# Iwasawa theory for the 2-components of ideal class groups 

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Mahiro Atsuta

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Graduate School of Science and Technology

Keio University

Mahiro Atsuta

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## Chapter 1

## Introduction

In number theory, the ideal class group of a number field is one of the most important arithmetic objects. The ideal class group $\mathrm{Cl}(K)$ of a number field $K$ is the quotient group $\mathrm{Cl}(K)=I(K) / P(K)$, where $I(K)$ is the group of fractional ideals of $K$ and $P(K)$ is the group of principal ideals of $K$. It is well known that $\mathrm{Cl}(K)$ is a finite abelian group. However, it is difficult in general to compute the order of it.

Iwasawa theory studies a relationship between arithmetic objects and special values of the zeta functions. The analytic class number formula is a famous theorem which relates the ideal class groups to the Dedekind zeta functions. More precisely, let $K$ be a number field, $\zeta_{K}(s)$ the Dedekind zeta function of $K$, and $h_{K}$ the order of $\mathrm{Cl}(K)$. The analytic class number formula says that

$$
\lim _{s \rightarrow 0} \frac{\zeta_{K}(s)}{s^{r_{1}+r_{2}-1}}=-\frac{h_{K} \operatorname{Reg}_{K}}{\sharp W(K)},
$$

where $r_{1}$ is the number of real places of $K, r_{2}$ is the number of complex places of $K, \operatorname{Reg}_{K}$ is the regulator of $K$ and $W(K)$ is the group of roots of unity in $K$. This formula means that the special value of the Dedekind zeta function knows the class number of $K$. In Iwasawa theory, the famous Iwasawa main conjecture is a refinement of the analytic class number formula (see §1.2.2).

### 1.1 Iwasawa theory for class groups

In this section, we introduce Iwasawa theory for class groups. Let $p$ be a prime, $k$ a totally real field and $K / k$ a finite abelian extension such that $K$ is a CM-field. Assume that for any number field $N$, we denote by $N_{\infty} / N$ the cyclotomic $\mathbb{Z}_{p}$-extension of $N$. Assume that $k_{\infty} \cap K=k$. We consider
the unramified Iwasawa module $X_{K_{\infty}}$ which is defined as the Galois group of the maximal unramified abelian pro- $p$ extension of $K_{\infty}$. By class field theory, we have

$$
X_{K_{\infty}} \simeq \lim _{\leftarrow}^{\leftrightarrows} A_{K_{n}},
$$

where $K_{n}$ is the $n$-th layer of $K_{\infty} / K$ and $A_{K_{n}}$ is the $p$-Sylow subgroup of the ideal class group of $K_{n}$. The projective limit is taken with respect to the relative norms. Since the Galois group $\operatorname{Gal}\left(K_{\infty} / k\right)$ naturally acts on $X_{K_{\infty}}$, it becomes a $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(K_{\infty} / k\right)\right]\right]$-module. Iwasawa proved that $X_{K_{\infty}}$ is a finitely generated torsion $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(K_{\infty} / K\right)\right]\right]$-module.

The Iwasawa main conjecture claims that the characteristic ideal of the $\chi$-component of $X_{K_{\infty}}$ coincides with the projective limit of the Stickelberger element for any odd character $\chi$ of $\operatorname{Gal}(K / k)$ (see Conjecture 2.4.1). This conjecture is proved when $k=\mathbb{Q}$ and $p=2$ by Mazur-Wiles [23], and for general totally real field $k$ and $p \neq 2$ by Wiles [34] (see Theorem 2.4.4).

As an application of the Iwasawa main conjecture, we get the information of the ideal class group of $K$ by descent theory. When $k=\mathbb{Q}$, Kurihara and Miura [21] proved that the Fitting ideal of the minus component of the ideal class group of $K$ coincides with the Stickelberger ideal (see Theorem 2.5.7) except for the 2 -component. Also when $k=\mathbb{Q}$ and the 2-Sylow subgroup of $\operatorname{Gal}(K / \mathbb{Q})$ is cyclic, Greither [13] determined the Fitting ideal of the minus component of the 2-part of the ideal class group of $K$ (see Theorem 2.5.8) under certain assumptions.

To study the Fitting ideal of the ideal class group, they firstly consider the Fitting ideal of the unramified Iwasawa module $X_{K_{\infty}}$. For any finitely generated torsion $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(K_{\infty} / K\right)\right]\right]$-module $M$, if $M$ has no non-trivial finite $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(K_{\infty} / K\right)\right]\right]$-submodule, the characteristic ideal of $M$ coincides with the Fitting ideal of $M$ (see Proposition 2.5.3). Therefore, a question of whether the Iwasawa module has non-trivial finite submodules or not is important in Iwasawa theory. If $p$ is an odd prime, Iwasawa proved that the minus component of $X_{K_{\infty}}$ has no non-trivial finite $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(K_{\infty} / K\right)\right]\right]-$ submodule (see Theorem 2.2.2). However, it sometimes has non-trivial finite submodules when $p=2$. For $p=2$, Ferrero [6] proved that if $K$ is an imaginary quadratic field that is not $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})$ and the prime above 2 ramifies in $K_{\infty} / \mathbb{Q}_{\infty}$, the maximal finite $\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(K_{\infty} / K\right)\right]\right]$-module of $X_{K_{\infty}}$ is isomorphic to $\mathbb{Z} / 2 \mathbb{Z}$.

In Iwasawa theory, one often assumes $p \neq 2$. The case for $p=2$ is totally different from the case that $p$ is an odd prime. For example, the Iwasawa main conjecture has not completely proved for $p=2$. Also, as we mentioned above, the minus component of $X_{K_{\infty}}$ sometimes has non-trivial
finite submodules for $p=2$. In this thesis, we study Iwasawa theory and the $p$-components of ideal class groups for $p=2$.

### 1.2 Main results

In this section, we state the main results in this thesis. Our main results are the following 3 theorems.

The first main result (Theorem 1.2.1) is on finite submodules of the minus quotient of the unramified Iwasawa module of a CM-field. This is a generalization of Fererro's result (see Theorem 2.2.3).

The second main result (Theorem 1.2.4) is on the Iwasawa main conjecture for $p=2$. Wiles [34] proved the Iwasawa main conjecture for $p=2$ only in the case where there is no trivial zero in the $p$-adic $L$-function (see Theorem 2.4.5). We remove this condition.

The third main result (Theorem 1.2.6) is on the Fitting ideals of ideal class groups. We determine the Fitting ideal of the minus quotient of the 2 -component of the ideal class group of a CM-field which is cyclic over a totally real field. This is a generalization of Greither's result (see Theorem 2.5.8).

### 1.2.1 The maximal finite submodules of Iwasawa modules

In this subsection, we introduce the first main theorem.
For any number field $K$, we denote by $K_{\infty} / K$ the cyclotomic $\mathbb{Z}_{2}$-extension of $K, K_{n}$ the $n$-th layer of $K_{\infty} / K$, and $\mathrm{Cl}(K)$ the ideal class group of $K$. We denote by $S_{2}(K), S_{\infty}(K)$ the set of primes of $K$ lying above $2, \infty$, respectively. Let $F$ be a CM-field and $F^{+}$the maximal real subfield of $F$. Put $\Lambda:=\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(F_{\infty} / F\right)\right]\right]$. We define the subset $\mathscr{S}_{2}\left(F^{+}\right)$of $S_{2}\left(F^{+}\right) \cup S_{\infty}\left(F^{+}\right)$ by

$$
\mathscr{S}_{2}\left(F^{+}\right)=\left\{v \in S_{2}\left(F^{+}\right) \mid v \text { ramifies in } F_{\infty} / F_{\infty}^{+}\right\} \cup S_{\infty}\left(F^{+}\right) .
$$

For any extension $K / F^{+}$, we denote by $\mathscr{S}_{2}(K)$ the set of primes of $K$ lying above $\mathscr{S}_{2}\left(F^{+}\right)$. We put

$$
d=\sharp\left(S_{2}\left(F_{\infty}\right) \cap \mathscr{S}_{2}\left(F_{\infty}\right)\right) .
$$

Using this particular $\mathscr{S}_{2}(K)$, we define $\mathrm{Cl}_{\mathscr{L}_{2}}(K)$ by the $\mathscr{S}_{2}(K)$-ideal class group of $K$, i.e

$$
\mathrm{Cl}_{\mathscr{\mathscr { L }}_{2}}(K)=\operatorname{coker}\left(K^{\times} \xrightarrow{\oplus \operatorname{ord}_{v}} \bigoplus_{v \notin \mathscr{\mathscr { V }}_{2}(K)} \mathbb{Z}\right) .
$$

We denote by $A_{K}$ (resp. $A_{K, \mathscr{S}_{2}}$ ) the 2-Sylow subgroup of the ideal class group $\mathrm{Cl}(K)$ (resp. $\mathrm{Cl}_{\mathscr{\mathscr { L }}_{2}}(K)$ ). By definition, we have $X_{F_{\infty}} \cong \lim _{\leftrightarrows} A_{F_{n}}$. There are several ways to define the minus quotient, but we adopt the following. Let $j$ be the complex conjugation. We define the minus quotient $X_{F_{\infty}}^{-}$by

$$
X_{F_{\infty}}^{-}=X_{F_{\infty}} /(1+j) X_{F_{\infty}} .
$$

We denote by $F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)$the maximal finite $\Lambda$-submodule of $X_{F_{\infty}}^{-}$. We define $X_{F_{\infty}, \mathscr{L}_{2}}, X_{F_{\infty}, \mathscr{S}_{2}}^{-}$by

$$
X_{F_{\infty}, \mathscr{S}_{2}}=\lim _{\leftrightarrows} A_{F_{n}, \mathscr{S}_{2}}, \quad X_{F_{\infty}, \mathscr{S}_{2}}^{-}=X_{F_{\infty}, \mathscr{Y}_{2}} /(1+j) X_{F_{\infty}, \mathscr{S}_{2}},
$$

where the projective limit is taken with respect to the norm maps.
We define

$$
\begin{gathered}
D_{n, \mathscr{S}_{2}}=\operatorname{ker}\left(A_{F_{n}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}\right), D_{n, \mathscr{S}_{2}}^{+}=\operatorname{ker}\left(A_{F_{n}^{+}} \longrightarrow A_{F_{n}^{+}, \mathscr{S}_{2}}\right), \\
\delta_{1}=\operatorname{rank}_{2}\left(\lim _{\leftrightarrows}^{\lim }\left(\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}\right)\right), \\
\delta_{2}=\operatorname{rank}_{2}\left(\lim _{\leftrightarrows} \operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right)\right),
\end{gathered}
$$

where $\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}$is the $\mathscr{S}_{2}\left(F_{n}\right)$-unit group of $F_{n}, \mathcal{O}_{F_{n}}^{\times}$the unit group of $F_{n}$, both projective limits are taken with respect to the norm maps, and $\operatorname{rank}_{2}(A)$ is the 2 -rank, namely the dimension of $A / 2 A$ as an $\mathbb{F}_{2}$-vector space. We note that $0 \leq \delta_{2} \leq \delta_{1} \leq 1$ and the 2 -rank of $\lim _{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$ is $d$ or $d-1$ (see Remark 3.3.5), where $d$ is defined in the previous paragraph (the number of certain 2 -adic primes). The first main theorem is the following.
Theorem 1.2.1. (Theorem 3.2.1, Theorem 3.3.1) (i) $X_{F_{\infty}, \mathscr{S}_{2}}^{-}$has no nontrivial finite $\Lambda$-submodule.
(ii) Assume that Leopoldt's conjecture is valid for $F^{+}$and the lifting maps $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ are injective for all sufficiently large $n \gg 0$. Then we have
$F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)=\lim _{\hookleftarrow} D_{n, \mathscr{\mathscr { C }}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong \begin{cases}(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d} & \left.\text { (if } \mu_{2^{\infty}} \not \subset F_{\infty}\right) \\ (\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-\delta_{1}+\delta_{2}} & \left.\text { if } \mu_{2 \infty} \subset F_{\infty}\right),\end{cases}$
where $d$ is the number of primes of $F_{\infty}$ above 2 which ramify in $F_{\infty} / F_{\infty}^{+}$and $\mu_{2 \infty}$ is the group of all 2 power roots of unity.

This is a generalization of the Ferrero's result (see Example 3.3.8). If all primes above 2 are unramified in $F_{\infty} / F_{\infty}^{+}$, the set $\mathscr{S}_{2}\left(F^{+}\right)$coincides with $S_{\infty}\left(F^{+}\right)$by definition. Therefore, we have $A_{F_{n}, \mathscr{S}_{2}}=A_{F_{n}}$, and $X_{F_{\infty}, \mathscr{Y}_{2}}^{-}=$ $X_{F_{\infty}}^{-}$. Thus Theorem 1.2.1 (i) implies the following result.

Corollary 1.2.2. Assume that all primes above 2 are unramified in $F_{\infty} / F_{\infty}^{+}$. Then, $X_{F_{\infty}}^{-}$has no non-trivial finite $\Lambda$-submodule.

Concerning the injectivity of the lifting map $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ for an imaginary abelian field $F$, we get lemma 3.3.7. Theorem 1.2.1 and lemma 3.3.7 imply the following result.

Corollary 1.2.3. Assume that $F$ is an imaginary abelian field and all primes above 2 ramify in $F_{\infty} / F_{\infty}^{+}$. If $F_{\infty}$ contains $\mu_{2^{\infty}}$ or Hasse's unit index $\left[\mathcal{O}_{F_{n}}^{\times}: \mu\left(F_{n}\right) \mathcal{O}_{F_{n}^{+}}^{\times}\right]=2$ for all sufficiently large $n \gg 0$, we have
$F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)=\lim _{\longleftarrow} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong \begin{cases}(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d} & \left(\text { if } \mu_{2 \infty} \not \subset F_{\infty}\right) \\ (\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-\delta_{1}+\delta_{2}} & \left(\text { if } \mu_{2 \infty} \subset F_{\infty}\right),\end{cases}$
where $d$ was defined on page 5 and in this case, it is the number of primes of $F_{\infty}$ above $2, \mu\left(F_{n}\right)$ is the group of roots of unity contained in $F_{n}$.

If $F$ is an imaginary abelian field and $\sqrt{-1} \in F$, we can determine the maximal finite $\Lambda$-submodule of $X_{K_{\infty}}^{-}$by Corollary 1.2.2 and Corollary 1.2.3.

For example, let $F^{+}$be a real abelian field which is unramified at 2 , and $F=F^{+}(\sqrt{-1})$. Then, we have

$$
F_{\Lambda}\left(X_{F_{\infty}}^{-}\right) \cong(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-1}
$$

and in this case, $d$ is the number of primes of $F$ lying above 2 (see Example 3.3.9).

### 1.2.2 Iwasawa main conjecture

In this subsection, we introduce the second main theorem. Let $k$ be a totally real field. We take a one-dimensional Artin character $\chi$ for $k$ and denote by $k^{\chi}$ the extension of $k$ attached to $\chi$, i.e., $\operatorname{Gal}\left(k^{\chi} / k\right)=\operatorname{Im}(\chi)$. Assume that $\chi$ is even, so $k^{\chi}$ is also totally real. We denote by $k_{\infty}^{\chi}$ the cyclotomic $\mathbb{Z}_{p}$-extension of $k^{\chi}$. Let $S$ be a set of primes of $k$, which contains all primes which ramify in $k_{\infty}^{\chi} / k$. We consider the Iwasawa module $\mathcal{X}_{k_{\infty}^{\chi}, S}$ which is defined by the Galois group of the maximal abelian pro- $p$ extension over $k_{\infty}^{\chi}$ unramified outside $S$. Put $\Lambda_{\chi}=\mathbb{Z}_{p}[\operatorname{Im}(\chi)]\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k^{\chi}\right)\right]\right]$. We consider the $\chi$-component $\mathcal{X}_{k_{\infty}^{\chi}, S}^{\chi}=\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes_{\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]} \Lambda_{\chi}$, which is a $\Lambda_{\chi}$-module.

The Iwasawa main conjecture (IMC) claims that the characteristic ideal of $\mathcal{X}_{k_{\infty}^{\chi}, S}^{\chi}$ coincides with the $p$-adic $L$-function (for more precise statement of (IMC), see Conjecture 2.4.2). If $p$ is an odd prime, Wiles (and Greenberg)
proved that (IMC) is valid in [34] (see Theorem 2.4.4). But for $p=2$, Wiles proved (IMC) only in the case that there is no trivial zero in the $p$-adic $L$ function (see Theorem 2.4.5). We can prove (IMC) for $p=2$ only assuming that the $\mu$-invariant vanishes and Leopoldt's conjecture is valid.

The second main theorem is the following result.
Theorem 1.2.4. (i) If the $\mu$-invariant of $k_{\infty}^{\chi}$ vanishes, then (IMC) for $p=2$ is valid for $\chi \neq \mathbf{1}$.
(ii) If the $\mu$-invariant of $k_{\infty}^{\chi}$ vanishes and Leopoldt's conjecture is valid for $k$, then (IMC) for $p=2$ is valid for $\chi=\mathbf{1}$.

Remark 1.2.5. In Iwasawa theory, one often assumes $p \neq 2$. One of the reasons is that the Iwasawa main conjecture (IMC) is not proved for the case $p=2$. Here, we give two applications of Theorem 1.2.4.
(i) An equivariant Iwasawa main conjecture has been formulated and proved by Ritter and Weiss for odd primes under the assumption $\mu=0$ in [27]. In [31], [32], Taleb extended this conjecture of Ritter and Weiss to all prime numbers (see Conjecture 5.1 in [32]) and proved it for $p=2$ assuming that the classical Iwasawa main conjecture (IMC) is valid (see Theorem 5.7 in [32]). Therefore, Theorem 1.2.4 implies that the equivariant Iwasawa main conjecture for $p=2$ holds by assuming that the $\mu$-invariant vanishes and Leopoldt's conjecture is valid.
(ii) Greither and Kurihara also proved an equivariant Iwasawa main conjecture which treats classical Iwasawa modules. Let $K / k$ be a finite abelian extension such that both $k$ and $K$ are totally real fields. We denote by $K_{\infty}$ the cyclotomic $\mathbb{Z}_{p}$-extension of $K$. When $p$ is an odd prime, Greither and Kurihara obtained the exact description of the Fitting ideal of the Iwasawa module $\mathcal{X}_{K_{\infty}, S}$ as a $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(K_{\infty} / k\right)\right]\right]$-module using the $p$-adic $L$-function under the assumption $\mu=0$ (see Theorem 4.1 in [15], see also Remark 0.4 (6) in [16]). In their theorem, the assumption $p \neq 2$ is used only for using (IMC). Using (IMC) for $p=2$, we get the same theorem for $p=2$ by the same method in [15]. Therefore, Theorem 1.2.4 implies that we can determine the Fitting ideal of the Iwasawa module $\mathcal{X}_{K_{\infty}, S}$, assuming that $\mu=0$ and Leopoldt's conjecture is valid.

### 1.2.3 Fitting ideal of the ideal class group

Let $k$ be a totally real number field, and $K$ a CM-field such that $K / k$ is a finite abelian extension. We denote by $K_{\infty}$ the cyclotomic $\mathbb{Z}_{2}$-extension of $K$. We consider the unramified Iwasawa module $X_{K_{\infty}}$ which is defined by the Galois group of the maximal unramified abelian pro-2 extension of
$K_{\infty}$. We denote by $A_{K}$ the 2-Sylow subgroup of the ideal class group of $K$. For any $\mathbb{Z}_{2}[\operatorname{Gal}(K / k)]$-module $M$, we define the minus quotient of $M$ by $M^{-}=M \otimes \mathbb{Z}_{2}[\operatorname{Gal}(K / k)] /(1+j)$, where $j$ is the complex conjugation in $\operatorname{Gal}(K / k)$. We put $d=[k: \mathbb{Q}]$. We write $G=\operatorname{Gal}(K / k)=G^{\prime} \times \Delta$, where $G^{\prime}$ is a 2 -group and the order of $\Delta$ is odd. We assume that $G^{\prime}$ is cyclic. Put $\mathbb{Z}_{2}[G]^{-}=\mathbb{Z}_{2}[G] /(1+j)$. Let $\psi$ be a faithful character of $G^{\prime}$. Since $G^{\prime}$ is cyclic, we have

$$
\mathbb{Z}_{2}[G]^{-} \simeq \bigoplus_{\chi \in \hat{\Delta} / \sim} \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]
$$

Therefore, $\mathbb{Z}_{2}[G]^{-}$is a direct product of discrete valuation rings. In this situation, we will define the Stickelberger ideal $\Theta_{K / k}^{-} \subset \mathbb{Z}_{2}[G]^{-}$in Definition 5.1.2 The definition of $\Theta_{K / k}^{-}$is similar to the definition of the Stickelberger ideal in Kurihara [19]. We denote by $K^{G^{\prime}}$ the fixed field $G^{\prime}$ in $K$, and by $k^{c l}$ the Galois closure of $k$ over $\mathbb{Q}$. Let $K^{+}$be the maximal real subfield of $K$. The third main theorem is the following.

Theorem 1.2.6. Assume that the following assumptions are satisfied.
(1) The $\mu$-invariant of $K_{\infty}$ vanishes.
(2) $K \cap k_{\infty}=k$.
(3) At least one of the following assumptions is satisfied.
(3a) No prime above 2 splits in $K / K^{+}$.
(3b) No prime above 2 ramifies in $K / K^{+}$and $k^{c l} \cap K^{G^{\prime}}=k$.
Then we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}^{-}\left(A_{K}^{-}\right)=\Theta_{K / k}^{-}
$$

If $k=\mathbb{Q}$ and 2 is unramified in $K / \mathbb{Q}$, Theorem 1.2 .6 was proved by Greither [13] (see Theorem 2.5.8). Greither proved this using the Iwasawa main conjecture and descent theory. In this paper, we also prove Theorem 1.2 .6 by a similar method as in Greither [13]. However, there are some difficulties in our general case. For example, if all primes above 2 are unramified in $K / k$, the minus quotient of $X_{K_{\infty}}$ has no non-trivial finite submodule (see Corollary 1.2.2). But, as we mentioned above, the minus quotient of $X_{K_{\infty}}$ sometimes has non-trivial finite submodules in general. In this case, the characteristic ideal does not coincide with the Fitting ideal of the $\chi \psi$ component of $X_{K_{\infty}}$.

If $k=\mathbb{Q}$, the assumption (3) in Theorem 1.2.6 is always satisfied and the $\mu$-invariant of $K_{\infty}$ vanishes by the theorem of Ferrero and Washington [7]. Therefore, Theorem 1.2.6 implies the following result.

Corollary 1.2.7. Assume that $K$ is an imaginary abelian field over $\mathbb{Q}$ and that the 2 -Sylow subgroup of $\operatorname{Gal}(K / \mathbb{Q})$ is cyclic. Then, we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)=\Theta_{K / \mathbb{Q}}^{-}
$$

For a general totally real field, Brumer gave a conjecture which generalizes the Stickelberger's theorem (cf. Conjecture 1 in [35]). Theorem 1.2.6 implies some affirmative result for the 2 -component of the conjecture of Brumer (cf. [35]) under the assumptions in Theorem 1.2.6.

Corollary 1.2.8. Assume that the assumptions (1), (2), (3) in Theorem 1.2.6 are satisfied. Then, we have

$$
2^{a_{1, K}} Q(K) \mathrm{Ann}_{\mathbb{Z}_{2}[G]}\left(W(K) \otimes \mathbb{Z}_{2}\right) \frac{1}{2^{d-1}} \theta_{K / k} \subset \operatorname{Ann}_{\mathbb{Z}_{2}[G]}\left(A_{K}\right),
$$

where $2^{a_{1, K}}=\sharp \operatorname{ker}\left(A_{K^{+}} \rightarrow A_{K}\right), d=[k: \mathbb{Q}]$ and $Q(K)$ is Hasse's unit index of $K$.

We note that both $2^{a_{1, K}}$ and $Q(K)$ are 1 or 2 . Therefore, if $d=[k:$ $\mathbb{Q}] \geq 3$, we have

$$
\operatorname{Ann}_{\mathbb{Z}_{2}[G]}\left(W(K) \otimes \mathbb{Z}_{2}\right) \theta_{K / k} \subset \operatorname{Ann}_{\mathbb{Z}_{2}[G]}\left(A_{K}\right)
$$

under the assumptions (1), (2), (3) in Theorem 1.2.6.

### 1.3 Key points in this thesis

To study the 2 -components of ideal class group is difficult. One of the reasons is that taking a minus component is not an exact functor for $p=2$. Let $k$ be a totally real field and $K / k$ a finite abelian extension. Assume that $K$ is a CM-field. If $p$ is an odd prime, 2 is invertible in $\mathbb{Z}_{p}[\operatorname{Gal}(K / k)]$. Therefore, $\frac{1 \pm j}{2}$ are in $\mathbb{Z}_{p}[\operatorname{Gal}(K / k)]$, and idempotents of the group ring, where $j$ is the complex conjugation in $\operatorname{Gal}(K / k)$. This implies that taking the minus component is an exact functor for the case $p \neq 2$. But this is not an exact functor for $p=2$, which makes all the arguments very complicated.

In Iwasawa theory, a question of whether the Iwasawa module has nontrivial finite submodules or not is important. The minus quotient of the unramified Iwasawa module sometimes has non-trivial finite submodules only in the case for $p=2$. To compute the Fitting ideal of the ideal class group, we need to determine this. We determine the size of the maximal finite
submodule of the minus quotient of the unramified Iwasawa module under some mild assumptions (see Theorem 1.2.1).

To prove Theorem 1.2.1, we use a result of Greenberg [11]. Greenberg gives sufficient conditions that Selmer groups have no non-trivial finite submodules. We can prove Theorem 1.2.1 (i), using Greenberg's result. However, as we mentioned above, the minus quotient of the unramified Iwasawa module sometimes has non-trivial finite submodules only in the case for $p=2$. This means that Selmer groups which have usual local conditions do not satisfy Greenberg's conditions in the case for $p=2$. Therefore, we need to choose some appropriate local conditions.

The Iwasawa main conjecture (IMC) claims that the characteristic ideal of the $\chi$-component of the Iwasawa module coincides with the $p$-adic $L$ function (see Conjecture 2.4.1, 2.4.2). If $p$ is an odd prime, Wiles proved that (IMC) is valid in [34]. But for $p=2$, Wiles proved (IMC) only in the case that there is no trivial zero in the $p$-adic $L$-function (see Theorem 11.1 in [34]). In this thesis, we prove (IMC) for $p=2$ assuming that the $\mu$-invariant vanishes and Leopoldt's conjecture is valid (see Theorem 1.2.4).

We show that the argument of Wiles to avoid the trivial zeros in [34] can be applied for $p=2$. The key lemmas of avoiding the trivial zeros by Wiles are Lemmas 10.1, 10.2 in [34]. Instead of these lemmas, assuming $\mu=0$, we prove Lemma 4.1.6 which is a similar statement of Lemma 10.1 in [34], and Lemma 4.1.1 which is the $p=2$ version of Wiles [34] Lemma 10.2.

Using Theorem 1.2.1, Theorem 1.2.4 and descent theory, we can get the information of the 2 -component of the ideal class group of a number field of finite degree. We determine the Fitting ideal of the minus quotient of the 2 -component of the ideal class group of a CM-field $K$ which is cyclic over a totally real field $k$.

If $k=\mathbb{Q}$, the $p$-adic $L$-function has at most one trivial zero, by which we can use the value of the first derivative of the $p$-adic $L$-function. But, in general, the $p$-adic $L$-function sometimes has more than two trivial zeros. In this case, we need the technical descent argument in Wiles [35].

To prove Theorem 1.2.6, we have to determine the order of the $\chi$ component of the ideal class group since $\mathbb{Z}_{2}[G]^{-}$is a direct product of discrete valuation rings. It might look easy to determine the order, but it is quite difficult in general. Even if we use the analytic class number formula, 2-power factors appear in several places. For example, Hasse's unit index in the class number formula is always 1 or 2 . We note that if $k=\mathbb{Q}$ and the 2-Sylow subgroup of $\operatorname{Gal}(K / \mathbb{Q})$ is cyclic, Hasse's unit index is always 1 but we can use this fact only for $k=\mathbb{Q}$ (see [22]). Also, the number of roots of unity of $K$ is always divided by 2 since $-1 \in K$. To study the order of
the $\chi$-component of the ideal class group, we need to consider these terms. We need to study these complicated factors to get the exact order of the $\chi$-component of the ideal class group.

### 1.4 Outline

The outline of this thesis is as follows. In Chapter 2, we introduce known results in Iwasawa theory. Also we state the Iwasawa main conjecture. In chapter 3, we study finite submodules of the minus quotient of the unramified Iwasawa modules of a CM-field for $p=2$ and prove Theorem 1.2.1. In chapter 4, we prove the Iwasawa main conjecture for $p=2$ (Theoren 1.2.4). In chapter 5, we study the Fitting ideal of the 2-component of the ideal class group of a CM-field and prove Theorem 1.2.6.

## Chapter 2

## Preliminary

### 2.1 Algebraic preliminary

Let $p$ be a prime number. Suppose that $\mathcal{O}$ is the integer ring of a finite extension of $\mathbb{Q}_{p}$ and put $\Lambda=\mathcal{O}[[T]]$. We denote by $\pi$ a uniformizer of $\mathcal{O}$.

Theorem 2.1.1. ( Theorem 7.3 in [33], p-adic Weierstrass preparation theorem) For any $0 \neq f(T) \in \Lambda$, we can write it (in a unique way) in the form

$$
f(T)=\pi^{\mu(f(T))} f^{*}(T) U(T),
$$

where $\mu(f(T)) \in \mathbb{Z}_{\geq 0}, f^{*}(T)$ is a distinguished polynomial and $U(T)$ is a unit power series.

Definition 2.1.2. (pseudo-isomorphic) Let $M_{1}$ and $M_{2}$ be finitely generated $\Lambda$-modules. We call $M_{1}$ and $M_{2}$ pseudo-isomorphic if there is a $\Lambda$ homomorphism $f: M_{1} \rightarrow M_{2}$ such that both $\operatorname{ker} f$ and coker $f$ are finite.

