Algebraic independence of the values of certain power series, infinite products, and Lambert type series

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Contents

List of Notation

$\mathbb{Z}_{\geq 0}$	The set of nonnegative integers
$\mathbb{Z}$	The ring of rational integers
$\mathbb{Q}$	The field of rational numbers
$\overline{\mathbb{Q}}$	The field of algebraic numbers
$\overline{\mathbb{Q}}^{\times}$	The set of nonzero algebraic numbers
$\mathbb{C},\ \overline{\mathbb{Q}}_{\infty},\ \mathbb{C}_{\infty}$	The field of complex numbers
$\mathbb{Q}_p$	The field of p-adic numbers for a prime number p
$\overline{\mathbb{Q}}_p$	The algebraic closure of $\mathbb{Q}_p$
$\mathbb{C}_p$	The completion of $\overline{\mathbb{Q}}_p$
$\mathcal{O}_K$	The ring of integers of a number field K
$f^{(l)}(z)$	The derivative of a function $f(z)$ of order l
$A \otimes B$	The Kronecker product of matrices $A = (a_{ij})$ and B, namely the block matrix $(a_{ij}B)$
I_n	The identity matrix of size n
$O_{m,n}$	The zero matrix of size $m \times n$
$\lvert\boldsymbol{\lambda}\rvert$	The sum of the components of a vector $\boldsymbol{\lambda}$ with nonnegative integer components
$\#S$	The cardinality of a set S
Y^X	The set of maps $X \to Y$ for sets X and Y
$[L : K]$	The degree of a finite extension L/K of fields
$N_{L/K}(\alpha)$	The norm of an element α of a field L with respect to a finite extension L/K
$[x]$	The integral part of a real number x, namely the largest integer not exceeding x

Let R and K be any commutative ring and any field, respectively.

$R[z_1, \dots, z_n],\ R[\boldsymbol{z}]$	The ring of polynomials in variables $z_1, \dots, z_n$ with coefficients in R
$R[[z_1, \dots, z_n]],\ R[[\boldsymbol{z}]]$	The ring of formal power series in variables $z_1, \dots, z_n$ with coefficients in R
$K(z_1, \dots, z_n),\ K(\boldsymbol{z})$	The field of rational functions in variables $z_1, \dots, z_n$ with coefficients in K
$K^{\times}$	The multiplicative group of nonzero elements of K

Chapter 1

Introduction

Algebraic independence properties of the values at algebraic numbers of analytic functions have been studied by various authors. In this thesis, we are interested in the necessary and sufficient condition on nonzero algebraic numbers $a_1, \ldots, a_r$ for the infinite set $\{f^{(l)}(a_i) \mid l \geq 0,\ 1 \leq i \leq r\}$ to be algebraically independent, where $f(z)$ is a given analytic function. If $a_i = a_j$ for some distinct i and j, then the set above is obviously algebraically dependent. The converse does not hold in general as shown by the following example:

Let $f_0(z) = \sum_{k=0}^{\infty} z^{k!}$ and let ζ be a d-th root of unity. Then we see that $f_0(\alpha) - f_0(\zeta\alpha) = \sum_{k=0}^{d-1} \alpha^{k!} - \sum_{k=0}^{d-1} (\zeta\alpha)^{k!} \in \overline{\mathbb{Q}}$ for any nonzero algebraic number α, which implies that the values $f_0(\alpha)$ and $f_0(\zeta\alpha)$ are algebraically dependent.

For the function $f_0(z)$ and its certain generalizations, the necessary and sufficient conditions mentioned above were obtained by Nishioka [19, 20, 21], which are more restrictive than the condition that $a_1, \ldots, a_r$ are distinct (see Theorems 1.1.4–1.1.6). On the other hand, some analytic functions are known to have the remarkable property that the infinite set above is algebraically independent as long as algebraic numbers $a_1, \ldots, a_r$ are distinct. Previous results on such functions will be introduced in Theorems 1.1.7, 1.1.8, 1.2.11, 1.2.14, and 1.2.15. These are the strongest results on the necessary and sufficient conditions mentioned above; however, they only deal with functions of one variable. In Theorem 1.3.1, the first main theorem of this thesis, we construct an entire function of two variables satisfying the following property:

The infinite set consisting of the values and the partial derivatives of any order at any distinct algebraic points is algebraically independent.

Theorem 1.3.1 is valid also in the case of p-adic numbers. Moreover, in the case of complex numbers, we prove Theorem 1.3.6 as the second main theorem of this thesis, which provides an infinite family of entire functions of two variables having the following property:

The infinite set consisting of the values and the partial derivatives of any order at any distinct algebraic points of all the functions belonging to the family is algebraically independent.

1.1 Transcendence and algebraic independence of the values of analytic functions

One of the main purposes of transcendental number theory is to determine the transcendency or the algebraic independency of given numbers. For example, in 1873, Hermite showed that e is a transcendental number. The proof was based on the properties of the exponential function e^z, in particular the properties that e^z satisfies the differential equation $(e^z)' = e^z$ and has the value 1 at $z = 0$. Extending Hermite's method and using the Euler's identity $e^{i\pi} = -1$, Lindemann proved, in 1882, that π is a transcendental number. He also proved the transcendency of e^α for any nonzero algebraic number α. Moreover, the following is known today as Lindemann-Weierstrass theorem:

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. If they are linearly independent over $\mathbb{Q}$, then the values $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.

We note that the converse is trivial. Hence the Lindemann-Weierstrass theorem gives the necessary and sufficient condition for the values of the exponential function at algebraic numbers to be algebraically independent.

Furthermore, Mahler and Nesterenko studied p-adic analogues of these results. Before stating their results, we introduce some notation and settings used throughout this thesis.

Let p be a prime number and $|\cdot|_p$ the p-adic absolute value on $\mathbb{Q}$ with the normalization condition $|p|_p = p^{-1}$. We denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ and by $\overline{\mathbb{Q}}_p$ the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ is extended uniquely to the algebraic closure $\overline{\mathbb{Q}}_p$ by

$$|\alpha|_p \coloneqq |N_{\mathbb{Q}_p(\alpha)/\mathbb{Q}_p}(\alpha)|_p^{1/[\mathbb{Q}_p(\alpha)\,:\,\mathbb{Q}_p]}$$

for any $\alpha \in \overline{\mathbb{Q}}_p$ (cf. Waldschmidt [35, Chapter 3]). Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. While we consider the field $\overline{\mathbb{Q}}$ of algebraic numbers as a subset of the field $\mathbb{C}$ of complex numbers, we also consider it as a subset of $\mathbb{C}_p$ by fixing an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$ for each prime number p. Several theorems or arguments in this thesis are valid not only in the case where the functions in question are considered as complex functions but also in the case where they are regarded as p-adic functions. In such situations, we will deal with the two cases simultaneously by stating the phrase *"Let v be ∞ or a prime number"* and by denoting $\mathbb{C}$ as $\overline{\mathbb{Q}}_\infty$ or $\mathbb{C}_\infty$ and the absolute value $|\alpha|$ of a complex number α as $|\alpha|_\infty$. (In the absence of such statements, we will discuss only the case of complex numbers.) Let v be ∞ or a prime number. Note that an element α of $\overline{\mathbb{Q}}_v$ is transcendental over $\mathbb{Q}$ if and only if it is transcendental over $\overline{\mathbb{Q}}$. Hence we simply say α is transcendental. For the similar reason, if elements $\alpha_1, \ldots, \alpha_n$ of $\overline{\mathbb{Q}}_v$ are algebraically independent over $\mathbb{Q}$, then we simply say $\alpha_1, \ldots, \alpha_n$ are algebraically independent. Moreover, an infinite subset S of $\overline{\mathbb{Q}}_v$ is said to be algebraically independent if any finite subset of S is algebaically independent.

Let p be a prime number. The p-adic exponential function $\exp_p(x)$ is defined as the power series $\sum_{n=0}^{\infty} x^n/n!$, which converges in the p-adic domain $\{x \in \mathbb{C}_p \mid |x|_p < p^{-1/(p-1)}\}$. In 1933, Mahler [15] proved that, for any algebraic number α with $0 < |\alpha|_p < p^{-1/(p-1)}$, the value $\exp_p(\alpha)$ is transcendental. The p-adic analogue of the 'full' Lindemann-Weierstrass theorem is still open. In 2008, Nesterenko [18] obtained the following 'half' result:

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers with $0 < |\alpha_i|_p < p^{-1/(p-1)}$ $(1 \le i \le n)$. If they form a basis of a finite extension of degree n of $\mathbb{Q}$, then the transcendence degree of $\mathbb{Q}(\exp_p(\alpha_1), \ldots, \exp_p(\alpha_n))$ over $\mathbb{Q}$ is at least $n/2$.

Beginning with the studies on the arithmetic properties of the values of the exponential function, various authors have investigated the transcendence and the algebraic independence of numbers given as the values of complex analytic functions at algebraic numbers and also their p-adic analogues.

As far back as 1844, Liouville proved the transcendency of the complex number $\sum_{k=0}^{\infty} 2^{-k!}$. This is not only the first example of transcendental numbers, but also the first proof of the existence of transcendental numbers. His proof is based on Diophantine approximation and can be summarized as follows. First, he proved the following theorem, which gives a lower bound for rational approximations of algebraic numbers.

Theorem 1.1.1 (Liouville's inequality)**.** *Let α be a real algebraic irrational number of degree n. Then there exists a positive constant c depending only on α such that*

$$\left| \alpha - \frac{p}{q} \right| > \frac{c}{q^n}$$

for any rational number p/q with $q > 0$.

Remark 1.1.2. Thue, Siegel, Dyson, and Roth refined the exponent n in the denominator of the right-hand side of Liouville's inequality. In 1955, Roth [25] proved the following: Let α be a real algebraic irrational number and ε a positive number. Then there exists a positive constant c depending only on α and ε such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{c}{q^{2+\varepsilon}}$$

for any rational number p/q with $q > 0$.

Secondly, he constructed a sharp rational approximation of the number $\sum_{k=0}^{\infty} 2^{-k!}$ so as to contradict the lower bound above. In fact, letting $p_m := 2^{m!} \sum_{k=0}^{m} 2^{-k!}$ and $q_m := 2^{m!}$, we obtain

$$0 < \left| \sum_{k=0}^{\infty} \frac{1}{2^{k!}} - \frac{p_m}{q_m} \right| < \frac{2}{q_m^{m+1}}$$

for any positive integer m, which contradicts the bound in Theorem 1.1.1 if $\sum_{k=0}^{\infty} 2^{-k!}$ is algebraic. Hence we conclude that $\sum_{k=0}^{\infty} 2^{-k!}$ is a transcendental number.

Because of this history, the power series $\sum_{k=0}^{\infty} z^{k!}$ is called Liouville series. Let v be ∞ or a prime number. Using the fundamental inequality, which gives a kind of lower bound for algebraic approximations of algebraic numbers and will be proved as Proposition 3.4.1 in this thesis, we can show the transcendency of the values of the Liouville series at any nonzero algebraic numbers inside the unit circle in $\mathbb{C}_v$. For the proof in the case of complex numbers, see Nishioka [23, Theorem 1.1.1]. Its proof is valid also in the case of p-adic numbers, since the fundamental inequality is extended to p-adic algebraic numbers (see Proposition 3.4.1, see also Waldschmidt [35, p. 84]). Modifying the proof of Nishioka [23, Theorem 1.1.1], for any sequence $\{e_k\}_{k\geq 0}$ of nonnegative integers satisfying $\lim_{k\to\infty} e_{k+1}/e_k = \infty$, we can prove the transcendency of the values of the power series $\sum_{k=0}^{\infty} z^{e_k}$ at any nonzero algebraic numbers inside the unit circle in $\mathbb{C}_v$.

More generally, we consider the function

$$\mathcal{F}(\{e_k\}; x, z) := \sum_{k=0}^{\infty} x^k z^{e_k}, \tag{1.1.1}$$

where $\{e_k\}_{k\geq 0}$ is a sequence of nonnegative integers satisfying $\liminf_{k\to\infty} e_{k+1}/e_k > 1$. This series converges at any point $(x, z) \in \mathbb{C}_v^2$ with $|z|_v < 1$. A specialization $\mathcal{F}(\{e_k\}; 1, z) = \sum_{k=0}^{\infty} z^{e_k}$ is called a *lacunary series*. By the so-called Hadamard's gap theorem (cf. Rudin [26, 16.6 Theorem]), every lacunary series has the unit circle in the complex plane as its natural boundary and hence they are transcendental functions.[1] Therefore, we may expect the transcendency of the values of any lacunary series at nonzero algebraic numbers inside the unit cicle. Indeed, in 2002, using Schmidt's subspace theorem, which is a generalization of Roth's theorem mentioned in Remark 1.1.2, Corvaja and Zannier established the following:

Theorem 1.1.3 (Corvaja and Zannier [4, A special case of Corollary 5])**.** *Let v be ∞ or a prime number. If a sequence $\{e_k\}_{k\geq 0}$ of nonnegative integers satisfies*

[1] It is known more generally that, if $\limsup_{k\to\infty}(e_{k+1}-e_k) = \infty$, then the power series $\sum_{k=0}^{\infty} z^{e_k}$ is transcendental over $\mathbb{C}(z)$ (see Mahler [16, p. 42]). Note that, some literature refers to a power series $\sum_{k=0}^{\infty} z^{e_k}$ as a *lacunary series* in the case where $\limsup_{k\to\infty}(e_{k+1} - e_k) = \infty$ and a *strongly lacunary series* in the case where $\liminf_{k\to\infty} e_{k+1}/e_k > 1$.

$\liminf_{k\to\infty} e_{k+1}/e_k > 1$, *then the number* $\mathcal{F}(\{e_k\}; 1, a) = \sum_{k=0}^{\infty} a^{e_k}$ *of* $\overline{\mathbb{Q}}_v$ *is transcendental for any algebraic number* a *with* $0 < |a|_v < 1$.

While it is difficult in general to determine the transcendency of the values of given analytic functions at algebraic numbers, it is much more difficult to determine their algebraic independency. In contrast with Theorem 1.1.3, there is no result which is applicable to all lacunary series and provides the algebraic independence of the values of those functions at distinct algebraic numbers. On the other hand, for lacunary series with rapidly increasing exponents such as the Liouville series, Nishioka obtained precise results on the algebraic independence by applying the method of Diophantine approximations. The most remarkable of her results is that they give necessary and sufficient conditions for the values of lacunary series, as well as their derivatives, at algebraic numbers to be algebraically independent. In 1986, she proved the following result on the Liouville series and its derivatives.

Theorem 1.1.4 (Nishioka [19, 20])**.** *Let* v *be* ∞ *or a prime number. Put* $f(z) := \mathcal{F}(\{k!\}; 1, z) = \sum_{k=0}^{\infty} z^{k!}$ *and let* $a_1, \ldots, a_r$ *be algebraic numbers with* $0 < |a_i|_v < 1$ $(1 \le i \le r)$. *Then the infinite subset* $\{f^{(l)}(a_i) \mid l \ge 0,\ 1 \le i \le r\}$ *of* $\overline{\mathbb{Q}}_v$ *is algebraically independent if and only if none of* a_i/a_j $(1 \le i < j \le r)$ *is a root of unity.*

Moreover, she established the following two theorems, from which we can deduce Theorem 1.1.4.

Theorem 1.1.5 (Nishioka [21, A special case of Theorem 1])**.** *Let* $\{e_k\}_{k\ge 0}$ *be a sequence of nonnegative integers satisfying* $\lim_{k\to\infty} e_{k+1}/e_k = \infty$. *Put* $f(z) := \mathcal{F}(\{e_k\}; 1, z) = \sum_{k=0}^{\infty} z^{e_k}$. *Let* $a_1, \ldots, a_r$ *be algebraic numbers with* $0 < |a_i| < 1$ $(1 \le i \le r)$. *Then the following three properties are equivalent:*

(i) *The infinite subset* $\{f^{(l)}(a_i) \mid l \ge 0,\ 1 \le i \le r\}$ *of* $\mathbb{C}$ *is algebraically dependent.*

(ii) *The* $r + 1$ *complex numbers* $1, f(a_1), \ldots, f(a_r)$ *are linearly dependent over* $\overline{\mathbb{Q}}$.

(iii) *There exist a nonempty subset* $\{a_{i_1}, \ldots, a_{i_s}\}$ *of* $\{a_1, \ldots, a_r\}$*, roots of unity* $\zeta_1, \ldots, \zeta_s$*, an algebraic number* γ *with* $a_{i_q} = \zeta_q \gamma$ $(1 \leq q \leq s)$*, and algebraic numbers* $\xi_1, \ldots, \xi_s$*, not all zero, such that*

$$\sum_{q=1}^{s} \xi_q \zeta_q^{e_k} = 0$$

for all sufficiently large k*.*

Theorem 1.1.6 (Nishioka [19, A special case of Theorem 1])**.** *Let* p *be a prime number and* $\{e_k\}_{k\geq 0}$ *a sequence of nonnegative integers satisfying* $\lim_{k\to\infty} e_{k+1}/e_k = \infty$*. Suppose that* 0 *is a limit point of* $\{e_k\}_{k\geq 0}$ *with respect to the* p*-adic norm. Put* $f(z) := \mathcal{F}(\{e_k\}; 1, z) = \sum_{k=0}^{\infty} z^{e_k}$ *and let* $a_1, \ldots, a_r$ *be algebraic numbers with* $0 < |a_i|_p < 1$ $(1 \leq i \leq r)$*. Then, if none of* a_i/a_j $(1 \leq i < j \leq r)$ *is a root of unity, then the infinite subset* $\{f^{(l)}(a_i) \mid l \geq 0,\ 1 \leq i \leq r\}$ *of* $\overline{\mathbb{Q}}_p$ *is algebraically independent.*

The following theorem was obtained as another corollary to Theorem 1.1.5.

Theorem 1.1.7 (Nishioka [21])**.** *Let* $f(z) := \mathcal{F}(\{k! + k\}; 1, z) = \sum_{k=0}^{\infty} z^{k!+k}$*. Then the infinite subset* $\{f^{(l)}(a) \mid l \geq 0,\ a \in \overline{\mathbb{Q}},\ 0 < |a| < 1\}$ *of* $\mathbb{C}$ *is algebraically independent.*

Furthermore, Nishioka also studied the algebraic independence of the values and the derivatives of an entire function defined as a power series having rapidly decreasing coefficients.

Theorem 1.1.8 (Nishioka [19, A special case of Theorem 4])**.** *Let* v *be* ∞ *or a prime number,* $\{e_k\}_{k\geq 0}$ *a sequence of nonnegative integers satisfying* $\lim_{k\to\infty} e_{k+1}/e_k = \infty$*, and* a *an algebraic number with* $0 < |a|_v < 1$*. Define* $F(x) := \mathcal{F}(\{e_k\}; x, a) = \sum_{k=0}^{\infty} a^{e_k} x^k$*. Then the infinite subset* $\{F^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ *of* $\overline{\mathbb{Q}}_v$ *is algebraically independent.*

As mentioned in the beginning of this chapter, Theorems 1.1.7 and 1.1.8 are two of the strongest results on the necessary and sufficient conditions for the values of analytic functions, as well as their derivatives, at algebraic numbers to be algebraically independent. Here we formulate the properties possessed by the power series $\sum_{k=0}^{\infty} z^{k!+k}$ in Theorem 1.1.7 and $\sum_{k=0}^{\infty} a^{e_k} x^k$ in Theorem 1.1.8 as follows:

Property 1.1.9. The infinite set consisting of all the values of a single analytic function and its derivatives of any order, at any nonzero algebraic numbers within its domain of existence, is algebraically independent.

Analytic functions having this property provide explicit examples of algebraically independent infinite subsets of the field of complex numbers or that of p-adic complex numbers, which may help us to investigate the structure of the set of transcendental numbers. In the next section we introduce further examples of complex analytic functions, particularly entire functions, which are known to have Property 1.1.9; however, those results only deal with functions of one variable. Moreover, it is quite difficult to determine the algebraic independency of the union of several sets, each of which is known to be algebraically independent. Therefore, it is a very interesting problem to construct a family of entire functions, which may have several variables, satisfying the following:

Property 1.1.10. The infinite set consisting of all the values of the functions belonging to the family and their partial derivatives of any order, at any algebraic points with nonzero components, is algebraically independent.

This property is so strong that, if a family of entire functions satisfies Property 1.1.10, then Property 1.1.9 is possessed by each of those functions. In this thesis, we will explicitly construct an infinite family of complex entire functions of two variables satisfying Property 1.1.10 (see Example 1.3.11).

1.2 Mahler's method

In this section we focus on the case where $1 < \lim_{k\to\infty} e_{k+1}/e_k < \infty$. We consider the power series $\mathcal{F}(\{e_k\}; x, z)$ defined by (1.1.1) and also the infinite product $\mathcal{G}(\{e_k\}; y, z)$ and the Lambert type series $\mathcal{H}(\{e_k\}; x, y, z)$ defined respectively by

$$\mathcal{G}(\{e_k\}; y, z) \coloneqq \prod_{k=0}^{\infty} (1 - y z^{e_k}), \qquad \mathcal{H}(\{e_k\}; x, y, z) \coloneqq \sum_{k=0}^{\infty} \frac{x^k z^{e_k}}{1 - y z^{e_k}}.$$

We note that $\mathcal{G}(\{e_k\}; y, z)$ converges at any point $(y, z) \in \mathbb{C}_v^2$ with $|z|_v < 1$ and $\mathcal{H}(\{e_k\}; x, y, z)$ converges at any point $(x, y, z) \in \mathbb{C}_v^3$ with $|z|_v < 1$ such that $1 - yz^{e_k} \neq 0$ for any $k \geq 0$. In the rest of this section, we consider $\mathcal{F}(\{e_k\}; x, z)$, $\mathcal{G}(\{e_k\}; y, z)$, and $\mathcal{H}(\{e_k\}; x, y, z)$ as complex functions. In the previous works, the transcendence and the algebraic independence of the values at algebraic numbers of these functions were dealt with mainly in the following two cases:

(A) The case where $x = y = 1$ and z runs through a finite set of algebraic numbers.

(B) The case where x, y run through infinite sets of algebraic numbers and z is fixed.

First we introduce the case (A). In this case, the power series $\mathcal{F}(\{e_k\}; 1, z) = \sum_{k=0}^{\infty} z^{e_k}$ is a lacunary series. A typical example is $f(z) := \mathcal{F}(\{d^k\}; 1, z) = \sum_{k=0}^{\infty} z^{d^k}$, where d is an integer greater than 1. This power series is called Fredholm series. Theorem 1.1.3 of course leads to the transcendency of the complex number $f(a)$ for any algebraic number a with $0 < |a| < 1$. For example, the complex number $\sum_{k=0}^{\infty} 2^{-2^k}$ is transcendental. However, in contrast with the case of the number $\sum_{k=0}^{\infty} 2^{-k!}$ mentioned in Section 1.1, it is not very easy to derive its transcendency by focusing on the rational approximation

$$0 < \left| \sum_{k=0}^{\infty} \frac{1}{2^{2^k}} - \frac{p_m}{q_m} \right| < \frac{2}{q_m^2},$$

where $p_m := 2^{2^m} \sum_{k=0}^{m} 2^{-2^k}$ and $q_m := 2^{2^m}$ for any positive integer m. In order to deduce the transcendency of $\sum_{k=0}^{\infty} 2^{-2^k}$ from this approximation, we need to use a refinement of Roth's theorem given by Ridout [24] in 1957. Therefore, the method of Diophantine approximation does not seem suitable for studying the arithmetic properties of the values of the Fredholm series at algebraic numbers. On the other hand, in 1929, Mahler [14] obtained the transcendency of the complex number $f(a)$ for any algebraic number a with $0 < |a| < 1$, which is the same conclusion as Theorem 1.1.3 for the Fredholm series. His proof is based on the properties of the function $f(z)$, in particular on the fact that $f(z)$ satisfies the functional equation

$$f(z) = f(z^d) + z. \tag{1.2.1}$$

This contrasts with Liouville's proof on the transcendency of $\sum_{k=0}^{\infty} 2^{-k!}$ in which the number was treated directly. Mahler's method for proving the transcendency of the number $f(a)$ consists of the following three steps. First, he proved the transcendency of the function $f(z)$ itself. Note that, while Hadamard's gap theorem is valid for this purpose, we can prove it by a more elementary argument using the functional equation (1.2.1) (cf. Nishioka [23, pp. 3–4]). Secondly, he constructed a polynomial $E(X, Y)$ with algebraic coefficients such that the auxiliary function $E(z, f(z))$ has a sufficiently high order at the origin but does not vanish at points $z = a^{d^k}$ for all sufficiently large k. Such a polynomial was constructed by linear algebra, while the requirement on the non-vanishing of the values was satisfied by the identity theorem and the transcendency of $f(z)$ proved in the first step. Finally, under the assumption that the number $f(a)$ is an algebraic number, he derived a contradiction by estimating a lower bound of the absolute value of a nonzero algebraic number $E(a^{d^k}, f(a^{d^k}))$ from the functional equation (1.2.1) and the fundamental inequality (Proposition 3.4.1). The most important and difficult of these three steps is the second one, construction of the auxiliary function.

Applying this method to the infinite product $g(z) := \mathcal{G}(\{d^k\}; 1, z) = \prod_{k=0}^{\infty}(1 - z^{d^k})$ and the Lambert series $h(z) := \mathcal{H}(\{d^k\}; 1, 1, z) = \sum_{k=0}^{\infty} z^{d^k}/(1 - z^{d^k})$, which satisfy the functional equations $g(z) = (1 - z)g(z^d)$ and $h(z) = h(z^d) + z/(1 - z)$, respectively, Mahler also proved the transcendency of the complex numbers $g(a)$ and $h(a)$ for any algebraic number a with $0 < |a| < 1$. Moreover, Mahler [14] also studied the transcendence of the values of functions of several variables at algebraic points. Here, those functions are assumed to satisfy functional equations under a transformation $\boldsymbol{z} \mapsto \Omega\boldsymbol{z}$ of variables defined in a certain way via a matrix Ω with nonnegative integer entries (see (3.1.1)). In the case of several variables, the second step above becomes more difficult, since the values of the auxiliary function at the sequence $\{\Omega^k\boldsymbol{\alpha}\}_{k\geq 0}$ converging to the origin, where $\boldsymbol{\alpha}$ is the algebraic point we are considering, may all vanish. In order to overcome this difficulty, he proved a theorem, now called Mahler's vanishing theorem, which gives a sufficient condition on Ω and $\boldsymbol{\alpha}$ for the values above do not vanish (see Lemma 3.1.1). Using this theorem, Mahler established Theorem 1.2.3 below, which includes the result above on the values of the Fredholm series.

Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 0), \tag{1.2.2}$$

where $R_0, \ldots, R_{n-1}$ are not all zero and $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. Define the polynomial associated with (1.2.2) by

$$\Phi(X) := X^n - c_1 X^{n-1} - \cdots - c_n. \tag{1.2.3}$$

Mahler assumed the following condition to apply his vanishing theorem.

Condition 1.2.1. $\Phi(X)$ is irreducible over $\mathbb{Q}$ and there exists an ordering of the roots $\rho_1, \ldots, \rho_n$ of $\Phi(X)$ such that $\rho_1 > \max\{1, |\rho_2|_\infty, \ldots, |\rho_n|_\infty\}$.

If $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.1, then

$$R_k = c\rho_1^k + o(\rho_1^k), \tag{1.2.4}$$

where c is a positive constant, and thus the power series $\mathcal{F}(\{R_k\}; 1, z) = \sum_{k=0}^{\infty} z^{R_k}$ is a lacunary series. He modified this series to $\sum_{k=0}^{\infty} \boldsymbol{z}^{\boldsymbol{R}_k}$, where $\boldsymbol{z}^{\boldsymbol{R}_k} := z_1^{R_{k+n-1}} \cdots z_n^{R_k}$. Then $\sum_{k=0}^{\infty} \boldsymbol{z}^{\boldsymbol{R}_k}$ satisfies a functional equation of the form $\sum_{k=0}^{\infty} \boldsymbol{z}^{\boldsymbol{R}_k} - \sum_{k=0}^{\infty} (\Omega_1 \boldsymbol{z})^{\boldsymbol{R}_k} \in \mathbb{Q}[\boldsymbol{z}]$, where Ω_1 is a matrix determined by the coefficients of $\Phi(X)$ (see (4.1.3)). Letting a be an algebraic number with $0 < |a| < 1$ and putting $\boldsymbol{\gamma}_1 := (1, \ldots, 1, a)$, we see that $\sum_{k=0}^{\infty} \boldsymbol{\gamma}_1^{\boldsymbol{R}_k} = \sum_{k=0}^{\infty} a^{R_k}$. In order to prove the transcendency of this number, Mahler used the following auxiliary result, which is deduced from Mahler's vanishing theorem. (For the entire statement of Mahler's vanishing theorem, see Lemma 3.1.1.)

Lemma 1.2.2 (Mahler [14])**.** *Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying* (1.2.2)*. Suppose that $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.1. Then, for any nonzero $f(\boldsymbol{z}) \in \mathbb{C}[[z_1, \ldots, z_n]]$ which converges in some neighborhood of the origin of $\mathbb{C}^n$, there are infinitely many positive integers k such that $f(\Omega_1^k \boldsymbol{\gamma}_1) \neq 0$.*

This lemma makes Mahler's method for proving the transcendence of the values of functions applicable to the function $\sum_{k=0}^{\infty} \boldsymbol{z}^{\boldsymbol{R}_k}$ as well as the Fredholm series $\sum_{k=0}^{\infty} z^{d^k}$. Using the functional equation of $\sum_{k=0}^{\infty} \boldsymbol{z}^{\boldsymbol{R}_k}$, the relation $\sum_{k=0}^{\infty} \boldsymbol{\gamma}_1^{\boldsymbol{R}_k} = \sum_{k=0}^{\infty} a^{R_k}$, and Lemma 1.2.2, Mahler proved the following:

Theorem 1.2.3 (Mahler [14])**.** *Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying* (1.2.2)*. Suppose that $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.1. Then, for any algebraic number a with $0 < |a| < 1$, the complex number $\sum_{k=0}^{\infty} a^{R_k}$ is transcendental.*

Example 1.2.4. Let $\{F_k\}_{k\geq 0}$ be the Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0).$$

Then $\{F_k\}_{k\geq 0}$ satisfies Condition 1.2.1 and thus the number $\sum_{k=0}^{\infty} a^{F_k}$ is transcendental for any algebraic number a with $0 < |a| < 1$.

In 1982, Masser [17] established a necessary and sufficient condition on a matrix Ω and a point $\boldsymbol{\alpha}$ under which the non-vanishing requirement on the values of auxiliary functions is satisfied (see Lemma 3.1.3). Using Masser's vanishing theorem, Tanaka relaxed Condition 1.2.1 on the linear recurrence $\{R_k\}_{k\geq 0}$ to the following:

Condition 1.2.5. $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity.

Since the polynomial $\Phi(X)$ is not assumed to be irreducible in Condition 1.2.5, it is weaker than Condition 1.2.1. It is also known that Condition 1.2.5 implies the asymptotic formula (1.2.4), which ensures the convergence of the series in Theorem 1.2.6 below. (For the proofs of these two statements, see Tanaka [29, Remarks 1 and 4].)

Theorem 1.2.6 (Tanaka [28, A special case of Theorem])**.** *Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying* (1.2.2)*. Suppose that $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.5. Then, for any algebraic number a with $0 < |a| < 1$, the complex number $\sum_{k=0}^{\infty} a^{R_k}$ is transcendental.*

Example 1.2.7 (cf. Tanaka [29, Example 1]). Let $\{R_k\}_{k\geq 0}$ be a linear recurrence satisfying

$$R_{k+3} = R_{k+2} + 16R_{k+1} + 20R_k \quad (k \geq 0) \tag{1.2.5}$$

with initial values $R_0 = 1$, $R_1 = 3$, and $R_2 = 33$. Then the associated polynomial is $\Phi(X) = X^3 - X^2 - 16X - 20 = (X-5)(X+2)^2$. Thus the linear recurrence $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.5 and so the number $\sum_{k=0}^{\infty} a^{R_k}$ is transcendental for any algebraic number a with $0 < |a| < 1$. In addition, we can verify that $R_k = 5^k + k(-2)^k$ for all $k \geq 0$, which implies that $\{R_k\}_{k\geq 0}$ does not satisfy any recurrence formula of the form (1.2.2) with $n \leq 2$.

Remark 1.2.8. Tanaka's results mentioned above include the case of geometric progressions, namely that of the Fredholm series. Indeed, Condition 1.2.5 admits geometric progressions even if we assume $n \geq 2$. For instance, the geometric progression $\{5^k\}_{k\geq 0}$ satisfies the recurrence formula (1.2.5) in Example 1.2.7. In this thesis we deal with linear recurrences satisfying Condition 1.2.5 which are not geometric progressions.

