On Finite Multiple Zeta Values and Finite Multiple Polylogarithms

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Part 1. **Introduction**

1. Introduction

In this thesis, we define and study two objects, a generalization of Kamano's finite multiple zeta values of Mordell–Tornheim type [Kam] and a finite sum analogue of multiple polylogarithms. More precisely, in the first half of this thesis, we introduce a combinatorial object, which we call a 2-colored rooted tree, and finite multiple zeta values associated to a 2-colored rooted tree. These values contain Kaneko–Zagier's classical finite multiple zeta values [KZ] and Kamano's finite multiple zeta values of Mordell–Tornheim type. We prove that with a certain assumption, finite multiple zeta values associated to a 2-colored rooted tree can be written explicitly as $\mathbb{Z}$-linear combinations of classical finite multiple zeta values. As a corollary, we give a new proof of the shuffle relation among classical finite multiple zeta values, which was first proved by Kaneko–Zagier [KZ]. In the second half of this thesis, we introduce a finite analogue of multiple polylogarithms in an analogous framework of Kaneko–Zagier's finite multiple zeta values, and we prove a "shuffle-like" relation among finite multiple polylogarithms. Using this relation, we give examples of the products of finite multiple polylogarithms with low depth which can be described in terms of sums of finite multiple polylogarithms.

At first, we will survey the history of multiple zeta values, in particular, of finite multiple zeta values and finite polylogarithms.

1.1. **Multiple zeta values.** Multiple zeta values are real numbers defined by

$$\zeta(k_1,\dots,k_r)=\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}.$$

Here, r is a non-negative integer and $k_1,\dots,k_r$ are positive integers. For the convergence, we assume that k_r is larger than 1. For an index $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}_{\geq1}^r$ we let $\mathrm{wt}(\mathbf{k}):=k_1+\cdots+k_r$ be the weight of $\mathbf{k}$ and $\mathrm{dep}(\mathbf{k}):=r$ be the depth of $\mathbf{k}$. Double zeta values were first studied by Euler [Eul]. The study of multiple zeta values with general depth was started by Hoffman [Hof2] and Zagier [Zag] independently in the 1990's. In particular, Zagier showed that multiple zeta values appear in many areas of mathematics. The theory of multiple zeta values has been rapidly developed after the Zagier's work.

In [Zag], Zagier conjectured the dimension of $\mathbb{Q}$-vector spaces $\mathcal{Z}_{\mathbb{R},k}$ generated by all the multiple zeta values of weight k for all non-negative integers k. Let $\{d_k\}_{k\geq0}$ be the sequence of positive integers defined by

$$\begin{cases} d_0 = 1, d_1 = 0, d_2 = 1, \\ d_k = d_{k-2} + d_{k-3}. \end{cases}$$

Then, Zagier conjectured the following equality:

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathbb{R},k} = d_k.$$

This conjecture has not yet been proved. Deligne–Goncharov [DG] and Terasoma [Ter], however, proved the following inequality

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathbb{R},k} \leq d_k$$

for all non-negative integers k. This result means that there exist many $\mathbb{Q}$-linear relations among multiple zeta values. The reverse direction of this inequality has not been proved. This problem seems to be very difficult for us because this problem contains the algebraic independence of multiple zeta values over $\mathbb{Q}$.

1.2. **Finite multiple zeta values.** There exist many variants of multiple zeta values, for example, the q-analogue, the p-adic version and the analogue for function field over a finite field. Hoffman [Hof2] and Zhao [Zha] considered independently the following truncated version of multiple zeta values for each prime p:

$$\zeta_{(p)}(k_1, \ldots, k_r) := \sum_{0<n_1<\cdots<n_r<p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p \in \mathbb{F}_p.$$

We call this value mod p multiple zeta value in this thesis.

After the pioneering work of Hoffman and Zhao, Kaneko–Zagier [KZ] proposed a new "adélic" framework for mod p multiple zeta values. In their framework, they considered mod p multiple zeta values as elements in the following ring:

$$\mathcal{A} := \left(\prod_p \mathbb{F}_p\right) \Big/ \left(\bigoplus_p \mathbb{F}_p\right).$$

Here, p runs through all the rational primes. Thus, an element of $\mathcal{A}$ is represented by a family $(a_p)_p$ of elements $a_p \in \mathbb{F}_p$, and two families $(a_p)_p$ and $(b_p)_p$ represent the same element of $\mathcal{A}$ if and only if $a_p = b_p$ for all but finitely many primes p. Note that $\mathcal{A}$ is a $\mathbb{Q}$-algebra.

Finite multiple zeta values are mod p multiple zeta values considered as elements in $\mathcal{A}$:

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r) := (\zeta_{(p)}(k_1, \ldots, k_r))_p = \left(\sum_{0<n_1<\cdots<n_r<p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p\right)_p.$$

In the following, we often denote an element $(a_p)_p$ in $\mathcal{A}$ simply by a_p omitting $(\ )_p$ if there is no fear of confusion. For example, the above definition is written as

$$\zeta_{\mathcal{A}}(k_1,\ldots,k_r)=\sum_{0<n_1<\cdots<n_r<p}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}.$$

There exist many researches of finite multiple zeta values, for example, [Mur], [Oy], [SS], [SW1], [SW2]. One of the main topics for the study of finite multiple zeta values is obtaining all the algebraic relations over $\mathbb{Q}$ among finite multiple zeta values. For example, finite single zeta values are 0 (see Proposition 3.15). Moreover, finite double zeta values can be written explicitly by using Bernoulli numbers:

$$\zeta_{\mathcal{A}}(k_1,k_2)=(-1)^{k_2}\binom{k_1+k_2}{k_2}\frac{B_{p-(k_1+k_2)}}{k_1+k_2}. \tag{1.1}$$

See Proposition 3.16.

Finite multiple zeta values satisfy many algebraic relations over $\mathbb{Q}$. For example, finite multiple zeta values satisfy the shuffle relation and the stuffle relation. To state these relations precisely, we introduce the algebraic setup of finite multiple zeta values due to Hoffman [Hof1].

Let $\mathfrak{H}$ be the noncommutative polynomial ring $\mathbb{Q}\langle x,y\rangle$ in the variables x and y over $\mathbb{Q}$, and we set $\mathfrak{H}^1:=\mathbb{Q}+y\mathfrak{H}$. Note that $\mathfrak{H}^1$ is generated by $z_k:=yx^{k-1}$ $(k=1,2,\ldots)$ as a $\mathbb{Q}$-algebra. For an index $\mathbf{k}=(k_1,\ldots,k_r)$, we write $z_{\mathbf{k}}:=z_{k_1}\cdots z_{k_r}$. We define the map $Z_{\mathcal{A}}:\mathfrak{H}^1\to\mathcal{A}$ by sending $z_{\mathbf{k}}$ to $\zeta_{\mathcal{A}}(\mathbf{k})$ and extend it $\mathbb{Q}$-linearly.

We define the shuffle product $ш:\mathfrak{H}\times\mathfrak{H}\to\mathfrak{H}$ on $\mathfrak{H}$ by the following rule and $\mathbb{Q}$-bilinearity.

(i) $w\,ш\,1=1\,ш\,w=w$ for all $w\in\mathfrak{H}$.
(ii) $(w_1u_1)\,ш\,(w_2u_2)=(w_1\,ш\,w_2u_2)u_1+(w_1u_1\,ш\,w_2)u_2$ for all $w_1,w_2\in\mathfrak{H}$ and $u_1,u_2\in\{x,y\}$.