We note that this is not an equivalence relation in the category of finitely generated $\Lambda$-modules. However, this is an equivalence relation in the category of finitely generated torsion $\Lambda$-modules.

Theorem 2.1.3. (Theorem 13.12 in [33], Structure theorem for torsion $\Lambda$ modules) Let $M$ be a finitely generated torsion $\Lambda$-module. Then there exists a pseudo-isomorphism

$$
M \longrightarrow \bigoplus_{i=1}^{s} \Lambda /\left(\pi^{m_{i}}\right) \oplus \bigoplus_{j=1}^{t} \Lambda /\left(f_{j}(T)^{n_{j}}\right)
$$

where $s, t$ are non-negative integers, $m_{i}, n_{j}$ are positive integers and $f_{j}(T)$ is an irreducible distinguished polynomial. The integers $s, t, m_{i}, n_{j}$ and the irreducible distinguished polynomial $f_{j}(T)$ are determined uniquely by $M$.

This theorem guarantees the following quantities are well-defined.
Definition 2.1.4. (Iwasawa invariants, characteristic ideal) With the notation of Theorem 2.1.3, we define

$$
\begin{aligned}
\mu(M) & =\sum_{i=1}^{s} m_{i} \\
\lambda(M) & =\sum_{j=1}^{t} n_{j} \operatorname{deg}\left(f_{j}(T)\right) \\
\operatorname{char}_{\Lambda}(M) & =\left(\pi^{\mu(M)} \prod_{j=1}^{t} f_{j}(T)^{n_{j}}\right)
\end{aligned}
$$

Let $\Gamma$ be a topological group which is isomorphic to the additive group $\mathbb{Z}_{p}$. For any $n \in \mathbb{Z}_{\geq 0}$, we denote by $\Gamma_{n}$ the subgroup of $\Gamma$ of index $p^{n}$.
Theorem 2.1.5. (Serre, Theorem 7.1 in [33]) Let $\gamma$ be a topological generator of $\Gamma$. Then there is a non-canonical isomorphism

$$
\begin{aligned}
\mathbb{Z}_{p}[[T]] & \simeq \mathbb{Z}_{p}[[\Gamma]]:={\underset{\gtrless}{n}}^{\lim _{p}}\left[\Gamma / \Gamma_{n}\right] \\
T & \mapsto \gamma-1
\end{aligned}
$$

of topological rings.

### 2.2 Iwasawa module

In this section, we mention some properties of Iwasawa modules.

### 2.2.1 unramified Iwasawa module

First of all, we consider the unramified Iwasawa module. Let $p$ be a prime number, $K$ a number field and $K_{\infty} / K$ the cyclotomic $\mathbb{Z}_{p}$-extension. Put $\Gamma=\operatorname{Gal}\left(K_{\infty} / K\right)$ and $\Lambda=\mathbb{Z}_{p}[[\Gamma]]$. We consider $L_{\infty}$ the maximal unramified abelian pro- $p$ extension of $K_{\infty}$. Put $X_{K_{\infty}}=\operatorname{Gal}\left(L_{\infty} / K_{\infty}\right)$. Then $\Gamma$ acts on $X_{K_{\infty}}$ by

$$
x^{\gamma}=\tilde{\gamma} x \tilde{\gamma}^{-1} \quad x \in X_{K_{\infty}}, \gamma \in \Gamma,
$$

where $\tilde{\gamma}$ is an extension of $\gamma$ to $\operatorname{Gal}\left(L_{\infty} / K\right)$. Since $X_{K_{\infty}}$ is an abelian, this action is well-defined. Therefore, $X_{K_{\infty}}$ becomes a $\Lambda$-module.

Theorem 2.2.1. (Iwasawa) $X_{K_{\infty}}$ is a finitely generated torsion $\Lambda$-module.
Assume that $K$ is a CM-field and $K^{+}$is the maximal real subfield of $K$. Let $j$ be the complex conjugation in $\operatorname{Gal}\left(K / K^{+}\right)$. We consider the minus quotient of Iwasawa module defined by

$$
X_{K_{\infty}}^{-}=X_{K_{\infty}} /(1+j) X_{K_{\infty}}
$$

Theorem 2.2.2. (Iwasawa, Proposition 13.28 in [33]) If $p$ is an odd prime, $X_{K_{\infty}}^{-}$has no non-trivial finite $\Lambda$-module.

However, when $p=2, X_{K_{\infty}}^{-}$sometimes has non-trivial finite submodule. Ferrero proved the following result.

Theorem 2.2.3. (Ferrero, Theorem 5 in [6]) Let $K$ be an imaginary quadratic field. Assume that the prime above 2 ramifies in $K_{\infty} / K_{\infty}^{+}$. If $p=2$, the maximal finite $\Lambda$-submodule of $X_{K_{\infty}}$ is isomorphic to $\mathbb{Z} / 2 \mathbb{Z}$.

### 2.2.2 $S$-ramified Iwasawa module

Next, we consider the $S$-ramified Iwasawa module. Let $k$ be a totally real field and $k_{\infty} / k$ the cyclotomic $\mathbb{Z}_{p}$-extension. We take $S$ a finite set of prime of $k$, which contains all primes above $p$. We denote by $M_{k_{\infty}, S}$ the maximal abelian pro-p extension of $k_{\infty}$ unramified outside $S$. Put $\mathcal{X}_{k_{\infty}, S}=\operatorname{Gal}\left(M_{k_{\infty}, S} / k_{\infty}\right)$. Since $\mathcal{X}_{k_{\infty}, S}$ is abelian, $\operatorname{Gal}\left(k_{\infty} / k\right)$ acts on $\mathcal{X}_{k_{\infty}, S}$ by conjugation as above.

Theorem 2.2.4. (Iwasawa, Theorem 13.31 in [33]) $\mathcal{X}_{k_{\infty}, S}$ is a finitely generated torsion $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty} / k\right)\right]\right]-$ module.

## 2.3 -adic $L$-function

In this section, we introduce the $p$-adic $L$-function of Deligne-Ribet.
Let $k$ be a totally real field, $F / k$ a finite abelian extension. We define in the usual way the partial zeta function for $\sigma \in \operatorname{Gal}(F / k)$ by

$$
\zeta(s, \sigma)=\sum_{(\mathfrak{a}, F / k)=\sigma} N(\mathfrak{a})^{-s}
$$

for $\operatorname{Re}(s)>1$ where the sum is taken over integral ideals $\mathfrak{a}$ of $k$ which are prime to the conductor ideal $\mathfrak{f}_{F / k}$ such that the Artin symbol $(\mathfrak{a}, F / k)$ is
equal to $\sigma(N(\mathfrak{a})$ is the norm of $\mathfrak{a})$. The partial zeta functions are meromorphically continued to the whole complex plane, and holomorphic except $s=1$. We consider the Stickelberger element defined by

$$
\theta_{F / k}=\sum_{\sigma \in \operatorname{Gal}(F / k)} \zeta(0, \sigma) \sigma^{-1}
$$

By Klingen and Siegel, we know that $\theta_{F / k} \in \mathbb{Q}[\operatorname{Gal}(F / k)]$.
Lemma 2.3.1. (cf. Lemma 2.1 in [19]) Let $M$ be a field such that $k \subset M \subset$ $F$. We denote by $S_{F}$ (resp. $S_{M}$ ) the set of primes of $k$ ramifying in $F / k$ (resp. M/k). Let

$$
c_{F / M}: \mathbb{Q}[\operatorname{Gal}(F / k)] \longrightarrow \mathbb{Q}[\operatorname{Gal}(M / k)]
$$

denote the natural homomorphism. Then we have

$$
c_{F / M}\left(\theta_{F / k}\right)=\left(\prod_{v \in S_{F} \backslash S_{M}}\left(1-\operatorname{Frob}_{v}^{-1}\right)\right) \theta_{M / k}
$$

where $\operatorname{Frob}_{v}$ is the Frobenius of $v$ in $\operatorname{Gal}(M / k)$.
Let $p$ be a prime number and $K / k$ a finite abelian extension. For any number field $N$, we denote by $N_{\infty}$ the cyclotomic $\mathbb{Z}_{p}$-extension of $N$ and $N_{n}$ the $n$-th layer of $N_{\infty} / N$. Assume that $K$ is also a totally real field and $K \cap k_{\infty}=k$. Put $L=K\left(\mu_{p}\right)\left(L=K\left(\mu_{4}\right)\right.$ if $\left.p=2\right), \mathcal{G}=\operatorname{Gal}\left(L_{\infty} / k\right)$.

Theorem 2.3.2. (Deligne-Ribet [4]) For any integral ideal $\mathfrak{c}$ of $k$ which is prime to conductor of $L_{n} / k$,

$$
\left(N \mathfrak{c}-\operatorname{Frob}_{\mathfrak{c}}\right) \theta_{L_{n} / k} \in 2^{d-1}(1-j) \mathbb{Z}\left[\operatorname{Gal}\left(L_{n} / k\right)\right]
$$

for all sufficiently large $n \gg 0$. Where $d=[k: \mathbb{Q}], N \mathfrak{c}=\sharp\left(\mathcal{O}_{k} / \mathfrak{c}\right)$ and $j$ is the complex conjugation.

We consider the restriction maps

$$
c_{L_{n+1} / L_{n}}: \mathbb{Q}_{p}\left[\operatorname{Gal}\left(L_{n+1} / k\right)\right] \rightarrow \mathbb{Q}_{p}\left[\operatorname{Gal}\left(L_{n} / k\right)\right]
$$

for all $n \geq 0$. Lemma 2.3.1 implies that the Stickelberger elements $\theta_{L_{n} / k}$ satisfy $c_{L_{n+1} / L_{n}}\left(\theta_{L_{n+1} / k}\right)=\theta_{L_{n} / k}$ for all sufficiently large $n \gg 0$. Theorem 2.3.2 implies that the existence $\theta_{L_{\infty} / k} \in \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right)$ which is the following properties, where $\mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right)$ is the totally quotient ring of $\mathbb{Z}_{p}[[\mathcal{G}]]$.
(1) $\theta_{L_{\infty} / k} \in\left\{\lambda \in \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right) \mid(\kappa(\sigma)-\sigma) \lambda \in 2^{d-1}(1-j) \mathbb{Z}_{p}[[\mathcal{G}]]\right.$ for all $\sigma \in$ $\mathcal{G}\}$, where $\kappa$ is the cyclotomic character.
(2) The canonical map $c_{L_{\infty} / L_{n}}: \mathbb{Z}_{p}[[\mathcal{G}]] \rightarrow \mathbb{Z}_{p}\left[\operatorname{Gal}\left(L_{n} / k\right)\right]$ extends to $c_{L_{\infty} / L_{n}}: \mathbb{Z}_{p}[[\mathcal{G}]] \theta_{L_{\infty} / k} \rightarrow \mathbb{Q}_{p}\left[\operatorname{Gal}\left(L_{n} / k\right)\right]$, and $c_{L_{\infty} / L_{n}}\left(\theta_{L_{\infty} / k}\right)=\theta_{L_{n} / k}$ for all sufficiently large $n \gg 0$.

We denote by $i$ the involution map $i: \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right) \rightarrow \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right)$ induced by $\sigma \rightarrow \sigma^{-1}$ for all $\sigma \in \mathcal{G}$ and by $j_{\kappa}$ the twist map $j_{\kappa}: \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right) \rightarrow$ $\mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right)$ induced by $\sigma \rightarrow \kappa(\sigma) \sigma$ for all $\sigma \in \mathcal{G}$. Put $\Phi_{L_{\infty} / k}=i \circ j_{\kappa}\left(\theta_{L_{\infty} / k}\right)$. The property (1) implies the following result.
$\Phi_{L_{\infty} / k} \in\left\{\lambda \in \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right) \mid(\sigma-1) \lambda \in 2^{d-1}(1+j) \mathbb{Z}_{p}[[\mathcal{G}]]\right.$ for all $\left.\sigma \in \mathcal{G}\right\}$.
Let $\chi$ be a character of $\operatorname{Gal}(L / k)$. We consider the classical $L$-function defined by

$$
L(\chi, s)=\sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N \mathfrak{a}^{s}},
$$

for $\operatorname{Re}(s)>1$ where the sum is taken over non-zero integral ideals $\mathfrak{a}$ of $k$. This is meromorphically continued to the complex plane, and holomorphic except $\chi=\mathbf{1}$ and $s=1$. We have

$$
L(\chi, s)=\prod_{\mathfrak{p}}\left(1-\chi(\mathfrak{p}) N(\mathfrak{p})^{-s}\right)^{-1},
$$

where $\mathfrak{p}$ ranges over prime ideals of $k$. Let $S$ be a finite set of finite prime of $k$, which contains all primes which ramify in $L_{\infty} / k$. Let $L_{S}(\chi, s)$ be defined by

$$
L_{S}(\chi, s)=\prod_{\mathfrak{p} \in S}\left(1-\chi(\mathfrak{p}) N(\mathfrak{p})^{-s}\right) L(\chi, s) .
$$

Theorem 2.3.3. (Deligne-Ribet [4]) There is an unique element $\Phi_{L_{\infty} / k, S} \in$ $\mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right)$ such that

$$
\chi \kappa^{n}\left(\Phi_{L_{\infty} / k, S}\right)=L_{S}(1-n, \chi)
$$

for all $n \in \mathbb{Z}_{>0}$ and all characters $\chi$ of $\operatorname{Gal}(L / k)$. In particular, if $S$ is the set of finite primes of $k$, which ramify in $L_{\infty} / k$, we have

$$
\Phi_{L_{\infty} / k, S}=\Phi_{L_{\infty} / k} .
$$

Put $\mathcal{G}^{+}=\operatorname{Gal}\left(K_{\infty} / k\right), \Gamma=\operatorname{Gal}\left(K_{\infty} / K\right)$. The canonical map $c_{L_{\infty} / K_{\infty}}$ : $\mathbb{Z}_{p}[[\mathcal{G}]] \rightarrow \mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]$extends to $c_{L_{\infty} / K_{\infty}}: \mathcal{Q}\left(\mathbb{Z}_{p}[[\mathcal{G}]]\right) \rightarrow \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]\right)$. We consider $\Phi_{K_{\infty} / k}=c_{L_{\infty} / K_{\infty}}\left(\Phi_{L_{\infty} / k}\right) \in \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]\right)$. The above inclusion implies that the following result.

Proposition 2.3.4. Put $I_{K_{\infty} / k}$ is the augmentation ideal of $\mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]$, i.e., $I_{K_{\infty} / k}=\operatorname{ker}\left(\mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right] \rightarrow \mathbb{Z}_{p}\right)$. Then we have

$$
\frac{1}{2^{d}} I_{K_{\infty} / k} \Phi_{K_{\infty} / k} \subset \mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right] .
$$

We write $G=\operatorname{Gal}(K / k)=G^{\prime} \times \Delta$ where $G^{\prime}$ is a $p$-group and the order of $\Delta$ is prime to $p$. We say two $\overline{\mathbb{Q}}_{p}$-valued character $\chi_{1}$ and $\chi_{2}$ are $\mathbb{Q}_{p}$-conjugate if $\sigma \chi_{1}=\chi_{2}$ for some $\sigma \in \operatorname{Gal}\left(\overline{\mathbb{Q}}_{p} / \mathbb{Q}_{p}\right)$. We consider this equivalence relation on $\hat{\Delta}$. Then we have

$$
\mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]=\mathbb{Z}_{p}[[G \times \Gamma]] \simeq \bigoplus_{\chi^{\prime} \in \hat{\Delta} / \sim} \mathbb{Z}_{p}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times \Gamma\right]\right] .
$$

For any element $x \in \mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]$, we denote by $x^{\chi^{\prime}} \in \mathbb{Z}_{p}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times \Gamma\right]\right]$ the $\chi^{\prime}$-component of $x$. Proposition 2.3.4 implies the following result.

Corollary 2.3.5. If $\chi^{\prime}$ is a non-trivial character of $\Delta$, we have

$$
\frac{1}{2^{d}} \Phi_{K_{\infty} / k}^{\chi^{\prime}} \in \mathbb{Z}_{p}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times \Gamma\right]\right] .
$$

If $\chi^{\prime}$ is the trivial character of $\Delta$, we have

$$
\frac{1}{2^{d}}(\sigma-1) \Phi_{K_{\infty} / k}^{\chi^{\prime}} \in \mathbb{Z}_{p}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times \Gamma\right]\right]
$$

for all $\sigma \in G^{\prime} \times \Gamma$.
Let $\chi$ be a character of $G$. We consider the map $f_{\chi}: \mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right] \rightarrow$ $\mathbb{Z}_{p}[\operatorname{Im}(\chi)][[\Gamma]]$ induced by $\chi$. For any element $x \in \mathbb{Z}_{p}\left[\left[\mathcal{G}^{+}\right]\right]$, we write $x^{\chi}=$ $f_{\chi}(x)$.
Theorem 2.3.6. ([4], Theorem 1.15 in [28]) If $\chi$ is a non-trivial character of $G$, we have

$$
\frac{1}{2^{d}} \Phi_{K_{\infty} / k}^{\chi} \in \mathbb{Z}_{p}[\operatorname{Im}(\chi)][[\Gamma]] .
$$

Assume that Leopoldt's conjecture is valid for $k$. Then we have

$$
\frac{1}{2^{d}} \Phi_{K_{\infty} / k}^{1} \notin \mathbb{Z}_{p}[[\Gamma]],
$$

where $\mathbf{1}$ is the trivial character of $G$.

### 2.4 Iwasawa main conjecture

In this section, we state the classical Iwasawa main conjecture (IMC)
Let $p$ be a prime number and $k$ a totally real field. We take a onedimensional Artin character $\chi$ for all $k$ and denote by $k^{\chi}$ the extension of $k$ attached to $\chi$. For any number field $N$, we denote by $N_{\infty} / N$ the cyclotomic $\mathbb{Z}_{p}$-extension. Assume that $k_{\infty} \cap k^{\chi}=k$. Put $\Gamma=\operatorname{Gal}\left(k_{\infty}^{\chi} / k^{\chi}\right)$. We fix a topological generator $\gamma$ of $\Gamma$ and put $T=\gamma-1$. By Theorem 2.1.5, we have $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right] \simeq \mathbb{Z}_{p}\left[\operatorname{Gal}\left(k^{\chi} / k\right)\right][[T]]$. Put $\Lambda_{\chi}=\mathbb{Z}_{p}[\operatorname{Im}(\chi)][[T]]$ on which $\operatorname{Gal}\left(k^{\chi} / k\right)$ acts via $\chi$. For any element $x \in \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]\right)$, we write $x^{\chi} \in \mathcal{Q}\left(\Lambda_{\chi}\right)$ the image of $x$ of the map $\mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]\right) \rightarrow \mathcal{Q}\left(\Lambda_{\chi}\right)$ induced by $\chi$. For any $\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]$-module $M$, we define the $\chi$-quotient by $M^{\chi}=M \otimes_{Z_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]} \Lambda_{\chi}$.

### 2.4.1 (IMC) for unramified Iwasawa module

First of all, we introduce (IMC) for the unramified Iwasawa module. Assume that $\chi$ is an odd character (i.e., $k^{\chi}$ is a CM-field). We consider the unramified Iwasawa module $X_{k_{\infty}^{\chi}}$ which is defined by the Galois group of the maximal unramified abelian pro-p extension of $k_{\infty}^{\chi}$ (see section 2.2). We denote by $\theta_{k_{\infty}^{\chi} / k} \in \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]\right)$ the projective limit of the Stickelberger elements defined in section 2.3. By Theorem 2.3.2, we have

$$
\begin{aligned}
\frac{1}{2^{d}} \theta_{k_{\infty}^{\chi} / k}^{\chi} & \in \Lambda_{\chi}(\chi \neq \omega) \\
(\kappa(\gamma)-\gamma) \frac{1}{2^{d}} \theta_{k_{\infty}^{\chi} / k}^{\chi} & \in \Lambda_{\chi}(\chi=\omega),
\end{aligned}
$$

where $d=[k: \mathbb{Q}], \omega$ is the Teichmüller character and

$$
\kappa: \Gamma \simeq \operatorname{Gal}\left(k^{\chi}\left(\mu_{p^{\infty}}\right) / k^{\chi}\left(\mu_{p}\right)\right) \rightarrow \mathbb{Z}_{p}^{\times}
$$

is the cyclotomic character.
Conjecture 2.4.1. (IMC, first form)

$$
\operatorname{char}_{\Lambda_{\chi}}\left(X_{k_{\infty}^{\chi}}^{\chi}\right)= \begin{cases}\left(\frac{1}{2^{d}} \theta_{k_{x}^{\chi} / k}^{\chi}\right) & (\chi \neq \omega) \\ \left((\kappa(\gamma)-\gamma) \frac{1}{2^{d}} \theta_{k_{\infty}^{\chi} / k}^{\chi}\right) & (\chi=\omega) .\end{cases}
$$

### 2.4.2 (IMC) for $S$-ramified Iwasawa module

Next we introduce (IMC) for $S$-ramified Iwasawa modules. Assume that $\chi$ is an even character (i.e., $k^{\chi}$ is a totally real field). Let $S$ be a finite set
of finite primes of $k$ which contains all ramifying primes in $k_{\infty}^{\chi} / k$. We consider the $S$-ramified Iwasawa module $\mathcal{X}_{k_{\infty}^{\chi}, S}$ which is defined by the Galois group of the maximal abelian pro- $p$ extension of $k_{\infty}^{\chi}$ unramified outside $S$ (see section 2.2). Put $L^{\chi}=k^{\chi}\left(\mu_{p}\right)$. Let $\Phi_{L_{\infty}^{\chi} / k, S} \in \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi} / k\right)\right]\right]\right)$ be the $p$-adic $L$-function of Deligne-Ribet in Theorem 2.3.3. We write $\Phi_{k_{\infty}^{\chi} / k, S} \in \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]\right)$ the image of $\Phi_{L_{\infty}^{\chi} / k, S}$ of the natural restriction map $\mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi} / k\right)\right]\right]\right) \rightarrow \mathcal{Q}\left(\mathbb{Z}_{p}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\infty} / k\right)\right]\right]\right)$. By Corollary 2.3.5 and Theorem 2.3.6, we have

$$
\begin{aligned}
\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k, S}^{\chi} \in \Lambda_{\chi}(\chi \neq \mathbf{1}) \\
T \frac{1}{2^{d}} \Phi_{k_{\infty} / k, S}^{\chi} \in \Lambda_{\chi}(\chi=\mathbf{1}),
\end{aligned}
$$

where $\mathbf{1}$ is the trivial character.
Conjecture 2.4.2. (IMC, second form)

$$
\operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S}^{\chi}\right)= \begin{cases}\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k, S}^{\chi}\right) & (\chi \neq 1) \\ \left(T \frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k, S}^{\chi}\right) & (\chi=1) .\end{cases}
$$

Remark 2.4.3. (1) Greenberg proved that the validity of (IMC, second form) for one $S$ implies the validity for others. Therefore, we can take $S$ minimal when we prove (IMC) for $p=2$ in chapter 4 .
(2) Let $\psi$ be an even character for $k$. Iwasawa proved that Conjecture 2.4.1 for $\psi^{-1} \omega$ is equivalent to Conjecture 2.4.2 for $\psi$.

Theorem 2.4.4. (Mazur-Wiles [23], Wiles [34], Greenberg [12]) If $p$ is an odd prime, (IMC) is valid.

When $p=2$, Wiles proved the partial result as following.
Theorem 2.4.5. (Wiles, Theorem 11.1 in [34]) Suppose that $\chi$ is an odd character for $k$ such that $k^{\chi} \cap k_{\infty}=k$ and $p=2$. If $\chi \psi(\mathfrak{p}) \neq 1$ for all characters $\psi$ of $\operatorname{Gal}\left(k_{\infty} / k\right)$ and all primes $\mathfrak{p}$ above 2 and the $\mu$-invariant of $k_{\infty}^{\chi}$ vanishes, (IMC, first form) is valid.

### 2.5 A refinement of Iwasawa theory

In this section, we discuss a refinement of Iwasawa theory.

### 2.5.1 Fitting ideal

Let $R$ be a commutative ring and $M$ a finitely presented $R$-module $M$ such that $R^{m} \xrightarrow{\phi} R^{n} \rightarrow M \rightarrow 0$ is exact.

Definition 2.5.1. (Fitting ideal) The (initial) Fitting ideal of $M$ is defined to be the ideal of $R$ generated by all $n \times n$ minors of the matrix which corresponds to $\phi$.

This does not depend on the choice of this exact sequence. We denote the Fitting ideal of $M$ over $R$ by $\operatorname{Fitt}_{R}(M)$.

It is well known the following results about the Fitting ideal.
Proposition 2.5.2. (see [25])
(1) $\operatorname{Ann}_{R}(M) \subset \operatorname{Fitt}_{R}(M)$.
(2) Let $M_{1}, M_{2}$ and $M_{3}$ be finitely presented $R$-modules and there is an exact sequence $M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$. Then we have

$$
\operatorname{Fitt}_{R}\left(M_{1}\right) \operatorname{Fitt}_{R}\left(M_{3}\right) \subset \operatorname{Fitt}_{R}\left(M_{2}\right)
$$

(3) Let $M_{1}, M_{2}$ and $M_{3}$ be finitely presented $R$-modules and there is an exact sequence $0 \rightarrow M_{1} \rightarrow M_{2} \rightarrow M_{3} \rightarrow 0$. If there exists an exact sequence $R^{n} \rightarrow R^{n} \rightarrow M \rightarrow 0$ for some integer $n>0$, we have

$$
\operatorname{Fitt}_{R}\left(M_{1}\right) \operatorname{Fitt}_{R}\left(M_{3}\right)=\operatorname{Fitt}_{R}\left(M_{2}\right)
$$

Suppose that $\mathcal{O}$ is the integer ring of a finite extension of $\mathbb{Q}_{p}$ with a prime $p$.

Proposition 2.5.3. (Corollary 2.8 in [26]) Let $R=\mathcal{O}[[T]]$ and $M$ a finitely generated torsion $R$-module. We denote the maximal finite $R$-submodule of $M$ by $M_{\text {fin }}$. Then we have

$$
\operatorname{Fitt}_{R}(M)=\operatorname{char}_{R}(M) \operatorname{Fitt}_{R}\left(M_{\mathrm{fin}}\right)
$$

In particular, if $M$ has no non-trivial finite $R$-submodule, we have

$$
\operatorname{Fitt}_{R}(M)=\operatorname{char}_{R}(M)
$$

### 2.5.2 Stickelberger ideal

Let $K / \mathbb{Q}$ be a finite imaginary abelian extension. For any subfield $F \subset K$, we consider the canonical homomorphism

$$
c_{K / F}: \mathbb{Q}[\operatorname{Gal}(K / \mathbb{Q})] \longrightarrow \mathbb{Q}[\operatorname{Gal}(F / \mathbb{Q})]
$$

induced by the natural restriction map. In this situation,

$$
\nu_{K / F}: \mathbb{Q}[\operatorname{Gal}(F / \mathbb{Q})] \longrightarrow \mathbb{Q}[\operatorname{Gal}(K / \mathbb{Q})]
$$

denotes the homomorphism defined by

$$
\sigma \mapsto \sum_{c_{K / F}(\tau)=\sigma} \tau
$$

for $\sigma \in \operatorname{Gal}(F / \mathbb{Q})$ where $\tau$ ranges over elements of $\operatorname{Gal}(F / \mathbb{Q})$ such that $c_{K / F}(\tau)=\sigma$.

Let $p_{1}, p_{2}, \ldots, p_{r}$ be all primes ramifying in $K / \mathbb{Q}$. We denote by $I_{p_{i}}$ the inertia subgroup of $p_{i}$ in $\operatorname{Gal}(K / \mathbb{Q})$. If $K / \mathbb{Q}$ satisfies the following condition

$$
\text { (A) } \quad \operatorname{Gal}(K / \mathbb{Q})=I_{p_{1}} \times \cdots \times I_{p_{r}},
$$

we say that $K / \mathbb{Q}$ satisfies the condition (A).
Lemma 2.5.4. (Lemma 2.3 in [19]) Let $K / \mathbb{Q}$ be a finite abelian extension. Then, there exists a unique abelian extension $K^{\prime} / \mathbb{Q}$ such that $K \subset K^{\prime}, K^{\prime} / K$ is unramified at all finite primes, and that $K^{\prime} / \mathbb{Q}$ satisfies the condition (A).
Definition 2.5.5. (Stickelberger ideal, Kurihara [19] ) Let $K / \mathbb{Q}$ be a finite imaginary abelian extension which satisfies the condition (A). We define a $\mathbb{Z}[\operatorname{Gal}(K / \mathbb{Q})]$-module $\Theta_{K / \mathbb{Q}}^{\prime}$ by

$$
\Theta_{K / \mathbb{Q}}^{\prime}=\left\langle\nu_{K / F}\left(\theta_{F / \mathbb{Q}}\right) \mid \mathbb{Q} \subset F \subset K\right\rangle_{\mathbb{Z}[\operatorname{Gal}(K / \mathbb{Q})]} \subset \mathbb{Q}[\operatorname{Gal}(K / \mathbb{Q})],
$$

where $\theta_{F / \mathbb{Q}}$ is the Stickelberger element defined in §2.3. In this situation, we define the Stickelberger ideal $\Theta_{K / \mathbb{Q}}$ by

$$
\Theta_{K / \mathbb{Q}}=\Theta_{K / \mathbb{Q}} \cap \mathbb{Z}[\operatorname{Gal}(K / \mathbb{Q})] .
$$

Next, we consider a finite imaginary abelian extension $K / \mathbb{Q}$ which does not satisfy the condition (A). By Lemma 2.5.4, there exists a unique abelian extension $K^{\prime} / \mathbb{Q}$ such that $K \subset K^{\prime}, K^{\prime} / K$ is unramified at all finite primes, and that $K^{\prime} / \mathbb{Q}$ satisfies the condition (A). In this situation, we define the Stickelberger ideal $\Theta_{K / \mathbb{Q}}$ by

$$
\Theta_{K / \mathbb{Q}}=c_{K^{\prime} / K}\left(\Theta_{K^{\prime} / \mathbb{Q}}\right) .
$$

Remark 2.5.6. If $K$ is a cyclotomic field, the Stickelberger ideal defined by Kurihara coincides with the Stickelberger ideal defined by Iwasawa and Sinnott in [29].