Analytic functions satisfying functional equations such as $f(z)$, $g(z)$, $h(z)$ or $\sum_{k=0}^{\infty} \boldsymbol{z}^{\boldsymbol{R}_k}$ above are called Mahler functions. Mahler's method is suitable for proving not only the transcendence of the values of Mahler functions but also their algebraic independence. For certain types of Mahler functions, Kubota [11] and Nishioka [22] proved, independently, the algebraic independence of the values under the assumption that the functions themselves are algebraically independent over the field of rational functions. Moreover, they also established necessary and sufficient conditions for the Mahler functions themselves to be algebraically independent. These results are so powerful that the algebraic independency of the values of Mahler functions can be reduced to the linear independency or the multiplicative independency of the functions themselves in many cases. For instance, applying Kubota's criterion, we can show that, if $a_1, \ldots, a_r$ are multiplicatively independent algebraic numbers with $0 < |a_i| < 1$ $(1 \leq i \leq r)$, then the $3r$ numbers $f(a_i) = \sum_{k=0}^{\infty} a_i^{d^k}$, $g(a_i) = \prod_{k=0}^{\infty}(1 - a_i^{d^k})$, and $h(a_i) = \sum_{k=0}^{\infty} a_i^{d^k}/(1 - a_i^{d^k})$ $(1 \leq i \leq r)$ are algebraically independent (cf. Nishioka [23, pp. 106–107]).

On the other hand, the values of $f(z)$, $g(z)$, and $h(z)$ at multiplicatively dependent algebraic numbers can be algebraically dependent. Indeed, by the functional equations we have $f(a) - f(a^d)$, $g(a)/g(a^d)$, $h(a) - h(a^d) \in \overline{\mathbb{Q}}^\times$ for any algebraic number a with $0 < |a| < 1$. For the Fredholm series $f(z)$, Loxton and van der Poorten [12, Theorem 3] obtained a necessary and sufficient condition on algebraic numbers $a_1, \ldots, a_r$ for the values $f(a_1), \ldots, f(a_r)$ to be algebraically independent. For the Lambert series $h(z)$, Bundschuh and Väänänen [2, Theorem 2] obtained the following:

Let a be an algebraic number with $0 < |a| < 1$ and let $m_1, \ldots, m_r$ be positive integers such that none of m_i/m_j $(1 \le i < j \le r)$ is a power of d. Then, for each choice of r signs, the r numbers $h(\pm a^{m_1}), \ldots, h(\pm a^{m_r})$ are algebraically independent.

Except for this result, there are no known results on the algebraic independence of the values at multiplicatively dependent algebraic numbers of the functions $g(z)$ and $h(z)$ so far. In particular, it is an open problem to determine the necessary and sufficient condition on algebraic numbers $a_1, \ldots, a_r$ for the $3r$ numbers $f(a_i)$, $g(a_i)$, and $h(a_i)$ $(1 \le i \le r)$ to be algebraically independent. On the other hand, if we replace the $\{d^k\}_{k \ge 0}$ appearing in $f(z)$, $g(z)$, and $h(z)$ with a linear recurrence which is not a geometric progression, then the situation on the algebraic independence of the values at algebraic numbers becomes quite different. Tanaka proved the following theorem, which gives the necessary and sufficient condition above for the functions generated by such a linear recurrence.

Theorem 1.2.9 (Tanaka [31, Theorem 1]). *Let $\{R_k\}_{k\ge0}$ be a linear recurrence of positive integers satisfying* (1.2.2). *Suppose that $\{R_k\}_{k\ge0}$ satisfies Condition 1.2.5. Assume that $\{R_k\}_{k\ge0}$ is not a geometric progression. Define $f^*(z) := \mathcal{F}(\{R_k\}; 1, z) = \sum_{k=0}^{\infty} z^{R_k}$, $g^*(z) := \mathcal{G}(\{R_k\}; 1, z) = \prod_{k=0}^{\infty}(1 - z^{R_k})$, and $h^*(z) := \mathcal{H}(\{R_k\}; 1, 1, z) = \sum_{k=0}^{\infty} z^{R_k}/(1 - z^{R_k})$. Let $a_1, \ldots, a_r$ be algebraic numbers with $0 < |a_i| < 1$ $(1 \le i \le r)$. Then the following three properties are equivalent:*

(i) *The $3r$ complex numbers $f^*(a_i)$, $g^*(a_i)$, and $h^*(a_i)$ $(1 \le i \le r)$ are algebraically dependent.*

(ii) *The* $r+1$ *complex numbers* $1, f^*(a_1), \ldots, f^*(a_r)$ *are linearly dependent over* $\overline{\mathbb{Q}}$.

(iii) *There exist a nonempty subset* $\{a_{i_1}, \ldots, a_{i_s}\}$ *of* $\{a_1, \ldots, a_r\}$*, roots of unity* $\zeta_1, \ldots, \zeta_s$*, an algebraic number* γ *with* $a_{i_q} = \zeta_q \gamma$ $(1 \leq q \leq s)$*, and algebraic numbers* $\xi_1, \ldots, \xi_s$*, not all zero, such that*

$$\sum_{q=1}^{s} \xi_q \zeta_q^{R_k} = 0$$

for all sufficiently large k.

In particular, there are no algebraic relations among the values of the functions $f^*(z)$, $g^*(z)$, and $h^*(z)$ at algebraic numbers such as a and a^d with $0 < |a| < 1$ and $d \geq 2$, in contrast with the case of geometric progressions.

Remark 1.2.10. Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying (1.2.2). Define $f_1(z) := \sum_{k=0}^{\infty} z^{R_k+k}$ and $f_2(z) := \sum_{k=0}^{\infty} z^{R_k-k}$. Tanaka, Toyama, and the author [10, Theorems 1.11 and 1.16] proved that, if the roots of $\Phi(X)$ satisfy suitable conditions, then each of $f_1(z)$ and $f_2(z)$ possesses Property 1.1.9, namely the infinite subset $\{f_i^{(l)}(a) \mid l \geq 0,\ a \in \overline{\mathbb{Q}},\ 0 < |a| < 1\}$ of $\mathbb{C}$ is algebraically independent for each $i = 1, 2$. In particular, if d is an integer greater than 1, then each of $\sum_{k=0}^{\infty} z^{d^k+k}$ and $\sum_{k=0}^{\infty} z^{d^k-k}$ has Property 1.1.9. As far as the author knows, these are the only examples of lacunary series $\sum_{k=0}^{\infty} z^{e_k}$ with $1 < \lim_{k\to\infty} e_{k+1}/e_k < \infty$ having Property 1.1.9. We note that the series $\sum_{k=0}^{\infty} z^{k!+k}$ in Theorem 1.1.7 is an example of lacunary series satisfying $\lim_{k\to\infty} e_{k+1}/e_k = \infty$ and Property 1.1.9. On the other hand, Tanuma [34, Corollary 2] proved that the exponential type Hecke-Mahler series $\sum_{k=1}^{\infty} z^{[k\omega]}$ has Property 1.1.9, where ω is a positive quadratic irrational number. This is the only result on power series $\sum_{k=0}^{\infty} z^{e_k}$ satisfying $\lim_{k\to\infty} e_{k+1}/e_k = 1$ and Property 1.1.9.

Next we introduce the case (B) mentioned at the beginning of this section. We fix an algebraic number a with $0 < |a| < 1$. First we consider the functions $\mathcal{F}(\{d^k\}; x; a^m)$ $(m = 1, 2, \ldots)$, $\mathcal{G}(\{d^k\}; y, a)$, and $\mathcal{H}(\{d^k\}; x, y, a)$, where d is an integer greater than 1. Nishioka proved that each of the entire functions $\mathcal{F}(\{d^k\}; x; a^m)$

($m = 1, 2, \ldots$) has Property 1.1.9, using her criterion on the algebraic independence of the values of Mahler functions.

Theorem 1.2.11 (Nishioka [22, Theorem 7])**.** *Let m be a positive integer and let $F_m(x) \coloneqq \mathcal{F}(\{d^k\}; x; a^m) = \sum_{k=0}^{\infty} a^{md^k} x^k$. Then the infinite subset $\{F_m^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ of $\mathbb{C}$ is algebraically independent for each fixed m.*

Remark 1.2.12. The family $\{F_m(x)\}_{m\geq 1}$ in Theorem 1.2.11 does not have Property 1.1.10. Indeed, the infinite set $\{F_m^{(l)}(\alpha) \mid l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ is algebraically dependent since $\alpha F_d(\alpha) + a = F_1(\alpha)$ for any nonzero algebraic number α.

Using Mahler's method, we can also derive the transcendency of the values of the infinite product $\mathcal{G}(\{d^k\}; y, a) = \prod_{k=0}^{\infty}(1 - a^{d^k} y)$ and the Lambert type series $\mathcal{H}(\{d^k\}; x, y, a) = \sum_{k=0}^{\infty} a^{d^k} x^k/(1 - a^{d^k} y)$. Let α and β be nonzero algebraic numbers with $\beta \notin \{a^{-d^k} \mid k \geq 0\}$. Then the values $\mathcal{G}(\{d^k\}; \beta, a)$ and $\mathcal{H}(\{d^k\}; \alpha, \beta, a)$ are transcendental if $(d, \beta) \neq (2, -1)$ and $(d, \alpha, \beta) \neq (2, 2, -1)$, respectively (cf. Nishioka [23, Theorems 1.2 and 1.3]). However, in contrast with Theorem 1.2.11, these values at distinct algebraic points are not always algebraically independent as shown below. Let $\gamma_1, \ldots, \gamma_d$ be the d-th roots of β. Then the $d + 1$ values $\mathcal{G}(\{d^k\}; \beta, a), \mathcal{G}(\{d^k\}; \gamma_1, a), \ldots, \mathcal{G}(\{d^k\}; \gamma_d, a)$ are algebraically dependent and so are $\mathcal{H}(\{d^k\}; 1, \beta, a), \mathcal{H}(\{d^k\}; 1, \gamma_1, a), \ldots, \mathcal{H}(\{d^k\}; 1, \gamma_d, a)$, since

$$(1 - a\beta) \prod_{i=1}^{d} \mathcal{G}(\{d^k\}; \gamma_i, a) = \mathcal{G}(\{d^k\}; \beta, a)$$

and

$$\sum_{i=1}^{d} \gamma_i \mathcal{H}(\{d^k\}; 1, \gamma_i, a) + ad\frac{\beta}{1 - a\beta} = d\beta\mathcal{H}(\{d^k\}; 1, \beta, a).$$

Remark 1.2.13. For the case where $(d, \beta) = (2, -1)$, we see that

$$\mathcal{G}(\{2^k\}; -1, a) = \prod_{k=0}^{\infty}(1 + a^{2^k}) = \frac{1}{1 - a} \in \overline{\mathbb{Q}}.$$

Moreover, for the case where $(d, \alpha, \beta) = (2, 2, -1)$, we can verify that

$$\mathcal{H}(\{2^k\}; 2, -1, a) = \sum_{k=0}^{\infty} \frac{2^k a^{2^k}}{1 + a^{2^k}} = \frac{a}{1 - a} \in \overline{\mathbb{Q}}.$$

Finally, we consider the functions

$$F_m(x) := \mathcal{F}(\{R_k\}; x; a^m) = \sum_{k=0}^{\infty} a^{mR_k} x^k \quad (m = 1, 2, \ldots), \tag{1.2.6}$$

$$G(y) := \mathcal{G}(\{R_k\}; y, a) = \prod_{k=0}^{\infty} (1 - a^{R_k} y), \tag{1.2.7}$$

and

$$H(x, y) := \mathcal{H}(\{R_k\}; x, y, a) = \sum_{k=0}^{\infty} \frac{a^{R_k} x^k}{1 - a^{R_k} y}, \tag{1.2.8}$$

where $\{R_k\}_{k \geq 0}$ is a linear recurrence of nonnegative integers satisfying (1.2.2). We suppose that $\{R_k\}_{k \geq 0}$ satisfies Condition 1.2.5 and that $\{R_k\}_{k \geq 0}$ is not a geometric progression (cf. Remark 1.2.8). For the algebraic independence of the values of the functions above, there are more remarkable results than the case of geometric progressions. In contrast with Remark 1.2.12, not only each function $F_m(x)$ has Property 1.1.9, but also the infinite family $\{F_m(x)\}_{m \geq 1}$ has Property 1.1.10.

Theorem 1.2.14 (Tanaka [30, A special case of Theorem 3]). *Let $\{R_k\}_{k \geq 0}$ be a linear recurrence of nonnegative integers satisfying* (1.2.2). *Suppose that $\{R_k\}_{k \geq 0}$ satisfies Condition 1.2.5. Assume that $\{R_k\}_{k \geq 0}$ is not a geometric progression. Then the infinite subset $\{F_m^{(l)}(\alpha) \mid l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times}\}$ of $\mathbb{C}$ is algebraically independent.*

This theorem was obtained by applying Nishioka's criterion for the algebraic independence of the values of Mahler functions. On the other hand, using Kubota's criterion, Tanaka [33, Theorem 2] proved the algebraic independency of the infinite set

$$\{G(\beta) \mid \beta \in \mathcal{B}\} \bigcup \left\{ \frac{\partial^m H}{\partial y^m}(1, \beta) \;\middle|\; m \geq 0,\ \beta \in \mathcal{B} \right\},$$

where $\mathcal{B}$ denotes the set of nonzero algebraic numbers different from the zeros of $G(y)$. From this, he deduced the following theorem by using the fact that the derivatives of $G(y)$ are expressed as polynomials with integral coefficients of $G(y)$, $H(1, y)$, and the partial derivatives of $H(1, y)$ with respect to y.

Theorem 1.2.15 (Tanaka [33, Theorem 1]). *Let* $\{R_k\}_{k\geq 0}$ *be a linear recurrence of nonnegative integers satisfying* (1.2.2). *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies Condition 1.2.5. Assume that* $\{R_k\}_{k\geq 0}$ *is not a geometric progression. Then the infinite subset* $\{G^{(m)}(\beta) \mid m \geq 0,\ \beta \in \mathcal{B}\}$ *of* $\mathbb{C}$ *is algebraically independent.*

This theorem asserts that the infinite product $G(y)$ has 'quasi' Property 1.1.9, however does not give an answer to whether the algebraic independency of a larger infinite set $\{G^{(m)}(\beta) \mid m \geq N_\beta,\ \beta \in \overline{\mathbb{Q}}^{\times}\}$ holds, where N_β is defined by

$$N_\beta := \#\{k \geq 0 \mid a^{-R_k} = \beta\} = \operatorname*{ord}_{y=\beta} G_i(y) \tag{1.2.9}$$

for each nonzero algebraic number β.

1.3 Main results

In this section we introduce two main theorems of this thesis and their corollaries.

In the first main theorem we deal with not only the complex case but also the p-adic case. Let v be ∞ or a prime number and $\{R_k\}_{k\geq 0}$ a linear recurrence of nonnegative integers satisfying (1.2.2). In the case where v is ∞, we suppose that $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.5. On the other hand, in the case where v is a prime number, we suppose the stronger Condition 1.2.1 on $\{R_k\}_{k\geq 0}$. In both of these two cases we assume further that $\{R_k\}_{k\geq 0}$ is not a geometric progression. We fix an algebraic number a with $0 < |a|_v < 1$ and consider the functions $F_m(x)$, $G(y)$, and $H(x,y)$ given by (1.2.6), (1.2.7), and (1.2.8), respectively. If v is a prime number p, we regard these functions as p-adic functions. Moreover, we define a two-variable function $\Theta(x,y)$ by

$$\Theta(x,y) := G(y)H(x,y) = \sum_{k=0}^{\infty} a^{R_k} x^k \prod_{\substack{k'=0,\\ k'\neq k}}^{\infty} (1 - a^{R_{k'}} y). \tag{1.3.1}$$

By the asymptotic formula (1.2.4), $\Theta(x,y)$ is an entire function on $\mathbb{C}_v \times \mathbb{C}_v$. For each nonzero algebraic number β, let N_β be the number defined by (1.2.9). Note that N_β is 0 or 1 for all but finitely many β. The following is the first main theorem of this thesis.

Theorem 1.3.1 (Ide [7, Theorem 1.7]). *Let v be ∞ or a prime number and $\{R_k\}_{k\geq 0}$ a linear recurrence of nonnegative integers satisfying* (1.2.2)*. Suppose that $\{R_k\}_{k\geq 0}$ satisfies Conditions 1.2.5 or 1.2.1 according respectively as v is ∞ or a prime number. Assume further that $\{R_k\}_{k\geq 0}$ is not a geometric progression. Then the infinite subset*

$$\left\{F_m^{(l)}(\alpha)\;\middle|\; l\geq 0,\ m\geq 1,\ \alpha\in\overline{\mathbb{Q}}^\times\right\}\bigcup\left\{G^{(N_\beta)}(\beta)\;\middle|\;\beta\in\overline{\mathbb{Q}}^\times\right\}$$
$$\bigcup\left\{\frac{\partial^{l+m}\Theta}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\; l\geq 0,\ m\geq N_\beta,\ \alpha\in\overline{\mathbb{Q}}^\times,\ \beta\in\overline{\mathbb{Q}}^\times\right\}$$

of $\overline{\mathbb{Q}}_v$ is algebraically independent.

We will see in Remark 2.2.3 that the algebraic independency of the infinite set $\{F_m^{(l)}(\alpha)\mid l\geq 0,\ m\geq 1,\ \alpha\in\overline{\mathbb{Q}}^\times\}$ is equivalent to that of $\{\partial^{l+m}\Theta/\partial x^l\partial y^m(\alpha,0)\mid l\geq 0,\ m\geq 0,\ \alpha\in\overline{\mathbb{Q}}^\times\}$. Hence the conclusion of Theorem 1.3.1 is equivalent to the following:

The infinite subset

$$\left\{G^{(N_\beta)}(\beta)\;\middle|\;\beta\in\overline{\mathbb{Q}}^\times\right\}\bigcup\left\{\frac{\partial^{l+m}\Theta}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\; l\geq 0,\ m\geq N_\beta,\ \alpha\in\overline{\mathbb{Q}}^\times,\ \beta\in\overline{\mathbb{Q}}\right\} \tag{1.3.2}$$

of $\overline{\mathbb{Q}}_v$ is algebraically independent, where $N_0 := 0$.

On the other hand, substituting $x=1$ into $\Theta(x,y)$, we see by the logarithmic derivation of $G(y)$ that $\Theta(1,y)=G(y)H(1,y)=-G'(y)$ and thus

$$\frac{\partial^m\Theta}{\partial y^m}(1,y)=-G^{(m+1)}(y) \tag{1.3.3}$$

for any $m\geq 0$. Therefore the set $\{F_m^{(l)}(\alpha)\mid l\geq 0,\ m\geq 1,\ \alpha\in\overline{\mathbb{Q}}^\times\}$ treated in Theorem 1.2.14 and the set $\{G^{(m)}(\beta)\mid m\geq N_\beta,\ \beta\in\overline{\mathbb{Q}}^\times\}$ mentioned immediately after Theorem 1.2.15 can be regarded as subsets of the set (1.3.2), as long as we are interested only in their algebraic independency. From this point of view, the sets treated in Theorems 1.2.14, 1.2.15, and 1.3.1 are represented respectively in Tables 1.1, 1.2, and 1.3 below. (In these tables, the function $\partial^{l+m}\Theta/\partial x^l\partial y^m(x,y)$ is written as $\Theta^{(l,m)}(x,y)$.)

			$x = \alpha \in \overline{\mathbb{Q}}^{\times}$			
			$\alpha_1 = 1$	α_2	α_3	$\cdots$
$y = \beta \in \overline{\mathbb{Q}}$	$\beta_0 = 0$		$F_m^{(l)}(1)$	$F_m^{(l)}(\alpha_2)$	$F_m^{(l)}(\alpha_3)$	$\cdots$
	$\beta \in \mathcal{B}$	β_1	$G^{(m)}(\beta_1)$	$\Theta^{(l,m)}(\alpha_2, \beta_1)$	$\Theta^{(l,m)}(\alpha_3, \beta_1)$	$\cdots$
		β_2	$G^{(m)}(\beta_2)$	$\Theta^{(l,m)}(\alpha_2, \beta_2)$	$\Theta^{(l,m)}(\alpha_3, \beta_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	$\beta \in \overline{\mathbb{Q}}^{\times} \setminus \mathcal{B}$ $= \{a^{-R_k} \mid k \geq 0\}$	β'_1	$G^{(m)}(\beta'_1)$	$\Theta^{(l,m)}(\alpha_2, \beta'_1)$	$\Theta^{(l,m)}(\alpha_3, \beta'_1)$	$\cdots$
		β'_2	$G^{(m)}(\beta'_2)$	$\Theta^{(l,m)}(\alpha_2, \beta'_2)$	$\Theta^{(l,m)}(\alpha_3, \beta'_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.1: The numbers treated in Theorem 1.2.14

			$x = \alpha \in \overline{\mathbb{Q}}^{\times}$			
			$\alpha_1 = 1$	α_2	α_3	$\cdots$
$y = \beta \in \overline{\mathbb{Q}}$	$\beta_0 = 0$		$F_m^{(l)}(1)$	$F_m^{(l)}(\alpha_2)$	$F_m^{(l)}(\alpha_3)$	$\cdots$
	$\beta \in \mathcal{B}$	β_1	$G^{(m)}(\beta_1)$	$\Theta^{(l,m)}(\alpha_2, \beta_1)$	$\Theta^{(l,m)}(\alpha_3, \beta_1)$	$\cdots$
		β_2	$G^{(m)}(\beta_2)$	$\Theta^{(l,m)}(\alpha_2, \beta_2)$	$\Theta^{(l,m)}(\alpha_3, \beta_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	$\beta \in \overline{\mathbb{Q}}^{\times} \setminus \mathcal{B}$ $= \{a^{-R_k} \mid k \geq 0\}$	β'_1	$G^{(m)}(\beta'_1)$	$\Theta^{(l,m)}(\alpha_2, \beta'_1)$	$\Theta^{(l,m)}(\alpha_3, \beta'_1)$	$\cdots$
		β'_2	$G^{(m)}(\beta'_2)$	$\Theta^{(l,m)}(\alpha_2, \beta'_2)$	$\Theta^{(l,m)}(\alpha_3, \beta'_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.2: The numbers treated in Theorem 1.2.15

			$x = \alpha \in \overline{\mathbb{Q}}^{\times}$			
			$\alpha_1 = 1$	α_2	α_3	$\cdots$
$y = \beta \in \overline{\mathbb{Q}}$	$\beta_0 = 0$		$F_m^{(l)}(1)$	$F_m^{(l)}(\alpha_2)$	$F_m^{(l)}(\alpha_3)$	$\cdots$
	$\beta \in \mathcal{B}$	β_1	$G^{(m)}(\beta_1)$	$\Theta^{(l,m)}(\alpha_2, \beta_1)$	$\Theta^{(l,m)}(\alpha_3, \beta_1)$	$\cdots$
		β_2	$G^{(m)}(\beta_2)$	$\Theta^{(l,m)}(\alpha_2, \beta_2)$	$\Theta^{(l,m)}(\alpha_3, \beta_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	$\beta \in \overline{\mathbb{Q}}^{\times} \setminus \mathcal{B}$ $= \{a^{-R_k} \mid k \geq 0\}$	β'_1	$G^{(m)}(\beta'_1)$	$\Theta^{(l,m)}(\alpha_2, \beta'_1)$	$\Theta^{(l,m)}(\alpha_3, \beta'_1)$	$\cdots$
		β'_2	$G^{(m)}(\beta'_2)$	$\Theta^{(l,m)}(\alpha_2, \beta'_2)$	$\Theta^{(l,m)}(\alpha_3, \beta'_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.3: The numbers treated in Theorem 1.3.1

Extracting the first row and the first column from Table 1.3, we obtain the following Corollary 1.3.2 and Table 1.4.

Corollary 1.3.2. *Let v be ∞ or a prime number and $\{R_k\}_{k\geq 0}$ as in Theorem 1.3.1. Then the infinite subset*

$$\left\{F_m^{(l)}(\alpha) \;\middle|\; l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^\times\right\} \bigcup \left\{G^{(m)}(\beta) \;\middle|\; m \geq N_\beta,\ \beta \in \overline{\mathbb{Q}}^\times\right\}$$

of $\overline{\mathbb{Q}}_v$ is algebraically independent.

			$x = \alpha \in \overline{\mathbb{Q}}^\times$			
			$\alpha_1 = 1$	α_2	α_3	$\cdots$
$y = \beta \in \overline{\mathbb{Q}}$	$\beta_0 = 0$		$F_m^{(l)}(1)$	$F_m^{(l)}(\alpha_2)$	$F_m^{(l)}(\alpha_3)$	$\cdots$
	$\beta \in \mathcal{B}$	β_1	$G^{(m)}(\beta_1)$	$\Theta^{(l,m)}(\alpha_2, \beta_1)$	$\Theta^{(l,m)}(\alpha_3, \beta_1)$	$\cdots$
		β_2	$G^{(m)}(\beta_2)$	$\Theta^{(l,m)}(\alpha_2, \beta_2)$	$\Theta^{(l,m)}(\alpha_3, \beta_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	$\beta \in \overline{\mathbb{Q}}^\times \setminus \mathcal{B}$ $= \{a^{-R_k} \mid k \geq 0\}$	β'_1	$G^{(m)}(\beta'_1)$	$\Theta^{(l,m)}(\alpha_2, \beta'_1)$	$\Theta^{(l,m)}(\alpha_3, \beta'_1)$	$\cdots$
		β'_2	$G^{(m)}(\beta'_2)$	$\Theta^{(l,m)}(\alpha_2, \beta'_2)$	$\Theta^{(l,m)}(\alpha_3, \beta'_2)$	$\cdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.4: The numbers treated in Corollary 1.3.2

Corollary 1.3.2 refines Theorems 1.2.14 and 1.2.15, namely we obtain the algebraic independency of the union of the infinite sets treated in the two theorems as well as the nonzero derivatives at the zeros of the infinite product $G(y)$.

As another corollary to Theorem 1.3.1, we obtain an entire function of two variables which possesses Property 1.1.9 for its partial derivatives. Such a function is defined by

$$\Xi(x, y) := \frac{\partial \Theta}{\partial y}(x, y) = G(y) \sum_{\substack{k_1, k_2 \geq 0, \\ k_1 \neq k_2}} \frac{-a^{R_{k_1} + R_{k_2}} x^{k_1}}{(1 - a^{R_{k_1}} y)(1 - a^{R_{k_2}} y)}.$$

Corollary 1.3.3. *Let v be ∞ or a prime number and $\{R_k\}_{k\geq 0}$ as in Theorem 1.3.1. Assume in addition that $\{R_k\}_{k\geq 0}$ is strictly increasing. Then the infinite subset*

$$\left\{\frac{\partial^{l+m}\Xi}{\partial x^l \partial y^m}(\alpha, \beta) \;\middle|\; l \geq 0,\ m \geq 0,\ \alpha \in \overline{\mathbb{Q}}^\times,\ \beta \in \overline{\mathbb{Q}}^\times\right\}$$

of $\overline{\mathbb{Q}}_v$ is algebraically independent.

Remark 1.3.4. Corollary 1.3.3 establishes the algebraic independence of the partial derivatives of the entire function $\Xi(x, y)$ of two variables at any distinct algebraic points. On the other hand, there are known results on the algebraic independence of the values of three-variable functions without their partial derivatives. For example, let $\{G_k\}_{k\geq 0}$ be the generalized Fibonacci numbers defined by

$$G_0 = 0, \quad G_1 = 1, \quad G_{k+2} = bG_{k+1} + G_k \quad (k \geq 0),$$

where b is a positive integer. We consider here the complex function

$$T(x, y, z) := \sum_{k=1}^{\infty} \frac{x^k z^{G_1+G_2+\cdots+G_k}}{(1 - yz^{G_1})(1 - yz^{G_2}) \cdots (1 - yz^{G_k})} = \sum_{k=1}^{\infty} \prod_{l=1}^{k} \frac{xz^{G_l}}{1 - yz^{G_l}}$$

of three variables, which converges in the union of the two domains

$$\{(x, y, z) \in \mathbb{C}^3 \mid |z| < 1,\ 1 - yz^{G_k} \neq 0 \text{ for any } k \geq 1\} \tag{1.3.4}$$

and

$$\{(x, y, z) \in \mathbb{C}^3 \mid |x| < |y|,\ 1 < |z|,\ 1 - yz^{G_k} \neq 0 \text{ for any } k \geq 1\}.$$

Let D be the subdomain of (1.3.4) defined by $D := \mathbb{C} \times \{|y| \leq 1\} \times \{|z| < 1\}$. For the function $T(x, y, z)$, Tanaka [32, Example 1] proved that the infinite set

$$\{T(\alpha, \beta, a) \mid (\alpha, \beta, a) \in (\overline{\mathbb{Q}}^{\times})^3 \cap D\} = \{T(\alpha, \beta, a) \mid \alpha, \beta, a \in \overline{\mathbb{Q}}^{\times},\ |\beta| \leq 1,\ |a| < 1\}$$

is algebraically independent. (More generally, he [32, Theorem] obtained a necessary and sufficient condition for the values at algebraic points of a certain class of three-variable functions including $T(x, y, z)$ to be algebraically independent.) This is a remarkable result that establishes the algebraic independence of the infinite set consisting of the values of the three-variable function $T(x, y, z)$ at distinct algebraic points; however, it should be noted that the domains of definition of y and z are restricted.

Let $\mathcal{B}$ be as in Theorem 1.2.15, namely

$$\mathcal{B} := \overline{\mathbb{Q}}^{\times} \setminus \{a^{-R_k} \mid k \geq 0\} = \{\beta \in \overline{\mathbb{Q}}^{\times} \mid G(\beta) \neq 0\}.$$

We note that $N_\beta = 0$ if and only if $\beta \in \mathcal{B}$. In Chapter 2 we deduce Theorem 1.3.1 from the following:

Theorem 1.3.5 (Ide [7, Theorem 1.11]). *Let v be ∞ or a prime number and $\{R_k\}_{k\geq 0}$ as in Theorem 1.3.1. Then the infinite subset*

$$\left\{F_m^{(l)}(\alpha) \;\middle|\; l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times}\right\} \bigcup \{G(\beta) \mid \beta \in \mathcal{B}\}$$
$$\bigcup \left\{\frac{\partial^{l+m} H}{\partial x^l \partial y^m}(\alpha, \beta) \;\middle|\; l \geq 0,\ m \geq 0,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \mathcal{B}\right\}$$

of $\overline{\mathbb{Q}}_v$ is algebraically independent.

We prove Theorem 1.3.5 in Chapter 4. In what follows, we consider only the case of complex numbers. In the second main theorem of this thesis, we merge the two cases (A) and (B) described in Section 1.2 by extending Theorem 1.3.5. Let $a_1, \ldots, a_r$ be algebraic numbers with $0 < |a_i| < 1$ $(1 \leq i \leq r)$. Inspired by Theorem 1.2.9, we define

$$F_{i,m}(x) := \mathcal{F}(\{R_k\}; x, a_i^m) = \sum_{k=0}^{\infty} a_i^{mR_k} x^k \quad (m = 1, 2, \ldots), \tag{1.3.5}$$

$$G_i(y) := \mathcal{G}(\{R_k\}; y, a_i) = \prod_{k=0}^{\infty} (1 - a_i^{R_k} y), \tag{1.3.6}$$

$$H_i(x, y) := \mathcal{H}(\{R_k\}; x, y, a_i) = \sum_{k=0}^{\infty} \frac{a_i^{R_k} x^k}{1 - a_i^{R_k} y}, \tag{1.3.7}$$

and

$$\mathcal{B}_i := \overline{\mathbb{Q}}^{\times} \setminus \{a_i^{-R_k} \mid k \geq 0\} = \{\beta \in \overline{\mathbb{Q}}^{\times} \mid G_i(\beta) \neq 0\}$$

for each i $(1 \leq i \leq r)$. Then the infinite set

$$\mathcal{T}_i := \left\{F_{i,m}^{(l)}(\alpha) \;\middle|\; l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times}\right\} \bigcup \{G_i(\beta) \mid \beta \in \mathcal{B}_i\}$$
$$\bigcup \left\{\frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha, \beta) \;\middle|\; l \geq 0,\ m \geq 0,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \mathcal{B}_i\right\}$$

is algebraically independent for each i $(1 \leq i \leq r)$ by Theorem 1.3.5. However, Theorem 1.3.5 provides no information on the algebraic independence of the union

$$\begin{aligned}\mathcal{T} :=& \bigcup_{i=1}^{r} \mathcal{T}_i \\ =& \left\{F_{i,m}^{(l)}(\alpha) \;\middle|\; 1 \leq i \leq r,\ l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times}\right\} \bigcup \{G_i(\beta) \mid 1 \leq i \leq r,\ \beta \in \mathcal{B}_i\} \\ & \bigcup \left\{\frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha, \beta) \;\middle|\; 1 \leq i \leq r,\ l \geq 0,\ m \geq 0,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \mathcal{B}_i\right\}.\end{aligned}$$

The main aim of this study is to obtain the necessary and sufficient condition on algebraic numbers $a_1, \ldots, a_r$ for the infinite set $\mathcal{T}$ to be algebraically independent.