For instance, we have

$$z_2\,ш\,z_2=yx\,ш\,yx=2yxyx+4y^2x^2=2z_2z_2+4z_1z_3.$$

It is known that $(\mathfrak{H},ш)$ is a commutative $\mathbb{Q}$-algebra [Reu, p.24].

We define the stuffle product $*:\mathfrak{H}^1\times\mathfrak{H}^1\to\mathfrak{H}^1$ on $\mathfrak{H}^1$ by the following rule and $\mathbb{Q}$-bilinearity.

(i) $w*1=1*w=w$ for all $w\in\mathfrak{H}^1$.
(ii) $(w_1z_k)*(w_2z_l)=(w_1*w_2z_l)z_k+(w_1z_k*w_2)z_l+(w_1*w_2)z_{k+l}$ for all $w_1,w_2\in\mathfrak{H}^1$ and $k,l\in\mathbb{Z}_{\geq1}$.

For instance, we have

$$z_2 * z_3 = z_2z_3 + z_3z_2 + z_5.$$

It is also known that $(\mathfrak{H}^1, *)$ is a commutative $\mathbb{Q}$-algebra [Hof1, Theorem 2.1].

The shuffle relation and the stuffle relation among finite multiple zeta values are stated as follows. For positive integers r, s and elements $w_1 := z_{k_1} \cdots z_{k_r}, w_2 := z_{l_1} \cdots z_{l_s}$ in $\mathfrak{H}^1$, we have

$$Z_{\mathcal{A}}(w_1 ш w_2) = (-1)^{l_1+\cdots+l_s} Z_{\mathcal{A}}(z_{k_1} \cdots z_{k_r} z_{l_s} \cdots z_{l_1}), \tag{1.2}$$

$$Z_{\mathcal{A}}(w_1 * w_2) = Z_{\mathcal{A}}(w_1) Z_{\mathcal{A}}(w_2). \tag{1.3}$$

See the first proof of Theorem 2.15. These equalities give many algebraic relations over $\mathbb{Q}$ among finite multiple zeta values. Indeed, it is conjectured by Kaneko-Zagier that all the algebraic relations over $\mathbb{Q}$ among finite multiple zeta values can be deduced from the shuffle relation and the stuffle relation.

Similar to the case of the multiple zeta values, Zagier conjectured the dimension of the $\mathbb{Q}$-vector space $\mathcal{Z}_{\mathcal{A},k}$ generated by all the finite multiple zeta values of weight k for all non-negative integers k. For example, $\mathcal{Z}_{\mathcal{A},0} = \mathbb{Q}, \mathcal{Z}_{\mathcal{A},1} = 0$. Then, Zagier conjectured the following identity for all non-negative integers k:

$$\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} = d_{k-3}.$$

Recently, Yasuda announced that $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} \leq d_{k-3}$ for all the non-negative integers k by using Akagi–Hirose–Yasuda's results and Jarrosay's results. Therefore, there also exist many linear relations over $\mathbb{Q}$ among finite multiple zeta values.

Moreover, Kaneko and Zagier conjectured a mysterious relation between finite multiple zeta values and the classical multiple zeta values. Set $\mathcal{Z}_{\bullet} := \bigoplus_{k \geq 0} \mathcal{Z}_{\bullet,k}$ for $\bullet \in \{\mathbb{R}, \mathcal{A}\}$. Kaneko and Zagier conjectured that there exists the well-defined map sending $\zeta_{\mathcal{A}}(k_1, \ldots, k_r)$ to $\zeta_{\mathcal{S}}(k_1, \ldots, k_r)$ which gives the isomorphism between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}_{\mathbb{R}}/\zeta(2)\mathcal{Z}_{\mathbb{R}}$ as $\mathbb{Q}$-algebras. Here $\zeta_{\mathcal{S}}(k_1, \ldots, k_r)$ is called symmetric multiple zeta values. Symmetric multiple zeta values are defined for positive integers $k_1, \ldots, k_r$, by using the regularized values of divergent multiple zeta values. We do not state the precise definition of symmetric multiple zeta values in this thesis. If we believe Kaneko and Zagier's conjecture, then the finite and symmetric multiple zeta values satisfy the same algebraic relations over $\mathbb{Q}$. Through Kaneko and Zagier's conjectural isomorphism, the study of finite multiple zeta values becomes more important.

1.3. **Multiple zeta values of Mordell–Tornheim type.** For non-negative integer r and positive integers $k_1, \ldots, k_r, k_{r+1}$, Multiple zeta values of Mordell-Tornheim type are defined as follows:

$$\zeta^{MT}(k_1, \ldots, k_r; k_{r+1}) := \sum_{m_1, \ldots, m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}} \in \mathbb{R}. \tag{1.4}$$

The sum of these types of multiple zeta values were studied first by Tornheim [Tor] and Mordell [Mor] in the case $r = 2$. Mordell proved [Mor, Theorem III] that $\zeta^{MT}(2k, 2k; 2k)$ is a rational multiple of π^{6k} for a positive integer k. Bradley–Zhou [BZ, Theorem 1.1] proved that Multiple zeta values of Mordell–Tornheim type can be written as a $\mathbb{Z}$-linear combination of the classical multiple zeta values. Tornheim proved [Tor] that $\zeta^{MT}(k_1, k_2; k_3)$ can be written as a $\mathbb{Q}$-linear combination of products of Riemann zeta values. Tsumura [Tsu, Theorem 1.1] generalized Tornheim's result to the case of general depth, that is, Tsumura proved that if the parities of r and $k_1 + \cdots + k_{r+1}$ are different, Multiple zeta values $\zeta^{MT}(k_1, \ldots, k_r; k_{r+1})$ of Mordell–Tornheim type can be written as a $\mathbb{Q}$-linear combination of products of Multiple zeta values of Mordell–Tornheim type with the smaller depth than r. Matsumoto studied (1.4) as a complex function of $(r+1)$-variables, and proved [Mat, Theorem 5] that (1.4) can be meromorphically continued as a complex function to the whole space $\mathbb{C}^{r+1}$.

1.4. **Finite multiple zeta values of Mordell–Tornheim type.** Kamano [Kam] defined a finite analogue of multiple zeta values of Mordell–Tornheim type as an element in $\mathcal{A}$ as follows:

$$\zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_r; k_{r+1}) := \sum_{\substack{m_1, \ldots, m_r \geq 1 \\ m_1 + \cdots + m_r \leq p-1}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}}.$$

We call this sum in this thesis the finite multiple zeta values of Mordell–Tornheim type. Similar to the case of the classical finite multiple zeta values, we call $k_1 + \cdots + k_{r+1}$ and r the weight and the depth of the finite multiple zeta values of Mordell–Tornheim type, respectively. Note that Kuba considered finite Mordell–Tornheim double zeta values before Kamano defined, and Kuba [Kub, Theorem 5] obtained the explicit formula of finite Mordell–Tornheim double zeta values by using binomial coefficients and Bernoulli numbers. This result is a Mordell–Tornheim type analogue of (1.1). Kamano [Kam, Theorem 1.2] gave an explicit formula of the finite multiple zeta values of Mordell–Tornheim type to write them as a $\mathbb{Z}$-linear combination of the classical finite multiple zeta values by using the language of the Hoffman algebra. By using this formula, Kamano obtained many $\mathbb{Q}$-linear relations among the finite multiple zeta values as corollaries [Kam, Theorem 3.2, Proposition 3.4]. Moreover, Kamano's method can be applied to the case of the classical multiple zeta values. Kamano also obtained an explicit formula of the multiple zeta values of Mordell–Tornheim type to write them as a $\mathbb{Z}$-linear combination of the classical multiple zeta values, which can be regarded as a refinement of a result obtained previously by Bradley–Zhou [Kam, Remark 2.2].