### 2.5.3 Fitting ideal of class groups

Let $p$ be a prime number, $K / \mathbb{Q}$ a finite imaginary abelian extension. Let $j$ denote the complex conjugation in $\operatorname{Gal}(K / \mathbb{Q})$. Put $\mathbb{Z}_{p}[\operatorname{Gal}(K / \mathbb{Q})]^{-}=$ $\mathbb{Z}_{p}[\operatorname{Gal}(K / \mathbb{Q})] /(1+j)$. For any $\mathbb{Z}_{p}[\operatorname{Gal}(K / \mathbb{Q})]$-module $M$, we define the $\pm$-component $M^{ \pm}$by

$$
M^{ \pm}=M /(1 \pm j) M
$$

If $p$ is an odd prime, we note that $M=M^{+} \oplus M^{-}$. Let $\mathrm{Cl}(K)$ be the ideal class group of $K$. Let

$$
f: \mathbb{Z}_{p}[\operatorname{Gal}(K / \mathbb{Q})] \longrightarrow \mathbb{Z}_{p}[\operatorname{Gal}(K / \mathbb{Q})]^{-}
$$

denote the natural homomorphism.
Theorem 2.5.7. (Kurihara-Miura, Theorem 0.1 in [21]) For any odd prime $p$, we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{p}[\operatorname{Gal}(K / \mathbb{Q})]^{-}}\left(\left(\operatorname{Cl}(K) \otimes \mathbb{Z}_{p}\right)^{-}\right)=f\left(\Theta_{K / \mathbb{Q}} \otimes \mathbb{Z}_{p}\right) .
$$

Theorem 2.5.8. (Greither, Theorem A in [13]) Assume that 2 is unramified in $K / \mathbb{Q}$ and the 2 -Sylow subgroup of $\operatorname{Gal}(K / \mathbb{Q})$ is cyclic. Then, we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}\left[\operatorname{Gal}(K / \mathbb{Q})^{-}\right.}\left(\left(\mathrm{Cl}(K) \otimes \mathbb{Z}_{2}\right)^{-}\right)=\frac{1}{2} f\left(\Theta_{K / \mathbb{Q}} \otimes \mathbb{Z}_{2}\right)
$$

## Chapter 3

## Finite submodules of Iwasawa modules for $p=2$

### 3.1 A result of Greenberg

In this section, we introduce the result of Greenberg. Greenberg describes his theorems in a much more general setting in [11]. However, we describe it in a restricted setting here.

Let $p$ be a prime. Suppose that $K$ is a finite extension of $\mathbb{Q}$ and that $\Sigma$ is a finite set of primes of $K$. Let $K_{\Sigma}$ be the maximal extension of $K$ unramified outside $\Sigma$. We assume that $\Sigma$ contains all archimedean primes and all primes lying above $p$. Put $\Lambda:=\mathbb{Z}_{p}[[T]]$ and let $\mathcal{T}$ be a $\operatorname{Gal}\left(K_{\Sigma} / K\right)$ module such that $\mathcal{T} \cong \Lambda$ as a group and $\operatorname{Gal}\left(K_{\Sigma} / K\right)$ acts on $\mathcal{T}$ continuously. We define $\mathcal{D}=\mathcal{T} \otimes_{\Lambda} \hat{\Lambda}$, where $\hat{\Lambda}=\operatorname{Hom}\left(\Lambda, \mathbb{Q}_{p} / \mathbb{Z}_{p}\right)$ is the Pontryagin dual of $\Lambda$. The Galois group $\operatorname{Gal}\left(K_{\Sigma} / K\right)$ acts on $\mathcal{D}$ through its action on the first factor $\mathcal{T}$.

We note that $\mathcal{D}$ is a discrete abelian group and the Galois cohomology group $H^{1}\left(K_{\Sigma} / K, \mathcal{D}\right)$ is a discrete $\Lambda$-module. Let $L\left(K_{v}, \mathcal{D}\right)$ be a $\Lambda$-submodule of $H^{1}\left(K_{v}, \mathcal{D}\right)$ for each $v \in \Sigma$, where $K_{v}$ is the completion of $K$ at $v$. Put $Q(K, \mathcal{D}):=\prod_{v \in \Sigma} H^{1}\left(K_{v}, \mathcal{D}\right) / L\left(K_{v}, \mathcal{D}\right)$. The natural global-to-local maps induce a map

$$
\phi: H^{1}\left(K_{\Sigma} / K, \mathcal{D}\right) \longrightarrow Q(K, \mathcal{D}) .
$$

The kernel of $\phi$ is denoted by $S(K, \mathcal{D})$. We define $\mathcal{T}^{*}=\operatorname{Hom}\left(\mathcal{D}, \mu_{p^{\infty}}\right)$, and

$$
\amalg^{2}(K, \Sigma, \mathcal{D})=\operatorname{ker}\left(H^{2}\left(K_{\Sigma} / K, \mathcal{D}\right) \longrightarrow \prod_{v \in \Sigma} H^{2}\left(K_{v}, \mathcal{D}\right)\right) .
$$

We say that a finitely generated $\Lambda$-module $M$ is reflexive if the map

$$
\begin{aligned}
M & \longrightarrow \operatorname{Hom}_{\Lambda}\left(\operatorname{Hom}_{\Lambda}(M, \Lambda), \Lambda\right) \\
m & \longmapsto[\alpha \mapsto \alpha(m)] .
\end{aligned}
$$

is an isomorphism. Suppose that $N$ is a discrete $\Lambda$-module and that its Pontryagin dual is finitely generated. We say that $N$ is almost $\Lambda$-divisible if there is a nonzero element $f(T) \in \Lambda$ such that $g(T) N=N$ for all irreducible elements $g(T) \in \Lambda$ not dividing $f(T)$.

Theorem 3.1.1. (Greenberg [11] Proposition 4.1.1) Suppose that the following assumptions are satisfied.
(a) The $\Lambda$-module $Ш^{2}(K, \Sigma, \mathcal{D})$ is $\Lambda$-cotorsion,
(b) The $\Lambda$-module $\mathcal{T}^{*} /\left(\mathcal{T}^{*}\right)^{G_{K v}}$ is reflexive for all $v \in \Sigma$,
(c) There exists a non-archimedean prime $v \in \Sigma$ such that $\left(\mathcal{T}^{*}\right)^{G_{K_{v}}}=0$,
(d) $\prod_{v \in \Sigma} L\left(K_{v}, \mathcal{D}\right)$ is almost $\Lambda$-divisible,
(e) $\operatorname{corank}_{\Lambda}\left(H^{1}\left(K_{\Sigma} / K, \mathcal{D}\right)\right)=\operatorname{corank}_{\Lambda}(S(K, \mathcal{D}))+\operatorname{corank}_{\Lambda}(Q(K, \mathcal{D}))$,
(f) At least one of the following additional assumptions is satisfied.

- $\mathcal{D}[\mathfrak{m}]$ has no subquotient isomorphic to $\mu_{p}$ for the action of $G_{K}=$ $\operatorname{Gal}(\bar{K} / K)$.
- $\mathcal{D}$ is a cofree $\Lambda$-module and $\mathcal{D}[\mathfrak{m}]$ has no quotient isomorphic to $\mu_{p}$ for the action of $G_{K}$.
- There is a prime $v \in \Sigma$ which satisfies (c) and such that $H^{1}\left(K_{v}, \mathcal{D}\right) / L\left(K_{v}, \mathcal{D}\right)$ is coreflexive as a $\Lambda$-module.

Then $S(K, \mathcal{D})$ is almost $\Lambda$-divisible.
In section 3.4, 3.5 in [11], Greenberg discussed the case that the assumption (f) is not satisfied. We can replace the assumption (f) to ( $\mathrm{f}^{*}$ ) as following.

Theorem 3.1.2. (Greenberg [11] Proposition 4.1.1 and section 3.4, 3.5) Suppose that the following assumptions are satisfied.
(a) The $\Lambda$-module $Ш^{2}(K, \Sigma, \mathcal{D})$ is $\Lambda$-cotorsion,
(b) The $\Lambda$-module $\mathcal{T}^{*} /\left(\mathcal{T}^{*}\right)^{G_{K v}}$ is reflexive for all $v \in \Sigma$,
(c) There exists a non-archimedean prime $v \in \Sigma$ such that $\left(\mathcal{T}^{*}\right)^{G_{K v}}=0$,
(d) $\prod_{v \in \Sigma} L\left(K_{v}, \mathcal{D}\right)$ is almost $\Lambda$-divisible,
(e) $\operatorname{corank}_{\Lambda}\left(H^{1}\left(K_{\Sigma} / K, \mathcal{D}\right)\right)=\operatorname{corank}_{\Lambda}(S(K, \mathcal{D}))+\operatorname{corank}_{\Lambda}(Q(K, \mathcal{D}))$,
(f*) $L\left(K_{v}, \mathcal{D}\right) \subset H^{1}\left(K_{v}, \mathcal{D}\right)_{\Lambda-\text { div }}$ for all $v \in \Sigma$.
Then $S(K, \mathcal{D})$ is almost $\Lambda$-divisible.
Remark 3.1.3. Let $M$ be a finitely generated $\Lambda$-module, and $N$ the Pontryagin dual of $M$ (i.e., $N=\operatorname{Hom}\left(M, \mathbb{Q}_{p} / \mathbb{Z}_{p}\right)$ ). Then, the following two statements are equivalent:

- $M$ has no non-trivial finite $\Lambda$-submodule.
- $N$ is almost $\Lambda$-divisible.

The proof of this fact can be found in Proposition 2.4 in Greenberg [10]

## $3.2 \quad \mathscr{S}_{2}$-modified Iwasawa module $X_{F_{\infty}, \mathscr{S}_{2}}$

For any number field $N$, we denote by $N_{\infty} / N$ the cyclotomic $\mathbb{Z}_{2}$-extension of $N$ and $N_{n}$ the $n$-th layer of $N_{\infty} / N$. We denote by $S_{2}(N), S_{\infty}(N)$ the set of primes of $N$ lying above $2, \infty$, respectively. Let $F$ be a CM-field and $F^{+}$ the maximal real subfield of $F$. Put $\Lambda:=\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(F_{\infty} / F\right)\right]\right]$. We define the subset $\mathscr{S}_{2}\left(F^{+}\right)$of $S_{2}\left(F^{+}\right) \cup S_{\infty}\left(F^{+}\right)$by

$$
\mathscr{S}_{2}\left(F^{+}\right)=\left\{v \in S_{2}\left(F^{+}\right) \mid v \text { ramifies in } F_{\infty} / F_{\infty}^{+}\right\} \cup S_{\infty}\left(F^{+}\right) .
$$

For any extension $K / F^{+}$, we denote by $\mathscr{S}_{2}(K)$ the set of primes of $K$ lying above $\mathscr{S}_{2}\left(F^{+}\right)$. Using this particular $\mathscr{S}_{2}(K)$, we define $\mathrm{Cl}_{\mathscr{S}_{2}}(K)$ by the $\mathscr{S}_{2}(K)$-ideal class group of $K$, i.e

$$
\mathrm{Cl}_{\mathscr{\mathscr { L }}_{2}}(K)=\operatorname{coker}\left(K^{\times} \xrightarrow{\oplus \operatorname{ord}_{v}} \bigoplus_{v \notin \mathscr{S}_{2}(K)} \mathbb{Z}\right) .
$$

We denote by $A_{K, \mathscr{S}_{2}}$ the 2-Sylow subgroup of the $\mathscr{S}_{2}(K)$-ideal class group $\mathrm{Cl}_{\mathscr{S}_{2}}(K)$. We define $X_{F_{\infty}, \mathscr{S}_{2}}, X_{F_{\infty}, \mathscr{S}_{2}}^{-}$by

$$
X_{F_{\infty}, \mathscr{S}_{2}}=\lim _{\leftrightarrows} A_{F_{n}, \mathscr{S}_{2}}, \quad X_{F_{\infty}, \mathscr{S}_{2}}^{-}=X_{F_{\infty}, \mathscr{S}_{2}} /(1+j) X_{F_{\infty}, \mathscr{S}_{2}},
$$

where the projective limit is taken with respect to the norm maps. In this section, we prove the following result.

Theorem 3.2.1. $X_{F_{\infty}, \mathscr{S}_{2}}^{-}$has no non-trivial finite $\Lambda$-submodule.

We prove Theorem 3.2.1, using Theorem 3.1.2 and taking $K=F^{+}, p=$ 2. We may assume that all primes above 2 are totally ramified in $F_{\infty} / F$ and $F_{\infty}^{+} / F^{+}$. We define

$$
\Sigma=S_{\mathrm{ram}}\left(F / F^{+}\right) \cup S_{\infty}\left(F^{+}\right) \cup S_{2}\left(F^{+}\right)
$$

where $S_{\mathrm{ram}}\left(F / F^{+}\right)$is the set of primes of $F^{+}$which ramify in $F / F^{+}$. Let $F_{\Sigma}^{+}$be the maximal extension of $F^{+}$unramified outside $\Sigma$. By definition, $F_{\infty} \subset F_{\Sigma}^{+}$. Put $\Gamma:=\operatorname{Gal}\left(F_{\infty}^{+} / F^{+}\right)$, and $\Lambda:=\mathbb{Z}_{2}[[\Gamma]] \cong \mathbb{Z}_{2}[[T]]$. Let $j$ be the complex conjugation. By definition, $\operatorname{Gal}\left(F / F^{+}\right)=\{1, j\}$. We take $\mathcal{T}$ to be a $\operatorname{Gal}\left(F_{\Sigma}^{+} / F^{+}\right)$-module such that $\mathcal{T} \cong \Lambda$ as a $\Lambda$-module, for which $j$ acts as -1 , and the group $\operatorname{Gal}\left(F_{\Sigma}^{+} / F^{+}\right)$acts on $\mathcal{T}$ through the natural map $\operatorname{Gal}\left(F_{\Sigma}^{+} / F^{+}\right) \rightarrow \operatorname{Gal}\left(F_{\infty} / F^{+}\right) \cong \operatorname{Gal}\left(F / F^{+}\right) \times \operatorname{Gal}\left(F_{\infty}^{+} / F^{+}\right)$. We define

$$
\mathcal{D}=\mathcal{T} \otimes_{\Lambda} \hat{\Lambda}, \quad \mathcal{T}^{*}=\operatorname{Hom}\left(\mathcal{D}, \mu_{2^{\infty}}\right)
$$

where $\hat{\Lambda}=\operatorname{Hom}\left(\Lambda, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right)$ is the Pontryagin dual of $\Lambda$. We define the $\Lambda$-submodule $L\left(F_{v}^{+}, \mathcal{D}\right)$ of $H^{1}\left(F_{v}^{+}, \mathcal{D}\right)$ for each $v \in \Sigma$

$$
\begin{aligned}
& L\left(F_{v}^{+}, \mathcal{D}\right) \\
= & \begin{cases}\operatorname{ker}\left(H^{1}\left(F_{v}^{+}, \mathcal{D}\right) \longrightarrow H^{1}\left(F_{v}^{+\mathrm{unr}}, \mathcal{D}\right)\right) & \left(\text { if } v \notin S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)\right) \\
0 & \left(\text { if } v \in S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)\right)\end{cases}
\end{aligned}
$$

where $F_{v}^{+ \text {unr }}$ is the maximal unramified extension of $F_{v}^{+}$and $S_{\text {ram }}\left(F_{\infty} / F_{\infty}^{+}\right)$ is the set of primes of $F^{+}$which ramify in $F_{\infty} / F_{\infty}^{+}$. Put $Q\left(F^{+}, \mathcal{D}\right):=$ $\prod_{v \in \Sigma} H^{1}\left(F_{v}^{+}, \mathcal{D}\right) / L\left(F_{v}^{+}, \mathcal{D}\right)$. The natural global-to-local maps induce a map

$$
\phi: H^{1}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right) \longrightarrow Q\left(F^{+}, \mathcal{D}\right)
$$

The kernel of $\phi$ is denoted by $S\left(F^{+}, \mathcal{D}\right)$. In this situation, we check the assumptions (a), (b), (c), (d), (e), (f*) in Theorem 3.1.2.

Proof of Theorem 3.2.1 (c) $\mathcal{T}^{*}=\operatorname{Hom}\left(\mathcal{D}, \mu_{2 \infty}\right)=\operatorname{Hom}\left(\mathcal{D}, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right) \otimes$ $\mathbb{Z}_{2}(1)=\Lambda(1)$ as $\Lambda$-modules. Since no prime splits completely in the cyclotomic $\mathbb{Z}_{2}$-extension in $F_{\infty}^{+} / F^{+}, G_{K_{v}}$ acts on $\Lambda(1)$ nontrivially for each $v$. Therefore, $\left(\mathcal{T}^{*}\right)^{G_{F_{v}^{+}}}=0$ for any non-archimedean prime $v \in \Sigma$.
(b) If $v$ is non-archimedean, $\mathcal{T}^{*} /\left(\mathcal{T}^{*}\right)^{G_{F_{v}^{+}}} \cong \Lambda(1)$. This $\Lambda$-module $\Lambda(1)$ is reflexive. If $v$ is archimedean, $G_{F_{v}^{+}}=\{1, j\}$. Since

$$
(j f)(x)=j\left(f\left(j^{-1} x\right)\right)=j(f(-x))=j\left(f(x)^{-1}\right)=f(x)
$$

for any $f \in \mathcal{T}^{*}$ and $x \in \mathcal{D}, j$ acts trivially on $\mathcal{T}^{*}$. Thus, $\mathcal{T}^{*} /\left(\mathcal{T}^{*}\right)^{G_{F_{v}^{+}}}=0$.
(d) We claim that

$$
L\left(F_{v}^{+}, \mathcal{D}\right) \cong \begin{cases}\mathbb{Q}_{2} / \mathbb{Z}_{2} & \left(\text { if } v \in S_{2}\left(F^{+}\right) \text {and } v \text { splits in } F_{\infty} / F_{\infty}^{+}\right) \\ 0 & \text { (otherwise) }\end{cases}
$$

for each $v \in \Sigma$. This fact implies that $\prod_{v \in \Sigma} L\left(K_{v}, \mathcal{D}\right)$ is almost $\Lambda$-divisible.
If $v \in S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)$, this is trivial by definition. Thus, we consider the case $v \notin S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)$. The inflation-restriction sequence shows that

$$
L\left(F_{v}^{+}, \mathcal{D}\right) \cong H^{1}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}, \mathcal{D}^{G_{F_{v}^{+}}}{ }^{\mathrm{unr}}\right)
$$

If $v$ is archimedean, $\operatorname{Gal}\left(F_{v}^{+ \text {unr }} / F_{v}^{+}\right)=1$ implies $L\left(F_{v}^{+}, \mathcal{D}\right)=0$.
If $v$ is non-archimedean and $v \notin S_{2}\left(F^{+}\right)$, then $v$ is unramified in $F_{\infty}^{+} / F^{+}$ and hence $F_{v, \infty}^{+} \subset F_{v}^{+ \text {unr }}$, where $F_{v, \infty}^{+}$is the cyclotomic $\mathbb{Z}_{2}$-extension of $F_{v}^{+}$. Thus, $\operatorname{Gal}\left(F_{v}^{+ \text {unr }} / F_{v}^{+}\right)$contains the unique subgroup $P_{v}$ which is isomorphic to $\mathbb{Z}_{2}$ and the restriction map $P_{v} \rightarrow \Gamma_{v}=\operatorname{Gal}\left(F_{v, \infty}^{+} / F_{v}^{+}\right)$is an isomorphism. The inflation-restriction sequence shows that the restriction map

$$
H^{1}\left(F_{v}^{+ \text {unr }} / F_{v}^{+}, \mathcal{D}^{G_{F_{v}^{+ \text {unr }}}}\right) \longrightarrow H^{1}\left(P_{v}, \mathcal{D}^{G_{F_{v}^{+ \text {unr }}}}\right)
$$

is injective. Hence, it suffices to show that $H^{1}\left(P_{v}, \mathcal{D}^{G_{F_{v}^{+}}}\right.$unr $)=0$. The action of $G_{F_{v}^{+}}$unr on $\mathcal{D}$ factors through $G_{F_{v}^{+}} \rightarrow \operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)=\{1, j\}$, where $w$ is a prime of $F$ lying above $v$. Since $j$ acts on $\mathcal{D}$ as -1 and $\mathcal{D}$ is a divisible group, we get an exact sequence

$$
0 \longrightarrow \mathcal{D}^{G_{F_{v}^{+}} \text {unr }} \longrightarrow \mathcal{D} \xrightarrow{1-j} \mathcal{D} \longrightarrow 0
$$

Taking Galois cohomology, we get an exact sequence

$$
\mathcal{D}^{P_{v}} \xrightarrow{\times 2} \mathcal{D}^{P_{v}} \longrightarrow H^{1}\left(P_{v}, \mathcal{D}^{G_{F_{v}^{+}} \mathrm{unr}}\right) \longrightarrow H^{1}\left(P_{v}, \mathcal{D}\right) .
$$

Let $\gamma_{v}$ be a topological generator of $\Gamma_{v}$. Then,

$$
\mathcal{D}^{P_{v}} \cong \operatorname{Hom}_{\Gamma_{v}}\left(\Lambda, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right) \cong \operatorname{Hom}\left(\Lambda /\left(1-\gamma_{v}\right), \mathbb{Q}_{2} / \mathbb{Z}_{2}\right) \cong\left(\mathbb{Q}_{2} / \mathbb{Z}_{2}\right)^{\oplus n}
$$

where, $n=\left[\Gamma: \Gamma_{v}\right]$. Thus, $\mathcal{D}^{P_{v}}$ is a divisible group and the map $\mathcal{D}^{P_{v}} \xrightarrow{\times 2}$ $\mathcal{D}^{P_{v}}$ is surjective. Therefore, the map $H^{1}\left(P_{v}, \mathcal{D}^{G_{F_{v}^{+}}{ }^{\text {unr }}}\right) \longrightarrow H^{1}\left(P_{v}, \mathcal{D}\right)$ is injective. Here, $H^{1}\left(P_{v}, \mathcal{D}\right) \cong \mathcal{D} /\left(1-\gamma_{v}\right) \mathcal{D}=0$ because $1-\gamma_{v}$ acts on $\mathcal{D}$ as
the multiplication by a nonzero element of $\Lambda$ and $\mathcal{D}$ is $\Lambda$-divisible. Thus, $H^{1}\left(P_{v}, \mathcal{D}^{G_{F_{v}^{+}}{ }^{\text {unr }}}\right)=0$ for each non-archimedean prime $v \notin S_{2}\left(F^{+}\right)$.

We consider the case that $v \in S_{2}\left(F^{+}\right)$and $v$ is inert in $F_{\infty} / F_{\infty}^{+}$. Let $P_{v}$ be the maximal subgroup of $\operatorname{Gal}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}\right)$which is isomorphic to $\mathbb{Z}_{2}$, and $\gamma_{v}$ a topological generator of $P_{v}$. The action of $\operatorname{Gal}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}\right)$on $\mathcal{D}$ factors through $\operatorname{Gal}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}\right) \rightarrow \operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)=\{1, j\}$, where $w$ is the prime of $F$ lying above $v$. Therefore, $\gamma_{v}$ acts on $\mathcal{D}$ as -1 . Thus,

$$
\begin{aligned}
H^{1}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}, \mathcal{D}^{G_{F_{v}^{+\mathrm{unr}}}}\right) & =H^{1}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}, A\right) \\
& =H^{1}\left(P_{v}, A\right) \\
& =A /\left(1-\gamma_{v}\right) A=0
\end{aligned}
$$

where $A$ is a $\operatorname{Gal}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}\right)$-module such that $A$ is isomorphic to $\mathbb{Q}_{2} / \mathbb{Z}_{2}$ as a group, for which $j$ acts as -1 , and $\operatorname{Gal}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}\right)$acts via $\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)=$ $\{1, j\}$.

If $v \in S_{2}\left(F^{+}\right)$and $v$ splits in $F_{\infty} / F_{\infty}^{+}$, the action of $\operatorname{Gal}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}\right)$on $\mathcal{D}$ is trivial since we assumed that all primes above 2 are totally ramified in $F_{\infty}^{+} / F^{+}$. Therefore,

$$
\begin{aligned}
H^{1}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}, \mathcal{D}^{G_{v}^{+\mathrm{unr}}}\right) & =H^{1}\left(F_{v}^{+\mathrm{unr}} / F_{v}^{+}, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right) \\
& =\operatorname{Hom}\left(\mathbb{Z}_{2}, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right) \\
& \cong \mathbb{Q}_{2} / \mathbb{Z}_{2}
\end{aligned}
$$

$\left(\mathrm{f}^{*}\right)$ Let $\gamma$ be a topological generator of $\Gamma$. We know that $L\left(F_{v}^{+}, \mathcal{D}\right)$ is a divisible group and annihilated by the ideal $(1-\gamma)$ for each $v \in \Sigma$. By Remark 3.5.2 in [11], we have the inclusion $L\left(F_{v}^{+}, \mathcal{D}\right) \subset H^{1}\left(F_{v}^{+}, \mathcal{D}\right)_{\Lambda-\text { div }}$.

Before we check the assumptions (a) and (e), we prove the following lemma.

Lemma 3.2.2. We have

$$
S\left(F^{+}, \mathcal{D}\right)=\operatorname{Hom}\left(X_{F_{\infty}, \mathscr{S}_{2}}^{-}, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right)
$$

where $X_{F_{\infty}, \mathscr{I}_{2}}^{-}$is the $\Lambda$-module defined on page $\mathcal{L}$.
Proof. For each $w$ lying above $v \in \Sigma$, we define $L\left(F_{w}, \mathcal{D}\right)$ by

$$
L\left(F_{w}, \mathcal{D}\right)=\operatorname{ker}\left(H^{1}\left(F_{w}, \mathcal{D}\right)^{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)} \rightarrow H^{1}\left(F_{w}^{\mathrm{unr}}, \mathcal{D}\right)\right)
$$

if $v \notin S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)$, and

$$
L\left(F_{w}, \mathcal{D}\right)=0
$$

if $v \in S_{2}\left(F^{+}\right) \cap S_{\text {ram }}\left(F_{\infty} / F_{\infty}^{+}\right)$.
At first, we claim that the map

$$
\begin{equation*}
\frac{H^{1}\left(F_{v}^{+}, \mathcal{D}\right)}{L\left(F_{v}^{+}, \mathcal{D}\right)} \longrightarrow \frac{H^{1}\left(F_{w}, \mathcal{D}\right)^{\mathrm{Gal}\left(F_{w} / F_{v}^{+}\right)}}{L\left(F_{w}, \mathcal{D}\right)} \tag{3.1}
\end{equation*}
$$

induced by the restriction map is injective for each $w$ lying above $v \in \Sigma$.
If $v \notin S_{2}\left(F^{+}\right)$or $v \in S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)$, we have $L\left(F_{v}^{+}, \mathcal{D}\right)=0$. Similarly, we have $L\left(F_{w}, \mathcal{D}\right)=0$. Since $\mathcal{D}^{G_{F_{w}}}$ is a divisible group and $j$ acts on $\mathcal{D}^{G_{F_{w}}}$ as -1 , we have

$$
H^{1}\left(F_{w} / F_{v}^{+}, \mathcal{D}^{G_{F w}}\right)=\frac{\operatorname{ker}\left(\mathcal{D}^{G_{F_{w}}} \xrightarrow{1+j} \mathcal{D}^{G_{F w}}\right)}{\left(\mathcal{D}^{G_{F_{w}}}\right)^{1-j}}=0 .
$$

Therefore the inflation-restriction sequence

$$
0 \longrightarrow H^{1}\left(F_{w} / F_{v}^{+}, \mathcal{D}^{G_{F_{w}}}\right) \longrightarrow H^{1}\left(F_{v}^{+}, \mathcal{D}\right) \longrightarrow H^{1}\left(F_{w}, \mathcal{D}\right)^{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}
$$

implies the above map (2) is injective.
If $v \in S_{2}\left(F^{+}\right)$and $v$ is unramified in $F_{\infty} / F_{\infty}^{+}$, then $F_{w}^{\mathrm{unr}}=F_{v}^{+\mathrm{unr}}$. We consider the commutative diagram


By the snake lemma, the map $\frac{H^{1}\left(F_{v}^{+}, \mathcal{D}\right)}{L\left(F_{v}^{+}, \mathcal{D}\right)} \longrightarrow \frac{H^{1}\left(F_{w}, \mathcal{D}\right)^{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}}{L\left(F_{w}, \mathcal{D}\right)}$ is injective. This completes the proof of the injectivity of (2).

Put

$$
Q(F, \mathcal{D})=\prod_{w \in \Sigma_{F}} \frac{H^{1}\left(F_{w}, \mathcal{D}\right)^{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}}{L\left(F_{w}, \mathcal{D}\right)}
$$

where $\Sigma_{F}$ is the set of primes of $F$ lying above $\Sigma$. We consider the commutative diagram


Here, we defined $g_{1}$ to be the restriction map $H^{1}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right) \longrightarrow$ $H^{1}\left(F_{\Sigma}^{+} / F, \mathcal{D}\right)^{\operatorname{Gal}\left(F / F^{+}\right)}$and $g_{2}$ the map $Q\left(F^{+}, \mathcal{D}\right) \rightarrow Q(F, \mathcal{D})$ induced by the restriction map. Next, we show that the map $\operatorname{coker} g_{1} \rightarrow \operatorname{coker} g_{2}$ is injective.