Suppose here that a_i and a_j are multiplicatively dependent for some i, j with $1 \leq i < j \leq r$. Then it is easily seen that the infinite set $\mathcal{T}$ is algebraically dependent. Indeed, assuming $a_1^{d_1} = a_2^{d_2}$ for some positive integers d_1 and d_2, we have $F_{1,d_1}(\alpha) = \sum_{k=0}^{\infty} a_1^{d_1 R_k} \alpha^k = \sum_{k=0}^{\infty} a_2^{d_2 R_k} \alpha^k = F_{2,d_2}(\alpha)$ for any nonzero algebraic number α, which implies that the infinite set $\mathcal{T}$ is algebraically dependent. In addition, algebraic relations also appear respectively among the values at several algebraic numbers of $G_i(y)$ $(i = 1, 2)$ and those of $H_i(1, y)$ $(i = 1, 2)$ in this case. To see this, let ζ_i $(i = 1, 2)$ be a primitive d_i-th root of unity. Then we have

$$\prod_{i=0}^{d_1-1} G_1(\zeta_1^i y^{d_2}) = \prod_{k=0}^{\infty} (1 - a_1^{d_1 R_k} y^{d_1 d_2}) = \prod_{k=0}^{\infty} (1 - a_2^{d_2 R_k} y^{d_1 d_2}) = \prod_{j=0}^{d_2-1} G_2(\zeta_2^j y^{d_1}).$$

Taking the logarithmic derivative of this equation and using the relations $H_i(1, y) = -G_i'(y)/G_i(y)$ $(i = 1, 2)$, we obtain

$$\sum_{i=0}^{d_1-1} d_2 \zeta_1^i y^{d_2} H_1(1, \zeta_1^i y^{d_2}) = \sum_{j=0}^{d_2-1} d_1 \zeta_2^j y^{d_1} H_2(1, \zeta_2^j y^{d_1}).$$

Hence, if a nonzero algebraic number β satisfies $\zeta_1^i \beta^{d_2} \in \mathcal{B}_1$ $(0 \leq i \leq d_1 - 1)$ and $\zeta_2^j \beta^{d_1} \in \mathcal{B}_2$ $(0 \leq j \leq d_2 - 1)$, then the $d_1 + d_2$ nonzero elements $G_1(\zeta_1^i \beta^{d_2}), G_2(\zeta_2^j \beta^{d_1})$ $(0 \leq i \leq d_1 - 1,\ 0 \leq j \leq d_2 - 1)$ of $\mathcal{T}$ are algebraically dependent and so are the $d_1 + d_2$ elements $H_1(1, \zeta_1^i \beta^{d_2}), H_2(1, \zeta_2^j \beta^{d_1})$ $(0 \leq i \leq d_1 - 1,\ 0 \leq j \leq d_2 - 1)$ of $\mathcal{T}$. Therefore the infinite set $\mathcal{T}$ is algebraically independent only if $a_1, \ldots, a_r$ are pairwise multiplicatively independent. The second main theorem of this thesis is the following:

Theorem 1.3.6 (Ide [8, Theorem 1.8])**.** *Let $\{R_k\}_{k \geq 0}$ be a linear recurrence of non-negative integers satisfying* (1.2.2)*. Suppose that $\{R_k\}_{k \geq 0}$ satisfies Condition 1.2.5. Assume that $\{R_k\}_{k \geq 0}$ is not a geometric progression. Then the infinite subset*

$$\mathcal{T} = \left\{ F_{i,m}^{(l)}(\alpha) \;\middle|\; 1 \leq i \leq r,\ l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \bigcup \{ G_i(\beta) \mid 1 \leq i \leq r,\ \beta \in \mathcal{B}_i \}$$
$$\bigcup \left\{ \frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha, \beta) \;\middle|\; 1 \leq i \leq r,\ l \geq 0,\ m \geq 0,\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \mathcal{B}_i \right\}$$

of $\mathbb{C}$ is algebraically independent if and only if $a_1, \ldots, a_r$ are pairwise multiplicatively independent.

In Chapter 4, we prove the *if* part of Theorem 1.3.6, namely the following:

Theorem 1.3.7 (Ide [8, Theorem 1.9])**.** *Let $\{R_k\}_{k\geq 0}$ be as in Theorem 1.3.6. Assume that $a_1, \ldots, a_r$ are pairwise multiplicatively independent. Then the infinite subset $\mathcal{T}$ of $\mathbb{C}$ is algebraically independent.*

In order to construct a family of entire functions of two variables having Property 1.1.10, we define

$$\Theta_i(x, y) \coloneqq G_i(y)H_i(x, y) = \sum_{k=0}^{\infty} a_i^{R_k} x^k \prod_{\substack{k'=0, \\ k'\neq k}}^{\infty} (1 - a_i^{R_{k'}} y) \tag{1.3.8}$$

and

$$N_{i,\beta} \coloneqq \#\{k \geq 0 \mid a_i^{-R_k} = \beta\} = \operatorname*{ord}_{y=\beta} G_i(y)$$

for each i ($1 \leq i \leq r$) and for each nonzero algebraic number β. Applying Theorem 1.3.7, we prove in Chapter 2 the following theorem, which is an extension of Theorem 1.3.1 in the case of complex numbers.

Theorem 1.3.8 (Ide [8, Theorem 1.12])**.** *Let $\{R_k\}_{k\geq 0}$ and $a_1, \ldots, a_r$ be as in Theorem 1.3.7. Then the infinite subset*

$$\begin{aligned} &\left\{ F_{i,m}^{(l)}(\alpha) \,\middle|\, 1 \leq i \leq r,\ l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \\ \cup &\left\{ G_i^{(N_{i,\beta})}(\beta) \,\middle|\, 1 \leq i \leq r,\ \beta \in \overline{\mathbb{Q}}^{\times} \right\} \\ \cup &\left\{ \frac{\partial^{l+m}\Theta_i}{\partial x^l \partial y^m}(\alpha, \beta) \,\middle|\, 1 \leq i \leq r,\ l \geq 0,\ m \geq N_{i,\beta},\ \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \overline{\mathbb{Q}}^{\times} \right\} \end{aligned}$$

of $\mathbb{C}$ is algebraically independent.

Similarly to Corollary 1.3.2, we can deduce the following from Theorem 1.3.8.

Corollary 1.3.9. *Let $\{R_k\}_{k\geq 0}$ and $a_1, \ldots, a_r$ be as in Theorem 1.3.7. Then the infinite subset*

$$\begin{aligned} &\left\{ F_{i,m}^{(l)}(\alpha) \,\middle|\, 1 \leq i \leq r,\ l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \\ \cup &\left\{ G_i^{(m)}(\beta) \,\middle|\, 1 \leq i \leq r,\ m \geq N_{i,\beta},\ \beta \in \overline{\mathbb{Q}}^{\times} \right\} \end{aligned}$$

of $\mathbb{C}$ is algebraically independent.

Moreover, similarly to Corollary 1.3.3, letting

$$\Xi_i(x,y) := \frac{\partial \Theta_i}{\partial y}(x,y) = G_i(y) \sum_{\substack{k_1,k_2\geq 0,\\ k_1\neq k_2}} \frac{-a_i^{R_{k_1}+R_{k_2}} x^{k_1}}{(1-a_i^{R_{k_1}}y)(1-a_i^{R_{k_2}}y)}$$

for each i ($1 \leq i \leq r$), we obtain the following:

Corollary 1.3.10. *Let $\{R_k\}_{k\geq 0}$ and $a_1,\ldots,a_r$ be as in Theorem 1.3.7. Assume in addition that $\{R_k\}_{k\geq 0}$ is strictly increasing. Then the infinite subset*

$$\left\{ \frac{\partial^{l+m}\Xi_i}{\partial x^l \partial y^m}(\alpha,\beta) \;\middle|\; 1\leq i\leq r,\ l\geq 0,\ m\geq 0,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times} \right\}$$

of $\mathbb{C}$ is algebraically independent.

Using Corollary 1.3.10, we exhibit a concrete example of an infinite family of entire functions of two variables having Property 1.1.10.

Example 1.3.11. Let $\{F_k\}_{k\geq 0}$ be the Fibonacci numbers as in Example 1.2.4. For any nonnegative integer i, letting $a_i = 2^{-1}3^{-i}$ and regarding $\{F_{k+2}\}_{k\geq 0}$ as $\{R_k\}_{k\geq 0}$, we define

$$\Xi_i(x,y) = \left(\prod_{k=2}^{\infty} (1-(2\cdot 3^i)^{-F_k} y) \right) \sum_{\substack{k_1,k_2\geq 2,\\ k_1\neq k_2}} \frac{-(2\cdot 3^i)^{-F_{k_1}-F_{k_2}} x^{k_1-2}}{(1-(2\cdot 3^i)^{-F_{k_1}}y)(1-(2\cdot 3^i)^{-F_{k_2}}y)}.$$

Then by Corollary 1.3.10 the infinite family $\{\Xi_i(x,y)\}_{i\geq 0}$ has Property 1.1.10, namely the infinite subset

$$\left\{ \frac{\partial^{l+m}\Xi_i}{\partial x^l \partial y^m}(\alpha,\beta) \;\middle|\; i\geq 0,\ l\geq 0,\ m\geq 0,\ \alpha\in\overline{\mathbb{Q}}^{\times},\ \beta\in\overline{\mathbb{Q}}^{\times} \right\}$$

of $\mathbb{C}$ is algebraically independent.

In the rest of this section we introduce another result on the algebraic independence of the values and the partial derivatives of $F_{i,m}(x), G_i(y)$, and $H_i(x,y)$. Assume here that none of a_i/a_j ($1\leq i<j\leq r$) is a root of unity. In this case, we cannot deduce the algebraic independency of the infinite set $\mathcal{T}$ itself since $a_1,\ldots,a_r$

are not always pairwise multiplicatively independent. On the other hand, we see by Theorem 1.2.9 that, if $1 \in \mathcal{B}_i$ $(1 \le i \le r)$, then the $3r$ elements $F_{i,1}(1) = \sum_{k=0}^{\infty} a_i^{R_k}$, $G_i(1) = \prod_{k=0}^{\infty}(1 - a_i^{R_k})$, and $H_i(1,1) = \sum_{k=0}^{\infty} a_i^{R_k}/(1 - a_i^{R_k})$ $(1 \le i \le r)$ of $\mathcal{T}$ are algebraically independent. Here we can extend this result to the following theorem, whose proof will be provided in Chapter 4.

Theorem 1.3.12 (Ide [8, Theorem 1.13])**.** *Let $\{R_k\}_{k\ge0}$ be as in Theorem 1.3.6. Assume that none of a_i/a_j $(1 \le i < j \le r)$ is a root of unity. Let m_0 be a positive integer. For each i $(1 \le i \le r)$, let β_i and β_i' be any elements of $\mathcal{B}_i$. Then the infinite subset*

$$\left\{ F_{i,m_0}^{(l)}(\alpha) \;\middle|\; 1 \le i \le r,\ l \ge 0,\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\} \bigcup \left\{ G_i(\beta_i) \mid 1 \le i \le r \right\}$$
$$\bigcup \left\{ \frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha, \beta_i') \;\middle|\; 1 \le i \le r,\ l \ge 0,\ m \ge 0,\ \alpha \in \overline{\mathbb{Q}}^{\times} \right\}$$

of $\mathcal{T}$ is algebraically independent.

This thesis is organized as follows. In Chapter 2, we establish Lemmas 2.1.1–2.1.3 and 2.2.1–2.2.2. The first three lemmas assert the existence of invertible linear relations among the values of a wider class of functions including those stated in Section 1.3, using which we can avoid the zeros of the infinite products $G_i(y)$. The latter two lemmas claim the existence of invertible algebraic relations among the above functions themselves, by which we can reduce the algebraic independency of the values and the partial derivatives of the entire functions $\Theta_i(x, y)$ to that of the Lambert type series $H_i(x, y)$. Using these lemmas and shifting the linear recurrence $\{R_k\}_{k\ge0}$ so as to avoid the zeros of the infinite products $G_i(y)$, we deduce Theorems 1.3.1 and 1.3.8 from Theorems 1.3.5 and 1.3.7, respectively. The proofs of Corollaries 1.3.2, 1.3.3, 1.3.9, and 1.3.10 are also provided in Chapter 2. In Chapter 3, we establish a criterion for the algebraic independence of the values of Mahler functions corresponding to those stated in Theorems 1.3.5, 1.3.7, and 1.3.12. Our criterion, which is valid not only in the complex case but also in the p-adic case, includes Nishioka's criterion and a special case of Kubota's one. In the last chapter, using our criterion, we reduce the negations of Theorems 1.3.5, 1.3.7, and 1.3.12

to the linear dependence or the multiplicative dependence of the Mahler functions above. Applying Tanaka's results on the nonexistence of nontrivial rational function solutions of certain types of functional equations, we obtain a linear dependence relation of certain power series of one variable, which easily leads to a contradiction in the case of Theorem 1.3.5. In the cases of Theorems 1.3.7 and 1.3.12, the situation on the linear dependence becomes more complicated. We prove Theorem 1.3.12 by showing directly that a term of sufficiently high order never vanishes in the linear combination. On the other hand, in the proof of Theorem 1.3.7, using the precise assumption on the algebraic numbers $a_1, \ldots, a_r$, we can reduce the situation on the linear dependence to that similar to Theorem 1.3.5. This observation allows us to establish the algebraic independency of the infinite set $\mathcal{T}$ larger than that treated in Theorem 1.3.12.

Chapter 2

Proofs of Theorems 1.3.1, 1.3.8, and their corollaries

2.1 Linear relations among the values of a wider class of functions

Let v be ∞ or a prime number. In this section and the next section we consider the functions $f(x)$, $g(y)$, $h(x,y)$, and $\theta(x,y)$ defined by

$$f(x) \coloneqq \sum_{k=0}^{\infty} a_k x^k, \qquad g(y) \coloneqq \prod_{k=0}^{\infty} (1 - a_k y), \qquad h(x,y) \coloneqq \sum_{k=0}^{\infty} \frac{a_k x^k}{1 - a_k y}, \tag{2.1.1}$$

and

$$\theta(x,y) \coloneqq g(y)h(x,y) = \sum_{k=0}^{\infty} a_k x^k \prod_{\substack{k'=0, \\ k' \neq k}}^{\infty} (1 - a_{k'} y), \tag{2.1.2}$$

where $\{a_k\}_{k \geq 0}$ is a sequence of algebraic numbers satisfying

$$\frac{1}{\limsup_{k \to \infty} |a_k|_v^{1/k}} =: r > 1. \tag{2.1.3}$$

Then $f(x)$, $g(y)$, $h(x,y)$, and $\theta(x,y)$ converge in $\{x \in \mathbb{C}_v \mid |x|_v < r\}$, in $\mathbb{C}_v$, in $\{x \in \mathbb{C}_v \mid |x|_v < r\} \times \{y \in \mathbb{C}_v \mid y \neq a_k^{-1} \text{ for any } k \geq 0 \text{ with } a_k \neq 0\}$, and in $\{x \in \mathbb{C}_v \mid |x|_v < r\} \times \mathbb{C}_v$, respectively. This framework includes the case of the functions $F_m(x)$, $G(y)$, $H(x,y)$, and $\Theta(x,y)$ defined by (1.2.6), (1.2.7), (1.2.8), and (1.3.1) as the special case of $a_k = a^{mR_k}$ or more simply that of $a_k = a^{R_k}$. In the same way as (1.3.3), we see that

$$\frac{\partial^m \theta}{\partial y^m}(1, y) = -g^{(m+1)}(y) \tag{2.1.4}$$

for any $m \geq 0$. Let α and β be any nonzero algebraic numbers with $|\alpha|_v < r$. Similarly to the number N_β defined by (1.2.9), we define the number n_β by

$$n_\beta := \#\{k \geq 0 \mid a_k = \beta^{-1}\} = \operatorname*{ord}_{y=\beta} g(y).$$

In this section we establish explicit invertible linear relations between the values and the (partial) derivatives of the functions above at $(x, y) = (\alpha, \beta)$ and those defined by a shifted sequence of $\{a_k\}_{k\geq 0}$.

Since $a_k \to 0$ as $k \to \infty$ by (2.1.3), there exists a sufficiently large integer k_0 depending on β such that $1 - a_k\beta \neq 0$ for all $k \geq k_0$. Put $\widetilde{a}_k := a_{k+k_0}$ $(k \geq 0)$. Let $\widetilde{f}(x)$, $\widetilde{g}(y)$, $\widetilde{h}(x, y)$, and $\widetilde{\theta}(x, y)$ be the functions given respectively by (2.1.1) and (2.1.2) with the sequence $\{\widetilde{a}_k\}_{k\geq 0}$ in place of $\{a_k\}_{k\geq 0}$. For each positive integer L, let $\mathcal{M}_L(\overline{\mathbb{Q}})$ be the multiplicative group of $L \times L$ lower triangular matrices with entries in $\overline{\mathbb{Q}}$ whose diagonal entries are nonzero. We note that, if $A \in \mathcal{M}_{L_1}$ and $B \in \mathcal{M}_{L_2}$, then the Kronecker product $A \otimes B$ belongs to $\mathcal{M}_{L_1 L_2}$. The following three lemmas, especially the latter two, play a crucial role in the proofs of Theorems 1.3.1 and 1.3.8.

Lemma 2.1.1 (Ide [7, Proof of Theorem 2.1])**.** *Let L be a nonnegative integer. Then there exists $A_\alpha \in \mathcal{M}_{L+1}(\overline{\mathbb{Q}})$ depending on α such that*

$$\begin{pmatrix} f(\alpha) \\ f'(\alpha) \\ \vdots \\ f^{(L)}(\alpha) \end{pmatrix} \equiv A_\alpha \begin{pmatrix} \widetilde{f}(\alpha) \\ \widetilde{f}'(\alpha) \\ \vdots \\ \widetilde{f}^{(L)}(\alpha) \end{pmatrix} \pmod{\overline{\mathbb{Q}}^{L+1}}.$$

Lemma 2.1.2 (Ide [7, Proof of Theorem 2.1])**.** *Let L be a nonnegative integer. Then there exists $B_\beta \in \mathcal{M}_{L+1}(\overline{\mathbb{Q}})$ depending on β such that*

$$\begin{pmatrix} g^{(n_\beta)}(\beta) \\ g^{(1+n_\beta)}(\beta) \\ \vdots \\ g^{(L+n_\beta)}(\beta) \end{pmatrix} = B_\beta \begin{pmatrix} \widetilde{g}(\beta) \\ \widetilde{g}'(\beta) \\ \vdots \\ \widetilde{g}^{(L)}(\beta) \end{pmatrix}.$$

Lemma 2.1.3 (Ide [7, Proof of Theorem 2.1])**.** *Let L be a nonnegative integer. Let $\boldsymbol{\theta}$ and $\widetilde{\boldsymbol{\theta}}$ be column vectors given respectively by sorting the values $\partial^{l+m}\theta/\partial x^l \partial y^m(\alpha,\beta)$ and $\partial^{l+m}\widetilde{\theta}/\partial x^l \partial y^m(\alpha,\beta)$ in the ascending lexicographical order of $(l,m) \in \{0,\dots,L\}^2$ such that*

$$(0,0) < (0,1) < \cdots < (0,L) < (1,0) < \cdots < (L,L).$$

Then

$$\boldsymbol{\theta} \equiv (A_\alpha \otimes B_\beta)\widetilde{\boldsymbol{\theta}} \pmod{W^{(L+1)^2}}, \tag{2.1.5}$$

where A_α and B_β are as in Lemmas 2.1.1 and 2.1.2, respectively, and W is the $\overline{\mathbb{Q}}$-vector space generated by $\{\widetilde{g}^{(m)}(\beta) \mid 0 \le m \le L+1\}$. In particular, for the case of $\alpha = 1$, let $\boldsymbol{\theta}'$ and $\widetilde{\boldsymbol{\theta}}'$ be column vectors given respectively by sorting the values $\partial^{l+m}\theta/\partial x^l \partial y^m(1,\beta)$ and $\partial^{l+m}\widetilde{\theta}/\partial x^l \partial y^m(1,\beta)$ in the ascending lexicographical order of $(l,m) \in \{1,\dots,L\} \times \{0,\dots,L\}$ such that

$$(1,0) < (1,1) < \cdots < (1,L) < (2,0) < \cdots < (L,L).$$

Then

$$\boldsymbol{\theta}' \equiv (A'_1 \otimes B_\beta)\widetilde{\boldsymbol{\theta}}' \pmod{W^{L(L+1)}}, \tag{2.1.6}$$

where $A'_1 \in \mathcal{M}_L(\overline{\mathbb{Q}})$ is the submatrix of A_1 given by

$$A_1 = \left(\begin{array}{c|c} 1 & O_{1,L} \\ \hline * & A'_1 \end{array}\right).$$

Remark 2.1.4. For each m ($0 \le m \le L+1$), we see by Lemma 2.1.2 that $g^{(m+n_\beta)}(\beta)$ is represented as a linear combination of $\widetilde{g}^{(\mu)}(\beta)$ ($0 \le \mu \le m$) and conversely $\widetilde{g}^{(m)}(\beta)$ is represented as that of $g^{(\mu+n_\beta)}(\beta)$ ($0 \le \mu \le m$). Hence $\{g^{(m+n_\beta)}(\beta) \mid 0 \le m \le L+1\}$ and $\{\widetilde{g}^{(m)}(\beta) \mid 0 \le m \le L+1\}$ generate the same $\overline{\mathbb{Q}}$-vector space W in Lemma 2.1.3.

Proof of Lemma 2.1.1. Let

$$R(x) := x^{k_0}. \tag{2.1.7}$$

Since

$$f(x) = \sum_{k=k_0}^{\infty} a_k x^k + \sum_{k=0}^{k_0-1} a_k x^k = R(x)\widetilde{f}(x) + \sum_{k=0}^{k_0-1} a_k x^k,$$

we have

$$f^{(l)}(\alpha) \equiv \sum_{h=0}^{l} \binom{l}{h} R^{(l-h)}(\alpha) \widetilde{f}^{(h)}(\alpha) \pmod{\overline{\mathbb{Q}}}$$

for any l $(0 \le l \le L)$. Hence, letting

$$A_\alpha := \begin{pmatrix} \alpha^{k_0} & & & \\ R'(\alpha) & \alpha^{k_0} & & \text{\Large 0} \\ \vdots & \ddots & \ddots & \\ R^{(L)}(\alpha) & \cdots & LR'(\alpha) & \alpha^{k_0} \end{pmatrix},$$

we obtain the lemma. □

Proof of Lemma 2.1.2. We define polynomials $P_\beta(y)$ and $Q_\beta(y)$ with algebraic coefficients by

$$P_\beta(y) := (1-\beta^{-1}y)^{n_\beta}, \qquad Q_\beta(y) := \prod_{\substack{k=0,\\ a_k \neq \beta^{-1}}}^{k_0-1} (1-a_k y). \tag{2.1.8}$$

Let

$$p_\beta := P_\beta^{(n_\beta)}(y), \qquad q_\beta := Q_\beta(\beta). \tag{2.1.9}$$

Then p_β and q_β are nonzero algebraic numbers. Since

$$g(y) = \prod_{k=0}^{k_0-1}(1-a_k y) \times \prod_{k=k_0}^{\infty}(1-a_k y) = P_\beta(y) Q_\beta(y) \widetilde{g}(y),$$

we see that, for any m $(0 \le m \le L)$,

$$g^{(m+n_\beta)}(\beta) = \sum_{h=0}^{m} \begin{pmatrix} & m+n_\beta & \\ n_\beta & m-h & h \end{pmatrix} p_\beta Q_\beta^{(m-h)}(\beta) \widetilde{g}^{(h)}(\beta).$$

Hence, letting

$$B_\beta := \begin{pmatrix} p_\beta q_\beta & & & \\ (1+n_\beta)p_\beta Q'_\beta(\beta) & (1+n_\beta)p_\beta q_\beta & & \text{\Large 0} \\ \vdots & \ddots & \ddots & \\ \binom{L+n_\beta}{L} p_\beta Q_\beta^{(L)}(\beta) & \cdots & \left(\begin{smallmatrix} & L+n_\beta & \\ n_\beta & 1 & L-1 \end{smallmatrix}\right) p_\beta Q'_\beta(\beta) & \binom{L+n_\beta}{L} p_\beta q_\beta \end{pmatrix},$$

we obtain the lemma. □

Proof of Lemma 2.1.3. We define

$$U_\beta(y) \coloneqq (1-\beta^{-1}y)^{\max\{n_\beta-1,0\}} \in \overline{\mathbb{Q}}[y]$$

and

$$V_\beta(x,y) \coloneqq \left(\sum_{k=0}^{k_0-1} a_k x^k \prod_{\substack{k'=0,\\ k'\neq k}}^{k_0-1} (1-a_{k'}y)\right) U_\beta(y)^{-1}.$$

Since $U_\beta(y)$ is a common divisor of $\prod_{k'=0,\ k'\neq k}^{k_0-1}(1-a_{k'}y)$ $(0 \le k \le k_0-1)$, we see that $V_\beta(x,y) \in \overline{\mathbb{Q}}[x,y]$. Let $R(x)$, $P_\beta(y)$, and $Q_\beta(y)$ be as in (2.1.7) and (2.1.8). Noting that $P_\beta(y)Q_\beta(y)$ and $U_\beta(y)V_\beta(x,y)$ do not depend on β, we see that

$$\theta(x,y) = R(x)P_\beta(y)Q_\beta(y)\widetilde{\theta}(x,y) + U_\beta(y)V_\beta(x,y)\widetilde{g}(y).$$

Since $\max\{n_\beta,1\}+\min\{1,n_\beta\} = n_\beta+1$ and so $n_\beta - \max\{n_\beta-1,0\} = \min\{1,n_\beta\}$, we have

$$\begin{aligned}
&\frac{\partial^{l+m+n_\beta}\theta}{\partial x^l \partial y^{m+n_\beta}}(\alpha,\beta)\\
&= \sum_{h_1=0}^{l} \binom{l}{h_1} R^{(l-h_1)}(\alpha) \sum_{h_2=0}^{m} \binom{m+n_\beta}{n_\beta\ m-h_2\ h_2} p_\beta Q_\beta^{(m-h_2)}(\beta) \frac{\partial^{h_1+h_2}\widetilde{\theta}}{\partial x^{h_1}\partial y^{h_2}}(\alpha,\beta)\\
&\quad + \sum_{h_3=0}^{m+\min\{1,n_\beta\}} \left(\binom{m+n_\beta}{\max\{n_\beta-1,0\}\quad m+\min\{1,n_\beta\}-h_3\quad h_3} \right.\\
&\qquad\qquad \left. \times u_\beta \frac{\partial^{l+m+\min\{1,n_\beta\}-h_3} V_\beta}{\partial x^l \partial y^{m+\min\{1,n_\beta\}-h_3}}(\alpha,\beta)\widetilde{g}^{(h_3)}(\beta)\right)
\end{aligned} \tag{2.1.10}$$

for any l $(0 \le l \le L)$ and m $(0 \le m \le L)$, where $p_\beta \in \overline{\mathbb{Q}}^\times$ is defined by (2.1.9) and $u_\beta \coloneqq U_\beta^{(\max\{n_\beta-1,0\})}(y) \in \overline{\mathbb{Q}}^\times$. Hence we obtain the linear relation (2.1.5).

Considering the case where $\alpha = 1$ in (2.1.10), we see by (2.1.4) that

$$\begin{aligned}
&\frac{\partial^{l+m+n_\beta}\theta}{\partial x^l \partial y^{m+n_\beta}}(1,\beta) \\
&= \sum_{h_1=1}^{l}\binom{l}{h_1}R^{(l-h_1)}(1)\sum_{h_2=0}^{m}\binom{m+n_\beta}{n_\beta\ m-h_2\ h_2}p_\beta Q_\beta^{(m-h_2)}(\beta)\frac{\partial^{h_1+h_2}\widetilde{\theta}}{\partial x^{h_1}\partial y^{h_2}}(1,\beta) \\
&\quad + \sum_{h_3=0}^{m+\min\{1,n_\beta\}}\left(\binom{m+n_\beta}{\max\{n_\beta-1,0\}\quad m+\min\{1,n_\beta\}-h_3\quad h_3}\right. \\
&\qquad\qquad \left.\times u_\beta \frac{\partial^{l+m+\min\{1,n_\beta\}-h_3}V_\beta}{\partial x^l \partial y^{m+\min\{1,n_\beta\}-h_3}}(1,\beta)\widetilde{g}^{(h_3)}(\beta)\right) \\
&\quad - R^{(l)}(1)\sum_{h_4=0}^{m}\binom{m+n_\beta}{n_\beta\quad m-h_4\quad h_4}p_\beta Q_\beta^{(m-h_4)}(\beta)\widetilde{g}^{(h_4+1)}(\beta)
\end{aligned}$$

for any l $(1 \le l \le L)$ and m $(0 \le m \le L)$, which implies the linear relation (2.1.6). □

2.2 Algebraic relations among the functions themselves

Next we provide invertible algebraic relations among the functions $g(y)$, $h(x,y)$, and $\theta(x,y)$ themselves defined by (2.1.1) and (2.1.2). In this section we write $\varphi^{(l,m)}(x,y) := \partial^{l+m}\varphi/\partial x^l \partial y^m(x,y)$ for any analytic function $\varphi(x,y)$ and nonnegative integers l, m. For each positive integer L, let $\mathcal{M}_L^*(\mathcal{R})$ be the multiplicative group of $L \times L$ lower triangular matrices with entries in the commutative ring $\mathcal{R}$ whose diagonal entries are 1's of $\mathcal{R}$.

Lemma 2.2.1. *Let l be a nonnegative integer and L a positive integer. Then there exists $C \in \mathcal{M}_{(L+1)^2}^*(\mathbb{Z}[\{g^{(m)}(y)/g(y) \mid 1 \le m \le L\}])$ such that*

$$\begin{pmatrix} \theta^{(l,0)}(x,y)/g(y) \\ \theta^{(l,1)}(x,y)/g(y) \\ \vdots \\ \theta^{(l,L)}(x,y)/g(y) \end{pmatrix} = C \begin{pmatrix} h^{(l,0)}(x,y) \\ h^{(l,1)}(x,y) \\ \vdots \\ h^{(l,L)}(x,y) \end{pmatrix}.$$

Proof. From (2.1.2), we see that

$$\frac{\theta^{(l,m)}(x,y)}{g(y)} = \sum_{\mu=0}^{m} \binom{m}{\mu} \frac{g^{(m-\mu)}(y)}{g(y)} h^{(l,\mu)}(x,y)$$

for any m ($0 \leq m \leq L$). Hence, letting

$$C \coloneqq \begin{pmatrix} 1 & & & \\ g'(y)/g(y) & 1 & & \text{\huge 0} \\ \vdots & \ddots & \ddots & \\ g^{(L)}(y)/g(y) & \cdots & Lg'(y)/g(y) & 1 \end{pmatrix},$$

we obtain the lemma. $\square$

Lemma 2.2.2 (Ide and Tanaka [9, Lemma 2.4]). *Let L be a positive integer. Then there exist $D_1 \in \mathcal{M}^*_{L+1}(\mathbb{Z}[\{h^{(0,m)}(1,y) \mid 0 \leq m \leq L-1\}])$ and $D_2 \in \mathcal{M}^*_{L+1}(\mathbb{Z}[\{g^{(m)}(y)/g(y) \mid 1 \leq m \leq L\}])$ such that*

$$\begin{pmatrix} -g'(y)/g(y) \\ -g''(y)/g(y) \\ \vdots \\ -g^{(L+1)}(y)/g(y) \end{pmatrix} = D_1 \begin{pmatrix} h(1,y) \\ h^{(0,1)}(1,y) \\ \vdots \\ h^{(0,L)}(1,y) \end{pmatrix}$$

and

$$\begin{pmatrix} h(1,y) \\ h^{(0,1)}(1,y) \\ \vdots \\ h^{(0,L)}(1,y) \end{pmatrix} = D_2 \begin{pmatrix} -g'(y)/g(y) \\ -g''(y)/g(y) \\ \vdots \\ -g^{(L+1)}(y)/g(y) \end{pmatrix}.$$

Proof. Since $g'(y) = -g(y)h(1,y)$ by (2.1.1), we see inductively that, for any $m \geq 0$,

$$g^{(m)}(y) = g(y)P_m\left(h(1,y), \ldots, h^{(0,m-1)}(1,y)\right), \tag{2.2.1}$$

where $P_0 \coloneqq 1$ and $P_m(X_0, \ldots, X_{m-1}) \in \mathbb{Z}[X_0, \ldots, X_{m-1}]$ ($m \geq 1$). Then again from the equation $-g'(y) = g(y)h(1,y)$, using the Leibniz rule and (2.2.1), we have

$$\begin{aligned} -\frac{g^{(m+1)}(y)}{g(y)} &= \sum_{\mu=0}^{m} \binom{m}{\mu} \frac{g^{(m-\mu)}(y)}{g(y)} h^{(0,\mu)}(1,y) \\ &= \sum_{\mu=0}^{m} \binom{m}{\mu} P_{m-\mu}\left(h(1,y), \ldots, h^{(0,m-\mu-1)}(1,y)\right) h^{(0,\mu)}(1,y) \end{aligned} \tag{2.2.2}$$

for any $m \geq 0$. On the other hand, we see inductively that, for any $m \geq 0$,

$$\frac{d^m}{dy^m}\left(\frac{1}{g(y)}\right) = \frac{1}{g(y)} Q_m\left(-\frac{g'(y)}{g(y)}, \dots, -\frac{g^{(m)}(y)}{g(y)}\right),$$

where $Q_0 := 1$ and $Q_m(Y_1, \dots, Y_m) \in \mathbb{Z}[Y_1, \dots, Y_m]$ $(m \geq 1)$. Since $h(1, y) = -g'(y)/g(y)$, we have

$$\begin{aligned} h^{(0,m)}(1, y) &= -\sum_{\mu=0}^{m} \binom{m}{\mu} g^{(\mu+1)}(y) \frac{d^{m-\mu}}{dy^{m-\mu}}\left(\frac{1}{g(y)}\right) \\ &= \sum_{\mu=0}^{m} \binom{m}{\mu} Q_{m-\mu}\left(-\frac{g'(y)}{g(y)}, \dots, -\frac{g^{(m-\mu)}(y)}{g(y)}\right)\left(-\frac{g^{(\mu+1)}(y)}{g(y)}\right) \end{aligned} \tag{2.2.3}$$

for any $m \geq 0$. Hence, letting

$$D_1 := \begin{pmatrix} 1 & & & 0 \\ P_1(h(1,y)) & 1 & & \\ \vdots & \ddots & \ddots & \\ P_L(h(1,y), \dots, h^{(0,L-1)}(1,y)) & \cdots & LP_1(h(1,y)) & 1 \end{pmatrix}$$

and

$$D_2 := \begin{pmatrix} 1 & & & 0 \\ Q_1(-g'(y)/g(y)) & 1 & & \\ \vdots & \ddots & \ddots & \\ Q_L(-g'(y)/g(y), \dots, -g^{(L)}(y)/g(y)) & \cdots & LQ_1(-g'(y)/g(y)) & 1 \end{pmatrix},$$

we obtain the equations in the lemma from (2.2.2) and (2.2.3). □

Remark 2.2.3. Here we consider the case of $a_k = a^{R_k}$ in Lemmas 2.2.1 and 2.2.2. Substituting $y = 0$ into the equation in Lemma 2.2.1 and using the relation

$$\frac{\partial^m H}{\partial y^m}(x, 0) = m! \sum_{k=0}^{\infty} \frac{a^{(m+1)R_k} x^k}{(1 - a^{R_k} y)^{m+1}}\Bigg|_{y=0} = m! \sum_{k=0}^{\infty} a^{(m+1)R_k} x^k = m! F_{m+1}(x), \tag{2.2.4}$$

we obtain

$$\begin{pmatrix} \Theta^{(l,0)}(x,0) \\ \Theta^{(l,1)}(x,0) \\ \vdots \\ \Theta^{(l,L)}(x,0) \end{pmatrix} = C' \begin{pmatrix} 0! F_1^{(l)}(x) \\ 1! F_2^{(l)}(x) \\ \vdots \\ L! F_{L+1}^{(l)}(x) \end{pmatrix},$$

where $C' \in \mathcal{M}^*_{L+1}(\mathbb{Z}[\{G^{(m)}(0) \mid 1 \leq m \leq L\}])$. Then by (1.3.3) we have $C' \in \mathcal{M}^*_{L+1}(\mathbb{Z}[\{\Theta^{(0,m)}(1,0) \mid 0 \leq m \leq L-1\}])$. On the other hand, by the first equation in Lemma 2.2.2 and (2.2.4) we see that

$$\begin{pmatrix} -G'(0) \\ -G''(0) \\ \vdots \\ -G^{(L)}(0) \end{pmatrix} = D'_1 \begin{pmatrix} 0!F_1(1) \\ 1!F_2(1) \\ \vdots \\ (L-1)!F_L(1) \end{pmatrix},$$

where $D'_1 \in \mathcal{M}^*_L(\mathbb{Z}[\{F_m(1) \mid 1 \leq m \leq L-1\}])$. Hence $C' \in \mathcal{M}^*_{L+1}(\mathbb{Z}[\{F_m(1) \mid 1 \leq m \leq L\}])$. Therefore the algebraic independency of the infinite set $\{\Theta^{(l,m)}(\alpha, 0) \mid l \geq 0,\ m \geq 0,\ \alpha \in \overline{\mathbb{Q}}^\times\}$ is equivalent to that of $\{F_m^{(l)}(\alpha) \mid l \geq 0,\ m \geq 1,\ \alpha \in \overline{\mathbb{Q}}^\times\}$, since their finite subsets $\{\Theta^{(l,m)}(\alpha_\lambda, 0) \mid 0 \leq l \leq L,\ 0 \leq m \leq L,\ 1 \leq \lambda \leq L\}$ and $\{F_m^{(l)}(\alpha_\lambda) \mid 0 \leq l \leq L,\ 1 \leq m \leq L+1,\ 1 \leq \lambda \leq L\}$ generate the same field over $\mathbb{Q}$ for any positive integer L and nonzero distinct L algebraic numbers $\alpha_1, \ldots, \alpha_L$ with $\alpha_1 = 1$.

