy to verify that φ_i is a proper 5-coloring of G such that every vertex other than v_i satisfies the odd condition.

Finally, we consider the case where B is a one-hat plus of G. Let $V(B) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0,3]\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, $N_G(u_1) = \{v_1, v_2, u_0, u_3\}$, $N_G(u_2) = \{v_2, v_3, u_0\}$ and $N_G(u_3) = \{v_1, u_0, u_1\}$. We let $\varphi_1(u_0) = 4$ and choose $\varphi_1(u_2) \in \{1,5\}$ so that v_3 satisfies the odd condition. By the argument in the last paragraph, we can choose $\varphi_1(u_1)$ and $\varphi_1(u_3)$ so that every vertex in $\{v_2, u_0, u_1, u_3\}$ satisfies the odd condition. It is easy to verify that every vertex of $V(B) \setminus \{v_1\}$ satisfies the odd condition with respect to φ_1. Similarly, we can extend φ' to a coloring φ_2 such that every vertex of $V(B) \setminus \{v_2\}$ satisfies the odd condition. We extend φ' to a coloring φ_3 of G by letting $\varphi_3(u_0) = 4$, $\varphi_3(u_2) = 5$, and choosing $\varphi_3(u_1)$ and $\varphi_3(u_3)$ so that every vertex of $\{v_1, v_2, u_1, u_3\}$ satisfies the odd condition. Since $d_G(u_0) = 6$ and distinct four colors appear in the neighbors of u_0, u_0 satisfies the odd condition with respect to φ_3. Thus, every vertex of $V(B) \setminus \{v_3\}$ satisfies the odd condition with respect to φ_3. □

6.4.2 Proof of Theorem 6.5

Let G be a 3-tree of order n. The proof goes by induction on n. If $n \le 5$, then trivially G is odd 5-colorable. We assume that $n \ge 6$, and that every 3-tree of order less than n is odd 5-colorable. We first consider the case where G has a one-hat or a one-hat plus with a vertex of specified degree in its root.

Claim 1. Suppose that G contains a branch $B = B(\{y_1, y_2, y_3\}, x)$ which is either a one-hat or a one-hat plus of G. If $d_G(y_1)$ is equal to 4 or an odd integer, then G is odd 5-colorable.

Proof. Let $G'' = G - V(B) \setminus \{y_1, y_2, y_3\}$. By the induction hypothesis, G'' admits an odd 5-coloring φ''. By using Lemma 6.11 to B, we can extend φ'' to a proper 5-coloring of G such that every vertex other than y_1 satisfies the odd condition. If $d_G(y_1)$ is an odd integer, then y_1 satisfies the odd condition with respect to any proper 5-coloring of G. Since every vertex of a 3-tree is contained in a clique of order 4, if $d_G(y_1) \le 5$, then y_1 satisfies the odd condition with respect to any proper 5-coloring of G. Hence, the extension of φ'' to G is an odd 5-coloring of G. □

By Lemma 6.10, G has a subgraph $H \in \mathcal{H}^{(3)}(G) \cup \mathcal{T}^{(3)}(G)$ with the root $\{v_1, v_2, v_3\}$ that is neither a one-hat nor a one-hat plus of G. Note that $H \notin \mathcal{H}^{(3)}_{1,0,0}(G)$ since H is not a one-hat of G. Let $G' = G - (V(H) \setminus \{v_1, v_2, v_3\})$. By the induction hypothesis, G' admits an odd 5-coloring φ'. Without loss of generality, we may assume that $\varphi'(v_i) = i$ for $i \in \{1, 2, 3\}$. Let $\varphi(x) = \varphi'(x)$ for every $x \in V(G')$.

Claim 2. If $H \in \mathcal{H}^{(3)}_{a,b,c}(G)$ for some tuple (a, b, c) of non-negative integers such that $1 \le a + b + c \le 3$ and $c \ge 1$, then G is odd 5-colorable.

Proof. Let $H = B(\{v_1, v_2, v_3\}, u_0) \in \mathcal{H}^{(3)}_{a,b,c}(G)$ and suppose that there is a vertex $u_1 \in V(H)$ such that $B_1 = B(\{v_1, v_2, u_0\}, u_1)$ is a one-hat plus of G. Let $B_2 = B(\{v_2, v_3, u_0\}, x)$ if there is a vertex $x \in (N_G(v_2) \cap N_G(v_3) \cap N_G(u_0)) \setminus \{v_1\}$ and let $B_2 = v_2v_3u_0v_2$ otherwise. Similarly, let $B_3 = B(\{v_3, v_1, u_0\}, x)$ if there is a vertex $x \in (N_G(v_3) \cap N_G(v_1) \cap N_G(u_0)) \setminus \{v_2\}$ and let $B_3 = v_3v_1u_0v_3$ otherwise. By the assumption that $H \in \mathcal{H}^{(3)}_{a,b,c}(G)$, each of B_2, B_3 is either an ear, a one-hat, a one-hat plus or a triangle of G. We first show the following subclaim.

Subclaim 2.1. If $d_{B_1}(u_0) = 4$, then G is odd 5-colorable.

Proof. By the symmetry of v_1 and v_2, we assume that $V(B_1) = \{v_1, v_2, u_0, u_1, u_2, u_3, u_4\}$ such that $N_G(u_1) = \{v_1, v_2, u_0, u_2, u_3, u_4\}$, $N_G(u_2) = \{v_1, v_2, u_1, u_4\}$, $N_G(u_3) = \{v_1, u_0, u_1\}$, and $N_G(u_4) = \{v_2, u_1, u_2\}$. If both B_2 and B_3 are triangles of G, then we have $d_G(u_0) = 5$, which falls into Claim 1. Hence, we assume that at least one of B_2 and B_3 is not a triangle of G.

Let $\varphi(u_0) = 4$, $\varphi(u_1) = 5$ and $\varphi(u_3) = 3$. Suppose that $d_G(u_0) \le 7$. Then, by using Lemma 6.11, we can choose colors for the vertices in $V(H) \setminus (V(B_1) \cup \{v_3\})$ so that every vertex of $V(H) \setminus V(B_1)$ satisfies the odd condition. Again, by Lemma 6.11, we choose $\varphi(u_2)$ and $\varphi(u_4)$ so that every vertex of $\{v_1, v_2, u_2, u_4\}$ satisfies the odd condition. By the choice of colors, it is easy to verify that every vertex other than u_0 and u_1 satisfies the odd condition. Furthermore, we have $d_G(u_0), d_G(u_1) \le 7$ and four distinct colors appear in the neighborhood of each of u_0 and u_1. Thus, φ is an odd 5-coloring of G.

We assume that $d_G(u_0) \ge 8$, which implies that at least one of B_1 and B_2 is a one-hat or a one-hat plus of G. By using Lemma 6.11 to B_2 and B_3, we can choose colors for the vertices in $V(H) \setminus (V(B_1) \cup \{v_3\})$ so that every vertex of $V(H) \setminus (V(B_1) \setminus \{u_0\})$ satisfies the odd condition. By Lemma 6.11, we can choose $\varphi(u_2)$ and $\varphi(u_4)$ so that every vertex of $\{v_1, v_2, u_2, u_4\}$ satisfies the odd condition, and the resulting coloring φ is an odd 5-coloring of G. □

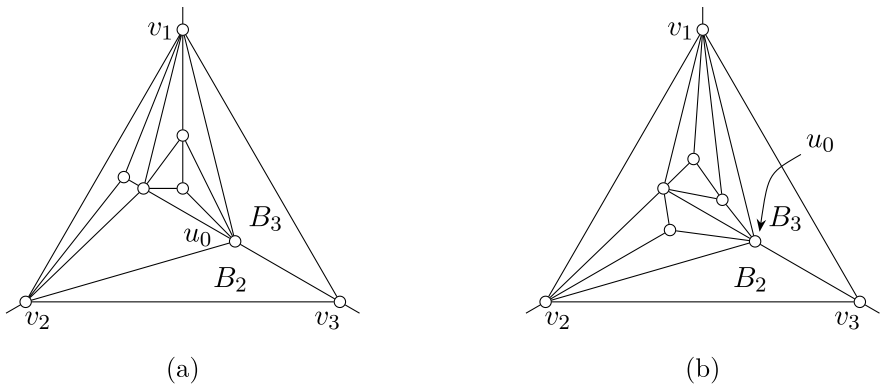


Figure 6.8: Graphs in Claim 2

Now we return to the proof of Claim 2. Let $V(B_1) = \{v_1, v_2, u_0, u_1, u_2, u_3, u_4\}$. By Subclaim 2.1 and the symmetry of v_1 and v_2, we may assume one of the followings:

(a) $N_G(u_1) = \{v_1, v_2, u_0, u_2, u_3, u_4\}$, $N_G(u_2) = \{v_1, u_0, u_1, u_4\}$, $N_G(u_3) = \{v_1, v_2, u_1\}$, and $N_G(u_4) = \{u_0, u_1, u_2\}$ (Figure 6.8a), or

(b) $N_G(u_1) = \{v_1, v_2, u_0, u_2, u_3, u_4\}$, $N_G(u_2) = \{v_1, u_0, u_1, u_4\}$, $N_G(u_3) = \{v_2, u_0, u_1\}$, and $N_G(u_4) = \{v_1, u_1, u_2\}$ (Figure 6.8b).

We give a strategy for a coloring of H that works for both of these two cases as follows. Suppose first that both B_2 and B_3 are triangles of G. We let $(\varphi(u_0), \varphi(u_1))$ be $(4, 5)$ or $(5, 4)$ so that v_3 satisfies the odd condition and choose $\varphi(u_3)$ so that v_2 satisfies the odd condition. By using Lemma 6.11 to $B(\{v_1, u_0, u_1\}, u_2)$, we choose $\varphi(u_2)$ and $\varphi(u_4)$ so that every vertex of $\{v_1, u_1, u_2, u_4\}$ satisfies the odd condition. It is easy to verify that every vertex other than u_0 satisfies the odd condition, and u_0 satisfies the odd condition since $d_G(u_0) = 6$ and four colors $\{1, 2, 3, 5\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G.

Next, we assume that neither B_2 nor B_3 is a triangle of G. In this case, we let $\varphi(u_0) = 4$, $\varphi(u_1) = 5$, and $\varphi(u_3) = 3$. By using Lemma 6.11 to B_2, we can color $V(B_2) \setminus \{v_2, v_3, u_0\}$ so that every vertex of $V(B_2) \setminus \{v_3, u_0\}$ satisfies the odd condition. Then again by using Lemma 6.11 to B_3, we can color B_3 so that every vertex of $V(B_3) \setminus \{v_1, u_0\}$ satisfies the odd condition. Finally, we color $B(\{v_1, u_0, u_1\}, u_2)$ so that every vertex of $\{v_1, u_0, u_2, u_4\}$ satisfies the odd condition. It is easy to verify that every vertex other than u_1 satisfies the odd condition, and u_1 satisfies the odd condition since $d_G(u_1) = 6$ and four colors $\{1, 2, 3, 4\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G.

Hence, we may assume that one of B_2 and B_3 is a triangle and the other is not. Then we have $d_G(u_0) \in \{7, 8, 9\}$, and we are done for the cases $d_G(u_0) \in \{7, 9\}$ by Claim 1. Thus, $d_G(u_0) = 8$, which implies that either $d_{B_2}(u_0) = 4$ and B_3 is a triangle, or $d_{B_3}(u_0) = 4$ and B_2 is a triangle. If $d_{B_i}(u_0) = 4$ and B_i is a one-hat plus of G for some $i \in \{2, 3\}$, then we are done by Subclaim 2.1. Hence, we may assume that one of B_2 and B_3 is a one-hat of G, and the other is a triangle. For each case, we extend φ to G as follows:

- When B_2 is a one-hat of G, we let $\varphi(u_0) = 4$, $\varphi(u_1) = 5$ and $\varphi(u_3) = 3$. We use Lemma 6.11 to B_2 to color $V(B_2) \setminus \{v_2, v_3, u_0\}$ so that every vertex of $V(B_2) \setminus \{u_0\}$ satisfies the odd condition and use Lemma 6.11 to $B(\{v_1, u_0, u_1\}, u_2)$ to choose $\varphi(u_2)$ and $\varphi(u_4)$ so that every vertex of $\{v_1, u_0, u_2, u_4\}$ satisfies the odd condition.

- When B_3 is a one-hat of G, we let $\varphi(u_0) = 4$, $\varphi(u_1) = 5$, $\varphi(u_2) = 3$, and choose $\varphi(u_3)$ so that v_2 satisfies the odd condition. Let $x \in \{v_1, u_0\}$ so that $N_G(u_4) = \{u_0, u_1, x\}$. We use Lemma 6.11 to B_3 to color $V(B_3) \setminus \{v_3, v_1, u_0\}$ so that every vertex of $V(B_3) \setminus \{x\}$ satisfies the odd condition, and choose $\varphi(u_4) \in \{1, 2, 4\} \setminus \{\varphi(x)\}$ so that x satisfies the odd condition.

In both cases, every vertex other than u_1 satisfies the odd condition by the choice of colors, and u_1 satisfies the odd condition since $d_G(u_1) = 6$ and four colors $\{1, 2, 3, 4\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G. This completes the proof of Claim 2. □

Claim 3. If $H \in \mathcal{H}^{(3)}_{2,0,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3, u_0, u_1, u_2\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2\}$, $N_G(u_1) = \{v_1, v_2, u_0\}$, and $N_G(u_2) = \{v_2, v_3, u_0\}$. As $d_G(u_0) = 5$, u_0 satisfies the odd condition with respect to any proper 5-coloring of G. If $|\varphi^{-1}(4) \cap N_{G'}(v_2)|$ is even, then we let $\varphi(u_0) = 4$, choose $\varphi(u_1) \in \{3, 5\}$ so that v_1 satisfies the odd condition, and choose $\varphi(u_2) \in \{1, 5\}$ so that v_3 satisfies the odd condition. Every vertex other than v_2 satisfies the odd condition by the choice of colors, and v_2 satisfies the odd condition since $|\varphi^{-1}(4) \cap N_G(v_2)| = |\varphi^{-1}(4) \cap N_{G'}(v_2)| + 1$ is odd. Thus, φ is an odd 5-coloring of G. Hence, we may assume that $|\varphi^{-1}(4) \cap N_{G'}(v_2)|$ is odd. If $|\varphi^{-1}(4) \cap N_{G'}(v_1)|$ is even, then we let $\varphi(u_0) = 4$, choose $\varphi(u_2) \in \{1, 5\}$ so that v_3 satisfies the odd condition, and choose $\varphi(u_1) \in \{3, 5\}$ so that v_2 satisfies the odd condition. Every vertex other than v_1 satisfies the odd condition by the choice of colors, and v_1 satisfies the odd condition since $|\varphi^{-1}(4) \cap N_G(v_1)| = |\varphi^{-1}(4) \cap N_{G'}(v_1)| + 1$ is odd. Thus, φ is an odd 5-coloring of G. Hence, we may assume that $|\varphi^{-1}(4) \cap N_{G'}(v_1)|$ is odd. By the symmetry of v_1 and v_3, we may also assume that $|\varphi^{-1}(4) \cap N_{G'}(v_3)|$ is odd. Then we let $\varphi(u_0) = 5$, $\varphi(u_1) = 3$ and $\varphi(u_2) = 1$. Since $|\varphi^{-1}(4) \cap N_G(v_i)| = |\varphi^{-1}(4) \cap N_{G'}(v_i)|$ is odd for each $i \in \{1, 2, 3\}$, φ is an odd 5-coloring of G. □

Claim 4. If $H \in \mathcal{H}^{(3)}_{3,0,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, 3]\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, $N_G(u_1) = \{v_1, v_2, u_0\}$, $N_G(u_2) = \{v_2, v_3, u_0\}$, and $N_G(u_3) = \{v_3, v_1, u_0\}$. If $|\varphi^{-1}(4) \cap N_{G'}(v_1)|$ is even, then we let $\varphi(u_0) = 4$, $\varphi(u_1) = 5$, choose $\varphi(u_2) \in \{1, 5\}$ so that v_2 satisfies the odd condition, and choose $\varphi(u_3) \in \{2, 5\}$ so that v_3 satisfies the odd condition. Every vertex other than $\{v_1, u_0\}$ satisfies the odd condition by the choice of colors. The vertex u_0 satisfies the odd condition since $d_G(u_0) = 6$ and four colors $\{1, 2, 3, 4\}$ appear in the neighborhood, and v_1 satisfies the odd condition since $|\varphi^{-1}(4) \cap N_G(v_1)| = |\varphi^{-1}(4) \cap N_{G'}(v_1)| + 1$ is odd. Thus, φ is an odd 5-coloring of G. Hence, we may assume that $|\varphi^{-1}(4) \cap N_{G'}(v_1)|$ is odd. By the symmetry of v_1, v_2 and v_3, we may also assume that

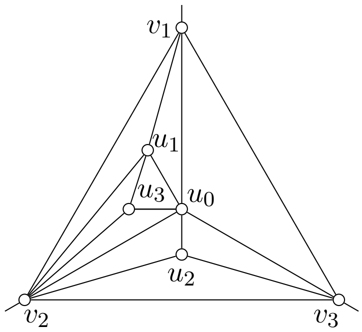


Figure 6.9: A graph in Claim 6.

$|\varphi^{-1}(4) \cap N_{G'}(v_i)|$ for each $i \in \{2, 3\}$. Then let $\varphi(u_0) = 5$ and $\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = 4$. The vertex u_0 satisfies the odd condition since $|\varphi^{-1}(4) \cap N_G(u_0)| = 3$. Furthermore, $|\varphi^{-1}(4) \cap N_G(v_i)| = |\varphi^{-1}(4) \cap N_{G'}(v_i)| + 2$ is odd for each $i \in \{1, 2, 3\}$. Hence, φ is an odd 5-coloring of G. □

Claim 5. If $H \in \mathcal{H}^{(3)}_{0,1,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3, u_0, u_1, u_2\}$ such that $B_1 = B(\{v_1, v_2, u_0\}, u_1)$ is a one-hat of G. Since $d_G(u_0) \in \{4, 5\}$, G is odd 5-colorable by Claim 1. □

Claim 6. If $H \in \mathcal{H}^{(3)}_{1,1,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, 3]\}$ such that $B(\{v_1, v_2, u_0\}, u_1)$ is a one-hat of G and $B(\{v_2, v_3, u_0\}, u_2)$ is an ear of G. Then we have $d_G(u_0) \in \{5, 6\}$. If $d_G(u_0) = 5$, then we are done by Claim 1. Hence, we assume that $d_G(u_0) = 6$. Since H is not a one-hat plus of G, we set $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, $N_G(u_1) = \{v_1, v_2, u_0, u_3\}$, $N_G(u_2) = \{v_2, v_3, u_0\}$ and $N_G(u_3) = \{v_2, u_0, u_1\}$ (Figure 6.9). Let $\varphi(u_0) = 4$. We choose $\varphi(u_2) \in \{1, 5\}$ so that v_3 satisfies the odd condition, and we use Lemma 6.11 to $B(\{v_1, v_2, u_0\}, u_1)$ to color $\{u_1, u_3\}$ so that every vertex of $\{v_1, v_2, u_1, u_3\}$ satisfies the odd condition. Every vertex other than u_0 satisfies the odd condition by the choice of colors, and u_0 satisfies the odd condition since $\varphi^{-1}(2) \cap N_G(u_0) = \{v_2\}$. Thus, φ is an odd 5-coloring of G. □

Claim 7. If $H \in \mathcal{H}^{(3)}_{2,1,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, 4]\}$ such that $B_1 = B(\{v_1, v_2, u_0\}, u_1)$ is a one-hat of G and both $B_2 = B(\{v_2, v_3, u_0\}, u_2)$ and $B_3 = B(\{v_3, v_1, u_0\}, u_3)$ are ears of G. We have $d_G(u_0) \in \{6, 7\}$, and we are done for case $d_G(u_1) = 7$ by Claim 1. Hence, we may assume that $d_G(u_0) = 6$, which implies that $d_{B_1}(u_0) = 3$. Then we set $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, $N_G(u_1) = \{v_1, v_2, u_0, u_4\}$, $N_G(u_2) = \{v_2, v_3, u_0\}$, $N_G(u_3) = \{v_3, v_1, u_0\}$, and $N_G(u_4) = \{v_1, v_2, u_1\}$. We let $\varphi(u_0) = 4$, $\varphi(u_2) = 5$, and choose $\varphi(u_3) \in \{2, 5\}$ so that v_3 satisfies the odd condition. By using Lemma 6.11 to B_1, we choose $\varphi(u_1)$ and $\varphi(u_4)$ so that every vertex of $\{v_1, v_2, u_1, u_4\}$ satisfies the odd condition. Every vertex other than u_0 satisfies the odd condition by the choice of colors, and u_0 satisfies the odd condition since $d_G(u_0) = 6$ and four colors $\{1, 2, 3, 5\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G. $\square$

Claim 8. If $H \in \mathcal{H}^{(3)}_{0,2,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, 4]\}$ such that $B_1 = B(\{v_1, v_2, u_0\}, u_1)$ and $B_2 = B(\{v_2, v_3, u_0\}, u_2)$ are both one-hats of G. We have $d_G(u_0) \in \{5, 6, 7\}$, and we are done for cases $d_G(u_0) \in \{5, 7\}$ by Claim 1. Suppose that $d_G(u_0) = 6$. Then by the symmetry of v_1 and v_3, we may assume that $V(B_1) = \{v_1, v_2, u_0, u_1, u_3\}$ such that $N_G(u_1) = \{v_1, v_2, u_0, u_3\}$ and $N_G(u_3) = \{v_1, v_2, u_1\}$. We let $\varphi(u_0) = 4$, $\varphi(u_1) = 5$, and choose $\varphi(u_3) \in \{3, 4\}$ so that v_1 satisfies the odd condition. By using Lemma 6.11 to B_2, we color $V(B_2) \setminus \{v_2, v_3, u_0\}$ so that every vertex of $V(B_2) \setminus \{u_0\}$ satisfies the odd condition. Every vertex other than u_0 satisfies the odd condition by the choice of colors, and u_0 satisfies the odd condition since $d_G(u_0) = 6$ and four colors $\{1, 2, 3, 5\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G. $\square$

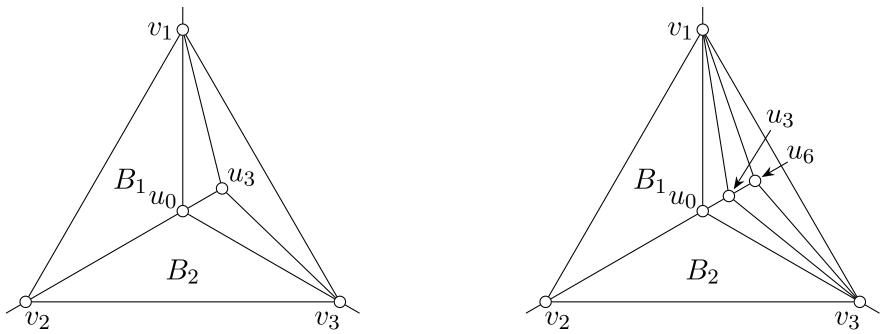
Claim 9. If $H \in \mathcal{H}^{(3)}_{1,2,0}(G) \cup \mathcal{H}^{(3)}_{0,3,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, t]\}$ for some $t \in \{5, 6\}$ such that both $B_1 = B(\{v_1, v_2, u_0\}, u_1)$ and $B_2 = B(\{v_2, v_3, u_0\}, u_2)$ are one-hats of G, and $B_3 = B(\{v_3, v_1, u_0\}, u_3)$ is either an ear or a one-hat of G. ($t = 5$ when B_3 is an ear and $t = 6$ when B_3 is a one-hat.) Let $V(B_1) = \{v_1, v_2, u_0, u_1, u_4\}$ and $V(B_2) = \{v_2, v_3, u_0, u_2, u_5\}$. Let $V(B_3) = \{v_3, v_1, u_0, u_3\}$ if B_3 is an ear, and let $V(B_3) = \{v_3, v_1, u_0, u_3, u_6\}$ if B_3 is a one-hat. We have $d_G(u_0) \in \{6, 7, 8\}$, and we are done for the case $d_G(u_0) = 7$ by Claim 1.

Suppose first that $d_G(u_0) = 6$. Then we have $d_{B_3}(u_0) = 3$. When B_3 is a one-hat, $N_G(u_6) = \{v_3, v_1, u_3\}$. Let $\varphi(u_0) = 4$, $\varphi(u_3) = 5$, and if B_3 is a one-hat, then let $\varphi(u_6) = 2$. We use Lemma 6.11 to B_1 to color $\{u_1, u_4\}$ so that every vertex of $V(B_1) \setminus \{u_0\}$ satisfies the odd condition, then we apply Lemma 6.11 to B_2 to color $\{u_2, u_5\}$ so that every vertex of $V(B_2) \setminus \{u_0\}$ satisfies the odd condition. Every vertex other than u_0 satisfies the odd condition by the choice of colors, and u_0 satisfies the odd condition since $d_G(u_0) = 6$ and four colors $\{1, 2, 3, 5\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G.

Suppose that $d_G(u_1) = 8$. We may assume that $d_{B_1}(u_0) = d_{B_2}(u_0) = 4$ and $d_{B_3}(u_0) = 3$; If B_3 is an ear, then we have $d_{B_3}(u_0) = 3$ and thus both $d_{B_1}(u_0)$ and $d_{B_2}(u_0)$ must be equal to 4 (Figure 6.10a). Otherwise, two of $\{d_{B_1}(u_0), d_{B_2}(u_0), d_{B_3}(u_0)\}$ are equal to 4 and the other is equal to 3. By the symmetry of B_1, B_2, B_3, we may assume that $d_{B_3}(u_0) = 3$ (Figure 6.10b). Note that every vertex of $V(H) \setminus \{u_0, u_6\}$ is adjacent to u_0, and thus a color assigned to u_0 appears exactly once in $N_G(v_2) \cap V(H)$ for any proper coloring of G.

We suppose that $|\varphi^{-1}(4) \cap N_{G'}(v_1)|$ is even. Then let $\varphi(u_0) = 4$, $\varphi(u_3) = 5$, and if B_3 is a one-hat, then let $\varphi(u_6) = 2$. We use Lemma 6.11 to B_2 to color $\{u_2, u_5\}$ so that every vertex of $\{v_2, v_3, u_2, u_5\}$ satisfies the odd condition, and we use Lemma 6.11 to B_1 to color $\{u_1, u_4\}$ so that every vertex of $\{v_2, u_0, u_1, u_4\}$ satisfies the odd condition. Every vertex other than v_1 satisfies the odd condition by the choice of colors, and v_1 satisfies the odd condition since $|\varphi^{-1}(4) \cap N_G(v_1)| = |\varphi^{-1}(4) \cap N_{G'}(v_1)| + 1$ is odd. Thus, φ is an odd



(a) The case B_3 is an ear. (b) The case B_3 is a one-hat.

Figure 6.10: Graphs in Claim 9.

5-coloring of G. Hence, we may assume that $|\varphi^{-1}(4) \cap N_{G'}(v_1)|$ is odd. By the symmetry of v_1 and v_3, we may also assume that $|\varphi^{-1}(4) \cap N_{G'}(v_3)|$ is odd.

Next, we suppose that $|\varphi^{-1}(4) \cap N_{G'}(v_2)|$ is even. Then let $\varphi(u_0) = 4$, $\varphi(u_3) = 5$, and if B_3 is a one-hat, then let $\varphi(u_6) = 2$. We use Lemma 6.11 to B_1 to color $\{u_1, u_4\}$ so that every vertex of $\{v_1, u_0, u_1, u_4\}$ satisfies the odd condition, and we use Lemma 6.11 to B_2 to color $\{u_2, u_5\}$ so that every vertex of $\{v_3, u_0, u_2, u_5\}$ satisfies the odd condition. Every vertex other than v_2 satisfies the odd condition by the choice of colors, and v_2 satisfies the odd condition since $|\varphi^{-1}(4) \cap N_G(v_2)| = |\varphi^{-1}(4) \cap N_{G'}(v_2)| + 1$ is odd. Thus, φ is an odd 5-coloring of G. Hence, we may assume that $|\varphi^{-1}(4) \cap N_{G'}(v_2)|$ is odd.

Now we let $\varphi(u_0) = 5$ and $\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = 4$. We choose $\varphi(u_4)$, $\varphi(u_5)$, and $\varphi(u_6)$ arbitrarily so that the resulting coloring of G is proper. For each $i \in \{1, 2, 3\}$, $|\varphi^{-1}(4) \cap N_G(v_i)| = |\varphi^{-1}(4) \cap N_{G'}(v_i)| + 2$ is odd and thus v_i satisfies the odd condition. Furthermore, since $|\varphi^{-1}(4) \cap N_G(u_0)| = 3$, u_0 satisfies the odd conditions. Thus, φ is an odd 5-coloring of G. □

Claim 10. If $H \in \mathcal{T}^{(3)}_{a,b,c}(G)$ for some tuple (a, b, c) of non-negative integers such that $a + b + c = 2$ and $c \geq 1$, then G is odd 5-colorable.

Proof. Let $V(H) = V(B_u) \cup V(B_w)$ where $B_u = B(\{v_1, v_2, v_3\}, u_0)$ is a one-hat plus of G and $B_w = B(\{v_1, v_2, v_3\}, w_0)$ is either an ear, a one-hat, or a one-hat plus of G. By the symmetry of v_1, v_2, and v_3, we may assume that $V(B_u) = \{v_1, v_2, v_3\} \cup \{u_i \mid i \in [0, 3]\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1, u_2, u_3\}$, $N_G(u_1) = \{v_1, v_2, u_0, u_3\}$, $N_G(u_2) = \{v_2, v_3, u_0\}$, and $N_G(u_3) = \{v_1, u_0, u_1\}$. We let $\varphi(u_0) = 4$ and $\varphi(u_2) = 5$. By using Lemma 6.11 to B_w, we choose colors for $V(B_w) \setminus \{v_1, v_2, v_3\}$ so that every vertex of $V(B_w) \setminus \{v_1, v_2\}$ satisfies the odd condition. Then we use Lemma 6.11 to a one-hat $B(\{v_1, v_2, u_0\}, u_1)$ to color u_1 and u_3 so that every vertex of $\{v_1, v_2, u_1, u_3\}$ satisfies the odd condition. Every vertex other than u_0 satisfies the odd condition by the choice of colors, and u_0 satisfies the odd condition since $d_G(u_0) = 6$ and four colors $\{1, 2, 3, 5\}$ appear in the neighborhood. Thus, φ is an odd 5-coloring of G. □

Claim 11. If $H \in \mathcal{T}^{(3)}_{2,0,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = \{v_1, v_2, v_3, u_0, w_0\}$ such that $N_G(u_0) = N_G(w_0) = \{v_1, v_2, v_3\}$. We let $\varphi(u_0) = \varphi(w_0) = 4$. Then, for every vertex x of G' and every color $i \in [5]$, $|\varphi^{-1}(i) \cap N_G(x)|$ and $|\varphi^{-1}(i) \cap N_{G'}(x)|$ have the same parity, and hence φ is an odd 5-coloring of G. $\square$

Claim 12. If $H \in \mathcal{T}^{(3)}_{1,1,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = V(B_u) \cup V(B_w)$ where $B_u = B(\{v_1, v_2, v_3\}, u_0)$ is a one-hat of G and $B_w = B(\{v_1, v_2, v_3\}, w_0)$ is an ear of G. By the symmetry of v_1, v_2 and v_3, we may assume that $V(B_u) = \{v_1, v_2, v_3, u_0, u_1\}$ such that $N_G(u_0) = \{v_1, v_2, v_3, u_1\}$ and $N_G(u_1) = \{v_1, v_2, u_0\}$. We first choose $\varphi(u_1) \in \{3, 4, 5\}$ so that $|\varphi^{-1}(i_1) \cap (N_G(v_1) \setminus \{u_0, w_0\})|$ is odd for some color $i_1 \in [5]$ and $|\varphi^{-1}(i_2) \cap (N_G(v_2) \setminus \{u_0, w_0\})|$ is odd for some color $i_2 \in [5]$. We let $\varphi(u_0) = \varphi(w_0) = j$ where $j \in \{4, 5\} \setminus \{\varphi(u_1)\}$ to obtain a coloring of G. For $i \in \{1, 2\}$, the vertex v_i satisfies the odd condition since $|\varphi^{-1}(i_1) \cap N_G(v_1)|$ is odd. Similarly, v_3 satisfies the odd condition since $|\varphi^{-1}(i) \cap N_G(v_3)|$ and $|\varphi^{-1}(i) \cap N_{G'}(v_3)|$ have the same parity for every color $i \in [5]$. Thus, φ is an odd 5-coloring of G. $\square$

Claim 13. If $H \in \mathcal{T}^{(3)}_{0,2,0}(G)$, then G is odd 5-colorable.