By definition, we have

$$
\operatorname{coker} g_{2}=\prod_{v \in \Sigma} \operatorname{coker}\left(\frac{H^{1}\left(F_{v}^{+}, \mathcal{D}\right)}{L\left(F_{v}^{+}, \mathcal{D}\right)} \rightarrow \bigoplus_{w \mid v} \frac{H^{1}\left(F_{w}, \mathcal{D}\right)^{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}}{L\left(F_{w}, \mathcal{D}\right)}\right)
$$

For any prime $v \in \Sigma$, put

$$
\left(\operatorname{coker}_{2}\right)_{v}=\operatorname{coker}\left(\frac{H^{1}\left(F_{v}^{+}, \mathcal{D}\right)}{L\left(F_{v}^{+}, \mathcal{D}\right)} \rightarrow \bigoplus_{w \mid v} \frac{H^{1}\left(F_{w}, \mathcal{D}\right)^{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}}{L\left(F_{w}, \mathcal{D}\right)}\right)
$$

It is sufficiently to show that the map $\operatorname{coker} g_{1} \rightarrow\left(\operatorname{coker} g_{2}\right)_{v}$ is injective for any $v \in S_{\infty}\left(F^{+}\right)$. Since $L\left(F_{v}^{+}, \mathcal{D}\right)=0, L\left(F_{w}, \mathcal{D}\right)=0$ for any $v \in S_{\infty}\left(F^{+}\right)$, the inflation-restriction sequence implies the commutative diagram


We show that the map $g_{3}$ is injective. We know that $H^{2}\left(F_{w} / F_{v}^{+}, \mathcal{D}^{G_{F_{w}}}\right)=$ $H^{2}\left(F / F^{+}, \mathcal{D}\right)$. Put $\mathcal{D}^{\prime}=\operatorname{coker}\left(\mathcal{D}^{\operatorname{Gal}\left(F_{\Sigma}^{+} / F\right)} \rightarrow \mathcal{D}\right)$. Since both $\mathcal{D}^{\operatorname{Gal}\left(F_{\Sigma}^{+} / F\right)}$ and $\mathcal{D}$ are divisible groups, $\mathcal{D}^{\prime}$ is also divisible. We consider the exact sequence

$$
H^{1}\left(F / F^{+}, \mathcal{D}^{\prime}\right) \longrightarrow H^{2}\left(F / F^{+}, \mathcal{D}^{\operatorname{Gal}\left(F_{\Sigma}^{+} / F\right)}\right) \xrightarrow{g_{3}} H^{2}\left(F / F^{+}, \mathcal{D}\right) .
$$

Since $\mathcal{D}^{\prime}$ is divisible, we have

$$
H^{1}\left(F / F^{+}, \mathcal{D}^{\prime}\right)=\frac{\operatorname{ker}\left(\mathcal{D}^{\prime} \stackrel{1+j}{\rightarrow} \mathcal{D}^{\prime}\right)}{\mathcal{D}^{\prime 1-j}}=\frac{\mathcal{D}^{\prime}}{2 \mathcal{D}^{\prime}}=0 .
$$

Therefore the map $g_{3}$ is injective. This implies that the map $\operatorname{coker} g_{1} \rightarrow$ coker $g_{2}$ is injective.

The map $g_{2}$ is injective by the injectivity of (2). And we have $H^{1}\left(F / F^{+}, \mathcal{D}^{\operatorname{Gal}\left(F_{\Sigma}^{+} / F\right)}\right)=0$. Thus, $S\left(F^{+}, \mathcal{D}\right)$ is isomorphic to $\operatorname{ker} f$ by the snake lemma. We also have $\mathcal{D}=\operatorname{Ind}_{\operatorname{Gal}\left(F_{\Sigma}^{+} / F\right)}^{\operatorname{Gal}\left(F_{\infty}^{+} / F_{\infty}\right)}(A)$, where $A$ is a $\operatorname{Gal}\left(F_{\Sigma}^{+} / F^{+}\right)$-module such that $A$ is isomorphic to $\mathbb{Q}_{2} / \mathbb{Z}_{2}$ as a group, for which $j$ acts as -1 , and $\operatorname{Gal}\left(F_{\Sigma}^{+} / F^{+}\right)$acts on $A$ via $\operatorname{Gal}\left(F / F^{+}\right)=\{1, j\}$. Thus, we have

$$
\begin{aligned}
H^{1}\left(F_{\Sigma}^{+} / F, \mathcal{D}\right)^{\operatorname{Gal}\left(F / F^{+}\right)} & \cong H^{1}\left(F_{\Sigma}^{+} / F_{\infty}, A\right)^{\operatorname{Gal}\left(F / F^{+}\right)} \\
& =\operatorname{Hom}_{\operatorname{Gal}\left(F / F^{+}\right)}\left(\operatorname{Gal}\left(F_{\Sigma}^{\mathrm{ab}} / F_{\infty}\right), A\right)
\end{aligned}
$$

by Shapiro's lemma, where $F_{\Sigma}^{\mathrm{ab}}$ is the maximal abelian pro-2-extension of $F$ unramified outside $\Sigma_{F}$. We may assume that all primes in $\Sigma_{F}$ does not split in $F_{\infty} / F$. We denote by $F_{w, \infty}$ the cyclotomic $\mathbb{Z}_{2}$-extension of $F_{w}$. Similarly, we have

$$
\begin{aligned}
& \frac{H^{1}\left(F_{w}, \mathcal{D}\right) \operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}{L\left(F_{w}, \mathcal{D}\right)} \\
& = \begin{cases}\operatorname{Hom}_{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}\left(I_{F_{w, \infty}}, A\right) & \left(v \notin S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)\right) \\
\operatorname{Hom}_{\operatorname{Gal}\left(F_{w} / F_{v}^{+}\right)}\left(G_{F_{w, \infty}, \infty}, A\right) & \left(v \in S_{2}\left(F^{+}\right) \cap S_{\mathrm{ram}}\left(F_{\infty} / F_{\infty}^{+}\right)\right)\end{cases}
\end{aligned}
$$

for each $w \mid v$, where $I_{F_{w, \infty}}$ is the inertia group in $G_{F_{w}, \infty}$. Therefore we have

$$
\operatorname{ker} f \cong \operatorname{Hom}_{\operatorname{Gal}\left(F / F^{+}\right)}\left(\operatorname{Gal}\left(L_{\infty}^{\prime} / F_{\infty}\right), A\right),
$$

where $L_{\infty}^{\prime}$ is the maximal unramified abelian pro-2-extension of $F_{\infty}$ in which the primes of $F_{\infty}$ lying above $S_{2}(F) \cap S_{\text {ram }}\left(F_{\infty} / F_{\infty}^{+}\right)$split completely. By class field theory, $\operatorname{Gal}\left(L_{\infty}^{\prime} / F_{\infty}\right)$ is isomorphic to $X_{F_{\infty}, \mathscr{S}_{2}}$. Thus, we have

$$
S\left(F^{+}, \mathcal{D}\right) \cong \operatorname{Hom}_{\operatorname{Gal}\left(F / F^{+}\right)}\left(\operatorname{Gal}\left(L_{\infty}^{\prime} / F_{\infty}\right), A\right)=\operatorname{Hom}\left(X_{F_{\infty}, \mathscr{S}_{2}}^{-}, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right) .
$$

This completes the proof of Lemma 3.2.2.

Finally, we check the assumptions (a) and (e) to complete the proof of Theorem 3.2.1.
(e) One has the following obvious inequality:

$$
\operatorname{corank}_{\Lambda}\left(S\left(F^{+}, \mathcal{D}\right)\right) \geq \operatorname{corank}_{\Lambda}\left(H^{1}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right)\right)-\operatorname{corank}_{\Lambda}\left(Q\left(F^{+}, \mathcal{D}\right)\right)
$$

The formulae in section 2.3 in [11] show that the $\Lambda$-corank of $H^{1}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right)$ is at least $\left[F^{+}: \mathbb{Q}\right]$ and the $\Lambda$-corank of $Q\left(F^{+}, \mathcal{D}\right)$ is equal to $\left[F^{+}: \mathbb{Q}\right]$. Iwasawa proved that $S\left(F^{+}, \mathcal{D}\right)$ is cotorsion $\Lambda$-module(Theorem 5 in [18]). This implies that the $\Lambda$-corank of $H^{1}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right)$ is equal to $\left[F^{+}: \mathbb{Q}\right]$ and (e) is satisfied.
(a) The formulae in section 2.3 in [11] also show that

$$
\operatorname{corank}_{\Lambda}\left(H^{1}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right)\right)=\operatorname{corank}_{\Lambda}\left(H^{2}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right)\right)+\left[F^{+}: \mathbb{Q}\right]
$$

This implies that $H^{2}\left(F_{\Sigma}^{+} / F^{+}, \mathcal{D}\right)$ is a cotorsion $\Lambda$-module and hence $\amalg^{2}(K, \Sigma, \mathcal{D})$ is also $\Lambda$-cotorsion.

Thus Theorem 3.1.2 implies that $S\left(F^{+}, \mathcal{D}\right)$ is almost $\Lambda$-divisible. Since $S\left(F^{+}, \mathcal{D}\right)=\operatorname{Hom}\left(X_{F_{\infty}, \mathscr{S}_{2}}^{-}, \mathbb{Q}_{2} / \mathbb{Z}_{2}\right)$ by Lemma 3.2 .2 , this is equivalent that $X_{F_{\infty}, \mathscr{S}_{2}}^{-}$has no non-trivial finite $\Lambda$-submodule (see Remark 3.1.3). This completes the proof of Theorem 3.2.1.

### 3.3 The maximal finite $\Lambda$-submodule of $X_{F_{\infty}}^{-}$

We use the same notation as in the previous section. For any number field $N$, we denote by $A_{N}$ the 2 -Sylow subgroup of the ideal class group $\mathrm{Cl}(N)$. We consider the unramified Iwasawa module $X_{F_{\infty}}$ defined by the Galois group of the maximal unramified abelian pro- 2 extension over $F_{\infty}$. By class field theory, we have $X_{F_{\infty}} \cong \lim _{\leftarrow} A_{F_{n}}$. Put $X_{F_{\infty}}^{-}=X_{F_{\infty}} /(1+j) X_{F_{\infty}}$. We denote by $F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)$the maximal finite $\Lambda$-submodule of $X_{F_{\infty}}^{-}$. We define

$$
\begin{gathered}
D_{n, \mathscr{S}_{2}}=\operatorname{ker}\left(A_{F_{n}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}\right), D_{n, \mathscr{S}_{2}}^{+}=\operatorname{ker}\left(A_{F_{n}^{+}} \longrightarrow A_{F_{n}^{+}, \mathscr{S}_{2}}\right), \\
\delta_{1}=\operatorname{rank}_{2}\left(\lim _{\leftrightarrows}\left(\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}\right)\right), \\
\delta_{2}=\operatorname{rank}_{2}\left(\lim _{\longleftarrow}^{\operatorname{ker}}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right)\right),
\end{gathered}
$$

where $\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}$is the $\mathscr{S}_{2}\left(F_{n}\right)$-unit group of $F_{n}, \mathcal{O}_{F_{n}}^{\times}$the unit group of $F_{n}$, both projective limits are taken with respect to the norm maps, and $\operatorname{rank}_{2}(A)$ is the 2 -rank, namely the dimension of $A / 2 A$ as an $\mathbb{F}_{2}$-vector space. In this section, we prove the following result.

Theorem 3.3.1. Assume that Leopoldt's conjecture is valid for $F^{+}$and the lifting maps $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ are injective for all sufficiently large $n \gg 0$. Then we have
$F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)=\lim _{\hookleftarrow} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong \begin{cases}(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d} & \left(\text { if } \mu_{2^{\infty}} \not \subset F_{\infty}\right) \\ (\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-\delta_{1}+\delta_{2}} & \left.\text { if } \mu_{2^{\infty}} \subset F_{\infty}\right),\end{cases}$
where $d$ is the number of primes of $F_{\infty}$ above 2 which ramify in $F_{\infty} / F_{\infty}^{+}$and $\mu_{2 \infty}$ is the group of all 2 power roots of unity.

We note that $0 \leq \delta_{2} \leq \delta_{1} \leq 1$ and the 2-rank of $\lim _{\Longleftarrow} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$ is $d$ or $d-1$ (see Remark 3.3.5 in this paper).

Lemma 3.3.2. We have an exact sequence

$$
\begin{aligned}
0 \longrightarrow \operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right) & \longrightarrow\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j} \\
\longrightarrow \bigoplus_{w \in \mathscr{L}_{2}\left(F_{n}\right) \cap S_{2}\left(F_{n}\right)} \mathbb{Z} / 2 \mathbb{Z} & \longrightarrow D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \longrightarrow 0
\end{aligned}
$$

of $\mathbb{F}_{2}$-vector spaces for all sufficiently large $n \gg 0$, where $j$ is the complex conjugation and $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j}=\left\{(1-j) x \mid x \in \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right\}$.
Proof. For any extension $K / F^{+}$, put $\mathscr{S}_{2 f}(K)=\mathscr{S}_{2}(K) \cap S_{2}(K)$. We take $n$ sufficiently large such that the primes above 2 are totally ramified in $F_{\infty} / F_{n}$ and $F_{\infty}^{+} / F_{n}^{+}$. We consider the following commutative diagram

where $f_{1}, f_{2}$ are homomorphisms induced by the natural maps $\mathcal{O}_{F_{n}^{+}, \mathscr{L}_{2}}^{\times} \longrightarrow$ $\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}$and $\mathrm{Cl}_{\mathscr{S}_{2}}\left(F_{n}^{+}\right) \longrightarrow \mathrm{Cl}_{\mathscr{S}_{2}}\left(F_{n}\right)$. By the snake lemma, we get an exact sequence

$$
0 \longrightarrow \operatorname{ker} f_{2} \longrightarrow \operatorname{coker} f_{1} \longrightarrow \bigoplus_{w \in \mathscr{S}_{2 f}\left(F_{n}\right)} \mathbb{Z} / 2 \mathbb{Z} \longrightarrow \operatorname{coker} f_{2} \longrightarrow 0
$$

Therefore, it suffices to show that coker $f_{1} \cong\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}$ and coker $f_{2} \cong D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$. We consider the following diagram


Since the map $f_{1}^{\prime}$ is injective and $\left(\mathcal{O}_{F_{n}, \mathscr{L}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}$ is a 2-group (see Remark 3.3.5 in this paper), we have

$$
\operatorname{coker} f_{1}=\operatorname{coker} f_{1}^{\prime} \otimes \mathbb{Z}_{2} \cong\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}
$$

by the snake lemma.
Next we show that coker $f_{2} \cong D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$. Since $D_{n, \mathscr{S}_{2}}$ is equal to $\operatorname{ker}\left(\mathrm{Cl}\left(F_{n}\right) \rightarrow \mathrm{Cl}_{\mathscr{\mathscr { S }}_{2}}\left(F_{n}\right)\right) \otimes \mathbb{Z}_{2}$, we have $\operatorname{coker} f_{2} \cong D_{n, \mathscr{S}_{2}} / D_{n, \mathscr{S}_{2}}^{+}$. We consider the following diagram,


Since all primes above 2 which are contained in $\mathscr{S}_{2}\left(F_{n}\right)$ ramify in $F_{n} / F_{n}^{+}$, the norm map $N_{F_{n} / F_{n}^{+}}: D_{n, \mathscr{S}_{2}} \longrightarrow D_{n, \mathscr{S}_{2}}^{+}$is surjective. This implies that $\operatorname{coker} f_{2} \cong D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$.

Lemma 3.3.3. (Corollary in [9]) Assume that Leopoldt's conjecture is valid for $F^{+}$. Then the order of $D_{n, \mathscr{S}_{2}}^{+}$remains bounded as $n \rightarrow \infty$.

Proof. Put $\Gamma_{n}=\operatorname{Gal}\left(F_{n}^{+} / F^{+}\right)$. If Leopoldt's conjecture is valid for $F^{+}$, the order of the Galois invariant $A_{F_{n}^{+}}^{\Gamma_{n}}$ remains bounded as $n \rightarrow \infty$ (see proposition 1 in [9]). This implies that the order of $D_{n, \mathscr{S}_{2}}^{+}$remains bounded as $n \rightarrow \infty$.

Proposition 3.3.4. Assume that Leopoldt's conjecture is valid for $F^{+}$. Then,

$$
\lim _{\longleftarrow} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong \begin{cases}(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d} & \left(\mu_{2^{\infty}} \not \subset F_{\infty}\right) \\ (\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-\delta_{1}+\delta_{2}} & \left(\mu_{2^{\infty}} \subset F_{\infty}\right)\end{cases}
$$

where $d$ is the number of primes of $F_{\infty}$ above 2 which ramify in $F_{\infty} / F_{\infty}^{+}$, and $\delta_{1}, \delta_{2}$ are defined just before Theorem 3.3.1.
Proof. Put $B_{n}=\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}$ for any $n \in \mathbb{Z}_{\geq 0}$. We consider the following commutative diagram which is obtained by Lemma 3.3.2 for $n \geq m \gg 0$,


Since the action of $j$ on $\mathscr{S}_{2}\left(F_{n}\right)$ is trivial, we have $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} \subset \mu\left(F_{n}\right)$ for all $n \geq 0$, where $\mu\left(F_{n}\right)$ is the set of roots of unity which contains in $F_{n}$ (see Lemma 1.6 in [33] ).

If $F_{\infty}$ does not contain $\mu_{2^{\infty}}$, the 2-Sylow subgroup of $\mu\left(F_{n}\right)$ is $\{ \pm 1\}$ for all $n \geq 0$. Therefore the norm map $\mu\left(F_{n}\right) \otimes \mathbb{Z}_{2} \rightarrow \mu\left(F_{m}\right) \otimes \mathbb{Z}_{2}$ is the 0 -map. This fact and $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} \subset \mu\left(F_{n}\right)$ imply that $B_{n} \rightarrow B_{m}$ is the 0 -map for all $n \geq m \geq 0$. Therefore we have $\bigoplus_{v \in \mathscr{S}_{2}\left(F_{n}\right) \cap S_{2}\left(F_{n}\right)} \mathbb{Z} / 2 \mathbb{Z} \cong$ $D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$ for all sufficiently large $n \gg 0$. Taking the projective limit, we have

$$
\lim _{\leftarrow} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d} .
$$

If $F_{\infty}$ contains $\mu_{2^{\infty}}$, we have $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j}=\mu\left(F_{n}\right)$ or $\mu\left(F_{n}\right)^{2}$ (see Theorem 4.12 in [33] ). Since the norm map $\mu\left(F_{n}\right) \rightarrow \mu\left(F_{m}\right)$ is surjective, the norm map $B_{n} \rightarrow B_{m}$ is surjective for all sufficiently large $n \geq m \gg 0$.

We claim that the norm map

$$
N_{F_{n}^{+} / F_{m}^{+}}: \operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right) \longrightarrow \operatorname{ker}\left(D_{m . \mathscr{S}_{2}}^{+} \rightarrow D_{m, \mathscr{S}_{2}}\right)
$$

is also surjective for all sufficiently large $n \geq m \gg 0$. We consider the commutative diagram

$$
\begin{gathered}
0 \longrightarrow \operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right) \longrightarrow D_{n, \mathscr{S}_{2}}^{+} \downarrow^{\downarrow_{F_{n}^{+} / F_{m}^{+}}} \\
0 \longrightarrow \operatorname{ker}\left(D_{m, \mathscr{S}_{2}}^{+} \rightarrow D_{m, \mathscr{S}_{2}}\right) \longrightarrow N_{F_{n}^{+} / F_{m}^{+}}^{+} \longrightarrow \mathscr{S}_{2} .
\end{gathered}
$$

Lemma 3.3.3 implies that the norm map $N_{F_{n}^{+} / F_{m}^{+}}: D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{m, \mathscr{S}_{2}}^{+}$is an isomorphism for all sufficiently large $n \geq m \gg 0$. Therefore the norm map $N_{F_{n}^{+} / F_{m}^{+}}: \operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right) \longrightarrow \operatorname{ker}\left(D_{m, \mathscr{S}_{2}}^{+} \rightarrow D_{m, \mathscr{S}_{2}}\right)$ is injective for all sufficiently large $n \geq m \gg 0$. Since the order of $\operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right)$ is 1 or 2 for all $n \geq 0$ (see Theorem 10.3 in [33]), the norm map $N_{F_{n}^{+} / F_{m}^{+}}$: $\operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right) \longrightarrow \operatorname{ker}\left(D_{m, \mathscr{S}_{2}}^{+} \rightarrow D_{m, \mathscr{S}_{2}}\right)$ is surjective for all sufficiently large $n \geq m \gg 0$. Therefore, taking the projective limit of the exact sequences obtained from in Lemma 3.3.2, we get an exact sequence

$$
\begin{aligned}
0 \longrightarrow \lim \left(\operatorname{ker}\left(D_{n, \mathscr{S}_{2}}^{+} \rightarrow D_{n, \mathscr{S}_{2}}\right)\right) & \longrightarrow \lim _{\leftrightarrows}\left(\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}\right) \\
& \longrightarrow(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d}
\end{aligned} \longrightarrow \underset{\lim _{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \longrightarrow 0}{ }
$$

of $\mathbb{F}_{2}$-vector spaces. Proposition is obtained by considering the 2 -rank of this exact sequence.
Remark 3.3.5. Since $\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}=\mu\left(F_{n}\right)$ or $\mu\left(F_{n}\right)^{2},\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}$ is isomorphic to 0 or $\mathbb{Z} / 2 \mathbb{Z}$ for all $n \geq 0$. Thus we have $0 \leq \delta_{2} \leq \delta_{1} \leq 1$ and the 2 -rank of $\lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}}$ is $d$ or $d-1$.
Lemma 3.3.6. We have

$$
\lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong\left(\lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}\right) /(1+j)\left(\lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}\right) .
$$

Proof. Put $D_{n, \mathscr{S}_{2}}^{\prime}:=\operatorname{ker}\left(D_{n, \mathscr{S}_{2}} \xrightarrow{1-j} D_{n, \mathscr{S}_{2}}\right)$. We consider the commutative diagram

$$
\begin{aligned}
& 0 \longrightarrow D_{m, \mathscr{S}_{2}}^{\prime} \longrightarrow D_{m, \mathscr{S}_{2}} \xrightarrow{1+j}(1+j) D_{m, \mathscr{S}_{2}} \longrightarrow 0 .
\end{aligned}
$$

Since $D_{n, \mathscr{S}_{2}}^{\prime}$ is finite for any $n \geq 0$, the system $\left(D_{n, \mathscr{S}_{2}}^{\prime}, N_{F_{n} / F_{n-1}}\right)$ satisfies the Mittag-Leffler property (see chapter 2, § 7 in [24] ). Therefore taking projective limits, we get an exact sequence

$$
0 \longrightarrow \lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}^{\prime} \longrightarrow \lim _{\rightleftarrows} D_{n, \mathscr{S}_{2}} \xrightarrow{1+j} \underset{\leftrightarrows}{\lim }(1+j) D_{n, \mathscr{S}_{2}} \longrightarrow 0 .
$$

Thus we have $\varliminf_{\rightleftarrows}(1+j) D_{n, \mathscr{S}_{2}} \cong(1+j) \lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}$. This implies that

$$
\lim _{亡} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong\left(\lim _{亡} D_{n, \mathscr{S}_{2}}\right) /(1+j)\left(\lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}\right) .
$$

Now we proceed to the proof of Theorem 3.3.1.
Proof of Theorem 3.3.1. We consider the commutative diagram

where $f_{1}, f_{2}, f_{3}$ are induced by $1+j$, and $X_{F_{\infty}, \mathscr{S}_{2}}=\lim _{\longleftarrow} A_{F_{n}, \mathscr{S}_{2}}$. By the snake lemma, we get an exact sequence

$$
\begin{align*}
\operatorname{ker} f_{2} \longrightarrow \operatorname{ker} f_{3} & \longrightarrow\left(\lim _{\rightleftarrows} D_{n, \mathscr{S}_{2}}\right) /(1+j)\left(\lim _{\rightleftarrows} D_{n, \mathscr{S}_{2}}\right)  \tag{3.2}\\
& \longrightarrow X_{F_{\infty}}^{-} \longrightarrow X_{F_{\infty}, \mathscr{S}_{2}}^{-} \longrightarrow 0
\end{align*}
$$

where $X_{F_{\infty}, \mathscr{S}_{2}}^{-}=X_{F_{\infty}, \mathscr{S}_{2}} /(1+j) X_{F_{\infty}, \mathscr{S}_{2}}$. We claim that the map ker $f_{2} \longrightarrow$ ker $f_{3}$ is surjective if $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ is injective for sufficiently large $n \gg 0$. We define

$$
\begin{aligned}
A_{F_{n}}^{\prime} & =\operatorname{ker}\left(A_{F_{n}} \xrightarrow{1+j} A_{F_{n}}\right) \\
A_{F_{n}, \mathscr{S}_{2}}^{\prime} & =\operatorname{ker}\left(A_{F_{n}, \mathscr{S}_{2}} \xrightarrow{1+j} A_{F_{n}, \mathscr{S}_{2}}\right) \\
D_{n, \mathscr{S}_{2}}^{\prime} & =\operatorname{ker}\left(D_{n, \mathscr{S}_{2}} \xrightarrow{1+j} D_{n, \mathscr{S}_{2}}\right)
\end{aligned}
$$

By definition, $\operatorname{ker} f_{2}=\lim _{\longleftarrow} A_{F_{n}}^{\prime}$ and $\operatorname{ker} f_{3}=\lim _{\longleftarrow} A_{F_{n}, \mathscr{S}_{2}}^{\prime}$. We consider the commutative diagram

$$
\begin{aligned}
& 0 \longrightarrow D_{n, \mathscr{S}_{2}} \longrightarrow A_{F_{n}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}} \longrightarrow 0 \\
& \downarrow N_{F_{n} / F_{n}^{+}} \quad \downarrow N_{F_{n} / F_{n}^{+}} \quad \downarrow N_{F_{n} / F_{n}^{+}} \\
& 0 \longrightarrow D_{n, \mathscr{S}_{2}}^{+} \longrightarrow A_{F_{n}^{+}} \longrightarrow A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow 0 \text {. }
\end{aligned}
$$

Since infinite primes ramify in $F_{n} / F_{n}^{+}$, all norm maps $N_{F_{n} / F_{n}^{+}}$are surjective by class field theory. Since we assumed that $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ is injective, $A_{F_{n}, \mathscr{S}_{2}}^{\prime}=\operatorname{ker}\left(A_{F_{n}, \mathscr{S}_{2}} \xrightarrow{N_{F_{n} / F_{n}^{+}}} A_{F_{n}^{+}, \mathscr{S}_{2}}\right)$. Put $D_{n, \mathscr{S}_{2}}^{\prime \prime}=\operatorname{ker}\left(D_{n, \mathscr{S}_{2}} \xrightarrow{N_{F_{n} / F_{n}^{+}}}\right.$ $\left.D_{n, \mathscr{S}_{2}}^{+}\right), A_{F_{n}}^{\prime \prime}=\operatorname{ker}\left(A_{F_{n}} \xrightarrow{N_{F_{n} / F_{n}^{+}}} A_{F_{n}^{+}}\right)$. By the snake lemma, we get an exact sequence,

$$
0 \longrightarrow D_{n, \mathscr{S}_{2}}^{\prime \prime} \longrightarrow A_{F_{n}}^{\prime \prime} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}^{\prime} \longrightarrow 0
$$

for all sufficiently large $n \gg 0$. We consider the commutative diagram


Since the map $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ is injective, the map $f_{4}$ is an isomorphism. Therefore we get an exact sequence,

$$
0 \longrightarrow D_{n, \mathscr{S}_{2}}^{\prime} \longrightarrow A_{F_{n}}^{\prime} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}^{\prime} \longrightarrow 0
$$

for all sufficiently large $n \gg 0$. Lemma 3.3.3 implies that the map $D_{n, \mathscr{S}_{2}}^{\prime}$ $\xrightarrow{N_{F_{n} / F_{n-1}}} D_{n-1, \mathscr{S}_{2}}^{\prime}$ is surjective for sufficiently $n \gg 0$. Therefore, taking the projective limit, we get an exact sequence

$$
0 \longrightarrow \lim _{\rightleftarrows} D_{n, \mathscr{S}_{2}}^{\prime} \longrightarrow \lim _{\rightleftarrows} A_{F_{n}}^{\prime} \longrightarrow \lim _{\longleftarrow} A_{F_{n}, \mathscr{S}_{2}}^{\prime} \longrightarrow 0
$$

This implies that ker $f_{2} \longrightarrow \operatorname{ker} f_{3}$ is surjective. Therefore, it follows from (1) that we have an exact sequence

$$
0 \longrightarrow\left(\lim _{\longleftarrow} D_{n, \mathscr{S}_{2}}\right) /(1+j)\left(\lim _{\longleftarrow} D_{n, \mathscr{S}_{2}}\right) \longrightarrow X_{F_{\infty}}^{-} \longrightarrow X_{F_{\infty}, \mathscr{S}_{2}}^{-} \longrightarrow 0
$$

If $X_{F_{\infty}, \mathscr{S}_{2}}^{-}$has no non-trivial finite $\Lambda$-submodule, we have

$$
F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)=\left(\lim _{\longleftarrow} D_{n, \mathscr{S}_{2}}\right) /(1+j)\left(\lim _{\longleftarrow} D_{n, \mathscr{S}_{2}}\right) .
$$

Proposition 3.3.4, Lemma 3.3.6 and the above equality imply that
$F_{\Lambda}\left(X_{F_{\infty}}^{-}\right)=\lim _{\longleftrightarrow} D_{n, \mathscr{S}_{2}} /(1+j) D_{n, \mathscr{S}_{2}} \cong \begin{cases}(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d} & \left(\text { if } \mu_{2 \infty} \not \subset F_{\infty}\right) \\ (\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-\delta_{1}+\delta_{2}} & \left.\text { (if } \mu_{2^{\infty}} \subset F_{\infty}\right),\end{cases}$
if $X_{F_{\infty}, \mathscr{S}_{2}}^{-}$has no non-trivial finite $\Lambda$-submodule. This completes the proof of Theorem 3.3.1.

Next we study certain conditions on the injectivity of the map $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow$ $A_{F_{n}, \mathscr{L}_{2}}$ for an imaginary abelian field $F$. Leopoldt's conjecture is valid for a real abelian field. Hence the following result implies Corollary 1.2.3.