2.3 Proofs of Theorem 1.3.8 and its corollaries

Using the lemmas proved in the previous two sections, we deduce Theorem 1.3.8 from Theorem 1.3.7.

Proof of Theorem 1.3.8. Let L be any positive integer and $\alpha_1, \ldots, \alpha_L$ any nonzero distinct L algebraic numbers with $\alpha_1 = 1$. Let $\beta_1, \ldots, \beta_L$ be any nonzero distinct L algebraic numbers. To simplify our notation, we write $N_{i,\mu} := N_{i,\beta_\mu}$ $(1 \leq i \leq r,\ 1 \leq \mu \leq L)$. It suffices to show that the finite set

$$\begin{aligned} &\left\{ F_{i,m}^{(l)}(\alpha_\lambda) \,\middle|\, 1 \leq i \leq r,\ 0 \leq l \leq L,\ 1 \leq m \leq L,\ 1 \leq \lambda \leq L \right\} \\ \bigcup &\left\{ G_i^{(N_{i,\mu})}(\beta_\mu) \,\middle|\, 1 \leq i \leq r,\ 1 \leq \mu \leq L \right\} \\ \bigcup &\left\{ \frac{\partial^{l+m+N_{i,\mu}}\Theta_i}{\partial x^l \partial y^{m+N_{i,\mu}}}(\alpha_\lambda, \beta_\mu) \,\middle|\, 1 \leq i \leq r,\ 0 \leq l, m \leq L,\ 1 \leq \lambda, \mu \leq L \right\} \end{aligned}$$

is algebraically independent. Using the lemmas in Sections 2.1 and 2.2, we reduce the algebraic independency of this set to that of other sets having the same cardinality.

By the equation (1.3.3), the algebraic independency of the above set is equivalent to that of

$$\left\{F_{i,m}^{(l)}(\alpha_\lambda)\;\middle|\; 1\le i\le r,\ 0\le l\le L,\ 1\le m\le L,\ 1\le\lambda\le L\right\}$$
$$\bigcup\left\{G_i^{(m+N_{i,\mu})}(\beta_\mu)\;\middle|\; 1\le i\le r,\ 0\le m\le L+1,\ 1\le\mu\le L\right\}$$
$$\bigcup\left\{\frac{\partial^{l+m+N_{i,\mu}}\Theta_i}{\partial x^l\partial y^{m+N_{i,\mu}}}(\alpha_\lambda,\beta_\mu)\;\middle|\;\begin{array}{l}1\le i\le r,\ l_0(\lambda)\le l\le L,\ 0\le m\le L,\\ 1\le\lambda,\mu\le L\end{array}\right\},$$

where

$$l_0(\lambda):=\begin{cases}1 & (\lambda=1),\\ 0 & (2\le\lambda\le L).\end{cases}$$

Since $R_k\to\infty$ as $k\to\infty$ by (1.2.4), there exists a sufficiently large integer k_0 such that $1-a_i^{R_k}\beta_\mu\neq 0$ ($1\le i\le r,\ 1\le\mu\le L$) for all $k\ge k_0$. Let $\widetilde{R}_k:=R_{k+k_0}$ ($k\ge 0$). Clearly, the linear recurrence $\{\widetilde{R}_k\}_{k\ge0}$ also satisfies Condition 1.2.5 and $\{\widetilde{R}_k\}_{k\ge0}$ is not a geometric progression. Let $\widetilde{F}_{i,m}(x)$, $\widetilde{G}_i(y)$, $\widetilde{H}_i(x,y)$, and $\widetilde{\Theta}_i(x,y)$ be the functions given respectively by (1.3.5), (1.3.6), (1.3.7), and (1.3.8) with $\{\widetilde{R}_k\}_{k\ge0}$ in place of $\{R_k\}_{k\ge0}$. Then Lemmas 2.1.1, 2.1.2, and 2.1.3, together with Remark 2.1.4, imply that the second set above and the set

$$\left\{\widetilde{F}_{i,m}^{(l)}(\alpha_\lambda)\;\middle|\; 1\le i\le r,\ 0\le l\le L,\ 1\le m\le L,\ 1\le\lambda\le L\right\}$$
$$\bigcup\left\{\widetilde{G}_i^{(m)}(\beta_\mu)\;\middle|\; 1\le i\le r,\ 0\le m\le L+1,\ 1\le\mu\le L\right\}$$
$$\bigcup\left\{\frac{\partial^{l+m}\widetilde{\Theta}_i}{\partial x^l\partial y^{m}}(\alpha_\lambda,\beta_\mu)\;\middle|\;\begin{array}{l}1\le i\le r,\ l_0(\lambda)\le l\le L,\ 0\le m\le L,\\ 1\le\lambda,\mu\le L\end{array}\right\}$$

generate the same $\overline{\mathbb{Q}}$-vector space. Hence the algebraic independency of these two sets are equivalent. Moreover, the algebraic independency of the third set is equivalent to that of

$$\left\{\widetilde{F}_{i,m}^{(l)}(\alpha_\lambda)\;\middle|\; 1\le i\le r,\ 0\le l\le L,\ 1\le m\le L,\ 1\le\lambda\le L\right\}$$
$$\bigcup\left\{\widetilde{G}_i^{(m)}(\beta_\mu)\;\middle|\; 1\le i\le r,\ 0\le m\le L+1,\ 1\le\mu\le L\right\}$$
$$\bigcup\left\{\frac{\partial^{l+m}\widetilde{H}_i}{\partial x^l\partial y^{m}}(\alpha_\lambda,\beta_\mu)\;\middle|\;\begin{array}{l}1\le i\le r,\ l_0(\lambda)\le l\le L,\ 0\le m\le L,\\ 1\le\lambda,\mu\le L\end{array}\right\}$$

since these two sets generate the same field over $\mathbb{Q}$ by Lemma 2.2.1. Furthermore, the algebraic independency of the fourth set is equivalent to that of

$$\left\{\widetilde{F}_{i,m}^{(l)}(\alpha_\lambda)\;\middle|\;1\le i\le r,\ 0\le l\le L,\ 1\le m\le L,\ 1\le\lambda\le L\right\}$$
$$\bigcup\left\{\widetilde{G}_i(\beta_\mu)\;\middle|\;1\le i\le r,\ 1\le\mu\le L\right\}$$
$$\bigcup\left\{\frac{\partial^{l+m}\widetilde{H}_i}{\partial x^l\partial y^m}(\alpha_\lambda,\beta_\mu)\;\middle|\;1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda,\mu\le L\right\}$$

since these two sets generate the same field over $\mathbb{Q}$ by Lemma 2.2.2. This completes the proof since Theorem 1.3.7 for the linear recurrence $\{\widetilde{R}_k\}_{k\ge0}$ asserts that the last set is algebraically independent. □

Proof of Corollary 1.3.9. By the relation (1.3.3), the infinite set treated in Corollary 1.3.9 coincides with

$$\left\{F_{i,m}^{(l)}(\alpha)\;\middle|\;1\le i\le r,\ l\ge0,\ m\ge1,\ \alpha\in\overline{\mathbb{Q}}^\times\right\}$$
$$\bigcup\left\{G_i^{(N_{i,\beta})}(\beta)\;\middle|\;1\le i\le r,\ \beta\in\overline{\mathbb{Q}}^\times\right\}$$
$$\bigcup\left\{-\frac{\partial^m\Theta_i}{\partial y^m}(1,\beta)\;\middle|\;1\le i\le r,\ m\ge N_{i,\beta},\ \beta\in\overline{\mathbb{Q}}^\times\right\},$$

whose algebraic independency is deduced from Theorem 1.3.8. □

Proof of Corollary 1.3.10. If $\{R_k\}_{k\ge0}$ is strictly increasing, then $N_{i,\beta}\le1$ for any i and nonzero algebraic number β. Thus, if $\{R_k\}_{k\ge0}$ is strictly increasing, then the infinite set

$$\left\{\frac{\partial^{l+m}\Xi_i}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\;1\le i\le r,\ l\ge0,\ m\ge0,\ \alpha\in\overline{\mathbb{Q}}^\times,\ \beta\in\overline{\mathbb{Q}}^\times\right\}$$
$$=\left\{\frac{\partial^{l+m}\Theta_i}{\partial x^l\partial y^m}(\alpha,\beta)\;\middle|\;1\le i\le r,\ l\ge0,\ m\ge1,\ \alpha\in\overline{\mathbb{Q}}^\times,\ \beta\in\overline{\mathbb{Q}}^\times\right\}$$

is a subset of the infinite set treated in Theorem 1.3.8, and hence it is algebraically independent. □

We omit the proofs of Theorem 1.3.1 and Corollaries 1.3.2 and 1.3.3, since their proofs are obtained by considering the case where $r=1$ in the proofs above and using Theorem 1.3.5 instead of Theorem 1.3.7.

Chapter 3

Mahler functions of several variables

Theorems 1.3.5, 1.3.7, and 1.3.12 will be proved in the next chapter by reducing the algebraic independency of the numbers in question to that of the values of certain Mahler functions of several variables. The aim of this chapter is to establish a criterion for algebraic independence of the values of Mahler functions.

3.1 Multiplicative transformation Ω

Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values on $\mathbb{C}$ of the eigenvalues of Ω is itself an eigenvalue of Ω (cf. Gantmacher [6, p. 66]). Let v be ∞ or a prime number. We define a multiplicative transformation $\Omega\colon \mathbb{C}_v^n \to \mathbb{C}_v^n$ by

$$\Omega \boldsymbol{z} := \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \prod_{j=1}^{n} z_j^{\omega_{2j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right) \tag{3.1.1}$$

for any $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}_v^n$. Then the iterates $\Omega^k \boldsymbol{z}$ $(k = 0, 1, 2, \ldots)$ are well-defined. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a point with $\alpha_1, \ldots, \alpha_n$ nonzero algebraic numbers. We consider the following four conditions on Ω and $\boldsymbol{\alpha}$.

(I) Ω is nonsingular and none of its eigenvalues is a root of unity, so that $\rho > 1$.

(II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.

(III)$_v$ If we put $\Omega^k \boldsymbol{\alpha} =: (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$, then

$$\log |\alpha_i^{(k)}|_v \leq -c\rho^k \quad (1 \leq i \leq n)$$

for all sufficiently large k, where c is a positive constant.

In the case where v is ∞, the fourth condition is the following:

(IV)$_\infty$ For any nonzero $f(\boldsymbol{z}) \in \mathbb{C}[[z_1, \ldots, z_n]]$ which converges in some neighborhood of the origin of $\mathbb{C}^n$, there are infinitely many positive integers k such that $f(\Omega^k \boldsymbol{\alpha}) \neq 0$.

On the other hand, in the case where v is a prime number p, the fourth condition becomes the following:

(IV)$_p$ For any nonzero $f(\boldsymbol{z}) \in \mathbb{C}_p[[z_1, \ldots, z_n]]$ which converges in some neighborhood of the origin of $\mathbb{C}_p^n$ and for any positive integer a, there are infinitely many positive integers k such that $f(\Omega^{ak} \boldsymbol{\alpha}) \neq 0$.

In the case where v is ∞, Mahler proved the following lemma, called Mahler's vanishing theorem, which gives a sufficient condition for a matrix Ω and a point $\boldsymbol{\alpha}$ to satisfy the four conditions (I)–(IV)$_\infty$.

Lemma 3.1.1 (Mahler [14], cf. Nishioka [23, Theorem 2.2])**.** *Let Ω be an $n \times n$ matrix with nonnegative integer entries. Suppose that the characteristic polynomial of Ω is irreducible over $\mathbb{Q}$ and that Ω has an eigenvalue $\rho > 1$ which is greater than the absolute values on $\mathbb{C}$ of any other eigenvalues. We denote by A_{ij} the (i, j)-cofactor of the matrix $\Omega - \rho I_n$. Then $A_{i1} \neq 0$ for all i $(1 \leq i \leq n)$. Moreover, if nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n$ satisfy*

$$\sum_{i=1}^{n} |A_{i1}|_\infty \log |\alpha_i|_\infty < 0,$$

then the matrix Ω and the point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ satisfy the four conditions (I)–(IV)$_\infty$.

Although the condition $(\mathrm{IV})_p$ for a prime number p is stronger than the condition $(\mathrm{IV})_\infty$, we can prove the following lemma, which is the p-adic analogue of Mahler's vanishing theorem.

Lemma 3.1.2. *Let Ω and A_{ij} be as in Lemma 3.1.1. Then $A_{i1} \neq 0$ for all i $(1 \leq i \leq n)$. Suppose that p is a prime number. If nonzero algebraic numbers $\alpha_1, \dots, \alpha_n$ satisfy*

$$\sum_{i=1}^{n} |A_{i1}|_\infty \log |\alpha_i|_p < 0,$$

then the matrix Ω and the point $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ satisfy the four conditions (I)–$(\mathrm{IV})_p$.

Proof. By Lemma 3.1.1, Ω satisfies the conditions (I) and (II). Moreover, in the same way as in the proof of Lemma 3.1.1, we see that $|A_{11}|_\infty, \dots, |A_{1n}|_\infty$ are linearly independent over $\mathbb{Q}$ and so are $|A_{11}|_\infty, \dots, |A_{n1}|_\infty$ (see Nishioka [23, pp. 36–38]). In particular, $A_{i1} \neq 0$ for all i $(1 \leq i \leq n)$.

We see that the (i,j)-entry of Ω^k is equal to $A_1 |A_{1i}|_\infty |A_{j1}|_\infty \rho^k + o(\rho^k)$, where $A_1 > 0$. For any index $\boldsymbol{h} = (h_1, \dots, h_n) \in \mathbb{Z}_{\geq 0}^n$, we have

$$\begin{aligned}
\log |(\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}}|_p &= \log |(\alpha_1^{(k)})^{h_1} \cdots (\alpha_n^{(k)})^{h_n}|_p \\
&= (h_1, \dots, h_n) \Omega^k \begin{pmatrix} \log |\alpha_1|_p \\ \vdots \\ \log |\alpha_n|_p \end{pmatrix} \\
&= \rho^k A_1 (h_1, \dots, h_n) \begin{pmatrix} |A_{11}|_\infty \sum_{j=1}^{n} |A_{j1}|_\infty \log |\alpha_j|_p \\ \vdots \\ |A_{1n}|_\infty \sum_{j=1}^{n} |A_{j1}|_\infty \log |\alpha_j|_p \end{pmatrix} + o(\rho^k) \\
&= \rho^k A_1 \left(\sum_{i=1}^{n} |A_{1i}|_\infty h_i \right) \left(\sum_{j=1}^{n} |A_{j1}|_\infty \log |\alpha_j|_p \right) + o(\rho^k).
\end{aligned}$$

In particular,

$$\log |\alpha_i^{(k)}|_p = \rho^k A_1 |A_{1i}|_\infty \sum_{j=1}^{n} |A_{j1}|_\infty \log |\alpha_j|_p + o(\rho^k).$$

By the assumption of the lemma, $\sum_{j=1}^{n} |A_{j1}|_\infty \log |\alpha_j|_p < 0$ and so the condition $(\mathrm{III})_p$ is satisfied.

Finally we show that Ω and $\boldsymbol{\alpha}$ satisfy the condition $(\mathrm{IV})_p$. Let $f(\boldsymbol{z}) = \sum_{\boldsymbol{h}} b_{\boldsymbol{h}} \boldsymbol{z}^{\boldsymbol{h}}$ be a nonzero power series of n variables with coefficients in $\mathbb{C}_p$ which converges in some neighborhood of the origin of $\mathbb{C}_p^n$. Since $|A_{11}|_\infty, \ldots, |A_{1n}|_\infty$ are linearly independent over $\mathbb{Q}$, if $(h_1, \ldots, h_n)$ and $(h'_1, \ldots, h'_n)$ are distinct indices in $\mathbb{Z}_{\geq 0}^n$, then

$$\sum_{i=1}^n |A_{1i}|_\infty h_i \neq \sum_{i=1}^n |A_{1i}|_\infty h'_i.$$

Let $\boldsymbol{H} = (H_1, \ldots, H_n)$ be the only index in $\mathbb{Z}_{\geq 0}^n$ such that

$$\min \left\{ \sum_{i=1}^n |A_{1i}|_\infty h_i \;\middle|\; b_{\boldsymbol{h}} \neq 0,\ \boldsymbol{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_{\geq 0}^n \right\} =: H = \sum_{i=1}^n |A_{1i}|_\infty H_i.$$

Let Λ_1 be the finite subset of $\mathbb{Z}_{\geq 0}^n$ consisting of the indices $(h_1, \ldots, h_n)$ such that $|A_{1i}|_\infty h_i \leq H$ for all i $(1 \leq i \leq n)$ and let Λ_2 be the complement of Λ_1 in $\mathbb{Z}_{\geq 0}^n$. If $\boldsymbol{h} = (h_1, \ldots, h_n) \neq \boldsymbol{H}$ with $b_{\boldsymbol{h}} \neq 0$, then

$$\begin{aligned} \log \left| \frac{(\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}}}{(\Omega^k \boldsymbol{\alpha})^{\boldsymbol{H}}} \right|_p &= \rho^k A_1 \left(\sum_{i=1}^n |A_{1i}|_\infty h_i - H \right) \left(\sum_{j=1}^n |A_{j1}|_\infty \log |\alpha_j|_p \right) + o(\rho^k) \\ &\to -\infty \quad (k \to \infty) \end{aligned}$$

and hence the finite sum $\sum_{\boldsymbol{h} \in \Lambda_1} b_{\boldsymbol{h}} (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}} / (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{H}}$ tends to the nonzero coefficient $b_{\boldsymbol{H}}$ as k tends to infinity. Assume that k is sufficiently large. Since the p-adic absolute value $|\cdot|_p$ is ultrametric, if $\sum_{\boldsymbol{h} \in \Lambda_2} b_{\boldsymbol{h}} (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}} \neq 0$, then there exists a finite subset Λ_3 of Λ_2 depending on k such that

$$\begin{aligned} & \left| \sum_{\boldsymbol{h} \in \Lambda_2} b_{\boldsymbol{h}} (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}} \right|_p \\ = & \left| \sum_{\boldsymbol{h} \in \Lambda_3} b_{\boldsymbol{h}} (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}} \right|_p \\ \leq & \max_{\boldsymbol{h} \in \Lambda_3} |b_{\boldsymbol{h}} (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}}|_p \leq \max_{\boldsymbol{h} \in \Lambda_3} c_1^{|\boldsymbol{h}|} |(\Omega^k \boldsymbol{\alpha})^{\boldsymbol{h}}|_p \leq c_2 \max_{1 \leq i \leq n} |\alpha_i^{(k)}|_p^{[H/|A_{1i}|_\infty]+1}, \end{aligned}$$

where c_1 and c_2 are positive constants independent of k. Since

$$\begin{aligned}
&\log\left|\frac{(\alpha_i^{(k)})^{[H/|A_{1i}|_\infty]+1}}{(\Omega^k\boldsymbol{\alpha})^{\boldsymbol{H}}}\right|_p \\
&= \rho^k A_1\left(|A_{1i}|_\infty\left(\left[\frac{H}{|A_{1i}|_\infty}\right]+1\right)-H\right)\left(\sum_{j=1}^n |A_{j1}|_\infty \log|\alpha_j|_p\right)+o(\rho^k) \\
&\to -\infty \qquad (k\to\infty)
\end{aligned}$$

for any $i = 1, \ldots, n$, we see that $\sum_{\boldsymbol{h}\in\Lambda_2} b_{\boldsymbol{h}}(\Omega^k\boldsymbol{\alpha})^{\boldsymbol{h}}/(\Omega^k\boldsymbol{\alpha})^{\boldsymbol{H}}$ tends to zero as k tends to infinity. Therefore $f(\Omega^k\boldsymbol{\alpha})/(\Omega^k\boldsymbol{\alpha})^{\boldsymbol{H}}$ tends to the nonzero coefficient $b_{\boldsymbol{H}}$ as k tends to infinity, which implies the condition $(\mathrm{IV})_p$. □

In the case where v is ∞, Masser established the following lemma, called Masser's vanishing theorem, which gives a necessary and sufficient condition for Ω and $\boldsymbol{\alpha}$ to satisfy the condition $(\mathrm{IV})_\infty$.

Lemma 3.1.3 (Masser [17]). *Let Ω be an $n\times n$ matrix with nonnegative integer entries satisfying the condition* (I). *Let $\boldsymbol{\alpha}$ be an n-dimensional vector whose components $\alpha_1, \ldots, \alpha_n$ are nonzero algebraic numbers such that $\Omega^k\boldsymbol{\alpha} = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}) \to (0, \ldots, 0)$ in $\mathbb{C}^n$ as k tends to infinity. Then the negation of the condition $(\mathrm{IV})_\infty$ is equivalent to the following: There exist integers $i_1, \ldots, i_n$, not all zero, and positive integers a, b such that*

$$(\alpha_1^{(k)})^{i_1}\cdots(\alpha_n^{(k)})^{i_n} = 1$$

for all $k = a + lb$ $(l = 0, 1, 2, \ldots)$.

The p-adic analogue of Masser's vanishing theorem is unsolved. This is the reason why we need to assume the stronger condition $(\mathrm{IV})_p$ in Theorems 3.2.1 and 3.2.4 below in the case where v is a prime number p (see Remark 3.4.5 stated in the proof of Theorem 3.2.4).

3.2 Criterion for algebraic independence

Mahler functions of several variables are analytic functions which satisfy certain types of functional equations under the transformation $\boldsymbol{z} \mapsto \Omega\boldsymbol{z}$ defined by (3.1.1).

Kubota [11] studied Mahler functions $g_1(\boldsymbol{z}),\dots,g_m(\boldsymbol{z})$ satisfying respective functional equations

$$\begin{pmatrix} g_1(\boldsymbol{z}) \\ \vdots \\ g_m(\boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} e_1(\boldsymbol{z}) & & 0 \\ & \ddots & \\ 0 & & e_m(\boldsymbol{z}) \end{pmatrix} \begin{pmatrix} g_1(\Omega\boldsymbol{z}) \\ \vdots \\ g_m(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} b_1(\boldsymbol{z}) \\ \vdots \\ b_m(\boldsymbol{z}) \end{pmatrix}, \tag{3.2.1}$$

where $e_h(\boldsymbol{z}),\ b_h(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1,\dots,z_n)$ $(1 \le h \le m)$, and established a criterion for the algebraic independence of their values as well as that of the functions themselves (see also Nishioka [23]). On the other hand, Nishioka [22] studied Mahler functions $f_{ij}(\boldsymbol{z})$ $(1 \le i \le l,\ 1 \le j \le n(i))$ satisfying a system of functional equations

$$\begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_l(\boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_l \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_1(\Omega\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_l(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{b}_l(\boldsymbol{z}) \end{pmatrix},$$

where

$$\boldsymbol{f}_i(\boldsymbol{z}) = {}^t(f_{i1}(\boldsymbol{z}),\dots,f_{in(i)}(\boldsymbol{z})) \quad (1 \le i \le l), \tag{3.2.2}$$

$$A_i = \begin{pmatrix} a_i & & & \\ a_{21}^{(i)} & a_i & & \text{\Large 0} \\ \vdots & \ddots & \ddots & \\ a_{n(i)1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \in GL_{n(i)}(\overline{\mathbb{Q}}), \quad a_i \ne 0, \quad a_{s\,s-1}^{(i)} \ne 0, \tag{3.2.3}$$

and

$$\boldsymbol{b}_i(\boldsymbol{z}) = {}^t(b_{i1}(\boldsymbol{z}),\dots,b_{in(i)}(\boldsymbol{z})) \in \overline{\mathbb{Q}}(z_1,\dots,z_n)^{n(i)} \quad (1 \le i \le l), \tag{3.2.4}$$

and established a criterion for the algebraic independence of their values as well as that of the functions themselves.

In order to prove Theorems 1.3.5, 1.3.7, and 1.3.12, we need the following criterion for the algebraic independence of the values of Mahler functions, which includes Nishioka's criterion and a special case of Kubota's one, that is, the case where $b_h(\boldsymbol{z}) = 0$ for any h $(1 \le h \le m)$ in the functional equation (3.2.1). In what follows, we call a subfield K of $\overline{\mathbb{Q}}$ a *number field* if K is a finite extension of $\mathbb{Q}$.

Theorem 3.2.1 (Ide [7, Theorem 4.3]). *Let v be ∞ or a prime number, K a number field, and Ω an $n \times n$ matrix with nonnegative integer entries. Let $f_{ij}(\boldsymbol{z}), g_h(\boldsymbol{z}) \in$*

$K[[z_1,\ldots,z_n]]$ $(1 \le i \le l,\ 1 \le j \le n(i),\ 1 \le h \le m)$ *with* $g_h(\mathbf{0}) \ne 0$ $(1 \le h \le m)$. *Suppose that they converge in an* n*-polydisc* U *around the origin of* $\mathbb{C}_v^n$ *and satisfy the system of functional equations*

$$\begin{pmatrix} \boldsymbol{f}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_l(\boldsymbol{z}) \\ g_1(\boldsymbol{z}) \\ \vdots \\ g_m(\boldsymbol{z}) \end{pmatrix} = \left(\begin{array}{ccc|ccc} A_1 & & 0 & & & \\ & \ddots & & & \mathbf{0} & \\ 0 & & A_l & & & \\ \hline & & & e_1(\boldsymbol{z}) & & 0 \\ & \mathbf{0} & & & \ddots & \\ & & & 0 & & e_m(\boldsymbol{z}) \end{array}\right) \begin{pmatrix} \boldsymbol{f}_1(\Omega\boldsymbol{z}) \\ \vdots \\ \boldsymbol{f}_l(\Omega\boldsymbol{z}) \\ g_1(\Omega\boldsymbol{z}) \\ \vdots \\ g_m(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{b}_l(\boldsymbol{z}) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\boldsymbol{f}_i(\boldsymbol{z})$, A_i, *and* $\boldsymbol{b}_i(\boldsymbol{z})$ $(1 \le i \le l)$ *are as in* (3.2.2), (3.2.3), *and* (3.2.4), *respectively, and* $e_h(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1,\ldots,z_n)$ $(1 \le h \le m)$. *Let* $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ *be a point in* U *whose components are nonzero algebraic numbers. Assume that* Ω *and* $\boldsymbol{\alpha}$ *satisfy the four conditions* (I)–(IV)$_v$. *Assume further that* $b_{ij}(\Omega^k\boldsymbol{\alpha})$ $(1 \le i \le l,\ 1 \le j \le n(i))$ *and* $e_h(\Omega^k\boldsymbol{\alpha})$ $(1 \le h \le m)$ *are defined and* $e_h(\Omega^k\boldsymbol{\alpha}) \ne 0$ $(1 \le h \le m)$ *for all* $k \ge 0$. *Then, if the numbers* $f_{ij}(\boldsymbol{\alpha})$ $(1 \le i \le l,\ 1 \le j \le n(i))$ *and* $g_h(\boldsymbol{\alpha})$ $(1 \le h \le m)$ *of* $\overline{\mathbb{Q}}_v$ *are algebraically dependent, then at least one of the following two conditions holds:*

(i) *There exist a non-empty subset* $\{i_1,\ldots,i_r\}$ *of* $\{1,\ldots,l\}$ *and nonzero algebraic numbers* $c_1,\ldots,c_r$ *such that*

$$a_{i_1} = \cdots = a_{i_r}$$

and

$$f(\boldsymbol{z}) := c_1 f_{i_1 1}(\boldsymbol{z}) + \cdots + c_r f_{i_r 1}(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1,\ldots,z_n).$$

(ii) *There exist integers* $d_1,\ldots,d_m$, *not all zero, and* $g(\boldsymbol{z}) \in \overline{\mathbb{Q}}(z_1,\ldots,z_n)^\times$ *such that*

$$g(\boldsymbol{z}) = \left(\prod_{h=1}^m e_h(\boldsymbol{z})^{d_h}\right) g(\Omega\boldsymbol{z}).$$

Remark 3.2.2. If the case (i) in Theorem 3.2.1 arises, then the rational function $f(\boldsymbol{z})$ satisfies the functional equation

$$f(\boldsymbol{z}) = a_{i_1} f(\Omega \boldsymbol{z}) + c_1 b_{i_1 1}(\boldsymbol{z}) + \cdots + c_r b_{i_r 1}(\boldsymbol{z}).$$

The proof of Theorem 3.2.1 consists of two parts. The first is Theorem 3.2.3 below, the algebraic independence over the field of rational functions of Mahler functions themselves.