Proof. Let $V(H) = V(B_u) \cup V(B_w)$ where both $B_u = B(\{v_1, v_2, v_3\}, u_0)$ and $B_w = B(\{v_1, v_2, v_3\}, w_0)$ are one-hats of G. Let $V(B_u) = \{v_1, v_2, v_3, u_0, u_1\}$ and $V(B_w) = \{v_1, v_2, v_3, w_0, w_1\}$. By the symmetry of v_1, v_2 and v_3, we may assume that

(a) $N_G(u_0) = \{v_1, v_2, v_3, u_1\}$, $N_G(u_1) = \{v_1, v_2, u_0\}$, $N_G(w_0) = \{v_1, v_2, v_3, w_1\}$, and $N_G(w_1) = \{v_1, v_2, w_0\}$, or

(b) $N_G(u_0) = \{v_1, v_2, v_3, u_1\}$, $N_G(u_1) = \{v_1, v_2, u_0\}$, $N_G(w_0) = \{v_1, v_2, v_3, w_1\}$, and $N_G(w_1) = \{v_2, v_3, w_0\}$.

If H satisfies (a), then we let $\varphi(u_0) = \varphi(w_0) = 4$ and $\varphi(u_1) = \varphi(w_1) = 5$. Then, for every vertex x of G' and every color $i \in [5]$, $|\varphi^{-1}(i) \cap N_G(x)|$ and $|\varphi^{-1}(i) \cap N_{G'}(x)|$ have the same parity, which implies that φ is an odd 5-coloring of G. Hence, we may assume that G satisfies H satisfies (b). Let $\varphi(u_0) = 4$ and choose $\varphi(w_0) \in \{4, 5\}$ so that $|\varphi^{-1}(4) \cap (N_G(v_1) \setminus \{u_1\})|$ is odd. We choose $\varphi(w_1) \in \{1, 4, 5\} \setminus \{\varphi(w_0)\}$ so that v_3 satisfies the odd condition, and then choose $\varphi(u_1) \in \{3, 5\}$ so that v_2 satisfies the odd condition. By the choice of colors, every vertex other than v_1 satisfies the odd condition. Since $\varphi(u_1) \neq 4$, $|\varphi^{-1}(4) \cap N_G(v_1)| = |\varphi^{-1}(4) \cap (N_G(v_1) \setminus \{u_1\})|$ is odd and v_1 satisfies the odd condition. Thus, φ is an odd 5-coloring of G. This completes the proof of Theorem 6.5. $\square$

Chapter 7

Brooks-type theorems and degree-choosability for proper conflict-free coloring

Brooks-type theorems for proper conflict-free coloring have been actively studied since Caro, Petruševski, and Škrekovski posed a conjecture on a Brooks-type statement for proper conflict-free coloring in [13]. There are many partial results towards the conjecture, but it is still widely open. A characterization of degree-choosable graphs given in Erdős, Rubin, and Taylor [27] and in Borodin [9] is a natural extension of Brooks' theorem. In many variations of coloring, analogous concepts of degree-choosability have been studied by many researchers. (degree-paintability in [21], degree-DP-colorability in [44], degree-AT in [34], etc.) From this background, we introduce proper conflict-free (degree+k)-choosability of graphs, which is an analogy of degree-choosability for proper conflict-free coloring. In Section 7.1, we see results on Brooks-type theorems for proper conflict-free colorings. In Section 7.2, we introduce the definition of proper conflict-free (degree + k)-choosability and show some results on it. We see classes of proper conflict-free (degree + 2)-choosable graphs in Section 7.4, and a class of proper conflict-free (degree + 3)-choosable graphs in Section 7.5. Most of the new results in this chapter are based on [41].

7.1 Brooks-type theorems for proper conflict-free coloring

Studies on Brooks-type theorem for proper conflict-free coloring begin with the following conjecture, which was posed in Caro, Petruševski, and Škrekovski [13].

Conjecture 7.1 (Caro, Petruševski, and Škrekovski [13])**.** For every connected graph G with $\Delta(G) \geq 3$, $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 1$.

It follows from (4.2.1) that the cycle C_n with $n \not\equiv 0 \mod 3$ satisfies $\Delta(C_n) = 2$ and $\chi_{\mathrm{pcf}}(C_n) \geq 4$, and thus the assumption $\Delta(G) \geq 3$ in Conjecture 7.1 is necessary. Liu and Yu proved the following, which implies that Conjecture 7.1 holds for the case $\Delta(G) = 3$.

Theorem 7.2 (Liu and Yu [48])**.** For every connected graph G with $\Delta(G) = 3$ and every list assignment L of G, if $|L(v)| \geq 4$ for every vertex v of G, then G is proper conflict-free L-colorable.

Note that they studied the concept in the name of "superlinear coloring", with a motivation from linear coloring of graphs. For graphs with large maximum degree, the asymptotic behavior of the proper conflict-free chromatic number was studied by using probabilistic methods. Cranston and Liu showed in [20] that every connected graph G with $\Delta(G) \geq 10^8$ satisfies $\chi_{\mathrm{pcf}}(G) \leq \left\lceil 1.6550826\Delta(G) + \sqrt{\Delta(G)} \right\rceil$, and Kamyczura and Przybylo showed in [46] that every regular graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + O(\ln \Delta(G))$. Liu and Reed [47] showed that $\chi_{\mathrm{pcf}}(G) \leq (1+o(1))\Delta(G)$, which implies that Conjecture 7.1 is asymptotically true.

For the general case, Caro, Petruševski, and Škrekovski showed in [13] that every graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq \left\lfloor \frac{5\Delta(G)}{2} \right\rfloor$, and the bound was improved in Cranston and Liu [20], where they showed that every graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq 2\Delta(G) + 1$.

Recently, Cho, Choi, Kwon, and Park showed the following Brooks-type theorem for proper h-conflict-free coloring. For a positive integer h, a mapping φ from $V(G)$ to positive integers is called a *proper h-conflict-free coloring* of G if φ is a proper coloring of G such that

> for every vertex v of G, there is a set of colors $C(v)$ such that $|C(v)| \geq \min\{h, d_G(v)\}$ and $|\varphi^{-1}(c) \cap N_G(v)| = 1$ for every color $c \in C(v)$.

For a positive integer k, a *proper h-conflict-free k-coloring* of G is a proper h-conflict-free coloring of G such that every vertex of G maps to an integer in $[k]$. The *proper h-conflict-free chromatic number* of a graph G, denoted by $\chi^h_{\mathrm{pcf}}(G)$ is the least integer k such that G admits a proper h-conflict-free k-coloring. Note that a proper conflict-free coloring of a graph G is a proper 1-conflict-free coloring of G.

Theorem 7.3 (Cho, Choi, Kwon, and Park [17])**.** Let h be a positive integer and let G be a connected graph with $\Delta(G) \geq h + 2$. Then $\chi^h_{\mathrm{pcf}}(G) \leq (h + 1)\Delta(G) - 1$.

As a corollary of Theorem 7.3, they showed that $\chi_{\mathrm{pcf}}(G) \leq 2\Delta(G) - 1$ for every connected graph G with $\Delta(G) \geq 3$, which improves the bound in [20]. For some special classes of graphs, they improved the bound of the proper conflict-free chromatic number in terms of the coefficient of the maximum degree as follows. A graph G is called a *quasi-line graph* if for every vertex v of G, $N_G(v)$ is covered by at most two cliques of G. A graph G is called *claw-free* if G does not have an induced subgraph that is isomorphic to the complete bipartite graph $K_{1,3}$.

Theorem 7.4 (Cho, Choi, Kwon, and Park [17])**.** Let G be a graph.

- If G is a quasi-line graph, then $\chi_{\mathrm{pcf}}(G) \leq \frac{3}{2}\Delta(G) + 2$.
- If G is claw-free, then $\chi_{\mathrm{pcf}}(G) \leq \frac{3}{2}\Delta(G) + \left\lceil \sqrt{\Delta(G)} \right\rceil$.

7.2 Proper conflict-free $(\text{degree} + k)$-choosability of graphs

For a non-negative integer k, we say that a graph G is *proper conflict-free* $(\text{degree} + k)$-*choosable* if G is proper conflict-free L-colorable for any list assignment L of G such that $|L(v)| \geq d_G(v) + k$ for every vertex v of G. If a graph G is proper conflict-free $(\text{degree}+k)$-choosable for some positive integer k, then obviously it follows that $\chi_{\text{pcf}}(G) \leq \Delta(G) + k$.

The following proposition can be easily proved by a greedy coloring algorithm. Recall that for a graph G and a (partial) coloring φ of G, $\mathcal{U}_\varphi(v, G)$ is the set of colors that appear exactly once at the neighborhood of a vertex v of G.

Proposition 7.5. Let G be a graph. If G is d-degenerate for a positive integer d, then G is proper conflict-free $(\text{degree} + d + 1)$-choosable.

Proof. Let G be a d-degenerate graph for a positive integer d. Let L be a list assignment of G such that $|L(v)| \geq d_G(v) + d + 1$ for every vertex v of G. We show that for every subgraph H of G, there is a proper conflict-free L-coloring φ of H such that $|\mathcal{U}_\varphi(v, G)| \geq \min\{1, |N_G(v) \cap V(H)|\}$ for every vertex v of G. The proof goes by induction on the order of H. The statement is trivial when $|V(H)| = 1$. Assume that $|V(H)| = n \geq 2$ and every subgraph of G with less than n vertices admits a desired L-coloring. Since G is d-degenerate, there is a vertex $v_0 \in V(H)$ such that $d_H(v_0) \leq d$. Let $H' = H - v_0$, and let φ' be a proper conflict-free L-coloring of H' such that $|\mathcal{U}_{\varphi'}(v, G)| \geq \min\{1, |N_G(v) \cap V(H')|\}$ for every vertex v of G. Let $X = \{x \in N_G(v_0) \mid N_G(x) \cap V(H') \neq \emptyset\}$. For each $x \in X$, let $c(x)$ be a color in $\mathcal{U}_{\varphi'}(x, G)$. Let $C(v_0) = \{\varphi'(v) \mid v \in N_H(v_0)\} \cup \{c(x) \mid x \in X\}$. Since $|C(v_0)| \leq d_H(v_0) + d_G(v_0) \leq d + d_G(v_0) < |L(v_0)|$, there is a color $\alpha \in L(v_0) \setminus C(v_0)$. We define a coloring φ of H by $\varphi(v) = \varphi'(v)$ for every vertex $v \in V(H) \setminus \{v_0\}$ and $\varphi(v_0) = \alpha$. By the choice of color α, it is easy to verify that φ is a proper conflict-free L-coloring of H such that $|\mathcal{U}_\varphi(v, G)| \geq \min\{1, |N_G(v) \cap V(H)|\}$ for every vertex v of G. $\square$

As a corollary of Proposition 7.5, it follows that

> for every graph G and every list assignment L of G such that $|L(v)| \geq 2d_G(v)+1$ for every vertex v of G, G is proper conflict-free L-colorable,

which implies the bound of proper conflict-free chromatic number shown in Cranston and Liu [20].

While it is conjectured that every connected graph G satisfies $\chi_{\text{pcf}}(G) \leq \Delta(G)+1$ unless G is not isomorphic to a cycle (Conjecture 7.1), there is a graph that is not isomorphic to a cycle and not proper conflict-free $(\text{degree} + 1)$-choosable. Indeed, let G be a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ with a chord v_1v_5, and let L be a list assignment of G such that $L(v_1) = L(v_5) = \{1, 2, 3, 4\}$ and $L(v_i) = \{1, 2, 3\}$ for every $i \in [8] \setminus \{1, 5\}$ (Figures 7.1 and 7.2). Suppose that G admits a proper conflict-free L-coloring φ. Since $v_1v_5 \in E(G)$, at least one of v_1 and v_5 receives a color in $\{1, 2, 3\}$. Without loss of generality, we may assume that $\varphi(v_1) = 1$. Since $\varphi(v_2) \neq \varphi(v_1)$ and $\varphi(v_i) \notin \{\varphi(v_{i-1}), \varphi(v_{i-2})\}$ for $i \in \{3, 4\}$, we conclude that $\{\varphi(v_2), \varphi(v_3)\} = \{2, 3\}$ and $\varphi(v_4) = 1$. Similarly, we have $\{\varphi(v_7), \varphi(v_8)\} = \{2, 3\}$ and $\varphi(v_6) = 1$. This is a contradiction since $N_G(v_5) = \{v_1, v_4, v_6\}$

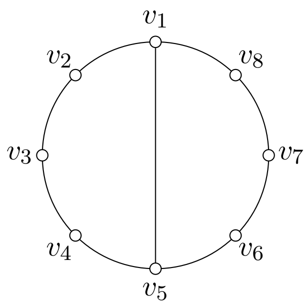


Figure 7.1: A graph G.

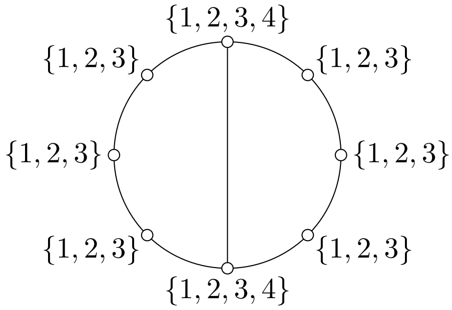


Figure 7.2: A list assignment L of G.

and all of them are colored with a color 1. Hence G is not proper conflict-free L-colorable, and thus G is not proper conflict-free (degree + 1)-choosable.

Furthermore, by the following theorem, there are infinitely many graphs which are not proper conflict-free (degree + 1)-choosable.

Theorem 7.6. For every connected graph H, there is a connected graph G such that H is an induced subgraph of G and G is not proper conflict-free (degree + 1)-choosable.

Hence, we focus on proper conflict-free (degree + k)-choosability for the cases $k = 2$ and $k = 3$. Since the cycle C_5 is not proper conflict-free 4-colorable, it follows that C_5 is not proper conflict-free (degree + 2)-choosable. On the other hand, we do not know any other graph which is not proper conflict-free (degree + 2)-choosable, which motivates us to pose the following conjecture.

Conjecture 7.7. Every connected graph other than C_5 is proper conflict-free (degree+2)-choosable.

The following three theorems give classes of proper conflict-free (degree + 2)-choosable graphs. Recall that $S(G)$ is a graph obtained from a graph G by subdividing each edge once.

Theorem 7.8. Every connected graph with maximum degree at most 3 other than C_5 is proper conflict-free (degree + 2)-choosable.

Theorem 7.9. For every graph G, $S(G)$ is proper conflict-free (degree + 2)-choosable.

Theorem 7.10. Every 2-connected outerplanar graph other than C_5 is proper conflict-free (degree + 2)-choosable.

Since C_5 is proper conflict-free (degree + 3)-choosable, a weaker conjecture than Conjecture 7.7 is that every connected graph is proper conflict-free (degree + 3)-choosable. Towards the weaker conjecture, we show the following theorem on proper conflict-free (degree + 3)-choosable graphs.

Theorem 7.11. Every graph with maximum degree at most 4 is proper conflict-free (degree + 3)-choosable.

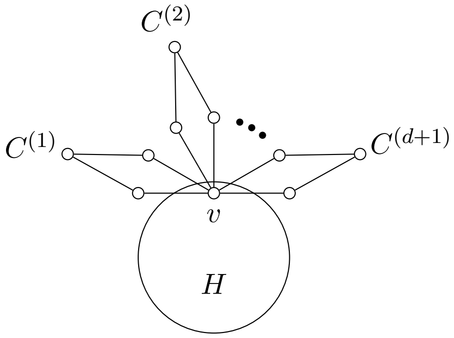


Figure 7.3: A graph G defined from (H, v_0) in the proof of Theorem 7.6.

7.3 Proof of Theorem 7.6

Let H be an arbitrary connected graph and let v_0 be a vertex of H. Let $d_H(v_0) = d$. For each $i \in [d+1]$, let $C^{(i)} = v_0^{(i)}v_1^{(i)}v_2^{(i)}v_3^{(i)}v_0^{(i)}$ be a cycle of length 4. Let G be a graph obtained from H and $C^{(1)}, C^{(2)}, \ldots, C^{(d+1)}$ by identifying v_0 and $v_0^{(1)}, v_0^{(2)}, \ldots, v_0^{(d+1)}$ into a vertex v (Figure 7.3). Then H is an induced subgraph of G. We define a list assignment L of G as follows.

- for each $i \in [d+1]$, let $L(v_1^{(i)}) = L(v_2^{(i)}) = L(v_3^{(i)}) = \{3i-2, 3i-1, 3i\}$,
- $L(v) = [3d+3]$, and
- for each vertex $u \in V(G) \setminus \{v\}$, let $L(u) = [d_G(u) + 1]$.

Since $d_G(v) = d_H(v_0) + 2(d+1) = d + 2(d+1) = 3d+2$, L is a list assignment of G such that $|L(u)| \geq d_G(u) + 1$ for every vertex u of G. We show that G is not proper conflict-free L-colorable. Assume to the contrary that G admits a proper conflict-free L-coloring φ. For each $i \in [d+1]$, since $\mathcal{U}_\varphi(v_2^{(i)}, G) \neq \emptyset$, we have $\{\varphi(v_1^{(i)}), \varphi(v_2^{(i)}), \varphi(v_3^{(i)})\} = \{3i-2, 3i-1, 3i\}$. In addition, since $\varphi(v_j^{(i)}) \neq \varphi(v)$ and $\mathcal{U}_\varphi(v_j^{(i)}, G) \neq \emptyset$ for $j = 1, 3$, we have $\varphi(v) \notin \{\varphi(v_1^{(i)}), \varphi(v_2^{(i)}), \varphi(v_3^{(i)})\} = \{3i-2, 3i-1, 3i\}$. Combining these, we have $\varphi(v) \notin \bigcup_{i=1}^{d+1}\{3i-2, 3i-1, 3i\} = [3d+3] = L(v)$, a contradiction. Hence G is not proper conflict-free L-colorable.

7.4 Proper conflict-free $(\text{degree} + 2)$-choosable graphs

The following lemmas are used in our proofs of Theorems 7.8, 7.9 and 7.10.

Lemma 7.12. Every cycle other than C_5 is proper conflict-free $(\text{degree} + 2)$-choosable.

Proof. Let $C = v_0v_1 \cdots v_{n-1}v_0$ be a cycle of length $n \neq 5$ and let L be a list assignment of G such that $|L(v_i)| = 4$ for every $i \in [0, n-1]$. Obviously C is proper conflict-free L-colorable when $n \leq 4$. We assume that $n \geq 6$. If $L(v_0) = L(v_1) = \cdots = L(v_{n-1})$, then it follows from a result in Caro, Petruševski, and Škrekovski [13] that C is proper conflict-free 4-colorable, and thus C is proper conflict-free L-colorable. Hence, we may assume that $L(v_i) \neq L(v_j)$ for some i and j with $0 \leq i < j \leq n-1$. Without loss of generality, we may assume that $L(v_0) \neq L(v_{n-1})$. Let α be a color in $L(v_0) \setminus L(v_{n-1})$. We define an L-coloring φ of C as follows. We first let $\varphi(v_0) = \alpha$, and then sequentially choose colors for $v_1, v_2, \dots, v_{n-3}$ so that $\varphi(v_i) \in L(v_i) \setminus \{\varphi(v_{i-2}), \varphi(v_{i-1})\}$ for each $i \in \{1, 2, \dots, n-3\}$. Finally, we choose $\varphi(v_{n-2}) \in L(v_{n-2}) \setminus \{\varphi(v_{n-4}), \varphi(v_{n-3}), \alpha\}$ and choose $\varphi(v_{n-1}) \in L(v_{n-1}) \setminus \{\varphi(v_{n-3}), \varphi(v_{n-2}), \varphi(v_1), \alpha\}$, where such a color $\varphi(v_{n-1})$ exists since $\alpha \notin L(v_{n-1})$. By the choice of colors, it follows that $\varphi(v_i) \neq \varphi(v_{i+2})$ and $\varphi(v_i) \neq \varphi(v_{i+2})$ for every $i \in [0, n-1]$ where the indices are on modulo n. Thus, φ is a proper conflict-free L-coloring of C. $\square$

While the cycle C_5 is not proper conflict-free 4-colorable, if every vertex has at least 4 available colors and one vertex has 5 available colors, then we can greedily color all vertices with distinct colors. Thus, the following lemma holds.

Lemma 7.13. Let $C = v_0v_1v_2v_3v_4v_0$ be a cycle of length 5, and let L be a list assignment of C. If $|L(v_i)| \geq 4$ for every $i \in [0, 4]$ and $|L(v_0)| \geq 5$, then C is proper conflict-free L-colorable.

7.4.1 Proof of Theorem 7.8

Let G be a connected graph with maximum degree at most 3, which is not isomorphic to C_5. Let L be a list assignment of G such that $|L(v)| \geq d_G(v) + 2$ for each $v \in V(G)$. Obviously G is proper conflict-free L-colorable when G is isomorphic to K_1 or K_2. We assume that G has at least three vertices.

If G is a cycle, then we are done by Lemma 7.12. Thus, we may assume that G is not a cycle. Furthermore, if G is a tree, then G is 1-degenerate and thus G is proper conflict-free L-colorable by Proposition 7.5. Thus, the *core* of G is not isomorphic to K_1, where the core of G is a subgraph obtained from G by repeatedly removing a vertex of degree 1 until there is no vertex of degree 1. It follows that if the core of G is proper conflict-free L-colorable, then G is proper conflict-free L-colorable as well. Indeed, suppose that H is a subgraph of G with at least two vertices, v be a vertex of H such that $d_H(v) = 1$, and u be the neighbor of v in H. Suppose that $H - v$ is proper conflict-free L-colorable. Let φ be a proper conflict-free L-coloring of $H - v$. Note that $|L(v)| \geq d_G(v) + 2 = 3$. Let $\varphi(u) = \alpha$ and let β be a color in $\mathcal{U}_\varphi(u, H - v)$. Then we extend φ to a proper conflict-free L-coloring of H by letting $\varphi(v) \in L(v) \setminus \{\alpha, \beta\}$. Thus, H is proper conflict-free L-colorable. Repetition of this argument leads us to the conclusion that if the core of G is proper conflict-free L-colorable, then G is proper conflict-free L-colorable.

Let G' be the core of G. Since G' is not isomorphic to K_1, by the definition of the core, we have $\delta(G') \geq 2$, and thus $|L(v)| \geq 4$ for every vertex v of G. If G' is not a cycle, then we have $\Delta(G') = 3$ and it follows from Theorem 7.2 that G' is proper conflict-free L-colorable. Thus, we assume that G' is a cycle $v_0v_1 \cdots v_{l-1}v_0$. Since $V(G) \setminus V(G') \neq \emptyset$, at least one vertex of G', say v_0, has degree 3 in G. Hence, we have $|L(v_i)| \geq 4$ for every $i \in [0, l-1]$, and $|L(v_0)| \geq 5$. By Lemmas 7.12 and 7.13, G' is proper conflict-free L-colorable. This completes the proof of Theorem 7.8.

7.4.2 Proof of Theorem 7.9

Let G be a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$. The proof goes by induction on n. If G is disconnected, then we prove for each component of $S(G)$ and simply combine them to obtain a desired coloring of $S(G)$. Thus, we assume that G is connected. Let $S(G)$ be labeled as $V(S(G)) = \{v_i \mid 1 \leq i \leq n\} \cup \{u_e \mid e \in E(G)\}$ and $E(S(G)) = \{v_i u_e \mid v_i \text{ incident with } e \text{ in } G\}$. Note that $d_{S(G)}(v_i) = d_G(v_i)$ for every $i \in \{1, 2, \ldots, n\}$ and $d_{S(G)}(u_e) = 2$ for every $e \in E(G)$. Let L be a list assignment of G such that $|L(v_i)| \geq d_G(v_i) + 2$ for each $i \in \{1, 2, \ldots, n\}$ and $|L(u_e)| = 4$ for every edge $e \in E(G)$. If $n \leq 2$, then obviously $S(G)$ is 1-degenerate, and thus $S(G)$ is proper conflict-free (degree + 2)-choosable by Proposition 7.5. Thus, we assume that $n \geq 3$ and $S(G')$ is proper conflict-free (degree + 2)-choosable for every graph G' of order less than n.

Suppose that G contains a vertex v_i with $d_G(v_i) = 1$. Let e be the only edge of G with an end vertex v_i. By the induction hypothesis, $S(G - v_i)$ admits a proper conflict-free L-coloring φ. Since $S(G - v)$ is obtained from $S(G)$ by sequentially removing v_i and u_e, by the argument in the proof of Theorem 7.8, φ can be extended to a proper conflict-free L-coloring of $S(G)$. We assume that $\delta(G) \geq 2$, and thus $\delta(S(G)) \geq 2$ as well.

If every vertex of G has degree 2, then $S(G)$ is a cycle of length $2n$, which is proper conflict-free L-colorable by Lemma 7.12. Thus, without loss of generality, we may assume that $d_G(v_1) \geq 3$ and $N_G(v_1) = \{v_2, v_3, \ldots, v_{d+1}\}$, where $d \geq 3$ is the degree of v_1 in G. Let $e_i = v_1 v_i \in E(G)$ for each $i \in \{2, 3, \ldots, d+1\}$. For each color $c \in L(v_1)$, let $U_c = \{u_{e_i} \mid 2 \leq i \leq d+1, c \in L(u_{e_i})\}$. Then there is a color $\alpha \in L(v_1)$ such that $|U_\alpha| \leq \min\{3, d-1\}$. Indeed, if $d = 3$, then since $|L(v_1)| \geq 5$ and $|L(u_{e_2})| = 4$, a color $\alpha \in L(v_1) \setminus L(u_{e_2})$ satisfies $U_\alpha \subseteq \{u_{e_3}, u_{e_4}\}$. If $d \geq 4$, then we have $|L(v_1)| \geq d+2$, which implies that there must be a color $\alpha \in L(v_1)$ such that $|U_\alpha| \leq 3$, for otherwise $4(d+2) \leq \sum_{c \in L(v_1)} |U_c| = \sum_{i=2}^{d+1} |L(u_{e_i})| = 4d$, a contradiction. Without loss of generality, we may assume that $u_{e_2} \notin U_\alpha$.

Let $G' = G - v_1$ and let L' be a list assignment of $S(G')$ such that $L'(v_i) = L(v_i) \setminus \{\alpha\}$ for each $i \in \{2, 3, \ldots, d+1\}$ and $L'(w) = L(w)$ for every vertex $w \in V(S(G')) \setminus \{v_i \mid 2 \leq i \leq d+1\}$. By the induction hypothesis, $S(G')$ admits a proper conflict-free L'-coloring φ. We extend φ to a proper conflict-free L-coloring of G. Note that G' has no isolated vertices since $\delta(G) \geq 2$. For each $i \in \{2, 3, \ldots, d+1\}$, let c_i be a color in $\mathcal{U}_\varphi(v_i, S(G'))$. Let $\varphi(v_1) = \alpha$, and we arbitrarily choose $\varphi(u_{e_i}) \in L(u_{e_i}) \setminus \{\varphi(v_i), c_i, \alpha\}$ for each $u_{e_i} \in U_\alpha$. We define a color β as follows: If $U_\alpha = \emptyset$, then let β be a color in $L(u_{e_2}) \setminus \{\varphi(v_2), c_2, \alpha\}$. If $U_\alpha \neq \emptyset$ and $\varphi(u_{e_i}) = \gamma$ for all vertices $u_{e_i} \in U_\alpha$ for some color γ, then since $\alpha \notin L(u_{e_2})$, let β be a color in $L(u_{e_2}) \setminus \{\varphi(v_2), c_2, \alpha, \gamma\}$. Otherwise, i.e. $U_\alpha \neq \emptyset$ and $|\{\varphi(u_{e_i}) \mid u_{e_i} \in U_\alpha\}| \geq 2$, the fact $|U_\alpha| \leq 3$ implies that some color appears exactly once at U_α. Let β be the color. Finally, for each uncolored vertex u_{e_i}, since $u_{e_i} \notin U_\alpha$, we can choose a color $L(u_{e_i}) \setminus \{\varphi(v_j), c_j, \alpha, \beta\}$ as $\varphi(u_{e_i})$. It is easy to verify that φ is a proper L-coloring of $S(G)$, and that $\mathcal{U}_\varphi(w, S(G)) = \mathcal{U}_\varphi(w, S(G')) \neq \emptyset$ for each $w \in V(S(G')) \setminus \{v_i \mid 2 \leq i \leq d+1\}$. Since $\varphi(u_{e_i}) \neq c_i$, $\mathcal{U}_\varphi(v_i, S(G))$ contains the color

c_i for each $i \in \{2, 3, \ldots, d+1\}$. By the definition of L', we have $\varphi(v_i) \neq \alpha = \varphi(v_1)$, and thus $\alpha \in \mathcal{U}_\varphi(u_{e_i}, S(G))$ for each $i \in \{2, 3, \ldots, d+1\}$. Furthermore, by the choice of colors $\{\varphi(u_{e_i}) \mid 2 \leq i \leq d+1\}$, we have $\beta \in \mathcal{U}_\varphi(v_1, S(G))$. Hence φ is a proper conflict-free L-coloring of $S(G)$. This completes the proof of Theorem 7.9.

7.4.3 Lemmas for Theorem 7.10

In this subsection, we show lemmas used in our proof of Theorem 7.10. We define an unavoidable set of 2-connected outerplanar graphs, which is smaller than that used in the proof of Theorem 5.10. Let G be a 2-connected outerplanar graph which is not a cycle.

- An *ear* H of G is a cycle $u_1u_2 \cdots u_ru_1$ such that $d_G(u_i) = 2$ for $i \in \{2, 3, \ldots, r-1\}$. The edge u_1u_r is the *root edge* of H. We say H is *good* if either $d_G(u_1) = 3$ or $d_G(u_r) = 3$.
- An *ear chain* H is a subgraph induced by $\bigcup_{i=1}^{s-1} V(H_i)$ ($s \geq 3$) where each H_i is an ear of G such that the root edge of the ears form a cycle $v_1v_2 \cdots v_sv_1$ (the root edge of H_i is v_iv_{i+1}) and $d_G(v_i) = 4$ for $i \in \{2, 3, \ldots, s-1\}$. The edge v_1v_s is the *root edge* of H.

Then the following holds.

Lemma 7.14. Let G be a 2-connected outerplanar graph of order at least 4. If G is not a cycle, then G contains a good ear or an ear chain.