Lemma 3.3.7. Assume that $F$ is an imaginary abelian field and all primes above 2 ramify in $F_{\infty} / F_{\infty}^{+}$. If $F_{\infty}$ contains $\mu_{2^{\infty}}$ or Hasse's unit index $\left[\mathcal{O}_{F_{n}}^{\times}\right.$: $\left.\mu\left(F_{n}\right) \mathcal{O}_{F_{n}^{+}}^{\times}\right]=2$ for all sufficiently large $n \gg 0$, the lifting map $A_{F_{n}^{+}, \mathscr{L}_{2}} \longrightarrow$ $A_{F_{n}, \mathscr{S}_{2}}$ is injective for all sufficiently large $n \gg 0$.

Proof. It is well known that the kernel of the map $A_{F_{n}^{+}, \mathscr{S}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ coincides with the kernel of the map

$$
H^{1}\left(F_{n} / F_{n}^{+}, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right) \longrightarrow H^{1}\left(F_{n} / F_{n}^{+}, \prod_{v \notin \mathscr{S}_{2}\left(F_{n}\right)} \mathcal{O}_{F_{n}, v}^{\times}\right)
$$

where $\mathcal{O}_{F_{n}, v}^{\times}$is the unit group of the completion of $F_{n}$ at $v$. Therefore it suffices to show that $H^{1}\left(F_{n} / F_{n}^{+}, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)=0$ for all sufficiently large $n \gg 0$.

If $F_{\infty}$ contains $\mu_{2^{\infty}}$, since all primes above 2 are contained in $\mathscr{S}_{2}\left(F_{n}\right)$, $\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}$contains $1-\zeta_{2^{m}}$ for all $2^{m}$ th roots of unity $\zeta_{2^{m}}$ in $\mu\left(F_{n}\right)$. This implies that $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j}=\mu\left(F_{n}\right)$ for all sufficiently large $n \gg 0$.

If Hasse's unit index $\left[\mathcal{O}_{F_{n}}^{\times}: \mu\left(F_{n}\right) \mathcal{O}_{F_{n}^{+}}^{\times}\right]=2$, we also have $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j}=$ $\mu\left(F_{n}\right)$ (see Satz 14 in [17]).

Therefore, we get an exact sequence for all sufficiently large $n \gg 0$,

$$
0 \longrightarrow \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times} \longrightarrow \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times} \stackrel{1-i}{\longrightarrow} \mu\left(F_{n}\right) \rightarrow 0
$$

Put $G=\operatorname{Gal}\left(F_{n} / F_{n}^{+}\right)$. Taking Galois cohomology, we get an exact sequence

$$
\begin{aligned}
0 \longrightarrow & \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times} \xrightarrow{f_{1}} \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times} \xrightarrow{f_{2}}\{ \pm 1\} \xrightarrow{f_{3}} H^{1}\left(G, \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right) \xrightarrow{f_{4}} H^{1}\left(G, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right) \\
& \xrightarrow{f_{5}} H^{1}\left(G, \mu\left(F_{n}\right)\right) \xrightarrow{f_{6}} H^{2}\left(G, \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right) \xrightarrow{f_{7}} H^{2}\left(G, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right) .
\end{aligned}
$$

Since $f_{1}$ is an isomorphism, $f_{2}$ is the 0 -map. Therefore $f_{3}$ is injective. Since $F_{n} / F_{n}^{+}$is a cyclic extension,

$$
H^{1}\left(G, \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)=\frac{\operatorname{ker}\left(1+j: \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times} \rightarrow \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)}{\left(\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)^{1-j}}=\{ \pm 1\}
$$

Thus $f_{3}$ is also an isomorphism and $f_{4}$ is the 0 -map. Therefore $f_{5}$ is injective. Since

$$
H^{1}\left(G, \mu\left(F_{n}\right)\right)=\frac{\operatorname{ker}\left(1+j: \mu\left(F_{n}\right) \rightarrow \mu\left(F_{n}\right)\right)}{\mu\left(F_{n}\right)^{1-j}}=\frac{\mu\left(F_{n}\right)}{\mu\left(F_{n}\right)^{2}}=\{ \pm 1\}
$$

we get an exact sequence

$$
0 \longrightarrow H^{1}\left(G, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right) \xrightarrow{f_{5}}\{ \pm 1\} \xrightarrow{f_{6}} H^{2}\left(G, \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right) \xrightarrow{f_{7}} H^{2}\left(G, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right) .
$$

If $f_{7}$ is not injective, $f_{6}$ is not the 0-map. This implies that $H^{1}\left(G, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)=$ 0 . We show that $f_{7}$ is not injective. Since $F_{n} / F_{n}^{+}$is a cyclic extension,

$$
\begin{gathered}
H^{2}\left(G, \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)=\hat{H}^{0}\left(G, \mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)=\frac{\left(\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)^{G}}{N_{F_{n} / F_{n}^{+}}\left(\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)}=\frac{\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}}{\left(\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}\right)^{2}}, \\
H^{2}\left(G, \mathcal{O}_{F_{n}, \mathscr{\mathscr { S }}_{2}}^{\times}\right)=\frac{\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}}{N_{F_{n} / F_{n}^{+}}\left(\mathcal{O}_{\left.F_{n}, \mathscr{L}_{2}\right)}^{\times}\right.} .
\end{gathered}
$$

If Hasse's unit index $\left[\mathcal{O}_{F_{n}}^{\times}: \mu\left(F_{n}\right) \mathcal{O}_{F_{n}^{+}}^{\times}\right]=2$, Satz 14 in [17] shows that $\left[\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1+j}:\left(\mathcal{O}_{F_{n}^{+}}^{\times}\right)^{2}\right]=2$. This implies that $f_{7}$ is not injective.

If $F_{\infty}$ contains $\mu_{2 \infty}, \mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}$contains $1+\zeta_{2^{l}}$, where $2^{l}$ is the order of the 2-Sylow subgroup of $\mu\left(F_{n}\right)$. Thus we have

$$
N_{F_{n} / F_{n}^{+}}\left(1+\zeta_{2^{l}}\right)=2+\zeta_{2^{l}}+\zeta_{2^{l}}^{-1} \in N_{F_{n} / F_{n}^{+}}\left(\mathcal{O}_{F_{n}, \mathscr{\mathscr { L }}_{2}}^{\times}\right) .
$$

Since $\sqrt{2+\zeta_{2^{l}}+\zeta_{2^{l}}^{-1}}= \pm\left(\zeta_{2^{l+1}}+\zeta_{2^{l+1}}^{-1}\right) \notin F_{n}^{\times}$, we have

$$
2+\zeta_{2^{l+1}}+\zeta_{2^{l+1}}^{-1} \notin\left(\mathcal{O}_{F_{n}^{+}, \mathscr{S}_{2}}^{\times}\right)^{2}
$$

This implies that $f_{7}$ is not injective. This completes the proof of Lemma 3.3.7.

We give some examples here.
Example 3.3.8. Let $F$ be an imaginary quadratic field that is not $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ and the prime above 2 ramifies in $F_{\infty} / F_{\infty}^{+}$. Then, Leopoldt's conjecture is valid for $F^{+}=\mathbb{Q}$. Since the class numbers of $F_{n}^{+}$are odd for all $n \geq 0$ (see Satz 6 in [17]), the lifting maps $A_{F_{n}^{+}, \mathscr{I}_{2}} \longrightarrow A_{F_{n}, \mathscr{S}_{2}}$ are injective for all $n \geq 0$. Since $F_{\infty}$ does not contain all $2^{n}$ th roots of unity for $n \geq 1$, Theorem 3.3.1 implies that

$$
F_{\Lambda}\left(X_{F_{\infty}}^{-}\right) \cong \mathbb{Z} / 2 \mathbb{Z}
$$

This was proved by Ferrero in [6] (see Theorem 2.2.3).
Example 3.3.9. Let $F^{+}$be a real abelian field which is unramified at 2, and $F=F^{+}(\sqrt{-1})$. Then, we have

$$
F_{\Lambda}\left(X_{F_{\infty}}^{-}\right) \cong(\mathbb{Z} / 2 \mathbb{Z})^{\oplus d-1}
$$

where $d$ is the number of primes of $F$ lying above 2 . In fact, Theorem 1 in [22] implies that $\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j}=\mu\left(F_{n}\right)^{2}$ for all $n \geq 0$. Since $1-\zeta_{2^{n+2}} \in$ $\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}$, we have $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j}=\mu\left(F_{n}\right)$ for all $n \geq 0$. Therefore we have $\left(\mathcal{O}_{F_{n}, \mathscr{S}_{2}}^{\times}\right)^{1-j} /\left(\mathcal{O}_{F_{n}}^{\times}\right)^{1-j} \cong \mathbb{Z} / 2 \mathbb{Z}$ for all $n \geq 0$ and $\delta_{1}=1$. Theorem 1 in [22] also implies that $\delta_{2}=0$.

## Chapter 4

## Iwasawa main conjecture for $p=2$

In this section, we prove Theorem 1.2.4.

### 4.1 Proof of Iwasawa main conjecture for $p=2$

If $l$ is a prime number and $l \equiv 1 \bmod 2^{n+1}$, let $\mathbb{Q}_{n, l}$ denote the cyclic extension of $\mathbb{Q}$ of degree $2^{n}$ contained in $\mathbb{Q}\left(\mu_{l}\right)$ and let $N_{n, l}=N \mathbb{Q}_{n, l}$ for any finite extension $N$ of $\mathbb{Q}$. We note that $\mathbb{Q}_{n, l}$ is a real abelian field since $l \equiv 1$ $\bmod 2^{n+1}$.

Lemma 4.1.1. Let $N$ be a finite abelian extension of $k$. There exists an $r \in \mathbb{Z}_{\geq 2}$ depending only on $N$ such that for any $n>r$ we can find infinitely many $l$ satisfying the following conditions.
(a) $l$ is unramified in $N / \mathbb{Q}$.
(b) $l \equiv 1 \bmod 2^{n+1}$.
(c) For any character $\left.\chi_{1} \in \operatorname{Gal} \widehat{\left(N_{\infty} /\right.} / k\right)$ and any character $\left.\chi_{2} \in \operatorname{Gal} \widehat{\left(k_{n, l} /\right.} / k\right)$ of order $>2^{r}$, we have $\chi_{1} \chi_{2}(v) \neq 1$ for any prime $v$ of $k$ above 2 .

Proof. The proof of this lemma goes by the same method of the proof of Lemma 10.2 in Wiles [34]. Let $r=\max \left\{m \in \mathbb{Z}_{\geq 1}\left|2^{m}\right|\left[N_{v}: \mathbb{Q}_{2}\right]\right.$ for all $v$ above 2$\}+$ 1 , where $N_{v}$ is the completion of $N$ at $v$. For any $n>r$, we can find infinitely many $l$ such that $l$ splits completely in $\mathbb{Q}\left(\mu_{2^{n+1}}\right) / \mathbb{Q}$ and is inert in $\mathbb{Q}\left(\mu_{2^{n+1}}, \sqrt[4]{2}\right) / \mathbb{Q}\left(\mu_{2^{n+1}}\right)$ by Chebotarev's density theorem. We can choose that $l$ is unramified in $N / \mathbb{Q}$ since the number of primes which ramify in $N / \mathbb{Q}$ is finite. Any such $l$ satisfies the conditions (a), (b). We show that any such $l$ satisfies the condition (c). Since $l$ is inert in $\mathbb{Q}\left(\mu_{2^{n+1}}, \sqrt[4]{2}\right) / \mathbb{Q}\left(\mu_{2^{n+1}}\right)$,
we have $\sqrt[4]{2} \notin \mathbb{F}_{l}, 2 \notin\left(\mathbb{F}_{l}^{\times}\right)^{4}$. This implies that the order of $\mathrm{Frob}_{2}$ in $\operatorname{Gal}\left(\mathbb{Q}_{n, l} / \mathbb{Q}\right)$ is equal to $2^{n-1}$, where $\mathrm{Frob}_{2}$ is the Frobenius map of 2. Since the restriction map $\operatorname{Gal}\left(k_{n, l} / k\right) \rightarrow \operatorname{Gal}\left(\mathbb{Q}_{n, l} / \mathbb{Q}\right)$ is an isomorphism, the order of $\operatorname{Frob}_{v} \in \operatorname{Gal}\left(k_{n . l} / k\right)$ is $2^{n-1}$ for all $v$ above 2 . This implies that any such $l$ satisfies the condition (c).

Let $\chi$ be a one-dimensional Artin character for $k$ and $k^{\chi}$ the extension of $k$ attached to $\chi$, i.e., $k^{\chi}=\bar{k}^{\mathrm{ker} \chi}$. Assume that $k^{\chi}$ is a totally real field and $k^{\chi} \cap k_{\infty}=k$. Let $S$ be the finite set of primes of $k$ which ramify in $k_{\infty}^{\chi} / k$. We consider $N=k^{\chi}$ in Lemma 4.1.1 and take $r, n, l$. Put $H=k^{\chi} \mathbb{Q}_{n, l}$. Let $S_{l}$ be the finite set of primes of $k$ above $l$.

We denote by $\mathcal{X}_{H_{\infty}, S \cup S_{l}}$ the Galois group of the maximal abelian pro-2extension over $H_{\infty}$ unramified outside $S \cup S_{l}$. Put $\Gamma=\operatorname{Gal}\left(H_{\infty} / H\right)$. We write $G=\operatorname{Gal}(H / k)=G^{\prime} \times \Delta$ where $G^{\prime}$ is a 2 -group and the order of $\Delta$ is odd. Then, we have

$$
\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(H_{\infty} / k\right)\right]\right]=\mathbb{Z}_{2}[[G \times \Gamma]] \simeq \bigoplus_{\chi^{\prime} \in \hat{\Delta} / \sim} \mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times \Gamma\right]\right]
$$

We write $\chi=\chi^{\prime} \psi$ where $\chi^{\prime} \in \hat{\Delta}$ and $\psi \in \hat{G}^{\prime}$. Put $\Lambda_{G^{\prime}}^{\chi^{\prime}}=\mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times\right.\right.$ $\Gamma]]$. For any $\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(H_{\infty} / k\right)\right]\right]$-module $M$, we put $M^{\chi^{\prime}}=M \otimes_{\mathbb{Z}_{2}\left[\left[G a l\left(H_{\infty} / k\right)\right]\right]}$ $\Lambda_{G^{\prime}}^{\chi^{\prime}}$. We take $\tau$ a generator of $\operatorname{Gal}\left(H_{\infty} / k_{\infty}^{\chi}\right)$ and put $\xi_{n, l}=\frac{\tau^{n}-1}{\tau^{r}-1} \in \Lambda_{G^{\prime}}^{\chi^{\prime}}$. We denote by $P$ the prime ideal of $\Lambda_{G^{\prime}}^{\chi^{\prime}}$ which is generated by 2 and $\sigma-1$ for all $\sigma \in G^{\prime}$. We note that $\xi_{n, l} \in P$. We consider the map $\mathcal{Q}\left(\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(H_{\infty} / k\right)\right]\right]\right) \rightarrow$ $\mathcal{Q}\left(\mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime} \times \Gamma\right]\right]\right)$ induced by $\chi^{\prime}$ and denote by $\Phi_{H_{\infty} / k}^{\chi^{\prime}}$ the image of $\Phi_{H_{\infty} / k} \in \mathcal{Q}\left(\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(H_{\infty} / k\right)\right]\right]\right)$, where $\Phi_{H_{\infty} / k}$ is defined in section 1.

Proposition 4.1.2. (i) Assume that $\chi^{\prime} \neq 1$. If

$$
\begin{equation*}
\operatorname{Fitt}_{\Lambda_{G^{\prime}, P}^{\prime}}^{\chi^{\prime}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right) \subset\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right), \tag{4.1}
\end{equation*}
$$

we have

$$
\operatorname{Fitt}_{\Lambda_{G^{\prime}}^{\chi^{\prime}}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right) \subset\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right)
$$

where $\Lambda_{G^{\prime}, P}^{\chi^{\prime}}$ is the localization of $\Lambda_{G^{\prime}}^{\chi^{\prime}}$ at $P$.
(ii) Assume that $\chi^{\prime}=1$. If

$$
\begin{equation*}
(\gamma-1) \text { Fitt }_{\Lambda_{G^{\prime}, P}^{\chi^{\prime}}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right) \subset\left((\gamma-1) \frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right) \tag{4.2}
\end{equation*}
$$

we have

$$
(\gamma-1) \operatorname{Fitt}_{\Lambda_{G^{\prime}}^{\chi^{\prime}}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right) \subset\left((\gamma-1) \frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right),
$$

where $\gamma$ is a topological generator of $\Gamma$.
Proof. At first, we consider the case $\chi^{\prime} \neq 1$, this is equivalent to saying that the order of $\chi$ is not 2-power. In this case, Corollary 2.3.5 implies that

$$
\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}} \in \Lambda_{G^{\prime}}^{\chi^{\prime}} .
$$

Let $\phi$ be a character of $\operatorname{Gal}\left(k_{n, l} / k\right)$ of order $>2^{r}$. Then Lemma 4.1.1 (c) implies that $\chi \phi \chi_{1}(v) \neq 1$ for any prime $v$ above 2 and any character $\left.\chi_{1} \in \operatorname{Gal} \widehat{\left(k_{\infty}^{x}\right.} / k\right)$. Thus Theorem 2.4.5 implies that

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im} \chi \phi][[\Gamma]] \otimes \mathbb{Q}_{2}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}} \otimes \mathbb{Z}_{2}[\operatorname{Im} \chi \phi][[\Gamma]] \otimes \mathbb{Q}_{2}\right)=\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi \phi}\right)
$$

for any character $\phi$ of $\operatorname{Gal}\left(k_{n, l} / k\right)$ of order $>2^{r}$. This implies that

$$
\operatorname{Fitt}_{\Lambda_{G^{\prime}}^{\chi^{\prime}} \otimes \mathbb{Q}_{2}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}} \otimes \mathbb{Q}_{2}\right) \subset\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right) .
$$

This inclusion and (4.1) imply that for any $a \in \operatorname{Fitt}_{\Lambda_{G^{\prime}}^{\prime}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right)$, there exist $j_{a} \in \mathbb{Z}_{\geq 0}$ and $f \in \Lambda_{G^{\prime}}^{\chi^{\prime}} \backslash P$ such that

$$
\left(f, 2^{j_{a}}\right) a \subset\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right)
$$

Put $A=\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right) \subset \Lambda_{G^{\prime}}^{\chi^{\prime}}$. Since $2 \in P$ and $f \notin P$, this inclusion implies that $\Lambda_{G^{\prime}}^{\chi^{\prime}} a+A / A$ is a finite submodule of $\Lambda_{G^{\prime}}^{\chi^{\prime}} / A$. If $\Lambda_{G^{\prime}}^{\chi^{\prime}} / A$ has no non-trivial finite $\Lambda_{G^{\prime}}^{\chi^{\prime}}$-submodule, we have $a \in A$ and complete the proof of this proposition for $\chi^{\prime} \neq \mathbf{1}$.

We claim that $\Lambda_{G^{\prime}}^{\chi^{\prime}} / A$ has no non-trivial finite $\Lambda_{G^{\prime}}^{\chi^{\prime}}$-submodule. Since all primes above $l$ totally ramify in $H / k^{\chi}$ and $l$ is unramified in $k^{\chi} / \mathbb{Q}$, we have $\operatorname{Gal}(H / k) \simeq \operatorname{Gal}\left(H / k^{\chi}\right) \times \operatorname{Gal}\left(k^{\chi} / k\right)$. Put $n^{\prime}=\operatorname{ord}_{2}\left(\sharp \operatorname{Gal}\left(k^{\chi} / k\right)\right)$. Then we have

$$
\Lambda_{G^{\prime}}^{\chi^{\prime}}\left(\xi_{n, l}\right) \simeq \mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[X_{1}, X_{2}, T\right]\right] /\left(\frac{X_{1}^{2^{n}}-1}{X_{1}^{2^{r}}-1}, X_{2}^{2^{n^{\prime}}}-1\right)
$$

We note that $\mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[X_{1}, X_{2}, T\right]\right]$ is a regular local ring of dimension 4. Since $\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}$ is the 2-adic $L$-function, we have $\phi\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}\right) \neq 0$
in $\mathbb{Z}_{2}\left[\operatorname{Im}\left(\phi \chi^{\prime}\right)\right][[\Gamma]]$ for any character $\phi \in G^{\prime}$. Therefore, $\left(\frac{X_{1}^{2^{n}}-1}{X_{1}^{2^{n}}-1}, X_{2}^{2^{n^{\prime}}}-\right.$ $\left.1, \frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}\right)$ is a regular sequence in $\Lambda_{G^{\prime}}^{\chi^{\prime}}$. Thus $\Lambda_{G^{\prime}}^{\chi^{\prime}} / A$ has depth one and there exists a non-unit $x \in \Lambda_{G^{\prime}}^{\chi^{\prime}}$ such that the map

$$
\Lambda_{G^{\prime}}^{\chi^{\prime}} / A \xrightarrow{\times x} \Lambda_{G^{\prime}}^{\chi^{\prime}} / A
$$

is injective. For any finite submodule $M$ of $\Lambda_{G^{\prime}}^{\chi^{\prime}} / A$, the map $M \xrightarrow{\times x} M$ is an isomorphism. By Nakayama's lemma, we have $M=0$. Therefore, $\Lambda_{G^{\prime}}^{\chi^{\prime}} / A$ has no non-trivial finite $\Lambda_{G^{\prime}}^{\chi^{\prime}}$-submodule.

Next, we consider the case $\chi^{\prime}=\mathbf{1}$. In this case, Corollary 2.3.5 implies that

$$
(\gamma-1) \frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}} \in \Lambda_{G^{\prime}}^{\chi^{\prime}}
$$

By a similar method, Lemma 4.1.1 (c) and Theorem 2.4.5 imply that

$$
(\gamma-1) \operatorname{Fitt}_{\Lambda_{G^{\prime}} \otimes \mathbb{Q}_{2}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}} \otimes \mathbb{Q}_{2}\right) \subset(\gamma-1)\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right) .
$$

We can also prove that $\Lambda_{G^{\prime}}^{\chi^{\prime}} /(\gamma-1)\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}, \xi_{n, l}\right)$ has no non-trivial finite $\Lambda_{G^{\prime}}^{\chi^{\prime} \text {-submodule similarly. Therefore, these fact and (4.2) imply the state- }}$ ment of Proposition 4.1.2.

We write $\operatorname{Gal}\left(k^{\chi} / k\right)=\Delta \times G^{\prime \prime}$ where $G^{\prime \prime}$ is the 2-Sylow subgroup of $\operatorname{Gal}\left(k^{\chi} / k\right)$. We know that $G^{\prime}=G^{\prime \prime} \times \operatorname{Gal}\left(H / k^{\chi}\right)$. Put $\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}=\mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime \prime} \times\right.\right.$ $\Gamma]]$. For any prime $\lambda$ of $k$ above $l$, we define $E_{\lambda} \in \mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]$ by

$$
E_{\lambda}=1-N \lambda^{-1} \mathrm{Frob}_{\lambda},
$$

where $\operatorname{Frob}_{\lambda}$ is the Frobenius map of $\lambda$ in $\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)$. We consider the natural map $\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right] \rightarrow \Lambda_{G^{\prime \prime}}^{\chi^{\prime}}$ induced by $\chi^{\prime}$ and denote by $E_{\lambda}^{\chi^{\prime}}$ the image of $E_{\lambda}$.

Corollary 4.1.3. (i) Assume that $\chi^{\prime} \neq \mathbf{1}$ and (4.1) is valid. Then we have

$$
\operatorname{Fitt}_{\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}} /(\tau-1) \mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right) \subset\left(\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi^{\prime}} \frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi^{\prime}}, 2^{n-r}\right)
$$

(ii) Assume that $\chi^{\prime}=\mathbf{1}$ and (4.2) is valid. Then we have

$$
(\gamma-1) \operatorname{Fitt}_{\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}}\left(\mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}} /(\tau-1) \mathcal{X}_{H_{\infty}, S \cup S_{l}}^{\chi^{\prime}}\right) \subset\left((\gamma-1) \prod_{\lambda \in S_{l}} E_{\lambda}^{\chi^{\prime}} \frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi^{\prime}}, 2^{n-r}\right)
$$

Proof. We consider the restriction map $c_{H_{\infty} / k_{\infty}^{\chi}}: \Lambda_{G^{\prime}}^{\chi^{\prime}} \rightarrow \Lambda_{G^{\prime \prime}}^{\chi^{\prime}}$. By the property of $p$-adic $L$-function, we have

$$
c_{H_{\infty} / k_{\infty}^{\chi}}\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}\right)=\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi^{\prime}} \frac{1}{2^{d}} \Phi_{k_{\infty} / k}^{\chi^{\prime}} .
$$

Therefore, Proposition 4.1.2 and Remark 2.2 in [26] imply these inclusions.

Let $\mathcal{X}_{k_{\infty}^{\chi}, S}$ (resp. $\mathcal{X}_{k_{\infty}^{\chi}, S \cup S_{l}}$ ) be the Galois group of the maximal abelian pro-2-extension over $k_{\infty}^{\infty}$ unramified outside $S$ (resp. $S \cup S_{l}$ ). We consider the restriction map $\mathcal{X}_{k_{\infty}^{\chi}, S \cup S_{l}} \rightarrow \mathcal{X}_{k_{\infty}^{\chi}, S}$ and denote by $J_{l}$ the kernel of this map. Put $\Lambda_{\chi}=\mathbb{Z}_{2}[\operatorname{Im} \chi][[\Gamma]] . \operatorname{Gal}\left(k^{\chi} / k\right)$ acts on $\Lambda_{\chi}$ via $\chi$. We consider the natural map $\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right] \rightarrow \Lambda_{\chi}$ induced by $\chi$ and denote by $E_{\lambda}^{\chi}$ the image of $E_{\lambda}$.

## Lemma 4.1.4.

$$
J_{l} \otimes_{\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]} \Lambda_{\chi} \simeq \bigoplus_{\lambda \in S_{l}} \Lambda_{\chi} /\left(E_{\lambda}^{\chi}\right) .
$$

Proof. For any $m \in \mathbb{Z}_{\geq 0}$, let $k_{m}^{\chi}$ be the $m$-th layer of $k_{\infty}^{\chi} / k^{\chi}, S_{m}$ (resp. $S_{m, l}$ ) the set of primes of $k_{m}^{\chi}$ above $S$ (resp. $S_{l}$ ). We denote $M_{m}\left(S_{m}\right)$ (resp. $M_{m}\left(S_{m} \cup S_{m, l}\right)$ ) by the maximal abelian pro-2 extension over $k_{m}^{\chi}$ unramified outside $S_{m}$ (resp. $S_{m} \cup S_{m, l}$ ). There is an exact sequence

$$
\begin{aligned}
0 & \rightarrow \operatorname{Gal}\left(M_{m}\left(S_{m} \cup S_{m, l}\right) / M_{m}\left(S_{m}\right)\right) \\
& \rightarrow \operatorname{Gal}\left(M_{m}\left(S_{m} \cup S_{m, l}\right) / k_{m}^{\chi}\right) \rightarrow \operatorname{Gal}\left(M_{m}\left(S_{m}\right) / k_{m}^{\chi}\right) \rightarrow 0 .
\end{aligned}
$$

By class field theory, we have

$$
\begin{aligned}
\operatorname{Gal}\left(M_{m}\left(S_{m} \cup S_{m, l}\right) / k_{m}^{\chi}\right) & \simeq J_{k_{m}^{\chi}} / k_{m}^{\chi} \prod_{v \notin S_{m} \cup S_{m, l}} E_{k_{m, v}^{\chi}} \otimes \mathbb{Z}_{2}, \\
\operatorname{Gal}\left(M_{m}\left(S_{m}\right) / k_{m}^{\chi}\right) & \simeq J_{k_{m}^{\chi}} / k_{m}^{\chi} \prod_{v \notin S_{m}} E_{k_{m, v}^{\chi}} \otimes \mathbb{Z}_{2},
\end{aligned}
$$

where $J_{k_{m}^{\chi}}$ is the idele group of $k_{m}^{\chi}, E_{k \chi_{m}, v}$ is the local unit group of $k_{m}^{\chi}$ at $v$. Therefore, we have

$$
\operatorname{Gal}\left(M_{m}\left(S_{m} \cup S_{m, l}\right) / M_{m}\left(S_{m}\right)\right) \simeq \prod_{v \in S_{m, l}} E_{k_{m, v}^{\chi}} /\left(k_{m}^{\chi} \cap \prod_{v \in S_{m, l}} E_{k_{m}^{\chi}, v}\right) \otimes \mathbb{Z}_{2} .
$$

Lemma 4.1.1 (b) implies that all 2-power roots of unity are contained in $k_{\infty, v}^{\chi}$ for any prime $v$ above $l$. Since $E_{k_{m}^{\chi}, v} \otimes \mathbb{Z}_{2}=\mu_{2 \infty}\left(k_{m, v}^{\chi}\right)$ for all $v \in S_{m, l}$ and $J_{l} \simeq \underset{m}{\lim _{m}} \operatorname{Gal}\left(M_{m}\left(S_{m} \cup S_{m, l}\right) / M_{m}\left(S_{m}\right)\right)$, we have
where $\mu_{2 \infty}\left(k_{m, v}^{\chi}\right)$ (resp. $\mu_{2^{\infty}}\left(k_{m}^{\chi}\right)$ ) is the all 2-power roots of unity contained in $k_{m, v}^{\chi}$ (resp. $k_{m}^{\chi}$ ). Since $\chi$ is an even character, $\mu_{2 \infty}\left(k_{m}^{\chi}\right)=\{ \pm 1\}$ for all $m$. Since

$$
\varliminf_{m} \bigoplus_{v \in S_{m, l}}\left(\mu_{2^{\infty}}\left(k_{m, v}^{\chi}\right)\right) \otimes \Lambda_{\chi} \simeq \bigoplus_{\lambda \in S_{l}} \Lambda_{\chi} /\left(E_{\lambda}^{\chi}\right)
$$

has no non-trivial finite $\Lambda_{\chi}$-submodule, we have $J_{l} \otimes_{\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]} \Lambda_{\chi} \simeq$ $\oplus_{\lambda \in S_{l}} \Lambda_{\chi} /\left(E_{\lambda}^{\chi}\right)$. This completes the proof of Lemma 4.1.4.