1.5. **Main result on finite multiple zeta values associated to 2-colored rooted trees.** In order to obtain his main result in [Kam], Kamano only used the following partial fraction decomposition:

$$\frac{1}{X_1\cdots X_s}=\frac{1}{X_1+\cdots+X_s}\sum_{i=1}^{s}\frac{1}{\underbrace{X_1\cdots X_s}_{\text{remove } i\text{-th}}}. \tag{1.5}$$

Here, s is a positive integer and $X_1,\dots,X_s$ are indeterminates. Therefore, there is a natural question how we can generalize Kamano's results to elements in $\mathcal{A}$ defined by more general finite sums. For example, consider the following element in $\mathcal{A}$:

$$\sum_{\substack{m_1,m_2,m_3\geq 1\\ m_1+m_2+m_3\leq p-1}}\frac{1}{m_1^{k_1}m_2^{k_2}m_3^{k_3}(m_1+m_2)^{l_2}(m_1+m_2+m_3)^{l_3}}. \tag{1.6}$$

Here, k_1,k_2,k_3,l_2,l_3 are all non-negative integers. Note that this element coincides with $\zeta_{\mathcal{A}}^{MT}(k_1,k_2,k_3;l_3)$ if k_1,k_2,k_3,l_3 are positive and $l_2=0$. Furthermore, if $k_1=k_2=k_3=l_2=1$, we can easily prove the following expression of (1.6) as a $\mathbb{Z}$-linear combination of the classical finite multiple zeta values by using 3 times the partial fraction decomposition (1.5):

$$\sum_{\substack{m_1,m_2,m_3\geq 1\\ m_1+m_2+m_3\leq p-1}}\frac{1}{m_1m_2m_3(m_1+m_2)(m_1+m_2+m_3)^{l_3}}=2\zeta_{\mathcal{A}}(1,2,l_3+1)+6\zeta_{\mathcal{A}}(1,1,l_3+2).$$

We can also prove that (1.6) can in general be written explicitly as a $\mathbb{Z}$-linear combination of the classical finite multiple zeta values.

In this thesis, inspired by Yamamoto's work [Y] on multiple integrals associated to 2-labeled posets, we introduce the finite multiple zeta value associated to a triple $X=(T,\mathrm{rt}_X,V_\bullet)$ consisting of the following three data, which we call a 2-colored rooted tree.

(i) $T=(V,E)$ is a tree (in the graph theoretic sense) such that $\#V(=\#E+1)<\infty$.

(ii) $\mathrm{rt}_X\in V$ is a vertex, called the root of T.

(iii) $V_\bullet$ is a subset of V containing all terminals of T.

We set $V_\circ := V\setminus V_\bullet$. Further, for a 2-colored rooted tree $X=(T,\mathrm{rt}_X,V_\bullet)$ and a map $k: E\to\mathbb{Z}_{\geq 0}$ from the set of edges of the given 2-colored rooted tree to the set of non-negative integers, a finite multiple zeta value associated to a 2-colored rooted tree is defined as an element in $\mathcal{A}$ as follows:

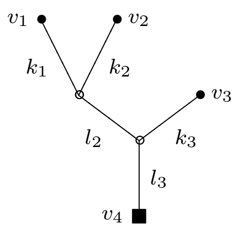
$$\zeta_{\mathcal{A}}(X,k) := \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{V_\bullet} \text{ s.t.} \\ \sum_{v\in V_\bullet} m_v = p}} \prod_{e\in E} L_e(\mathrm{rt}_X, (m_v))^{-k(e)}$$

Here, for a 2-colored rooted tree $(T, \mathrm{rt}_X, V_\bullet)$ and an edge e of T and $(m_v) \in \mathbb{Z}_{\geq 1}^{V_\bullet}$, we set

$$L_e(\mathrm{rt}_X, (m_v)) := \sum_{v\in V_\bullet \text{ s.t. } e\in P(\mathrm{rt}_X, v)} m_v,$$

where $P(\mathrm{rt}_X, v)$ is the path from rt_X to v and $e \in P(\mathrm{rt}_X, v)$ denotes that the edge e is on the path $P(\mathrm{rt}_X, v)$. The map k is called an index on the 2-colored rooted tree X. For example, (1.6) is the finite multiple zeta value associated to the following 2-colored rooted tree and an index:

(1.7)



The classical finite multiple zeta values and finite multiple zeta values of Mordell–Tornheim type are also examples of finite multiple zeta values associated to 2-colored rooted trees. See Example 2.1.

In the case of the finite multiple zeta values of Mordell–Tornheim type, Kamano obtained the explicit formula of them as a $\mathbb{Z}$-linear combination of the classical finite multiple zeta values using the language of the Hoffman algebra. Using the above partial fraction decomposition (1.5), with an assumption on the index on the 2-colored rooted tree, we can obtain an explicit formula of the finite multiple zeta values associated to 2-colored rooted trees as a $\mathbb{Z}$-linear combination of the classical finite multiple zeta values. This is the first main result of this thesis.

Theorem 1.1 ([On, Theorem 1.4]). *Let $X = (T, \mathrm{rt}_X, V_\bullet)$ be a 2-colored rooted tree and k an index on X. Suppose that $\sum_{e\in P(v,v')} k(e)$ is positive for any $v, v' \in V_\bullet$ with $v \neq v'$. Then, the finite multiple zeta value $\zeta_{\mathcal{A}}(X,k)$ can be written explicitly as a $\mathbb{Z}$-linear combination of the classical finite multiple zeta values.*

Note that for a 2-colored rooted tree X giving finite multiple zeta values of Mordell–Tornheim type, our first main result above coincides with Kamano's result [Kam, Theorem 1.2].

As a corollary of our first main result, we give a new proof of the shuffle relation among finite multiple zeta values. This new proof means that the class of the $\mathbb{Z}$-linear relation among finite multiple zeta values coming from 2-colored rooted trees contains the class of the shuffle relation among finite multiple zeta values. Moreover, our $\mathbb{Z}$-linear relation among finite multiple zeta values can be regarded as a simultaneous generalization of both Kamano's relation and the shuffle relation among finite multiple zeta values. We should note that this is very surprising since there were no obvious connections between these two classes of relations.

1.6. **Polylogarithms.** From this subsection, we shift our interest to the second main object, finite multiple polylogarithms. In order to explain the second main result on the shuffle relation among finite multiple polylogarithms, we first explain the shuffle relation among usual multiple polylogarithms. The (one variable) multiple polylogarithm is the following power series

$$\mathrm{Li}_{k_1,\ldots,k_r}(T) = \sum_{l_1,\ldots,l_r\in\mathbb{Z}_{\geq 1}} \frac{T^{l_1+\cdots+l_r}}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}}.$$

Here $k_1,\ldots,k_r$ are positive integers. A multiple polylogarithm $\mathrm{Li}_{k_1,\ldots,k_r}(T)$ converges if $|T|<1$ and $\lim_{T\to 1}\mathrm{Li}_{k_1,\ldots,k_r}(T) = \mathrm{Li}_{k_1,\ldots,k_r}(1) = \zeta(k_1,\ldots,k_r)$ holds if $k_r>1$. Multiple polylogarithms are related to many areas including number theory ([Bro], [DG], [Ter]), topology ([LM]) and quantum theory ([BK], [Dri]), and has been studied by many authors ([BBBL], [Gon], [MPV], [Rac], to name a few examples).