Proof. Let G be a 2-connected outerplanar graph of order $n \geq 4$ that is not a cycle. Let F be the outercycle of G. Let E_1 be the set of edges which is the root edge of an ear of G, and let G_1 be a subgraph of G induced by E_1. We first suppose that $E(G) = E(F) \cup E_1$. By the definition of E, G_1 is either a cycle or a disjoint union of paths. If G_1 is a cycle $v_1v_2v_3 \cdots v_lv_1$, where H_i is an ear with the root edge v_iv_{i+1} for each $i \in [l-1]$, then a subgraph induced by $\bigcup_{i=1}^{l-1} V(H_i)$ is an ear chain of G with the root edge v_1v_l. Otherwise, there is an edge v_1v_2 of G_1 such that $d_{G_1}(v_1) = 1$, and let H_1 be the ear of G with the root edge v_1v_2. Since $d_G(v_1) = d_{G_1}(v_1) + 2 = 3$, H_1 is a good ear of G.

Hence, we suppose that $E(G) \setminus (E(F) \cup E_1) \neq \emptyset$. For each edge $xy \in E(G) \setminus (E(F) \cup E_1)$, let H_{xy} be a subgraph of G induced by the vertices of an xy-subpath of F so that $|V(H_{xy})| \leq \frac{1}{2}(|V(G)| + 2)$. We take an edge $xy \in E(G) \setminus (E(F) \cup E_1)$ so that the order of H_{xy} is as small as possible. By the choice of xy, we have $E(H_{xy}) \subseteq E(F) \cup E_1 \cup \{xy\}$. Let G_2 be a subgraph of G induced by $E_1 \cap E(H_{xy})$. Then G_2 is either a xy-path or a disjoint union of paths. If G_2 is an xy-path, then H_{xy} is an ear chain of G. Otherwise, there is an edge $v'_1v'_2$ of G_2 such that $d_{G_2}(v'_1) = 1$, and let H'_1 be the ear of G with the root edge $v'_1v'_2$. Since $d_G(v'_1) = d_{G_2}(v'_1) + 2 = 3$, H'_1 is a good ear of G. $\square$

The following lemma is used repeatedly in our proof of Theorem 7.10.

Lemma 7.15. Let G be a 2-connected outerplanar graph and let L be a list assignment of G such that $|L(v)| \geq d_G(v) + 2$ for every vertex $v \in V(G)$. Let $H = u_1u_2 \cdots u_ru_1$ be an ear of G with the root edge u_1u_r, and let G' be a subgraph of $G - (V(H) \setminus \{u_1, u_r\})$ such that $u_1, u_r \in V(G')$. Suppose that there is a proper L-coloring φ' of G' such that either $\mathcal{U}_{\varphi'}(u_1, G') \neq \emptyset$ or $N_{G'}(u_1)$ is colored with one color. Then φ' can be extended to a proper L-coloring φ of $G' \cup H$ such that $\mathcal{U}_{\varphi}(u_i, G) \neq \emptyset$ for each $i \in [r-1]$.

Proof. Let φ' be a proper L-coloring of G' with the assumption of Lemma 7.15. Without loss of generality, we may assume that $\varphi'(u_1) = 1$. We may also assume that either $2 \in \mathcal{U}_{\varphi}(u_1, G')$ or every vertex of $N_{G'}(u_1)$ is colored with a color 2. Let $\varphi(v) = \varphi'(v)$ for every vertex $v \in V(G')$, and define $\varphi(u_i) \in L(u_i)$ for each $i \in \{2, 3, \ldots, r-1\}$ as follows. Let $\varphi(u_2) \in L(u_2) \setminus \{1, 2\}$, and for $i = 3, 4, \ldots, r-3$, we sequentially choose $\varphi(u_i) \in L(u_i) \setminus \{\varphi(u_{i-2}), \varphi(u_{i-1})\}$. Then we choose a color in $L(u_{r-2}) \setminus \{\varphi(u_r), \varphi(u_{r-4}), \varphi(u_{r-3})\}$ as $\varphi(u_{r-2})$, and choose a color in $L(u_{r-1}) \setminus \{\varphi(u_r), \varphi(u_{r-3}), \varphi(u_{r-2})\}$ as $\varphi(u_{r-1})$. By the choice of colors, we have $\varphi(u_{i-1}) \in \mathcal{U}_{\varphi}(u_i, G' \cup H)$ for each $i \in \{2, 3, \ldots, r-1\}$. Furthermore, we have $2 \in \mathcal{U}_{\varphi}(u_1, G' \cup H)$ if $2 \in \mathcal{U}_{\varphi'}(u_1, G')$ and $\varphi(u_2) \in \mathcal{U}_{\varphi}(u_1, G' \cup H)$ otherwise. Hence φ is a desired coloring of $G' \cup H$. $\square$

7.4.4 Proof of Theorem 7.10

The proof goes by induction on the number of vertices. Let G be a 2-connected outerplanar graph of order n. Let L be a list assignment of G such that $|L(v)| \geq d_G(v) + 2$ for every vertex $v \in V(G)$. We may assume that $|L(v)| = d_G(v) + 2$ for every vertex $v \in V(G)$. When $n = 3$, G is isomorphic to K_3 and proper conflict-free L-colorable. Thus, we assume that $n \geq 4$ and that every 2-connected outerplanar graph of order less than n is proper conflict-free (degree $+\,2$)-choosable. If G is a cycle, then we are done by Lemma 7.12. We assume that G is not a cycle. By Lemma 7.14, G contains either a good ear or an ear chain.

Case 1. G contains a good ear.

Let $H = u_1u_2 \cdots u_ru_1$ be a good ear of G with the root edge u_1u_r such that $d_G(u_i) = 2$ for each $i \in \{2, 3, \ldots, r-1\}$ and $d_G(u_r) = 3$. Let $G' = G - (V(H) \setminus \{u_1, u_r\})$. Note that $|L(u_1)| = d_G(u_1) + 2 = d_{G'}(u_1) + 3$. By the induction hypothesis and Lemma 7.13, G' admits a proper conflict-free L-coloring φ. By Lemma 7.15, φ can be extended to a proper L-coloring of G such that $\mathcal{U}_{\varphi}(v, G) \neq \emptyset$ for every vertex v other than u_r. Since $d_{G'}(u_r) = 2$ and $\mathcal{U}_{\varphi}(u_r, G') \neq \emptyset$, we have $|\mathcal{U}_{\varphi}(u_r, G')| = 2$. Thus, it follows that $|\mathcal{U}_{\varphi}(u_r, G)| \geq |\mathcal{U}_{\varphi}(u_r, G')| - 1 \geq 1$, which implies that φ is a proper conflict-free L-coloring of G.

Before going to the case where G contains an ear chain, we consider the case where G contains an ear with at least 6 vertices.

Case 2. G contains an ear with at least 6 vertices.

Let $H = u_1u_2 \cdots u_ru_1$ be an ear of G with the root edge u_1u_r such that $d_G(u_i) = 2$ for each $i \in \{2, 3, \ldots, r-1\}$, and suppose that $r \geq 6$. Let $G' = G - \{v_i \mid 2 \leq i \leq l-1\}$. Note that $|L(u_1)| = d_G(u_1) + 2 = d_{G'}(u_1) + 3$. By the induction hypothesis and Lemma 7.13, G' admits a proper conflict-free L-coloring φ. Without loss of generality, we may assume

$\varphi(u_1) = 1$ and $\varphi(u_r) = 2$. Let α be a color in $\mathcal{U}_\varphi(u_1, G')$, and let β be a color in $\mathcal{U}_\varphi(u_r, G')$. Note that $r - 2 > 3$ by the assumption of case. Let $\varphi(u_2) \in L(u_2) \setminus \{1, \alpha\}$. We consider the following three subcases.

Subcase 2.1. $|L(u_{r-1}) \cap \{2, \beta\}| \leq 1$.

For $i = 3, 4, \ldots, r-3$, we sequentially choose a color in $L(u_i) \setminus \{\varphi(u_{i-2}), \varphi(u_{i-1})\}$ as $\varphi(u_i)$. Then we choose a color $\varphi(u_{r-2}) \in L(u_{r-2}) \setminus \{2, \varphi(u_{r-4}), \varphi(u_{r-3})\}$. Since $|L(u_{r-1}) \cap \{2, \beta\}| \leq 1$, we can choose $\varphi(u_{r-1}) \in L(u_{r-1}) \setminus \{2, \beta, \varphi(u_{r-3}), \varphi(u_{r-2})\}$. It is easy to verify that φ is a proper conflict-free L-coloring of G.

Subcase 2.2. $\{2, \beta\} \subseteq L(u_{r-1})$ and $L(u_{r-2}) \setminus L(u_{r-1}) \neq \emptyset$.

Let $\varphi(u_{r-2}) \in L(u_{r-2}) \setminus L(u_{r-1})$. Note that $\varphi(u_{r-2}) \neq 2$ since $2 \in L(u_{r-1})$. For $i = 3, 4, \ldots, r-3, r-1$, we sequentially choose a color $\varphi(u_i) \in L(u_i)$ as follows. For $i \in \{3, 4, \ldots, r-5\}$, we let $\varphi(u_i) \in L(u_i) \setminus \{\varphi(u_{i-2}), \varphi(u_{i-1})\}$. For $i \in \{r-4, r-3\}$, we let $\varphi(u_i) \in L(u_i) \setminus \{\varphi(u_{r-2}), \varphi(u_{i-2}), \varphi(u_{i-1})\}$. Since $\varphi(u_{r-2}) \notin L(u_{r-1})$, we can choose $\varphi(u_{r-1}) \in L(u_{r-1}) \setminus \{2, \beta, \varphi(u_{r-3}), \varphi(u_{r-2})\}$. It is easy to verify that φ is a proper conflict-free L-coloring of G.

Subcase 2.3. $\{2, \beta\} \subseteq L(u_{r-1})$ and $L(u_{r-2}) \setminus L(u_{r-1}) = \emptyset$.

As $|L(u_{r-1})| = |L(u_{r-2})| = 4$, the subcase assumption forces $L(u_{r-1}) = L(u_{r-2})$, and thus $\beta \in L(u_{r-2})$. Let $\varphi(u_{r-2}) = \beta$. For $i = 3, 4, \ldots, r-3, r-1$, we sequentially choose a color $\varphi(u_i) \in L(u_i)$ as follows. For $i \in \{3, 4, \ldots, r-5\}$, we let $\varphi(u_i) \in L(u_i) \setminus \{\varphi(u_{i-2}), \varphi(u_{i-1})\}$. For $i \in \{r-4, r-3\}$, we let $\varphi(u_i) \in L(u_{r-2}) \setminus \{\beta, \varphi(u_{i-2}), \varphi(u_{i-1})\}$. Finally, we choose a color in $L(u_{r-1}) \setminus \{2, \beta, \varphi(u_{r-3})\}$ as $\varphi(u_{r-1})$. It is easy to verify that φ is a proper conflict-free L-coloring of G. This completes the proof for Case 2.

Now we consider the case where G contains an ear chain.

Case 3. G contains an ear chain.

Let H be an ear chain of G consisting of ears $H_1, H_2, \ldots, H_{s-1}$ ($s \geq 3$) of G such that the root edge of H_i is $v_i v_{i+1}$ for each $i \in [s-1]$. By Case 2, we may assume that $|V(H_i)| \leq 5$ for each $i \in [s-1]$. Note that $|L(v_i)| = 6$ for each $i \in \{2, 3, \ldots, s-1\}$ and $|L(v)| = 4$ for every vertex $v \in V(H) \setminus \{v_i \mid 2 \leq i \leq r-1\}$. We consider the following four subcases.

Subcase 3.1. $s \geq 4$.

Let $G' = G - (V(H) \setminus \{v_1, v_s\})$. Note that $|L(v_1)| = d_G(v_1) + 2 = d_{G'}(v_1) + 4$. By the induction hypothesis and Lemma 7.13, G' admits a proper conflict-free L-coloring φ. Without loss of generality, we may assume that $\varphi(v_1) = 1$ and $\varphi(v_s) = 2$. Let α be a color in $\mathcal{U}_\varphi(v_1, G')$ and let β be a color in $\mathcal{U}_\varphi(v_s, G')$. Let $H_{s-1} = v_{s-1} u_2 u_3 \cdots u_{r-1} v_s v_{s-1}$, where $r \in \{3, 4, 5\}$. We give a coloring of $\{v_i \mid 2 \leq i \leq s-1\} \cup V(H_{s-1})$ as follows. We first choose $\varphi(u_{r-1}) \in L(u_{r-1}) \setminus \{2, \beta\}$, and for $j = r-2, r-3, \ldots, 2$, we sequentially choose a color in $L(u_j) \setminus \{2, \varphi(u_{j+1}), \varphi(u_{j+2})\}$ as $\varphi(u_j)$, where $u_r = v_s$. Let $\varphi(v_{s-1}) \in L(v_{s-1}) \setminus \{2, \beta, \varphi(u_2), \varphi(u_3)\}$. Since $|L(v_i)| = 6$ for each $i \in \{2, 3, \ldots, s-2\}$, we can choose

$\varphi(v_i) \in L(v_i)$ so that $\varphi(v_2) \notin \{1, \alpha\}$, $\varphi(v_{s-2}) \notin \{2, \varphi(v_{s-1}), \varphi(u_2)\}$, and $\varphi(v_i) \neq \varphi(v_{i-1})$ for every $i \in \{2, 3, \ldots, s-2\}$. For the remaining ears $H_1, H_2, \ldots, H_{s-2}$, we sequentially apply Lemma 7.15 to extend φ to a proper L-coloring of G such that $\mathcal{U}_\varphi(v, G) \neq \emptyset$ for every vertex $v \in V(H_i) \setminus \{v_{i+1}\}$. Indeed, when we color H_i with the root edge v_iv_{i+1}, the vertex v_i has three colored neighbors. Thus, either $\mathcal{U}_\varphi(v_i, G - V(H_i)) \neq \emptyset$ or all three neighbors receive the same color, and hence the assumptions of Lemma 7.15 are satisfied for each H_i. By the choice of colors, it is easy to verify that $\mathcal{U}_\varphi(v, G) \neq \emptyset$ for every vertex $v \in V(G) \setminus \{v_{s-1}\}$. Since $d_G(v_{s-1}) = 4$ and $\varphi(v_{s-2})$, $\varphi(v_s)$ and $\varphi(u_2)$ are distinct three colors, we have $\mathcal{U}_\varphi(v_{s-1}, G) \neq \emptyset$. Thus, φ is a proper conflict-free L-coloring of G.

Subcase 3.2. $s = 3$ and $|V(H_2)| = 3$.

Let $H_2 = v_2u_2v_3v_2$. Let $G' = G - (V(H) \setminus \{v_1, v_3\})$. Note that $|L(v_1)| = d_G(v_1) + 2 = d_{G'}(v_1) + 4$. By the induction hypothesis and Lemma 7.13, G' admits a proper conflict-free L-coloring φ. Without loss of generality, we may assume that $\varphi(v_1) = 1$ and $\varphi(v_3) = 2$. Let α be a color in $\mathcal{U}_\varphi(v_1, G')$ and let β be a color in $\mathcal{U}_\varphi(v_3, G')$. We choose $\varphi(u_2) \in L(u_2) \setminus \{1, 2, \beta\}$, and choose $\varphi(v_2) \in L(v_2) \setminus \{1, 2, \alpha, \beta, \varphi(u_2)\}$. Then we apply Lemma 7.15 to the ear H_1 to extend φ to a proper L-coloring of G such that $\mathcal{U}_\varphi(v, G) \neq \emptyset$ for every vertex $v \in V(H_1) \setminus \{v_2\}$. By the choice of colors, it is easy to verify that $\mathcal{U}_\varphi(v, G) \neq \emptyset$ for every vertex $v \in V(G) \setminus \{v_2\}$. Since $d_G(v_2) = 4$ and $\varphi(v_1)$, $\varphi(v_3)$ and $\varphi(u_2)$ are distinct three colors, we have $\mathcal{U}_\varphi(v_2, G) \neq \emptyset$, and thus φ is a proper conflict-free L-coloring of G.

Subcase 3.3. $s = 3$ and $|V(H_2)| = 4$.

Let $H_2 = v_2u_2u_3v_3v_2$. We fix a color γ such that $\gamma \in L(u_2) \setminus L(u_3)$ if $L(u_2) \setminus L(u_3) \neq \emptyset$ and $\gamma \in L(u_2)$ otherwise. Let $G' = G - (V(H) \setminus \{v_1, v_3\})$ and let L' be a list assignment of G' such that $L'(v_1) = L(v_1) \setminus \{\gamma\}$, $L'(v_3) = L(v_3) \setminus \{\gamma\}$, and $L'(v) = L(v)$ for every vertex $v \in V(G') \setminus \{v_1, v_3\}$. Since $|L'(v)| \geq d_{G'}(v) + 2$ for every vertex $v \in V(G')$ and $|L'(v_1)| \geq |L(v_1)| - 1 = d_G(v_1) + 1 = d_{G'}(v_1) + 3$, by the induction hypothesis and Lemma 7.13, G' admits a proper conflict-free L'-coloring φ. Without loss of generality, we may assume that $\varphi(v_1) = 1$ and $\varphi(v_3) = 2$. Let α be a color in $\mathcal{U}_\varphi(v_1, G')$ and let β be a color in $\mathcal{U}_\varphi(v_3, G')$. Note that $\gamma \notin \{1, 2\}$ by the definition of L'.

Suppose first that $\beta = 1$. We let $\varphi(u_2) \in L(u_2) \setminus \{1, 2\}$, $\varphi(u_3) \in L(u_3) \setminus \{1, 2, \varphi(u_2)\}$, and $\varphi(v_2) \in L(v_2) \setminus \{1, 2, \alpha, \varphi(u_2), \varphi(u_3)\}$. Then we use Lemma 7.15 to the ear H_1 to extend φ to a proper L-coloring of G such that $\mathcal{U}_\varphi(v, G) \neq \emptyset$ for every vertex $v \in V(G) \setminus \{v_2\}$. Since $d_G(v_2) = 4$ and $\varphi(v_1)$, $\varphi(v_3)$, $\varphi(u_2)$ are three distinct colors, we have $\mathcal{U}_\varphi(v_2, G) \neq \emptyset$, and thus φ is a proper conflict-free L-coloring of G.

Thus, we may assume that $\beta \neq 1$. Then there is a color $\gamma' \in L(u_2) \setminus \{1, 2\}$ (possibly $\gamma' = \gamma$) such that $|L(u_3) \setminus \{2, \beta, \gamma'\}| \geq 2$. Indeed, if $L(u_2) \setminus L(u_3) \neq \emptyset$, then we know $\gamma \in L(u_2) \setminus \{1, 2\}$ and $|L(u_3) \setminus \{2, \beta, \gamma\}| = |L(u_3) \setminus \{2, \beta\}| \geq 2$ by the choice of γ. Otherwise, since $L(u_2) = L(u_3)$ and $\{1, 2\} \neq \{2, \beta\}$, either $|L(u_3) \setminus \{2, \beta\}| \geq 3$, or $|L(u_3) \setminus \{2, \beta\}| = 2$ and $L(u_2) \setminus \{1, 2\} \neq L(u_3) \setminus \{2, \beta\}$. In the former case, we have $|L(u_3) \setminus \{2, \beta, \gamma\}| \geq |L(u_3) \setminus \{2, \beta\}| - 1 \geq 2$. In the latter case, there is a color $\gamma' \in (L(u_2) \setminus \{1, 2\}) \setminus (L(u_3) \setminus \{2, \beta\})$, and we have $|L(u_3) \setminus \{2, \beta, \gamma'\}| = |L(u_3) \setminus \{2, \beta\}| \geq 2$. We let $\varphi(u_2) = \gamma'$ and choose

$\varphi(v_2) \in L(v_2) \setminus \{1, 2, \alpha, \beta, \gamma'\}$. Since $|L(u_3) \setminus \{2, \beta, \gamma', \varphi(v_2)\}| \geq |L(u_3) \setminus \{2, \beta, \gamma'\}| - 1 \geq 1$, we can choose a color in $L(u_3) \setminus \{2, \beta, \gamma', \varphi(v_2)\}$ as $\varphi(u_3)$. By using Lemma 7.15 to the ear H_1, we extend φ to a proper L-coloring of G such that $\mathcal{U}_\varphi(v_2, G) \neq \emptyset$ for every vertex $v \in V(G) \setminus \{v_2\}$. Since $d_G(v_2) = 4$ and $\varphi(v_1)$, $\varphi(v_3)$, $\varphi(u_2)$ are three distinct colors, we have $\mathcal{U}_\varphi(v_2, G) \neq \emptyset$, and thus φ is a proper conflict-free L-coloring of G.

Subcase 3.4. $s = 3$ and $|V(H_2)| = 5$.

Let $H_2 = v_2u_2u_3u_4v_3v_2$. Since $|L(v_1)| = d_G(v_1) + 2 = d_{G'}(v_1) + 4$, by the induction hypothesis and Lemma 7.13, G' admits a proper conflict-free L-coloring φ. Without loss of generality, we may assume that $\varphi(v_1) = 1$ and $\varphi(v_3) = 2$. Let α be a color in $\mathcal{U}_\varphi(v_1, G')$ and let β be a color in $\mathcal{U}_\varphi(v_3, G')$.

First, we choose $\tilde{L}(v_2) \subseteq L(v_2) \setminus \{1, 2, \alpha, \beta\}$, $\tilde{L}(u_3) \subseteq L(u_3) \setminus \{2\}$ and $\tilde{L}(u_4) \subseteq L(u_4) \setminus \{2, \beta\}$ so that $|\tilde{L}(v_2)| = |\tilde{L}(u_4)| = 2$ and $|\tilde{L}(u_3)| = 3$. Next, choose a color $\varphi(v_2) \in \tilde{L}(v_2)$ and $\varphi(u_4) \in \tilde{L}(u_4)$ as follows. If $\tilde{L}(v_2) \cap \tilde{L}(u_4) \neq \emptyset$, then we let $\varphi(v_2) = \varphi(u_4)$ be a color in $\tilde{L}(v_2) \cap \tilde{L}(u_4)$. Otherwise, since $|\tilde{L}(u_3)| = 3$ and $|\tilde{L}(v_2) \cup \tilde{L}(u_4)| = 4$, we choose $\varphi(v_2) \in \tilde{L}(v_2)$ and $\varphi(u_4) \in \tilde{L}(u_4)$ so that $\varphi(v_2)$ or $\varphi(u_4)$ does not belong to $\tilde{L}(u_3)$. In both cases, we have $|\tilde{L}(u_3) \setminus \{\varphi(v_2), \varphi(u_4)\}| \geq 2$.

Now we apply Lemma 7.15 to the ear H_1 to extend φ to a proper L-coloring of $G - \{u_2, u_3\}$ so that $\mathcal{U}_\varphi(v, G) \neq \emptyset$ for every vertex $v \in V(G) \setminus \{v_2, u_2, u_3\}$. Since v_2 has three colored neighbors and $\varphi(v_1) \neq \varphi(v_3)$, there is a color $\gamma \in \mathcal{U}_\varphi(v_2, G)$. We choose $\varphi(u_2) \in L(u_2) \setminus \{\varphi(v_2), \varphi(u_4), \gamma\}$. Since $|\tilde{L}(u_3) \setminus \{\varphi(v_2), \varphi(u_4)\}| \geq 2$, we can choose $\varphi(u_3) \in \tilde{L}(u_3) \setminus \{\varphi(v_2), \varphi(u_2), \varphi(u_4)\}$. It is easy to verify that φ is a proper conflict-free L-coloring of G.

This completes the proof for Case 3 and the proof of Theorem 7.10.

7.5 Proper conflict-free (degree + 3)-choosable graphs

In this section, we give a proof of Theorem 7.11. We adapt the technique of the DP-coloring, which was introduced in Dvořák and Postle [24].

Suppose that the statement is false. Let G be a minimum counterexample in terms of the order. Let L be a list assignment of G such that $|L(v)| \geq d_G(v) + 3$ for each $v \in V(G)$ and G is not proper conflict-free L-colorable. Clearly, G is connected and has at least 5 vertices as every vertex has at least 4 colors.

Claim 1. The minimum degree of G is at least 3.

Proof. If there is a vertex v of degree 1 in G with the neighbor u, then by the minimality of G, there is a proper conflict-free L-coloring φ of $G - v$. We can extend φ to a proper conflict-free L-coloring of G by setting $\varphi(v) \in L(v) \setminus \{\varphi(u), \alpha\}$, where $\alpha \in \mathcal{U}_\varphi(u, G)$. Thus, we have $\delta(G) \geq 2$.

Now we assume that $v \in V(G)$ is a vertex of degree 2 with neighbors u_1 and u_2. Let G' be a graph obtained from $G - v$ by adding an edge u_1u_2. As $\Delta(G') \leq 4$, by the minimality of G, G' admits a proper conflict-free L-coloring φ. For each $i \in \{1, 2\}$, we define a color c_i as follows: If $|\mathcal{U}_\varphi(u_i, G-v)| = 0$, then all neighbors of u_i in $G-v$ receive the same color, so let c_i be the color. Otherwise, since $|\mathcal{U}_\varphi(u_i, G-v)| \geq 1$, we choose a color $c_i \in \mathcal{U}_\varphi(u_i, G-v)$. Let $\varphi(v)$ be a color in $L(v) \setminus \{\varphi(u_1), \varphi(u_2), c_1, c_2\}$ to extend φ to an L-coloring of G. It is easy to verify that $\mathcal{U}_\varphi(x, G) \neq \emptyset$ for every vertex $x \in V(G) \setminus \{v, u_1, u_2\}$, and $\mathcal{U}_\varphi(v, G) \neq \emptyset$ since $\varphi(u_1) \neq \varphi(u_2)$. For each $i \in \{1, 2\}$, we have $c_i \in \mathcal{U}_\varphi(u_i, G)$ if $\mathcal{U}_\varphi(u_i, G-v) \neq \emptyset$ and $\varphi(v) \in \mathcal{U}_\varphi(u_i, G)$ otherwise. Thus, φ is a proper conflict-free L-coloring of G, a contradiction. $\square$

Let $C = v_0v_1 \cdots v_{k-1}v_0$ be a shortest cycle of G, and let $G' = G - V(C)$. For each $i \in [0, k-1]$, let $N_G(v_i) \setminus V(C) = \{u_i, w_i\}$ if $d_G(v_i) = 4$, and let $N_G(v_i) \setminus V(C) = \{u_i\}$ if $d_G(v_i) = 3$.

In the rest of the proof, we will first give an L-coloring (not necessarily a proper conflict-free) φ of G', and then we extend φ to a proper conflict-free L-coloring of G. For a coloring φ of G', a pair of colors (α, β) and a vertex $v_i \in V(C)$, we say v_i *blocks* (α, β) *with respect to* φ if one of the following holds.

- $d_G(v_i) = 4$, $\alpha \neq \beta$ and $\{\varphi(u_i), \varphi(w_i)\} = \{\alpha, \beta\}$,
- $d_G(v_i) = 4$, $\alpha = \beta$ and $\varphi(u_i) = \varphi(w_i)$, or
- $d_G(v_i) = 3$ and $\varphi(u_i) = \alpha = \beta$.

If (α, β) is not blocked by v_i, then we say (α, β) is *feasible* for v_i. Observe that if v_i blocks (α, β), then we cannot extend φ to G as $\{\varphi(v_{i-1}), \varphi(v_{i+1})\} = \{\alpha, \beta\}$ because such an extension forces the vertex v_i to have $\mathcal{U}_\varphi(v_i, G) = \emptyset$.

By the definition, we have the following observation.

Observation 2. Every vertex blocks at most two color pairs. If v_i blocks both (α_1, β_1) and (α_2, β_2), then either $(\alpha_1, \beta_1) = (\beta_2, \alpha_2)$, or $\alpha_1 = \beta_1 \neq \alpha_2 = \beta_2$.

Claim 3. $k \geq 4$, i.e., G contains no triangle.

Proof. Assume $k = 3$. By the minimality of G, G' admits a proper conflict-free L-coloring φ. If a vertex $u \in V(G')$ is an isolated vertex of G', then we have $N_G(u) = \{v_0, v_1, v_2\}$, which implies that $\mathcal{U}_\varphi(u, G) \neq \emptyset$ for any proper L-coloring φ of G. For each vertex $u \in \bigcup_{v_i \in V(C)} N_{G'}(v_i)$ which is non-isolated in G', we arbitrarily choose a color $c_u \in \mathcal{U}_\varphi(u, G')$. Let L_C be a list assignment of C defined by

$$L_C(v_i) = L(v_i) \setminus (\{c_u \mid u \in N_{G'}(v_i), u \text{ is non-isolated in } G'\} \cup \{\varphi(u) \mid u \in N_{G'}(v_i)\})$$

for each $i \in \{0, 1, 2\}$. Note that $|L_C(v_i)| \geq |L(v_i)| - 2d_{G'}(v) \geq (d_G(v) + 3) - 2(d_G(v) - 2) = 7 - d_G(v) \geq 3$ for each $i \in \{0, 1, 2\}$. Without loss of generality, we may assume that one of the followings holds.

(a) $L_C(v_0) = L_C(v_1) = L_C(v_2)$;

(b) $L_C(v_0) \cap L_C(v_1) \cap L_C(v_2) \neq \emptyset$ and $L_C(v_0) \neq L_C(v_1)$;

(c) $L_C(v_0) \cap L_C(v_1) \cap L_C(v_2) = \emptyset$ and $L_C(v_0) \cap L_C(v_1) \neq \emptyset$; or

(d) $L_C(v_0) \cap L_C(v_1) = \emptyset$ and $L_C(v_1) \cap L_C(v_2) = \emptyset$ and $L_C(v_2) \cap L_C(v_0) = \emptyset$.

For each case, we show that φ can be extended to an L-coloring of G such that $\varphi(v_i) \in L_C(v_i)$ and $\mathcal{U}_\varphi(v_i, G) \neq \emptyset$ for each $i \in [0, 2]$, which derives a contradiction. If (a) occurs, then we extend φ to an L-coloring of G such that the restriction of φ to $V(C)$ is a proper L_C-coloring of G. Since $\varphi(v_1)$ and $\varphi(v_2)$ are two distinct colors that do not appear in $N_{G'}(v_0)$, there are at least three distinct colors that appear in the neighborhood of v_0 in G, which implies that $\mathcal{U}_\varphi(v_0, G) \neq \emptyset$. Similarly, we have $\mathcal{U}_\varphi(v_1, G) \neq \emptyset$ and $\mathcal{U}_\varphi(v_2, G) \neq \emptyset$, and thus φ is a proper conflict-free L-coloring of G, a contradiction.

When (b) occurs, we may assume that $1 \in L_C(v_0) \cap L_C(v_1) \cap L_C(v_2)$. Let $\varphi(v_2) = 1$. Since $|L_C(v_0) \setminus \{1\}| \geq 2$, $|L_C(v_1) \setminus \{1\}| \geq 2$ and $L_C(v_0) \setminus \{1\} \neq L_C(v_1) \setminus \{1\}$, by Observation 2, we can choose $\varphi(v_0) \in L_C(v_0) \setminus \{1\}$ and $\varphi(v_1) \in L_C(v_1) \setminus \{1\}$ so that $\varphi(v_0) \neq \varphi(v_1)$ and the pair $(\varphi(v_0), \varphi(v_1))$ is feasible for v_2. By the choice of colors, we have $\mathcal{U}_\varphi(v_2, G) \neq \emptyset$. As $1 \in \mathcal{U}_\varphi(v_i, G)$ for $i = 0, 1$, φ is a proper conflict-free L-coloring of G, a contradiction.