Theorem 3.2.3 (Ide [7, Theorem 4.4])**.** *Let C be a field of characteristic 0 and M the quotient field of $C[[z_1, \ldots, z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries satisfying the condition* (I)*. Suppose that $f_{ij}(\boldsymbol{z}) \in M$ $(1 \le i \le l,\ 1 \le j \le n(i))$ satisfy the system of functional equations*

$$\begin{pmatrix} f_{i1}(\Omega \boldsymbol{z}) \\ \vdots \\ \vdots \\ f_{in(i)}(\Omega \boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} a_i & & & 0 \\ a_{21}^{(i)} & a_i & & \\ \vdots & \ddots & \ddots & \\ a_{n(i)1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \begin{pmatrix} f_{i1}(\boldsymbol{z}) \\ \vdots \\ \vdots \\ f_{in(i)}(\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} b_{i1}(\boldsymbol{z}) \\ \vdots \\ \vdots \\ b_{in(i)}(\boldsymbol{z}) \end{pmatrix}, \tag{3.2.5}$$

where a_i, $a_{st}^{(i)} \in C$, $a_i \neq 0$, $a_{s\,s-1}^{(i)} \neq 0$, and $b_{ij}(\boldsymbol{z}) \in C(z_1, \ldots, z_n)$. Assume that $g_h(\boldsymbol{z}) \in M^{\times}$ $(1 \le h \le m)$ satisfy the functional equations

$$g_h(\Omega \boldsymbol{z}) = e_h(\boldsymbol{z}) g_h(\boldsymbol{z}) \quad (1 \le h \le m), \tag{3.2.6}$$

where $e_h(\boldsymbol{z}) \in C(z_1, \ldots, z_n)$ $(1 \le h \le m)$. Then, if the functions $f_{ij}(\boldsymbol{z})$ $(1 \le i \le l,\ 1 \le j \le n(i))$ and $g_h(\boldsymbol{z})$ $(1 \le h \le m)$ are algebraically dependent over $C(z_1, \ldots, z_n)$, then at least one of the following two conditions holds:

(i) *There exist a non-empty subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, l\}$ and nonzero elements $c_1, \ldots, c_r$ of C such that*

$$a_{i_1} = \cdots = a_{i_r}$$

and

$$f(\boldsymbol{z}) := c_1 f_{i_1 1}(\boldsymbol{z}) + \cdots + c_r f_{i_r 1}(\boldsymbol{z}) \in C(z_1, \ldots, z_n).$$

(ii) *There exist integers* $d_1, \ldots, d_m$*, not all zero, and* $g(\boldsymbol{z}) \in C(z_1, \ldots, z_n)^\times$ *such that*

$$g(\Omega \boldsymbol{z}) = \left(\prod_{h=1}^{m} e_h(\boldsymbol{z})^{d_h} \right) g(\boldsymbol{z}).$$

The second part, Theorem 3.2.4 below, asserts the algebraic independence of the values of Mahler functions under the assumption that the Mahler functions themselves are algebraically independent over the field of rational functions.

Theorem 3.2.4 (Ide [7, Theorem 4.5])**.** *Let* v *be* ∞ *or a prime number,* K *a number field, and* Ω *an* $n \times n$ *matrix with nonnegative integer entries. Let* $f_1(\boldsymbol{z}), \ldots, f_l(\boldsymbol{z})$*,* $g_1(\boldsymbol{z}), \ldots, g_m(\boldsymbol{z}) \in K[[z_1, \ldots, z_n]]$ *with* $g_h(\mathbf{0}) \neq 0$ $(1 \leq h \leq m)$*. Suppose that they converge in an* n*-polydisc* U *around the origin of* $\mathbb{C}_v^n$ *and that* $f_i(\boldsymbol{z})$ $(1 \leq i \leq l)$ *satisfy the system of functional equations*

$$\begin{pmatrix} f_1(\boldsymbol{z}) \\ \vdots \\ f_l(\boldsymbol{z}) \end{pmatrix} = A \begin{pmatrix} f_1(\Omega\boldsymbol{z}) \\ \vdots \\ f_l(\Omega\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} b_1(\boldsymbol{z}) \\ \vdots \\ b_l(\boldsymbol{z}) \end{pmatrix}, \tag{3.2.7}$$

where A *is an* $l \times l$ *matrix with entries in* K *and* $b_i(\boldsymbol{z}) \in K(z_1, \ldots, z_n)$ $(1 \leq i \leq l)$*. Assume that* $g_h(\boldsymbol{z})$ $(1 \leq h \leq m)$ *satisfy the functional equations*

$$g_h(\boldsymbol{z}) = e_h(\boldsymbol{z}) g_h(\Omega\boldsymbol{z}) \quad (1 \leq h \leq m), \tag{3.2.8}$$

where $e_h(\boldsymbol{z}) \in K(z_1, \ldots, z_n)$ $(1 \leq h \leq m)$*. Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *be a point in* U *whose components are nonzero algebraic numbers. Suppose that* Ω *and* $\boldsymbol{\alpha}$ *satisfy the four conditions* (I)–(IV)$_v$*. Assume further that* $b_i(\Omega^k\boldsymbol{\alpha})$ $(1 \leq i \leq l)$ *and* $e_h(\Omega^k\boldsymbol{\alpha})$ $(1 \leq h \leq m)$ *are defined and* $e_h(\Omega^k\boldsymbol{\alpha}) \neq 0$ $(1 \leq h \leq m)$ *for all* $k \geq 0$*. Then, if the functions* $f_i(\boldsymbol{z})$ $(1 \leq i \leq l)$ *and* $g_h(\boldsymbol{z})$ $(1 \leq h \leq m)$ *are algebraically independent over* $K(z_1, \ldots, z_n)$*, then the numbers* $f_i(\boldsymbol{\alpha})$ $(1 \leq i \leq l)$ *and* $g_h(\boldsymbol{\alpha})$ $(1 \leq h \leq m)$ *of* $\overline{\mathbb{Q}}_v$ *are algebraically independent.*

We prove Theorems 3.2.3 and 3.2.4 in Sections 3.3 and 3.4, respectively.

3.3 Algebraic independence of Mahler functions themselves

In this section we prove Theorem 3.2.3. First we introduce the following:

Lemma 3.3.1. *Let F be a field and $u_1, \ldots, u_n$ elements of an extension field of F. Suppose that the transcendence degree of $F(\boldsymbol{u})$ over F equals $n-1$. Let P be an irreducible polynomial of $F[X_1, \ldots, X_n]$ such that $P(\boldsymbol{u}) = 0$. Then, if $A \in F[X_1, \ldots, X_n]$ satisfies $A(\boldsymbol{u}) = 0$, then P divides A in $F[X_1, \ldots, X_n]$.*

Proof. We may assume that $u_1, \ldots, u_{n-1}$ are algebraically independent over F. Then we have the ring isomorphism $F[X_1, \ldots, X_{n-1}] \simeq F[u_1, \ldots, u_{n-1}] =: \mathcal{R}$, which induces $F[X_1, \ldots, X_n] \simeq \mathcal{R}[X_n]$. Let $\overline{P}$ and $\overline{A}$ be the polynomials of $\mathcal{R}[X_n]$ corresponding to P and A, respectively. Since $P(\boldsymbol{u}) = A(\boldsymbol{u}) = 0$, we have $\overline{P}(u_n) = \overline{A}(u_n) = 0$. Since P is irreducible in $F[X_1, \ldots, X_n]$, so is $\overline{P}$ in $\mathcal{R}[X_n]$. Letting $\mathcal{K}$ be the quotient field of $\mathcal{R}$, we see by Gauss's lemma that $\overline{P}$ is also irreducible in $\mathcal{K}[X_n]$. Then $\overline{P}$ is the minimal polynomial of u_n over the field $\mathcal{K}$ and hence, $\overline{P}$ divides $\overline{A}$ in $\mathcal{K}[X_n]$. Noting that $\overline{P}$ is primitive in $\mathcal{R}[X_n]$ since it is irreducible in $\mathcal{R}[X_n]$, we see by Gauss's lemma that $\overline{P}$ divides $\overline{A}$ in $\mathcal{R}[X_n]$. Therefore the isomorphism $F[X_1, \ldots, X_n] \simeq \mathcal{R}[X_n]$ implies the lemma. □

Let C be a field of characteristic 0, L the rational function field $C(z_1, \ldots, z_n)$, and M the quotient field of $C[[z_1, \ldots, z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries satisfying the condition (I). Then we can define an endomorphism $\tau \colon M \to M$ by

$$f^{\tau}(\boldsymbol{z}) := f(\Omega \boldsymbol{z})$$

for any $f \in M$. Suppose that $f_{ij} \in M$ $(1 \le i \le l,\ 1 \le j \le n(i))$ satisfy (3.2.5). Let X_{ij} $(1 \le i \le l,\ 1 \le j \le n(i))$ be variables and let $\boldsymbol{X} := \{X_{ij}\}_{1 \le i \le l,\, 1 \le j \le n(i)}$. We define an endomorphism T of $L[\boldsymbol{X}]$ by

$$Ta := a^{\tau} \quad (a \in L)$$

and

$$\begin{pmatrix} TX_{i1} \\ \vdots \\ \vdots \\ TX_{in(i)} \end{pmatrix} := \begin{pmatrix} a_i & & & \text{\huge 0} \\ a_{21}^{(i)} & a_i & & \\ \vdots & \ddots & \ddots & \\ a_{n(i)1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \begin{pmatrix} X_{i1} \\ \vdots \\ \vdots \\ X_{in(i)} \end{pmatrix} + \begin{pmatrix} b_{i1} \\ \vdots \\ \vdots \\ b_{in(i)} \end{pmatrix}$$

for each i $(1 \le i \le l)$. From the proof of Theorem 3 of Nishioka [22, pp. 56–58], we obtain the following:

Claim 3.3.2. *If there exists $F \in L[\boldsymbol{X}] \setminus L$ such that F divides TF in $L[\boldsymbol{X}]$, then the condition* (i) *of Theorem 3.2.3 holds.*

Proof of Theorem 3.2.3. If the functions f_{ij} $(1 \le i \le l,\ 1 \le j \le n(i))$ are algebraically dependent over L, then the condition (i) holds by Theorem 3 of Nishioka [22]. Thus we may assume that f_{ij} $(1 \le i \le l,\ 1 \le j \le n(i))$ are algebraically independent over L. Let L^* be the subfield of M generated by f_{ij} $(1 \le i \le l,\ 1 \le j \le n(i))$ over L. In what follows, we prove inductively on $m \ge 0$ that, if the functions $g_1, \ldots, g_m$ are algebraically dependent over L^*, then the condition (ii) holds. If $m = 0$, then the assertion is trivial. Hence we may suppose that $m \ge 1$ and that $g_1, \ldots, g_{m-1}$ are algebraically independent over L^*. Since $g_1, \ldots, g_m$ are algebraically dependent over L^*, there exists an irreducible $G \in L^*[\boldsymbol{Y}]$ such that $G(\boldsymbol{g}) = 0$, where $\boldsymbol{Y} := (Y_1, \ldots, Y_n)$ is a vector of variables and $\boldsymbol{g} := (g_1, \ldots, g_m)$. Put

$$G(\boldsymbol{Y}) =: \sum_{\boldsymbol{I}} b_{\boldsymbol{I}} \boldsymbol{Y}^{\boldsymbol{I}}, \quad b_{\boldsymbol{I}} \in L^*.$$

We may assume that $b_{\boldsymbol{J}} = 1$ for some $\boldsymbol{J} = (j_1, \ldots, j_m)$. Noting the functional equation (3.2.5), we define an endomorphism T^* of $L^*[\boldsymbol{Y}]$ by

$$T^* a := a^\tau \in L^* \quad (a \in L^*), \qquad T^* Y_h := e_h Y_h \quad (1 \le h \le m).$$

By (3.2.6), we have $(T^*G)(\boldsymbol{g}) = G(\boldsymbol{g})^\tau = 0^\tau = 0$. Therefore, by Lemma 3.3.1, we see that G divides T^*G in $L^*[\boldsymbol{Y}]$. Since τ is injective and since $e_h \ne 0$ $(1 \le h \le m)$, we see that

$$T^*G = \boldsymbol{e}^{\boldsymbol{J}} G,$$

where $\boldsymbol{e}^{\boldsymbol{J}} := e_1^{j_1} \cdots e_m^{j_m}$. Comparing the coefficients of both sides above, we get

$$b_{\boldsymbol{I}}^{\tau} \boldsymbol{e}^{\boldsymbol{I}} = \boldsymbol{e}^{\boldsymbol{J}} b_{\boldsymbol{I}} \tag{3.3.1}$$

for any $\boldsymbol{I}$. Since $g_h \neq 0$ $(1 \leq h \leq m)$, there exists $\boldsymbol{I}$ distinct from $\boldsymbol{J}$ such that $b_{\boldsymbol{I}} \neq 0$. Then we have a representation

$$b_{\boldsymbol{I}} = \frac{A(\boldsymbol{f})}{B(\boldsymbol{f})},$$

where $A, B \in L[\boldsymbol{X}] \setminus \{0\}$ are coprime and $\boldsymbol{f} := \{f_{ij}\}_{1 \leq i \leq l,\, 1 \leq j \leq n(i)}$. By the functional equation (3.2.5) and the definition of T we get $A(\boldsymbol{f})^{\tau} = (TA)(\boldsymbol{f})$ and $B(\boldsymbol{f})^{\tau} = (TB)(\boldsymbol{f})$. Hence, by (3.3.1), we obtain

$$B(\boldsymbol{f}) \cdot (TA)(\boldsymbol{f}) = \boldsymbol{e}^{\boldsymbol{J}-\boldsymbol{I}} A(\boldsymbol{f}) \cdot (TB)(\boldsymbol{f}).$$

Since f_{ij} $(1 \leq i \leq l,\ 1 \leq j \leq n(i))$ are algebraically independent over L, we have

$$B \cdot (TA) = \boldsymbol{e}^{\boldsymbol{J}-\boldsymbol{I}} A \cdot (TB) \tag{3.3.2}$$

and therefore, A and B divide TA and TB in $L[\boldsymbol{X}]$, respectively. If either A or B do not belong to L, then by the Claim 3.3.2 the condition (i) holds, which contradicts the algebraic independency of f_{ij} $(1 \leq i \leq l,\ 1 \leq j \leq n(i))$. Hence we conclude that $A, B \in L^{\times}$ and we see by (3.3.2) that

$$\left(\frac{A}{B}\right)^{\tau} = \boldsymbol{e}^{\boldsymbol{J}-\boldsymbol{I}} \frac{A}{B},$$

which implies that the condition (ii) holds since $\boldsymbol{J} - \boldsymbol{I} \in \mathbb{Z}^m \setminus \{\boldsymbol{0}\}$. $\square$

3.4 Algebraic independence of the values of Mahler functions

In this section we prove Theorem 3.2.4. Let us first introduce some notation which will be used in the proof. For any algebraic number α, we denote by $\overline{|\alpha|}$ the maximum of the absolute values on $\mathbb{C}$ of the conjugates of α and by $\mathrm{den}(\alpha)$ the least positive integer d such that $d\alpha$ is an algebraic integer. We define

$$\|\alpha\| := \max\{\overline{|\alpha|},\ \mathrm{den}(\alpha)\}.$$

It is easily seen that

$$\left\| \sum_{i=1}^{n} \alpha_i \right\| \leq n \prod_{i=1}^{n} \|\alpha_i\|$$

and

$$\left\| \prod_{i=1}^{n} \alpha_i \right\| \leq \prod_{i=1}^{n} \|\alpha_i\|$$

for any algebraic numbers $\alpha_1, \dots, \alpha_n$. Moreover, by Lemma 2.10.2 of Nishioka [23], we have

$$\|\alpha^{-1}\| \leq \|\alpha\|^{2[\mathbb{Q}(\alpha)\,:\,\mathbb{Q}]}$$

for any nonzero algebraic number α. The following proposition plays a fundamental role in proving the transcendency or the algebraic independency of given numbers.

Proposition 3.4.1 (Fundamental inequality)**.** *Let v be ∞ or a prime number. For any nonzero algebraic number α, we have*

$$|\alpha|_v \geq \|\alpha\|^{-2[\mathbb{Q}(\alpha)\,:\,\mathbb{Q}]}.$$

Proof. Put $n := [\mathbb{Q}(\alpha) : \mathbb{Q}]$ and $N(\alpha) := N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)$. First we consider the case where α is a nonzero algebraic integer. Then $N(\alpha)$ is a nonzero rational integer. If v is ∞, then we see that

$$1 \leq |N(\alpha)|_\infty \leq |\alpha|_\infty \lceil \alpha \rceil^{n-1},$$

which implies

$$|\alpha|_\infty \geq \lceil \alpha \rceil^{-n+1}. \tag{3.4.1}$$

Suppose that v is a prime number p. Since the conjugates of α are algebraic integers, their p-adic absolute values are less than or equal to 1, which implies $|N(\alpha)|_p \leq |\alpha|_p$ (cf. Waldschmidt [35, Corollary 3.2]). Since $N(\alpha)$ is a rational integer, we have $|N(\alpha)|_{p'} \leq 1$ for any prime number p'. Hence by the product formula we get

$$1 = |N(\alpha)|_\infty |N(\alpha)|_p \prod_{\substack{p'\colon \text{prime number},\\ p' \neq p}} |N(\alpha)|_{p'} \leq |N(\alpha)|_\infty |\alpha|_p \leq \lceil \alpha \rceil^n |\alpha|_p,$$

and thus

$$|\alpha|_p \geq \lceil \alpha \rceil^{-n}. \tag{3.4.2}$$

Next we prove the proposition for a general α. Let $D \coloneqq \operatorname{den}\alpha$. Then $D\alpha$ is a nonzero algebraic integer. If v is ∞, then we see by (3.4.1) that

$$D|\alpha|_\infty = |D\alpha|_\infty \geq \lceil D\alpha \rceil^{-n+1} = D^{-n+1}\lceil \alpha \rceil^{-n+1}$$

and hence

$$|\alpha|_\infty \geq D^{-n}\lceil \alpha \rceil^{-n+1} \geq \|\alpha\|^{-2n+1} \geq \|\alpha\|^{-2n}.$$

If v is a prime number p, then $|D|_p \leq 1$ and therefore, by (3.4.2),

$$|\alpha|_p \geq |D\alpha|_p \geq \lceil D\alpha \rceil^{-n} = D^{-n}\lceil \alpha \rceil^{-n} \geq \|\alpha\|^{-2n}.$$

□

The following lemma is proved in a way similar to Nishioka [23, p. 6, Remark]. We give the proof for the sake of readers.

Lemma 3.4.2. *Let C be a field and F a subfield of C. Let $f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z}) \in F[[z_1, \dots, z_n]]$. Then they are algebraically dependent over $C(z_1, \dots, z_n)$ if and only if they are algebraically dependent over $F(z_1, \dots, z_n)$.*

Proof. Assume that $f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z})$ are algebraically dependent over $C(z_1, \dots, z_n)$. Then there exist a non-empty finite subset Λ of $\mathbb{Z}_{\geq 0}^m$ and nonzero polynomials $a_{\boldsymbol{I}}(\boldsymbol{z}) \in C[z_1, \dots, z_n] \setminus \{0\}$ ($\boldsymbol{I} \in \Lambda$) such that

$$\sum_{\boldsymbol{I} \in \Lambda} a_{\boldsymbol{I}}(\boldsymbol{z}) \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}} = 0,$$

where $\boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}} \coloneqq f_1(\boldsymbol{z})^{i_1} \cdots f_m(\boldsymbol{z})^{i_m}$ for $\boldsymbol{I} = (i_1, \dots, i_m)$. Let $\{b_1, \dots, b_N\}$ be a maximal subset of the set of all the coefficients of $a_{\boldsymbol{I}}(\boldsymbol{z})$ ($\boldsymbol{I} \in \Lambda$) which is linearly independent over F. For each $\boldsymbol{I} \in \Lambda$, we can write

$$a_{\boldsymbol{I}}(\boldsymbol{z}) = \sum_{j=1}^{N} a_{\boldsymbol{I}j}(\boldsymbol{z}) b_j,$$

where $a_{\boldsymbol{I}j}(\boldsymbol{z}) \in F[z_1, \ldots, z_n]$ $(1 \le j \le N)$. Then we have

$$\sum_{j=1}^{N} \left(\sum_{\boldsymbol{I} \in \Lambda} a_{\boldsymbol{I}j}(\boldsymbol{z}) \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}} \right) b_j = 0.$$

Since $\sum_{\boldsymbol{I} \in \Lambda} a_{\boldsymbol{I}j(\boldsymbol{z})} \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}} \in F[[z_1, \ldots, z_n]]$ $(1 \le j \le N)$, by the linear independency of $b_1, \ldots, b_N$ we obtain

$$\sum_{\boldsymbol{I} \in \Lambda} a_{\boldsymbol{I}j}(\boldsymbol{z}) \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}} = 0 \qquad (1 \le j \le N).$$

Let $\boldsymbol{I}_0$ be an element of Λ. Since $a_{\boldsymbol{I}_0}(\boldsymbol{z}) \neq 0$, there exists j_0 with $1 \le j_0 \le N$ such that $a_{\boldsymbol{I}_0 j_0}(\boldsymbol{z}) \neq 0$. Hence the equation

$$\sum_{\boldsymbol{I} \in \Lambda} a_{\boldsymbol{I}j_0}(\boldsymbol{z}) \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}} = a_{\boldsymbol{I}_0 j_0}(\boldsymbol{z}) \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{I}_0} + \cdots = 0$$

implies that $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z})$ are algebraically dependent over $F(z_1, \ldots, z_n)$. The converse is trivial. $\square$

The following lemma plays a crucial role in the proof of Theorem 3.2.4.

Lemma 3.4.3. *Let v be ∞ or a prime number. Let Ω be an $n \times n$ matrix with nonnegative integer entries and $\boldsymbol{\alpha}$ an n-dimensional vector whose components are nonzero algebraic numbers. Suppose that Ω and $\boldsymbol{\alpha}$ satisfy the four conditions* (I)–(IV)$_v$. *Define the function*

$$\psi(\boldsymbol{z}; x) := \sum_{i=1}^{q} \sum_{j=1}^{d_i} x^{j-1} \gamma_i^x h_{ij}(\boldsymbol{z}),$$

where $\gamma_1, \ldots, \gamma_q$ are nonzero distinct elements of $\mathbb{C}_v$ and $h_{ij}(\boldsymbol{z}) \in \mathbb{C}_v[[z_1, \ldots, z_n]]$ $(1 \le i \le q,\ 1 \le j \le d_i)$ converge in an n-polydisc U around the origin of $\mathbb{C}_v^n$. Then, if $\psi(\Omega^k \boldsymbol{\alpha}; k) = 0$ for all sufficiently large k, then $h_{ij}(\boldsymbol{z}) = 0$ for every i, j.

This lemma was proved by Loxton and van der Poorten [13, Lemma 2] in the case where v is ∞. For the proof, see Nishioka [22, Lemma 3]. The proof is valid also in the case where v is a prime number p since the condition (IV)$_p$ is stronger than the condition (IV)$_\infty$. For the reader's convenience, we prove the lemma in these two cases simultaneously. In the proof we use the following:

Lemma 3.4.4 (Nishioka [22, Lemma 1]). *Let C be a field and Ω an $n \times n$ matrix with nonnegative integer entries satisfying the condition* (I). *Then, if an element $f(\boldsymbol{z})$ of the quotient field of $C[[z_1, \dots, z_n]]$ satisfies the constant coefficient equation*

$$f(\Omega \boldsymbol{z}) = af(\boldsymbol{z}) + b \qquad (a, b \in C),$$

then $f(\boldsymbol{z}) \in C$.

Proof of Lemma 3.4.3. We prove the lemma by induction on $\sum_{i=1}^{q_i} d_i$. If $\sum_{i=1}^{q_i} d_i = 1$, then the lemma is true by the condition $(\mathrm{IV})_v$. Let $\sum_{i=1}^{q} d_i > 1$. By the induction hypothesis, it suffices to prove that $h_{q\,d_q}(\boldsymbol{z}) = 0$. Suppose on the contrary that $h(\boldsymbol{z}) := h_{q\,d_q}(\boldsymbol{z}) \neq 0$. Replacing γ_i by γ_i/γ_q for each i $(1 \leq i \leq q)$, we may assume $\gamma_q = 1$. Consider

$$\begin{aligned} \xi(\boldsymbol{z}; x) &:= h(\Omega \boldsymbol{z})\psi(\boldsymbol{z}; x) - h(\boldsymbol{z})\psi(\Omega \boldsymbol{z}; x+1) \\ &= \sum_{i=1}^{q-1} \sum_{j=1}^{d_i} x^{j-1} \gamma_i^x h_{ij}^*(\boldsymbol{z}) + \sum_{j=1}^{d_q - 1} x^{j-1} h_j^*(\boldsymbol{z}), \end{aligned}$$

where

$$h_j^*(\boldsymbol{z}) := h(\Omega \boldsymbol{z}) h_{qj}(\boldsymbol{z}) - h(\boldsymbol{z}) \sum_{s=j}^{d_q} \binom{s-1}{j-1} h_{qs}(\Omega \boldsymbol{z}) \qquad (1 \leq j \leq d_q - 1)$$

and

$$h_{ij}^*(\boldsymbol{z}) := h(\Omega \boldsymbol{z}) h_{ij}(\boldsymbol{z}) - \gamma_i h(\boldsymbol{z}) \sum_{s=j}^{d_i} \binom{s-1}{j-1} h_{is}(\Omega \boldsymbol{z}) \qquad (1 \leq i \leq q-1,\ 1 \leq j \leq d_i).$$

Then $\xi(\Omega^k \boldsymbol{\alpha}; k) = h(\Omega^{k+1}\boldsymbol{\alpha})\psi(\Omega^k \boldsymbol{\alpha}; k) - h(\Omega^k \boldsymbol{\alpha})\psi(\Omega^{k+1}\boldsymbol{\alpha}; k+1) = 0$ for all sufficiently large k. Applying the induction hypothesis to the function $\xi(\boldsymbol{z}; x)$, we know that $h_j^*(\boldsymbol{z})$ and $h_{ij}^*(\boldsymbol{z})$ are all identically zero. In particular,

$$h_{d_q - 1}^*(\boldsymbol{z}) = h(\Omega \boldsymbol{z}) h_{q\,d_q - 1}(\boldsymbol{z}) - h(\boldsymbol{z}) \left(h_{q\,d_q - 1}(\Omega \boldsymbol{z}) + (d_q - 1) h(\Omega \boldsymbol{z}) \right) = 0.$$

Since $h(\boldsymbol{z})h(\Omega \boldsymbol{z}) \neq 0$, we have

$$\frac{h_{q\,d_q-1}(\boldsymbol{z})}{h(\boldsymbol{z})} = \frac{h_{q\,d_q-1}(\Omega \boldsymbol{z})}{h(\Omega \boldsymbol{z})} + d_q - 1.$$

By Lemma 3.4.4, $h_{q\,d_q-1}(\boldsymbol{z})/h(\boldsymbol{z}) \in \mathbb{C}_v$ and so $d_q - 1 = 0$. By the assumption $\sum_{i=1}^{q} d_i > 1$, we get $q \geq 2$ and hence, we know in particular that

$$h^*_{1\,d_1}(\boldsymbol{z}) = h(\Omega\boldsymbol{z})h_{1\,d_1}(\boldsymbol{z}) - \gamma_1 h(\boldsymbol{z})h_{1,d_1}(\Omega\boldsymbol{z}) = 0.$$

Thus $h_{1\,d_1}(\boldsymbol{z})/h(\boldsymbol{z}) \in \mathbb{C}_v$ by Lemma 3.4.4. Since $\gamma_1 \neq 1$, we have $h_{1\,d_1}(\boldsymbol{z}) = 0$. Applying the induction hypothesis to the function $\psi(\boldsymbol{z}; x)$, we see that $h_{ij}(\boldsymbol{z})$ are all identically zero. In particular $h_{q\,d_q}(\boldsymbol{z}) = 0$, which is a contradiction. □

Proof of Theorem 3.2.4. We may assume that $\alpha_1, \ldots, \alpha_n$ and the eigenvalues of A are all contained in K. Since $f_1(\boldsymbol{z}), \ldots, f_l(\boldsymbol{z})$ are algebraically independent over $K(z_1, \ldots, z_n)$, we see that $\det A \neq 0$. Let $\boldsymbol{f}(\boldsymbol{z}) := {}^t(f_1(\boldsymbol{z}), \ldots, f_l(\boldsymbol{z}))$, $\boldsymbol{b}(\boldsymbol{z}) := {}^t(b_1(\boldsymbol{z}), \ldots, b_l(\boldsymbol{z}))$, and $\boldsymbol{g}(\boldsymbol{z}) := {}^t(g_1(\boldsymbol{z}), \ldots, g_m(\boldsymbol{z}))$. Iterating the functional equations (3.2.7) and (3.2.8), we have

$$\boldsymbol{f}(\boldsymbol{z}) = A^k \boldsymbol{f}(\Omega^k \boldsymbol{z}) + \boldsymbol{b}^{(k)}(\boldsymbol{z}) \quad (k \geq 0) \tag{3.4.3}$$

and

$$g_h(\boldsymbol{z}) = e_h^{(k)}(\boldsymbol{z}) g_h(\Omega^k \boldsymbol{z}) \quad (1 \leq h \leq m,\ k \geq 0), \tag{3.4.4}$$

where

$$\boldsymbol{b}^{(k)}(\boldsymbol{z}) = {}^t(b_1^{(k)}(\boldsymbol{z}), \ldots, b_l^{(k)}(\boldsymbol{z})) := \sum_{j=0}^{k-1} A^j \boldsymbol{b}(\Omega^j \boldsymbol{z}) \in K(z_1, \ldots, z_n)^l \tag{3.4.5}$$

and

$$e_h^{(k)}(\boldsymbol{z}) := \prod_{j=0}^{k-1} e_h(\Omega^j \boldsymbol{z}) \in K(z_1, \ldots, z_n). \tag{3.4.6}$$

We note here that, any power of Ω and the point $\boldsymbol{\alpha}$ also satisfy the four conditions (I)–(IV)$_v$. Indeed, it is clear that they satisfy the conditions (I)–(III)$_v$. If v is ∞, then we see by Lemma 3.1.3 that they satisfy the condition (IV)$_\infty$, and if v is a prime number p, then it is obvious that they satisfy the condition (IV)$_p$. Therefore, taking a sufficiently large integer k_0 and replacing Ω, A, $b_i(\boldsymbol{z})$, and $e_h(\boldsymbol{z})$ with Ω^{k_0}, A^{k_0}, $b_i^{(k_0)}(\boldsymbol{z})$, and $e_h^{(k_0)}(\boldsymbol{z})$, respectively, we may assume that $\Omega^k \boldsymbol{\alpha} \in U$ for all $k \geq 0$ and that the multiplicative subgroup G of $K^\times$ generated by the eigenvalues of A is torsion free.

Remark 3.4.5. We need the stronger condition $(\mathrm{IV})_p$ for this argument to be valid in the case where v is a prime number p.

Since $e_h(\Omega^k\boldsymbol{\alpha}) \neq 0$ $(1 \leq h \leq m)$ for all $k \geq 0$, by the functional equation (3.2.8), the condition $(\mathrm{III})_v$, and the assumption that $g_h(\mathbf{0}) \neq 0$ $(1 \leq h \leq m)$, we see that $g_h(\Omega^k\boldsymbol{\alpha}) \neq 0$ $(1 \leq h \leq m)$ for all $k \geq 0$.