One of the important properties of the multiple polylogarithms is the shuffle relation

$$\mathrm{Li}_{\mathbf{k}}(T)\mathrm{Li}_{\mathbf{k}'}(T) = \mathrm{Li}_{\mathbf{k}\,ш\,\mathbf{k}'}(T), \tag{1.8}$$

where $\mathbf{k}\,ш\,\mathbf{k}'$ denotes the shuffle product of indices $\mathbf{k}$ and $\mathbf{k}'$, and $\mathrm{Li}_{\mathbf{k}\,ш\,\mathbf{k}'}(T)$ is the corresponding finite sum of multiple polylogarithms. That is, $\mathbf{k}\,ш\,\mathbf{k}'$ is a formal sum of indices corresponding to $z_{\mathbf{k}\,ш\,\mathbf{k}'}$, and $\mathrm{Li}_{\mathbf{k}\,ш\,\mathbf{k}'}(T)$ is the formal sum of multiple polylogarithms corresponding to $\mathbf{k}\,ш\,\mathbf{k}'$. For example, since we can calculate $z_2\,ш\,z_3 = z_{(3,2)}+3z_{(2,3)}+6z_{(1,4)}$, the definition of $\mathrm{Li}_{(2)ш(3)}(T)$ says that

$$\mathrm{Li}_{(2)ш(3)}(T) = \mathrm{Li}_{(3,2)}(T) + 3\mathrm{Li}_{(2,3)}(T) + 6\mathrm{Li}_{(1,4)}(T).$$

The shuffle relation among multiple polylogarithms is proved by using the iterated integral expression of multiple polylogarithms:

$$\mathrm{Li}_{k_1,\dots,k_r}(T) = \int_0^T \underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}\circ\frac{dt}{1-t}}_{k_1}\circ\cdots\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}\circ\frac{dt}{1-t}}_{k_r}.$$

Komori, Matsumoto and Tsumura [KMT, Theorem 2] gave a different proof of the shuffle relation among multiple polylogarithms only using the partial fraction decomposition as follows:

$$\frac{1}{X^\alpha Y^\beta} = \sum_{\tau=0}^{\beta-1}\binom{\alpha-1+\tau}{\tau}\frac{1}{(X+Y)^{\alpha+\tau}Y^{\beta-\tau}} + \sum_{\tau=0}^{\alpha-1}\binom{\beta-1+\tau}{\tau}\frac{1}{(X+Y)^{\beta+\tau}X^{\alpha-\tau}}. \tag{1.9}$$

Here α and β are positive integers. This is proved by differentiating both hand sides of the equation in the case $\alpha=\beta=1$ by X at α times and Y at β times [Wei].

1.7. **Finite polylogarithms.** There is a finite sum analogue of the single polylogarithms, which was introduced by Kontsevich [Kon] and Elbaz-Vincent and Gangl [EG]. For a fixed prime p, Kontsevich considered a finite sum analogue of $\mathrm{Li}_1(x) = -\log(1-x)$ as a function on $\mathbb{Z}/p\mathbb{Z}$, which he called $1\frac{1}{2}$-logarithm, and he proved its functional equation and a cohomological interpretation of this functional equation.

On the other hand, Cathelineau [Cat1] introduced a certain "infinitesimal version" of dilogarithms, and proved [Cat1, Theorèm 1] that the infinitesimal dilogarithms satisfy a similar functional equation as that of finite logarithms. Moreover, Cathelineau extended this result to higher infinitesimal polylogarithms, and as a byproduct Cathelineau [Cat2, Corollarie 1] obtained the functional equation of an infinitesimal trilogarithm in three variables, which contains 22-terms.

Kontsevich posed a question how to obtain a functional equation of finite dilogarithms. Elbaz-Vincent and Gangl [EG, Theorem 5.12] proved that finite dilogarithms satisfy the same functional equation of infinitesimal trilogarithms via the analogy of finite logarithms and inifinitesimal dilogarithms.

Fix a prime p. Elbaz-Vincent and Gangl [EG] defined a finite analogue of polylogarithms as follows:

$$\pounds_n(T) := \sum_{k=1}^{p-1}\frac{T^k}{k^n} \in \mathbb{F}_p[T].$$

Here n is a positive integer. $\pounds_1(T)$ is Kontsevich's $1\frac{1}{2}$-logarithms. Elbaz-Vincent and Gangl [EG, PART II, 5] obtained functional equations and distribution relations of finite polylogarithms $\pounds_n(T)$.

On the other hand, it is known that Wojtkowiak [Woj, Proposition 4.4] showed that Coleman's p-adic polylogarithms [Col] satisfy the same functional equations of classical polylogarithms. It was hoped that there would exist a variant of p-adic polylogarithms whose certain derivative coincides with finite polylogarithms. This problem was solved by Besser [Bes, Theorem 1.1]. Besser proved that the certain special values of the differential of a certain $\mathbb{Q}$-linear combination of the p-adic polylogarithm coincide with the certain special values of finite polylogarithm.

1.8. **Main result on the shuffle relation among finite multiple polylogarithms.** We have a natural question: how to define a multiple version of finite polylogarithms satisfying the shuffle relation? In the second part of this thesis, we give an answer to this question. That is, we define a multiple version of finite polylogarithms $\mathrm{li}_{\mathbf{k}}(T)$ and prove that there exists a finite analogue of the shuffle relation.

As in the case of finite multiple zeta values, we define a finite multiple polylogarithm as an element in the following adélic ring $\mathcal{B}$:

$$\mathcal{B} := \left(\prod_p \mathbb{F}_p[T]\right) \Big/ \left(\bigoplus_p \mathbb{F}_p[T]\right).$$

Here, p runs through all the rational primes. Then $\mathcal{B}$ becomes a $\mathbb{Q}$-algebra containing $\mathcal{A}$ as a $\mathbb{Q}$-subalgebra.

Next, for a non-negative integer r and an index $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$, we define a finite multiple polylogarithm as follows:

$$\mathrm{li}_{\mathbf{k}}(T) = \mathrm{li}_{k_1,\dots,k_r}(T) = \begin{cases} \displaystyle\sum_{0<l_1,\dots,l_r<p}{}' \frac{T^{l_1+\cdots+l_r}}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} & (r \geq 1), \\ 1 & (r = 0). \end{cases}$$

Here, $\sum'$ denotes the sum over the terms whose denominators are prime to p. We also denote such an element of $\mathcal{B}$ simply by f_p omitting $(\ )_p$ if there is no fear of confusion. For instance, T^p denotes an element of $\mathcal{B}$ whose p-component is $T^p \in \mathbb{F}_p[T]$. Note that if $r = 1$ then the p-component of $\mathrm{li}_k(T)$ coincides with the classical finite polylogarithms $\pounds_k(T) \in \mathbb{F}_p[T]$.

Our second main result is the following:

Theorem 1.2 (=Theorem 3.11, [OY, Theorem 1.3])**.** *For non-negative integers r, r' and indices $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$, $\mathbf{k}' = (k'_1, \dots, k'_{r'}) \in (\mathbb{Z}_{\geq 1})^{r'}$, set $k := k_1 + \cdots + k_r, k' :=$*

$k'_1 + \cdots + k'_{r'}$. *Then we have*

$$\mathrm{li}_{\mathbf{k}}(T)\mathrm{li}_{\mathbf{k}'}(T) \equiv \mathrm{li}_{\mathbf{k}ш\mathbf{k}'}(T) \pmod{\mathcal{R}_{k+k',k+k'-1}}.$$

Moreover, $\mathrm{li}_{\mathbf{k}}(T)\mathrm{li}_{\mathbf{k}'}(T) - \mathrm{li}_{\mathbf{k}ш\mathbf{k}'}(T) \in \mathcal{R}_{k+k',k+k'-1}$ *can be calculated explicitly in terms of* $\mathrm{li}_{\mathbf{k}}(T)$ *and* $\mathrm{li}_{\mathbf{k}'}(T)$.