When (c) occurs, we may assume that $1 \in L_C(v_0) \cap L_C(v_1)$. We let $\varphi(v_0) = 1$ and choose a color $\varphi(v_1) \in L_C(v_1) \setminus \{1\}$ so that the pair $(1, \varphi(v_1))$ is feasible for v_2 Since $1 \notin L_C(v_2)$, we can choose a color in $L_C(v_2) \setminus \{1, \varphi(v_1)\}$ as $\varphi(v_2)$ so that the pair $(\varphi(v_1), \varphi(v_2))$ is feasible for v_0. By the choice of colors, we have $\mathcal{U}_\varphi(v_i, G)$ for each $i \in [0, 2]$, and thus φ is a proper conflict-free L-coloring of G, a contradiction.

When (d) occurs, for each $i \in [0, 2]$, let $\varphi(v_i) \in L_C(v_i)$ be a color that does not appear in $N_{G'}(v_{i+1})$ where the indices are on modulo 3. Then the resulting L-coloring of G is proper, and we have $\varphi(v_i) \in \mathcal{U}_\varphi(v_{i+1}, G)$ for each $i \in [0, 2]$. Thus, φ is a proper conflict-free L-coloring of G, a contradiction. $\square$

By Claim 3, we conclude that G' has no isolated vertices. Indeed, if u is an isolated vertex of G', then since $\delta(G) \geq 3$, there exist vertices $v_a, v_b, v_c \in V(C) \cap N_G(u)$ with $0 \leq a < b < c \leq k - 1$. Claim 3 implies that $b - a \geq 2$ and $c - b \geq 2$, and thus $v_a u v_c v_{c+1} \cdots v_{a-1} v_a$ is a shorter cycle than C, a contradiction.

Let G'' be the graph obtained from G' by adding all the edges $u_i w_i$ for each $i \in [0, k-1]$ if $d_G(v_i) = 4$, and removing the multiple edges. Then we have $\Delta(G'') \leq 4$. By the minimality of G, G'' admits a proper conflict-free L-coloring φ with $\varphi(u_i) \neq \varphi(w_i)$ for every $i \in [0, k-1]$ with $d_G(v_i) = 4$. Note that for each $u \in N_G(V(C))$, $\mathcal{U}_\varphi(u, G'') \neq \emptyset$, but it is possible that $\mathcal{U}_\varphi(u, G') = \emptyset$. For each $u \in \bigcup_{v_i \in V(C)} N_{G'}(v_i)$, we define a color c_u as follows: If $\mathcal{U}_\varphi(u, G') \neq \emptyset$, then let c_u be an arbitrary color in it. Otherwise, only one color appears in $N_{G'}(u)$, and let c_u be the color.

Let L_C be a list assignment of C defined by $L_C(v_i) = L(v_i) \setminus \bigcup_{u \in N_{G'}(v_i)} \{\varphi(u), c_u\}$ for each $i \in [0, k-1]$. Note that $|L_C(v_i)| \geq 3$ for each $i \in [0, k-1]$ and that $|L_C(v_i)| \geq 4$ if $d_G(v_i) = 3$. Since $\varphi(u_i) \neq \varphi(w_i)$ for every $v_i \in V(C)$ with $d_G(v_i) = 4$, the following observation holds.

Observation 4. Every vertex of C blocks at most two color pairs. If v_i blocks two distinct pairs (α_1, β_1) and (α_2, β_2), then $\alpha_1 \neq \beta_1$ and $(\alpha_1, \beta_1) = (\beta_2, \alpha_2)$.

In order to consider the case $k = 4$, we prepare the following two observations.

Observation 5. Let $C = xzywx$, and let $L'_C(x) \subseteq L_C(x)$ and $L'_C(y) \subseteq L_C(y)$ be two nonempty sets of colors. If $|L'_C(x)| + |L'_C(y)| \geq 4$, then there exists a pair of colors in $L'_C(x) \times L'_C(y)$ that is feasible for both z and w.

Proof. If z blocks two pairs of colors $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in L'_C(x) \times L'_C(y)$, then by Observation 4, we have $\alpha_1 \neq \beta_1$ and $(\alpha_1, \beta_1) = (\beta_2, \alpha_2)$. Then we have $\{\alpha_1, \beta_1\} \subseteq L'_C(x) \cap L'_C(y)$, and thus at least one of (α_1, α_1) and (β_1, β_1) is feasible for both w and z. By the symmetry of z and w, we may assume that both z and w block at most one color pair in $L'_C(x) \times L'_C(y)$. Since the condition $|L'_C(x)| + |L'_C(y)| \geq 4$ implies $|L'_C(x) \times L'_C(y)| \geq 3$, there is a desired pair in $L'_C(x) \times L'_C(y)$. □

The following observation immediately follows from the definition of feasibility.

Observation 6. Let $C = xzywx$. If there is a color pair $(\alpha, \beta) \in L_C(x) \times L_C(y)$ that is feasible for both z and w (possibly $\alpha = \beta$), and a color pair in $(L_C(z) \setminus \{\alpha, \beta\}) \times (L_C(w) \setminus \{\alpha, \beta\})$ that is feasible for both x and y, then φ can be extended to a proper conflict-free L-coloring of G.

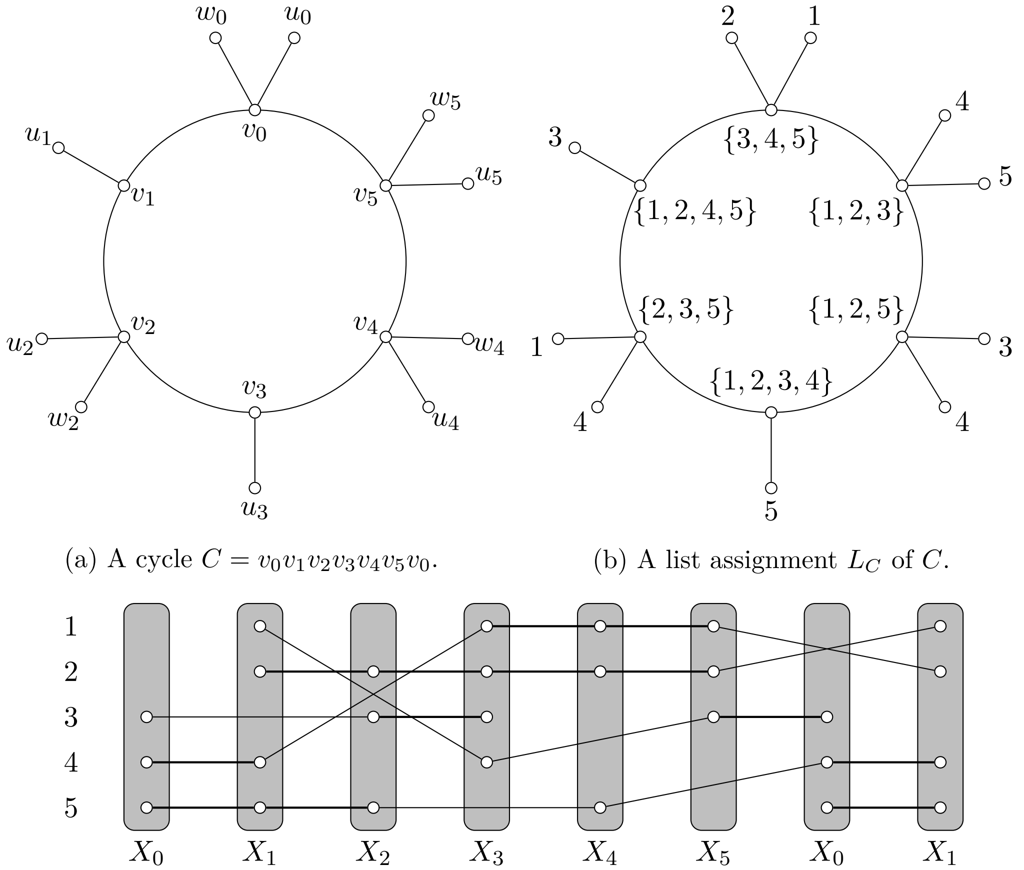
Claim 7. $k \geq 5$, i.e., G contains no cycle of length 4.

Proof. Suppose to the contrary, $k = 4$. By Observation 4 and the fact that $L_C(v_1) \times L_C(v_3)$ contains at least six pairs of distinct colors, there must be a pair of distinct colors, say $(1, 2) \in L_C(v_1) \times L_C(v_3)$, which is feasible for both v_0 and v_2. By Observations 4, 6 and the fact that $|L_C(v_i)| \geq 3$, we have $1 \leq |L_C(v_i) \setminus \{1, 2\}| \leq 2$ for $i \in \{0, 2\}$, and at least one of $|L_C(v_0) \setminus \{1, 2\}|$ and $|L_C(v_2) \setminus \{1, 2\}|$ is equal to 1. Without loss of generality, we may assume that $1 \in L_C(v_0)$ and $\{1, 2\} \subseteq L_C(v_2)$. We deduce that $d_G(v_3) = 3$ and $\varphi(u_3) = 1$, for otherwise $(1, 1) \in L_C(v_0) \times L_C(v_2)$ is feasible for both v_1 and v_3 and $|L_C(v_1) \setminus \{1\}| + |L_C(v_3) \setminus \{1\}| \geq 4$, a contradiction by Observation 6. Thus, we have $1 \notin L_C(v_3)$. Observe that $(1, 2)$ is feasible for both v_1 and v_3, this implies that $2 \in L_C(v_1)$ and $|L_C(v_1)| = 3$ again by Observation 6. We deduce that $d_G(v_0) = 3$ and $\varphi(u_0) = 2$ since $(2, 2) \in L_C(v_1) \times L_C(v_3)$ must be blocked by v_0. Therefore, we have $2 \notin L_C(v_0)$. Since $2 \in L_C(v_1)$ and $|L_C(v_1)| = 3$, there exists a color $\alpha \in L_C(v_0) \setminus L_C(v_1)$. Then $(\alpha, 1) \in L_C(v_0) \times L_C(v_2)$ is feasible for both v_1 and v_3, and $|L_C(v_1) \setminus \{\alpha, 1\}| + |L_C(v_3) \setminus \{\alpha, 1\}| \geq 2 + 2 \geq 4$, a contradiction by Observation 6. □

By Claims 3 and 7, G has no cycle of length less than 5. This together with the assumption that C is a shortest cycle implies that $N_{G'}(v_i) \cap N_{G'}(v_j) = \emptyset$ for any distinct $i, j \in [0, k-1]$.

In the rest of the proof, the indices $i \in [0, k-1]$ are on modulo k. We construct an auxiliary graph H with $V(H) = \bigcup_{i=1}^{k} X_i$ where $X_i = \{v_i\} \times L_C(v_i)$, and $E(H) = E_0 \cup E_1 \cup E_2$, where

$$E_0 = \{(v_i, \alpha)(v_i, \beta) \mid \alpha, \beta \in L_C(v_i), \alpha \neq \beta, i \in [0, k-1]\},$$
$$E_1 = \{(v_i, \alpha)(v_{i+1}, \alpha) \mid \alpha \in L_C(v_i) \cap L_C(v_{i+1}), i \in [0, k-1]\}, \text{and}$$
$$E_2 = \{(v_i, \alpha)(v_{i+2}, \beta) \mid (\alpha, \beta) \in L_C(v_i) \times L_C(v_{i+2}) \text{ is blocked by } v_{i+1}, i \in [0, k-1]\}.$$



(a) A cycle $C = v_0v_1v_2v_3v_4v_5v_0$.

(b) A list assignment L_C of C.

(c) A graph H. The edges belong to E_0 are not drawn. Thick edges represent edges in E_1 and thin edges represent edges in E_2.

Figure 7.4: An example of C and H.

The edges in E_1 (resp. E_2) are called *short* (resp. *long*) edges. The edges joining $(v_i, \alpha) \in X_i$ and the vertices in $X_{i+1} \cup X_{i+2}$ (resp. $X_{i-1} \cup X_{i-2}$) are called *forward* (resp. *backward*) edges of (v_i, α). See Figure 7.4 for an example of C and H.

The next observation follows immediately from the definition of H.

Observation 8. For each $i \in [0, k-1]$, there are at most two long edges between X_{i-1} and X_{i+1}. Furthermore, if there are two long edges between X_{i-1} and X_{i+1}, then they must be $(v_{i-1}, \alpha)(v_{i+1}, \beta)$ and $(v_{i-1}, \beta)(v_{i+1}, \alpha)$ for some distinct colors α and β in $L_C(v_{i-1}) \cap L_C(v_{i+1})$.

Claim 9. H has no independent set of order $|V(C)| = k$.

Proof. Suppose I is an independent set of H with $|I| = k$. We extend φ to G so that for each $(v, \alpha) \in I$, $\varphi(v) = \alpha$. Since $|I| = k$ and I contains at most one vertex of X_i for each $i \in [0, k-1]$, all the vertices of C are colored in this way. Clearly, φ is proper, $\varphi(v_i) \in L_C(v_i)$, and $\mathcal{U}_\varphi(v_i, G) \neq \emptyset$ for each $v_i \in V(C)$. By the definition of L_C, $\mathcal{U}_\varphi(u, G) \neq \emptyset$ for each $u \in V(G) \setminus V(C)$, which implies that φ is a proper conflict-free L-coloring of G, a contradiction. $\square$

Claim 10. For every $i \in [0, k-1]$ and every $\alpha \in L_C(v_i)$, there are at most one forward edge and at most one backward edge from (v_i, α).

Proof. Suppose that there is a forward short edge from (v_i, α), thus $\alpha \in L_C(v_{i+1})$. Then α does not appear in $N_{G'}(v_{i+1})$, and hence there is no long forward edge with an end vertex (v_i, α) by the definition of H. This implies that there is at most one forward edge from (v_i, α). By symmetry, the statement holds for backward edges as well. $\square$

An *extendable structure* (U, I) of H is a pair of a set $U \subseteq X_i \cup X_{i+t}$ for some $i \in [k]$ and integer $t \in \{1, 2\}$, and an independent set I of H satisfying the followings:

(C1) $|U \cap X_i| \geq 3$ and $|U \cap X_{i+t}| = 2$.

(C2) No backward long edges or no backward short edges incident with the vertices of $U \cap X_{i+t}$.

(C3) If $t = 1$, then $|I| = 2$ and $|I \cap X_j| = 1$ for $j \in \{i+2, i+3\}$. If $t = 2$, then $|I| = 3$ and $|I \cap X_j| = 1$ for $j \in \{i+1, i+3, i+4\}$.

(C4) There is no backward edges from the vertices in I to $U \cup X_{i-1}$.

As $k \geq 5$ by Claim 7, an extendable structure is a local structure contained in $\bigcup_{i \leq j \leq i+4} X_j$ for some $i \in [0, k-1]$. If H contains an extendable structure, then it can be extended to an independent set of H with k vertices as in the proof of the next claim.

Claim 11. H contains no extendable structure.

Proof. Suppose, to the contrary, (U, I) is an extendable structure of H. Without loss of generality, we may assume that $U \subseteq X_0 \cup X_t$ for some $t \in \{1, 2\}$. Since every vertex of H has at most one forward edge by Claim 10 and $|L_C(v_i)| \geq 3$ for each $i \in [0, k-1]$, we can greedily add vertices to I from each of $X_{t+3}, X_{t+4}, \ldots, X_{k-1}, X_0$ to obtain an independent set J^- of size $k-1$. Observe that $J^- \cap X_t = \emptyset$. By the condition (C2), at least one of the two vertices in $X_t \cap U$ has no neighbors in $I \cap (X_{t-1} \cup X_{t-2})$. We add one of such vertex to J^- to have a set J of vertices. By the condition (C4), J is still an independent set of H and $|J| = k$, a contradiction by Claim 9. $\square$

The following observation is frequently used in the sequel.

Observation 12. Suppose that $\alpha, \beta \in L_C(v_i)$ for some $i \in [k]$ and $\alpha \neq \beta$. Then there exist a color $\gamma \in L_C(v_{i+1}) \setminus \{\alpha, \beta\}$ and a color $\tau \in L_C(v_{i+2}) \setminus \{\gamma\}$ such that (v_{i+2}, τ) is adjacent to neither (v_i, α) nor (v_i, β). (Possibly $\tau \in \{\alpha, \beta\}$.)

Proof. Suppose that one of (v_i, α) and (v_i, β), say (v_i, α), has a neighbor in X_{i+2}. As $|L_C(v_{i+2})| \geq 3$, there exists a color $\tau \in L_C(v_{i+2})$ such that (v_{i+2}, τ) is adjacent to neither (v_i, α) nor (v_i, β). By Claim 10, we know $\alpha \notin L_C(v_{i+1})$, and thus we can choose a color γ from $L_C(v_{i+1}) \setminus \{\alpha, \beta, \tau\}$, as desired. Hence, we may assume that neither (v_i, α) nor (v_i, β) has neighbors in X_{i+2}. Then we can arbitrarily choose a color $\gamma \in L_C(v_{i+1}) \setminus \{\alpha, \beta\}$ and a color $\tau \in L_C(v_{i+2}) \setminus \{\gamma\}$, as desired. $\square$

Claim 13. Every vertex of H is incident with exactly one forward edge and one backward edge.

Proof. Suppose, to the contrary, (v_0, α) has no backward edge for some $\alpha \in L_C(v_0)$. The argument for the forward edge is symmetric. First, we assume that there exists a color $\beta \in L_C(v_1) \setminus \{\alpha\}$ such that (v_1, β) has no neighbors in X_{k-1}. As $|L_C(v_{k-1})| \geq 3$, by the Pigeon-Hole principle, there exist two colors, say $1, 2 \in L_C(v_{k-1})$, such that backward long edges or backward short edges are missed from $(v_{k-1}, 1), (v_{k-1}, 2) \in X_{k-1}$. Then $U = X_{k-2} \cup \{(v_{k-1}, 1), (v_{k-1}, 2)\}$ and $I = \{(v_0, \alpha), (v_1, \beta)\}$ form an extendable structure with $t = 1$: The conditions (C1) and (C3) are obvious. The condition (C2) follows from the choice of $(v_{k-1}, 1)$ and $(v_{k-1}, 2)$, and the condition (C4) is satisfied since (v_0, α) has no backward edge and (v_1, β) has no backward edge to X_{k-1}, a contradiction.

Thus, by the definition of H, we deduce that there exist two distinct colors, say 1 and 2, such that $L_C(v_1) = \{\alpha, 1, 2\}$ and both $(v_1, 1)$ and $(v_1, 2)$ are incident with long backward edges to X_{k-1}. By Observation 12, there exists a color $\gamma \in L_C(v_2) \setminus \{1, 2\}$ and $\tau \in L_C(v_3) \setminus \{\gamma\}$ such that (v_3, τ) has no neighbors in $\{(v_1, 1), (v_1, 2)\}$. Then $U = X_{k-1} \cup \{(v_1, 1), (v_1, 2)\}$ and $I = \{(v_0, \alpha), (v_2, \gamma), (v_3, \tau)\}$ form an extendable structure with $t = 2$: The conditions (C1), (C2), and (C3) are obvious. The condition (C4) is satisfied since (v_0, α) has no backward edge, and both (v_2, γ) and (v_3, τ) have no backward edge to $\{(v_1, 1), (v_1, 2)\}$, a contradiction. $\square$

Claim 14. For every $i \in [0, k-1]$, there is at most one forward short edge from vertices in X_i.

Proof. Suppose that there are two forward short edges from the vertices of X_0. Without loss of generality, we assume that $\{1, 2, 3\} \subseteq L_C(v_0)$ and both $(v_0, 1)$ and $(v_0, 2)$ incident with forward short edges. Let $U = \{(v_0, 1), (v_0, 2), (v_0, 3), (v_1, 1), (v_1, 2)\}$. If $(v_0, 3)$ is incident with forward short edge, then no vertex of $\{(v_0, 1), (v_0, 2), (v_0, 3)\}$ has a neighbor in X_2. By Observation 12, there exist colors $\gamma \in L_C(v_2) \setminus \{1, 2\}$ and $\tau \in L_C(v_3) \setminus \{\gamma\}$ such that neither $(v_1, 1)$ nor $(v_1, 2)$ is adjacent to (v_3, τ). Then U and $I = \{(v_2, \gamma), (v_3, \tau)\}$ form an extendable structure with $t = 1$, a contradiction. Hence $(v_0, 3)$ is adjacent to (v_2, α) for some color $\alpha \in L_C(v_2) \setminus \{1, 2\}$.

We first assume that there exists a color $\beta \in L_C(v_2) \setminus \{1, 2, \alpha\}$. Clearly, (v_2, β) has no neighbors in U. If there exists a color $\gamma \in L_C(v_3) \setminus \{\beta\}$ such that (v_3, γ) has no neighbors in $\{(v_1, 1), (v_1, 2)\}$, then U and $I = \{(v_2, \beta), (v_3, \gamma)\}$ form an extendable structure with $t = 1$, a contradiction. Thus, we deduce that $L_C(v_3) = \{1, 2, \beta\}$ and $\{\varphi(u_2), \varphi(w_2)\} = \{1, 2\}$, and hence $1, 2 \notin L_C(v_2)$. As there is a color $\beta' \in L_C(v_2) \setminus \{1, 2, \alpha, \beta\}$, U and $I = \{(v_2, \beta'), (v_3, \beta)\}$ form an extendable structure with $t = 1$, a contradiction.

Hence, we have $L_C(v_2) = \{1, 2, \alpha\}$. By using the same argument sequentially to $X_1, X_2, \ldots, X_{k-1}$, we conclude that $\{1, 2\} \subseteq L_C(v_i)$ for every $i \in [0, k-1]$. Then $I = \{(v_{2m+1}, 1) \mid 0 \leq m \leq \lfloor (k-1)/2 \rfloor\} \cup \{(v_{2m}, 2) \mid 0 \leq m \leq \lfloor (k-1)/2 \rfloor\} \cup \{(v_0, 3)\}$ is an independent set of H with $|I| = k$, a contradiction. $\square$

If $|X_i| \geq 4$ for some $i \in [0, k-1]$, then by Claims 13 and 14, there are at least three long edges between X_i and X_{i+2}, a contradiction by Observation 8. Thus, for each $i \in [0, k-1]$, $|X_i| = 3$, two of X_i incident with forward long edges, and the rest incident with a forward short edge.

If all the short edges form a matching of H, then we can easily find an independent set I of H with $|I| = k$ by choosing vertices incident with forward short edges, a contradiction by Claim 9. Therefore, without loss of generality, there must exist a color, say α, which appears in $L_C(v_0) \cap L_C(v_1) \cap L_C(v_2)$. Then there are two distinct colors in $L_C(v_0) \setminus \{\alpha\}$, say 1 and 2, and there are two distinct colors in $L_C(v_1) \setminus \{\alpha, 1, 2\}$, say 3 and 4 such that all of $\{(v_0, 1), (v_0, 2), (v_1, 3), (v_1, 4)\}$ are incident with forward long edges. By Observation 8 and the fact $|L_C(v_i)| = 3$ for every $i \in [0, k-1]$, we know that $L_C(v_0) = L_C(v_2) = \{\alpha, 1, 2\}$ and $L_C(v_1) = \{\alpha, 3, 4\}$, $L_C(v_3) = \{\beta, 3, 4\}$ for some color $\beta \notin \{3, 4\}$. If $\beta \neq \alpha$, then $U = X_0 \cup \{(v_1, 3), (v_1, 4)\}$ and $I = \{(v_2, \alpha), (v_3, \beta)\}$ form an extendable structure with $t = 1$: The conditions (C1), (C2) and (C3) are obvious by the choice of vertices, and the condition (C4) is satisfied since (v_2, α) has a backward short edge to $(v_1, \alpha) \notin U$ and the long edges between X_1 and X_3 are $(v_1, 3)(v_3, 4)$ and $(v_1, 4)(v_3, 3)$, which are not joined to (v_3, β), a contradiction. Thus, $L_C(v_3) = \{\alpha, 3, 4\}$. Using the same argument sequentially to $X_4, X_5, \ldots, X_{k-1}$, we conclude that k is even, $L_C(v_j) = \{\alpha, 1, 2\}$ if j is even, and $L_C(v_j) = \{\alpha, 3, 4\}$ if j is odd. Then $I = \{(v_j, 1) \mid j \in [k] \text{ and } j \text{ is even}\} \cup \{(v_j, 3) \mid j \in [k] \text{ and } j \text{ is odd}\}$ is an independent set of H with $|I| = k$, a contradiction. This completes the proof of Theorem 7.11.

Part II

Degree conditions for some vertex partition problems

Chapter 8

Fundamental properties of minimum degree sum on large independent sets

In Part II, we investigate new types of degree conditions named "minimum degree sum on large independent sets", which was partly introduced in Furuya, Kashima, and Ota [31]. As preparations for Chapters 9 and 10, in this chapter, we introduce concepts of minimum degree sum on large independent sets and see some fundamental properties of them. We define minimum degree sum on large independent sets and see relations to other degree conditions in Section 8.1. In Section 8.2, we give a proof of Theorem 8.3, which is a result on the complexity of determining the minimum degree sum on large independent sets. In Section 8.3, we give a proof of Theorem 8.4, which is on an extremal problem of minimum degree sum on large independent sets.

8.1 Minimum degree sum on large independent sets

For a graph G, let $\mathcal{I}(G)$ denote the family of independent sets of G. For each independent set I of G, let $\delta_G(I) = \min\{d_G(v) \mid v \in I\}$ and let $w_G(I) = \sum_{v\in I} d_G(v)$. For a graph G, we define two invariants $\sigma_0^*(G)$ and $\sigma_1^*(G)$ as

$$\sigma_0^*(G) = \min\{w_G(I) \mid I \in \mathcal{I}(G), |I| \geq \delta_G(I)\} \text{ and}$$
$$\sigma_1^*(G) = \min\{w_G(I) \mid I \in \mathcal{I}(G), |I| \geq \delta_G(I) + 1\}.$$

If every independent set $I \in \mathcal{I}(G)$ satisfies $|I| \leq \delta_G(I) - 1$, then we define $\sigma_0^*(G) = \infty$. Similarly, if every independent set $I \in \mathcal{I}(G)$ satisfies $|I| \leq \delta_G(I)$, then we define $\sigma_1^*(G) = \infty$. For $t \in \{0, 1\}$, when $\sigma_t^*(G)$ have a finite value, i.e. there is an independent set $I \in \mathcal{I}(G)$ such that $|I| \geq \delta_G(I) + t$, we write $\sigma_t^*(G) < \infty$.

The following relations between $\sigma_0^*(G)$ and $\sigma_1^*(G)$ follow directly from the definitions.

Proposition 8.1. For every graph G, $\sigma_0^*(G) \leq \sigma_1^*(G)$. In particular, if $\sigma_0^*(G) = \infty$, then $\sigma_1^*(G) = \infty$.

Recall that $\delta(G)$ denotes the minimum degree of a graph G and $\pi_2(G)$ denotes the minimum degree product of a graph G. The minimum degree sum on large independent sets is bounded by some functions of the minimum degree and the minimum degree product as follows.

Proposition 8.2. For every non-complete graph G, $\sigma_0^*(G) \geq \delta(G)^2$ and $\sigma_0^*(G) \geq \pi_2(G) + \delta(G) - \frac{\pi_2(G)}{\delta(G)}$. For every non-complete graph G, $\sigma_1^*(G) \geq \delta(G)(\delta(G)+1)$ and $\sigma_1^*(G) \geq \pi_2(G) + \delta(G)$.

Proof. We first show the former statement. Let G be a non-complete graph, and let $I \in \mathcal{I}(G)$ be an independent set such that $|I| \geq \delta_G(I)$. The statement is trivial when $\delta(G) \leq 1$. Thus, we assume that $\delta(G) \geq 2$. Let u be a vertex in I such that $d_G(u) = \delta_G(I)$. Since $|I| \geq \delta_G(I) \geq d_G(u) \geq \delta(G)$, we have $w_G(I) = \sum_{v \in I} d_G(v) \geq |I| \cdot \delta(G) \geq \delta(G)^2$. Since every vertex $v \in I \setminus \{u\}$ satisfies $d_G(u) \cdot d_G(v) \geq \pi_2(G)$, we have $d_G(v) \geq \frac{\pi_2(G)}{d_G(u)}$ and thus

$$\begin{aligned} w_G(I) = d_G(u) + \sum_{v \in I \setminus \{u\}} d_G(v) &\geq d_G(u) + (|I|-1)\frac{\pi_2(G)}{d_G(u)} \\ &\geq d_G(u) + (d_G(u)-1)\frac{\pi_2(G)}{d_G(u)} = \pi_2(G) + d_G(u) - \frac{\pi_2(G)}{d_G(u)} \geq \pi_2(G) + \delta(G) - \frac{\pi_2(G)}{\delta(G)}. \end{aligned}$$

The choice of an independent set I is arbitrary, which implies that $\sigma_0^*(G) \geq \delta(G)^2$ and $\sigma_0^*(G) \geq \pi_2(G) + \delta(G) - \frac{\pi_2(G)}{\delta(G)}$.

Now we show the latter statement. Let G be a non-complete graph, and let $I \in \mathcal{I}(G)$ be an independent set such that $|I| \geq \delta_G(I) + 1$. By the relation between $\delta(G)$ and $\pi_2(G)$, it suffices to show that $\sigma_1^*(G) \geq \pi_2(G) + \delta(G)$. Let u be a vertex in I such that $d_G(u) = \delta_G(I)$. Using the same argument as above, we have

$$\begin{aligned} w_G(I) = d_G(u) + \sum_{v \in I \setminus \{u\}} d_G(v) \geq d_G(u) + (|I|-1)\frac{\pi_2(G)}{d_G(u)} &\geq d_G(u) + d_G(u)\frac{\pi_2(G)}{d_G(u)} \\ &\geq \pi_2(G) + \delta(G). \end{aligned}$$

Thus we have $\sigma_1^*(G) \geq \pi_2(G) + \delta(G)$. □

Hence, the assumption on the minimum degree sum on large independent sets can be regarded as a relaxation of a minimum degree condition or a minimum degree product condition.

While a graph G satisfies $\sigma_2(G) = \infty$ or $\pi_2(G) = \infty$ only when G is a complete graph, there are wide varieties of graphs with $\sigma_0^*(G) = \infty$ or $\sigma_1^*(G) = \infty$. To support this fact, in the rest of this section, we see two results on the graphs with $\sigma_0^*(G) = \infty$ or $\sigma_1^*(G) = \infty$.

The first result is on the complexity to determine whether a given graph G satisfies $\sigma_0^*(G) = \infty$ ($\sigma_1^*(G) = \infty$) or not. We introduce two decision problems Sigma Star-0 and Sigma Star-1 as follows.

- Sigma Star-0 asks whether a given graph G satisfies $\sigma_0^*(G) < \infty$ or not.

- Sigma Star-1 asks whether a given graph G satisfies $\sigma_1^*(G) < \infty$ or not.

Then it follows that both Sigma Star-0 and Sigma Star-1 are NP-complete problems.

Theorem 8.3. Both Sigma Star-0 and Sigma Star-1 are NP-complete.

Thus, if we believe $\mathsf{P} \neq \mathsf{NP}$, then it is difficult to give an easy characterization of the graphs with $\sigma_0^*(G) < \infty$ or $\sigma_1^*(G) < \infty$.

The second result is an extremal problem on the minimum degree sum on large independent sets. The assumption that $\sigma_1^*(G) = \infty$ implies that G has many edges, but the minimum number of edges of a graph G of order n with $\sigma_0^*(G) = \infty$ is far from that of K_n.