To prove the theorem, we assume on the contrary that $f_i(\boldsymbol{\alpha})$ $(1 \leq i \leq l)$ and $g_h(\boldsymbol{\alpha})$ $(1 \leq h \leq m)$ are algebraically dependent. Then there exist a positive integer L and integers $\tau_{\boldsymbol{\lambda\mu}}$ $(\boldsymbol{\lambda} \in \mathcal{L},\ \boldsymbol{\mu} \in \mathcal{M})$, not all zero, such that

$$\sum_{\boldsymbol{\lambda}\in\mathcal{L},\,\boldsymbol{\mu}\in\mathcal{M}} \tau_{\boldsymbol{\lambda\mu}} \boldsymbol{f}(\boldsymbol{\alpha})^{\boldsymbol{\lambda}} \boldsymbol{g}(\boldsymbol{\alpha})^{\boldsymbol{\mu}} = 0,$$

where $\mathcal{L} \coloneqq \{\boldsymbol{\lambda} \in \mathbb{Z}_{\geq 0}^l \mid |\boldsymbol{\lambda}| \leq L\}$ and $\mathcal{M} \coloneqq \{0, 1, \ldots, L\}^m$. Let x_{ij} $(1 \leq i, j \leq l)$, w_i $(1 \leq i \leq l)$, y_i $(1 \leq i \leq l)$, x'_h $(1 \leq h \leq m)$, w'_h $(1 \leq h \leq m)$, and $t_{\boldsymbol{\lambda\mu}}$ $(\boldsymbol{\lambda} \in \mathcal{L},\ \boldsymbol{\mu} \in \mathcal{M})$ be variables and let

$$X \coloneqq \begin{pmatrix} x_{11} & \cdots & x_{1l} \\ \vdots & & \vdots \\ x_{l1} & \cdots & x_{ll} \end{pmatrix}, \quad \boldsymbol{w} \coloneqq \begin{pmatrix} w_1 \\ \vdots \\ w_l \end{pmatrix}, \quad \boldsymbol{y} \coloneqq \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix},$$

$$\boldsymbol{x}' \coloneqq \begin{pmatrix} x'_1 \\ \vdots \\ x'_m \end{pmatrix}, \quad \boldsymbol{w}' \coloneqq \begin{pmatrix} w'_1 \\ \vdots \\ w'_m \end{pmatrix}, \quad \boldsymbol{x}'\boldsymbol{w}' \coloneqq \begin{pmatrix} x'_1 w'_1 \\ \vdots \\ x'_m w'_m \end{pmatrix},$$

and

$$F(\boldsymbol{z}; \boldsymbol{t}) \coloneqq \sum_{\boldsymbol{\lambda}\in\mathcal{L},\,\boldsymbol{\mu}\in\mathcal{M}} t_{\boldsymbol{\lambda\mu}} \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\lambda}} \boldsymbol{g}(\boldsymbol{z})^{\boldsymbol{\mu}}.$$

We define $T_{\boldsymbol{\lambda\mu}}(\boldsymbol{t}; X; \boldsymbol{y}; \boldsymbol{x}')$ $(\boldsymbol{\lambda} \in \mathcal{L},\ \boldsymbol{\mu} \in \mathcal{M})$ by the equality

$$\sum_{\boldsymbol{\lambda}\in\mathcal{L},\,\boldsymbol{\mu}\in\mathcal{M}} t_{\boldsymbol{\lambda\mu}} (X\boldsymbol{w} + \boldsymbol{y})^{\boldsymbol{\lambda}} (\boldsymbol{x}'\boldsymbol{w}')^{\boldsymbol{\mu}} \eqqcolon \sum_{\boldsymbol{\lambda}\in\mathcal{L},\,\boldsymbol{\mu}\in\mathcal{M}} T_{\boldsymbol{\lambda\mu}}(\boldsymbol{t}; X; \boldsymbol{y}; \boldsymbol{x}') \boldsymbol{w}^{\boldsymbol{\lambda}} \boldsymbol{w}'^{\boldsymbol{\mu}}, \tag{3.4.7}$$

namely,

$$\begin{aligned} &T_{\boldsymbol{\lambda\mu}}(\boldsymbol{t}; X; \boldsymbol{y}; \boldsymbol{x}') \\ &\coloneqq \boldsymbol{x}'^{\boldsymbol{\mu}} \sum_{\substack{\boldsymbol{\nu}=(\nu_1,\ldots,\nu_l)\in\mathbb{Z}_{\geq 0}^l, \\ |\boldsymbol{\lambda}|\leq|\boldsymbol{\nu}|\leq L}} t_{\boldsymbol{\nu\mu}} \sum_{\substack{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_l\in\mathbb{Z}_{\geq 0}^{l+1}, \\ \boldsymbol{\nu}_i=(\nu_{i0},\nu_{i1},\ldots,\nu_{il}), \\ |\boldsymbol{\nu}_i|=\nu_i\ (1\leq i\leq l), \\ \sum_{i=1}^l \nu_{ij}=\lambda_j\ (1\leq j\leq l)}} \prod_{i=1}^{l} \binom{\nu_i}{\nu_{i0}\ \nu_{i1}\ \cdots\ \nu_{il}} y_i^{\nu_{i0}} x_{i1}^{\nu_{i1}} \cdots x_{il}^{\nu_{il}} \end{aligned}$$

for any $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_l) \in \mathcal{L}$ and $\boldsymbol{\mu} \in \mathcal{M}$. Letting

$$\boldsymbol{e}^{(k)}(\boldsymbol{z}) := {}^t(e_1^{(k)}(\boldsymbol{z}), \dots, e_m^{(k)}(\boldsymbol{z}))$$

and

$$\boldsymbol{e}^{(k)}(\boldsymbol{z})\boldsymbol{g}(\Omega^k \boldsymbol{z}) := {}^t(e_1^{(k)}(\boldsymbol{z})g_1(\Omega^k \boldsymbol{z}), \dots, e_m^{(k)}(\boldsymbol{z})g_m(\Omega^k \boldsymbol{z})),$$

by the functional equations (3.4.3) and (3.4.4), we have

$$\begin{aligned} F(\boldsymbol{z}; \boldsymbol{t}) &= \sum_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}} \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\lambda}} \boldsymbol{g}(\boldsymbol{z})^{\boldsymbol{\mu}} \\ &= \sum_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}} (A^k \boldsymbol{f}(\Omega^k \boldsymbol{z}) + \boldsymbol{b}^{(k)}(\boldsymbol{z}))^{\boldsymbol{\lambda}} (\boldsymbol{e}^{(k)}(\boldsymbol{z})\boldsymbol{g}(\Omega^k \boldsymbol{z}))^{\boldsymbol{\mu}} \\ &= \sum_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} T_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\boldsymbol{t}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{z}); \boldsymbol{e}^{(k)}(\boldsymbol{z})) \boldsymbol{f}(\Omega^k \boldsymbol{z})^{\boldsymbol{\lambda}} \boldsymbol{g}(\Omega^k \boldsymbol{z})^{\boldsymbol{\mu}} \\ &= F(\Omega^k \boldsymbol{z}; \boldsymbol{T}(\boldsymbol{t}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{z}); \boldsymbol{e}^{(k)}(\boldsymbol{z}))) \end{aligned}$$

for all $k \geq 0$. Hence

$$F(\Omega^k \boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) = F(\boldsymbol{\alpha}; \boldsymbol{\tau}) = 0 \quad (k \geq 0). \tag{3.4.8}$$

We define an ideal $V(\boldsymbol{\tau})$ of $K[\boldsymbol{t}]$ by

$$V(\boldsymbol{\tau}) := \{Q(\boldsymbol{t}) \in K[\boldsymbol{t}] \mid Q(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}')) = 0 \text{ for all } k \geq 0\}.$$

Lemma 3.4.6. *$V(\boldsymbol{\tau})$ is a prime ideal of $K[\boldsymbol{t}]$.*

For the proof we use the following:

Lemma 3.4.7 (Skolem-Lech-Mahler, cf. Cassels [3, Theorem 1.1], see also Nishioka [23, Theorem 2.5.3])**.** *Let C be a field of characteristic 0. Let $\gamma_1, \dots, \gamma_s$ be nonzero distinct elements of C and $P_1(X), \dots, P_s(X) \in C[X]$ nonzero polynomials. Then, if $\{k \in \mathbb{Z}_{\geq 0} \mid \sum_{i=1}^s P_i(k)\gamma_i^k = 0\}$ is an infinite set, then γ_i/γ_j is a root of unity for some distinct i, j.*

Proof of Lemma 3.4.6. Recall that G is a torsion free subgroup of $K^\times$ generated by the eigenvalues of A. Let $\mathcal{R}_1$ be the subset of $(K[\boldsymbol{y}; \boldsymbol{x}'])^{\mathbb{Z}_{\geq 0}}$ consisting of the sequences of the form $\{\sum_{\gamma\in\Gamma} p_\gamma(k)\gamma^k\}_{k\geq 0}$, where Γ is a finite subset of G, independent of k, and $p_\gamma(Y)$ ($\gamma \in \Gamma$) are polynomials with coefficients in $K[\boldsymbol{y};\boldsymbol{x}']$. Then $\mathcal{R}_1$ forms a commutative ring including $K[\boldsymbol{y};\boldsymbol{x}']$ under termwise addition and multiplication. If we put $A^k =: (a_{ij}^{(k)})$, then $\{a_{ij}^{(k)}\}_{k\geq 0} \in \mathcal{R}_1$ for any $1 \leq i,j \leq l$. Since $T_{\boldsymbol{\lambda\mu}}(\boldsymbol{\tau}; X; \boldsymbol{y}; \boldsymbol{x}') \in (\mathbb{Z}[\boldsymbol{y};\boldsymbol{x}'])[\{x_{ij}\}]$, we have $\{T_{\boldsymbol{\lambda\mu}}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}')\}_{k\geq 0} \in \mathcal{R}_1$ for any $\boldsymbol{\lambda} \in \mathcal{L}$ and $\boldsymbol{\mu} \in \mathcal{M}$. Therefore, if $P(\boldsymbol{t}) \in K[\boldsymbol{t}]$, then $\{P(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}'))\}_{k\geq 0} \in \mathcal{R}_1$, so that there exist a finite subset $\Gamma = \Gamma(P)$ of G and nonzero polynomials $p_\gamma(Y) \in (K[\boldsymbol{y};\boldsymbol{x}'])[Y]$ ($\gamma \in \Gamma$) such that

$$P(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}')) = \sum_{\gamma\in\Gamma} p_\gamma(k)\gamma^k$$

for all $k \geq 0$.

To prove the lemma, let $P_1(\boldsymbol{t}), P_2(\boldsymbol{t}) \in K[\boldsymbol{t}]$ and suppose that $P_1(\boldsymbol{t})P_2(\boldsymbol{t}) \in V(\boldsymbol{\tau})$. Since $P_1(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}'))P_2(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}')) = 0$ for all $k \geq 0$, we may assume that $P_1(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}')) = 0$ for infinitely many k. Hence, if $\Gamma(P_1) \neq \emptyset$, then Lemma 3.4.7 implies that there exist distinct $\gamma, \gamma' \in \Gamma(P_1)$ such that γ/γ' is a root of unity, which contradicts the fact that G is torsion free. Thus $\Gamma(P_1) = \emptyset$ and $P_1(\boldsymbol{t}) \in V(\boldsymbol{\tau})$. □

Proposition 3.4.8. *The following two conditions are equivalent for any* $P(\boldsymbol{z};\boldsymbol{t}) \in K[\boldsymbol{z};\boldsymbol{t}]$.

(i) $P(\Omega^k\boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) = 0$ *for all sufficiently large* k.

(ii) *If we put* $P(\boldsymbol{z};\boldsymbol{t}) =: \sum_{\boldsymbol{\eta}\in\mathcal{H}} Q_{\boldsymbol{\eta}}(\boldsymbol{t})\boldsymbol{z}^{\boldsymbol{\eta}}$, *where* $Q_{\boldsymbol{\eta}}(\boldsymbol{t}) \in K[\boldsymbol{t}]$ ($\eta \in \mathcal{H}$) *and* $\mathcal{H}$ *is a finite subset of* $\mathbb{Z}_{\geq 0}^n$, *then* $Q_{\boldsymbol{\eta}}(\boldsymbol{t}) \in V(\boldsymbol{\tau})$ *for any* $\boldsymbol{\eta} \in \mathcal{H}$.

Proof. We only prove that the condition (i) implies (ii) since the converse is trivial. Let $\mathcal{R}_2$ be the subset of $(\overline{\mathbb{Q}}_v[w_1, \ldots, w_l, \frac{1}{w_1'}, \ldots, \frac{1}{w_m'}])^{\mathbb{Z}_{\geq 0}}$ consisting of the sequences of the form $\{\sum_{\gamma\in\Gamma} q_\gamma(k)\gamma^k\}_{k\geq 0}$, where Γ is a finite subset of G, independent of k, and $q_\gamma(Y)$ ($\gamma \in \Gamma$) are polynomials with coefficients in $\overline{\mathbb{Q}}_v[w_1, \ldots, w_l, \frac{1}{w_1'}, \ldots, \frac{1}{w_m'}]$. Then $\mathcal{R}_2$ forms a commutative ring including $\overline{\mathbb{Q}}_v[w_1, \ldots, w_l, \frac{1}{w_1'}, \ldots, \frac{1}{w_m'}]$ under termwise addition and multiplication. In the same way as in the proof of Lemma 3.4.6, we see that $\{Q_{\boldsymbol{\eta}}(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{f}(\boldsymbol{\alpha}) - A^k\boldsymbol{w}; \boldsymbol{g}(\boldsymbol{\alpha})/\boldsymbol{w}'))\}_{k\geq 0} \in \mathcal{R}_2$ for any $\boldsymbol{\eta} \in \mathcal{H}$, where

$\boldsymbol{g}(\boldsymbol{\alpha})/\boldsymbol{w}' := {}^t(g_1(\boldsymbol{\alpha})/w'_1, \ldots, g_m(\boldsymbol{\alpha})/w'_m)$. Hence there exist finite sets $\mathcal{N} \subset \mathbb{Z}_{\geq 0}^l$ and $\mathcal{X} \subset \mathbb{Z}_{\geq 0}^m$, distinct elements $\gamma_1, \ldots, \gamma_q$ of G, and positive integers $d_1, \ldots, d_q$ such that

$$Q_{\boldsymbol{\eta}}(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{f}(\boldsymbol{\alpha}) - A^k \boldsymbol{w}; \boldsymbol{g}(\boldsymbol{\alpha})/\boldsymbol{w}')) = \sum_{\boldsymbol{\nu} \in \mathcal{N}, \boldsymbol{\xi} \in \mathcal{X}} R_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}}(k) \boldsymbol{w}^{\boldsymbol{\nu}} \boldsymbol{w}'^{-\boldsymbol{\xi}}$$

for all $k \geq 0$ and $\boldsymbol{\eta} \in \mathcal{H}$, where

$$R_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}}(k) = \sum_{i=1}^{q} \sum_{j=1}^{d_i} r_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}ij} k^{j-1} \gamma_i^k, \quad r_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}ij} \in \overline{\mathbb{Q}}_v.$$

We claim that every $\{R_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}}(k)\}_{k \geq 0}$ is the null sequence. Since $g_h(\mathbf{0}) \neq 0$ $(1 \leq h \leq m)$,

$$h_{ij}(\boldsymbol{z}) := \sum_{\boldsymbol{\eta} \in \mathcal{H}} \sum_{\boldsymbol{\nu} \in \mathcal{N}, \boldsymbol{\xi} \in \mathcal{X}} r_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}ij} \boldsymbol{f}(\boldsymbol{z})^{\boldsymbol{\nu}} \boldsymbol{g}(\boldsymbol{z})^{-\boldsymbol{\xi}} \boldsymbol{z}^{\boldsymbol{\eta}} \quad (1 \leq i \leq q, \ 1 \leq j \leq d_i)$$

are formal power series in the variables $z_1, \ldots, z_n$ with coefficients in $\overline{\mathbb{Q}}_v$ which converge in an n-polydisc around the origin of $\mathbb{C}_v^n$. Define

$$\psi(\boldsymbol{z}; x) := \sum_{i=1}^{q} \sum_{j=1}^{d_i} x^{j-1} \gamma_i^x h_{ij}(\boldsymbol{z}).$$

By the condition (i) of the proposition and the functional equations (3.4.3) and (3.4.4), we see that

$$\begin{aligned}
0 &= P(\Omega^k \boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) \\
&= \sum_{\boldsymbol{\eta} \in \mathcal{H}} Q_{\boldsymbol{\eta}}(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))(\Omega^k \boldsymbol{\alpha})^{\boldsymbol{\eta}} \\
&= \sum_{\boldsymbol{\eta} \in \mathcal{H}} \left(\sum_{\boldsymbol{\nu} \in \mathcal{N}, \boldsymbol{\xi} \in \mathcal{X}} R_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}}(k) \boldsymbol{f}(\Omega^k \boldsymbol{\alpha})^{\boldsymbol{\nu}} \boldsymbol{g}(\Omega^k \boldsymbol{\alpha})^{-\boldsymbol{\xi}} \right) (\Omega^k \boldsymbol{\alpha})^{\boldsymbol{\eta}} \\
&= \psi(\Omega^k \boldsymbol{\alpha}; k)
\end{aligned}$$

for all sufficiently large k. Then Lemma 3.4.3 implies that $h_{ij}(\boldsymbol{z}) = 0$ for any $1 \leq i \leq q$ and $1 \leq j \leq d_i$. Therefore, since $f_1(\boldsymbol{z}), \ldots, f_l(\boldsymbol{z}), g_1(\boldsymbol{z}), \ldots, g_m(\boldsymbol{z})$ are algebraically independent over $\overline{\mathbb{Q}}_v(z_1, \ldots, z_n)$ by Lemma 3.4.2, we have $r_{\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\xi}ij} = 0$ for any $\boldsymbol{\eta}$, $\boldsymbol{\nu}$, $\boldsymbol{\xi}$, i, and j. This proves our claim.

By the claim we have

$$Q_{\boldsymbol{\eta}}(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{f}(\boldsymbol{\alpha}) - A^k \boldsymbol{w}; \boldsymbol{g}(\boldsymbol{\alpha})/\boldsymbol{w}')) = 0$$

for all $k \geq 0$ and $\boldsymbol{\eta} \in \mathcal{H}$. Noting that $\det A \neq 0$ and that $g_h(\boldsymbol{\alpha}) \neq 0$ $(1 \leq h \leq m)$, we obtain

$$Q_{\boldsymbol{\eta}}(\boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{y}; \boldsymbol{x}')) = 0$$

for all $k \geq 0$ and $\boldsymbol{\eta} \in \mathcal{H}$, which implies the condition (ii) of the proposition. □

Definition 3.4.9. For $P(\boldsymbol{z}; \boldsymbol{t}) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}^n_{\geq 0}} p_{\boldsymbol{\eta}}(\boldsymbol{t}) \boldsymbol{z}^{\boldsymbol{\eta}} \in (K[\boldsymbol{t}])[[z_1, \dots, z_n]]$ we define

$$\operatorname{ind} P(\boldsymbol{z}; \boldsymbol{t}) := \min\{|\boldsymbol{\eta}| \mid p_{\boldsymbol{\eta}}(\boldsymbol{t}) \notin V(\boldsymbol{\tau})\},$$

where $\min \emptyset := \infty$.

By Lemma 3.4.6, we have

$$\operatorname{ind}(P_1(\boldsymbol{z}; \boldsymbol{t}) P_2(\boldsymbol{z}; \boldsymbol{t})) = \operatorname{ind} P_1(\boldsymbol{z}; \boldsymbol{t}) + \operatorname{ind} P_2(\boldsymbol{z}; \boldsymbol{t}). \tag{3.4.9}$$

Lemma 3.4.10. $\operatorname{ind} F(\boldsymbol{z}; \boldsymbol{t}) < \infty$.

Proof. Since $f_1(\boldsymbol{z}), \dots, f_l(\boldsymbol{z}), g_1(\boldsymbol{z}), \dots, g_m(\boldsymbol{z})$ are algebraically independent, we see that $F(\boldsymbol{z}; \boldsymbol{\tau}) \neq 0$. By the condition $(\mathrm{IV})_v$, there exists k_0 such that $F(\Omega^{k_0} \boldsymbol{\alpha}; \boldsymbol{\tau}) \neq 0$. Let

$$F(\boldsymbol{z}; \boldsymbol{t}) =: \sum_{\boldsymbol{\eta} \in \mathbb{Z}^n_{\geq 0}} p_{\boldsymbol{\eta}}(\boldsymbol{t}) \boldsymbol{z}^{\boldsymbol{\eta}},$$

where $p_{\boldsymbol{\eta}}(\boldsymbol{t}) \in K[\boldsymbol{t}]$. Suppose on the contrary that $\operatorname{ind} F(\boldsymbol{z}; \boldsymbol{t}) = \infty$. Then $p_{\boldsymbol{\eta}}(\boldsymbol{t}) \in V(\boldsymbol{\tau})$ for every $\boldsymbol{\eta} \in \mathbb{Z}^n_{\geq 0}$. Noting that $\boldsymbol{\tau} = \boldsymbol{T}(\boldsymbol{\tau}; I_l; \boldsymbol{0}; \boldsymbol{1}) = \boldsymbol{T}(\boldsymbol{\tau}; A^0; \boldsymbol{0}; \boldsymbol{1})$ by (3.4.7), we have

$$F(\Omega^{k_0} \boldsymbol{\alpha}; \boldsymbol{\tau}) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}^n_{\geq 0}} p_{\boldsymbol{\eta}}(\boldsymbol{T}(\boldsymbol{\tau}; A^0; \boldsymbol{0}; \boldsymbol{1}))(\Omega^{k_0} \boldsymbol{\alpha})^{\boldsymbol{\eta}} = 0,$$

which is a contradiction. □

Let N be a nonnegative integer, $R(N)$ the K-vector space of polynomials in $K[\boldsymbol{t}]$ of degree at most N in each $t_{\boldsymbol{\lambda\mu}}$, and $d(N)$ the dimension over K of the quotient space $\overline{R}(N) := R(N)/(R(N) \cap V(\boldsymbol{\tau}))$. The coset containing a polynomial $P(\boldsymbol{t})$ of $R(N)$ in $\overline{R}(N)$ is denoted by $\overline{P(\boldsymbol{t})}$.

Lemma 3.4.11. $d(2N) \leq 2^{(L+1)^{l+m}} d(N)$.

Proof. Let $\{\overline{Q_1(\boldsymbol{t})}, \ldots, \overline{Q_{d(N)}(\boldsymbol{t})}\}$ be a basis of $\overline{R}(N)$ over K, where $Q_1(\boldsymbol{t}), \ldots, Q_{d(N)}(\boldsymbol{t}) \in R(N)$. Let $Q(\boldsymbol{t}) \in R(2N)$. Then $Q(\boldsymbol{t})$ is written in the form

$$Q(\boldsymbol{t}) = \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^{\mathcal{L} \times \mathcal{M}}} \left(\prod_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}(\boldsymbol{\lambda},\boldsymbol{\mu})N} \right) Q_{\boldsymbol{\varepsilon}}(\boldsymbol{t}),$$

where $Q_{\boldsymbol{\varepsilon}}(\boldsymbol{t}) \in R(N)$. For each $\boldsymbol{\varepsilon} \in \{0,1\}^{\mathcal{L} \times \mathcal{M}}$, there exist $q_{\boldsymbol{\varepsilon}1}, \ldots, q_{\boldsymbol{\varepsilon}d(N)} \in K$ such that

$$\overline{Q_{\boldsymbol{\varepsilon}}(\boldsymbol{t})} = \sum_{i=1}^{d(N)} q_{\boldsymbol{\varepsilon}i} \overline{Q_i(\boldsymbol{t})}$$

in $\overline{R}(N)$. Then we can check that

$$\begin{aligned} \overline{\left(\prod_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}(\boldsymbol{\lambda},\boldsymbol{\mu})N} \right) Q_{\boldsymbol{\varepsilon}}(\boldsymbol{t})} &= \overline{\left(\prod_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}(\boldsymbol{\lambda},\boldsymbol{\mu})N} \right) \sum_{i=1}^{d(N)} q_{\boldsymbol{\varepsilon}i} Q_i(\boldsymbol{t})} \\ &= \sum_{i=1}^{d(N)} q_{\boldsymbol{\varepsilon}i} \overline{\left(\prod_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}(\boldsymbol{\lambda},\boldsymbol{\mu})N} \right) Q_i(\boldsymbol{t})} \end{aligned}$$

in $\overline{R}(2N)$. Hence

$$\overline{Q(\boldsymbol{t})} = \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^{\mathcal{L} \times \mathcal{M}}} \sum_{i=1}^{d(N)} q_{\boldsymbol{\varepsilon}i} \overline{\left(\prod_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}(\boldsymbol{\lambda},\boldsymbol{\mu})N} \right) Q_i(\boldsymbol{t})}$$

in $\overline{R}(2N)$, which implies that

$$\left\{ \overline{\left(\prod_{\boldsymbol{\lambda} \in \mathcal{L},\, \boldsymbol{\mu} \in \mathcal{M}} t_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\varepsilon}(\boldsymbol{\lambda},\boldsymbol{\mu})N} \right) Q_i(\boldsymbol{t})} \;\middle|\; \boldsymbol{\varepsilon} \in \{0,1\}^{\mathcal{L} \times \mathcal{M}},\ 1 \leq i \leq d(N) \right\}$$

generates $\overline{R}(2N)$ over K. Therefore we have

$$d(2N) \leq 2^{\#(\mathcal{L} \times \mathcal{M})} d(N) \leq 2^{\#\{0,1,\ldots,L\}^{l+m}} d(N) = 2^{(L+1)^{l+m}} d(N).$$

□

In what follows, $c_1, c_2, \ldots$ denote positive constants independent of N and k. If they depend on N, then we denote them by $c_1(N), c_2(N), \ldots$.

Proposition 3.4.12. *Let N be a sufficiently large positive integer. Then there exist $N+1$ polynomials $P_0(\boldsymbol{z};\boldsymbol{t}),\dots,P_N(\boldsymbol{z};\boldsymbol{t}) \in \mathcal{O}_K[\boldsymbol{z};\boldsymbol{t}]$ with degree at most N in each of the variables z_i $(1 \le i \le n)$ and $t_{\boldsymbol{\lambda\mu}}$ $(\boldsymbol{\lambda} \in \mathcal{L},\ \boldsymbol{\mu} \in \mathcal{M})$ such that the following two conditions are satisfied.*

(i) $\operatorname{ind} P_0(\boldsymbol{z};\boldsymbol{t}) < \infty$.

(ii) $\operatorname{ind}(\sum_{h=0}^{N} P_h(\boldsymbol{z};\boldsymbol{t})F(\boldsymbol{z};\boldsymbol{t})^h) \ge c_1(N+1)^{1+1/n}$.

Proof. Let $\{\overline{Q_1^{(N)}(\boldsymbol{t})},\dots,\overline{Q_{d(N)}^{(N)}(\boldsymbol{t})}\}$ and $\{\overline{Q_1^{(2N)}(\boldsymbol{t})},\dots,\overline{Q_{d(2N)}^{(2N)}(\boldsymbol{t})}\}$ be a basis over K of $\overline{R}(N)$ and that of $\overline{R}(2N)$, respectively. We may assume that $Q_1^{(N)}(\boldsymbol{t}),\dots,Q_{d(N)}^{(N)}(\boldsymbol{t}) \in R(N) \cap \mathcal{O}_K[\boldsymbol{t}]$ and that $Q_1^{(2N)}(\boldsymbol{t}),\dots,Q_{d(2N)}^{(2N)}(\boldsymbol{t}) \in R(2N) \cap \mathcal{O}_K[\boldsymbol{t}]$. For each i and j with $1 \le i,j \le d(N)$, there exist $g_{ijk} \in K$ $(1 \le k \le d(2N))$ such that

$$\overline{Q_i^{(N)}(\boldsymbol{t})Q_j^{(N)}(\boldsymbol{t})} = \sum_{k=1}^{d(2N)} g_{ijk}\overline{Q_k^{(2N)}(\boldsymbol{t})} \tag{3.4.10}$$

in $\overline{R}(2N)$. For each h $(0 \le h \le N)$, put

$$F(\boldsymbol{z};\boldsymbol{t})^h =: \sum_{\boldsymbol{\nu} \in \mathbb{Z}_{\ge 0}^n} F_{h\boldsymbol{\nu}}(\boldsymbol{t})\boldsymbol{z}^{\boldsymbol{\nu}}.$$

Then $F_{h\boldsymbol{\nu}}(\boldsymbol{t}) \in R(h) \subset R(N)$ for any $\boldsymbol{\nu} \in \mathbb{Z}_{\ge 0}^n$ and hence, there exist $f_{h\boldsymbol{\nu}j} \in K$ $(1 \le j \le d(N))$ such that

$$\overline{F_{h\boldsymbol{\nu}}(\boldsymbol{t})} = \sum_{j=1}^{d(N)} f_{h\boldsymbol{\nu}j}\overline{Q_j^{(N)}(\boldsymbol{t})} \tag{3.4.11}$$

in $\overline{R}(N)$.

Let $p_{h\boldsymbol{\xi}i}$ $(0 \le h \le N,\ \boldsymbol{\xi} \in \{0,1,\dots,N\}^n, 1 \le i \le d(N))$ be unknowns in $\mathcal{O}_K$. Put

$$P_{h\boldsymbol{\xi}}(\boldsymbol{t}) := \sum_{i=1}^{d(N)} p_{h\boldsymbol{\xi}i}Q_i^{(N)}(\boldsymbol{t}) \in R(N) \cap \mathcal{O}_K[\boldsymbol{t}],$$

$$P_h(\boldsymbol{z};\boldsymbol{t}) := \sum_{\boldsymbol{\xi} \in \{0,1,\dots,N\}^n} P_{h\boldsymbol{\xi}}(\boldsymbol{t})\boldsymbol{z}^{\boldsymbol{\xi}} \in \mathcal{O}_K[\boldsymbol{z};\boldsymbol{t}],$$

and

$$E(\boldsymbol{z};\boldsymbol{t}) := \sum_{h=0}^{N} P_h(\boldsymbol{z};\boldsymbol{t})F(\boldsymbol{z};\boldsymbol{t})^h =: \sum_{\boldsymbol{\eta}\in\mathbb{Z}_{\geq 0}^n} E_{\boldsymbol{\eta}}(\boldsymbol{t})\boldsymbol{z}^{\boldsymbol{\eta}}.$$

Then

$$E_{\boldsymbol{\eta}}(\boldsymbol{t}) = \sum_{h=0}^{N}\sum_{\boldsymbol{\xi}\in\mathcal{X}_{\boldsymbol{\eta}}} P_{h\boldsymbol{\xi}}(\boldsymbol{t})F_{h\,\boldsymbol{\eta}-\boldsymbol{\xi}}(\boldsymbol{t}) \in R(2N)$$

for each $\boldsymbol{\eta}\in\mathbb{Z}_{\geq 0}^n$, where $\mathcal{X}_{\boldsymbol{\eta}}$ denotes the set of $\boldsymbol{\xi}\in\{0,1,\ldots,N\}^n$ whose components do not exceed the corresponding component of $\boldsymbol{\eta}$. Using (3.4.10) and (3.4.11), we can check that

$$\begin{aligned}\overline{E_{\boldsymbol{\eta}}(\boldsymbol{t})} &= \sum_{h=0}^{N}\sum_{\boldsymbol{\xi}\in\mathcal{X}_{\boldsymbol{\eta}}} \overline{P_{h\boldsymbol{\xi}}(\boldsymbol{t})F_{h\,\boldsymbol{\eta}-\boldsymbol{\xi}}(\boldsymbol{t})}\\ &= \sum_{k=1}^{d(2N)}\left\{\sum_{h=0}^{N}\sum_{\boldsymbol{\xi}\in\mathcal{X}_{\boldsymbol{\eta}}}\sum_{i=1}^{d(N)}\left(\sum_{j=1}^{d(N)} f_{h\,\boldsymbol{\eta}-\boldsymbol{\xi}\,j}\,g_{ijk}\right)p_{h\boldsymbol{\xi}i}\right\}\overline{Q_k^{(2N)}(\boldsymbol{t})}\end{aligned}$$

in $\overline{R}(2N)$ for any $\boldsymbol{\eta}\in\mathbb{Z}_{\geq 0}^n$. Choose N so large that $J := [2^{-(L+1)^{l+m}/n}(N+1)^{1+1/n}]-1$ is a positive integer. Then $\operatorname{ind} E(\boldsymbol{z};\boldsymbol{t}) \geq J$ holds if and only if

$$\sum_{h=0}^{N}\sum_{\boldsymbol{\xi}\in\mathcal{X}_{\boldsymbol{\eta}}}\sum_{i=1}^{d(N)}\left(\sum_{j=1}^{d(N)} f_{h\,\boldsymbol{\eta}-\boldsymbol{\xi}\,j}\,g_{ijk}\right)p_{h\boldsymbol{\xi}i} = 0 \tag{3.4.12}$$

for any $k\in\{1,\ldots,d(2N)\}$ and $\boldsymbol{\eta}\in\mathbb{Z}_{\geq 0}^n$ with $|\boldsymbol{\eta}|\leq J-1$. Here, the number of the unknowns $p_{h\boldsymbol{\xi}i}$ is equal to $(N+1)^{n+1}d(N)$ and that of the equations is equal to $\binom{J+n-1}{n}d(2N)$, which do not exceed $J^n d(2N)$. Since $J^n < 2^{-(L+1)^{l+m}}(N+1)^{n+1}$, we see by Lemma 3.4.11 that

$$J^n d(2N) \leq J^n 2^{(L+1)^{l+m}} d(N) < (N+1)^{n+1}d(N).$$

Hence the system (3.4.12) has nontrivial solutions $\underline{p_{h\boldsymbol{\xi}i}}$ in $\mathcal{O}_K$. Using these $p_{h\boldsymbol{\xi}i}$, we construct the polynomials $P_h(\boldsymbol{z};\boldsymbol{t})$ above. Since $\overline{Q_1^{(N)}(\boldsymbol{t})},\ldots,\overline{Q_{d(N)}^{(N)}(\boldsymbol{t})}$ are linearly independent over K, we see that $\operatorname{ind} P_h(\boldsymbol{z};\boldsymbol{t}) < \infty$ for some h with $0\leq h\leq N$. Let r be the least one among such h and put

$$E_0(\boldsymbol{z};\boldsymbol{t}) := \sum_{h=r}^{N} P_h(\boldsymbol{z};\boldsymbol{t})F(\boldsymbol{z};\boldsymbol{t})^{h-r}.$$

By (3.4.9) and the fact that $\operatorname{ind} P_h(\boldsymbol{z};\boldsymbol{t}) = \infty$ for any h with $0 \le h \le r-1$, we see that

$$\begin{aligned}\operatorname{ind} E(\boldsymbol{z};\boldsymbol{t}) = \operatorname{ind}\left(\sum_{h=r}^{N} P_h(\boldsymbol{z};\boldsymbol{t})F(\boldsymbol{z};\boldsymbol{t})^h\right) &= \operatorname{ind}(F(\boldsymbol{z};\boldsymbol{t})^r E_0(\boldsymbol{z};\boldsymbol{t}))\\ &= r \operatorname{ind} F(\boldsymbol{z};\boldsymbol{t}) + \operatorname{ind} E_0(\boldsymbol{z};\boldsymbol{t}).\end{aligned}$$

Using Lemma 3.4.10, we have

$$\begin{aligned}\operatorname{ind} E_0(\boldsymbol{z};\boldsymbol{t}) &= \operatorname{ind} E(\boldsymbol{z};\boldsymbol{t}) - r \operatorname{ind} F(\boldsymbol{z};\boldsymbol{t})\\ &\ge J - N \operatorname{ind} F(\boldsymbol{z};\boldsymbol{t})\\ &> 2^{-(L+1)^{l+m}/n}(N+1)^{1+1/n} - 2 - N \operatorname{ind} F(\boldsymbol{z};\boldsymbol{t}).\end{aligned}$$

Hence, letting c_1 be a positive constant less than $2^{-(L+1)^{l+m}/n}$, we obtain $\operatorname{ind} E_0(\boldsymbol{z};\boldsymbol{t}) \ge c_1(N+1)^{1+1/n}$ for any sufficiently large N. The proposition is proved. □

Let $E(\boldsymbol{z};\boldsymbol{t})$ be the $\sum_{h=0}^{N} P_h(\boldsymbol{z};\boldsymbol{t})F(\boldsymbol{z};\boldsymbol{t})^h$ in Proposition 3.4.12 and ρ the maximum of the absolute values of the eigenvalues of Ω.