Here, $\mathrm{li}_{\mathbf{k}ш\mathbf{k}'}(T)$ is the formal sum of finite multiple polylogarithms corresponding to $\mathbf{k} ш \mathbf{k}'$, and $\mathcal{R}_{a,b}$ is a certain $\mathbb{Q}$-vector subspace of $\mathcal{B}$ defined in Definition 3.9.

Taking the usual complex case into account, I think that the "correct" finite analogue of multiple polylogarithm should coincide with the finite multiple zeta values at $T = 1$, and satisfy the shuffle relation. Unfortunately, since we can prove that $\mathrm{li}_{\mathbf{k}}(1) = 0$, our definition is unsatisfactory. It is important, however, that our finite multiple polylogarithms satisfy the above approximation of the shuffle relation. We hope that the study in this thesis would be helpful to find a better definition.

Recently, Sakugawa and Seki introduced other types of finite multiple polylogarithms with multi-variables [SS, Definition 3.8] as elements in a multi-variable version $\mathcal{A}_{\mathbb{Z}[T_1,\ldots,T_r]}$ of $\mathcal{B}$ which coincides with $\mathcal{B}$ in the case of $r = 1$. Their finite multiple polylogarithms evaluate the classical finite multiple zeta values when all variables are 1. They obtained functional equations [SS, §3] of their finite multiple polylogarithms and calculated special values [SS, §4] of their finite multiple polylogarithms. Moreover, they clarified the relation between our finite multiple polylogarithms and their finite multiple polylogarithms [SS, Proposition 3.26] and calculated special values of our finite multiple polylogarithms in some cases.

The contents of this thesis are as follows. In Part 2, we discuss the finite multiple zeta values associated to 2-colored rooted trees. In subsection 2.1, we give two examples and prove some properties of finite multiple zeta values associated to 2-colored rooted trees. We can see that the classical finite multiple zeta values and the finite multiple zeta values of Mordell-Tornheim type are special cases of finite multiple zeta values associated to 2-colored rooted trees. The key concept "harvestable" to prove our first main result will be introduced in this subsection. In subsection 2.3, we prove our first main result. In subsection 2.4, we give two proofs of the shuffle relation among finite multiple zeta values. The first proof is the one given by Kaneko and Zagier, and the second is a new proof given by our first main result.

In Part 3, we discuss the finite multiple polylogarithms. In subsection 3.1, we introduce a variant $\zeta_{\mathcal{A}}^{(i)}(k_1, \ldots, k_r)$ of finite multiple zeta values and prove that $\zeta_{\mathcal{A}}^{(i)}(k_1, \ldots, k_r)$ is expressed as a sum of finite multiple zeta values of the same weight. If $i = 1$, we see easily that $\zeta_{\mathcal{A}}^{(i)} = \zeta_{\mathcal{A}}$, so $\zeta_{\mathcal{A}}^{(i)}$ is a generalization of the classical finite multiple zeta values in some sense. In subsection 3.2, we prove our second main result. In order to prove our second result, we give the definition of finite multiple polylogarithms $\mathrm{li}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) \in \mathcal{B}$ of type $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ for indices $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$. This is a common generalization of finite multiple polylogarithms and a product of two finite multiple polylogarithms. In the subsection 3.3, we give proofs of some known facts on finite multiple zeta values. We use these facts in the next subsection. In the subsection 3.4, we will give examples of the products of two finite multiple polylogarithms of low depth. In the final subsection 3.5, we give an algebraic interpretation of our second main result.

1.9. **Terminology from graph theory.** We quote terminology on the graph theory from [Die], which will be used in the next section.

A *graph* is a pair $G = (V, E)$ such that V is a finite set and E is a subset of the set of 2-element subsets of V. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges set by $E(G)$.

A vertex v is *incident* with an edge e if $v \in e$; then e is an edge *at* v. An edge $\{x, y\}$ is usually written as xy or yx. The set of all the edges in E at a vertex v is denoted by $E(v)$.

The *degree* $\deg(v)$ of a vertex v of G is the number $\#E(v)$ of edges at v. A *branched point* is a vertex whose degree is larger than or equal to 3.

For a graph $G = (V, E)$ and distinct vertices $v_0, \ldots, v_n$ ($n \in \mathbb{Z}_{\geq 1}$), a *path* $P(v_0, v_n)$ from v_0 to v_n is a subset

$$P(v_0, v_n) := \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$$

of $E(G)$, and a *cycle* C is a subset

$$C := \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n, v_nv_0\}$$

consisting of $n + 1$ distinct edges of $E(G)$.

A non-empty graph $G = (V, E)$ is called *connected* if for any distinct two vertices of $V(G)$ there exists a path from one to another. A *tree* is a connected graph not containing any cycles. Note that for any pair of vertices u and v in a tree, the path $P(u, v)$ from u to v is uniquely determined. The vertices of degree 1 in a tree are its *terminals*.

A pair $X = (T, \mathrm{rt}_X)$ consisting of a tree $T = (V, E)$ and a vertex $\mathrm{rt}_X \in V$ is called a *rooted tree* and we call the distinguished vertex rt_X a *root*.

For a rooted tree $X = (T, \mathrm{rt}_X)$ and vertices $v, v' \in V(T)$, v is a *child* of v' if $vv' \in E(T)$ and the path $P(\mathrm{rt}_X, v)$ contains the edge vv'.

Let $e = xy$ be an edge of $G = (V, E)$. By G/e we denote the graph obtained from G by contracting the edge e into a new vertex v_e, which becomes adjacent to all the former neighbours of x and y. Formally, G/e is a graph (V', E') with vertex set $V' := (V \setminus \{x, y\}) \cup \{v_e\}$ and edge set

$$E' := \{vw \in E \mid \{v, w\} \cap \{x, y\} = \emptyset\} \cup \{v_e w \mid xw \in E \setminus \{e\} \text{ or } yw \in E \setminus \{e\}\}.$$

Here, v_e is the new vertex.

Using these terminology, for a certain edge of a given 2-colored rooted tree, we define the edge contraction of a 2-colored rooted tree.

Definition 1.3. For a 2-colored rooted tree $X = (T, \mathrm{rt}_X, V_\bullet)$ and an edge $e = v_1v_2 \in E$ of T with $v_2 \in V_\circ$, let $\overline{T} := T/e$ be the tree obtained from T by contracting e. Consider the triple $\overline{X} := (\overline{T}, \mathrm{rt}_{\overline{X}}, \overline{V}_\bullet)$ consisting of $\overline{T}$,

$$\mathrm{rt}_{\overline{X}} := \begin{cases} \mathrm{rt}_X & \text{if } \mathrm{rt}_X \neq v_1 \text{ and } v_2, \\ v_e & \text{if } \mathrm{rt}_X = v_1 \text{ or } v_2, \end{cases}$$

and

$$\overline{V}_\bullet := \begin{cases} V_\bullet & \text{if } v_1 \notin V_\bullet, \\ (V_\bullet \setminus \{v_1\}) \cup \{v_e\} & \text{if } v_1 \in V_\bullet. \end{cases}$$

Then $\overline{X}$ is a 2-colored rooted tree. $\overline{X}$ is called in this thesis the 2-colored rooted tree obtained from X by contracting an edge e.