Theorem 8.4. Let G be a graph of order $n \geq 3$. If $\sigma_1^*(G) = \infty$, then $|E(G)| \geq \frac{1}{2}n\left\lfloor\sqrt{n+1}\right\rfloor$.

The lower bound in Theorem 8.4 is almost tight. Indeed, for a positive integer k, let G be a disjoint union of k copies of the complete graph K_{k+1}. Then G satisfies $\delta(G) = k$ and $\alpha(G) = k$, and thus $|I| \leq \alpha(G) = k = \delta(G) \leq \delta_G(I)$ for every independent set $I \in \mathcal{I}(G)$. Thus, $\sigma_1^*(G) = \infty$. On the other hand, we have $n = |V(G)| = k(k+1)$ and $|E(G)| = k \cdot \binom{k+1}{2} = \frac{1}{2}k^2(k+1) = \frac{1}{2}n(\sqrt{n} - o(1))$.

8.2 Proof of Theorem 8.3

To prove Theorem 8.3, we give a reduction from a well-known decision problem Independent Set. An instance of Independent Set is a pair of a graph G and a positive integer k with $k \leq |V(G)|$, and an output of Independent Set is whether $\alpha(G) \geq k$ or not. The following proposition is well known. (For example, see Theorem 3.3 in Garey and Johnson [33].)

Proposition 8.5 (Garey and Johnson [33])**.** The problem Independent Set is NP-complete.

Now we give a proof of Theorem 8.3. We first show that both Sigma Star-0 and Sigma Star-1 belong to NP. Suppose that G is an yes-instance of Sigma Star-0 and an independent set $I \in \mathcal{I}(G)$ is given. Then, it can be checked in polynomial time that I satisfies $|I| \geq \delta_G(I)$, thus Sigma Star-0 belongs to NP. Similarly, we can show that Sigma Star-1 belongs to NP.

Now we show that there is a polynomial-time reduction from Independent Set to Sigma Star-0 and Sigma Star-1. Let a pair (G, k) of a graph G of order n and a positive integer k at most n be an instance of Independent Set. We define two new graphs H_0 and H_1 as follows:

- A graph H_0 is obtained from G by adding new vertices $\{u_i \mid 1 \leq i \leq n+1\} \cup \{v\}$ such that $\{u_i \mid 1 \leq i \leq n+1\}$ is a clique of H_0 joined to all vertices of G and $N_{H_0}(v) = \{u_i \mid 1 \leq i \leq k+1\}$ (Figure 8.1).

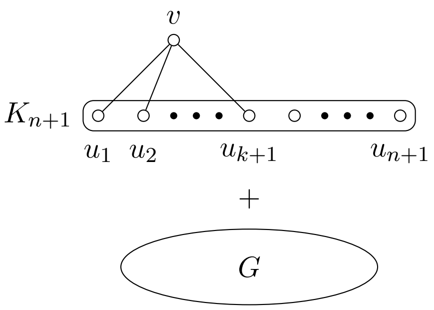


Figure 8.1: A graph H_0

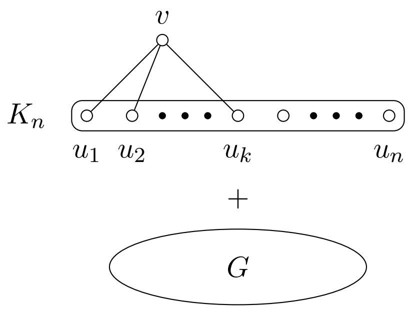


Figure 8.2: A graph H_1

- A graph H_1 is obtained from G by adding new vertices $\{u_i \mid 1 \leq i \leq n\} \cup \{v\}$ such that $\{u_i \mid 1 \leq i \leq n\}$ is a clique of H_1 joined to all vertices of G and $N_{H_1}(v) = \{u_i \mid 1 \leq i \leq k\}$ (Figure 8.2).

We shall show that $\sigma_0^*(H_0) < \infty$ if and only if $\alpha(G) \geq k$. If G contains an independent set I of order k, then $I \cup \{v\}$ is an independent set of H_0 with $\delta_{H_0}(I \cup \{v\}) \leq d_{H_0}(v) = k+1 = |I \cup \{v\}|$, and hence it follows that $\sigma_0^*(H_0) < \infty$. Conversely, suppose that $\alpha(G) < k$. By the definition of H_0, we have $\delta(H_0) = k+1$ and $\alpha(H_0) = \alpha(G) + 1$. Thus, for every independent set I of H_0, $|I| \leq \alpha(H_0) = \alpha(G) + 1 \leq k = \delta(H_0) - 1 \leq \delta_{H_0}(I) - 1$, which implies that $\sigma_0^*(H_0) = \infty$. Thus, there is a polynomial reduction from Independent Set to Sigma Star-0, and hence Sigma Star-0 is NP-complete. Similarly, it follows that $\sigma_1^*(H_1) < \infty$ if and only if $\alpha(G) \geq k$. Thus, Sigma Star-1 is NP-complete as well.

8.3 Proof of Theorem 8.4

In our proof of Theorem 8.4, we use the following fundamental result in extremal graph theory. For positive integers n and r with $n \geq r-1$, the *Turán graph* $T^{r-1}(n)$ is a complete $(r-1)$-partite graph with n vertices such that any two partite sets differ in size at most by 1. An upper bound of the number of edges of the Turán graph is known as follows. (For example, see p.201 in Diestel [22].)

Proposition 8.6 ([22])**.** For any positive integers n and r with $n \geq r-1$,

$$|E(T^{r-1}(n))| \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right). \tag{8.3.1}$$

Proof. For a given pair (n, r) of positive integers with $n \geq r-1$, let p and q be integers such that $n = p(r-1)+q$ and $0 \leq q \leq r-2$. We shall show that $n^2\left(1 - \frac{1}{r-1}\right) \geq 2|E(T^{r-1}(n))|$. The Turán graph $T^{r-1}(n)$ consists of q partite sets of order $p+1$ and $(r-1-q)$ partite sets of order p. Thus, we have

$$
\begin{aligned}
2|E(T^{r-1}(n))| = \sum_{v\in V(T^{r-1}(n))} d(v) &= p(r-1-q)(n-p)+(p+1)q(n-p-1) \\
&= p(r-1)(n-p)+q(n-2p-1) \\
&= (n-q)(n-p)+q(n-2p-1) \\
&= n^2-pn-pq-q \\
&= n^2-p^2(r-1)-2pq-q,
\end{aligned}
$$

and hence

$$
\begin{aligned}
n^2\left(1-\frac{1}{r-1}\right)-2|E(T^{r-1}(n))| &= n^2-\frac{n^2}{r-1}-n^2+p^2(r-1)+2pq+q \\
= p^2(r-1)+2pq+q-\frac{1}{r-1}&(p^2(r-1)^2+2pq(r-1)+q^2) = q\left(1-\frac{q}{r-1}\right) \geq 0,
\end{aligned}
$$

as desired. □

Turán showed the following in [60].

Theorem 8.7 (Turán [60])**.** Let G be a graph graph of order n without a clique of order at least r. Then $|E(G)| \leq |E(T^{r-1}(n))|$. The equality holds only when G is isomorphic to $T^{r-1}(n)$.

Taking a complement of a graph, we immediately have the following corollary.

Corollary 8.8. Let G be a graph graph of order n with $\alpha(G) < r$. Then $|E(G)| \geq \binom{n}{2} - |E(T^{r-1}(n))|$.

Now we prove Theorem 8.4. Let G be a graph of order n with $\sigma_1^*(G) = \infty$, and let $r = \lfloor \sqrt{n+1} \rfloor \geq 2$.

Let δ be the minimum degree of G, and let v_0 be a vertex with $d_G(v_0) = \delta$. If $\delta \geq r$, then it follows that $|E(G)| = \frac{1}{2}\sum_{v\in V(G)} d_G(v) \geq \frac{1}{2}n\delta \geq \frac{1}{2}nr$, as desired. Hence, we assume that $\delta \leq r-1$. Let $G' = G - N_G[v_0]$. Then we have $|V(G')| = n-\delta-1$. By the assumption that $\sigma_1^*(G) = \infty$, we have $\alpha(G') < \delta$; for otherwise, there is an independent set $I' \in \mathcal{I}(G')$ of order δ, and hence $I := I' \cup \{v_0\} \in \mathcal{I}(G)$ satisfies $|I| = \delta+1 = d_G(v_0)+1 \geq \delta_G(I)+1$, a contradiction. Let G'' be a disjoint union of G' and the complete graph K of order $\delta+1$. Obviously, $V(G'') = (n-\delta-1)+(\delta+1) = n$ and $\alpha(G'') = \alpha(G')+1 < \delta+1 \leq r$. By Corollary 8.8, we have $|E(G'')| \geq \binom{n}{2} - |E(T^{r-1}(n))|$.

Since every edge with an end point in $N_G[v_0]$ is contained in $E(G) \setminus E(G')$, we have $|E(G) \setminus E(G')| \geq \frac{1}{2}\sum_{v\in N_G[v_0]} d_G(v) \geq \frac{1}{2}\delta(\delta+1) = \binom{\delta+1}{2}$. This together with (8.3.1) implies that

$$
\begin{aligned}
|E(G)| = |E(G')| + |E(G) \setminus E(G')| &\geq |E(G')| + \binom{\delta+1}{2} = |E(G'')| \\
&\geq \binom{n}{2} - |E(T^{r-1}(n))| \geq \frac{n(n-1)}{2} - \frac{n^2}{2}\left(1-\frac{1}{r-1}\right) = \frac{n}{2}\left(\frac{n}{r-1}-1\right).
\end{aligned}
$$

Since $(r+1)(r-1) = r^2-1 \leq n$, we have $\frac{n}{r-1} - 1 \geq r$, and thus $|E(G)| \geq \frac{1}{2}nr$.

Chapter 9

Vertex partition into 2-connected subgraphs

Ferrara, Magnant, and Wenger investigated the existence of a vertex partition into highly connected subgraphs in [30]. The concept was later named k-proper partition, and some degree conditions for a graph to have a k-proper partition have been studied. In this chapter, we focus on degree conditions for a graph to have a 2-proper partition. In Section 9.1, we see results on k-proper partition, and in Section 9.2, we see the relationship between theorems on the existence of a 2-proper partition. In Section 9.4, we give a proof of Theorem 9.7, which implies all the known results on the existence of a 2-proper partition. All the new results in this chapter are based on [31].

9.1 Studies on k-proper partitions of graphs

For a graph G and an integer $k \geq 2$, a *k-proper partition* of G is a partition $\mathcal{P}$ of $V(G)$ such that $G[P]$ is k-connected for every $P \in \mathcal{P}$. This concept was introduced in Ferrara, Magnant and Wenger [30] with a motivation from a classical result that every graph G with the average degree at least $4k$ contains a k-connected subgraph, which was shown in Mader [49]. The first result on k-proper partition is the following, which was shown in Ferrara, Magnant, and Wenger [30].

Theorem 9.1 (Ferrara, Magnant and Wenger [30])**.** Let G be a graph of order n and let k be an integer at least 2. If $\delta(G) \geq 2k\sqrt{n}$, then G has a k-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{2kn}{\delta(G)}$.

Later, the minimum degree condition for a graph to have a k-proper partition was improved in Borozan et al. [10].

Theorem 9.2 (Borozan et al. [10])**.** Let G be a graph of order n, and let k be an integer at least 2. If $\delta(G) \geq \sqrt{c(k-1)n}$, then G contains a k-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{cn}{\delta(G)}$, where $c = \frac{2123}{180}$.

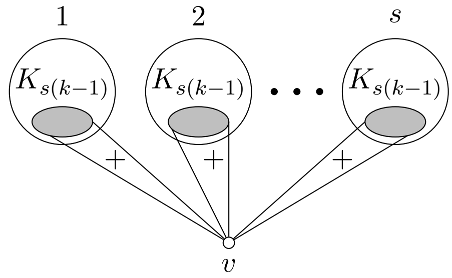


Figure 9.1: A graph G_1

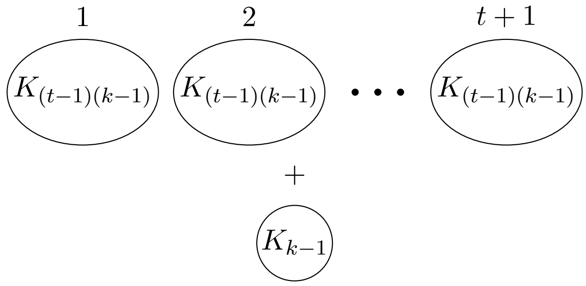


Figure 9.2: A graph G_2

The constant $c = \frac{2123}{180}$ depends on the following result in Yuster [66] on the edge density of graphs without k-connected subgraphs.

Theorem 9.3 (Yuster [66])**.** Let k be an integer at least 2, and let G be a graph of order at least $\frac{9}{4}(k-1)$. If G does not contain a k-connected subgraph, then $|E(G)| \leq \frac{193}{120}(k-1)(n-k+1)$.

This is a partial result towards the conjecture that every graph of order n without k-connected subgraphs has at most $\left(\frac{3}{2}k-2\right)(n-k+1)$ edges, which was posed in Mader [50]. Borozan et al. posed the following conjecture in [10], which states that the constant c in Theorem 9.2 can be reduced to 1.

Conjecture 9.4 (Borozan et al. [10])**.** Let G be a graph of order n, and let k be an integer at least 2. If $\delta(G) \geq \sqrt{(k-1)n}$, then G has a k-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{n-k+1}{\delta(G)-k+2}$.

Both the lower bound of $\delta(G)$ and the upper bound of $|\mathcal{P}|$ in Conjecture 9.4 is almost best possible. Indeed, for a given integers $k \geq 2$ and $s \geq 2$, let G_1 be a graph obtained from a disjoint union of s copies of the complete graph $K_{s(k-1)}$ by adding a new vertex v that is adjacent to the exact $k-1$ vertices of each copy of $K_{s(k-1)}$ (Figure 9.1). Then we have $n_1 := |V(G_1)| = s^2(k-1)+1$ and $\delta(G_1) = s(k-1)-1 = \sqrt{(k-1)(n-1)}-1$, but G_1 does not have a k-proper partition since there is no k-connected subgraph of G_1 which contains v. For given integers $k \geq 2$ and $t \leq 2$, let G_2 be a graph obtained from a disjoint union of $(t+1)$ copies of the complete graph $K_{(t-1)(k-1)}$ by joining the complete graph K_{k-1} (Figure 9.2). Then we have $n_2 := |V(G_2)| = (t+1)(t-1)(k-1)+(k-1) = t^2(k-1)$ and $\delta(G_2) = (t-1)(k-1)-1+(k-1) = t(k-1)-1$. It is easy to see that every k-proper partition of G_2 has at least $t+1$ parts, and thus $\frac{n_2-k+1}{\delta(G_2)-k+2} = \frac{t^2(k-1)-k+1}{t(k-1)-1-k+2} = \frac{(t^2-1)(k-1)}{(t-1)(k-1)} = t+1$.

Borozan et al. showed the following result on the existence of 2-proper partition.

Theorem 9.5 (Borozan et al. [10])**.** Let G be a graph of order n. If $\delta(G) \geq \sqrt{n}$, then G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{n-1}{\delta(G)}$.

As a result, Conjecture 9.4 holds for the case $k=2$. Their proof of Theorem 9.5 gives a method to construct a 2-proper partition from the block decomposition of a given graph. Using the same idea, in [15], Chen, Guo, and Yang extended Theorem 9.5 to a result with a minimum degree sum condition as follows.

Theorem 9.6 (Chen, Guo and Yang [15])**.** Let G be a non-complete graph of order n with the minimum degree δ. If either $\delta \geq \sqrt{n}$ or $1 \leq \delta \leq \sqrt{n} - 1$ and $\sigma_2(G) \geq \frac{n}{\delta} + \delta - 1$, then G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{2(n-1)}{\sigma_2(G)}$.

It is easy to see that Theorem 9.6 implies Theorem 9.5, which we see in the next section. The original paper does not assume the condition that G is non-complete. However, if G is complete, then we have $\sigma_2(G) = \infty$ and the upper bound of $|\mathcal{P}|$ is incorrect. In particular, since their proof of Theorem 9.6 uses an induction on the order, it needs a small modification when a reduced graph is complete. Hence, we added the assumption that G is non-complete. Note that Theorem 9.6 says nothing about a graph G with $\sqrt{|V(G)|} - 1 < \delta(G) < \sqrt{|V(G)|}$, and there are infinitely many graphs with the inequality.

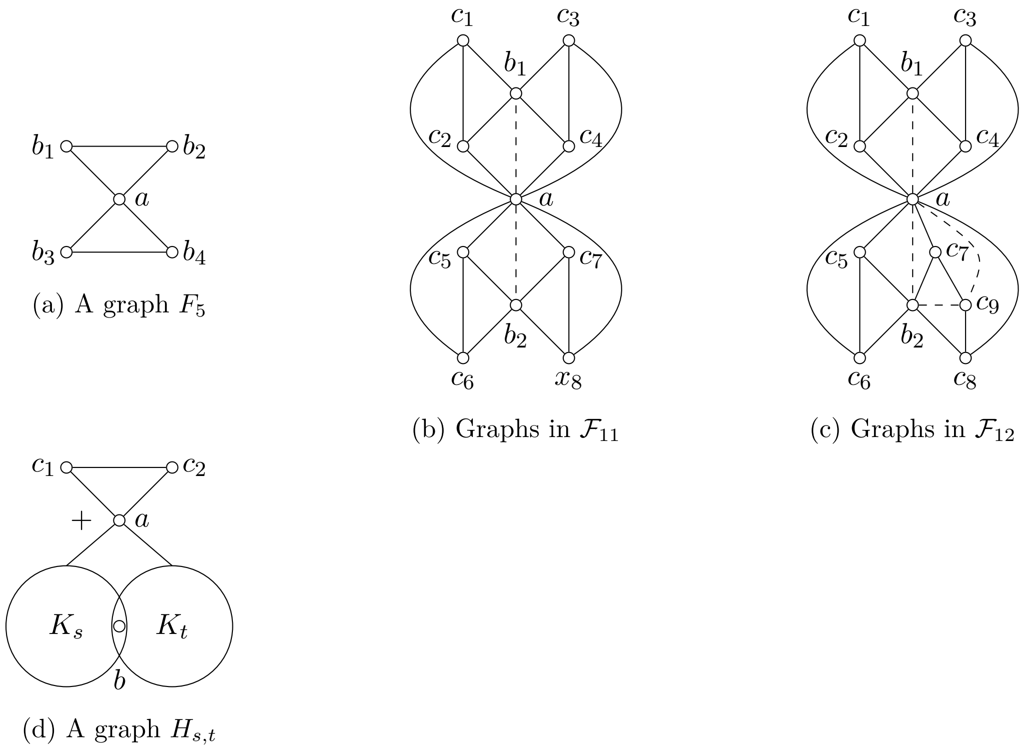
We extend these results by giving a condition on the minimum degree sum on large independent sets for a graph to have a 2-proper partition. For a graph G, we say that an independent set $I \in \mathcal{I}(G)$ is *light* if $w_G(I) \leq |V(G)| - 1$. The *light independence number* of G, denoted by $\alpha^*(G)$, is the maximum order of a light independent set of G. We show the following.

Theorem 9.7. Let G be a graph of order n. If $\sigma_1^*(G) \geq n$, then either G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \alpha^*(G)$, or G is isomorphic to a graph in $\{K_2, F_5\} \cup \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{H}_n$, where the exceptional graphs are defined below.

We define exceptional graphs in Theorem 9.7 (Figure 9.3).

- A graph F_5 is defined by $V(F_5) = \{a\} \cup \{b_i \mid i \in [4]\}$ and $E(F_5) = \{ab_i \mid i \in [4]\} \cup \{b_1b_2, b_3b_4\}$.
- Let F be a graph having the vertex set $V(F) = \{a, b_1, b_2\} \cup \{c_i \mid i \in [8]\}$ and the edge set $E(F) = \{ac_i \mid i \in [8]\} \cup \{b_1c_i \mid i \in [4]\} \cup \{b_2c_i \mid i \in [8] \setminus [4]\} \cup \{c_1c_2, c_3c_4, c_5c_6, c_7c_8\} \cup L$ where L is a subset of $\{ab_1, ab_2\}$. Let $\mathcal{F}_{11}$ be the family of such graphs F. Note that $\mathcal{F}_{11}$ consists of three graphs up to isomorphic.
- Let F be a graph having the vertex set $\{a, b_1, b_2\} \cup \{c_i \mid i \in [9]\}$ and the edge set $E(F) = \{ac_i \mid i \in [8]\} \cup \{b_1c_i \mid i \in [4]\} \cup \{b_2c_i \mid i \in [8] \setminus [4]\} \cup \{c_1c_2, c_3c_4, c_5c_6, c_7c_9, c_8c_9\} \cup L$ where L is a subset of $\{ab_1, ab_2, ac_9, b_2c_9\}$ with $L \cap \{ac_9, b_2c_9\} \neq \emptyset$. Let $\mathcal{F}_{12}$ be the family of such graphs F. Note that $\mathcal{F}_{12}$ consists of twelve graphs.
- Let s and t be two integers with $2 \leq s \leq t$. Let S_1 and S_2 be vertex disjoint complete graphs of order $s - 1$ and $t - 1$, respectively. Let $H_{s,t}$ be the graph obtained from S_1 and S_2 by adding four new vertices a, b, c_1 and c_2 and the edge set $\{ax, bx \mid x \in V(S_1) \cup V(S_2)\} \cup \{ab, ac_1, ac_2, c_1c_2\}$. Let $H_{s,t}^- = H_{s,t} - ab$. Note that $|H_{s,t}| = |H_{s,t}^-| = s + t + 2$. For an integer $n \geq 6$, let $\mathcal{H}_n = \{H_{s,t}, H_{s,t}^- \mid 2 \leq s \leq t, s + t + 2 = n\}$.

We remark that the case $\sigma_1^*(G) = \infty$ is allowed in Theorem 9.7. The lower bound of $\sigma_1^*(G)$ and the upper bound of $|\mathcal{P}|$ in Theorem 9.7 are both tight. Indeed, for a positive integer l, let G_3 be a graph obtained from l disjoint copies $H_1, H_2, \dots, H_l$ of the complete graph K_{l+1} by adding a new vertex v which is adjacent to a vertex $u_i \in V(H_i)$ for each



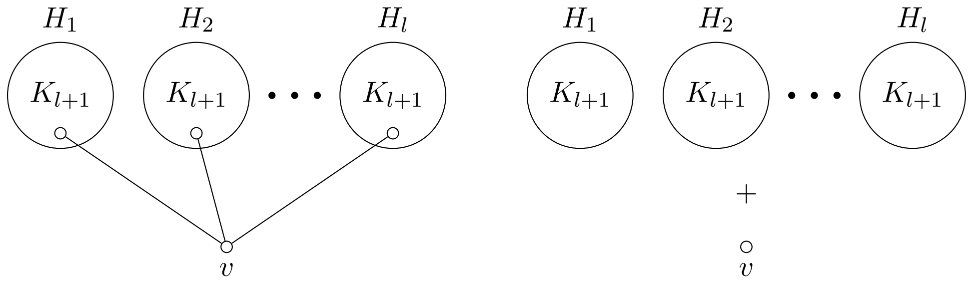
(a) A graph F_5

(b) Graphs in $\mathcal{F}_{11}$

(c) Graphs in $\mathcal{F}_{12}$

(d) A graph $H_{s,t}$

Figure 9.3: Exceptional graphs in Theorem 9.7. Dotted edges may or may not exist.



(a) A graph G_3.

(b) A graph G_4.

Figure 9.4: Graphs to show the tightness of Theorem 9.7.

$i \in [l]$ (Figure 9.4a). Then we have $n_3 := |V(G_3)| = l(l+1)+1$ and $\delta(G_3) = l$. By Proposition 8.2, we have $\sigma_1^*(G_3) \geq l(l+1) = n_3 - 1$, and the equality is attained by an independent set consisting of v and $w_i \in V(H_i)\setminus\{u_i\}$ for each $i \in [d]$. Since no 2-connected subgraph of G_3 contains v, G_3 has no 2-proper partition. Thus, the lower bound of $\sigma_1^*(G)$ in Theorem 9.7 is tight. For a positive integer l, let G_4 be a graph obtained from l disjoint copies $H_1, H_2, \ldots, H_l$ of the complete graph K_{l+1} by adding a new vertex v that is joined to all vertices in $\bigcup_{i=1}^{l} V(H_i)$ (Figure 9.4b). Then we have $n_4 := |V(G_4)| = l(l+1)+1$ and $\delta(G_4) = l+1$. Letting $I \in \mathcal{I}(G_4)$ be an independent set that contains exactly one vertex of H_i for each $i \in [d]$, we have $w_{G_4}(I) = l(l+1) = n_4 - 1$, and thus $\alpha^*(G_4) = |I| = l$. It is easy to verify that every 2-proper partition of G_4 consists of at least l parts. Thus, the upper bound of $|\mathcal{P}|$ in Theorem 9.7 is tight.

As corollaries of Theorem 9.7, we have the following two results on the existence of a 2-proper partition. For an even integer $n \geq 6$, we define $\mathcal{H}_n^{=}$ by $\mathcal{H}_n^{=} = \{H_{s,s}, H_{s,s}^{-}\}$ where $s = \frac{n-2}{2}$. Note that $\mathcal{H}_n^{=}$ is a subset of $\mathcal{H}_n$. The first corollary is on a minimum degree product condition for a graph to have a 2-proper partition.

Corollary 9.8. Let G be a graph of order n with the minimum degree δ. If $\pi_2(G) \geq n-\delta$, then either G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \alpha^*(G)$, or G is isomorphic to a graph in $\{K_2, F_5\} \cup \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{H}_n^{=}$.

Proof. Let G be a graph of order n with the minimum degree δ. Suppose that $\pi_2(G) \geq n - \delta$. By Proposition 8.2, we have $\sigma_1^*(G) \geq \pi_2(G) + \delta \geq n$. Then, Theorem 9.7 implies that either G contains a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \alpha^*(G)$, or G is isomorphic to a graph in $\{K_2, F_5\} \cup \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{H}_n$. It suffices to show that G is not isomorphic to a graph in $\mathcal{H}_n \setminus \mathcal{H}_n^{=}$. Suppose that G is isomorphic to $H_{s,t}$ for some integers s and t with $2 \leq s \leq t$ and $s+t = n-2$. Since G contains two non-adjacent vertices x and y such that $d_G(x) = 2$ and $d_G(y) = s$, we have $n-2 \leq \pi_2(G) \leq d_G(x)d_G(y) = 2s$, which implies that $s = t = \frac{n-2}{2}$. Thus, $G \in \mathcal{H}_n^{=}$. The case G is isomorphic to $H_{s,t}^{-}$ for some integers s and t goes similarly. ☐

The second corollary is on the lower bound of $\sigma_0^*(G)$ for a graph G to have a 2-proper partition.

Corollary 9.9. Let G be a graph of order n. If $\sigma_0^*(G) \geq n$, then G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \alpha^*(G)$.

Proof. Let G be a graph of order n and suppose that $\sigma_0^*(G) \geq n$. Since we have $\sigma_1^*(G) \geq \sigma_0^*(G) \geq n$ by Proposition 8.1, it suffices to show that G is not isomorphic to any exceptional graphs in Theorem 9.7. We label the vertices of G as in the definition of exceptional graphs. If G is isomorphic to K_2, then a single vertex is an independent set that violates the assumption. If G is isomorphic to F_5, then $I_1 = \{b_1, b_3\}$ satisfies $\delta_G(I_1) = 2 \leq |I_1|$ and $w_G(I_1) = 4 < n$, a contradiction. If G is isomorphic to a graph in $\mathcal{F}_{11} \cup \mathcal{F}_{12}$, then $I_2 = \{c_1, c_3, c_5\}$ satisfies $\delta_G(I_2) = 3 \leq |I_2|$ and $w_G(I_2) = 9 < n$, a contradiction. If G is isomorphic to $H_{s,t}$ for some positive integers s and t such that $2 \leq s \leq t$ and $s+t = n-2$, then $I_4 = \{c_1, b\}$ satisfies $\delta_G(I_4) = 2 \leq |I_4|$ and $w_G(I_4) = 2+(s+t-1) = n-1$, a contradiction. ☐

It is easy to see that every exceptional graph in Theorem 9.7 can be partitioned into 2-connected subgraphs except for one part which is isomorphic to K_2. (In particular, if G is isomorphic to $H_{s,t}$ or $H^-_{s,t}$, then G has a partition $\mathcal{P} = \{\{c_1, c_2\}, V(G) \setminus \{c_1, c_2\})\}$ with $\alpha^*(G) = 2$ parts.) Thus, we introduce the following relaxed concept of the 2-proper partition. An *almost* 2*-proper partition* of a graph G is a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of $V(G)$ such that

- $G[P_1]$ is either 2-connected or isomorphic to K_2, and
- $G[P_i]$ is 2-connected for every $i \in \{2, 3, \ldots, r\}$.

By the definition, every 2-proper partition of a graph G is also an almost 2-proper partition of G. By the observation above, Theorem 9.7 immediately implies the following.

Corollary 9.10. Let G be a graph of order n. If $\sigma_1^*(G) \geq n$, then G has an almost 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \alpha^*(G)$.

9.2 Relationship between results

In this section, we see the relationship between Theorems 9.5, 9.6, 9.7 Corollaries 9.8 and 9.9. First, we see the relationship between the degree conditions in the results.

Proposition 9.11. Let G be a graph of order n with the minimum degree $\delta \geq 1$.

- If $\delta \geq \sqrt{n}$, then $\sigma_2(G) > \frac{n}{\delta} + \delta - 1$.
- If $\delta \geq \sqrt{n}$, then $\sigma_0^*(G) \geq n$.
- If $\sigma_2(G) \geq \frac{n}{\delta} + \delta - 1$, then $\pi_2(G) \geq n - \delta$.
- If $\pi_2(G) \geq n - \delta$, then $\sigma_1^*(G) \geq n$.
- If $\sigma_0^*(G) \geq n$, then $\sigma_1^*(G) \geq n$.

Proof. The statements are trivial when G is complete. We assume that G is non-complete. The second, fourth and fifth statements follow directly from Propositions 8.1 and 8.2. We show only the first and the third. Suppose that $\delta \geq \sqrt{n}$. Then we have $\delta + 1 > \delta \geq \frac{n}{\delta}$, which implies that $2\delta > \frac{n}{\delta} + \delta - 1$. Since $\sigma_2(G) \geq 2\delta$ by the definition of $\sigma_2(G)$, we have $\sigma_2(G) \geq 2\delta > \frac{n}{\delta} + \delta - 1$, as desired.

We suppose that $\sigma_2(G) \geq \frac{n}{\delta} + \delta - 1$. Let u and v be non-adjacent vertices of G such that $d_G(u) \leq d_G(v)$ and $d_G(u)d_G(v) = \pi_2(G)$. If $d_G(u) \leq \frac{\sigma_2(G)}{2}$, then we have $\delta \leq d_G(u) \leq \frac{\sigma_2(G)}{2}$ and

$$d_G(u)d_G(v) \geq d_G(u)(\sigma_2(G) - d_G(u)) \geq \delta(\sigma_2(G) - \delta).$$

Otherwise, we have

$$d_G(u)d_G(v) > \left(\frac{\sigma_2(G)}{2}\right)^2 \geq \delta(\sigma_2(G) - \delta).$$

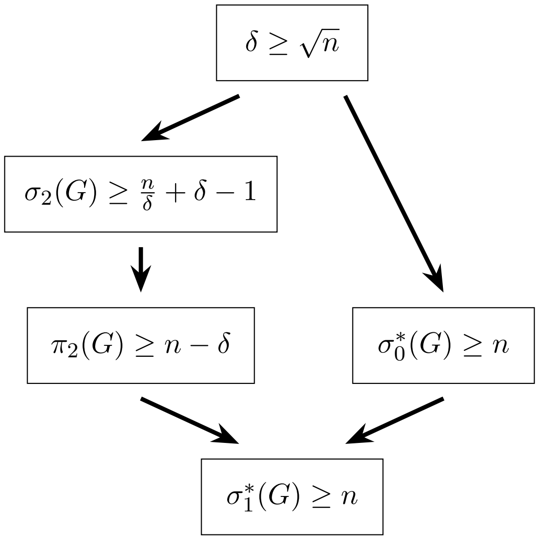


Figure 9.5: Relationship between degree conditions for a graph G of order n with the minimum degree δ to have a 2-proper partition. An arrow from an assumption A to an assumption B means assumption A implies assumption B.