$$
\text { Put } M_{\infty}(S)=\bigcup_{m=1}^{\infty}\left(M_{m}\left(S_{m}\right)\right) \text { and } M_{\infty}\left(S \cup S_{l}\right)=\bigcup_{m=1}^{\infty}\left(M_{m}\left(S_{m} \cup S_{m, l}\right)\right) \text {. }
$$

By definition, we have

$$
\mathcal{X}_{k_{\infty}^{\chi}, S} \simeq \operatorname{Gal}\left(M_{\infty}(S) / k_{\infty}^{\chi}\right), \mathcal{X}_{k_{\infty}^{\chi}, S \cup S_{l}} \simeq \operatorname{Gal}\left(M_{\infty}\left(S \cup S_{l}\right) / k_{\infty}^{\chi}\right)
$$

Since only primes above $l$ ramify in $H_{\infty} / k_{\infty}^{\chi}$, we know that $H_{\infty} M_{\infty}(S) \subset$ $M_{\infty}\left(S \cup S_{l}\right)$. Put $I_{l}=\operatorname{Gal}\left(M_{\infty}\left(S \cup S_{l}\right) / H_{\infty} M_{\infty}(S)\right)$.
Lemma 4.1.5. There exists an $a \in \mathbb{Z} \geq 0$ independent of $l$, $n$ in Lemma 4.1.1, we have

$$
2^{a} \prod_{\lambda \in S_{l}} E_{\lambda}^{\chi} \in \operatorname{Fitt}_{\Lambda_{\chi}}\left(I_{l} \otimes \Lambda_{\chi}\right)
$$

Proof. By definition, we get an exact sequence

$$
0 \longrightarrow I_{l} \longrightarrow J_{l} \longrightarrow \operatorname{Gal}\left(H_{\infty} M_{\infty}(S) / M_{\infty}(S)\right) \longrightarrow 0
$$

Since $\operatorname{Gal}\left(H_{\infty} / k_{\infty}\right)$ acts on $\operatorname{Gal}\left(H_{\infty} M_{\infty}(S) / M_{\infty}(S)\right) \simeq \mathbb{Z} / 2^{n} \mathbb{Z}$ trivially, we have

$$
\operatorname{Gal}\left(H_{\infty} M_{\infty}(S) / M_{\infty}(S)\right) \otimes \Lambda_{G^{\prime \prime}}^{\chi^{\prime}}=0
$$

if $\chi^{\prime} \neq \mathbf{1}$. Therefore, Lemma 4.1.4 implies that

$$
I_{l} \otimes_{\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(k_{\infty}^{\chi} / k\right)\right]\right]} \Lambda_{\chi} \simeq \bigoplus_{\lambda \in S_{l}} \Lambda_{\chi} /\left(E_{\lambda}^{\chi}\right)
$$

if $\chi^{\prime} \neq \mathbf{1}$. Therefore, we consider only the case for $\chi^{\prime}=\mathbf{1}$.
If $\psi=\mathbf{1}$ (i.e., $\chi=\mathbf{1}$ and $k^{\chi}=k$ ), we have $\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}=\Lambda_{\chi}$ and we get an exact sequence

$$
0 \longrightarrow I_{l} \otimes \Lambda_{\chi} \longrightarrow J_{l} \otimes \Lambda_{\chi} \longrightarrow \mathbb{Z} / 2^{n} \mathbb{Z} \longrightarrow 0
$$

Since $J_{l} \otimes \Lambda_{\chi}$ has no non-trivial finite $\Lambda_{\chi}$-submodule by Lemma 4.1.4, $I_{l} \otimes \Lambda_{\chi}$ has too. Therefore, we have

$$
\begin{aligned}
\operatorname{Fitt}_{\Lambda_{\chi}}\left(I_{l} \otimes \Lambda_{\chi}\right) & =\operatorname{char}_{\Lambda_{\chi}}\left(I_{l} \otimes \Lambda_{\chi}\right) \\
& =\operatorname{char}_{\Lambda_{\chi}}\left(J_{l} \otimes \Lambda_{\chi}\right) \\
& =\left(\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi}\right)
\end{aligned}
$$

We consider the case $\psi \neq 1$. Put $\mathfrak{a}$ is an ideal of $\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}$ generated by $2^{n}$ and $\sigma-1$ for all $\sigma \in G^{\prime \prime} \times \Gamma$. Then we have,

$$
\operatorname{Gal}\left(H_{\infty} M_{\infty}(S) / M_{\infty}(S)\right) \simeq \Lambda_{G^{\prime \prime}}^{\chi^{\prime}} / \mathfrak{a} \simeq \mathbb{Z} / 2^{n} \mathbb{Z}
$$

Consider the $\operatorname{map} \Lambda_{G^{\prime \prime}}^{\chi^{\prime}} \xrightarrow{\chi} \Lambda_{\chi}$ and we denote by $\mathfrak{q}$ the kernel of this map. Consider $\otimes_{\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}} \Lambda_{\chi}$, we get an exact sequence

$$
\operatorname{Tor}_{1}^{\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}}\left(\Lambda / \mathfrak{a}, \Lambda_{\chi}\right) \longrightarrow I_{l} \otimes \Lambda_{\chi} \longrightarrow J_{l} \otimes \Lambda_{\chi} \longrightarrow \Lambda_{\chi} / \mathfrak{a}+\mathfrak{q} \longrightarrow 0
$$

Since $\psi \neq 1, \psi$ is a faithful character of $G^{\prime \prime}$. Therefore, we have $\Lambda_{\chi} / \mathfrak{a}+\mathfrak{q} \simeq$ $\mathbb{Z} / 2 \mathbb{Z}$. We note that $\operatorname{Tor}_{1}^{\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}}\left(\Lambda / \mathfrak{a}, \Lambda_{\chi}\right) \simeq \mathfrak{a} \cap \mathfrak{q} / \mathfrak{a q}$ is annihilated by $\mathfrak{a}$ and $\mathfrak{q}$. Since the number of generators of $\mathfrak{a} \cap \mathfrak{q}$ as $\Lambda_{G^{\prime \prime \prime}}^{\chi^{\prime} \text {-module is independent }}$ of $l$, there is an $a \in \mathbb{Z}_{\geq 0}$ independent of $l, n$ in Lemma 4.1.1 such that $(\mathbb{Z} / 2 \mathbb{Z})^{a} \rightarrow \operatorname{Tor}_{1}^{\Lambda_{G^{\prime \prime}}^{\chi^{\prime}}}\left(\Lambda / \mathfrak{a}, \Lambda_{\chi}\right)$ is surjective. Since $J_{l} \otimes \Lambda_{\chi}$ has no non-trivial finite $\Lambda_{\chi}$-submodule by Lemma 4.1.4, we get an exact sequence

$$
(\mathbb{Z} / 2 \mathbb{Z})^{a} \longrightarrow I_{l} \otimes \Lambda_{\chi} \longrightarrow I_{l} \otimes \Lambda_{\chi} / \mathcal{F}_{\Lambda_{\chi}}\left(I_{l} \otimes \Lambda_{\chi}\right) \longrightarrow 0
$$

 term on the right has Fitting ideal equal to $\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi}$. By Lemma 7.1 in [19], we have

$$
2^{a} \prod_{\lambda \in S_{l}} E_{\lambda}^{\chi} \in \operatorname{Fitt}_{\Lambda_{\chi}}\left(I_{l} \otimes \Lambda_{\chi}\right)
$$

This completes the proof of Lemma 4.1.5.

Finally, we prove the Iwasawa main conjecture for $p=2$ assuming (4.1) and (4.2) in Proposition 4.1.2 for any $n, l$.

Proof. We consider the following exact sequence

$$
0 \longrightarrow I_{l} \longrightarrow \mathcal{X}_{H_{\infty}, S \cup S_{l}} /(\tau-1) \mathcal{X}_{H_{\infty}, S \cup S_{l}} \longrightarrow \mathcal{X}_{k_{\infty}^{\chi}, S} \longrightarrow 0 .
$$

Considering $\otimes_{\left.\mathbb{Z}_{2}\left[k \chi_{\infty}^{\chi} / k\right]\right]} \Lambda_{\chi}$, we get an exact sequence

$$
I_{l} \otimes \Lambda_{\chi} \longrightarrow \mathcal{X}_{H_{\infty}, S \cup S_{l}} /(\tau-1) \mathcal{X}_{H_{\infty}, S \cup S_{l}} \otimes \Lambda_{\chi} \longrightarrow \mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi} \longrightarrow 0 .
$$

We consider the case $\chi^{\prime} \neq \mathbf{1}$. Corollary 4.1.3 and Lemma 4.1.5 imply that

$$
2^{a} \prod_{\lambda \in S_{l}} E_{\lambda}^{\chi} \operatorname{Fitt}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset\left(\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi} \frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}, 2^{n-r}\right) .
$$

Since $\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi}$ is not a zero divisor, we have

$$
2^{a} \operatorname{Fitt}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}, 2^{n-r}\right)
$$

for all $n>r$. Consider $n \rightarrow \infty$, we have

$$
2^{a} \operatorname{Fitt}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) .
$$

Since there is a $b \in \mathbb{Z}_{\geq 1}$ such that

$$
2^{b} \operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset \operatorname{Fitt}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right),
$$

we have

$$
\begin{equation*}
2^{a+b} \operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) \tag{4.3}
\end{equation*}
$$

for $\chi^{\prime} \neq 1$.
Next, we consider the case $\chi^{\prime}=\mathbf{1}$. By a similar method, we have

$$
2^{a+b}(\gamma-1) \operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset(\gamma-1)\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) .
$$

If $\chi \neq \mathbf{1}$, Theorem 2.3.6 implies that

$$
\begin{equation*}
2^{a+b} \operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) . \tag{4.4}
\end{equation*}
$$

We consider the case $\chi=\mathbf{1}$. Assume that Leopoldt's conjecture is valid. Then Theorem 2.3.6 implies that $\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi} \notin \Lambda_{\chi}$. Therefore, $(\gamma-1)$ does
not divide $\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}$ in $\Lambda_{\chi}$. Since both $2^{a+b}(\gamma-1) \operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right)$ and $(\gamma-1)\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right)$ are principal ideals and $\Lambda_{\chi}$ is UFD, we have

$$
\begin{equation*}
2^{a+b} \operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \subset(\gamma-1)\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) \tag{4.5}
\end{equation*}
$$

for $\chi=1$.
For any finitely generated torsion $\Lambda_{\chi}$-module $M$, we denote by $\lambda(M)$ (resp. $\mu(M))$ the $\lambda$-invariant (resp. $\mu$-invariant) of $M$. Similarly, for any element $f \in \Lambda_{\chi}$, we denote by $\lambda(f)$ (resp. $\mu(f)$ ) the $\lambda$-invariant (resp. $\mu$ invariant) of $f$. For any finite abelian extension $K / k$ such that $K$ is a totally real field, we have

$$
\begin{aligned}
& \sum_{\chi \in \operatorname{Gal(K/k)}} \lambda\left(\mathcal{X}_{k_{\infty}, S} \otimes \Lambda_{\chi}\right) \\
&=\sum_{\substack{\chi \in \operatorname{Gal(K/k)} \\
\chi \neq 1}} \lambda\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right)+\lambda\left((\gamma-1) \frac{1}{2^{d}} \Phi_{k_{\infty} / k}^{1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{\chi \in \operatorname{Gal(K/k)}} \mu\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right) \\
= & \sum_{\substack{\chi \in \widehat{\operatorname{Gal}(K / k)} \\
\chi \neq 1}} \mu\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right)+\mu\left((\gamma-1) \frac{1}{2^{d}} \Phi_{k_{\infty} / k}^{1}\right) .
\end{aligned}
$$

by the analytic class number formula. (see [5]). Therefore, (4.3), (4.4) and (4.5) imply that

$$
\begin{gathered}
\operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right)=\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) \quad(\chi \neq \mathbf{1}) \\
\operatorname{char}_{\Lambda_{\chi}}\left(\mathcal{X}_{k_{\infty}^{\chi}, S} \otimes \Lambda_{\chi}\right)=(\gamma-1)\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right) \quad(\chi=\mathbf{1})
\end{gathered}
$$

This completes the proof of the Iwasawa main conjecture for $p=2$ assuming (4.1) and (4.2) in Proposition 4.1.2 for any $n, l$ in Lemma 4.1.1.

Lemma 4.1.6. (i) Assume that $\chi^{\prime} \neq 1$. If $\mu\left(\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right)=0$, then the conditions (4.1) in Proposition 4.1.2 is valid for any $n, l$ in Lemma 4.1.1.
(ii) Assume that $\chi^{\prime}=1$. If $\mu\left((\gamma-1) \frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi}\right)=0$, then the conditions (4.2) in Proposition 4.1.2 is valid for any $n, l$ in Lemma 4.1.1.

Proof. We denote $P^{\prime}$ by the prime ideal of $\mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime \prime} \times \Gamma\right]\right]$ generated by 2 and $\sigma-1$ for all $\sigma \in G^{\prime \prime}$. By the assumptions, we have

$$
\begin{aligned}
\frac{1}{2^{d}} \Phi_{k_{\infty}^{\chi} / k}^{\chi^{\prime}} & \notin P^{\prime} \quad\left(\text { if } \chi^{\prime} \neq 1\right) \\
(\gamma-1) \frac{1}{2^{d}} \Phi_{k_{\infty} / k}^{\chi^{\prime}} & \notin P^{\prime}\left(\text { if } \chi^{\prime}=1\right) .
\end{aligned}
$$

We consider the restriction map $c_{H_{\infty} / k_{\infty}^{\chi}}: \Lambda_{G^{\prime}}^{\chi^{\prime}} \rightarrow \mathbb{Z}_{2}\left[\operatorname{Im}\left(\chi^{\prime}\right)\right]\left[\left[G^{\prime \prime} \times \Gamma\right]\right]$. Then, we have

$$
c_{H_{\infty} / k_{\infty}^{\chi}}\left(\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}}\right)=\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi^{\prime}} \frac{1}{2^{d}} \Phi_{k_{\infty} / k}^{\chi^{\prime}}
$$

and $c_{H_{\infty} / k_{\infty}^{\chi}}(P) \subset P^{\prime}$. Since $\prod_{\lambda \in S_{l}} E_{\lambda}^{\chi^{\prime}} \notin P^{\prime}$, we have

$$
\begin{aligned}
\frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}} & \notin P\left(\text { if } \chi^{\prime} \neq 1\right) \\
(\gamma-1) \frac{1}{2^{d}} \Phi_{H_{\infty} / k}^{\chi^{\prime}} & \notin P\left(\text { if } \chi^{\prime}=1\right)
\end{aligned}
$$

These imply that the conditions (4.1), (4.2) in Proposition 4.1.2 are valid for any $n, l$ in Lemma 4.1 .1 for the simple reason that the right hand sides in $(4.1),(4.2)$ are then the unit ideals.

## Chapter 5

## Fitting ideal of the 2-part of the ideal class group

### 5.1 Stickelberger ideal

In this section, we define the Stickelberger ideal $\Theta_{K / k}^{-}$, which is in fact similar to the Stickelberger ideal defined by Kurihara in [19]. Let $k$ be a totally real field, and $K / k$ a finite abelian extension. Assume that $K$ is a CM-field. Put $G=\operatorname{Gal}(K / k)$. For any subfield $k \subset M \subset K$ and any ring $R$, we denote by $c_{K / M}$ the map $c_{K / M}: R[\operatorname{Gal}(K / k)] \rightarrow R[\operatorname{Gal}(M / k)]$ induced by the natural restriction map $\operatorname{Gal}(K / k) \rightarrow \operatorname{Gal}(M / k)$. In this situation,

$$
\nu_{K / M}: R[\operatorname{Gal}(M / k)] \longrightarrow R[G]
$$

denotes the $R$-linear homomorphism defined by

$$
\sigma \longmapsto \sum_{c_{K / M}(\tau)=\sigma} \tau
$$

for $\sigma \in \operatorname{Gal}(M / k)$ where $\tau$ ranges over the elements of $G$ such that $c_{K / M}(\tau)=$ $\sigma$. Let $j$ be the complex conjugation in $G$.

Definition 5.1.1. We define
$\Theta_{K / k}^{\prime}=\left\langle\frac{1}{2^{d-1}} \nu_{K / M}\left(\theta_{M / k}\right)\right| k \subset M \subset K, M$ is a CM field $\rangle \subset(1-j) \mathbb{Q}_{2}[G]$,
where $d=[k: \mathbb{Q}], \theta_{M / K}$ is the Stickelberger element defined in section 1 . We note that $\Theta_{K / k}^{\prime}$ is a $\mathbb{Z}_{2}[G]$-submodule of $\mathbb{Q}_{2}[G]$.

We write $G=\operatorname{Gal}(K / k)=G^{\prime} \times \Delta$ where $G^{\prime}$ is a 2-group and the order of $\Delta$ is odd. Assume that $G^{\prime}$ is cyclic. We denote by $W(K)$ all roots of unity contained in $K$. Let $K^{+}$be the maximal real subfield of $K$. For any number field $N$, we denote by $A_{N}$ the 2-part of the ideal class group of $N$. Let $a_{1, K}=0$ (resp., 1) if the ideal extension map $A_{K^{+}} \rightarrow A_{K}$ is injective (resp., not injective), and $a_{2, K}=0$ (resp., 1) if Hasse's unit index $Q(K)$ is equal to 1 (resp., 2). Put $\mathbb{Z}_{2}[G]^{-}=\mathbb{Z}_{2}[G] /(1+j)$. For any element $a \in \mathbb{Z}_{2}[G]$, we denote by $\bar{a}$ the image of $a$ of the natural surjective map $\mathbb{Z}_{2}[G] \rightarrow \mathbb{Z}_{2}[G]^{-}$. We denote by $I_{G}$ the ideal of $\mathbb{Z}_{2}[G]$ generated by 2 and $\sigma-1$ for all $\sigma \in G$. We consider the map $f$

$$
\begin{aligned}
f:(1-j) \mathbb{Z}_{2}[G] & \longrightarrow \mathbb{Z}_{2}[G]^{-} \\
(1-j) a & \longmapsto \bar{a} .
\end{aligned}
$$

This map is well defined and an isomorphism of $\mathbb{Z}_{2}[G]$-modules. We define

$$
\Theta_{K / k}=I_{G}^{a_{1, K}+a_{2, K}} \mathrm{Ann}_{\mathbb{Z}_{2}[G]}\left(W(K) \otimes \mathbb{Z}_{2}\right) \Theta_{K / k}^{\prime}
$$

By Deligne and Ribet [4], we know that $\Theta_{K / k} \subset(1-j) \mathbb{Z}_{2}[G]$.
Definition 5.1.2. We define

$$
\Theta_{K / k}^{-}=f\left(\Theta_{K / k}\right) .
$$

Remark 5.1.3. If $k=\mathbb{Q}$ and $a_{1, K}=0$ (i.e., the map $A_{K^{+}} \rightarrow A_{K}$ is injective), $\Theta_{K / \mathbb{Q}}$ coincides with the Stickelberger ideal defined by Kurihara (see section 2 in [19]).

We fix a faithful character $\psi$ of $G^{\prime}$. Since $G^{\prime}$ is cyclic, we have

$$
\mathbb{Z}_{2}[G]^{-} \simeq \bigoplus_{\chi \in \hat{\Delta} / \sim} \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)] .
$$

For any $\chi \in \hat{\Delta}=\operatorname{Hom}\left(\Delta, \overline{\mathbb{Q}}_{p}\right)$, we denote by $f_{\chi}$ the map

$$
f_{\chi}: \mathbb{Z}_{2}[G]^{-} \longrightarrow \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]
$$

induced by $\chi \psi$. For any $\chi \in \hat{\Delta}$, put $K^{\chi \psi}=K^{\operatorname{ker} \chi \psi}$. We consider the value of $L$-function at $s=0$ as the following

$$
L\left(0, \chi^{-1} \psi^{-1}\right)=\sum_{\sigma \in \operatorname{Gal}(K \chi \psi / k)} \zeta(0, \sigma) \chi^{-1} \psi^{-1}(\sigma),
$$

where $\zeta(0, \sigma)$ is the partial zeta function defined in section 1 .

Lemma 5.1.4. If $\chi \neq 1$, we have

$$
f_{\chi}\left(\Theta_{K / k}^{-}\right)=\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)
$$

If $\chi=1$, we have

$$
f_{\chi}\left(\Theta_{K / k}^{-}\right)=\left(\pi^{a_{1, K}+a_{2, K}+\operatorname{ord}_{2}\left(\sharp W_{K}\right)} \frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right),
$$

where $\pi$ is a uniformizer of $\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]$.
Proof. The maps $f$ and $f_{\chi}$ extend respectively to $f:(1-j) \mathbb{Q}_{2}[G] \longrightarrow$ $\mathbb{Q}_{2}[G] /(1+j)$ and $f_{\chi}: \mathbb{Q}_{2}[G] /(1+j) \longrightarrow \mathbb{Q}_{2}(\operatorname{Im}(\chi \psi))$. Since $\psi$ is a faithful character of $G^{\prime}$, we know that $K^{\chi \psi}$ is a CM-field. Therefore, we have

$$
\frac{1}{2^{d-1}} \nu_{K / K \chi \psi}\left(\theta_{K \chi \psi / k}\right) \in \Theta_{K / k}^{\prime}
$$

by definition. Since $\left[K: K^{\chi \psi}\right]$ is odd, we have

$$
\left(f_{\chi} \circ f\left(\frac{1}{2^{d-1}} \nu_{K / K \chi \psi}\left(\theta_{K \chi \psi}\right)\right)\right)=\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)
$$

We denote by $g$ the natural map $g: \mathbb{Z}_{2}[G] \rightarrow \mathbb{Z}_{2}[G]^{-}$. If $\chi \neq \mathbf{1}$, we have

$$
f_{\chi} \circ g\left(I_{G}^{a_{1, K}+a_{2, K}} \operatorname{Ann}_{\mathbb{Z}_{2}[G]}\left(W(K) \otimes \mathbb{Z}_{2}\right)\right)=(1)
$$

Thus we have

$$
f_{\chi}\left(\Theta_{K / k}^{-}\right) \supset\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)
$$

If $\chi=1$, we have
$f_{\chi} \circ g\left(I_{G}^{a_{1, K}+a_{2, K}}\right)=\left(\pi^{a_{1, K}+a_{2, K}}\right), \quad f_{\chi} \circ g\left(\operatorname{Ann}_{\mathbb{Z}_{2}[G]}\left(W(K) \otimes \mathbb{Z}_{2}\right)\right)=\left(\pi^{\operatorname{ord}_{2}(\sharp W(K))}\right)$.
Therefore, we have

$$
f_{\chi}\left(\Theta_{K / k}^{-}\right) \supset\left(\pi^{a_{1, K}+a_{2, K}+\operatorname{ord}_{2}\left(\nexists W_{K}\right)} \frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)
$$

For any CM-field $M$ such that $k \subset M \subset K$, Lemma 2.3.1 implies that

$$
f_{\chi} \circ f\left(\frac{1}{2^{d-1}} \nu_{K / M}\left(\theta_{M / k}\right)\right) \in\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)
$$

Therefore, we have

$$
f_{\chi}\left(\Theta_{K / k}^{-}\right)=\left\{\begin{array}{l}
\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)(\chi \neq \mathbf{1}) \\
\left(\pi^{a_{1, K}+a_{2, K}+\operatorname{ord}_{2}\left(\not\left(W_{K}\right)\right.} \frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right) \quad(\chi=\mathbf{1})
\end{array}\right.
$$

This completes the proof of Lemma 5.1.4.

$$
\text { Put } A_{K}^{-}=A_{K} \otimes \mathbb{Z}_{2}[G]^{-} .
$$

Corollary 5.1.5. We have

$$
\left[\mathbb{Z}_{2}[G]^{-}: \operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right]=\left[\mathbb{Z}_{2}[G]^{-}: \Theta_{K / k}^{-}\right] .
$$

Proof. We consider the following exact sequence

$$
0 \longrightarrow \operatorname{ker}\left(A_{K^{+}} \rightarrow A_{K}\right) \longrightarrow A_{K^{+}} \longrightarrow A_{K} \longrightarrow A_{K}^{-} \longrightarrow 0 .
$$

Since $G^{\prime}$ is cyclic, $\mathbb{Z}_{2}[G]^{-}$is a direct product of discrete valuation rings. Therefore, we have

$$
\left[\mathbb{Z}_{2}[G]^{-}: \operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right]=2^{a_{1, K}} \frac{\sharp A_{K}}{\sharp A_{K^{+}}} .
$$

The analytic class number formula implies that

$$
\frac{\sharp A_{K}}{\sharp A_{K^{+}}}=2^{\operatorname{ord}_{2}}\left(\Pi_{\chi \in \Delta} \frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)+a_{2, K}+\operatorname{ord}_{2}(\sharp W(K)) .
$$

Lemma 5.1.4 implies that

$$
\begin{aligned}
{\left[\mathbb{Z}_{2}[G]^{-}: \Theta_{K / k}^{-}\right] } & =2^{\operatorname{ord}_{2}\left(\Pi_{\chi \in \Delta} \frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)+a_{1, K}+a_{2, K}+\operatorname{ord}_{2}(\sharp W(K))} \\
& =2^{a_{1, K}} \frac{\sharp A_{K}}{\sharp A_{K^{+}}} .
\end{aligned}
$$

Proposition 5.1.6. Assume that $G^{\prime}$ is cyclic. Let $\mathbf{1}$ be the trivial character of $\Delta$. Then we have

$$
f_{\mathbf{1}}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right)=f_{\mathbf{1}}\left(\Theta_{K / k}^{-}\right) .
$$

Proof. Since $\mathbb{Z}_{2}[\operatorname{Im}(\psi)]$ is a discrete valuation ring, it is sufficiently to show that

$$
\left[\mathbb{Z}_{2}[\operatorname{Im}(\psi)]: f_{\mathbf{1}}\left(\Theta_{K / k}^{-}\right)\right]=\left[\mathbb{Z}_{2}[\operatorname{Im}(\psi)]: f_{\mathbf{1}}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right)\right] .
$$

Since $\sharp \Delta$ is odd, we have $\left(A_{K}^{-}\right)_{\Delta} \simeq A_{K^{\Delta}}^{-}$by the norm argument, where $K^{\Delta}$ is the subfield of $K$ fixed by $\Delta$. Therefore, we have

$$
f_{1}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]-}\left(A_{K}^{-}\right)\right)=\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\psi)]}\left(A_{K^{\Delta}}^{-}\right) .
$$

Since $\mathbb{Z}_{2}[\operatorname{Im}(\psi)]$ is a discrete valuation ring, we have

$$
\left[\mathbb{Z}_{2}[\operatorname{Im}(\psi)]: \operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\psi)]}\left(A_{K^{\Delta}}^{-}\right)\right]=\sharp A_{K^{\Delta}}^{-}
$$

Let $K^{\Delta,+}$ be the maximal real subfield of $K^{\Delta}$ and $h_{K^{\Delta}}\left(\right.$ resp. $\left.h_{K^{\Delta,+}}\right)$ the class number of $K^{\Delta}\left(\right.$ resp. $\left.K^{\Delta,+}\right)$. By the analytic class number formula, we have

$$
\frac{h_{K^{\Delta}}}{h_{K^{\Delta,+}}}=Q\left(K^{\Delta}\right) \sharp W\left(K^{\Delta}\right) \prod_{\substack{\psi \in G^{\prime} \\ \psi: \text { odd }}} \frac{1}{2^{d}} L(0, \psi),
$$

where $Q\left(K^{\Delta}\right)$ is Hasse's unit index of $K^{\Delta}$ and $W\left(K^{\Delta}\right)$ is the group of all roots of unity contained in $K^{\Delta}$. Since $\sharp \Delta$ is odd, we have $W\left(K^{\Delta}\right) \otimes \mathbb{Z}_{2}=$ $W(K) \otimes \mathbb{Z}_{2}, Q\left(K^{\Delta}\right)=Q(K)$ by Proposition $1(\mathrm{f})$ in [22] and

$$
\operatorname{ker}\left(A_{K^{+}} \rightarrow A_{K}\right) \simeq \operatorname{ker}\left(A_{K^{\Delta,+}} \rightarrow A_{K^{\Delta}}\right)
$$

by the norm argument. Thus Lemma 5.1.4 implies that

$$
\begin{aligned}
& {\left[\mathbb{Z}_{2}[\operatorname{Im}(\psi)]: f_{\mathbf{1}}\left(\Theta_{K / k}^{-}\right)\right] } \\
= & 2 \\
= & \frac{h_{K^{\Delta}}}{\operatorname{ord}_{2}\left(Q\left(K^{\Delta}\right) \sharp W\left(K^{\Delta}\right) \prod_{\psi \not \psi \in G^{\prime}} \frac{1}{2^{d}} L(0, \psi) \sharp \operatorname{ker}\left(A_{K^{\Delta,+}} \rightarrow A_{K^{\Delta}}\right)\right)} \\
= & \sharp A_{K^{\Delta}}^{-} \\
= & {\left[\mathbb{Z}_{2}[\operatorname{Im}(\psi)]: f_{K^{\Delta,+}} \rightarrow A_{K^{\Delta}}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right)\right] . }
\end{aligned}
$$

### 5.2 Descent theory

In this section, we prove Theorem 1.2.6 in the case (3a). As we mentioned in chapter 1 , we prove Theorem 1.2 .6 in the case (3a) by a similar method as in Greither [13]. For any number field $N$, let $N_{\infty}$ be the cyclotomic $\mathbb{Z}_{2^{-}}$ extension, $N_{n}$ the $n$-th layer. Let $k$ be a totally real field, and $K / k$ a finite abelian extension. We assume that $K$ is a CM-field and $K \cap k_{\infty}=k$. We write $G=\operatorname{Gal}(K / k)=G^{\prime} \times \Delta$ where $G^{\prime}$ is a 2 -group and the order of $\Delta$ is odd. In this section, we do not assume that $G^{\prime}$ is cyclic. Let $L / k$ be a finite abelian extension such that $L$ is a CM-field, $K \subset L$, and $L / K$ is
a 2 -extension. Let $L^{+}$be the maximal real subfield of $L$. We denote by $\operatorname{Gal}(L / k)_{2}$ the 2-Sylow subgroup of $\operatorname{Gal}(L / k)$. Then we have

$$
\begin{gathered}
\mathbb{Z}_{2}[\operatorname{Gal}(L / k)] \simeq \bigoplus_{\chi \in \hat{\Delta} / \sim} \mathbb{Z}_{2}[\operatorname{Im} \chi]\left[\operatorname{Gal}(L / k)_{2}\right] \\
\mathbb{Z}_{2}[\operatorname{Gal}(L / k)] /(1+j) \simeq \bigoplus_{\chi \in \hat{\Delta} / \sim} \mathbb{Z}_{2}[\operatorname{Im} \chi]\left[\operatorname{Gal}(L / k)_{2}\right] /(1+j)
\end{gathered}
$$

For any $\mathbb{Z}_{2}[\operatorname{Gal}(L / k)]$-module $M$ and any character $\chi$ of $\Delta$, put $M^{\chi}=$ $M \otimes_{\mathbb{Z}_{2}[\operatorname{Gal}(L / k)]} \mathbb{Z}_{2}[\operatorname{Im} \chi]\left[\operatorname{Gal}(L / k)_{2}\right], M^{-}=M \otimes_{\mathbb{Z}_{2}[\operatorname{Gal}(L / k)]} \mathbb{Z}_{2}[\operatorname{Gal}(L / k)] /(1+$ $j)$ and $M^{-, \chi}=M \otimes_{\mathbb{Z}_{2}[\operatorname{Gal}(L / k)]} \mathbb{Z}_{2}[\operatorname{Im} \chi]\left[\operatorname{Gal}(L / k)_{2}\right] /(1+j)$.