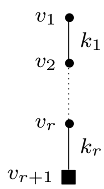
Part 2. **On finite multiple zeta values**

2. Finite multiple zeta value associated to 2-colored rooted trees

2.1. **Finite multiple zeta value associated to 2-colored rooted trees.** In this subsection, we will give two examples of 2-colored rooted trees and finite multiple zeta values associated to them. These examples show that the finite multiple zeta value associated to a 2-colored rooted tree is a generalization of the usual finite multiple zeta value and the finite multiple zeta value of Mordell-Tornheim type. Next, we prove the three basic properties of the finite multiple zeta values associated to 2-colored rooted trees. The first and second properties are about contracting certain edges of 2-colored rooted trees and the third is about changing the roots of the given 2-colored rooted trees. Using these properties, we define the notion "harvestable" for a pair consisting of a 2-colored rooted tree and an index on it. The proof of our first main theorem will be reduced to the case when the pair is harvestable.

Example 2.1. We use diagrams to indicate 2-colored rooted trees $X = (T, \mathrm{rt}_X, V_\bullet)$, with symbols $\circ$ and $\bullet$ corresponding to the vertices in $V_\circ$ or $V_\bullet$ which are not the root, respectively, and we use the symbols $\square$ or $\blacksquare$ whether the root is in $V_\circ$ or $V_\bullet$.

(i) Let $X = (T, \mathrm{rt}_X, V_\bullet)$ be a 2-colored rooted tree and k an index on X as follows.

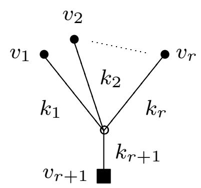


Here, $k_i := k(e_i)$ and $e_i \in E$ is an edge of T and $\mathrm{rt}_X = v_{r+1}$. If we set $m_i := m_{v_i}$ $(1 \le i \le r+1)$, since $L_{e_i}(\mathrm{rt}_X, (m_v)) = m_1 + \cdots + m_i$ $(1 \le i \le r)$, we obtain

$$\begin{aligned}
\zeta_{\mathcal{A}}(X, k) &= \sum_{\substack{m_1, \ldots, m_{r+1} \ge 1 \\ m_1 + \cdots + m_{r+1} = p}} (m_1 + \cdots + m_r)^{-k_r} \cdots (m_1 + m_2)^{-k_2} m_1^{-k_1} \\
&= \sum_{\substack{m_1, \ldots, m_r \ge 1 \\ m_1 + \cdots + m_r \le p-1}} \frac{1}{m_1^{k_1}(m_1 + m_2)^{k_2} \cdots (m_1 + \cdots + m_r)^{k_r}} \\
&= \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \\
&= \zeta_{\mathcal{A}}(k_1, \ldots, k_r).
\end{aligned}$$

Thus, the usual finite multiple zeta value coincides with the finite multiple zeta value associated to the above 2-colored rooted tree.

(ii) Next, consider the following 2-colored rooted tree $X = (T, \mathrm{rt}_X, V_\bullet)$ and the index k on X.

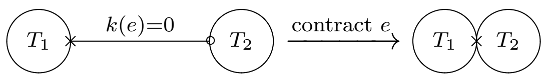


Assume that $\mathrm{rt}_X = v_{r+1}$ and $k_i \geq 1 (1 \leq i \leq r)$. Since $L_{e_i}(\mathrm{rt}_X, (m_v)) = m_i$ $(1 \leq i \leq r)$ and $L_{e_{r+1}}(\mathrm{rt}_X, (m_v)) = m_1 + \cdots + m_r$, we obtain

$$\begin{aligned}\zeta_{\mathcal{A}}(X, k) =& \sum_{\substack{m_1,\ldots,m_{r+1}\geq 1\\ m_1+\cdots+m_{r+1}=p}} m_1^{-k_1}\cdots m_r^{-k_r}(m_1+\cdots+m_r)^{-k_{r+1}}\\ =& \sum_{\substack{m_1,\ldots,m_r\geq 1\\ m_1+\cdots+m_r\leq p-1}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}(m_1+\cdots+m_r)^{k_{r+1}}}\\ =&\zeta_{\mathcal{A}}^{MT}(k_1,\ldots,k_r;k_{r+1}).\end{aligned}$$

Thus we see that the finite multiple zeta value of Mordell-Tornheim type is a special case of the finite multiple zeta value associated to the 2-colored rooted trees.

Proposition 2.2. Let $X = (T, \mathrm{rt}_X, V_\bullet)$ be a 2-colored rooted tree and k be an index on X. Assume that there exists an edge $e = v_1v_2 \in E$ satisfying that v_2 is in $V_\circ$ and $k(e) = 0$. Let $\overline{X} = (\overline{T}, \mathrm{rt}_{\overline{X}}, \overline{V}_\bullet)$ be the 2-colored rooted tree obtained from X contracting e defined in Definition 1.3. These situations can be written as the following figures.



Let $\overline{k} : \overline{E} \to \mathbb{Z}_{\geq 0}$ be the index on $\overline{X}$ defined by $\overline{k}(f) := k(f)$ for $f \in \overline{E}$. Then we have

$$\zeta_{\mathcal{A}}(X, k) = \zeta_{\mathcal{A}}(\overline{X}, \overline{k}).$$

Proof. Since $k(e)=0$, we have

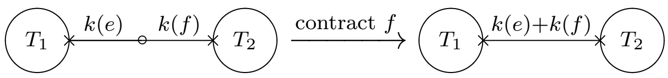
$$\begin{aligned}
\zeta_{\mathcal{A}}(X,k) =& \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{V_\bullet} \text{ s.t.} \\ \sum_{v\in V_\bullet} m_v = p}} L_e(\mathrm{rt}_X,(m_v))^{-k(e)} \prod_{f\in E\setminus\{e\}} L_f(\mathrm{rt}_X,(m_v))^{-k(f)} \\
=& \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{\overline{V}_\bullet} \text{ s.t.} \\ \sum_{v\in \overline{V}_\bullet} m_v = p}} \prod_{f\in \overline{E}} L_f(\mathrm{rt}_{\overline{X}},(m_v))^{-\overline{k}(f)} \\
=& \zeta_{\mathcal{A}}(\overline{X},\overline{k}),
\end{aligned}$$

which completes the proof. $\square$

Proposition 2.3. Let $X=(T,\mathrm{rt}_X,V_\bullet)$ be a 2-colored rooted tree and k an index on X. Assume that there exist edges $e=v_1v_2$, $f=v_2v_3\in E$ satisfying that e and f are incident at a vertex $v_2\in V_\circ$ with $\deg(v_2)=2$. Let $\overline{X}=(\overline{T},\mathrm{rt}_{\overline{X}},\overline{V}_\bullet)$ be the 2-colored rooted tree obtained from X by contracting f defined in Definition 1.3. Let $\overline{k}:\overline{E}\to\mathbb{Z}_{\geq 0}$ be the index on $\overline{X}$ defined by, for $g\in\overline{E}$,

$$\overline{k}(g) := \begin{cases} k(e)+k(f) & \text{if } g=e, \\ k(g) & \text{otherwise.} \end{cases}$$

These situations can be also written as the following figures.