In either case, we have $\pi_2(G) = d_G(u)d_G(v) \geq \delta(\sigma_2(G) - \delta) \geq \delta\left(\frac{n}{\delta} - 1\right) = n - \delta$, as desired. $\square$

We show the relationship between the degree conditions of the theorems and the corollaries in Figure 9.5. As we can see in the figure, the degree condition $\sigma_1^*(G) \geq n$ is implied by any other degree conditions. Furthermore, there are infinitely many graphs G that satisfy $\sigma_1^*(G) \geq |V(G)|$ but do not satisfy $\pi_2(G) \geq |V(G)| - \delta(G)$ or $\sigma_0^*(G) \geq |V(G)|$. For an integer $n \geq 6$, let G_5 be a graph obtained from the complete graph K_{n-3} with vertices $\{v_1, v_2, \ldots, v_{n-3}\}$ by adding three new vertices u_1, u_2 and u_3 such that both $u_1u_2v_1u_1$ and $u_1u_3v_1u_1$ are triangles of G_5 (Figure 9.6). Then we have $|V(G_5)| = n$ and $\delta(G_5) = 2$. Since $d_{G_5}(u_2) = d_{G_5}(u_3) = 2$, we have $\pi_2(G_5) \leq d_{G_5}(u_2)d_{G_5}(u_3) = 4$ and $\sigma_0^*(G_5) \leq w_{G_5}(\{u_2, u_3\}) = d_{G_5}(u_2) + d_{G_5}(u_3) = 4$, and thus G_5 satisfies neither $\pi_2(G_5) \geq n - 2$ nor $\sigma_0^*(G_5) \geq n$. Suppose that $I \in \mathcal{I}(G_5)$ is an independent set with $|I| \geq \delta_{G_5}(I) + 1$ and $\sigma_1^*(G_5) = w_{G_5}(I)$. Since $\delta_{G_5}(I) \geq \delta(G_5) = 2$, we have $|I| \geq 3$, and thus $I = \{u_2, u_3, v_i\}$ for some $i \in \{2, 3, \ldots, n-3\}$. Thus, we have $\sigma_1^*(G_5) = w_{G_5}(I) = d_{G_5}(u_2) + d_{G_5}(u_3) + d_{G_5}(v_i) = 2 + 2 + n - 4 = n$, and hence G_5 has a 2-proper partition by Theorem 9.7.

Next, we see the relationship between the upper bounds of the number of parts in the theorems and the corollaries.

Proposition 9.12. Let G be a non-complete graph of order n with the minimum degree $\delta \geq 1$. Then $\alpha^*(G) \leq \frac{2(n-1)}{\sigma_2(G)} \leq \frac{n-1}{\delta}$.

Proof. The second inequality follows from the fact $\sigma_2(G) \geq 2\delta$. We show the first inequality. Since $\sigma_2(G) < 2(n-1)$, i.e. $\frac{2(n-1)}{\sigma_2(G)} > 1$, the statement holds when $\alpha^*(G) = 1$. Hence,

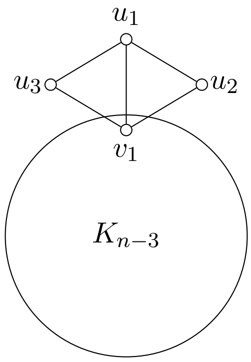


Figure 9.6: A graph G_5.

we assume $r := \alpha^*(G) \geq 2$. Let $I = \{u_0, u_1, \ldots, u_{r-1}\} \in \mathcal{I}(G)$ be an independent set with $w_G(I) \leq n-1$. Then we have

$$2(n-1) \geq 2w_G(I) = \sum_{i=0}^{r-1}(d_G(u_i) + d_G(u_{i+1})) \geq \sum_{i=0}^{r-1} \sigma_2(G) = r\sigma_2(G)$$

where $u_r = u_0$. Thus, $\alpha^*(G) = r \leq \frac{2(n-1)}{\sigma_2(G)}$. □

By Propositions 9.11 and 9.12, Theorem 9.7 implies both Theorems 9.5 and 9.6.

9.3 Lemmas

Before we go to the next section, we see some properties of $\sigma_1^*(G)$ and α^* that are used in our proof of Theorem 9.7.

Lemma 9.13. Let G be a graph and $G_1, G_2, \ldots, G_k$ be the components of G. Then $\sigma_1^*(G) \leq \min\{\sigma_1^*(G_i) \mid i \in [k]\}$.

Proof. Fix $i \in [k]$ and we prove $\sigma_1^*(G) \leq \sigma_1^*(G_i)$. If $\sigma_1^*(G_i) = \infty$, then we are done. Otherwise, let $I \in \mathcal{I}(G_i)$ be an independent set such that $|I| \geq \delta_{G_i}(I) + 1$ and $w_{G_i}(I) = \sigma_1^*(G_i)$. Then I is an independent set of G as well, and I satisfies $|I| \geq \delta_{G_i}(I) + 1 = \delta_G(I) + 1$. Hence $\sigma_1^*(G) \leq w_G(I) = w_{G_i}(I) = \sigma_1^*(G_i)$. □

For the light independence number $\alpha^*(G)$, we can easily verify the following properties.

Lemma 9.14. Let G be a graph, and let $G_1, G_2, \ldots, G_k$ be the components of G. Then $\alpha^*(G) \geq \sum_{i=1}^k \alpha^*(G_i)$. As a result, if a graph G has k components, then $\alpha^*(G) \geq k$.

Proof. For each $i \in [k]$, let I_i be a light independent set of G_i with $|I_i| = \alpha^*(G_i)$. Then, $I = \bigcup_{i=1}^k I_i$ is an independent set of G with $w_G(I) = \sum_{i=1}^k w_{G_i}(I_i) \leq \sum_{i=1}^k(|V(G_i)| - 1) \leq |V(G)| - 1$. Hence $\alpha^*(G) \geq |I| = \sum_{i=1}^k |I_i| = \sum_{i=1}^k \alpha^*(G_i)$. □

If $\alpha^*(G) \leq 1$, then either G is complete, or $d_G(u) + d_G(v) \geq |V(G)|$ for every non-adjacent pair of vertices $u, v \in V(G)$, and hence G is hamiltonian by Ore's theorem in [52]. Thus, the following lemma holds.

Lemma 9.15. Let G be a connected graph. If G has a cut vertex, then $\alpha^*(G) \geq 2$.

9.4 Proof of Theorem 9.7

Let $\mathcal{F}$ be the set of exceptional graphs in Theorem 9.7. Suppose that the statement is false. Let G be a counterexample of the minimum order, and let $n = |V(G)|$. If $n = 1$, i.e. $G \simeq K_1$, then $\sigma_1^*(G) = 0 < n$, which is a contradiction. Thus $n \geq 2$. If G has a vertex u with degree at most 1, then there is a vertex $v \in V(G) \setminus N_G(u)$ since $G \not\simeq K_2$. However, $I = \{u, v\}$ is an independent set of G with $|I| = 2 \geq d_G(u) \geq \delta_G(I)$, and thus $\sigma_1^*(G) \leq w_G(I) = d_G(u) + d_G(v) \leq 1 + (n-2) < n$, a contradiction. Thus the minimum degree of G is at least 2.

Claim 1. G is connected.

Proof. Suppose that G is disconnected and let $G_1, G_2, \ldots, G_k$ be the components of G. First, we show that G_i is not isomorphic to any exceptional graph for each $i \in [k]$. Suppose that G_i is isomorphic to a graph in $\mathcal{F}$ for some $i \in [k]$. As $\delta(G) \geq 2$, G_i is isomorphic to a graph in $\mathcal{F} \setminus \{K_2\}$. Then we can take an independent set I_i of G_i such that $|I_i| = \delta_{G_i}(I_i) = \delta_G(I_i)$ and $w_G(I_i) = w_{G_i}(I_i) \leq |V(G_i)|$. Let $u \in V(G) \setminus V(G_i)$ and set $I = I_i \cup \{u\}$. Then I is an independent set of G with $|I| = |I_i| + 1 = \delta_G(I_i) + 1 \geq \delta_G(I) + 1$. This forces $\sigma_1^*(G) \leq w_G(I) = w_G(I_i) + d_G(u) \leq |V(G_i)| + (n - |V(G_i)| - 1) = n - 1$, a contradiction. Hence G_i is not isomorphic to any graph in $\mathcal{F}$ for each $i \in [k]$. Since $\sigma_1^*(G_i) \geq \sigma_1^*(G) \geq n > |V(G_i)|$ by Lemma 9.13, G_i has a 2-proper partition $\mathcal{P}_i$ with $|\mathcal{P}_i| \leq \alpha^*(G_i)$. Then $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$ is a 2-proper partition of G. By Lemma 9.14, $\mathcal{P}$ satisfies $|\mathcal{P}| = \sum_{i=1}^k |\mathcal{P}_i| \leq \sum_{i=1}^k \alpha^*(G_i) \leq \alpha^*(G)$, a contradiction. □

When G is 2-connected, $\{V(G)\}$ is a desired 2-proper partition. Hence G has a cut vertex. Let $\mathcal{B}$ be the family of blocks of G, and let U be the set of cut vertices of G. The *block-cut vertex graph* T of G is defined by $V(T) = \mathcal{B} \cup U$ and $E(T) = \{Bu \mid B \in \mathcal{B}, u \in U, u \in V(B)\}$. By the definition of T, the graph T is a tree and every leaf of T belongs to $\mathcal{B}$. A block $B \in \mathcal{B}$ which corresponds to a leaf of T is an *end-block* of G. Since $\delta(G) \geq 2$, every end-block of G has at least 3 vertices.

Claim 2. Every end-block $B \in \mathcal{B}$ has at least 4 vertices.

Proof. Suppose that B is an end-block of G and $|V(B)| = 3$. As $\delta(G) \geq 2$, B is isomorphic to K_3. Write $V(B) = \{u_1, u_2, u_3\}$ where u_3 is a cut vertex of G. If $G - V(B)$ is 2-connected, then $\mathcal{P} = \{V(B), V(G) \setminus V(B)\}$ is a 2-proper partition of G and $|\mathcal{P}| = 2 \leq \alpha^*(G)$ by Lemma 9.15, a contradiction. Hence $G - V(B)$ is not 2-connected. Since $G \not\simeq F_5$, we have $G - V(B) \not\simeq K_2$. Now we define two subsets S_1 and S_2 of $V(G) \setminus V(B)$ as follows. If $G - V(B)$ is disconnected, then let S_1 be the vertex set of a component of $G - V(B)$

and let $S_2 = V(G) \setminus (V(B) \cup S_1)$; if $G - V(B)$ is connected, then let x be a cut-vertex of $G - V(B)$, and let $S_1, S_2 \subseteq V(G) \setminus V(B)$ be such that $G[S_1]$ and $G[S_2]$ are distinct components of $G - (V(B) \cup \{x\})$. Let $v_i \in S_i$ for $i \in \{1,2\}$, and define $I = \{u_1, v_1, v_2\}$. Then I is an independent set of G with $|I| = 3 = d_G(u_1) + 1 \geq \delta_G(I) + 1$. It follows that $w_G(I) \geq \sigma_1^*(G) \geq n$, and thus $d_G(v_1) + d_G(v_2) = w_G(I) - 2 \geq n - 2$. Note that $|N_{G-V(B)}[v_1] \cap N_{G-V(B)}[v_2]| \leq 1$, and the equality holds if and only if $G - V(B)$ is connected and $v_1x, v_2x \in E(G)$. Furthermore, for each $i \in \{1,2\}$, $|N_{G-V(B)}[v_i]| \geq d_G(v_i)$ and the equality holds if and only if $v_iu_3 \in E(G)$. Hence

$$\begin{aligned}n - 2 &= |V(G - \{u_1, u_2\})| \geq |N_{G-V(B)}[v_1] \cup N_{G-V(B)}[v_2] \cup \{u_3\}| \\ &= |N_{G-V(B)}[v_1]| + |N_{G-V(B)}[v_2]| - |N_{G-V(B)}[v_1] \cap N_{G-V(B)}[v_2]| + 1 \geq d_G(v_1) + d_G(v_2).\end{aligned}$$

This forces $d_G(v_1) + d_G(v_2) = n - 2$, and all equalities hold in the above inequalities. As we mentioned above, $G - V(B)$ is connected, $v_1x, v_1u_3, v_2x, v_2u_3 \in E(G)$. Since $|V(G - \{u_1, u_2\})| = |N_{G-V(B)}[v_1] \cup N_{G-V(B)}[v_2] \cup \{u_3\}|$, we have $V(G) = V(B) \cup S_1 \cup S_2 \cup \{x\}$ and $S_i \subseteq N_G[v_i]$ for each $i \in \{1,2\}$. Since $v_i \in S_i$ is arbitrary, we conclude that both $S_1 \cup \{x\}$ and $S_2 \cup \{x\}$ are cliques of G and $S_1 \cup S_2 \subseteq N_G(u_3)$. By the symmetry of S_1 and S_2, we may assume that $|S_1| \leq |S_2|$. Let $s = |S_1| + 1$ and $t = |S_2| + 1$. Then G is isomorphic to either $H_{s,t}$ or $H^-_{s,t}$ according to $u_3x \in E(G)$ or not. In particular, G is isomorphic to a graph in $\mathcal{H}_n$, which is a contradiction. □

For each $B \in \mathcal{B}$, let $X_B = V(B) \setminus U$.

Claim 3. Let $B \in \mathcal{B}$ with $X_B \neq \emptyset$. If there is a vertex $u \in V(B) \setminus X_B$ for which no block of $B - u$ contains X_B, then $G - V(B)$ is isomorphic to a graph in $\mathcal{F} \setminus \{K_2\}$.

Proof. Suppose that $X_B \neq \emptyset$ and that no block of $B - u$ contains X_B for some $u \in V(B) \setminus X_B$. Then, there are distinct vertices $x_1, x_2 \in X_B$ such that $x_1x_2 \notin E(G)$ and $|N_{B-u}[x_1] \cap N_{B-u}[x_2]| \leq 1$. Without loss of generality, we assume that $d_G(x_1) \leq d_G(x_2)$. Considering these two vertices, we have

$$\begin{aligned}|V(B)| \geq |N_{B-u}[x_1] \cup N_{B-u}[x_2] \cup \{u\}| &\geq |N_{B-u}[x_1]| + |N_{B-u}[x_2]| - 1 + 1 \\ &\geq d_G(x_1) + d_G(x_2) \geq 2d_G(x_1). \qquad (9.4.1)\end{aligned}$$

Let $G' = G - V(B)$. Then $d_{G'}(v) \geq d_G(v) - 1$ for every $v \in V(G')$. We shall show that $\sigma_1^*(G') \geq |V(G')|$. Let $I' = \{u_1, u_2, \ldots, u_t\} \in \mathcal{I}(G')$ be any independent set of G' with $|I'| \geq \delta_{G'}(I') + 1$. We may assume that $\delta_{G'}(I') = d_{G'}(u_1)$. Let $l = \min\{d_G(u_1), d_G(x_1)\}$. Since $t = |I'| \geq d_{G'}(u_1) + 1 \geq d_G(u_1) \geq l$, the set $I = \{u_1, u_2, \ldots, u_l, x_1\}$ is well-defined. Since I is an independent set of G with $|I| = l + 1 = \min\{d_G(u_1), d_G(x_1)\} + 1 \geq \delta_G(I) + 1$, we have $w_G(I) \geq n$. Consequently, it follows from (9.4.1) that

$$\begin{aligned}w_{G'}(I') = \sum_{i=1}^{t} d_{G'}(u_i) \geq \sum_{i=1}^{l} d_{G'}(u_i) \geq \sum_{i=1}^{l}(d_G(u_i) - 1) &\geq w_G(I) - l - d_G(x_1) \\ &\geq w_G(I) - 2d_G(x_1) \\ &\geq n - |V(B)| = |V(G')|,\end{aligned}$$

and we conclude that $\sigma_1^*(G') \geq |V(G')|$.

Suppose that G' is not isomorphic to any graph in $\mathcal{F}$. Then G' has a 2-proper partition $\mathcal{P}'$ with $|\mathcal{P}'| \leq \alpha^*(G')$. Since B is 2-connected, $\mathcal{P} = \mathcal{P}' \cup \{V(B)\}$ is a 2-proper partition of G. We shall show that $|\mathcal{P}| \leq \alpha^*(G)$. Let $J' = \{v_1, v_2, \dots, v_r\}$ be a light independent set of G' with $r = \alpha^*(G')$. If $r \geq d_G(x_1)$, then $J_{x_1} := \{x_1, v_1, v_2, \dots, v_{d_G(x_1)}\}$ is an independent set of G with $|J_{x_1}| \geq \delta_G(J_{x_1}) + 1$. Then by (9.4.1),

$$\begin{aligned}\sigma_1^*(G) \leq w_G(J_{x_1}) = d_G(x_1) + \sum_{i=1}^{d_G(x_1)} d_G(v_i) \leq d_G(x_1) + \sum_{i=1}^{d_G(x_1)} (d_{G'}(v_i) + 1) \\ \leq d_G(x_1) + w_{G'}(J') + d_G(x_1) \leq |V(G')| - 1 + 2d_G(x_1) \leq n - 1,\end{aligned}$$

which contradicts the assumption of the theorem. Thus, $r \leq d_G(x_1) - 1$. Let $J = J' \cup \{x_1\}$. Again by (9.4.1),

$$\begin{aligned}w_G(J) = d_G(x_1) + \sum_{i=1}^{r} d_G(v_i) \leq d_G(x_1) + \sum_{i=1}^{r} (d_{G'}(v_i) + 1) = d_G(x_1) + w_{G'}(J') + r \\ \leq |V(G')| - 1 + 2d_G(x_1) - 1 < n - 1,\end{aligned}$$

and in particular, J is a light independent set of G. Thus, $\alpha^*(G) \geq |J| = r+1 = \alpha^*(G')+1$ and hence $|\mathcal{P}| = |\mathcal{P}'| + 1 \leq \alpha^*(G') + 1 \leq \alpha^*(G)$, contradicting the fact that G is a counterexample of the theorem. Therefore, G' is isomorphic to a graph in $\mathcal{F}$. Moreover, Claim 2 implies that $G' \not\simeq K_2$, and thus G' is isomorphic to a graph in $\mathcal{F} \setminus \{K_2\}$. ☐

Claim 4. For every $B \in \mathcal{B}$ and every $u \in V(B) \setminus X_B$, X_B is contained in a block of $B - u$.

Proof. Let $B_1 \in \mathcal{B}$ and $u \in V(B_1) \setminus X_{B_1}$. Suppose that no block of $B_1 - u$ contains X_{B_1}, and let $B_1^- = B_1 - u$. By Claim 3, $G - V(B_1)$ is isomorphic to a graph in $\mathcal{F} \setminus \{K_2\}$. Since every graph in $\mathcal{F}$ is connected, B_1 is an end-block of G, which implies that $X_{B_1} = V(B_1^-)$. This together with the assumption of the proof implies that B_1^- is not 2-connected. Now we prove the following subclaims.

Subclaim 4.1. If $G - V(B_1)$ is isomorphic to a graph in $\{F_5, H_{2,2}, H_{2,2}^-\}$, then u is adjacent to all vertices of $G - V(B_1)$ having degree 2.

Proof. We may assume that $G - V(B_1)$ belongs to $\mathcal{F} \setminus \{K_2\}$. (For example, if $G - V(B_1)$ is isomorphic to a graph in $\mathcal{F}_{11}$, then we may assume that $V(G - V(B_1)) = \{a_1, b_2, b_3\} \cup \{c_i \mid i \in [8]\}$.) Since B_1^- is not 2-connected, there is a vertex $v \in V(B_1^-)$ such that $d_G(v) \leq |V(B_1)| - 2$.

Suppose that $G - V(B_1) \in \{F_5, H_{2,2}, H_{2,2}^-\}$. Then $|V(G) \setminus V(B_1)| \geq 5$. Suppose that there exists a vertex $x \in V(G) \setminus V(B_1)$ such that $d_{G-V(B_1)}(x) = 2$ and $xu \notin E(G)$. Then we can easily verify that there exists a vertex $y \in V(G) \setminus (V(B_1) \cup \{x\})$ such that $d_{G-V(B_1)}(y) = 2$ and $xy \notin E(G)$. Let $I = \{x, y, v\}$. Then I is an independent set of G with $|I| \geq \delta_G(I) + 1$, and hence

$$\begin{aligned}\sigma_1^*(G) \leq w_G(I) = d_G(x) + d_G(y) + d_G(v) \leq 2 + 3 + (|V(B_1)| - 2) \\ \leq |V(G) \setminus V(B_1)| + |V(B_1)| - 2 = n - 2,\end{aligned}$$

a contradiction. Thus u is adjacent to all vertices of $G - V(B_1)$ having degree 2. $\square$

Subclaim 4.2. The graph $G - V(B_1^-)$ is a block of G.

Proof. We assume that $G - V(B_1)$ belongs to $\mathcal{F} \setminus \{K_2\}$, and let v be a vertex of B_1^- such that $d_G(v) \leq |V(B_1)| - 2$. If $G - V(B_1) \simeq F_5$, then it follows from Subclaim 4.1 that $G - V(B_1^-)$ is a block of G. Thus, we may assume that $G - V(B_1)$ is isomorphic to a graph in $\mathcal{F} \setminus \{K_2, F_5\}$.

Case 1. $G - V(B_1) \in \mathcal{F}_{11} \cup \mathcal{F}_{12}$.

Suppose that $c_i u \notin E(G)$ for some $i \in [8]$. Then there exist two distinct indices j and k with $j, k \in [8] \setminus \{i\}$ such that $\{c_i, c_j, c_k\}$ is an independent set of G (for example, if $i = 1$, then $j = 3$, $k = 5$ are desired indices). Let $I = \{c_i, c_j, c_k, v\}$. Then I is an independent set of G with $|I| = 4 = d_G(c_i) + 1 \geq \delta_G(I) + 1$, and hence

$$\begin{aligned}\sigma_1^*(G) \leq w_G(I) = d_G(c_i) + d_G(c_j) + d_G(c_k) + d_G(v) &\leq 3 + 4 + 4 + (|V(B_1)| - 2)\\ &\leq |V(G) \setminus V(B_1)| + |V(B_1)| - 2 = n - 2,\end{aligned}$$

a contradiction. Thus, $\{c_i \mid i \in [8]\} \subseteq N_G(u)$, which implies that $G - V(B_1^-)$ is a block of G.

Case 2. $G - V(B_1) \in \mathcal{H}_l$, where $l = n - |V(B_1)|$.

Let s and t be integers such that $2 \leq s \leq t$, $l = s + t + 2$ and $G - V(B_1)$ is isomorphic to either $H_{s,t}$ or $H_{s,t}^-$. By the definition of $H_{s,t}$ and $H_{s,t}^-$, we have $V(G) \setminus V(B_1) = V(S_1) \cup V(S_2) \cup \{a, b, c_1, c_2\}$. Suppose that $c_i u \notin E(G)$ for some $i \in \{1, 2\}$. Let $I = \{c_i, b, v\}$. Then I is an independent set of G with $|I| = 3 = d_G(c_i) + 1 \geq \delta_G(I) + 1$, and hence

$$\sigma_1^*(G) \leq w_G(I) \leq 2 + (s + t) + (|V(B_1)| - 2) = |V(G) \setminus V(B_1)| + |V(B_1)| - 2 = n - 2,$$

a contradiction. Thus, $\{c_1, c_2\} \subseteq N_G(u)$.

Suppose that $(V(S_1) \cup V(S_2)) \cap N_G(u) = \emptyset$. Let $I' = \{c_1, x_1, x_2, v\}$, where x_j is a vertex in $V(S_j)$ for each $j \in \{1, 2\}$. Then I' is an independent set of G with $|I'| = 4 = d_G(c_1) + 1 \geq \delta_G(I') + 1$, and hence

$$\begin{aligned}\sigma_1^*(G) \leq w_G(I') = d_G(c_1) + d_G(x_1) + d_G(x_2) + d_G(v) &\leq 3 + s + t + (|V(B_1)| - 2)\\ &= (|V(G) \setminus V(B_1)| + 1) + |V(B_1)| - 2 = n - 1,\end{aligned}$$

a contradiction. Thus, $(V(S_1) \cup V(S_2)) \cap N_G(u) \neq \emptyset$, which implies that $G - V(B_1^-)$ is a block of G. $\square$

Let $B_2 = G - V(B_1^-) \in \mathcal{B}$ and $B_2^- = B_2 - u$ $(= G - V(B_1))$. Since B_2^- is isomorphic to a graph in $\mathcal{F} \setminus \{K_2\}$, B_2^- is not 2-connected. It follows from Subclaim 4.2 that B_2 is an end-block of G, and hence, $X_{B_2} = V(B_2) \setminus \{u\}$. This implies that no block of $B_2 - u$ contains X_{B_2}. Applying Claim 3 with $B = B_2$, we conclude that $G - V(B_2)$ $(= B_1^-)$ is isomorphic to a graph in $\mathcal{F} \setminus \{K_2\}$. In particular, the roles of B_1 and B_2 are symmetric. Furthermore, by exchanging the role of B_1 and B_2, the following subclaim holds.

Subclaim 4.3. If $G - V(B_2)$ $(= B_1^-)$ is isomorphic to a graph in $\{F_5, H_{2,2}, H_{2,2}^-\}$, then u is adjacent to all vertices of $G - V(B_2)$ having degree 2.

We may assume that $|V(B_1^-)| \leq |V(B_2^-)|$.

Subclaim 4.4. If B_1^- is isomorphic to F_5, then B_2^- is isomorphic to a graph in $\mathcal{F} \setminus \{K_2, F_5, H_{2,2}, H_{2,2}^-\}$.

Proof. If B_1^- is isomorphic to F_5 and B_2^- is isomorphic to a graph in $\{F_5, H_{2,2}, H_{2,2}^-\}$, then by Subclaims 4.1 and 4.3, G is isomorphic to a graph in $\mathcal{F}_{11} \cup \mathcal{F}_{12}$, which is a contradiction. $\square$

Subclaim 4.5. If B_1^- is isomorphic to a graph in $\{H_{2,2}, H_{2,2}^-\}$, then B_2^- is isomorphic to a graph in $\mathcal{F} \setminus \{K_2, F_5, H_{2,2}, H_{2,2}^-\}$.

Proof. Assume that B_1^- is isomorphic to a graph in $\{H_{2,2}, H_{2,2}^-\}$. Since $6 = |V(B_1^-)| \leq |V(B_2^-)|$, $B_2^- \not\simeq F_5$. Suppose that B_2^- is isomorphic to a graph in $\{H_{2,2}, H_{2,2}^-\}$. Then for $i \in \{1, 2\}$, there exist two non-adjacent vertices $x_1^{(i)}$ and $x_2^{(i)}$ of B_i^- with $d_{B_i^-}(x_1^{(i)}) = d_{B_i^-}(x_2^{(i)}) = 2$. Let $I = \{x_j^{(i)} \mid i \in \{1, 2\}, j \in \{1, 2\}\}$. Note that $d_G(x_j^{(i)}) \leq d_{B_i^-}(x_j^{(i)}) + 1 = 3$. Then I is an independent set of G with $|I| \geq \delta_G(I) + 1$, and hence

$$\sigma^*(G) \leq w_G(I) = \sum_{i\in\{1,2\}, j\in\{1,2\}} d_G(x_j^{(i)}) \leq 4 \cdot 3 < 13 = n,$$

a contradiction. $\square$

Since $|V(B_1^-)| \leq |V(B_2^-)|$, it follows from Subclaims 4.4 and 4.5 that B_2^- is isomorphic to a graph in $\mathcal{F} \setminus \{K_2, F_5, H_{2,2}, H_{2,2}^-\}$. Now we define an independent set I_2 of B_2^- as follows. If B_2^- is isomorphic to a graph in $\mathcal{F}_{11} \cup \mathcal{F}_{12}$, then let I_2 be the set of vertices of B_2^- corresponding to c_1 and c_3; If B_2^- is isomorphic to a graph in $\{H_{s,t}, H_{s,t}^-\}$ for some s and t with $2 \leq s \leq t$, then let I_2 be the set of vertices of B_2^- corresponding to c_1 and a vertex in $V(S_1)$. In the former case, we have $w_{B_2^-}(I_2) = 6 < |V(B_2^-)| - 3$; in the latter case, since $t \geq 3$, we have $w_{B_2^-}(I_2) = s + 2 \leq s + t - 1 = |V(B_2^-)| - 3$. In either case, we obtain $w_{B_2^-}(I_2) \leq |V(B_2^-)| - 3$.

Next, we define the independent set I_1 of B_1^- as follows. If $B_1^- \simeq F_5$, then let I_1 be the set of vertices corresponding to b_1 and b_3; if B_1^- is isomorphic to a graph in $\mathcal{F}_{11} \cup \mathcal{F}_{12}$, then let I_1 be the vertices corresponding to c_1, c_3 and c_5; if B_1^- is isomorphic to a graph in $\{H_{s,t}, H_{s,t}^-\}$ for some s and t with $2 \leq s \leq t$, then let I_1 be the vertices corresponding to b and c_1. Then we have $|I_1| = \delta_{B_1^-}(I_1)$ and $w_{B_1^-}(I_1) \leq |V(B_1^-)| - |I_1| + 1$.

Let $\tilde{I} = I_1 \cup I_2$. Then $\tilde{I}$ is an independent set of G with $|\tilde{I}| = |I_1| + 2 \geq (\delta_{B_1^-}(I_1) + 1) + 1 \geq \delta_G(I_1) + 1 \geq \delta_G(\tilde{I}) + 1$, and hence

$$\begin{aligned}\sigma_1^*(G) \leq w_G(\tilde{I}) = w_G(I_1) + w_G(I_2) &\leq w_{B_1^-}(I_1) + |I_1| + w_{B_2^-}(I_2) + |I_2| \\ &\leq (|V(B_1^-)| - |I_1| + 1) + |I_1| + (|V(B_2^-)| - 3) + 2 = n - 1,\end{aligned}$$

which is a contradiction. This completes the proof of Claim 4. $\square$

Let $X = \bigcup_{B \in \mathcal{B}} X_B$.

Claim 5. For every $u \in V(G)$, $N_G(u) \cap X \neq \emptyset$. In particular, for every $B \in \mathcal{B}$, if $X_B \neq \emptyset$, then $|X_B| \geq 2$.