Proposition 3.4.13. *If $k > c_2(N)$, then*

$$\log |E(\Omega^k \boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))|_v \le -c_3(N+1)^{1+1/n}\rho^k.$$

Proof. Since $f_j(\Omega^k\boldsymbol{\alpha}) \to f_j(\boldsymbol{0})$ $(k \to \infty)$ for $1 \le j \le l$, by the functional equation (3.4.3) we have $|b_i^{(k)}(\boldsymbol{\alpha})|_v \le c_4^k$ for $1 \le i \le l$. Similarly, since $g_h(\Omega^k\boldsymbol{\alpha}) \to g_h(\boldsymbol{0}) \ne 0$ $(k \to \infty)$ for $1 \le h \le m$, by the functional equation (3.4.4) we have $|e_h^{(k)}(\boldsymbol{\alpha})|_v \le c_5$ for $1 \le h \le m$. Hence $|T_{\boldsymbol{\lambda\mu}}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))|_v \le c_6^k$ for $\boldsymbol{\lambda} \in \mathcal{L}$ and $\boldsymbol{\mu} \in \mathcal{M}$. We note that $E(\boldsymbol{z};\boldsymbol{t})$ is a polynomial in the variables $t_{\boldsymbol{\lambda\mu}}$ $(\boldsymbol{\lambda} \in \mathcal{L},\ \boldsymbol{\mu} \in \mathcal{M})$ with degree at most $2N$ in each variable whose coefficients are power series convergent in U. Let

$$E(\boldsymbol{z};\boldsymbol{t}) =: \sum_{\boldsymbol{\nu} \in \{0,1,\dots,2N\}^s} h_{\boldsymbol{\nu}}(\boldsymbol{z})\boldsymbol{t}^{\boldsymbol{\nu}}, \qquad h_{\boldsymbol{\nu}}(\boldsymbol{z}) =: \sum_{\boldsymbol{\xi} \in \mathbb{Z}_{\ge 0}^n} h_{\boldsymbol{\nu\xi}}\boldsymbol{z}^{\boldsymbol{\xi}} \in K[[\boldsymbol{z}]],$$

where $s := \#\mathcal{L} \times \#\mathcal{M} = \binom{L+l}{l}(L+1)^m$. Then we have

$$|h_{\boldsymbol{\nu\xi}}|_v \le c_7(N)c_8^{|\boldsymbol{\xi}|} \quad (\boldsymbol{\nu} \in \{0,1,\dots,2N\}^s,\ \boldsymbol{\xi} \in \mathbb{Z}_{\ge 0}^n)$$

and

$$E(\boldsymbol{z};\boldsymbol{t}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n_{\geq 0}} \left(\sum_{\boldsymbol{\nu}\in\{0,1,\dots,2N\}^s} h_{\boldsymbol{\nu}\boldsymbol{\xi}} \boldsymbol{t}^{\boldsymbol{\nu}} \right) \boldsymbol{z}^{\boldsymbol{\xi}}.$$

Therefore

$$|E(\Omega^k\boldsymbol{\alpha};\boldsymbol{T}(\boldsymbol{\tau};A^k;\boldsymbol{b}^{(k)}(\boldsymbol{\alpha});\boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))|_v \leq c_9(N)c_{10}^{Nk} \sum_{\substack{\boldsymbol{\xi}\in\mathbb{Z}^n_{\geq 0},\\ |\boldsymbol{\xi}|\geq I}} c_8^{|\boldsymbol{\xi}|}|(\Omega^k\boldsymbol{\alpha})^{\boldsymbol{\xi}}|_v,$$

where $I := \operatorname{ind} E(\boldsymbol{z};\boldsymbol{t})$. By the condition (III), there exists a positive constant $\theta < 1$ such that $|\alpha_i^{(k)}|_v \leq \theta^{\rho^k}$ for $1 \leq i \leq n$ and for all sufficiently large k. Hence

$$\begin{aligned} |E(\Omega^k\boldsymbol{\alpha};\boldsymbol{T}(\boldsymbol{\tau};A^k;\boldsymbol{b}^{(k)}(\boldsymbol{\alpha});\boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))|_v &\leq c_9(N)c_{10}^{Nk} \sum_{i=1}^{n} \sum_{\substack{\boldsymbol{\xi}=(\xi_1,\dots,\xi_n)\in\mathbb{Z}^n_{\geq 0},\\ \xi_i\geq I/n}} (c_8\theta^{\rho^k})^{|\boldsymbol{\xi}|} \\ &\leq nc_9(N)c_{10}^{Nk}(c_8\theta^{\rho^k})^{I/n}/(1-c_8\theta^{\rho^k})^n. \end{aligned}$$

Since $I \geq c_1(N+1)^{1+1/n}$ by the condition (ii) of Proposition 3.4.12, we see that, if $k \geq c_2(N)$, then

$$\log |E(\Omega^k\boldsymbol{\alpha};\boldsymbol{T}(\boldsymbol{\tau};A^k;\boldsymbol{b}^{(k)}(\boldsymbol{\alpha});\boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))|_v \leq -c_3(N+1)^{1+1/n}\rho^k.$$

□

Proposition 3.4.14. *If $k > c_{11}(N)$, then*

$$\log \|E(\Omega^k\boldsymbol{\alpha};\boldsymbol{T}(\boldsymbol{\tau};A^k;\boldsymbol{b}^{(k)}(\boldsymbol{\alpha});\boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))\| \leq c_{12}N\rho^k.$$

Proof. From (3.4.8) we have

$$E(\Omega^k\boldsymbol{\alpha};\boldsymbol{T}(\boldsymbol{\tau};A^k;\boldsymbol{b}^{(k)}(\boldsymbol{\alpha});\boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) = P_0(\Omega^k\boldsymbol{\alpha};\boldsymbol{T}(\boldsymbol{\tau};A^k;\boldsymbol{b}^{(k)}(\boldsymbol{\alpha});\boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) \in K.$$

Letting $A^k =: (a_{ij}^{(k)})$, we have $\|a_{ij}^{(k)}\| \leq c_{13}^k$ for $1 \leq i,j \leq l$. By the condition (II) we see that $\|b_i(\Omega^k\boldsymbol{\alpha})\| \leq c_{14}^{\rho^k}$ for $1 \leq i \leq l$ and that $\|e_h(\Omega^k\boldsymbol{\alpha})\| \leq c_{15}^{\rho^k}$ for $1 \leq h \leq m$. Hence we have

$$\|b_i^{(k)}(\boldsymbol{\alpha})\| \leq kl\prod_{j=0}^{k-1}(c_{13}^j c_{14}^{\rho^j})^l \leq c_{16}^{\rho^k} \quad (1 \leq i \leq l)$$

and

$$\|e_h^{(k)}(\boldsymbol{\alpha})\| \le \prod_{j=0}^{k-1} c_{15}^{\rho^j} \le c_{17}^{\rho^k} \quad (1 \le h \le m)$$

by (3.4.5) and (3.4.6), respectively. Therefore

$$\|T_{\boldsymbol{\lambda}\boldsymbol{\mu}}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))\| \le c_{18}^{\rho^k}$$

for $\boldsymbol{\lambda} \in \mathcal{L}$ and $\boldsymbol{\mu} \in \mathcal{M}$. Since the degree of each variable of $P_0(\boldsymbol{z}; \boldsymbol{t}) \in \mathcal{O}_K[\boldsymbol{z}; \boldsymbol{t}]$ is at most N, we obtain

$$\|P_0(\Omega^k \boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha})))\| \le c_{19}(N) c_{20}^{N\rho^k}.$$

This implies the proposition. □

Completion of the proof of Theorem 3.2.4. By the condition (i) of Proposition 3.4.12 together with Proposition 3.4.8, there exists a positive integer k greater than both $c_2(N)$ and $c_{11}(N)$ such that

$$E(\Omega^k \boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) = P_0(\Omega^k \boldsymbol{\alpha}; \boldsymbol{T}(\boldsymbol{\tau}; A^k; \boldsymbol{b}^{(k)}(\boldsymbol{\alpha}); \boldsymbol{e}^{(k)}(\boldsymbol{\alpha}))) \ne 0.$$

Therefore, by Propositions 3.4.1, 3.4.13, and 3.4.14, we have

$$-c_3(N+1)^{1+1/n}\rho^k \ge -2[K:\mathbb{Q}]c_{12}N\rho^k.$$

Hence

$$c_3(N+1)^{1+1/n} \le 2[K:\mathbb{Q}]c_{12}N,$$

which is a contradiction if N is large. □

Chapter 4

Proofs of Theorems 1.3.5, 1.3.7, and 1.3.12

4.1 Lemmas

In this section we prepare several lemmas necessary for proving Theorems 1.3.5, 1.3.7, and 1.3.12. In the proofs of these theorems, we represent the numbers in question as the values at a single algebraic point of Mahler functions satisfying a system of functional equations as in Theorem 3.2.1. In the proofs of the latter two theorems, we use Lemmas 4.1.1 and 4.1.3 below to construct such Mahler functions.

Lemma 4.1.1 (Loxton and van der Poorten [12, Lemma 3]). *Let $a_1, \ldots, a_r$ be algebraic numbers with $0 < |a_i| < 1$ $(1 \le i \le r)$. Then there exist multiplicatively independent algebraic numbers $\gamma_1, \ldots, \gamma_s$ with $0 < |\gamma_j| < 1$ $(1 \le j \le s)$ such that*

$$a_i = \zeta_i \prod_{j=1}^{s} \gamma_j^{d_{ij}} \quad (1 \le i \le r), \tag{4.1.1}$$

where ζ_i $(1 \le i \le r)$ are roots of unity and d_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ are nonnegative integers.

Remark 4.1.2. The most important assertion of Lemma 4.1.1 is that d_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ are nonnegative. In particular, at least one of $d_{i1}, \ldots, d_{is}$ is positive for any i.

Lemma 4.1.3. *Let* $\{R_k\}_{k\geq 0}$ *be a linear recurrence of nonnegative integers satisfying* (1.2.2) *and* N *a positive integer. Then there exist a positive integer* p *and a nonnegative integer* q *such that*

$$R_{k+p} \equiv R_k \pmod{N} \tag{4.1.2}$$

for any $k \geq q$.

Proof. By the pigeonhole principle, we can choose distinct integers k_1 and k_2 with $0 \leq k_1 < k_2 \leq N^n$ such that $R_{k_1+k} \equiv R_{k_2+k} \pmod{N}$ for any k with $0 \leq k \leq n-1$. Letting $p := k_2 - k_1$ and $q := k_1$, we obtain (4.1.2) from the recurrence formula (1.2.2). □

In the rest of this thesis, let

$$\Omega_1 := \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{4.1.3}$$

where $c_1, \ldots, c_n$ are the coefficients of the polynomial $\Phi(X)$ defined by (1.2.3). In order to apply Theorem 3.2.1, we need to ensure the four conditions stated in Section 3.1. In the case where v is a prime number p, the p-adic analogue of Mahler's vanishing theorem (Lemma 3.1.2) implies that, if the polynomial $\Phi(X)$ satisfies the stronger Condition 1.2.1, then the matrix Ω_1 and the point

$$\boldsymbol{\gamma}_1 := (\underbrace{1, \ldots, 1}_{n-1}, a),$$

where a is an algebraic number with $0 < |a|_p < 1$, satisfy the four conditions (I)–(IV)$_p$. In the case where v is ∞, Masser's vanishing theorem (Lemma 3.1.3) induces the following lemma, which ensures the four conditions (I)–(IV)$_\infty$ for more general matrices and algebraic points under the weaker Condition 1.2.5 on $\Phi(X)$.

Lemma 4.1.4 (Tanaka [29, Lemma 4, Proof of Theorem 2])**.** *Suppose that* $\Phi(X)$ *satisfies Condition 1.2.5. Let* $\gamma_1, \ldots, \gamma_s$ *be multiplicatively independent algebraic numbers with* $0 < |\gamma_j| < 1$ $(1 \le j \le s)$*. Let* p *be a positive integer and put*

$$\Omega_2 := \operatorname{diag}(\underbrace{\Omega_1^p, \ldots, \Omega_1^p}_{s}).$$

Then the matrix Ω_2 *and the point*

$$\boldsymbol{\gamma}_2 := (\underbrace{1, \ldots, 1}_{n-1}, \gamma_1, \ldots, \underbrace{1, \ldots, 1}_{n-1}, \gamma_s)$$

satisfy the four conditions (I)–(IV)$_\infty$ *stated in Section 3.1.*

Let $\{R_k\}_{k\ge0}$ be a linear recurrence of nonnegative integers satisfying (1.2.2). We define a monomial

$$P(\boldsymbol{z}) := z_1^{R_{n-1}} \cdots z_n^{R_0}, \tag{4.1.4}$$

which is denoted similarly to (3.1.1) by

$$P(\boldsymbol{z}) = (R_{n-1}, \ldots, R_0)\boldsymbol{z}. \tag{4.1.5}$$

It follows from (1.2.2), (3.1.1), (4.1.3), and (4.1.5) that

$$P(\Omega_1^k \boldsymbol{z}) = z_1^{R_{k+n-1}} \cdots z_n^{R_k} \quad (k \ge 0). \tag{4.1.6}$$

In what follows, let $\overline{C}$ be an algebraically closed field of characteristic 0. The following two lemmas are central to the proofs of the three theorems.

Lemma 4.1.5 (Tanaka [30, Theorem 1])**.** *Let* $\{R_k\}_{k\ge0}$ *be a linear recurrence of non-negative integers satisfying* (1.2.2)*. Suppose that* $\{R_k\}_{k\ge0}$ *satisfies Condition 1.2.5. Assume that* $\{R_k\}_{k\ge0}$ *is not a geometric progression. Assume further that* $f(\boldsymbol{z}) \in \overline{C}[[z_1, \ldots, z_n]]$ *satisfies the functional equation of the form*

$$f(\boldsymbol{z}) = \alpha f(\Omega_1^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} Q_k(P(\Omega_1^k \boldsymbol{z})),$$

where $\alpha \ne 0$ *is an element of* $\overline{C}$*,* $p > 0$*,* $q \ge 0$ *are integers, and* $Q_k(X) \in \overline{C}(X)$ $(q \le k \le p+q-1)$ *are defined at* $X = 0$*. Then, if* $f(\boldsymbol{z}) \in \overline{C}(z_1, \ldots, z_n)$*, then* $f(\boldsymbol{z}) \in \overline{C}$ *and* $Q_k(X) = Q_k(0)$ $(q \le k \le p+q-1)$*.*

Lemma 4.1.6 (Tanaka [30, Theorem 2])**.** *Let $\{R_k\}_{k\geq 0}$ be as in Lemma 4.1.5. Suppose that $g(\boldsymbol{z})$ is a nonzero element of the quotient field of $\overline{C}[[z_1,\dots,z_n]]$ satisfying the functional equation of the form*

$$g(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} Q_k(P(\Omega_1^k \boldsymbol{z}))\right) g(\Omega_1^p \boldsymbol{z}),$$

where p, q, and $Q_k(X)$ are as in Lemma 4.1.5. Assume in addition that $Q_k(0) \neq 0$. Then, if $g(\boldsymbol{z}) \in \overline{C}(z_1,\dots,z_n)^{\times}$, then $g(\boldsymbol{z}) \in \overline{C}^{\times}$ and $Q_k(X) = Q_k(0)$ $(q \leq k \leq p+q-1)$.

In the proofs of Theorems 1.3.7 and 1.3.12, we apply Kronecker type specialization to rational functions with more variables than those treated in Lemmas 4.1.5 and 4.1.6. The following lemma ensures the non-vanishing of the denominators of those rational functions.

Lemma 4.1.7 (Nishioka [22, Lemma 4])**.** *Let L be a subfield of $\mathbb{C}$ and let*

$$f(\boldsymbol{z}) \in \mathbb{C}[[z_1,\dots,z_n]] \cap L(z_1,\dots,z_n).$$

Then there exist polynomials $A(\boldsymbol{z}), B(\boldsymbol{z}) \in L[z_1,\dots,z_n]$ such that

$$f(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z}), \quad B(\boldsymbol{0}) \neq 0.$$

4.2 Proof of Theorem 1.3.5

Theorem 1.3.7 includes Theorem 1.3.5 in the case of complex numbers. In this section we give the proof of Theorem 1.3.5, which also provides an outline of the proof of Theorem 1.3.7.

Proof of Theorem 1.3.5. Let L be any positive integer, $\alpha_1,\dots,\alpha_L$ any nonzero distinct L algebraic numbers, and $\beta_1,\dots,\beta_L$ any distinct L elements of $\mathcal{B}$. It suffices to show that the finite set

$$\begin{aligned} &\left\{F_{m+1}^{(l)}(\alpha_\lambda) \;\middle|\; 0 \leq l, m \leq L,\ 1 \leq \lambda \leq L\right\} \\ &\bigcup \{G(\beta_\mu) \mid 1 \leq \mu \leq L\} \\ &\bigcup \left\{\frac{\partial^{l+m} H}{\partial x^l \partial y^m}(\alpha_\lambda, \beta_\mu) \;\middle|\; 0 \leq l, m \leq L,\ 1 \leq \lambda, \mu \leq L\right\} \end{aligned} \tag{4.2.1}$$

is algebraically independent. Let $z_1, \dots, z_n$ be variables and let $\boldsymbol{z} \coloneqq (z_1, \dots, z_n)$. Put $\beta_0 \coloneqq 0$. We define

$$f_{m\mu}(x; \boldsymbol{z}) \coloneqq \sum_{k=0}^{\infty} x^k \left(\frac{P(\Omega_1^k \boldsymbol{z})}{1 - \beta_\mu P(\Omega_1^k \boldsymbol{z})} \right)^{m+1} \quad (0 \le m \le L,\ 0 \le \mu \le L)$$

and

$$g_\mu(\boldsymbol{z}) \coloneqq \prod_{k=0}^{\infty} (1 - \beta_\mu P(\Omega_1^k \boldsymbol{z})) \quad (1 \le \mu \le L),$$

where Ω_1 and $P(\boldsymbol{z})$ are given by (4.1.3) and (4.1.4), respectively. Moreover, define

$$f_{lm\lambda\mu}(\boldsymbol{z}) \coloneqq \frac{\partial^l f_{m\mu}}{\partial x^l}(\alpha_\lambda; \boldsymbol{z}) \quad (0 \le l, m \le L,\ 1 \le \lambda \le L,\ 0 \le \mu \le L).$$

Letting

$$\boldsymbol{\gamma}_1 \coloneqq (\underbrace{1, \dots, 1}_{n-1}, a),$$

we see by (4.1.6) that

$$F_{m+1}^{(l)}(\alpha_\lambda) = f_{lm\lambda 0}(\boldsymbol{\gamma}_1) \quad (0 \le l, m \le L,\ 1 \le \lambda \le L),$$
$$\frac{\partial^{l+m} H}{\partial x^l \partial y^m}(\alpha_\lambda, \beta_\mu) = m! f_{lm\lambda\mu}(\boldsymbol{\gamma}_1) \quad (0 \le l, m \le L,\ 1 \le \lambda, \mu \le L),$$

and

$$G(\beta_\mu) = g_\mu(\boldsymbol{\gamma}_1) \quad (1 \le \mu \le L).$$

Hence the algebraic independency of the set (4.2.1) is equivalent to that of

$$\{f_{lm\lambda\mu}(\boldsymbol{\gamma}_1) \mid 0 \le l, m \le L,\ 1 \le \lambda \le L,\ 0 \le \mu \le L\}$$
$$\cup \{g_\mu(\boldsymbol{\gamma}_1) \mid 1 \le \mu \le L\}. \qquad (4.2.2)$$

Here we see that

$$f_{m\mu}(x; \boldsymbol{z}) = x f_{m\mu}(x; \Omega_1 \boldsymbol{z}) + \left(\frac{P(\boldsymbol{z})}{1 - \beta_\mu P(\boldsymbol{z})} \right)^{m+1} \quad (0 \le m \le L,\ 0 \le \mu \le L)$$

and thus

$$\frac{\partial^l f_{m\mu}}{\partial x^l}(x; \boldsymbol{z}) = x \frac{\partial^l f_{m\mu}}{\partial x^l}(x; \Omega_1 \boldsymbol{z}) + l \frac{\partial^{l-1} f_{m\mu}}{\partial x^{l-1}}(x; \Omega_1 \boldsymbol{z})$$
$$(1 \le l \le L,\ 0 \le m \le L,\ 0 \le \mu \le L).$$

Hence, for each m, λ, μ $(0 \le m \le L,\ 1 \le \lambda \le L,\ 0 \le \mu \le L)$, the functions $f_{lm\lambda\mu}(\boldsymbol{z})$ $(0 \le l \le L)$ satisfy the functional equation

$$\boldsymbol{f}_{m\lambda\mu}(\boldsymbol{z}) = A_\lambda \boldsymbol{f}_{m\lambda\mu}(\Omega_1 \boldsymbol{z}) + \boldsymbol{b}_{m\mu}(\boldsymbol{z}), \tag{4.2.3}$$

where

$$\begin{aligned} \boldsymbol{f}_{m\lambda\mu}(\boldsymbol{z}) &:= {}^t(f_{0m\lambda\mu}(\boldsymbol{z}), f_{1m\lambda\mu}(\boldsymbol{z}), \dots, f_{Lm\lambda\mu}(\boldsymbol{z})), \\ \boldsymbol{b}_{m\mu}(\boldsymbol{z}) &:= {}^t((P(\boldsymbol{z})/(1-\beta_\mu P(\boldsymbol{z})))^{m+1}, 0, \dots, 0), \end{aligned}$$

and

$$A_\lambda := \begin{pmatrix} \alpha_\lambda & & & & \\ 1 & \alpha_\lambda & & & \text{\Large 0} \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ \text{\Large 0} & & & L & \alpha_\lambda \end{pmatrix}.$$

Moreover, for each μ $(1 \le \mu \le L)$, the function $g_\mu(\boldsymbol{z})$ satisfies the functional equation

$$g_\mu(\boldsymbol{z}) = (1 - \beta_\mu P(\boldsymbol{z})) g_\mu(\Omega_1 \boldsymbol{z}). \tag{4.2.4}$$

Furthermore, applying Lemmas 3.1.2 or 4.1.4, we can verify that the matrix Ω_1 and the point $\boldsymbol{\gamma}_1$ satisfy the four conditions (I)–(IV)$_v$ stated in Section 3.1. Now we assume on the contrary that the set (4.2.2) is algebraically dependent. Noting that $\alpha_1, \dots, \alpha_\lambda$ are distinct, we see by Theorem 3.2.1, by Remark 3.2.2, and by the functional equations (4.2.3) and (4.2.4) that at least one of the following two cases arises:

(i) There exist $\lambda \in \{1, \dots, L\}$, algebraic numbers $c_{m\mu}$ $(0 \le m \le L,\ 0 \le \mu \le L)$, not all zero, and $f(\boldsymbol{z}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]] \cap \overline{\mathbb{Q}}(\boldsymbol{z})$ such that

$$f(\boldsymbol{z}) = \alpha_\lambda f(\Omega_1 \boldsymbol{z}) + \sum_{m=0}^{L} \sum_{\mu=0}^{L} c_{m\mu} \left(\frac{P(\boldsymbol{z})}{1 - \beta_\mu P(\boldsymbol{z})} \right)^{m+1}. \tag{4.2.5}$$

(ii) There exist integers e_μ $(1 \le \mu \le L)$, not all zero, and $g(\boldsymbol{z}) \in \overline{\mathbb{Q}}(\boldsymbol{z})^\times$ such that

$$g(\boldsymbol{z}) = \left(\prod_{\mu=1}^{L} (1 - \beta_\mu P(\boldsymbol{z}))^{e_\mu} \right) g(\Omega_1 \boldsymbol{z}). \tag{4.2.6}$$

If the functional equation (4.2.5) is satisfied, then by Lemma 4.1.5

$$\sum_{m=0}^{L}\sum_{\mu=0}^{L} c_{m\mu}\left(\frac{X}{1-\beta_\mu X}\right)^{m+1} = 0 \tag{4.2.7}$$

holds, where X is a variable. If the functional equation (4.2.6) is satisfied, then by Lemma 4.1.6

$$\prod_{\mu=1}^{L}(1-\beta_\mu X)^{e_\mu} = 1$$

holds. Taking the logarithmic derivative of this equation and then multiplying both sides by $-X$, we get

$$\sum_{\mu=1}^{L}\beta_\mu e_\mu \frac{X}{1-\beta_\mu X} = 0,$$

which is a special case of (4.2.7) since $\beta_\mu e_\mu$ $(1 \le \mu \le L)$ are not all zero. It is easily seen that (4.2.7) does not hold since β_μ $(0 \le \mu \le L)$ are distinct and $c_{m\mu}$ $(0 \le j \le t,\ 0 \le m \le M)$ are not all zero. Therefore neither the case (i) nor (ii) arises, which is a contradiction. □

4.3 Proof of Theorem 1.3.7

Generalizing the proof of Theorem 1.3.5, we prove Theorem 1.3.7.

Proof of Theorem 1.3.7. Let L be any positive integer and $\alpha_1, \ldots, \alpha_L$ any nonzero distinct L algebraic numbers. For each i $(1 \le i \le r)$, let $\beta_1^{(i)}, \ldots, \beta_L^{(i)}$ be any distinct L elements of $\mathcal{B}_i$. It suffices to show that the finite set

$$\begin{aligned}
&\left\{F_{i,m+1}^{(l)}(\alpha_\lambda) \;\middle|\; 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L\right\} \\
&\bigcup \left\{G_i(\beta_\mu^{(i)}) \;\middle|\; 1 \le i \le r,\ 1 \le \mu \le L\right\} \\
&\bigcup \left\{\frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha_\lambda, \beta_\mu^{(i)}) \;\middle|\; 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda, \mu \le L\right\}
\end{aligned} \tag{4.3.1}$$

is algebraically independent. Let ζ_i, γ_j, and d_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ be as in Lemma 4.1.1. Since $a_1, \ldots, a_r$ are pairwise multiplicatively independent, we see by (4.1.1) that the s-tuples $(d_{i1}, \ldots, d_{is})$ $(1 \le i \le r)$ are pairwise non-proportional, namely $(d_{i1}:\cdots:d_{is})\neq(d_{j1}:\cdots:d_{js})$ in $P^{s-1}(\mathbb{Q})$ if $1\le i<j\le r$. Take a positive integer N such that $\zeta_i^N=1$ for any i $(1\le i\le r)$. We choose a positive integer p and a nonnegative integer q by Lemma 4.1.3. Let $y_{j1},\dots,y_{jn}$ $(1\le j\le s)$ be variables and let $\boldsymbol{y}_j:=(y_{j1},\dots,y_{jn})$ $(1\le j\le s)$, $\boldsymbol{y}:=(\boldsymbol{y}_1,\dots,\boldsymbol{y}_s)$. Put $\beta_0^{(i)}:=0$ $(1\le i\le r)$. We define

$$f_{im\mu}(x;\boldsymbol{y}):=\sum_{k=q}^{\infty}x^k\left(\frac{\zeta_i^{R_k}\prod_{j=1}^{s}P(\Omega_1^k\boldsymbol{y}_j)^{d_{ij}}}{1-\beta_\mu^{(i)}\zeta_i^{R_k}\prod_{j=1}^{s}P(\Omega_1^k\boldsymbol{y}_j)^{d_{ij}}}\right)^{m+1}$$
$$(1\le i\le r,\ 0\le m\le L,\ 0\le\mu\le L)$$

and

$$g_{i\mu}(\boldsymbol{y}):=\prod_{k=q}^{\infty}\left(1-\beta_\mu^{(i)}\zeta_i^{R_k}\prod_{j=1}^{s}P(\Omega_1^k\boldsymbol{y}_j)^{d_{ij}}\right)\quad(1\le i\le r,\ 1\le\mu\le L),$$

where Ω_1 and $P(\boldsymbol{z})$ are given by (4.1.3) and (4.1.4), respectively. Moreover, define

$$f_{ilm\lambda\mu}(\boldsymbol{y}):=\frac{\partial^l f_{im\mu}}{\partial x^l}(\alpha_\lambda;\boldsymbol{y})\quad(1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda\le L,\ 0\le\mu\le L).$$

Letting

$$\boldsymbol{\gamma}_2:=(\underbrace{1,\dots,1}_{n-1},\gamma_1,\dots,\underbrace{1,\dots,1}_{n-1},\gamma_s),$$

we see by (4.1.1) and (4.1.6) that

$$F_{i,m+1}^{(l)}(\alpha_\lambda)-f_{ilm\lambda 0}(\boldsymbol{\gamma}_2)\in\overline{\mathbb{Q}}\quad(1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda\le L),$$
$$\frac{\partial^{l+m}H_i}{\partial x^l\partial y^m}(\alpha_\lambda,\beta_\mu^{(i)})-m!f_{ilm\lambda\mu}(\boldsymbol{\gamma}_2)\in\overline{\mathbb{Q}}\quad(1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda,\mu\le L),$$

and

$$G_i(\beta_\mu^{(i)})/g_{i\mu}(\boldsymbol{\gamma}_2)\in\overline{\mathbb{Q}}^{\times}\quad(1\le i\le r,\ 1\le\mu\le L).$$

Hence the algebraic independency of the set (4.3.1) is equivalent to that of

$$\{f_{ilm\lambda\mu}(\boldsymbol{\gamma}_2)\mid 1\le i\le r,\ 0\le l,m\le L,\ 1\le\lambda\le L,\ 0\le\mu\le L\}$$
$$\cup\{g_{i\mu}(\boldsymbol{\gamma}_2)\mid 1\le i\le r,\ 1\le\mu\le L\}.\tag{4.3.2}$$

Let

$$\Omega_2:=\mathrm{diag}(\underbrace{\Omega_1^p,\dots,\Omega_1^p}_{s}).$$

Noting that $\Omega_2 \boldsymbol{y} = (\Omega_1^p \boldsymbol{y}_1, \ldots, \Omega_1^p \boldsymbol{y}_s)$, we have

$$f_{im\mu}(x; \boldsymbol{y}) = x^p f_{im\mu}(x; \Omega_2 \boldsymbol{y}) + b_{im\mu}(x; \boldsymbol{y}) \quad (1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L)$$

by (4.1.2), where

$$b_{im\mu}(x; \boldsymbol{y}) := \sum_{k=q}^{p+q-1} x^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^s P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^s P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1}$$
$$(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L).$$

Hence, for each i, m, λ, μ $(1 \le i \le r,\ 0 \le m \le L,\ 1 \le \lambda \le L,\ 0 \le \mu \le L)$, the functions $f_{ilm\lambda\mu}(\boldsymbol{y})$ $(0 \le l \le L)$ satisfy the functional equation

$$\boldsymbol{f}_{im\lambda\mu}(\boldsymbol{y}) = A_\lambda \boldsymbol{f}_{im\lambda\mu}(\Omega_2 \boldsymbol{y}) + \boldsymbol{b}_{im\lambda\mu}(\boldsymbol{y}), \tag{4.3.3}$$

where

$$\boldsymbol{f}_{im\lambda\mu}(\boldsymbol{y}) := {}^t(f_{i0m\lambda\mu}(\boldsymbol{y}), f_{i1m\lambda\mu}(\boldsymbol{y}), \ldots, f_{iLm\lambda\mu}(\boldsymbol{y})),$$
$$\boldsymbol{b}_{im\lambda\mu}(\boldsymbol{y}) := {}^t\!\left(b_{im\mu}(\alpha_\lambda; \boldsymbol{y}), \frac{\partial b_{im\mu}}{\partial x}(\alpha_\lambda; \boldsymbol{y}), \ldots, \frac{\partial^L b_{im\mu}}{\partial x^L}(\alpha_\lambda; \boldsymbol{y}) \right),$$

and

$$A_\lambda := \begin{pmatrix} \alpha_\lambda^p & & & & 0 \\ p\alpha_\lambda^{p-1} & \alpha_\lambda^p & & & \\ & 2p\alpha_\lambda^{p-1} & \ddots & & \\ & & \ddots & \ddots & \\ * & & & Lp\alpha_\lambda^{p-1} & \alpha_\lambda^p \end{pmatrix}.$$

Moreover, for each i, μ $(1 \le i \le r,\ 1 \le \mu \le L)$, the function $g_{i\mu}(\boldsymbol{y})$ satisfies the functional equation

$$g_{i\mu}(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1} \left(1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^s P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right) \right) g_{i\mu}(\Omega_2 \boldsymbol{y}) \tag{4.3.4}$$

by (4.1.2). Now we assume on the contrary that the set (4.3.2) is algebraically dependent. Then by Theorem 3.2.1, Remark 3.2.2, Lemma 4.1.4, and the functional equations (4.3.3) and (4.3.4), at least one of the following two cases arises:

(i) There exist a non-empty subset $\{\lambda_1, \ldots, \lambda_\nu\}$ of $\{1, \ldots, L\}$, algebraic numbers $c_{im\mu\sigma}$ $(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L,\ 1 \le \sigma \le \nu)$, not all zero, and $f(\boldsymbol{y}) \in \overline{\mathbb{Q}}[[\boldsymbol{y}]] \cap \overline{\mathbb{Q}}(\boldsymbol{y})$ such that

$$\alpha_{\lambda_1}^p = \cdots = \alpha_{\lambda_\nu}^p \tag{4.3.5}$$

and

$$f(\boldsymbol{y}) = \alpha_{\lambda_1}^p f(\Omega_2 \boldsymbol{y}) + \sum_{i,m,\mu,\sigma} c_{im\mu\sigma} b_{im\mu}(\alpha_{\lambda_\sigma}; \boldsymbol{y}).$$