Then we have

$$\zeta_{\mathcal{A}}(X,k)=\zeta_{\mathcal{A}}(\overline{X},\overline{k}).$$

Proof. Since v is a vertex in $V_\circ$, we have

$$\{v\in V_\bullet \mid e\in P(\mathrm{rt}_X,v)\} = \{v\in V_\bullet \mid e'\in P(\mathrm{rt}_X,v)\}.$$

Therefore, we obtain

$$
\begin{aligned}
\zeta_{\mathcal{A}}(X,k) &= \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{V_\bullet} \text{ s.t.} \\ \sum_{v\in V_\bullet} m_v = p}} \prod_{e\in E} L_e(\mathrm{rt}_X,(m_v))^{-k(e)} \\
&= \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{V_\bullet} \text{ s.t.} \\ \sum_{v\in V_\bullet} m_v = p}} L_e(\mathrm{rt}_X,(m_v))^{-(k(e)+k(f))} \prod_{g\in E\setminus\{e,f\}} L_g(\mathrm{rt}_X,(m_v))^{-k(g)} \\
&= \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{\overline{V}_\bullet} \text{ s.t.} \\ \sum_{v\in \overline{V}_\bullet} m_v = p}} L_e(\mathrm{rt}_{\overline{X}},(m_v))^{-\overline{k}(e)} \prod_{g\in \overline{E}\setminus\{e\}} L_g(\mathrm{rt}_{\overline{X}},(m_v))^{-\overline{k}(g)} \\
&= \zeta_{\mathcal{A}}(\overline{X},\overline{k}),
\end{aligned}
$$

which completes the proof. □

The following proposition, which is a generalization of [Kam, Lemma 3.1], is a key in obtaining non-trivial relations among the classical finite multiple zeta values.

Proposition 2.4. For a tree $T=(V,E)$, vertices $v_1,v_2\in V$ and a subset $V_\bullet$ of V, let X_1 (resp. X_2) be the 2-colored rooted tree consisting of T, $\mathrm{rt}_{X_1}=v_1$ (resp. $\mathrm{rt}_{X_2}=v_2$) and $V_\bullet$. Then we have

$$\zeta_{\mathcal{A}}(X_1,k) = (-1)^{k(P(v_1,v_2))}\zeta_{\mathcal{A}}(X_2,k)$$

for an index k on X. Here, we set $k(P(v_1,v_2)) := \sum_{e\in P(v_1,v_2)} k(e)$.

Proof. Consider the path $P(v_1,v_2)$ from v_1 to v_2. If $e\in P(v_1,v_2)$, $V_\bullet$ is divided into two subsets as follows:

$$V_\bullet = \{v\in V_\bullet \mid e\in P(v_1,v)\} \sqcup \{v\in V_\bullet \mid e\in P(v_2,v)\}.$$

Therefore, we have $L_e(v_1,(m_v)) = p - L_e(v_2,(m_v))$ for $e\in P(v_1,v_2)$. On the other hand, if $e\notin P(v_1,v_2)$, we see that

$$\{v\in V_\bullet \mid e\in P(v_1,v)\} = \{v\in V_\bullet \mid e\in P(v_2,v)\}.$$

Thus, we have $L_e(v_1,(m_v)) = L_e(v_2,(m_v))$ for $e \notin P(v_1,v_2)$. Therefore, we obtain

$$
\begin{aligned}
&\zeta_{\mathcal{A}}(X_1,k)\\
=&\sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{V_\bullet}\text{ s.t.}\\ \sum_{v\in V_\bullet} m_v=p}} \prod_{e\in P(v_1,v_2)} \frac{1}{(p-L_e(v_2,(m_v)))^{k(e)}} \prod_{e\notin P(v_1,v_2)} \frac{1}{L_e(v_2,(m_v))^{k(e)}}\\
=&(-1)^{\sum_{e\in P(v_1,v_2)} k(e)} \sum_{\substack{(m_v)\in\mathbb{Z}_{\geq 1}^{V_\bullet}\text{ s.t.}\\ \sum_{v\in V_\bullet} m_v=p}} \prod_{e\in P(v_1,v_2)} \frac{1}{L_e(v_2,(m_v))^{k(e)}} \prod_{e\notin P(v_1,v_2)} \frac{1}{L_e(v_2,(m_v))^{k(e)}}\\
=&(-1)^{k(P(v_1,v_2))}\zeta_{\mathcal{A}}(X_2,k),
\end{aligned}
$$

which completes the proof. □

Example 2.5. Consider the 2-colored rooted tree X and the index k in Example 2.1 (ii) with $\mathrm{rt}_X = v_{r+1}$. Then $\zeta_{\mathcal{A}}(X,k)$ coincides with $\zeta_{\mathcal{A}}^{MT}(k_1,\dots,k_r;k_{r+1})$. Let Y be the 2-colored rooted tree whose root is v_1. Then we have $\zeta_{\mathcal{A}}(Y,k) = \zeta_{\mathcal{A}}^{MT}(k_{r+1},k_2,\dots,k_r;k_1)$. Therefore, by Proposition 2.4, we have

$$
\begin{aligned}
\zeta_{\mathcal{A}}^{MT}(k_1,\dots,k_r;k_{r+1}) = \zeta_{\mathcal{A}}(',k) &= (-1)^{k(P(v_{r+1},v_1))}\zeta_{\mathcal{A}}(Y,k)\\
&= (-1)^{k_1+k_{r+1}}\zeta_{\mathcal{A}}^{MT}(k_{r+1},k_2,\dots,k_r;k_1),
\end{aligned}
$$

which is Kamano's result [Kam, Lemma 3.1].

For the proof of our main theorem, we need the following definitions that the pair consisting of a 2-colored rooted tree and an index on it is harvestable and that an index on a 2-colored rooted tree is essentially positive.

Definition 2.6. Let $X = (T,\mathrm{rt}_X,V_\bullet)$ be a 2-colored rooted tree and k an index on X. We say that the pair (X,k) is harvestable if the following conditions on (X,k) hold.

- **(H1):** The root rt_X is a terminal of T. In particular, rt_X is in $V_\bullet$.
- **(H2):** $\deg(v) \leq 2$ for all v in $V_\bullet$ and $\deg(v) \geq 3$ for all v in $V_\circ$.
- **(H3):** If an edge e connects a branched point v in $V_\circ$ and a child of v in $V_\bullet$, $k(e)$ is positive.

Definition 2.7. For a 2-colored rooted tree $X = (T,\mathrm{rt}_X,V_\bullet)$, an index k on X is essentially positive if $k(P(v,v'))$ is positive for any vertices $v \neq v'$ in $V_\bullet$.

Remark 2.8. If we delete the edges of 2-colored rooted trees which connect the branched point nearest to the root, the 2-colored rooted tree decomposes into many parts. We take the upper part, and the parts under the branched point again become 2-colored rooted trees satisfying the conditions **(H1)**, **(H2)** and **(H3)** after adding new roots to each part. We call this operation of taking the upper part as "harvest", and this is the reason why we call 2-colored rooted trees satisfying **(H1)**, **(H2)** and **(H3)** as being harvestable.

By using Propositions 2.2, 2.3 and 2.4, we see in Proposition 2.9 that for a pair consisting of a 2-colored rooted tree and an essentially positive index on it, there exists a harvestable pair such that finite multiple zeta values associated to them coincides up to sign.