Proof. Let $u \in V(G)$. Suppose that $N_G(u) \cap X = \emptyset$. Write $N_G(u) = \{v_1, v_2, \ldots, v_r\}$. For each $i \in [r]$, let D_i be a component of $G - v_i$ which does not contain u. Then

$$V(D_i) \cap V(D_j) = \emptyset \text{ for all } i, j \in [r] \text{ with } i \neq j. \tag{9.4.2}$$

Let y_i be a vertex of D_i for each $i \in [r]$. Then we have $|V(D_i)| \geq d_G(y_i)$. Let $I = \{u, y_1, y_2, \ldots, y_r\}$. The condition (9.4.2) implies that I is an independent set of G with $|I| \geq \delta_G(I) + 1$, and hence $w_G(I) \geq n$. Again, by (9.4.2), we have

$$|V(G)| \geq |N_G[u]| + \sum_{i=1}^{r} |V(D_i)| \geq (d_G(u) + 1) + \sum_{i=1}^{r} d_G(y_i) = w_G(I) + 1 \geq n + 1,$$

a contradiction. $\square$

Fix an end-block B_0 of G. We regard the block-cut vertex graph T of G as a rooted tree with the root B_0. For each $B \in \mathcal{B} \setminus \{B_0\}$, let u_B be the parent of B in T. By Claim 4, for each $B \in \mathcal{B} \setminus \{B_0\}$ with $X_B \neq \emptyset$, there is a block A_B of $B - u_B$ that contains X_B.

Claim 6. Let $B \in \mathcal{B} \setminus \{B_0\}$ with $X_B \neq \emptyset$. If $u \in V(B) \setminus (V(A_B) \cup \{u_B\})$, then $N_G(u) \cap (X \setminus X_B) \neq \emptyset$.

Proof. Let B be a block in $\mathcal{B} \setminus \{B_0\}$ and let $u \in V(B) \setminus (V(A_B) \cup \{u_B\})$. Write $N_G(u) = \{v_1, v_2, \ldots, v_r\}$. Suppose that $N_G(u) \cap (X \setminus X_B) = \emptyset$. By Claim 5, we have $N_G(u) \cap X_B \neq \emptyset$. Since $u \notin V(A_B)$ and $X_B \subseteq V(A_B)$, we have $|N_G(u) \cap X_B| = 1$. Without loss of generality, we may assume that $v_r \in X_B$ and $v_i \notin X$ for all $i \in [r-1]$. By Claim 5, there is a vertex $x \in X_B \setminus \{v_r\}$. Since $u \notin V(A_B)$, we have

$$|N_G(u) \cap N_G(x)| \leq |\{u_B, v_r\}| = 2. \tag{9.4.3}$$

Fix $i \in [r-1]$. Let D_i be a component of $G - v_i$ which does not contain u. Since B is a block of G, $B - v_i$ is a connected graph containing u if $v_i \in V(B)$. This leads to $V(D_i) \cap V(B) = \emptyset$. Let $y_i \in V(D_i)$ be a vertex. Then we have

$$|V(D_i)| \geq |N_G[y_i] \setminus \{v_i\}| \geq d_G(y_i). \tag{9.4.4}$$

Let $I = \{u, x, y_1, y_2, \ldots, y_{r-1}\}$. Then I is an independent set of G with $|I| \geq \delta_G(I) + 1$, and hence $w_G(I) \geq n$. Recall that $V(D_i) \cap V(B) = \emptyset$ for all $i \in [r-1]$. Hence by (9.4.3) and (9.4.4), we have

$$\begin{aligned} n = |V(G)| &\geq |N_G[u] \cup N_G[x]| + \sum_{i=1}^{r-1} |V(D_i)| \geq |N_G[u]| + |N_G[x]| - 2 + \sum_{i=1}^{r-1} d_G(y_i) \\ &= d_G(u) + d_G(x) + \sum_{i=1}^{r-1} d_G(y_i) = w_G(I) \geq n. \end{aligned}$$

This together with (9.4.4) forces $y_iv_i \in E(G)$ and $V(D_i) = N_G[y_i]\setminus\{v_i\}$ for each $i \in [r-1]$. Since $y_i \in V(D_i)$ is arbitrary, $V(D_i) \cup \{v_i\}$ is a clique of G. Furthermore, $V(G)$ is the disjoint union of $V(B)$, $N_G(u) \setminus V(B)$, and $\bigcup_{i=1}^{r-1} V(D_i)$. For each $i \in [r-1]$, we define V_i as $V_i = V(D_i)\cup\{v_i\}$ if $v_i \notin V(B)$ and $V_i = V(D_i)$ otherwise. By Claim 2, $|V_i| \geq 3$ for each $i \in [r-1]$. In addition, since $w_G(I \setminus \{x\}) < w_G(I) = n$, $I \setminus \{x\}$ is a light independent set of G, and hence $\alpha^*(G) \geq |I \setminus \{x\}| = r$. Combining these, we obtain a 2-proper partition $\mathcal{P} := \{V(B), V_2, V_3, \dots, V_r\}$ of G with $|\mathcal{P}| = r \leq \alpha^*(G)$, a contradiction. $\square$

Claim 7. For every $B \in \mathcal{B} \setminus \{B_0\}$ with $X_B \neq \emptyset$, A_B has at least 3 vertices. In particular, A_B is 2-connected.

Proof. If $B \in \mathcal{B} \setminus \{B_0\}$ is an end-block of G, then $|V(A_B)| = |V(B - u_B)| \geq 3$ by Claims 2 and 4. Suppose that $B \in \mathcal{B} \setminus \{B_0\}$ is not an end-block, $X_B \neq \emptyset$ and $|V(A_B)| \leq 2$. By Claim 5, we know that $V(A_B) = X_B$ and $|A_B| = 2$. Write $X_B = \{x_1, x_2\}$. Since A_B is a connected graph, we have $x_1x_2 \in E(G)$. Since B is not an end-block of G, one of x_1 and x_2, say x_1, is adjacent to a vertex $v_1 \in V(B)\setminus(X_B\cup\{u_B\})$. Write $N_G(x_2) = \{x_1, v_2, v_3, \dots, v_r\}$ where $r = d_G(x_2)$. Since $v_1 \notin V(A_B)$, v_1 is not adjacent to x_2. For each $i \in [r]$, let D_i be a component of $G - v_i$ which does not contain x_2, and let $y_i \in V(D_i)$. Then we have $|V(D_i)| \geq d_G(y_i)$. Since $V(D_i) \cap V(B) = \emptyset$ for every $i \in [r]$ and $V(D_i) \cap V(D_j) = \emptyset$ for all distinct $i, j \in [r]$, $I = \{x_2, y_1, y_2, \dots, y_r\}$ is an independent set of G with $|I| \geq \delta_G(I) + 1$. Hence

$$|V(G)| \geq |N_G[x_2]| + \sum_{i=1}^{r} |V(D_i)| \geq (d_G(x_2) + 1) + \sum_{i=1}^{r} d_G(y_i) = w_G(I) + 1 \geq n + 1,$$

which is a contradiction. $\square$

For each $B \in \mathcal{B}$, let $G(B)$ be a subgraph of G induced by the vertices of B and its descendant blocks with respect to T. In addition, for each $B \in \mathcal{B}$, let $\mathcal{B}(B) = \{B' \in \mathcal{B} \mid B' \subseteq G(B)\}$ and let $\tilde{\mathcal{B}}(B) = \{B' \in \mathcal{B}(B) \mid X_{B'} \neq \emptyset\}$.

Claim 8. Let $B \in \mathcal{B} \setminus \{B_0\}$. Then $G(B) - u_B$ has a 2-proper partition $\mathcal{P}_B^-$ with $|\mathcal{P}_B^-| = |\tilde{\mathcal{B}}(B)|$. Furthermore, if $X_B \neq \emptyset$, then $G(B)$ has a 2-proper partition $\mathcal{P}_B$ with $|\mathcal{P}_B| = |\tilde{\mathcal{B}}(B)|$.

Proof. The proof goes by induction on the height h of the block-cut vertex graph of $G(B)$ with the root B. When $h = 0$, B is an end-block of G, which implies that $G(B) = B$, $X_B = V(B) \setminus \{u_B\} \neq \emptyset$ and $|\tilde{\mathcal{B}}(B)| = 1$. By Claim 4, $\{X_B\}$ is a desired 2-proper partition of $G(B) - u_B$, and $\{V(B)\}$ is a desired 2-proper partition of $G(B)$. Hence, we assume $h \geq 1$, that is, B has a child in T. For each $x \in V(B) \setminus (X_B \cup \{u_B\})$, let $\mathcal{C}(x)$ be the family of blocks in $\mathcal{B} \setminus \{B\}$ containing x, and let $\mathcal{C}(B) = \bigcup_{x \in V(B)\setminus(X_B\cup\{u_B\})} \mathcal{C}(x)$.

By the induction hypothesis, the statements of the claim hold for every $B' \in \mathcal{C}(B)$. For each block $B' \in \mathcal{C}(B)$, let $\mathcal{P}_{B'}^-$ be a 2-proper partition of $G(B') - u_{B'}$ with $|\mathcal{P}_{B'}^-| = |\tilde{\mathcal{B}}(B')|$. In addition, for each block $B' \in \mathcal{C}(B)$ with $X_{B'} \neq \emptyset$, let $\mathcal{P}_{B'}$ be a 2-proper partition of $G(B')$ with $|\mathcal{P}_{B'}| = |\tilde{\mathcal{B}}(B')|$.

First, we assume $X_B = \emptyset$. For each $x \in V(B) \setminus \{u_B\}$, there is a block $B_x \in \mathcal{C}(x)$ with $X_{B_x} \neq \emptyset$ by Claim 5. Then,

$$\mathcal{P}_B^- = \bigcup_{x \in V(B) \setminus \{u_B\}} \left(\mathcal{P}_{B_x} \cup \bigcup_{B' \in \mathcal{C}(x) \setminus \{B_x\}} \mathcal{P}_{B'}^- \right)$$

is a 2-proper partition of $G(B) - u_B$ with $|\mathcal{P}_B^-| = \sum_{B' \in \mathcal{C}(B)} |\tilde{\mathcal{B}}(B')| = |\tilde{\mathcal{B}}(B)|$.

Next, we assume that $X_B \neq \emptyset$. Then B is 2-connected, and hence

$$\mathcal{P}_B = \{V(B)\} \cup \left(\bigcup_{B' \in \mathcal{C}(B)} \mathcal{P}_{B'}^- \right)$$

is a 2-proper partition of $G(B)$ with $|\mathcal{P}_B| = 1 + \sum_{B' \in \mathcal{C}(B)} |\tilde{\mathcal{B}}(B')| = |\tilde{\mathcal{B}}(B)|$, which proves the second statement of the claim. For each $x \in V(B) \setminus (V(A_B) \cup \{u_B\})$, there is a block $B_x \in \mathcal{C}(x)$ with $X_{B_x} \neq \emptyset$ by Claim 6. By Claim 7, A_B is 2-connected, and hence

$$\mathcal{P}_B^- = \{V(A_B)\} \cup \left(\bigcup_{x \in V(B) \setminus (V(A_B) \cup \{u_B\})} \mathcal{P}_{B_x} \right) \cup \left(\bigcup_{B' \in \mathcal{C}(B) \setminus \{B_x | x \in V(B) \setminus (V(A_B) \cup \{u_B\})\}} \mathcal{P}_{B'}^- \right)$$

is a 2-proper partition of $G(B) - u_B$ with $|\mathcal{P}_B^-| = 1 + \sum_{B' \in \mathcal{C}(B)} |\tilde{\mathcal{B}}(B')| = |\tilde{\mathcal{B}}(B)|$, which proves the first statement of the claim. $\square$

Claim 9. For every $B \in \mathcal{B}$, $|\tilde{\mathcal{B}}(B)| \leq \alpha^*(G(B))$.

Proof. Let $B \in \mathcal{B}$. For each $B' \in \tilde{\mathcal{B}}(B)$, take a vertex $x_{B'} \in X_{B'}$. Let $I = \{x_{B'} \mid B' \in \tilde{\mathcal{B}}(B)\}$. Then

$$\begin{aligned} w_{G(B)}(I) = \sum_{B' \in \tilde{\mathcal{B}}(B)} d_G(x_{B'}) \leq \sum_{B' \in \tilde{\mathcal{B}}(B)} (|V(B')| - 1) &\leq \sum_{B' \in \mathcal{B}(B)} (|V(B')| - 1) \\ &= |V(G(B))| - 1. \end{aligned}$$

This together with the definition of $X_{B'}$ implies that I is a light independent set of $G(B)$. Consequently, $\alpha^*(G(B)) \geq |I| = |\tilde{\mathcal{B}}(B)|$. $\square$

Recall that B_0 is an end-block of G. Let u_0 be the unique cut-vertex of G belonging to B_0, and let $\mathcal{B}_0$ be the family of blocks in $\mathcal{B} \setminus \{B_0\}$ of G containing u_0. For each $B \in \mathcal{B}_0$, let $\mathcal{P}_B^-$ be as in Claim 8. Then $\mathcal{P}_0 := \{V(B_0)\} \cup (\bigcup_{B \in \mathcal{B}_0} \mathcal{P}_B^-)$ is a 2-proper partition of G. Since B_0 is an end-block of G, $X_{B_0} \neq \emptyset$. This together with Claim 9 leads to $|\mathcal{P}_0| = 1 + \sum_{B \in \mathcal{B}_0} |\tilde{\mathcal{B}}(B)| = |\tilde{\mathcal{B}}(B_0)| \leq \alpha^*(G(B_0)) = \alpha^*(G)$, which is a contradiction. This completes the proof of Theorem 9.7.

Chapter 10

2-factors

For a positive integer r, an r-regular spanning subgraph of a graph is called an r-factor. Finding an r-factor in a given graph is a classical problem in graph theory and has been intensively studied for a long time. In particular, a 2-factor of a graph is a relaxation of a hamilton cycle, and thus sufficient conditions for a graph to have a 2-factor are important towards the studies of the hamiltonicity of graphs. In this chapter, we focus on degree conditions for a graph to have a 2-factor. We introduce results on degree conditions for a graph to have a 2-factor in Section 10.1, and we show the relationship between the degree conditions in Section 10.2. In Section 10.3, we give proofs of Theorems 10.5 and 10.6, which are assumptions on the minimum degree sum on large independent sets for a graph to have a 2-factor. In Section 10.4, we give a proof of Theorem 10.9, which is a result on the existence of a 2-factor with a bounded number of components. All the new results in this chapter are based on [38].

10.1 Studies on degree conditions for the existence of a 2-factor

Let G be a graph. A 2-*factor* of G is a 2-regular spanning subgraph of G. In other words, a 2-factor of G is a union of pairwise vertex-disjoint cycles that cover all vertices of G. A *hamilton cycle* of G, which is a cycle that passes through all vertices of G, is a 2-factor of G with exactly one component. A graph with a hamilton cycle is called *hamiltonian*. The following well-known theorems, respectively, in Dirac [23] and Ore [52] give degree conditions for a graph to have a 2-factor.

Theorem 10.1 (Dirac [23]). Let G be a graph of order n. If $\delta(G) \geq \frac{n}{2}$, then G has a hamilton cycle.

Theorem 10.2 (Ore [52]). Let G be a graph of order n. If $\sigma_2(G) \geq n$, then G has a hamilton cycle.

The degree conditions in Theorems 10.1 and 10.2 are tight even for the existence of a 2-factor. Indeed, the complete bipartite graph $K_{m,m+1}$ satisfies $n := |V(K_{m,m+1})| = 2m+1$,

$\delta(K_{m,m+1}) = m = \frac{n-1}{2}$, and $\sigma_2(K_{m,m+1}) = 2m = n-1$, but $K_{m,m+1}$ does not have a 2-factor.

Another well-known result on a sufficient condition for a graph to have a hamilton cycle is the following, which was shown in Chvátal and Erdős [18].

Theorem 10.3 (Chvátal and Erdős [18])**.** Let G be a 2-connected graph. If $\alpha(G) \leq \kappa(G)$, then G has a hamilton cycle.

While the assumption $\alpha(G) \leq \kappa(G)$ is not a degree condition, it is well known that Theorem 10.3 implies Theorem 10.2 as follows. The following proof was given in Bondy [8].

Proposition 10.4. Let G be a graph of order n. If $\sigma_2(G) \geq n$, then $\alpha(G) \leq \kappa(G)$.

Proof. Let G be a graph of order n. Suppose that $\sigma_2(G) \geq n$. If G is complete, then the statement is trivial. We assume that G is not complete. Let $S \subseteq V(G)$ be a minimum cut set of G, and let U and V be two disjoint sets of vertices such that $U \cup V = V(G) \setminus S$ and $E(U,V) = \emptyset$. Let I be a maximum independent set of G, and assume that $|I| > |S|$. Note that $|S| = \kappa(G)$ and $|I| = \alpha(G)$. Since $|I| > |S|$, we have $(U \cup V) \cap I \neq \emptyset$. By symmetry, we may assume that $I \cap U \neq \emptyset$.

We first suppose that $I \cap V = \emptyset$. Let u be a vertex in $U \cap I$ and v be a vertex in V. Since $|I \cap (U \cup S)| = |I|$, we have

$$\begin{aligned}\sigma_2(G) \leq d_G(u) + d_G(v) &\leq (|U \cup S| - |I \cap (U \cup S)|) + (|V \cup S| - 1)\\ &= |U| + |S| - |I| + |V| + |S| - 1\\ &< |U| + |S| - |S| + |V| + |S| - 1 = n-1,\end{aligned}$$

a contradiction. Next, we suppose that $I \cap V \neq \emptyset$. Let u be a vertex in $U \cap I$ and let v be a vertex in $V \cap I$. Since $|I \cap (U \cup S)| + |I \cap (V \cup S)| = |I \cap U| + 2|I \cap S| + |I \cap V| \geq |I| > |S|$, we have

$$\begin{aligned}\sigma_2(G) \leq d_G(u) + d_G(v) &\leq (|U \cup S| - |I \cap (U \cup S)|) + (|V \cup S| - |I \cap (V \cup S)|)\\ &= |U| + |S| - |I \cap (U \cup S)| + |V| + |S| - |I \cap (V \cup S)|\\ &< |U| + |S| + |V| + |S| - |S| = n,\end{aligned}$$

a contradiction. Thus, $\alpha(G) = |I| \leq |S| = \kappa(G)$. $\square$

From the perspective of vertex partitions, a 2-factor can be regarded as a special 2-proper partition. If a graph G has a 2-factor consisting of pairwise disjoint k cycles $C^{(1)}, C^{(2)}, \ldots, C^{(k)}$, then a partition $\mathcal{P} = \{V(C^{(1)}), V(C^{(2)}), \ldots, V(C^{(k)})\}$ of $V(G)$ is a 2-proper partition of G. This motivates us to investigate the assumption on the minimum degree sum on large independent sets for a graph to have a 2-factor. We prove the following two theorems. Recall that $\sigma_0^*(G) = \infty$ if every independent set $I \in \mathcal{I}(G)$ satisfies $|I| \leq \delta_G(I) - 1$, and $\sigma_1^*(G) = \infty$ if every independent set $I \in \mathcal{I}(G)$ satisfies $|I| \leq \delta_G(I)$.

Theorem 10.5. Let G be a graph. If $\sigma_0^*(G) = \infty$, then G has a 2-factor.

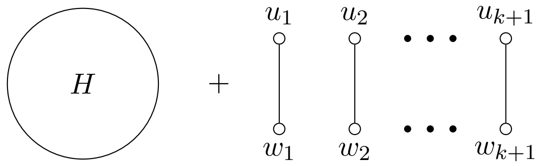


Figure 10.1: A graph in $\mathcal{G}_k$. A graph H is an arbitrary graph of order k.

Theorem 10.6. Let G be a graph. If $\sigma_1^*(G) = \infty$, then either G has a 2-factor, or G is isomorphic to a graph in $\mathcal{G}$ where the exceptional graphs are defined as below.

We define the set of exceptional graphs of Theorem 10.6. For an integer $k \geq 0$, let $\mathcal{G}_k$ be a set of graphs G with $3k+2$ vertices $\{v_i \mid i \in [k]\} \cup \{u_j \mid j \in [k+1]\} \cup \{w_j \mid j \in [k+1]\}$ such that

- $H = G[\{v_i \mid i \in [k]\}]$ is an arbitrary graph of order k, and
- $E(G) = E(H) \cup \{u_j w_j \mid j \in [k+1]\} \cup \{v_i u_j \mid i \in [k], j \in [k+1]\} \cup \{v_i w_j \mid i \in [k], j \in [k+1]\}$ (Figure 10.1).

We let $\mathcal{G} = \bigcup_{k=0}^{\infty} \mathcal{G}_k$. Thus, $\mathcal{G}$ is a class of infinitely many graphs. Note that every graph in $\mathcal{G}$ does not have a 2-factor. Indeed, suppose that G is a graph in $\mathcal{G}_k$ for some integer $k \geq 0$ labeled as in the definition of $\mathcal{G}_k$. Suppose that G has a 2-factor consisting of pairwise disjoint cycles $C^{(1)}, C^{(2)}, \dots, C^{(l)}$. Since $G - V(H)$ is a disjoint union of $k+1$ copies of K_2, if a cycle C of G covers s vertices of $\{u_j \mid j \in [k+1]\}$, then C covers at least s vertices of $V(H)$. This implies that

$$
\begin{aligned}
k = |V(H)| = \sum_{t=1}^{l} |V(C^{(t)}) \cap V(H)| &\geq \sum_{t=1}^{l} |V(C^{(t)}) \cap \{u_j \mid j \in [k+1]\}| \\
&= |\{u_j \mid j \in [k+1]\}| = k+1,
\end{aligned}
$$

a contradiction.

As we mentioned before, a hamilton cycle is a 2-factor with exactly one component. Thus, we are interested in the existence of a 2-factor with a bounded number of components. Considering Theorem 9.7 and Corollary 9.9, which are on the existence of a 2-proper partition with bounded number of parts, we pose the following two conjectures. Recall that $\alpha^*(G)$ is the order of a maximum independent set $I \in \mathcal{I}(G)$ with $w_G(I) \leq |V(G)| - 1$.

Conjecture 10.7. Let G be a graph. If $\sigma_0^*(G) = \infty$, then G has a 2-factor with at most $\alpha^*(G)$ components.

Conjecture 10.8. Let G be a graph. If $\sigma_1^*(G) = \infty$, then either G has a 2-factor with at most $\alpha^*(G)$ components, or G is isomorphic to a graph in $\mathcal{G}$.

The number of components in Conjectures 10.7 and 10.8 are best possible. Let $k \geq 3$ be an integer. Let $H_1, H_2, \dots, H_{2k-2}$ be pairwise disjoint copies of the complete graph K_k,

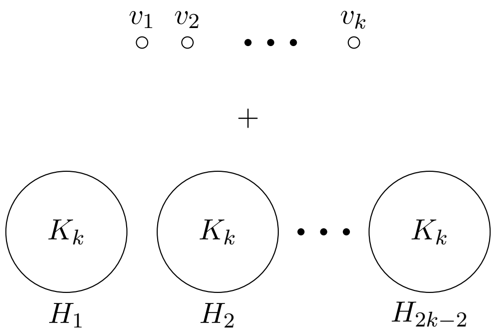


Figure 10.2: A graph G_1.

and let G_1 be a graph obtained from a disjoint union of $H_1, H_2, \dots, H_{2k-2}$ by adding new k vertices $\{v_1, v_2, \dots, v_k\}$ which are joined to all vertices of $\bigcup_{i=1}^{2k-2} V(H_i)$ (Figure 10.2). Then we have $n := |V(G_1)| = k + k(2k-2) = k(2k-1)$ and $\delta(G_1) = 2k-1$, which is attained at the vertices of $\bigcup_{i=1}^{2k-2} V(H_i)$. Since $\alpha(G_1) = 2k-2$, every independent set $I \in \mathcal{I}(G_1)$ satisfies $|I| \leq \alpha(G_1) = 2k-2 = \delta(G_1) - 1 \leq \delta_{G_1}(I) - 1$. Hence, we have $\sigma_0^*(G_1) = \infty$ and $\sigma_1^*(G_1) = \infty$. For any independent set $I \in \mathcal{I}(G_1)$ of order at least k, we have $w_{G_1}(I) \geq |I| \cdot \delta(G_1) \geq k(2k-1) = n$. Thus, we have $\alpha^*(G_1) \leq k-1$, and the equality is attained by taking an independent set consisting of $u_i \in V(H_i)$ for each $i \in [k-1]$. Suppose that G_1 has a 2-factor consisting of pairwise disjoint cycles $C^{(1)}, C^{(2)}, \dots, C^{(l)}$. For each $j \in [l]$, let $s_j = |\{H_i \mid i \in [2k-2], V(H_i) \cap V(C^{(j)}) \neq \emptyset\}|$ and $t_j = |V(C^{(j)}) \cap \{v_1, v_2, \dots, v_k\}|$. Then we have $s_j = 1$ if $t_j = 0$ and $s_j \leq t_j$ otherwise. Note that $t_j \geq 1$ for at least one $j \in [l]$. Since $\sum_{j=1}^{l} s_j \geq 2k-2$ and $\sum_{j=1}^{l} t_j = k$,

$$2k-2 \leq \sum_{j=1}^{l} s_j = \sum_{1 \leq j \leq l, t_j = 0} s_j + \sum_{1 \leq j \leq l, t_j \geq 1} s_j \leq \sum_{1 \leq j \leq l, t_j = 0} 1 + \sum_{1 \leq j \leq l, t_j \geq 1} t_j \leq (l-1) + k,$$

which implies that $l \geq k-1$. Thus, every 2-factor of G_1 has at least $k-1 = \alpha^*(G_1)$ components.

By the definition of $\alpha^*(G)$, it follows that a graph G satisfies $\alpha^*(G) \leq k$ if and only if $\sigma_{k+1}(G) \geq |V(G)|$. Thus, Theorem 10.2 implies that every graph G with $\alpha^*(G) \leq 1$ has a 2-factor with one component, and hence Conjectures 10.7 and 10.8 hold for graphs with $\alpha^*(G) \leq 1$. Focusing on this relation between $\alpha^*(G)$ and $\sigma_{k+1}(G)$, we prove the following theorem.

Theorem 10.9. Let $k \geq 2$ be an integer and let G be a graph of order n. Suppose that $\sigma_0^*(G) = \infty$ and $\sigma_k(G) \geq n$. If there is a cut set $S \subseteq V(G)$ of G such that $\omega(G-S) \geq |S| + k - 2$, then G has a 2-factor with exactly $k-1$ components.

Note that if a graph G with a 2-factor has a cut set $S \subseteq V(G)$ with $\omega(G-S) \geq |S| + k - 2$, then every 2-factor of G has at least $k-1$ components.

The following theorem on a dominating cycle of a graph was shown in Bauer, Veldman, Morgana, and Schmeichel [6]. A cycle C of G is called a *dominating cycle* if $G - V(C)$ has no edge. A graph G is 1-*tough* if every cut set $S \subseteq V(G)$ of G satisfies $\omega(G-S) \le |S|$.

Theorem 10.10 (Bauer, Veldman, Morgana and Schmeichel [6])**.** Let G be a graph of order n. If G is 1-tough and $\sigma_3(G) \ge n$, then every longest cycle of G is a dominating cycle of G.

By Theorems 10.9 and 10.10, we have the following corollary, which is a partial result of Conjecture 10.7.

Corollary 10.11. Let G be a graph. If $\sigma_0^*(G) = \infty$ and $\alpha^*(G) \le 2$, then G has a 2-factor with at most $\alpha^*(G)$ components.

10.2 Relationship between degree conditions

In this section, we see the relationship between degree conditions for a graph to have a 2-factor or a hamilton cycle. The following proposition holds.

Proposition 10.12. Let G be a graph of order n.

- If $\delta(G) \ge \frac{n}{2}$, then either $\sigma_0^*(G) = \infty$, or n is even and G contains a spanning subgraph that is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$.
- If $\alpha(G) \le \kappa(G)$, then $\sigma_1^*(G) = \infty$.

Proof. First, suppose that $\delta(G) \ge \frac{n}{2}$ and $\sigma_0^*(G) < \infty$. Let $I \in \mathcal{I}(G)$ be an independent set such that $|I| \ge \delta_G(I)$, and fix $v \in I$. We have $|I| \ge \delta_G(I) \ge \delta(G) \ge \lceil \frac{n}{2} \rceil$, and this forces

$$\left\lceil \frac{n}{2} \right\rceil \le d_G(v) = |N_G(v)| \le |V(G) \setminus I| = n - |I| \le n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor .$$

Thus, the all inequalities are attained by equalities, which implies that n is even and G contains a complete bipartite graph with partite sets I and $V(G) \setminus I$ as a spanning subgraph. Since $|I| = |V(G) \setminus I| = \frac{n}{2}$, it is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$, as desired.

Next, we assume that $\alpha(G) \le \kappa(G)$. Since $\delta(G) \ge \kappa(G)$, every independent set $I \in \mathcal{I}(G)$ satisfies

$$|I| \le \alpha(G) \le \kappa(G) \le \delta(G) \le \delta_G(I).$$

Thus, we have $\sigma_1^*(G) = \infty$. $\square$

Note that the situation that $n = |V(G)|$ is even and G has a spanning subgraph that is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$ is easy to deal with in terms of 2-factors or hamilton cycles. By combining Proposition 10.12 and other relations we have already seen, we show the relationship between degree conditions for a graph to have a 2-factor in Figure 10.3. As we see in the figure, the degree condition $\sigma_1^*(G) = \infty$ is implied by any other conditions. Furthermore, there are infinitely many graphs which satisfy $\sigma_1^*(G) = \infty$ but which neither satisfy $\alpha(G) \le \kappa(G)$ nor $\sigma_0^*(G) = \infty$. Let $k \ge 4$ be an integer. A graph G_2 is obtained by

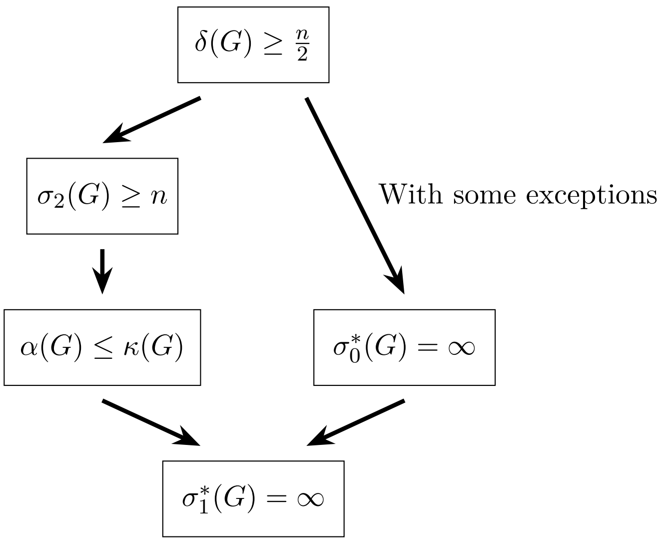


Figure 10.3: Relationship between degree conditions for a graph G of order n to have a 2-factor. An arrow from an assumption A to an assumption B means assumption A implies assumption B.

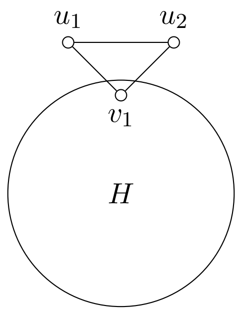


Figure 10.4: A graph G_2.

a complete graph H with k vertices $\{v_1, v_2, \ldots, v_k\}$ by adding two new vertices u_1, u_2 and edges $\{v_1u_1, v_1u_2, u_1u_2\}$. Then we have $\delta(G_2) = 2$ and $\alpha(G_2) = 2$. Since every independent set $I \in \mathcal{I}(G_2)$ satisfies $|I| \leq \alpha(G_2) = 2 = \delta(G_2) \leq \delta_{G_2}(I)$, we have $\sigma_1^*(G_2) = \infty$. On the other hand, since G_2 has a cut vertex v_1, we have $\kappa(G_2) = 1 < \alpha(G_2)$. Considering an independent set $I' = \{u_1, v_2\} \in \mathcal{I}(G_2)$, we have $|I'| = 2 = d_G(u_1) \leq \delta_{G_2}(I')$, and thus $\sigma_0^*(G_2) < \infty$.