Lemma 5.2.1. Let $\chi$ be a non-trivial character of $\Delta$. Then the norm map of the ideal class group

$$
A_{L_{n}}^{\chi} \longrightarrow A_{L}^{\chi}
$$

is surjective for all $n \geq 0$.
Proof. To prove this Lemma, it is sufficient to show that the norm map $A_{L_{n+1}}^{\chi} \rightarrow A_{L_{n}}^{\chi}$ is surjective for all $n \in \mathbb{Z}_{\geq 0}$. Let $H_{n}$ be the unramified extension of $L_{n}$ corresponding to $A_{L_{n}}$.

If there is a prime of $L_{n}$ which ramifies in $L_{n+1} / L_{n}$, we have $H_{n} \cap L_{n+1}=$ $L_{n}$. Therefore, we see that the norm map $A_{L_{n+1}} \rightarrow A_{L_{n}}$ is surjective.

If $L_{n+1} / L_{n}$ is an unramified extension, we have $L_{n+1} \subset H_{n}$. Therefore, we get an exact sequence

$$
A_{L_{n+1}} \longrightarrow A_{L_{n}} \longrightarrow \operatorname{Gal}\left(L_{n+1} / L_{n}\right) \longrightarrow 0
$$

Since $L_{n+1} / k$ is an abelian extension, $\Delta$ acts on $\operatorname{Gal}\left(L_{n+1} / L_{n}\right)$ trivially. Therefore, we have $\operatorname{Gal}\left(L_{n+1} / L_{n}\right)^{\chi}=0$ for any non-trivial character $\chi$ of $\Delta$. This implies that the norm map $A_{L_{n}}^{\chi} \rightarrow A_{L}^{\chi}$ is surjective for any nontrivial character $\chi$ of $\Delta$.

For any number field $N$, we denote by $P_{N}$ (resp. $\left.P_{N, \text { fin }}\right)$ the set of primes (resp. finite primes) of $N$.

Consider the map

$$
i_{L / K}: \hat{H}^{0}\left(L^{+} / K^{+}, \prod_{w^{+} \in P_{L^{+}}} E_{L^{+}, w^{+}}\right) \longrightarrow \hat{H}^{0}\left(L / K, \prod_{w \in P_{L}} E_{L, w}\right)
$$

induced by the natural injective map

$$
\prod_{w^{+} \in P_{L^{+}}} E_{L^{+}, w^{+}} \longrightarrow \prod_{w_{n} \in P_{L}} E_{L, w},
$$

where $E_{L, w}$ (resp. $E_{L^{+}, w^{+}}$) is the local unit group of $L$ at $w$ (resp. $L^{+}$at $\left.w^{+}\right)$. For any $\mathbb{Z}_{2}[\operatorname{Gal}(L / K)]$-module $M$, we denote by $M_{\operatorname{Gal}(L / K)}$ the Galois coinvariant of $M$.

Proposition 5.2.2. For any non-trivial character $\chi$ of $\Delta$, we get an exact sequence

$$
0 \longrightarrow \operatorname{coker} i_{L / K}^{\chi} \longrightarrow\left(A_{L}^{-, \chi}\right)_{\operatorname{Gal}(L / K)} \longrightarrow A_{K}^{-, \chi} \longrightarrow 0 .
$$

Proof. We consider the following commutative diagram

where the map $N_{L / K}$ is the norm map of ideal class group and $N_{\operatorname{Gal}(L / K)}$ : $\mathrm{Cl}\left(K_{n}\right)_{\operatorname{Gal}(L / K)} \rightarrow \mathrm{Cl}(L)$ is the map defined by $a \mapsto \sum_{\sigma \in \operatorname{Gal}(L / K)} \sigma a$. This implies that there is an exact sequence

$$
\begin{aligned}
0 \rightarrow \operatorname{ker}\left(\mathrm{Cl}(L)_{\mathrm{Gal}(L / K)} \xrightarrow{N_{\mathrm{Gal}(L / K)}}\right. & \mathrm{Cl}(K)) \\
& \\
\hat{H}^{-1}(L / K, & \mathrm{Cl}(L))
\end{aligned} \longrightarrow \operatorname{ker}(\mathrm{Cl}(K) \rightarrow \mathrm{Cl}(L)) .
$$

It is well known that

$$
\operatorname{ker}(\mathrm{Cl}(K) \rightarrow \mathrm{Cl}(L)) \simeq \operatorname{ker}\left(H^{1}\left(L / K, E_{K_{n}}\right) \rightarrow H^{1}\left(L / K, \prod_{w \in P_{L}} E_{L, w}\right)\right)
$$

where $E_{L}$ is the unit group of $L$. Therefore, we have

$$
\begin{gather*}
\operatorname{ker}\left(\mathrm{Cl}(L)_{\mathrm{Gal}(L / K)} \xrightarrow[\rightarrow]{N_{\mathrm{Gal}(L / K)}} \mathrm{Cl}(K)\right) \\
\simeq \operatorname{ker}\left(\hat{H}^{-1}(L / K, \mathrm{Cl}(L)) \rightarrow H^{1}\left(L / K, E_{L}\right)\right) . \tag{5.1}
\end{gather*}
$$

We consider the following exact sequence

$$
0 \longrightarrow E_{L} \longrightarrow \prod_{w \in P_{L}} E_{L, w} \longrightarrow C_{L} \longrightarrow \mathrm{Cl}(L) \longrightarrow 0
$$

where $C_{L}$ is the idele class group of $L$. Put $M=\operatorname{ker}\left(C_{L} \rightarrow \mathrm{Cl}(L)\right)$. By Tate-Nakayama's theorem, we have

$$
\hat{H}^{-1}\left(L / K, C_{L}\right)=\hat{H}^{-3}(L / K, \mathbb{Z})=H_{2}(L / K, \mathbb{Z})=\wedge^{2} \operatorname{Gal}(L / K)
$$

By class field theory, we have $\hat{H}^{0}\left(L / K, C_{K_{n}}\right) \simeq \operatorname{Gal}(L / K)$. Thus taking Tate cohomology of $A=\operatorname{Gal}(L / K)$, we get the following commutative diagram,


Since $G=\operatorname{Gal}(K / k)$ acts on $A=\operatorname{Gal}(L / K)$ trivially, we have $A^{\chi}=$ $\operatorname{Gal}(L / K)^{\chi}=0$ and $\wedge^{2} A^{\chi}=0$ for any non-trivial character $\chi$ of $\Delta$. Therefore, we get an exact sequence

$$
\begin{aligned}
\hat{H}^{0}\left(L / K, E_{L} \otimes \mathbb{Z}_{2}\right)^{\chi} & \longrightarrow \hat{H}^{0}\left(L / K, \prod_{w \in P_{L}} E_{L, w} \otimes \mathbb{Z}_{2}\right)^{\chi} \longrightarrow \\
\hat{H}^{-1}\left(L / K, A_{L}\right)^{\chi} & \longrightarrow H^{1}\left(L / K, E_{L} \otimes \mathbb{Z}_{2}\right)^{\chi}
\end{aligned}
$$

for any non-trivial character $\chi$ of $\Delta$.
By this exact sequence, (5.1) and Lemma 5.2.1, we get an exact sequence,

$$
\begin{aligned}
\hat{H}^{0}\left(L / K, E_{L} \otimes \mathbb{Z}_{2}\right)^{\chi} & \longrightarrow \hat{H}^{0}\left(L / K, \prod_{w \in P_{L}} E_{L, w} \otimes \mathbb{Z}_{2}\right)^{\chi} \\
& \longrightarrow\left(A_{L}^{\chi}\right)_{\operatorname{Gal}(L / K)} \xrightarrow{N_{\operatorname{Gal}(L / K)}} A_{K}^{\chi} \longrightarrow 0
\end{aligned}
$$

Put

$$
\begin{gathered}
S_{L}=\hat{H}^{0}\left(L / K, E_{L} \otimes \mathbb{Z}_{2}\right), \\
S_{L^{+}}=\hat{H}^{0}\left(L^{+} / K^{+}, E_{L^{+}} \otimes \mathbb{Z}_{2}\right),
\end{gathered}
$$

$$
\begin{gathered}
T_{L}=\hat{H}^{0}\left(L / K, \prod_{w \in P_{L}} E_{L, w} \otimes \mathbb{Z}_{2}\right) \\
T_{L^{+}}=\hat{H}^{0}\left(L^{+} / K^{+}, \prod_{w^{+} \in P_{L^{+}}} E_{L^{+}, w^{+}} \otimes \mathbb{Z}_{2}\right)
\end{gathered}
$$

By the same argument, we get an exact sequence for $L^{+} / K^{+}$. Therefore, we get the following commutative diagram,

for any non-trivial character $\chi$ of $\Delta$. Since $\operatorname{ker}\left(A_{K^{+}} \rightarrow A_{K}\right)$ is isomorphic to 0 or $\mathbb{Z} / 2 \mathbb{Z}, G=\operatorname{Gal}(K / k)$ acts on $\operatorname{ker}\left(A_{K^{+}} \rightarrow A_{K}\right)$ trivially. This implies that $A_{K^{+}}^{\chi} \rightarrow A_{K}^{\chi}$ is injective for any non-trivial character $\chi$ of $\Delta$. We consider an exact sequence

$$
0 \longrightarrow E_{L^{+}} \otimes \mathbb{Z}_{2} \longrightarrow E_{L} \otimes \mathbb{Z}_{2} \xrightarrow{1-j} E_{L}^{1-j} \otimes \mathbb{Z}_{2} \longrightarrow 0
$$

Since $E_{L}^{1-j} \subset W(L)$, we have $\left(E_{L}^{1-j} \otimes \mathbb{Z}_{2}\right)^{\chi}=0$ for any non-trivial character $\chi$ of $\Delta$. Therefore, we have $S_{L^{+}}^{\chi}=\hat{H}^{0}\left(L^{+} / K^{+}, E_{L^{+}} \otimes \mathbb{Z}_{2}\right)^{\chi} \simeq \hat{H}^{0}\left(L / K, E_{L} \otimes\right.$ $\left.\mathbb{Z}_{2}\right)^{\chi}=S_{L}^{\chi}$. These imply that we get the following exact sequence by the snake lemma,

$$
0 \longrightarrow \operatorname{coker}^{\chi}{ }_{L / K}^{\chi} \longrightarrow\left(A_{L}^{-}\right)_{\Gamma_{n}}^{\chi} \longrightarrow A_{K}^{-, \chi} \longrightarrow 0
$$

Put $\Gamma_{n}=\operatorname{Gal}\left(K_{n} / K\right), \Gamma_{n}^{+}=\operatorname{Gal}\left(K_{n}^{+} / K^{+}\right)$. We consider the set of primes $\mathscr{S}_{2}^{\prime}(k)$ of $k$ lying above 2 which ramify in $K / K^{+}$but do not ramify in $K_{\infty} / K_{\infty}^{+}$. For any prime $v$ of $k$ above 2 , we take $n_{v} \geq 0$ such that all primes above $v$ totally ramify in $K_{\infty} / K_{n_{v}}$ and are unramified in $K_{n_{v}} / K$.

Lemma 5.2.3. For all sufficiently large $n \gg 0$, we have

$$
\operatorname{coker} i_{K_{n} / K} \simeq \bigoplus_{\substack{v \notin \mathscr{S}_{2}^{\prime}(k) \\ v \mid 2, \text { prime of } k}}\left(\mathbb{Z} / 2^{n-n_{v}} \mathbb{Z}\left[G / D_{v}\right]\right)^{-}
$$

where $D_{v}$ is the decomposition group of $v$ in $G=\operatorname{Gal}(K / k)$.

Proof. Since only primes above 2 ramify in $K_{\infty} / K$ and $K_{\infty}^{+} / K^{+}$, we have

$$
\begin{aligned}
\hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \in P_{K_{n}}} E_{K_{n}, w_{n}}\right) & \simeq \prod_{\substack{v \mid 2 \\
\text { prime of } k}} \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right) \\
\hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \in P_{K_{n}^{+}}} E_{K_{n}^{+}, w_{n}^{+}}\right) & \simeq \prod_{\substack{v \mid 2 \\
\text { prime of } k}} \hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \mid v} E_{K_{n}^{+}, w_{n}^{+}}\right) .
\end{aligned}
$$

By local class field theory, we have

$$
\hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right) \simeq \bigoplus_{\substack{w \mid v \\ \text { prime of } K}} \operatorname{Gal}\left(K_{n} / K_{n_{v}}\right)
$$

for any $n \geq n_{v}$.
If $v \notin \mathscr{S}_{2}^{\prime}(k)$, we know that all primes above $v$ totally ramify in $K_{\infty}^{+} / K_{n_{v}}^{+}$ and are unramified in $K_{n_{v}}^{+} / K^{+}$. Therefore, by local class field theory, we have

$$
\hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \mid v} E_{K_{n}^{+}, w_{n}^{+}}\right) \simeq \bigoplus_{\substack{w^{+} \mid v \\ \text { prime of } K^{+}}} \operatorname{Gal}\left(K_{n}^{+} / K_{n_{v}}^{+}\right)
$$

for any $n \geq n_{v}$. Consider the map

$$
j_{K_{n} / K}: \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \in P_{K_{n}}} E_{K_{n}, w_{n}}\right) \longrightarrow \hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \in P_{K_{n}^{+}}} E_{K_{n}^{+}, w_{n}^{+}}\right)
$$

induced by the norm map

$$
\prod_{w_{n} \in P_{K_{n}}} E_{K_{n}, w_{n}} \longrightarrow \prod_{w_{n}^{+} \in P_{K_{n}^{+}}} E_{K_{n}^{+}, w_{n}^{+}}
$$

By local class field theory, the map $j_{K_{n} / K}$ is the restriction map. Therefore, $j_{K_{n} / K}$ is an isomorphism for all $n \geq n_{v}$. Since $i_{K_{n} / K} \circ j_{K_{n} / K}(x)=(1+j) x$
for any $x \in \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right)$, we have

$$
\begin{aligned}
& \operatorname{coker}\left(i_{K_{n} / K}: \hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \mid v} E_{K_{n}^{+}, w_{n}^{+}}\right) \longrightarrow \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right)\right) \\
= & \operatorname{coker}\left(i_{K_{n} / K} \circ j_{K_{n} / K}: \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right) \longrightarrow\left(\hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right)\right)\right. \\
= & \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right) /(1+j) \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right) \\
\simeq & \bigoplus_{\substack{\text { prime of } K \\
w \mid v}} \operatorname{Gal}\left(K_{n} / K_{n_{v}}\right) /(1+j) \bigoplus_{\substack{\text { prime of } K \\
w \mid v}} \operatorname{Gal}\left(K_{n} / K_{n_{v}}\right) \\
\simeq & \left(\mathbb{Z} / 2^{n-n_{v}} \mathbb{Z}\left[G / D_{v}\right]\right)^{-}
\end{aligned}
$$

for all $n \geq n_{v}$.
If $v \in \mathscr{S}_{2}^{\prime}(k)$, all primes above $v$ totally ramify in $K_{\infty}^{+} / K_{n_{v}-1}^{+}$and are unramified in $K_{n_{v}-1}^{+} / K^{+}$. By local class field theory, we have

$$
\hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \mid v} E_{K_{n}^{+}, w_{n}^{+}}\right) \simeq \bigoplus_{\substack{w^{+} \mid v \\ \text { prime of } K^{+}}} \operatorname{Gal}\left(K_{n}^{+} / K_{n_{v}-1}^{+}\right),
$$

for any $n \geq n_{v}$. In this case, the map

$$
i_{K_{n} / K}: \hat{H}^{0}\left(\Gamma_{n}^{+}, \prod_{w_{n}^{+} \mid v} E_{K_{n}^{+}, w_{n}^{+}}\right) \longrightarrow \hat{H}^{0}\left(\Gamma_{n}, \prod_{w_{n} \mid v} E_{K_{n}, w_{n}}\right)
$$

is surjective. This completes the proof of Lemma 5.2.3.

$$
\text { Put } \Gamma=\operatorname{Gal}\left(K_{\infty} / K\right), \mathcal{G}=\operatorname{Gal}\left(K_{\infty} / k\right), \Omega=\mathbb{Z}_{2}[[\mathcal{G}]], \Omega^{-}=\Omega /(1+j) .
$$

Then we have

$$
\Omega \simeq \bigoplus_{\chi \in \hat{\Delta} / \sim} \mathbb{Z}_{2}[\operatorname{Im} \chi]\left[\left[\Gamma \times G^{\prime}\right]\right] .
$$

For any $\Omega$-module $M$ and any character $\chi$ of $\Delta$, put $M^{\chi}=M \otimes_{\Omega} \mathbb{Z}_{2}[\operatorname{Im} \chi][[\Gamma \times$ $\left.\left.G^{\prime}\right]\right], M^{-}=M \otimes_{\Omega} \Omega^{-}$and $M^{-, \chi}=M \otimes_{\Omega} \mathbb{Z}_{2}[\operatorname{Im} \chi]\left[\left[\Gamma \times G^{\prime}\right]\right] /(1+j)$. We consider the unramified Iwasawa module $X_{K_{\infty}}=\lim A_{n}$, where the projective limit is taken by the norm map of the ideal class group.

Corollary 5.2.4. For any non-trivial character $\chi$ of $\Delta$, there is an exact sequence

$$
0 \longrightarrow \underset{\substack{v \notin \mathscr{Y}^{\prime}(k) \\ v \mid 2, \text { prime of } k}}{ } \mathbb{Z}_{2}\left[G / D_{v}\right]^{-, \chi} \longrightarrow\left(X_{K_{\infty}}^{-, \chi}\right)_{\Gamma} \longrightarrow A_{K}^{-, \chi} \longrightarrow 0
$$

Proof. By Proposition 5.2 .2 for $L=K_{n}$, we get an exact sequence

$$
0 \longrightarrow \operatorname{coker} i_{K_{n} / K}^{\chi} \longrightarrow\left(A_{K_{n}}^{-, \chi}\right)_{\Gamma_{n}} \longrightarrow A_{K}^{-, \chi} \longrightarrow 0
$$

Taking the projective limit of this exact sequence and Lemma 5.2.3, we get the exact sequence in Corollary 5.2.4.

Let $\mathscr{S}_{2}(k)$ be the set of primes of $k$ above 2 , which ramify in $K_{\infty} / K_{\infty}^{+}$. We denote by $X_{K_{\infty}, \text { fin }}^{-, \chi}$ the maximal finite $\mathbb{Z}_{2}[\operatorname{Im}(\chi)]\left[\left[G^{\prime} \times \Gamma\right]\right]$-submodule of $X_{K_{\infty}}^{-, \chi}$.

Lemma 5.2.5. For any non-trivial character $\chi$ of $\Delta$, we have

$$
X_{K_{\infty}, \operatorname{ini}}^{-, \chi} \simeq \bigoplus_{v \in \mathscr{S}_{2}(k)} \mathbb{Z} / 2 \mathbb{Z}\left[\mathcal{G} / D_{v, \infty}\right]^{\chi}
$$

where $D_{v, \infty}$ is the decomposition group of $v$ in $\mathcal{G}$.
Proof. For any extension $N / k$, we denote by $\mathscr{S}_{2}(N)$ the set of prime of $N$ lying above $\mathscr{S}_{2}(k)$. Put

$$
\begin{gathered}
A_{K_{n}, \mathscr{S}_{2}}=\operatorname{coker}\left(K_{n}^{\times} \stackrel{\oplus \operatorname{ord}_{w_{n}}}{\longrightarrow} \bigoplus_{w_{n} \notin \mathscr{S}_{2}\left(K_{n}\right) \cup S_{\infty}\left(K_{n}\right)} \mathbb{Z}\right) \otimes \mathbb{Z}_{2}, \\
A_{K_{n}^{+}, \mathscr{S}_{2}}=\operatorname{coker}\left(K_{n}^{+\times} \stackrel{\oplus \operatorname{ord}_{w_{n}^{+}}}{\longrightarrow} \bigoplus_{w_{n}^{+} \notin \mathscr{S}_{2}\left(K_{n}^{+}\right) \cup S_{\infty}\left(K_{n}^{+}\right)} \mathbb{Z}\right) \otimes \mathbb{Z}_{2}, \\
D_{n, \mathscr{S}_{2}}=\operatorname{ker}\left(A_{K_{n}} \rightarrow A_{\left.K_{n}, \mathscr{S}_{2}\right),} D_{n, \mathscr{S}_{2}}^{+}=\operatorname{ker}\left(A_{K_{n}^{+}} \rightarrow A_{K_{n}^{+}, \mathscr{S}_{2}}\right),\right.
\end{gathered}
$$

where $S_{\infty}\left(K_{n}\right)$ (resp. $S_{\infty}\left(K_{n}^{+}\right)$) is the set of infinity primes of $K_{n}$ (resp. $K_{n}^{+}$).

It is well known that the kernel of the map $A_{K_{n}^{+}, \mathscr{L}_{2}} \rightarrow A_{K_{n}, \mathscr{S}_{2}}$ coincides with the kernel of the map

$$
H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}, \mathscr{S}_{2}}\right) \longrightarrow H^{1}\left(K_{n} / K_{n}^{+}, \prod_{w \notin \mathscr{S}_{2}\left(K_{n}\right) \cup S_{\infty}\left(K_{n}\right)} E_{K_{n}, w_{n}}\right),
$$

where $E_{K_{n}, \mathscr{S}_{2}}$ is the $\mathscr{S}_{2}\left(K_{n}\right) \cup S_{\infty}\left(K_{n}\right)$-unit group of $K_{n}$. Since $\operatorname{Gal}\left(K_{n} / K_{n}^{+}\right)$ acts on $\mathscr{S}_{2}\left(K_{n}\right)$ trivially, we have $E_{K_{n}, \mathscr{S}_{2}}^{1-j} \subset W\left(K_{n}\right)$. We consider the following exact sequence,

$$
0 \longrightarrow E_{K_{n}^{+}, \mathscr{S}_{2}} \longrightarrow E_{K_{n}, \mathscr{S}_{2}} \longrightarrow E_{K_{n}, \mathscr{S}_{2}}^{1-j} \longrightarrow 0
$$

Taking Galois cohomology of $\operatorname{Gal}\left(K_{n} / K_{n}^{+}\right)$, we get an exact sequence

$$
H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}^{+}, \mathscr{Y}_{2}}\right) \longrightarrow H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}, \mathscr{S}_{2}}\right) \longrightarrow H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}, \mathscr{S}_{2}}^{1-j}\right) .
$$

Since $K_{n} / K_{n}^{+}$is a cyclic extension, we have

$$
H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}^{+}, \mathscr{\mathscr { C }}_{2}}\right)=\frac{\operatorname{ker}\left(1+j: E_{K_{n}^{+}, \mathscr{S}_{2}} \rightarrow E_{K_{n}^{+}, \mathscr{S}_{2}}\right)}{E_{K_{n}^{+}, \mathscr{S}_{2}}^{1-}} .
$$

Since $\operatorname{ker}\left(1+j: E_{K_{n}^{+}, \mathscr{L}_{2}} \rightarrow E_{K_{n}^{+}, \mathscr{L}_{2}}\right)=\{ \pm 1\}$ and $E_{K_{n}, \mathscr{S}_{2}}^{1-j} \subset W\left(K_{n}\right)$, we have
$H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}^{+}, \mathscr{S}_{2}}\right)^{\chi}=H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}, \mathscr{S}_{2}}\right)^{\chi}=H^{1}\left(K_{n} / K_{n}^{+}, E_{K_{n}, \mathscr{Y}_{2}}^{1-j}\right)^{\chi}=0$ for any non-trivial character $\chi$ of $\Delta$. Therefore, the map $A_{K_{n}^{+}, \mathscr{S}_{2}}^{\chi} \rightarrow A_{K_{n}, \mathscr{Y}_{2}}^{\chi}$ is injective.

We consider the following commutative diagram,


By the snake lemma, we get an exact sequence

$$
0 \longrightarrow D_{n, \mathscr{S}_{2}}^{-, \chi} \longrightarrow A_{K_{n}}^{-, \chi} \longrightarrow A_{K_{n}, \mathscr{S}_{2}}^{-, \chi} \longrightarrow 0
$$

for any non-trivial character $\chi$ of $\Delta$. Taking the projective limit, we get an exact sequence

$$
0 \longrightarrow \lim _{\check{2}} D_{n, \mathscr{S}_{2}}^{-, \chi} \longrightarrow X_{K_{\infty}}^{-, \chi} \longrightarrow \lim _{\check{2}} A_{K_{n}, \mathscr{S}_{2}}^{-, \chi} \longrightarrow 0 .
$$

Theorem 1.2.1 (i) implies that $\varliminf_{\succsim} A_{K_{n}, \mathscr{S}_{2}}^{-, \chi}$ has no non-trivial finite $\mathbb{Z}_{2}[\operatorname{Im}(\chi)]\left[\left[G^{\prime} \times\right.\right.$ $\Gamma]$ ]-submodule. Therefore, we have

$$
X_{K_{\infty}, \text { fin }}^{-, \chi} \simeq \lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}^{-, \chi} .
$$

Lemma 2.1 and Remark 2.4 in [1] imply that

$$
\lim _{\leftrightarrows} D_{n, \mathscr{S}_{2}}^{-, \chi} \simeq \bigoplus_{v \in \mathscr{\mathscr { A }}_{2}(k)} \mathbb{Z} / 2 \mathbb{Z}\left[\mathcal{G} / D_{v, \infty}\right]^{\chi}
$$

for any non-trivial character $\chi$ of $\Delta$. This completes the proof of lemma 5.2.5

Now we prove Theorem 1.2.6 in the case (3a).
Theorem 5.2.6. Assume that the following assumptions are satisfied.
(1) The $\mu$-invariant of $K_{\infty}$ vanishes.
(2) $G^{\prime}$ is cyclic.
(3a) No prime above 2 splits in $K / K^{+}$.
Then we have

$$
f_{\chi}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right)=f_{\chi}\left(\Theta_{K / k}^{-}\right)
$$

for any non-trivial character $\chi$ of $\Delta$, where $f_{\chi}$ is the map defined in section 5.1.

Proof. Let $\psi$ be a faithful character of $G^{\prime}, K^{\chi}$ is the subfield of $K$ attached to $\chi \psi$, i.e., $\operatorname{Gal}\left(K^{\chi} / k\right) \simeq \operatorname{Im}(\chi \psi)$. We consider the projective limit of the Stickelberger element $\theta_{K_{\infty}^{\chi} / k} \in \mathcal{Q}\left(\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(K_{\infty}^{\chi} / k\right)\right]\right]\right)$ (see section 1).

Put $\Lambda_{\chi}=\mathbb{Z}_{2}[\operatorname{Im} \chi \psi]\left[\left[\operatorname{Gal}\left(K_{\infty}^{\chi} / K^{\chi}\right)\right]\right]$. Since $\chi$ is a non-trivial character of $\Delta$, we have $\frac{1}{2^{2}} \theta_{K_{\infty}^{x} / k}^{\chi \chi} \in \Lambda_{\chi}$.