(ii) There exist integers $e_{i\mu}$ $(1 \le i \le r,\ 1 \le \mu \le L)$, not all zero, and $g(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y})^\times$ such that

$$g(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1} \prod_{i,\mu} \left(1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right)^{e_{i\mu}} \right) g(\Omega_2 \boldsymbol{y}). \tag{4.3.6}$$

Suppose first that the case (i) arises. By (4.3.5) we have $\nu \le p$ since $\alpha_1, \ldots, \alpha_L$ are distinct. Changing the indices λ $(1 \le \lambda \le L)$ if necessary, we may assume that $\lambda_\sigma = \sigma$ $(1 \le \sigma \le \nu)$. Then $f(\boldsymbol{y})$ satisfies the functional equation

$$f(\boldsymbol{y}) = \alpha_1^p f(\Omega_2 \boldsymbol{y}) + \sum_{k=q}^{p+q-1} \sum_{i,m,\mu,\sigma} c_{im\mu\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^s P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} \prod_{j=1}^s P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1}. \tag{4.3.7}$$

Let M be any positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \ldots, y_{jn}) = (z_1^{M^j}, \ldots, z_n^{M^j}) \quad (1 \le j \le s).$$

Note that, by Lemma 4.1.7, the denominator of

$$f^*(\boldsymbol{z}) := f(z_1^M, \ldots, z_n^M, \ldots, z_1^{M^s}, \ldots, z_n^{M^s})$$

does not vanish and so $f^*(\boldsymbol{z}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]] \cap \overline{\mathbb{Q}}(\boldsymbol{z})$. Then the functional equation (4.3.7) is specialized to

$$f^*(\boldsymbol{z}) = \alpha_1^p f^*(\Omega_1^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} \sum_{i,m,\mu,\sigma} c_{im\mu\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}} \right)^{m+1},$$

where $D_i := \sum_{j=1}^{s} d_{ij} M^j > 0$ $(1 \le i \le r)$. Hence, by Lemma 4.1.5, we see that

$$\sum_{i,m,\mu,\sigma} c_{im\mu\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} X^{D_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} X^{D_i}} \right)^{m+1} = 0 \tag{4.3.8}$$

for any k $(q \le k \le p+q-1)$. For each k $(q \le k \le p+q-1)$, we define

$$Q_k(\boldsymbol{X}) := \sum_{i,m,\mu,\sigma} c_{im\mu\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}} \right)^{m+1} \in \overline{\mathbb{Q}}(X_1, \ldots, X_s),$$

where $\boldsymbol{X}^{\boldsymbol{d}_i} := X_1^{d_{i1}} \cdots X_s^{d_{is}}$ $(1 \le i \le r)$. Then the left-hand side of (4.3.8) is equal to $Q_k(X^M, \ldots, X^{M^s})$. We assert that $Q_k(\boldsymbol{X}) = 0$ for any k $(q \le k \le p+q-1)$. Indeed, if $Q_{k'}(\boldsymbol{X}) \ne 0$ for some k', then there exist nonzero polynomials $A(\boldsymbol{X}), B(\boldsymbol{X}) \in \overline{\mathbb{Q}}[X_1, \ldots, X_s]$ with $B(\boldsymbol{0}) = 1$ such that $Q_{k'}(\boldsymbol{X}) = A(\boldsymbol{X})/B(\boldsymbol{X})$. We take M so large that $M > \max_{1 \le j \le s} \deg_{X_j} A(\boldsymbol{X})$. Then, by the uniqueness of the M-ary expression for nonnegative integers, we see that $A(X^M, \ldots, X^{M^s}) \ne 0$. Hence $Q_{k'}(X^M, \ldots, X^{M^s}) \ne 0$, which contradicts (4.3.8), and so the assertion is proved. For each i $(1 \le i \le r)$ and k $(q \le k \le p+q-1)$, define

$$Q_{ik}(Y) := \sum_{m,\mu,\sigma} c_{im\mu\sigma} \alpha_\sigma^k \left(\frac{Y}{1 - \beta_\mu^{(i)} Y} \right)^{m+1}.$$

Then $Q_{ik}(Y) \in Y\overline{\mathbb{Q}}[[Y]]$ $(1 \le i \le r,\ q \le k \le p+q-1)$ and

$$Q_k(\boldsymbol{X}) = \sum_{i=1}^{r} Q_{ik}(\zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}) = 0 \quad (q \le k \le p+q-1).$$

Since $\boldsymbol{d}_i = (d_{i1}, \ldots, d_{is})$ $(1 \le i \le r)$ are pairwise non-proportional, we see that $Q_{ik}(\zeta_i^{R_k} \boldsymbol{X}^{\boldsymbol{d}_i}) = 0$ and hence

$$Q_{ik}(Y) = \sum_{m=0}^{L} \sum_{\mu=0}^{L} \left(\sum_{\sigma=1}^{\nu} c_{im\mu\sigma} \alpha_\sigma^k \right) \left(\frac{Y}{1 - \beta_\mu^{(i)} Y} \right)^{m+1} = 0$$

for any i, k $(1 \le i \le r,\ q \le k \le p+q-1)$. Noting that $\beta_\mu^{(i)}$ $(0 \le \mu \le L)$ are distinct for each i $(1 \le i \le r)$, we obtain

$$\sum_{\sigma=1}^{\nu} c_{im\mu\sigma} \alpha_\sigma^k = 0 \quad (q \le k \le p+q-1)$$

for any i, m, μ $(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L)$. Since $\nu \le p$ and since α_σ $(1 \le \sigma \le \nu)$ are distinct and nonzero, by the non-vanishing of Vandermonde determinant, we see that $c_{im\mu\sigma} = 0$ for any i, m, μ, σ $(1 \le i \le r,\ 0 \le m \le L,\ 0 \le \mu \le L,\ 1 \le \sigma \le \nu)$, which is a contradiction.

Suppose next that the case (ii) arises. By Lemma 3.4.4 and by the functional equations (4.3.4) and (4.3.6), we see that $g(\boldsymbol{y})/\prod_{i,\mu} g_{i\mu}(\boldsymbol{y})^{e_{i\mu}} \in \overline{\mathbb{Q}}^{\times}$. Then $g(\boldsymbol{y}), g(\boldsymbol{y})^{-1} \in \overline{\mathbb{Q}}[[\boldsymbol{y}]]$ and hence, by Lemma 4.1.7,

$$g^*(\boldsymbol{z}) \coloneqq g(z_1^M, \ldots, z_n^M, \ldots, z_1^{M^s}, \ldots, z_n^{M^s}) \in \overline{\mathbb{Q}}(\boldsymbol{z})^{\times}$$

for any positive integer M. Letting $\boldsymbol{y}_j = (y_{j1}, \ldots, y_{jn}) = (z_1^{M^j}, \ldots, z_n^{M^j})$ $(1 \le j \le s)$ in (4.3.6), we have

$$g^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \prod_{i,\mu} \left(1 - \beta_\mu^{(i)} \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i} \right)^{e_{i\mu}} \right) g^*(\Omega_1^p \boldsymbol{z}),$$

where D_i $(1 \le i \le r)$ are the positive integers as in the case (i) above. Hence, by Lemma 4.1.6, we see in particular that

$$\prod_{i,\mu} (1 - \beta_\mu^{(i)} \zeta_i^{R_q} X^{D_i})^{e_{i\mu}} = 1.$$

Taking the logarithmic derivative of this equation and then multiplying both sides by $-X$, we get

$$\sum_{i,\mu} e_{i\mu} \frac{D_i \beta_\mu^{(i)} \zeta_i^{R_q} X^{D_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_q} X^{D_i}} = 0.$$

Let

$$R(\boldsymbol{X}) \coloneqq \sum_{i,\mu} e_{i\mu} \frac{D_i \beta_\mu^{(i)} \zeta_i^{R_q} \boldsymbol{X}^{\boldsymbol{d}_i}}{1 - \beta_\mu^{(i)} \zeta_i^{R_q} \boldsymbol{X}^{\boldsymbol{d}_i}} \in \overline{\mathbb{Q}}(X_1, \ldots, X_s).$$

Although D_i $(1 \le i \le r)$ depend on M, the maximum of the partial degrees of the numerator of $R(\boldsymbol{X})$ is bounded by a constant independent of M. Hence, similarly to the case (i), we see that $R(\boldsymbol{X}) = 0$ for any sufficiently large M. Using the fact that $\boldsymbol{d}_i$ $(1 \le i \le r)$ are pairwise non-proportional for any i, j with $1 \le i < j \le r$, we obtain

$$\sum_{\mu=1}^{L} e_{i\mu} \frac{D_i \beta_\mu^{(i)} Y}{1 - \beta_\mu^{(i)} Y} = 0$$

for any i $(1 \le i \le r)$. Since $\beta_\mu^{(i)}$ $(1 \le \mu \le L)$ are distinct and nonzero for each i $(1 \le i \le r)$, we see that $e_{i\mu} = 0$ for any i, μ $(1 \le i \le r,\ 1 \le \mu \le L)$, which is a contradiction. This completes the proof of Theorem 1.3.7. □

4.4 Proof of Theorem 1.3.12

Proof of Theorem 1.3.12. Let L be any positive integer and $\alpha_1, \ldots, \alpha_L$ any nonzero distinct L algebraic numbers. It suffices to show that the finite set

$$\begin{aligned}
&\left\{ F_{i,m_0}^{(l)}(\alpha_\lambda) \;\middle|\; 1 \le i \le r,\ 0 \le l \le L,\ 1 \le \lambda \le L \right\} \\
&\bigcup \{G_i(\beta_i) \mid 1 \le i \le r\} \\
&\bigcup \left\{ \frac{\partial^{l+m} H_i}{\partial x^l \partial y^m}(\alpha_\lambda, \beta_i') \;\middle|\; 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L \right\}
\end{aligned} \tag{4.4.1}$$

is algebraically independent. Let ζ_i, γ_j, and d_{ij} $(1 \le i \le r,\ 1 \le j \le s)$ be as in Lemma 4.1.1. Then the s-tuples $(d_{i1}, \ldots, d_{is})$ $(1 \le i \le r)$ are distinct since none of a_i/a_j $(1 \le i < j \le r)$ is a root of unity. In what follows, let $N, p, q, P(\boldsymbol{z}), \Omega_1, \Omega_2$, and $\boldsymbol{\gamma}_2$ be as in the proof of Theorem 1.3.7. Define

$$f_i(x; \boldsymbol{y}) := \sum_{k=q}^{\infty} x^k \left(\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right)^{m_0} \quad (1 \le i \le r),$$

$$h_{im}(x; \boldsymbol{y}) := \sum_{k=q}^{\infty} x^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_i' \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1} \quad (1 \le i \le r,\ 0 \le m \le L),$$

and

$$g_i(\boldsymbol{y}) := \prod_{k=q}^{\infty} \left(1 - \beta_i \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right) \quad (1 \le i \le r).$$

Then the algebraic independency of the set (4.4.1) is equivalent to that of

$$\begin{aligned}
&\left\{ \frac{\partial^l f_i}{\partial x^l}(\alpha_\lambda; \boldsymbol{\gamma}_2) \;\middle|\; 1 \le i \le r,\ 0 \le l \le L,\ 1 \le \lambda \le L \right\} \\
&\bigcup \left\{ \frac{\partial^l h_{im}}{\partial x^l}(\alpha_\lambda; \boldsymbol{\gamma}_2) \;\middle|\; 1 \le i \le r,\ 0 \le l, m \le L,\ 1 \le \lambda \le L \right\} \\
&\bigcup \{g_i(\boldsymbol{\gamma}_2) \mid 1 \le i \le r\}.
\end{aligned}$$

Assume on the contrary that this set is algebraically dependent. Similarly to the proof of Theorem 1.3.7, changing the indices λ $(1 \le \lambda \le L)$ if necessary, we see by Theorem 3.2.1, by Remark 3.2.2, and by Lemma 4.1.4 that at least one of the following two cases arises:

(i) There exist a positive integer ν with $\nu \le L$, algebraic numbers $b_{i\sigma}$ $(1 \le i \le r,\ 1 \le \sigma \le \nu)$, $c_{im\sigma}$ $(1 \le i \le r,\ 0 \le m \le L,\ 1 \le \sigma \le \nu)$, not all zero, and $f(\boldsymbol{y}) \in \overline{\mathbb{Q}}[[\boldsymbol{y}]] \cap \overline{\mathbb{Q}}(\boldsymbol{y})$ such that

$$\alpha_1^p = \cdots = \alpha_\nu^p$$

and

$$\begin{aligned} f(\boldsymbol{y}) = \alpha_1^p f(\Omega_2 \boldsymbol{y}) &+ \sum_{k=q}^{p+q-1} \sum_{i,\sigma} b_{i\sigma} \alpha_\sigma^k \left(\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right)^{m_0} \\ &+ \sum_{k=q}^{p+q-1} \sum_{i,m,\sigma} c_{im\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}}{1 - \beta_i' \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}}} \right)^{m+1}. \end{aligned} \tag{4.4.2}$$

(ii) There exist integers e_i $(1 \le i \le r)$, not all zero, and $g(\boldsymbol{y}) \in \overline{\mathbb{Q}}(\boldsymbol{y})^\times$ such that

$$g(\boldsymbol{y}) = \left(\prod_{k=q}^{p+q-1} \prod_{i=1}^{r} \left(1 - \beta_i \zeta_i^{R_k} \prod_{j=1}^{s} P(\Omega_1^k \boldsymbol{y}_j)^{d_{ij}} \right)^{e_i} \right) g(\Omega_2 \boldsymbol{y}). \tag{4.4.3}$$

Let M be a positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{M^j}, \dots, z_n^{M^j}) \quad (1 \le j \le s).$$

Since $(d_{i1}, \dots, d_{is})$ $(1 \le i \le r)$ are distinct, we can take M so large that the following two properties are both satisfied:

(A) $D_i := \sum_{j=1}^{s} d_{ij} M^j$ $(1 \le i \le r)$ are distinct positive integers.

(B) $D_i \ge m_0$ $(1 \le i \le r)$ and $D_1 \cdots D_r \ge L$.

Then by (4.4.2), (4.4.3), Lemmas 3.4.4 and 4.1.7, at least one of the following two conditions holds:

(i) $f^*(\boldsymbol{z}) \coloneqq f(z_1^M, \ldots, z_n^M, \ldots, z_1^{M^s}, \ldots, z_n^{M^s}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]] \cap \overline{\mathbb{Q}}(\boldsymbol{z})$ satisfies

$$f^*(\boldsymbol{z}) = \alpha_1^p f^*(\Omega_1^p \boldsymbol{z}) + \sum_{k=q}^{p+q-1} \sum_{i,\sigma} b_{i\sigma} \alpha_\sigma^k \left(\zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}\right)^{m_0} + \sum_{k=q}^{p+q-1} \sum_{i,m,\sigma} c_{im\sigma} \alpha_\sigma^k \left(\frac{\zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}}{1 - \beta_i' \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}} \right)^{m+1}.$$

(ii) $g^*(\boldsymbol{z}) \coloneqq g(z_1^M, \ldots, z_n^M, \ldots, z_1^{M^s}, \ldots, z_n^{M^s}) \in \overline{\mathbb{Q}}(\boldsymbol{z})^\times$ satisfies

$$g^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \prod_{i=1}^{r} \left(1 - \beta_i \zeta_i^{R_k} P(\Omega_1^k \boldsymbol{z})^{D_i}\right)^{e_i} \right) g^*(\Omega_1^p \boldsymbol{z}).$$

Hence by Lemmas 4.1.5 and 4.1.6, at least one of the following two properties is satisfied:

(i) For any k $(q \le k \le p+q-1)$,

$$\begin{aligned} &\sum_{i=1}^{r} \left(B_i(k)(\zeta_i^{R_k} X^{D_i})^{m_0} + \sum_{m=0}^{L} C_{im}(k) \left(\frac{\zeta_i^{R_k} X^{D_i}}{1 - \beta_i' \zeta_i^{R_k} X^{D_i}} \right)^{m+1} \right) \\ &= \sum_{i=1}^{r} \left(B_i(k)(\zeta_i^{R_k} X^{D_i})^{m_0} + \sum_{m=0}^{L} C_{im}(k) \sum_{h=0}^{\infty} \binom{h+m}{m} \beta_i'^h (\zeta_i^{R_k} X^{D_i})^{h+m+1} \right) \\ &= \sum_{i=1}^{r} \left(B_i(k)(\zeta_i^{R_k} X^{D_i})^{m_0} + \sum_{h'=0}^{\infty} \left(\sum_{m=0}^{\min\{L,h'\}} C_{im}(k) \binom{h'}{m} \beta_i'^{h'-m} \right) (\zeta_i^{R_k} X^{D_i})^{h'+1} \right) \\ &= 0, \end{aligned} \tag{4.4.4}$$

where $B_i(k) \coloneqq \sum_{\sigma=1}^{\nu} b_{i\sigma} \alpha_\sigma^k$ $(1 \le i \le r)$ and $C_{im}(k) \coloneqq \sum_{\sigma=1}^{\nu} c_{im\sigma} \alpha_\sigma^k$ $(1 \le i \le r,\ 0 \le m \le L)$.

(ii) For any k $(q \le k \le p+q-1)$,

$$\prod_{i=1}^{r} (1 - \beta_i \zeta_i^{R_k} X^{D_i})^{e_i} = 1. \tag{4.4.5}$$

Suppose first that (i) is satisfied. We show that $C_{im}(k) = 0$ for any i $(1 \le i \le r)$, m $(0 \le m \le L)$, and k $(q \le k \le p+q-1)$. Assume on the contrary that $C_{im}(k')$

$(1 \leq i \leq r,\ 0 \leq m \leq L)$ are not all zero for some k'. Let

$$S \coloneqq \{i \in \{1, \ldots, r\} \mid C_{im}(k')\ (0 \leq m \leq L) \text{ are not all zero}\}$$

and let $i' \in S$ be the index such that $D_{i'} < D_i$ for any $i \in S \setminus \{i'\}$. Note that $C_{i'm}(k')$ $(0 \leq m \leq L)$ are determined independently of M. Hence, replacing M if necessary, we may assume that the following property (C) is satisfied in addition to the properties (A) and (B) above.

(C) $\sum_{m=0}^{L} C_{i'm}(k') \binom{D_1 \cdots D_r}{m} \beta_{i'}'^{D_1 \cdots D_r - m} \neq 0$.

Indeed, let m' be the maximum of $m \in \{0, \ldots, L\}$ such that $C_{i'm}(k') \neq 0$. Since

$$\binom{x}{m} = \frac{x^m}{m!} + o(x^m) \qquad (\mathbb{Z} \ni x \to \infty)$$

for each $m \in \{0, \ldots, m'\}$, we see that

$$\begin{aligned}\sum_{m=0}^{L} C_{i'm}(k') \binom{x}{m} \beta_{i'}'^{x-m} &= C_{i'm'}(k') \binom{x}{m'} \beta_{i'}'^{x-m'} + \sum_{m=0}^{m'-1} C_{i'm}(k') \binom{x}{m} \beta_{i'}'^{x-m} \\ &= \frac{C_{i'm'}(k')}{m'! \beta_{i'}'^{m'}} x^{m'} \beta_{i'}'^{x} + o(x^{m'} \beta_{i'}'^{x}) \qquad (\mathbb{Z} \ni x \to \infty).\end{aligned}$$

Thus the property (C) is satisfied if M is sufficiently large. Noting the fact that $(D_1 \cdots D_r + 1) D_{i'}$ is not divided by any D_i with $i \in S \setminus \{i'\}$, we see by the properties (B) and (C) that the term

$$\left(\sum_{m=0}^{L} C_{i'm}(k') \binom{D_1 \cdots D_r}{m} \beta_{i'}'^{D_1 \cdots D_r - m} \right) (\zeta_{i'}^{R_{k'}} X^{D_{i'}})^{D_1 \cdots D_r + 1}$$

does not cancel in (4.4.4), which is a contradiction. Hence $C_{im}(k) = 0$ $(1 \leq i \leq r,\ 0 \leq m \leq L,\ q \leq k \leq p + q - 1)$. Then, since $D_1, \ldots, D_r$ are distinct by the property (A), we have $B_i(k) = 0$ $(1 \leq i \leq r,\ q \leq k \leq p + q - 1)$ by (4.4.4). Therefore, noting that $\nu \leq p$, we see that $b_{i\sigma} = 0$ $(1 \leq i \leq r,\ 1 \leq \sigma \leq \nu)$ and $c_{im\sigma} = 0$ $(1 \leq i \leq r,\ 0 \leq m \leq L,\ 1 \leq \sigma \leq \nu)$, which is also a contradiction.

Next suppose that (ii) is satisfied. Taking the logarithmic derivative of (4.4.5) and then multiplying both sides by $-X$, we see in particular that

$$\sum_{i=1}^{r} e_i \frac{D_i \beta_i \zeta_i^{R_q} X^{D_i}}{1 - \beta_i \zeta_i^{R_q} X^{D_i}} = 0.$$

This is a contradiction since $\mathrm{ord}_{X=0}\, D_i\beta_i\zeta_i^{R_q}X^{D_i}/(1-\beta_i\zeta_i^{R_q}X^{D_i}) = D_i$ $(1 \leq i \leq r)$, and the theorem is proved. $\square$

Appendix A

Further examples of linear recurrences satisfying Conditions 1.2.1 and 1.2.5

In order to obtain a wealth of examples of the results stated in Section 1.3, it is important to explicitly construct linear recurrences which satisfy Conditions 1.2.1 or 1.2.5. In this section we show the existence of linear recurrences satisfying Condition 1.2.1, and thus Condition 1.2.5 (cf. Tanaka [29, Remark 1]), for any large length n of the recurrence formula (1.2.2). For this purpose, the following result of Tamura gives an answer.

Proposition A.1 (Tamura [27, Lemma 10])**.** *Let $n \geq 2$ and $a_1 \geq \cdots \geq a_{n-1}$ be positive integers. Then the polynomial $f(X) \coloneqq X^n - a_1 X^{n-1} - \cdots - a_{n-1}X - 1$ is the minimal polynomial of a Pisot number α with $a_1 < \alpha < a_1 + 1$, i.e.,*

(i) *$f(X)$ is irreducible over $\mathbb{Q}$,*

(ii) *there exists only one real root α of $f(X)$ with $\alpha > 1$,*

(iii) *$|\beta| < 1$ for every algebraic conjugate $\beta (\neq \alpha)$ of α.*

Example A.2. For any integer $n \geq 2$, we consider the linear recurrence $\{R_k\}_{k \geq 0}$ of nonnegative integers satisfying

$$R_{k+n} = a_1 R_{k+n-1} + \cdots + a_{n-1} R_{k+1} + R_k \quad (k \geq 0),$$

where $a_1, \ldots, a_{n-1}$ are as in Proposition A.1. The associated polynomial defined by (1.2.3) is $\Phi(X) = X^n - a_1 X^{n-1} - \cdots - a_{n-1}X - 1$. Then $\Phi(X)$ is irreducible over $\mathbb{Q}$ and there exists an ordering of the roots $\rho_1, \ldots, \rho_n$ of $\Phi(X)$ such that $\rho_1 > 1 > \max\{|\rho_2|, \ldots, |\rho_n|\}$ by Proposition A.1. Hence the linear recurrence $\{R_k\}_{k \geq 0}$ satisfies Condition 1.2.1.

The roots $\rho_1, \ldots, \rho_n$ of the polynomial $\Phi(X) = X^n - a_1 X^{n-1} - \cdots - a_{n-1}X - 1$ in Example A.2 are multiplicatively dependent, since $\rho_1 \cdots \rho_n = (-1)^n \Phi(0) = (-1)^{n+1}$ and so $\rho_1^2 \cdots \rho_n^2 = 1$. On the other hand, if the roots of a given polynomial $\Phi(X)$ of the form (1.2.3) are multiplicatively independent, then the ratio of any pair of distinct roots of $\Phi(X)$ is clearly not a root of unity. In that case, if in addition $\Phi(X)$ is irreducible over $\mathbb{Q}$, then any linear recurrence $\{R_k\}_{k \geq 0}$ associated with the $\Phi(X)$ satisfies not only Condition 1.2.5 but also Condition 1.2.1 (cf. Tanaka [29, Proof of Lemma 4]). Therefore, in order to show further examples of linear recurrences satisfying Condition 1.2.1, it suffices to give conditions for the roots of an irreducible polynomial to be multiplicatively independent. One of such conditions was obtained by Becker and Töpfer.

Proposition A.3 (Becker and Töpfer [1, Lemma 5, Proof of Theorem 2])**.** *Let $f(X) = X^n - a_1 X^{n-1} - \cdots - a_n$ be a polynomial of degree $n \geq 2$ with integral coefficients. Suppose that $f(X)$ is irreducible over $\mathbb{Q}$. Then, if $a_n \neq \pm 1$ and $(a_{n-1}, a_n) = 1$, then the roots of $f(X)$ are multiplicatively independent.*

In Proposition A.3, the coefficient a_{n-1} of the first degree term of $f(X)$ is assumed to be nonzero, which excludes the case where $f(X)$ is represented as $f(X) = g(X^m)$ for some polynomial $g(X)$ and integer $m \geq 2$. We note that, if the polynomial $\Phi(X)$ associated with a linear recurrence $\{R_k\}_{k \geq 0}$ is represented as $\Phi(X) = \Psi(X^m)$ for some polynomial $\Psi(X)$ and integer $m \geq 2$, then $\{R_k\}_{k \geq 0}$ does not satisfy Condition 1.2.5. Thus we need to exclude such cases, as in Proposition A.3. The following result proved by Drmota and Skałba leads to our desired conclusion under a weaker assumption than Proposition A.3 in the case where the degree of a polynomial is restricted to an odd prime number.

Proposition A.4 (Drmota and Skałba [5, Theorem 3]). *Let p be an odd prime number and let $f(X) = X^p - a_1X^{p-1} - \cdots - a_p$ be a polynomial with rational coefficients. Suppose that $f(X)$ is irreducible over $\mathbb{Q}$. Then, if $a_p \neq \pm 1$ and if $a_i \neq 0$ for at least one $i \in \{1, \ldots, p-1\}$, then the roots of $f(X)$ are multiplicatively independent.*

Example A.5. Let p be any odd prime number, q a prime number, and $\{R_k\}_{k\geq 0}$ a linear recurrence of nonnegative integers satisfying

$$R_{k+p} = qR_{k+1} + qR_k \quad (k \geq 0).$$

The associated polynomial $\Phi(X) = X^p - qX - q$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. Then by Proposition A.4 the roots of $\Phi(X)$ are multiplicatively independent and hence $\{R_k\}_{k\geq 0}$ satisfies Condition 1.2.1. In this example, the dominant root ρ_1 appearing in Condition 1.2.1 and in the asymptotic formula (1.2.4) can be arbitrary close to 1 by taking a sufficiently large prime number p, as shown below. We fix a prime number q and define $\Phi_p(X) \coloneqq X^p - qX - q$ for any odd prime number p. Since $\Phi_p'(X) = pX^{p-1} - q$ has the only positive real root $\sqrt[p-1]{q/p}$ and since $\Phi_p(0) = -q < 0$, the polynomial $\Phi_p(X)$ has the only positive real root $\rho(p)$, which appears as ρ_1 in Condition 1.2.1 and so $\rho(p) > 1$ for any odd prime number p. We show that $\rho(p) \to 1$ as $p \to \infty$. Let p_1 and p_2 be odd prime numbers with $p_1 < p_2$. Then $\Phi_{p_2}(\rho(p_1)) - \Phi_{p_1}(\rho(p_1)) = \rho(p_1)^{p_2} - \rho(p_1)^{p_1} > 0$ and so $\Phi_{p_2}(\rho(p_1)) > \Phi_{p_1}(\rho(p_1)) = 0$. Since $\rho(p_2)$ is the only positive real root of $\Phi_{p_2}(X)$, we have $\rho(p_2) < \rho(p_1)$. Thus the sequence $\{\rho(p)\}_p$ is strictly decreasing with respect to p and hence it converges to some real number greater than or equal to 1. By the fact that $\rho(p)^p - q\rho(p) - q = 0$ for any odd prime number p, we see that $\{\rho(p)\}_p$ must converge to 1.

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Bibliography

[1] P.-G. Becker and T. Töpfer, *Transcendency results for sums of reciprocals of linear recurrences*, Math. Nachr. **168** (1994), 5–17.

[2] P. Bundschuh and K. Väänänen, *Algebraic independence results on the generating Lambert series of the powers of a fixed integer*, Hardy-Ramanujan J. **38** (2015), 36–44.

[3] J. W. S. Cassels, Local Fields, Cambridge University Press, 1986.

[4] P. Corvaja and U. Zannier, *Some new applications of the subspace theorem*, Compositio Math. **131** (2002), 319–340.

[5] M. Drmota and M. Skałba, *On multiplicative and linear independence of polynomial roots*, Contributions to General Algebra **7** (1991), 127–135.

[6] F. R. Gantmacher, Applications of the Theory of Matrices, Interscience, 1959.

[7] H. Ide, *Algebraic independence of certain entire functions of two variables generated by linear recurrences*, Acta Arith. **202** (2022), 303–336.

[8] H. Ide, *Algebraic independence results for a certain family of power series, infinite products, and Lambert type series*, Osaka J. Math., (to appear).

[9] H. Ide and T. Tanaka, *Algebraic independence of the partial derivatives of certain functions with arbitrary number of variables*, preprint, arXiv:2208.01788, (2022).

[10] H. Ide, T. Tanaka, and K. Toyama, *Algebraic independence of the values of power series and their derivatives generated by linear recurrences*, Tokyo J. Math. **45** (2022), 519–545.

[11] K. K. Kubota, *On the algebraic independence of holomorphic solutions of certain functional equations and their values*, Math. Ann. **227** (1977), 9–50.

[12] J. H. Loxton and A. J. van der Poorten, *Algebraic independence properties of the Fredholm series*, J. Austral. Math. Soc. Ser. A **26** (1978), 31–45.

[13] J. H. Loxton and A. J. van der Poorten, *Arithmetic properties of the solutions of a class of functional equations*, J. reine angew. Math. **330** (1982), 159–172.

[14] K. Mahler, *Arithmetische Eigenschaften der Losungen einer Klasse von Funktionalgleichungen*, Math. Ann. **101** (1929), 342–366.

[15] K. Mahler, *Ein Beweis der Transzendenz der p-adischen Exponentialfunktion*, J. reine angew. Math. **169** (1933), 61–66.

[16] K. Mahler, Lectures on Transcendental Numbers, Lecture Notes in Mathematics No. **546**, Springer, 1976.

[17] D. W. Masser, *A vanishing theorem for power series*, Invent. math. **67** (1982), 275–296.

[18] Yu. V. Nesterenko, *Algebraic independence of p-adic numbers*, Izv. Math. **72** (2008), 565–579.

[19] K. Nishioka, *Algebraic independence of certain power series of algebraic numbers*, J. Number Theory **23** (1986), 354–364.

[20] K. Nishioka, *Proof of Masser's conjecture on the algebraic independence of values of Liouville series*, Proc. Japan Acad. **62** (1986), 219–222.

[21] K. Nishioka, *Conditions for algebraic independence of certain power series of algebraic numbers*, Compositio Math. **62** (1987), 53–61.

[22] K. Nishioka, *Algebraic independence of Mahler functions and their values*, Tohoku Math. J. **48** (1996), 51–70.

[23] K. Nishioka, Mahler functions and transcendence, Lecture Notes in Mathematics No. **1631**, Springer, 1996.

[24] D. Ridout, *Rational approximations to algebraic numbers*, Mathematika **4** (1957), 125–131.

[25] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20.

[26] W. Rudin, Real and Complex Analysis, Third edition, McGraw-Hill, 1987.

[27] J. Tamura, *A class of transcendental numbers having explicit g-adic and Jacobi–Perron expansions of arbitrary dimension*, Acta Arith. **71** (1995), 301–329.

[28] T. Tanaka, *Algebraic independence of certain numbers defined by linear recurrences*, Keio Sci. Tech. Reports **47** (1994), 11–20.

[29] T. Tanaka, *Algebraic independence of the values of power series generated by linear recurrences*, Acta Arith. **74** (1996), 177–190.

[30] T. Tanaka, *Algebraic independence results related to linear recurrences*, Osaka J. Math. **36** (1999), 203–227.

[31] T. Tanaka, *Algebraic independence of the values of power series, Lambert series, and infinite products generated by linear recurrences*, Osaka J. Math. **42** (2005), 487–497.

[32] T. Tanaka, *Conditions for the algebraic independence of certain series involving continued fractions and generated by linear recurrences*, J. Number Theory **129** (2009), 3081–3093.

[33] T. Tanaka, *Algebraic independence properties related to certain infinite products*, AIP Conference Proceedings **1385** (2011), 116–123.

[34] Y. Tanuma, *Algebraic independence of the values of certain series and their derivatives involving the Hecke-Mahler series*, J. Number Theory **210** (2020), 231–248.

[35] M. Waldschmidt, Diophantine Approximation on Linear Algebraic Groups, Grundlehren der mathematischen Wissenschaften **326**, Springer, 2000.