10.3 Existence of a 2-factor

In this section, we give proofs of Theorems 10.5 and 10.6. We introduce some notation used in the proofs. Let G be a graph. Let C be a cycle of G. We set one orientation of the cycle C, and write $\overrightarrow{C}$ as the oriented cycle. For a vertex v of $\overrightarrow{C}$, let v^+ (resp. v^-)

denote the vertex which comes after (resp. before) v along the orientation of $\overrightarrow{C}$. For two vertices u and v of $\overrightarrow{C}$, let $u\overrightarrow{C}v$ be the subpath of $\overrightarrow{C}$ from u to v, including its orientation. We write $u\overleftarrow{C}v$ as the subpath of $\overrightarrow{C}$ from u to v with the opposite orientation to $\overrightarrow{C}$. For a uv-path P_1 of G and a vw-path P_2 of G which have no common vertices other than v, we write P_1P_2 as a uw-path of G which starts from u, goes along P_1 to v, and then goes along P_2 to w.

The following lemma is essential in our proof of Theorem 10.5.

Lemma 10.13. Let k be a positive integer and let G be a graph. Suppose that $\sigma_0^*(G) = \infty$. If there is a set of vertices $S \subseteq V(G)$ such that $G[S]$ is a forest and $G - S$ has a 2-factor with at most k components, then G has a 2-factor with at most k components.

Proof. Let G be a graph with $\sigma_0^*(G) = \infty$. Suppose that there is a set $S \subseteq V(G)$ such that $G[S]$ is a forest and $G - S$ has a 2-factor consisting of r disjoint cycles $C^{(1)}, C^{(2)}, \dots, C^{(r)}$ for some positive integer $r \le k$. We set one orientation of $C^{(i)}$ and write $\overrightarrow{C}^{(i)}$ for each $i \in [r]$. The proof goes by induction on the order of S. If $S = \emptyset$, then there is nothing to prove. Suppose that $|S| = s \ge 1$ and that the statement holds for any set of vertices of order less than s. Since $G[S]$ is a forest, there is a vertex $v \in S$ such that $d_{G[S]}(v) \le 1$. Let $X = N_G(v) \setminus S$. For each $x \in X$, we define x^+ along a cycle $\overrightarrow{C}^{(i)}$, where $\overrightarrow{C}^{(i)}$ is a cycle that contains x. Let $X^+ = \{x^+ \mid x \in X\}$. Note that $|X^+| = |X| = |N_G(v) \setminus S| \ge d_G(v) - 1$. If $I = \{v\} \cup X^+$ is an independent set of G, then we have $|I| = |X^+| + 1 \ge (d_G(v) - 1) + 1 \ge \delta_G(I)$, a contradiction by the assumption $\sigma_0^*(G) = \infty$. Hence $\{v\} \cup X^+$ is not an independent set of G, which implies that one of the following holds.

(i) There is a vertex $x^+ \in X^+$ such that $x^+v \in E(G)$.

(ii) There are vertices $x_1^+, x_2^+ \in X^+$ and an index $i \in [r]$ such that $x_1^+, x_2^+ \in V(C^{(i)})$ and $x_1^+x_2^+ \in E(G)$.

(iii) There are vertices $x_1^+, x_2^+ \in X^+$ and two distinct indices $i, j \in [r]$ such that $x_1^+ \in V(C^{(i)})$, $x_2^+ \in V(C^{(j)})$ and $x_1^+x_2^+ \in E(G)$.

We shall show that $G - (S \setminus \{v\})$ has a 2-factor with at most r components for each case.

When (i), without loss of generality, we may assume that $x^+ \in V(C^{(1)})$. By setting $C' = x^+\overrightarrow{C}^{(1)}xvx^+$, $C' \cup \bigcup_{i=2}^r C^{(i)}$ is a 2-factor of $G - (S \setminus \{v\})$ with r components. When (ii), without loss of generality, we may assume that $x_1^+, x_2^+ \in V(C^{(1)})$. By setting $C' = x_1^+\overrightarrow{C}^{(1)}x_2vx_1\overleftarrow{C}^{(1)}x_2^+x_1^+$, $C' \cup \bigcup_{i=2}^r C^{(i)}$ is a 2-factor of $G - (S \setminus \{v\})$ with r components. When (iii), without loss of generality, we may assume that $x_1^+ \in V(C^{(1)})$ and $x_2^+ \in V(C^{(2)})$. By setting $C' = x_1^+\overrightarrow{C}^{(1)}x_1vx_2\overleftarrow{C}^{(2)}x_2^+x_1^+$, $C' \cup \bigcup_{i=3}^r C^{(i)}$ is a 2-factor of $G - (S \setminus \{v\})$ with $r - 1$ components. Since $G - (S \setminus \{v\})$ has a 2-factor with at most r components and $G[S \setminus \{v\}]$ is a forest, by the induction hypothesis, G has a 2-factor with at most r components, as desired. $\square$

10.3.1 Proof of Theorem 10.5

Let G be a graph with $\sigma_0^*(G) = \infty$. If G has a vertex v with $d_G(v) \leq 1$, then $I = \{v\}$ is an independent set of G with $|I| = 1 \geq d_G(v) = \delta_G(v)$, a contradiction by the assumption $\sigma_0^*(G) = \infty$. Hence $\delta(G) \geq 2$ and thus G contains a cycle. Let G' be a maximal induced subgraph of G which has a 2-factor, and suppose that $V(G) \setminus V(G') \neq \emptyset$. Let $S = V(G) \setminus V(G')$. By the maximality of G', $G[S]$ is a forest; otherwise, $G[S]$ has a cycle C and $G[V(G') \cup V(C)]$ has a 2-factor, a contradiction. Thus, by Lemma 10.13, G has a 2-factor, a contradiction by the maximality of G'.

10.3.2 Tutte's 2-factor theorem

In this section, we introduce a theorem in Tutte [61], which gives an equivalent condition for a graph to have a 2-factor. Tutte showed a result with a more general setting called an f-factor, but in this thesis, we only consider a 2-factor. For a graph G and sets of vertices $S \subseteq V(G)$ and $T \subseteq V(G)$ with $S \cap T = \emptyset$, an *odd component* (resp. an *even component*) of $(G; S, T)$ is a component D of $G - (S \cup T)$ such that $e(T, V(D))$ is an odd integer (resp. an even integer). Let $\mathcal{H}(S, T)$ be the set of odd components of $(G; S, T)$, and let $h(S, T) = |\mathcal{H}(S, T)|$. Let $\delta(S, T) = 2|S| - 2|T| - \sum_{v \in T} d_{G-S}(v) - h(S, T)$. Then the following holds.

Theorem 10.14 (Tutte [61])**.** A graph G has a 2-factor if and only if $\delta(S, T) \geq 0$ for every pair (S, T) of disjoint sets of vertices of G.

For a graph G, a pair (S, T) of disjoint sets of vertices of G is called a *barrier* of G if $\delta(S, T) < 0$. Then by Theorem 10.14, every graph without a 2-factor has a barrier. Furthermore, in Tutte [61] it is observed that $\delta(S, T)$ is even for any graph G and any pair (S, T) of disjoint sets of vertices, and hence every barrier (S, T) of a graph G satisfies $\delta(S, T) \leq -2$. For a graph G without a 2-factor, a barrier (S, T) is called a *minimal barrier* of G if every pair (S', T') of disjoint sets of vertices of G with $|S' \cup T'| < |S \cup T|$ is not a barrier of G. The following properties of a minimal barrier is often used in literature. (See Lemmas 1 and 2 in [3] for example.)

Lemma 10.15 (Aldred et al. [3])**.** Let G be a graph without a 2-factor, and let (S, T) be a minimal barrier of G. Then the followings hold.

1. T is an independent set of G,
2. for every even component D of $(G; S, T)$, $e(T, V(D)) = 0$,
3. for every odd component D of $(G; S, T)$ and every vertex $v \in T$, $e(v, V(D)) \leq 1$, and
4. $|T| > |S|$.

10.3.3 Proof of Theorem 10.6

Suppose that G satisfies $\sigma_1^*(G) = \infty$ but G does not have a 2-factor. We shall show that $G \in \mathcal{G}$. Let $\mathcal{I}(G)$ be the family of independent sets of G. By Theorem 10.14, G contains a barrier. Let (S, T) be a minimal barrier of G. Let $t = |T|$ and $T = \{v_1, v_2, \ldots, v_t\}$.

Since $T \in \mathcal{I}(G)$ by Lemma 10.15, we have $\delta_G(T) \geq t$, and thus $d_G(v) \geq t$ for every vertex $v \in T$. Without loss of generality, we may assume that $d_{G-S}(v_1) \leq d_{G-S}(v_i)$ for all $i \in \{2, 3, \ldots, t\}$. Let $a = t - |S| > 0$.

Suppose first that $d_{G-S}(v_1) \geq a + 1$. Then we have $\sum_{v \in T} d_{G-S}(v) \geq (a+1)t$ and

$$-2 \geq \delta(S, T) = 2|S| - 2|T| + \sum_{v \in T} d_{G-S}(v) - h(S, T) \geq -2a + (a+1)t - h(S, T),$$

which implies that

$$h(S, T) \geq (a+1)t - 2(a-1) = (a-1)(t-2) + 2t \geq 2t > |S| + t, \tag{10.3.1}$$

where the third inequality holds since $1 \leq a \leq t$.

If every odd component $D \in \mathcal{H}(S, T)$ has at least $t + 1$ vertices, then Lemma 10.15 implies that there is a vertex $x_D \in V(D) \setminus N_G(T)$ for every $D \in \mathcal{H}(S, T)$, and $I = T \cup \{x_D \mid D \in \mathcal{H}(S, T)\}$ is an independent set of G such that $|I| = t + h(S, T) > |S| + h(S, T) \geq d_G(v_1) \geq \delta_G(I)$, a contradiction by the assumption $\sigma_1^*(G) = \infty$. Hence, there is an odd component $D_1 \in \mathcal{H}(S, T)$ such that $|V(D_1)| \leq t$. Let x_1 be a vertex of D_1 such that $e(T, x_1)$ is minimum among all vertices of D_1. Then we have

$$d_{G-S}(x_1) = d_{D_1}(x) + e(T, x_1) \leq |V(D_1)| - 1 + \frac{t}{|V(D_1)|} \leq t,$$

where the last inequality follows from $1 \leq |V(D_1)| \leq t$. Thus, $d_G(x_1) \leq |S| + d_{G-S}(x_1) \leq |S| + t$. Now we let $I' = \{x_1\} \cup \{x_D \mid D \in \mathcal{H}(S, T) \setminus \{D_1\}\}$ where $x_D \in V(D)$ is an arbitrary vertex for every $D \in \mathcal{H}(S, T) \setminus \{D_1\}$. It is obvious that I' is an independent set of G, and by (10.3.1), we have $|I'| = \mathcal{H}(S, T) \geq |S| + t \geq d_G(x_1) \geq \delta_G(I')$, a contradiction by the assumption $\sigma_1^*(G) = \infty$. Hence, we conclude that $d_{G-S}(v_1) \leq a$.

Since $d_{G-S}(v_1) \geq d_G(v_1) - |S| \geq t - |S| = a$, we have $d_{G-S}(v_1) = a$ and thus $d_G(v_1) = t$. If there is a vertex $x \in V(G) \setminus (S \cup T \cup N_G(T))$, then $T \cup \{x\}$ is an independent set of G with $|T \cup \{x\}| > t = d_G(v_1) \geq \delta_G(T \cup \{x\})$, a contradiction. Hence $V(G) = S \cup T \cup N_G(T)$. In particular, there is no even component of $(G; S, T)$ by Lemma 10.15. By Lemma 10.15, there are odd components $D_1, D_2, \ldots, D_a$ of $(G; S, T)$ such that $N_G(v_0) \setminus S = \{x_1, x_2, \ldots, x_a\}$ where $x_i \in V(D_i)$ for every $i \in [a]$. Let $\mathcal{H}'(S, T) = \mathcal{H}(S, T) \setminus \{D_1, D_2, \ldots, D_a\}$. If there are more than $t - 1$ odd components in $\mathcal{H}'(S, T)$, then $\{v_0\} \cup \{x_D \mid D \in \mathcal{H}'(S, T)\}$ is an independent set of G where x_D is an arbitrary vertex of D, and it violates the assumption $\sigma_1^*(G) = \infty$, a contradiction. Thus, we have $h(S, T) = a + |\mathcal{H}'(S, T)| \leq a + t - 1$. Since (S, T) is a barrier of G, we have

$$\begin{aligned} -2 \geq \delta(S, T) = 2|S| - 2|T| + \sum_{v \in T} d_{G-S}(v) - h(S, T) &\geq -2a + \sum_{v \in T} d_{G-S}(v) - (a + t - 1) \\ &= -3a - t + 1 + \sum_{v \in T} d_{G-S}(v), \end{aligned}$$

which implies that $\sum_{v\in T} d_{G-S}(v) \le t+3a-3$. By the choice of v_0, we have $at \le \sum_{v\in T} d_{G-S}(v) \le t+3a-3$, and thus

$$(t-3)(a-1) = at - t - 3a + a \le 0 \tag{10.3.2}$$

This together with the fact $1 \le a \le t$ implies that either $a = 1$, or $(t, a) \in \{(2, 2), (3, 2)\}$. Furthermore, if $a = 1$ or $t = 3$, then (10.3.2) holds by the equation.

Suppose first that $(t, a) = (2, 2)$. Then we have $S = \emptyset$ and $d_G(v_1) = a = 2$. Let D_1 and D_2 be odd components of $(G; S, T)$ such that there is a vertex $x_i \in N_G(v_1) \cap V(D_i)$ for $i \in \{1, 2\}$. If both D_1 and D_2 have at least 2 vertices, then there is a vertex $y_i \in V(D_i)\backslash\{x_i\}$ for each $i \in \{1, 2\}$ and $\{v_1, y_1, y_2\}$ is an independent set of G violating the assumption $\sigma_1^*(G) = \infty$, a contradiction. Hence, without loss of generality, we may assume that $V(D_1) = \{x_1\}$. Since G has the minimum degree at least 2, x_1 has a neighbor other than v_1, which must be v_2. This implies that $e(T, V(D_1)) = 2$, a contradiction by the assumption that D_1 is an odd component of $(G; S, T)$.

Next, we assume that $(t, a) = (3, 2)$. Since (10.3.2) holds by equality, we know that $h(S, T) = 2+3-1 = 4$ and $d_{G-S}(v) = 2$ for every $v \in T$. Let $\mathcal{H}(S, T) = \{D_1, D_2, D_3, D_4\}$. Then we have $e(T, V(D_1) \cup V(D_2) \cup V(D_3) \cup V(D_4)) = 6$ and one of the odd components, say D_1, satisfies $e(T, V(D_1)) = 1$. Let $V(D_1) = \{x_1\}$, and without loss of generality, we may assume that $v_1x_1 \in E(G)$. Since $|S| = 1$, we have $d_G(x_1) = 2$, and thus $\{x_1, v_2, v_3\}$ is an independent set that violates the assumption $\sigma_1^*(G) = \infty$, a contradiction.

Thus, we have $a = 1$. Since (10.3.2) holds by equation, we know that $h(S, T) = a + t - 1 = t$ and $d_{G-S}(v) = 1$ for every $v \in T$. Hence, we may assume that $\mathcal{H}(S, T) = \{D_1, D_2, \ldots, D_t\}$ and $N_G(v_i) \setminus S = \{x_i\} \subseteq V(D_i)$ for every $i \in [t]$. As $\bigcup_{i=1}^{t} V(D_i) \subseteq N_G(T)$, we know that $V(D_i) = \{x_i\}$ for every $i \in [t]$. Since $\{x_1, x_2, \ldots, x_t\}$ is an independent set of G, we have $d_G(x_i) \ge t$ for each $i \in [t]$. Thus, each vertex of $T \cup \{x_1, x_2, \ldots, x_t\}$ is adjacent to all vertices of S, implying that G belongs to the class $\mathcal{G}_{t-1}$. This completes the proof of Theorem 10.6.

10.4 2-factor with bounded number of components

10.4.1 Proof of Corollary 10.11

We first give a proof of Corollary 10.11 by using Theorems 10.9, 10.10 and Lemma 10.13.

Let G be a graph of order n with $\sigma_0^*(G) = \infty$ and $\alpha^*(G) \le 2$. As we mentioned before, if $\alpha^*(G) = 1$, then we have $\sigma_2(G) \ge n$, and thus it follows from Theorem 10.2 that G has a 2-factor with one component. We assume that $\alpha^*(G) = 2$. Then we have $\sigma_3(G) \ge n$. Note that $\delta(G) \ge 2$, and thus G has a cycle.

If there is a cut set $S \subseteq V(G)$ of G such that $\omega(G - S) \ge |S| + 1$, then Theorem 10.9 implies that G has a 2-factor with exactly 2 components. Hence we may assume that every cut set $S \subseteq V(G)$ of G satisfies $\omega(G - S) \le |S|$, equivalently, G is 1-tough. By Theorem 10.10, there is a cycle C of G such that $G - V(C)$ has no edge. By Lemma 10.13, we obtain a 2-factor of G with one component from C. This completes the proof of Corollary 10.11.

10.4.2 Lemmas for Theorem 10.9

In this subsection, we introduce a well-known result on the hamiltonicity of a graph shown by Ore. For a graph G, a *hamilton path* of G is a path that passes through all vertices of G. A graph G is *hamilton-connected* if there is a hamilton uv-path of G for every pair (u, v) of distinct vertices of G. By the definitions, if G is hamilton-connected, then by considering a hamilton uv-path of G for adjacent two vertices u and v, it follows that G has a hamilton cycle. If G has a hamilton cycle, then by deleting one edge of the cycle, we obtain a hamilton path of G. Ore shows the following theorem on these concepts in [52].

Theorem 10.16 (Ore [52])**.** Let G be a graph of order n. If $\sigma_2(G) \geq n-1$, then G has a hamilton path. If $\sigma_2(G) \geq n+1$, then G is hamilton-connected.

The following lemma follows easily from Theorem 10.16.

Lemma 10.17. Let G be a graph of order n. If $\sigma_2(G) = n-1$ and G is not hamiltonian, then G has a hamilton xy-path such that $d_G(x) + d_G(y) = n-1$.

10.4.3 Proof of Theorem 10.9

We fix an integer $k \geq 3$. Let G be a graph of order n which satisfies the assumptions of Theorem 10.9. We take a cut set $S = \{v_1, v_2, \dots, v_s\}$ of G such that $\omega(G-S) \geq s+k-2$. Let $G_1, G_2, \dots, G_{s+k-2+j}$ be the set of components of $G-S$ where j is a non-negative integer, and let $n_i = |V(G_i)|$ for each $i \in [s+k-2+j]$.

Claim 1. $\delta(G-S) \geq k-1$. In particular, for every $i \in [s+k-2+j]$, $n_i \geq k$.

Proof. Since $\omega(G-S) \geq s+k-2$, for every $x \in V(G) \setminus S$, there is an independent set I_x of G with $s+k-2$ vertices, including x. By our assumption that $\sigma_0^*(G) = \infty$, we have $d_G(x) \geq \delta_G(I_x) \geq |I_x| + 1 = s+k-1$, which implies $d_{G-S}(x) \geq d_G(x) - s \geq k-1$. Thus, $\delta(G-S) \geq k-1$ and every component of $G-S$ has at least k vertices. □

Now we consider the case $s = 1$. Suppose first that $j \geq 1$, and thus $s+k-2+j \geq k$. For each $i \in [k]$, we arbitrarily choose $x_i \in V(G_i)$ and consider an independent set $\{x_1, x_2, \dots, x_k\}$ of G. Since $d_G(x_i) \leq n_i - 1 + s = n_i$ for each $i \in [k]$, we have $\sigma_k(G) \leq \sum_{i=1}^{k} d_G(x_i) \leq \sum_{i=1}^{k} n_i \leq n-1$, a contradiction. Hence we have $j = 0$ and thus $G-S$ has exactly $k-1$ components. If every component is hamiltonian, then obviously $G - v_1$ contains a 2-factor with exactly $k-1$ components, and G has a k-factor with at most $k-1$ components by Lemma 10.13. Similarly to a graph G_1 in Section 10.1, it is easy to show that every 2-factor of G has at least $k-1$ components, and hence G has a 2-factor with exactly $k-1$ components. Thus, we may assume that there is a non-hamiltonian component. Without loss of generality, we may assume that G_1 is not hamiltonian. As $n_1 \geq k \geq 3$, G_1 is not complete, and thus there are non-adjacent vertices x_1 and y_1 of G_1 such that $d_{G_1}(x_1) + d_{G_1}(y_1) = \sigma_2(G_1) \leq n_1 - 1$ by Theorem 10.2. For each $i \in \{2, 3, \dots, k-1\}$, let x_i be an arbitrary vertex of G_i. By our assumption of $\sigma_k(G) \geq n$, we have

$$\sum_{i=1}^{k-1} n_i + 1 = n \leq d_G(x_1) + d_G(y_1) + \sum_{i=2}^{k-1} d_G(x_i)$$
$$\leq (d_{G_1}(x) + 1) + (d_{G_1}(y) + 1) + \sum_{i=2}^{k-1} n_i \leq \sum_{i=1}^{k-1} n_i + 1.$$

Thus, every inequality above holds by equality, which derives the followings:

- $\sigma_2(G_1) = n - 1$,
- if non-adjacent vertices x and y of G_1 satisfies $d_{G_1}(x) + d_{G_1}(y) = n - 1$, then $x, y \in N_G(v_1)$, and
- for every $i \in \{2, 3, \dots k - 1\}$, G_i is a complete graph of order at least k.

The first two conditions imply that $G[V(G_1) \cup \{v_1\}]$ is hamiltonian by Lemma 10.17, and the last condition implies that G_i is hamiltonian for every $i \in \{2, 3, \dots k - 1\}$. Combining these, we obtain a 2-factor of G with exactly $k - 1$ components.

In the rest of the proof, we may assume that $s \geq 2$, and thus $s + k - 2 + j \geq k$.

Claim 2. $j = 0$ and $s \leq k$. Furthermore, if $s \geq 3$, then every component of $G - S$ is hamilton-connected.

Proof. For each $i \in [k]$, let x_i be a vertex of G_i. Since $\sigma_k(G) \geq n$, we have

$$\sum_{i=1}^{s+k-2+j} n_i + s = n \leq \sum_{i=1}^{k} d_G(x_i) \leq \sum_{i=1}^{k} (n_i - 1 + s) = \sum_{i=1}^{k} n_i - k + ks,$$

so

$$ks - k - s \geq \sum_{i=k+1}^{s+k-2+j} n_i \geq k(s + j - 2) \tag{10.4.1}$$

where the last inequality follows from Claim 1. Thus, we have $kj \leq k - s$, which implies $j = 0$ and $s \leq k$.

When $s \geq 3$, we have $s + k - 2 \geq k + 1$, and without loss of generality, we may assume that $n_i \leq n_{k+1}$ for every $i \in [s + k - 2]$. By the first inequality of 10.4.1, we have

$$\sum_{i=k+1}^{s+k-2} (n_i - k) = \sum_{i=k+1}^{s+k-2} n_i - k(s - 2) \leq ks - k - s - k(s - 2) = k - s.$$

This together with Claim 1 implies that $n_{k+1} \leq k + k - s \leq 2k - 3$. By the choice of n_{k+1}, we have $n_i \leq 2k - 3$ for every $i \in [s + k - 2]$. Since $\sigma_2(G_i) \geq 2\delta(G - S) \geq 2(k - 1) \geq n_i + 1$ for each $i \in [s + k - 2]$, we conclude that every component of $G - S$ is hamilton-connected by Theorem 10.16. $\square$

We consider the case where there is a component of $G - S$ which is not hamilton-connected. By Claim 2, we know $s = 2$, and thus $G - S$ has exactly k components. Without loss of generality, we may assume that G_1 is not hamilton-connected. As $n_1 \geq k \geq 3$, G_1 is not complete and thus $\sigma_2(G_1) \leq n_1$ by Theorem 10.16. Let x_1 and y_1 be non-adjacent vertices of G_1 such that $d_{G_1}(x_1) + d_{G_1}(y_1) = \sigma_2(G_1) \leq n_1$. We fix $i \in \{2, 3, \ldots, k\}$ arbitrarily, and for each $j \in [k] \setminus \{1, i\}$, let x_j be a vertex of G_j. By the assumption that $\sigma_k(G) \geq n$, we have

$$\begin{aligned}\sum_{j=1}^{k} n_j + 2 = n &\leq d_G(x_1) + d_G(y_1) + \sum_{j \in [k] \setminus \{1,i\}} d_G(x_j) \\ &\leq (d_{G_1}(x_1) + 2) + (d_{G_1}(y_1) + 2) + \sum_{j \in [k] \setminus \{1,i\}} (n_j - 1 + 2) \\ &\leq \sum_{j \in [k] \setminus \{i\}} n_j + k + 2,\end{aligned}$$

which forces $n_i \leq k$. Thus, every inequality above holds by equality, which implies that $\sigma_2(G_1) = n_1$. By Theorem 10.16, G_1 is hamiltonian. Furthermore, since the choice of i is arbitrary, we conclude that G_i is complete and $V(G_i) \subseteq N_G(v_1) \cap N_G(v_2)$ for every $i \in \{2, 3, \ldots, k\}$. Combining a hamilton cycle of $G[\{v_1, v_2\} \cup V(G_2) \cup V(G_3)]$ and hamilton cycles of other components $G_1, G_4, G_5, \ldots, G_k$, we obtain a 2-factor of G with exactly $k-1$ components.

In the rest of the proof, we assume that every component of $G - S$ is hamilton-connected. We define a bipartite graph H by $V(H) = S \cup \{u_1, u_2, \ldots, u_{s+k-2}\}$ and $E(H) = \{v_i u_j \mid N_G(v_i) \cap V(G_j) \neq \emptyset\}$.

Claim 3. *H has a cycle that contains all vertices of S.*

Proof. For each $i \in [k]$, let x_i be a vertex of G_i, and we consider an independent set $\{x_1, x_2, \ldots, x_k\}$ of G. Since we have $d_G(x_i) \leq n_i - 1 + d_H(u_i)$, by our assumption $\sigma_k(G) \geq n$, we have

$$\sum_{i=1}^{s+k-2} n_i + s = n \leq \sum_{i=1}^{k} d_G(x_i) \leq \sum_{i=1}^{k} (n_i - 1 + d_H(u_i)) = \sum_{i=1}^{k} n_i - k + \sum_{i=1}^{k} d_H(u_i),$$

so

$$\sum_{i=1}^{k} d_H(u_i) \geq \sum_{i=k+1}^{s+k-2} n_i + k + s \geq k(s-2) + k + s = ks - k + s.$$

Let t be the number of vertices in $\{u_1, u_2, \ldots, u_k\}$ whose degrees in H are equal to s. Then we have $ks - k + s \leq \sum_{i=1}^{k} d_H(u_i) \leq ts + (k-t)(s-1) = ks - k + t$, which implies that $t \geq s$. Hence, there are at least s vertices in $\{u_1, u_2, \ldots u_k\}$, each of which is adjacent to all vertices in S. Combining S and such vertices, we obtain a subgraph of H that is isomorphic to the complete bipartite graph $K_{s,s}$. It is obvious that the subgraph contains a desired cycle of H. □

Let $C^{(0)}$ be a cycle of H as in Claim 3. Without loss of generality, we may assume that the vertices of S appear in the order $v_1, v_2, \dots, v_s$ along $C^{(0)}$.

Claim 4. Let $C = v_1 u_{a_1} v_2 u_{a_2} v_3 \cdots v_s u_{a_s} v_1$ be a cycle of H where $a_1, a_2, \dots, a_s$ are s distinct elements of $[s+k-2]$. For each $i \in [s]$, there is an index $b_i \in [s+k-2] \setminus \{a_j \mid j \in [s] \setminus \{i\}\}$ and a $v_i v_{i+1}$-path Q_i of G such that $V(Q_i) = \{v_i, v_{i+1}\} \cup V(G_{b_i})$, where $v_{s+1} = v_1$.

Proof. By the symmetry of the indices, it suffices to prove for the case $i = 1$. Let $\bar{A}$ denote the set $[s+k-2] \setminus \{a_2, a_3, \dots, a_s\}$. We assume the contrary, that is, for every $j \in \bar{A}$, G does not have a $v_1 v_2$-path whose vertex set is $\{v_1, v_2\} \cup V(G_j)$. As G_j is hamilton-connected, G_j satisfies one of the followings:

(a) either $N_G(v_1) \cap V(G_j) = \emptyset$ or $N_G(v_2) \cap V(G_j) = \emptyset$,

(b) there is a vertex x_j of $V(G_j)$ such that $N_G(v_1) \cap V(G_j) = N_G(v_2) \cap V(G_j) = \{x_j\}$.

In particular, since $\{v_1, v_2\} \subseteq N_G(V(G_{a_1}))$, G_{a_1} satisfies (b). For each $j \in \bar{A}$, we define a vertex y_j of G_j as follows: If G_j satisfies (a), then let y_j be an arbitrary vertex of G_j. If G_j satisfies (b), then let $y_j \in V(G_j) \setminus \{x_j\}$. Then we have $d_G(y_j) \le n_i - 1 + (s-1) = n_i - 2 + s$ for every $j \in \bar{A}$, and in particular, we have $d_G(y_{a_1}) \le n_{a_1} - 1 + (s-2) = n_{a_1} - 3 + s$. Let y_{a_2} be a vertex of G_{y_2}, and consider an independent set $I = \{y_j \mid j \in \bar{A} \cup \{a_2\}\}$. As $|I| = s + k - 2 - (s-2) = k$, we have

$$\begin{aligned}\sum_{l=1}^{s+k-2} n_l + s = n \le \sigma_k(G) &\le \sum_{j \in \bar{A} \cup \{a_2\}} d_G(y_j) \\ &\le (n_{a_1} - 3 + s) + (n_{a_2} - 1 + s) + \sum_{j \in \bar{A} \setminus \{a_1\}} (n_j - 2 + s) = \sum_{j \in \bar{A} \cup \{a_2\}} n_j - 2k + ks,\end{aligned}$$

so

$$k(s-2) \le \sum_{l \in \{a_3, a_4, \dots, a_s\}} n_l \le ks - 2k - s,$$

a contradiction. $\square$

Applying Claim 4 to $C^{(0)}$ for each $i = 1, 2, \dots, s$ sequentially, we obtain a cycle $C^{(k)} = v_1 u_{b_1} v_2 u_{b_2} v_3 \cdots v_s u_{b_s} v_1$ of H such that for every $i \in [s]$, there is a $v_i v_{i+1}$-path Q_i of G with $V(Q_i) = \{v_i, v_{i+1}\} \cup V(G_{b_i})$ $(v_{s+1} = v_1)$. Then a cycle $Q = v_1 Q_1 v_2 Q_2 v_3 \cdots v_s Q_s v_1$ is a hamilton cycle of $G[S \cup \bigcup_{i=1}^{s} V(G_{b_i})]$. Combining Q and a hamilton cycle of each component G_j with $j \in [s+k-2] \setminus \{b_i \mid i \in [s]\}$, we obtain a 2-factor of G with exactly $k-1$ components. This completes the proof of Theorem 10.9.

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