Since the $\mu(K)$-invariant vanishes, Theorem 1.2.4 and Proposition 2.5.3 imply that

$$
\operatorname{Fitt}_{\Lambda_{\chi}}\left(X_{K_{\infty}^{x}}^{-, \chi} / X_{K_{\infty}^{x}, \text { fin }}^{-\chi}\right)=\left(\frac{1}{2^{d}} \theta_{K_{\infty}^{\chi} / k}^{\chi \chi}\right) .
$$

By Lemma 5.2.5 and Proposition 2.5.3, we have

$$
\begin{equation*}
\operatorname{Fitt}_{\Lambda_{\chi}}\left(X_{K_{\infty}^{\prime}}^{-, \chi}\right)=\operatorname{Fitt}_{\Lambda_{\chi}}\left(\bigoplus_{v \in \mathscr{\mathscr { S }}_{2}(k)} \mathbb{Z} / 2 \mathbb{Z}\left[\mathcal{G} / D_{v, \infty}\right]^{\chi}\right)\left(\frac{1}{2^{d}} \theta_{K \infty}^{\chi \chi}\right) . \tag{5.2}
\end{equation*}
$$

We consider the natural restriction map $c_{\chi}: \Lambda_{\chi} \rightarrow \mathbb{Z}_{2}[\operatorname{Im} \chi \psi]$ defined by $\sigma \mapsto$ 1 for all $\sigma \in \operatorname{Gal}\left(K_{\infty}^{\chi} / K^{\chi}\right)$. Since $\left[K: K^{\chi}\right]$ is odd, we have $X_{K_{\infty}^{\chi}}^{-\chi} \simeq X_{K_{\infty}}^{-, \chi}$ by the norm argument. Let $S_{2}(k)$ be the set of primes of $k$ lying above 2. For any $v \in S_{2}(k)$, we denote by $\sigma_{v}$ a generator of the decomposition group of $v$ in $\operatorname{Gal}\left(K^{\chi} / k\right)$. Since $\mathbb{Z}_{2}[\operatorname{Im} \chi \psi]$ is a discrete valuation ring, Corollary 5.2.4 implies that

$$
\begin{aligned}
& c_{\chi}\left(\operatorname{Fitt}_{\Lambda_{\chi}}\left(X_{K \infty}^{-, \chi}\right)\right)=\operatorname{Fitt}_{\left.\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]\right]}\left(\left(X_{K_{\infty}}^{-, \chi}\right)_{\Gamma}\right) \\
= & \operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K}^{-, \chi}\right) \operatorname{Fitt}_{\left.\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]\right]}\left(\bigoplus_{\substack{v \in S_{2}(k) \\
v \not \mathscr{S}_{2}^{\prime}(k)}} \mathbb{Z}_{2}\left[G / D_{v}\right]^{-, \chi}\right) \\
= & \prod_{v \in S_{2}(k) \backslash \mathscr{S}_{2}^{\prime}(k)}\left(1-\chi \psi\left(\sigma_{v}\right)\right) \operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K}^{-, \chi}\right) .
\end{aligned}
$$

We denote by $S_{2, \text { unr }}^{\chi}$ the set of primes of $k$ above 2 which are unramified in $K^{\chi} / k$. Since $c_{\chi}\left(\theta_{K_{\infty}^{\chi} / k}^{\chi \psi}\right)=\prod_{v \in S_{2, \text { unr }}^{\chi}}\left(1-\chi \psi\left(\operatorname{Frob}_{v}^{-1}\right)\right) \theta_{K \chi / k}^{\chi \psi}$ (see Lemma 2.3.1), (5.2) implies that

$$
\begin{aligned}
& c_{\chi}\left(\operatorname{Fitt}_{\Lambda_{\chi}}\left(X_{K_{\infty}^{\chi}}^{-, \chi}\right)\right) \\
& =\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(\bigoplus_{v \in \mathscr{S}_{2}(k)} \mathbb{Z} / 2 \mathbb{Z}\left[G / D_{v}\right]^{\chi}\right) \prod_{v \in S_{2, \text { unr }}^{\chi}}\left(1-\chi \psi\left(\operatorname{Frob}_{v}^{-1}\right)\right)\left(\frac{1}{2^{d}} \theta_{K / k}^{\chi \psi}\right) \\
& =\prod_{v \in S_{2, \text { unr }}^{\chi} \cup \mathscr{S}_{2}(k)}\left(1-\chi \psi\left(\sigma_{v}\right)\right)\left(\frac{1}{2^{d}} \theta_{K \chi / k}^{\chi \psi}\right) .
\end{aligned}
$$

Put

$$
\mathscr{S}_{2}^{\prime}(k)^{\prime}=S_{2}(k) \backslash\left(S_{2, \mathrm{unr}}^{\chi} \cup \mathscr{S}_{2}(k) \cup \mathscr{S}_{2}^{\prime}(k)\right)
$$

We note that $\mathscr{S}_{2}^{\prime}(k)^{\prime}$ is the set of primes of $k$ above 2 which are unramified in $K_{\chi} / K_{\chi}^{+}$but ramify in $K_{\chi} / k$ and $\left(\mathscr{S}_{2}(k) \cup S_{2, \text { unr }}^{\chi}\right) \subset\left(S_{2}(k) \backslash \mathscr{S}_{2}^{\prime}(k)\right)$. Since $1-\chi \psi\left(\sigma_{v}\right)$ is not 0 for any $v \in S_{2}(k)$ by the assumption (3a), we have

$$
\left(\frac{1}{2^{d}} \theta_{K \chi / k}^{\chi \psi}\right)=\prod_{v \in \mathscr{S}_{2}^{\prime}(k)^{\prime}}\left(1-\chi \psi\left(\sigma_{v}\right)\right) \operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K}^{-, \chi}\right)
$$

By definition of $\mathscr{S}_{2}^{\prime}(k)^{\prime}$, the order of the decomposition group of $v$ in $\operatorname{Gal}\left(K^{\chi} / k\right)$ is not 2-power for any $v$ in $\mathscr{S}_{2}^{\prime}(k)^{\prime}$. Since $\chi \psi$ is a faithful character of $\operatorname{Gal}\left(K^{\chi} / k\right), 1-\chi \psi\left(\sigma_{v}\right)$ is a unit in $\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]$ for any $v$ in $\mathscr{S}_{2}^{\prime}(k)^{\prime}$. Therefore, we have

$$
\left(\frac{1}{2^{d}} \theta_{K \chi / k}^{\chi \psi}\right)=\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K}^{-, \chi}\right)
$$

Since $\frac{1}{2^{d}} \theta_{K}^{\chi \chi / k}=\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)$, Lemma 5.1.4 implies that

$$
f_{\chi}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]}\left(A_{K}^{-}\right)\right)=f_{\chi}\left(\Theta_{K / k}^{-}\right)
$$

Theorem 5.2.6, Corollary 5.1.5 and Proposition 5.1.6 imply Theorem 1.2 .6 in the case (3a).

### 5.3 Avoiding the Trivial Zero

In this section, we prove Theorem 1.2.6 in the case (3b). We use the same notation as in the previous section. Assume that $G^{\prime}$ is cyclic and $K^{G^{\prime}} \cap k^{c l}=$ $k$. Let $\psi$ be a faithful character of $G^{\prime}$.

If $l$ is an prime number and $l \equiv 1 \bmod 2^{n+1}$, let $\mathbb{Q}_{n, l}$ denote the cyclic extension of $\mathbb{Q}$ of degree $2^{n}$ contained in $\mathbb{Q}\left(\zeta_{l}\right)$ and let $N_{n, l}=N \mathbb{Q}_{n, l}$ for any finite extension $N$ of $\mathbb{Q}$. We note that $\mathbb{Q}_{n, l}$ is a real abelian field since $l \equiv 1$ $\bmod 2^{n+1}$.

Lemma 5.3.1. For any non-trivial character $\chi$ of $\Delta$, there exists an $r$ depending only on $K$ and $\chi$ such that for any $n>r$ we can find infinitely many $l$ satisfying the following conditions.
(a) $l$ is unramified in $K / \mathbb{Q}$.
(b) $l \equiv 1 \bmod 2^{n+1}$.
(c) For any character $\phi \in \operatorname{Gal} \widehat{\left(K_{n, l} / K\right)}$ of order $>2^{r}$, we have $\chi \psi \phi(v) \neq$ 1 for any prime $v$ of $k$ above 2 .
(d) For any prime $\lambda$ of $k$ above $l$, we have $\chi(\lambda) \neq 1$.

Proof. The proof of this lemma goes by the same method of the proof of the Lemma 4.1.1. Let $r=\max \left\{m \in \mathbb{Z}_{\geq 1}\left|2^{m}\right|\left[K_{v}: \mathbb{Q}_{2}\right]\right.$ for all v above 2$\}+1$, where $K_{v}$ is the completion of $K$ at $v$. Let $k^{\chi}$ be the subfield of $K^{G^{\prime}}$ attached to $\chi$ and $k^{\chi, c l}$ be the Galois closure of $k^{\chi}$ over $\mathbb{Q}$. Since $K^{G^{\prime}} \cap k^{c l}=k$ and $k^{c l}$ is a totally real field, the restriction map

$$
\text { res : } \operatorname{Gal}\left(k^{\chi, c l}\left(\zeta_{2^{n+1}}, \sqrt[4]{2}\right) / k^{c l}\left(\zeta_{2^{n+1}}, \sqrt[4]{2}\right)\right) \longrightarrow \operatorname{Gal}\left(k^{\chi}\left(\zeta_{2^{n+1}}, \sqrt[4]{2}\right) / k\left(\zeta_{2^{n+1}}\right)\right)
$$

is surjective for any $n>r$. Since $\left[k^{\chi}: k\right]$ is odd and $\left[k^{\chi}\left(\zeta_{2^{n+1}}, \sqrt[4]{2}\right)\right.$ : $\left.k\left(\zeta_{2^{n+1}}\right)\right]=2$, we can find infinitely many $l$ such that $l$ splits completely in $k^{c l}\left(\zeta_{2^{n+1}}\right) / \mathbb{Q}$ and for any prime $\lambda$ above $l$, $\operatorname{res}\left(\operatorname{Frob}_{\lambda}\right)$ is a generator of $\operatorname{Gal}\left(k^{\chi}\left(\zeta_{2^{n+1}}, \sqrt[4]{2}\right) / k\left(\zeta_{2^{n+1}}\right)\right)$ by Chebotarev's density theorem. We can choose that $l$ is unramified in $K / \mathbb{Q}$ since the number of primes which ramify in $K / \mathbb{Q}$ is finite. Any such $l$ satisfies the conditions (a), (b), (c), (d).

Next, we introduce Corollary 4.2 in Kurihara [19].
Let $p$ be a prime number, $R$ a complete discrete valuation ring of mixed characteristic $(0, p)$ and $\Lambda_{R}=R[[T]]$. Let $H$ be a cyclic $p$-group. For any subgroup $H^{\prime \prime} \subset H^{\prime} \subset H$, we consider the canonical map

$$
c_{H^{\prime \prime}, H^{\prime}}: \Lambda_{R}\left[H / H^{\prime \prime}\right] \longrightarrow \Lambda_{R}\left[H / H^{\prime}\right]
$$

induced by natural restriction map $H / H^{\prime \prime} \rightarrow H / H^{\prime}$. We also consider the map

$$
\nu_{H^{\prime}, H^{\prime \prime}}: \Lambda_{R}\left[H / H^{\prime}\right] \longrightarrow \Lambda_{R}\left[H / H^{\prime \prime}\right]
$$

induced by $\sigma \mapsto \sum_{c_{H^{\prime \prime}, H^{\prime}}(\tau)=\sigma} \tau$ for any $\sigma \in H / H^{\prime}$. Let $\phi$ be a character of $H$, namely a homomorphism from $H$ to the multiplicative group of an algebraic closure of the fractional field of $R$. We consider the ring homomorphism

$$
\phi_{\Lambda_{R}[H]}: \Lambda_{R}[H] \longrightarrow \Lambda_{R}[\operatorname{Im}(\phi)]
$$

induced by $\sigma \mapsto \phi(\sigma)$ for $\sigma \in H$.
Lemma 5.3.2. (Kurihara, Corollary 4.2 in [19]) Suppose that for any subgroup $H^{\prime} \subset H$, two ideals $I_{H / H^{\prime}}$ and $J_{H / H^{\prime}}$ of $\Lambda_{R}\left[H / H^{\prime}\right]$ are given and satisfy the following properties.
(1) For any subgroup $H^{\prime} \subset H$ and any faithful character $\phi$ of $H / H^{\prime}$,

$$
\phi_{\Lambda_{R}\left[H / H^{\prime}\right]}\left(I_{H / H^{\prime}}\right)=\phi_{\Lambda_{R}\left[H / H^{\prime}\right]}\left(J_{H / H^{\prime}}\right)
$$

(2) For any subgroup $H^{\prime} \subset H$ and any faithful character $\phi$ of $H / H^{\prime}$, $\phi_{\Lambda_{R}\left[H / H^{\prime}\right]}\left(I_{H / H^{\prime}}\right)$ is a free $R[\operatorname{Im}(\phi)]$-module of finite rank.
(3) For any subgroup $H^{\prime \prime} \subset H^{\prime} \subset H$, we have

$$
c_{H^{\prime \prime}, H^{\prime}}\left(I_{H / H^{\prime \prime}}\right) \subset I_{H / H^{\prime}}, \quad c_{H^{\prime \prime}, H^{\prime}}\left(J_{H / H^{\prime \prime}}\right) \subset J_{H / H^{\prime}}
$$

(4) For any subgroup $H^{\prime \prime} \subset H^{\prime} \subset H$, we have

$$
\nu_{H^{\prime}, H^{\prime \prime}}\left(I_{H / H^{\prime}}\right) \subset I_{H / H^{\prime \prime}}, \quad \nu_{H^{\prime}, H^{\prime \prime}}\left(J_{H / H^{\prime}}\right) \subset J_{H / H^{\prime \prime}}
$$

Then, we have $I_{H}=J_{H}$.
For any non-trivial character $\chi$ of $\Delta$, we consider $r, n, l$ in Lemma 5.3.1. Put $L^{\chi}=K_{n, l}^{\chi}, H=\operatorname{Gal}\left(L^{\chi} / K^{\chi}\right) \simeq \mathbb{Z} / 2^{n} \mathbb{Z}$.

For any subgroup $H^{\prime} \subset H$, we denote by $L^{\chi}, H^{\prime}$ the subfield of $L^{\chi}$ attached to $H^{\prime}$, and put $\Lambda_{\chi, H / H^{\prime}}=\mathbb{Z}_{2}[\operatorname{Im} \chi \psi]\left[\left[H / H^{\prime} \times \operatorname{Gal}\left(L_{\infty}^{\chi, H^{\prime}} / L^{\chi, H^{\prime}}\right)\right]\right]$. We consider the projective limit of the Stickelberger element $\theta_{L_{\infty}^{\chi, H^{\prime} / k}} \in$ $\mathcal{Q}\left(\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi, H^{\prime}} / k\right)\right]\right]\right)$ defined in section 1 . Since $\chi$ is a non-trivial character, we have $\frac{1}{2^{d}} \theta_{L_{\infty}^{\chi, H^{\prime}} / k}^{\chi \chi} \in \Lambda_{\chi, H / H^{\prime}}$. For any subgroup $H^{\prime \prime} \subset H$, we define $\Theta_{L_{\infty}^{\chi, H^{\prime \prime}} / k}^{\chi \psi}$ by

$$
\Theta_{L_{\infty}^{\chi, H^{\prime \prime}} / k}^{\chi \psi}=\left(\left.\nu_{H^{\prime}, H^{\prime \prime}}\left(\frac{1}{2^{d}} \theta_{L_{\infty}^{\chi, H^{\prime} / k}}^{\chi \psi}\right) \right\rvert\, H^{\prime \prime} \subset H^{\prime} \subset H\right) \subset \Lambda_{\chi, H / H^{\prime \prime}}
$$

For any $\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi} / k\right)\right]\right]$-module $M$, put

$$
M^{-, \chi}=M \otimes_{\left.\mathbb{Z}_{2}\left[\left[L_{\infty}^{\chi} / k\right)\right]\right]} \Lambda_{\chi, H}
$$

We consider the unramified Iwasawa module $X_{L_{\infty}^{\chi}}=\lim _{\leftarrow} A_{L_{n}^{\chi}}$.
Lemma 5.3.3. Assume that all primes above 2 are unramified in $K / K^{+}$ and the $\mu$-invariant of $K_{\infty}$ vanishes. Then we have

$$
\Theta_{L_{\infty}^{\chi} / k}^{\chi \psi}=\operatorname{Fitt}_{\Lambda_{\chi, H}}\left(X_{L_{\infty}^{\chi}}^{-\chi}\right)
$$

for any non-trivial character $\chi$ of $\Delta$.
Proof. To prove this lemma, we use Lemma 5.3.2 for $R=\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)], \Lambda_{R}\left[H / H^{\prime}\right]=$ $\Lambda_{\chi, H / H^{\prime}}, I_{H / H^{\prime}}=\Theta_{L_{\infty}^{\chi, H^{\prime}} / k}^{\chi \psi}, J_{H / H^{\prime}}=\operatorname{Fitt}_{\Lambda_{\chi, H / H^{\prime}}}\left(X_{L_{\infty}^{H^{\prime}, \chi}}^{-, \chi}\right)$. We check that the condisions (1), (2), (3), (4) in Lemma 5.3.2 are satisfied.

For any subgroup $H^{\prime} \subset H$ and any faithful character $\phi$ of $H / H^{\prime}$, we denote $L^{\chi \psi \phi}$ by the subfield of $L^{\chi, H^{\prime}}$ attached to $\chi \psi \phi$ and consider the map $\phi_{\Lambda_{\chi, H / H^{\prime}}}: \Lambda_{\chi, H / H^{\prime}} \rightarrow \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi \phi)]\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi \psi \phi} / L^{\chi \psi \phi}\right)\right]\right]$ induced by $\phi$. Since all primes are unramified in $L^{H^{\prime}} / L^{\chi \psi \phi}$, we have

$$
\phi_{\Lambda_{\chi, H / H^{\prime}}}\left(\Theta_{L_{\infty}^{\chi, H^{\prime}} / k}^{\chi \psi}\right)=\left(\frac{1}{2^{d}} \theta_{L_{\infty}^{\chi \psi \phi} / k}^{\chi \psi \phi}\right) .
$$

Since the $\mu$-invariant of $K$ is vanished,

$$
\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi \phi)]\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi \psi \phi} / L^{\chi \psi \phi}\right)\right]\right] /\left(\frac{1}{2^{d}} \theta_{L_{\infty}^{\chi \psi} / k}^{\chi \psi \phi}\right)
$$

is a free $\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi \phi)]$-module of finite rank. Therefore, the condition (2) is satisfied.

Since all primes are unramified in $L^{H^{\prime}} / L^{\chi \psi \phi}$, Proposition 5.2.2 implies that

$$
\left(X_{L_{\infty}^{\chi, H^{\prime}}}^{-, \chi}\right)_{\operatorname{Gal}\left(L^{\chi}, H^{\prime} / L^{\chi} \psi \phi\right)} \simeq X_{L \chi \psi \phi}^{-, \chi} .
$$

Since all primes above 2 are unramified in $K / K^{+}$, Lemma 5.2 .5 implies that $X_{L \chi \psi \phi}^{-, \chi}=X_{L \chi \psi \phi} \otimes_{\mathbb{Z}_{2}\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi \psi \phi} / k\right)\right]\right]} \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi \phi)]\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi \psi \phi} / L^{\chi \psi \phi}\right)\right]\right]$ has no non-trivial finite $\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi \phi)]\left[\left[\operatorname{Gal}\left(L_{\infty}^{\chi \psi \phi} / L^{\chi \psi \phi}\right)\right]\right]$-submodule. Therefore, the Iwasawa main conjecture implies that

$$
\phi_{\Lambda_{\chi, H / H^{\prime}}}\left(\operatorname{Fitt}_{\Lambda_{\chi, H / H^{\prime}}}\left(X_{L_{\infty}^{\chi, H^{\prime}}}^{-, \chi}\right)\right)=\left(\frac{1}{2^{d}} \theta_{L_{\infty}^{\chi \psi \phi} / k}^{\chi \psi}\right) .
$$

Therefore, the condition (1) is satisfied.
Since only primes above $l$ ramify in $L^{\chi, H^{\prime \prime}} / L^{\chi, H^{\prime}}$ and all primes above $l$ are totally ramified in $L^{\chi, H^{\prime \prime}} / L^{\chi, H^{\prime}}$ for any subgroup $H^{\prime \prime} \subsetneq H^{\prime} \subset H$, we have

$$
c_{H^{\prime \prime}, H^{\prime}}\left(\theta_{L_{\infty}^{\chi, H^{\prime \prime} / k}}^{\chi \psi}\right)=\left\{\begin{array}{lr}
\theta_{L_{\infty}^{\chi, H^{\prime} / k}}^{\chi \psi} & \left(\text { if } H^{\prime} \neq H\right) \\
\prod_{\lambda \mid l}\left(1-\chi \psi\left(\operatorname{Frob}_{\lambda}^{-1}\right) \gamma_{\lambda}^{-1}\right) \theta_{K_{\infty}^{x} / k}^{\chi \psi} & \left(\text { if } H^{\prime}=H\right),
\end{array}\right.
$$

where $\operatorname{Frob}_{\lambda}\left(\right.$ resp. $\left.\gamma_{\lambda}\right)$ is the Frobenius map of $\lambda$ in $\operatorname{Gal}\left(K^{\chi} / k\right)$ (resp. $\left.\operatorname{Gal}\left(K_{\infty}^{\chi} / K^{\chi}\right)\right)$. This implies that $c_{H^{\prime \prime}, H^{\prime}}\left(\Theta_{L_{\infty}^{\chi, H^{\prime \prime} / k}}^{\chi \psi}\right) \subset \Theta_{L_{\infty}^{\chi \chi, H^{\prime} / k}}^{\chi \psi}$. By definition, we have $\nu_{H^{\prime}, H^{\prime \prime}}\left(\Theta_{L_{\infty}^{\chi, H^{\prime} / k}}^{\chi \chi}\right) \subset \Theta_{L_{\infty}^{\chi, H^{\prime \prime} / k}}^{\chi \psi,}$ for any subgroup $H^{\prime \prime} \subsetneq H^{\prime} \subset H$.

By Proposition 5.2.2, we get the following exact sequence

$$
0 \longrightarrow \lim _{n}\left(\operatorname{coker} i_{L_{n}^{\chi, H^{\prime \prime}} / L_{n}^{\chi, H^{\prime}}}\right)^{\chi} \longrightarrow\left(X_{L_{\infty}^{\chi, H^{\prime \prime}}}^{-, \chi}\right)_{H^{\prime} / H^{\prime \prime}} \xrightarrow{N_{L_{\infty}^{\chi, H^{\prime \prime}}} L_{L}^{\chi \chi H^{\prime}}} X_{L_{\infty}^{\chi}}^{-, \chi} \longrightarrow 0
$$

for any subgroup $H^{\prime \prime} \subsetneq H^{\prime} \subset H$. Since the only primes above $l$ ramify in $L_{\infty}^{\chi, H^{\prime \prime}} / L_{\infty}^{\chi, H^{\prime}}$, the condition (d) in Lemma 5.3.1 implies that

$$
{\underset{n}{\mid}}_{\lim _{n}}\left(\operatorname{coker} i_{L_{n}^{\chi, H^{\prime \prime}} / L_{n}^{\chi, H^{\prime}}}\right)^{\chi}=0 .
$$

Therefore, we have $c_{H^{\prime \prime}, H^{\prime}}\left(\operatorname{Fitt}_{\Lambda_{\chi, H / H^{\prime \prime}}}\left(X_{L_{\infty}^{\chi, H^{\prime \prime}}}^{-\chi}\right)\right) \subset \operatorname{Fitt}_{\Lambda_{\chi, H / H^{\prime}}}\left(X_{L_{\infty}^{\chi, H^{\prime}}}^{-\chi}\right)$ and $\nu_{H^{\prime}, H^{\prime \prime}}\left(\operatorname{Fitt}_{\Lambda_{\chi, H / H^{\prime}}}\left(X_{L_{\infty}^{\chi}, H^{\prime}}^{--\chi}\right)\right) \subset \operatorname{Fitt}_{\Lambda_{\chi, H / H^{\prime \prime}}}\left(X_{L_{\infty}^{\chi, ~}}^{-\chi H^{\prime \prime}}\right)$. Therefore, the conditions (3), (4) are satisfied. This completes the proof of Lemma 5.3.3.

Finally, we prove the case ( 3 b ) of Theorem 1.2.6 in chapter 1.
Theorem 5.3.4. Assume that the following assumptions are satisfied.
(1) The $\mu$-invariant of $K_{\infty}$ vanishes.
(2) $G^{\prime}$ is cyclic.
(3b) All primes above 2 are unramified in $K / K^{+}$and $k^{c l} \cap K^{G^{\prime}}=k$. Then we have

$$
f_{\chi}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]-}\left(A_{K}^{-}\right)\right)=f_{\chi}\left(\Theta_{K / k}^{-}\right)
$$

for any non-trivial character $\chi$ of $\Delta$.
Proof. For any non-trivial character $\chi$ of $\Delta$, we define $\Theta_{L^{x / k}}^{\prime}$ by

$$
\left.\Theta_{L^{\chi} / k}^{\prime}=\left\langle\frac{1}{2^{d-1}} \nu_{L^{\chi} / M}\left(\theta_{M / k}\right)\right| k \subset M \subset L^{\chi}, M \text { is a CM field }\right\rangle,
$$

where $d=[k: \mathbb{Q}], \theta_{M / K}$ is the Stickelberger element defined in section 1 , $\nu_{L^{\chi} / M}$ is the map defined in section 3. We note that $\Theta_{L^{\chi} / k}^{\prime} \subset \mathbb{Q}_{2}\left[\operatorname{Gal}\left(L^{\chi} / k\right)\right]$. We consider the map $\mathbb{Z}_{2}\left[\operatorname{Gal}\left(L^{\chi} / k\right)\right] \rightarrow \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H]$ induced by $\chi \psi$. For any ideal $I$ of $\mathbb{Z}_{2}\left[\operatorname{Gal}\left(L^{\chi} / k\right)\right]$, we denote by $I^{\chi \psi}$ the image of $I$ of this map.

We consider the restriction map $c_{L_{\infty}^{\chi} / L^{\chi}}: \Lambda_{\chi, H} \rightarrow \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H]$ defined by $\sigma \mapsto 1$ for all $\sigma \in \operatorname{Gal}\left(L_{\infty}^{\chi} / L^{\chi}\right)$. Lemma 5.3.3 implies that

$$
\begin{aligned}
c_{L_{\infty}^{\chi} / L^{\chi}}\left(\operatorname{Fitt}_{\Lambda_{\chi, H}}\left(X_{L_{\infty}^{\chi}}^{-, \chi}\right)\right) & =c_{L_{\infty}^{\chi} / L^{\chi}}\left(\Theta_{L_{\infty}^{\chi} / k}^{\chi \psi}\right) \\
& =\left(\prod_{v \in S_{2, \mathrm{unr}}^{\chi}}\left(1-\operatorname{Frob}_{v}^{-1}\right) \Theta_{L^{\chi} / k}^{\prime}\right)^{\chi \psi}
\end{aligned}
$$

where $S_{2, \text { unr }}^{\chi}$ is the set of primes of $k$ lying above 2 which are unramified in $K^{\chi} / k$ and $\operatorname{Frob}_{v}$ is the Frobenius map of $v$ in $\operatorname{Gal}\left(L^{\chi} / k\right)$. Since all primes above 2 are unramified in $K / K^{+}$, we have $\mathscr{S}_{2}^{\prime}(k)=\emptyset$. Since ramification index of any prime of $k$ lying above 2 which ramifies in $L^{\chi} / k$ is odd,

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H]}\left(\mathbb{Z}_{2}\left[\operatorname{Gal}\left(L^{\chi} / k\right) / D_{v}\left(L^{\chi} / k\right)\right]^{-, \chi}\right)=(1)
$$

for any prime $v \notin S_{2, \mathrm{unr}}^{\chi}$, where $D_{v}\left(L^{\chi} / k\right)$ is the decomposition group of $v$ in $\operatorname{Gal}\left(L^{\chi} / k\right)$. Therefore, Corollary 5.2.4 implies that

$$
\prod_{v \in S_{2, \text { unr }}^{\chi}}\left(1-\operatorname{Frob}_{v}^{-1}\right)^{\chi \psi} \operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H]}\left(A_{L \chi}^{-, \chi}\right) \subset c_{L_{\infty}^{\chi} / L \chi}\left(\operatorname{Fitt}_{\Lambda_{\chi, H}}\left(X_{L_{\infty}^{\chi}}^{-, \chi}\right)\right)
$$

Let $\tau$ be a generator of $H$ and put $\nu=\frac{\tau^{2^{n}}-1}{\tau^{2^{r}}-1}$. The condition (c) in Lemma 5.3.1 implies that $\left(1-\operatorname{Frob}_{v}^{-1}\right)^{\chi \psi}$ is not a zero divisor in $\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H] /(\nu)$ for any $v \in S_{2, \text { unr }}^{\chi}$. Therefore, we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H]}\left(A_{L \chi}^{-, \chi}\right) \subset \Theta_{L \chi / k}^{\prime \chi \psi} \quad(\bmod \nu)
$$

We consider the map $c_{L \chi / K \chi}: \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)][H] / \nu \rightarrow \mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)] /\left(2^{n-r}\right)$ defined by $\tau \mapsto 1$. Proposition 5.2.2 implies that

$$
\prod_{\lambda \mid l}\left(1-\chi \psi\left(\operatorname{Frob}_{\lambda}^{-1}\right)\right) \operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K \chi}^{-, \chi}\right) \subset \prod_{\lambda \mid l}\left(1-\chi \psi\left(\operatorname{Frob}_{\lambda}^{-1}\right)\right) \Theta_{K \chi / k}^{\prime \chi \psi} \quad\left(\bmod 2^{n-r}\right) .
$$

The condition (d) in Lemma 5.3.1 implies that $\prod_{\lambda \mid l}\left(1-\chi \psi\left(\right.\right.$ Frob $\left._{\lambda}^{-1}\right)$ is a unit in $\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]$. Therefore, we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K \chi}^{-, \chi}\right) \subset \Theta_{K \chi / k}^{\prime \chi \psi} \quad\left(\bmod 2^{n-r}\right)
$$

for all $n \geq r$. Since $\Theta_{K \chi / k}^{\prime \chi \psi}=\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)$, we have

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K \chi}^{-, \chi}\right) \subset\left(\frac{1}{2^{d}} L\left(0, \chi^{-1} \psi^{-1}\right)\right)
$$

Since $\left[K: K^{\chi}\right]$ is odd, $\left(A_{K}^{-, \chi}\right)_{\operatorname{Gal}\left(K / K^{\chi}\right)} \simeq A_{K_{\chi}^{\chi}}^{-, \chi}$ by norm argument. This implies that

$$
\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K}^{-, \chi}\right)=\operatorname{Fitt}_{\mathbb{Z}_{2}[\operatorname{Im}(\chi \psi)]}\left(A_{K \chi}^{-, \chi}\right) .
$$

Therefore, Lemma 5.1.4 implies that

$$
f_{\chi}\left(\operatorname{Fitt}_{\mathbb{Z}_{2}[G]^{-}}\left(A_{K}^{-}\right)\right) \subset f_{\chi}\left(\Theta_{K / k}^{-}\right)
$$

for any non-trivial character $\chi$ of $\Delta$.
Theorem 5.3.4, Corollary 5.1.5 and Proposition 5.1.6 imply Theorem 1.2 .6 in the case (3b).

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