Proposition 2.9. Let $X = (T, \mathrm{rt}_X, V_\bullet)$ be a 2-colored rooted tree and k an essentially positive index on X. Then, there exists a harvestable pair $(X_\mathrm{h}, k_\mathrm{h})$ of the 2-colored rooted tree $X_\mathrm{h} = (T_\mathrm{h} = (V(X_\mathrm{h}), E(X_\mathrm{h})), \mathrm{rt}_{X_\mathrm{h}}, V_\bullet(X_\mathrm{h}))$ and the index k_h on X_h satisfying

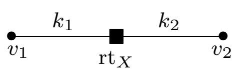
$$\zeta_{\mathcal{A}}(X, k) = (-1)^{k_\mathrm{h}(P(\mathrm{rt}_X, \mathrm{rt}_{X_\mathrm{h}}))} \zeta_{\mathcal{A}}(X_\mathrm{h}, k_\mathrm{h}). \tag{2.1}$$

We understand $k_\mathrm{h}(P(\mathrm{rt}_X, \mathrm{rt}_{X_\mathrm{h}})) = 0$ if $\mathrm{rt}_X = \mathrm{rt}_{X_\mathrm{h}}$.

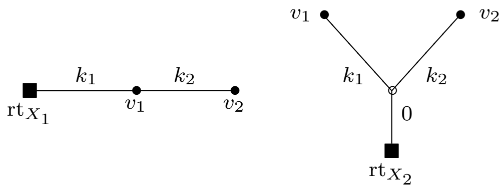
Proof. By using Propositions 2.2 and 2.3 to contract the edges one of whose end points is in $V_\circ$ and $k(e) = 0$, and the edges connecting $v' \in V_\circ$ with $\deg(v') = 2$, and again using Proposition 2.2 to insert edges e' with $k(e') = 0$ and vertices in $V_\circ$ into vertices in $V_\bullet$ whose degree is greater than or equal to 3, we obtain a pair (X', k') of a 2-colored rooted tree X' and an index k' on X' satisfying the conditions **(H2)**, **(H3)** and that finite multiple zeta values associated to them coincides. Further, by using Proposition 2.4 to move the root rt_X to a terminal, we obtain a desired harvestable pair $(X_\mathrm{h}, k_\mathrm{h})$ satisfying $\zeta_{\mathcal{A}}(X, k) = (-1)^{k_\mathrm{h}(P(\mathrm{rt}_X, \mathrm{rt}_{X_\mathrm{h}}))} \zeta_{\mathcal{A}}(X_\mathrm{h}, k_\mathrm{h})$, which completes the proof. $\square$

Definition 2.10. For a 2-colored rooted tree X and an essentially positive index k on X, we define a harvestable form of the pair (X, k) as the harvestable pair $(X_\mathrm{h}, k_\mathrm{h})$ satisfying Proposition 2.9.

Remark 2.11. For a 2-colored rooted tree X and an essentially positive index k on X, a harvestable pair $(X_\mathrm{h}, k_\mathrm{h})$ of (X, k) is not unique. For example, consider the following 2-colored rooted tree $X = (T, \mathrm{rt}_X, V_\bullet)$ and an essentially positive index k on X.



Then, we can take the following two 2-colored rooted trees $X_1 = (T_1, \mathrm{rt}_{X_1}, V_\bullet(X_1)), X_2 = (T_2, \mathrm{rt}_{X_2}, V_\bullet(X_2))$ and indices as harvestable forms of (X, k).



2.2. **First main result.** In this subsection, we will prove our first main theorem (Theorem 1.1). If the pair (X, k) consisting of a 2-colored rooted tree X and an essentially positive index k on X is harvestable, our first main theorem will be proved by induction on the sum of the index at the edges in paths from the branched point nearest to the root to all terminals (Proposition 2.12). The general case will be deduced to the harvestable case by using Proposition 2.9.

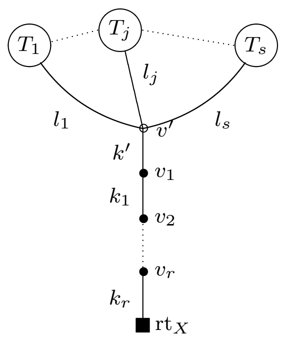
The next proposition is the harvestable case of our first main theorem.

Proposition 2.12. Let $X = (T, \mathrm{rt}_X, V_\bullet)$ be a 2-colored rooted tree and k be an essentially positive index on X. Assume that the pair (X, k) is harvestable. Then $\zeta_{\mathcal{A}}(X, k)$ coincides with the image $Z_{\mathcal{A}}(w)$ of $w \in \mathfrak{H}^1$ constructed by the following inductive method.

First, if (X, k) has no branched point, (X, k) coincides with the 2-colored rooted tree in Example. In this case, take

$$w := z_{k_1} \cdots z_{k_r}.$$

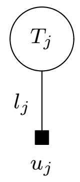
Next, assume that (X, k) has n branched point ($n \in \mathbb{Z}_{\geq 1}$). Since (X, k) is a harvestable pair, (X, k) has the following shape.



Here, r and s are positive integers and $k_i := k(e_i)$ $(1 \leq i \leq r), l_j := k(f_j)$ $(1 \leq j \leq s), k' := k(e')$ for edges e_i, f_j and e' of T and $T_1, \ldots, T_s$ are subtrees of T. Then we take

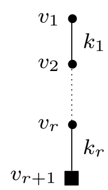
$$w := \Big(\mathop{\text{Ш}}_{j=1}^{s} w_j \Big) x^{k'} z_{k_1} \cdots z_{k_r}.$$

Here, $w_j \in \mathfrak{H}^1$ $(1 \leq j \leq s)$ are elements corresponding to the following harvestable pair $(X_j, k^{(j)})$.

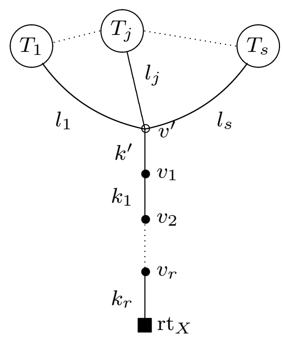


Note that since (X, k) is harvestable and k is essentially positive, $(X_j, k^{(j)})$ is harvestable and $k^{(j)}$ is essentially positive for any $1 \leq j \leq s$. Further, since X has n branched points, X_j has $n-1$ branched points.

Proof. Let $v' \in V$ be the branched point nearest to the root rt_X in all the branched points. As (X, k) is harvestable, v' is an element in $V_\circ$. Set $S = S(X, k) := \sum_e k(e)$, where e runs through edges in paths from v' to all terminals. If there exists no branched point, we set $S = 0$. We prove the statement by the induction on $S \geq 0$. If $S = 0$, as (X, k) is harvestable, we see that $\zeta_{\mathcal{A}}(X, k)$ coincides with that associated to the following 2-colored rooted tree and the index on it with the root v_{r+1}:



Therefore, from Example 2.1 (i), we have $\zeta_{\mathcal{A}}(X, k) = \zeta_{\mathcal{A}}(k_1, \ldots, k_r) = Z_{\mathcal{A}}(z_{k_1} \cdots z_{k_r})$, which completes the proof of the case $S = 0$. Next, assume that $S > 0$ and the statement holds for all the non-negative integers less than S. The assumption $S > 0$ means that there exists at least one branched point. Then the given 2-colored rooted tree $X = (T, \mathrm{rt}_X, V_\bullet)$ and the index k on X can be written as follows:



Then, if we set

$$M_j := \sum_{\substack{v\in V_\bullet \text{ s.t.} \\ f_j\in P(\mathrm{rt}_X,v)}} m_v = L_{f_j}(\mathrm{rt}_X,(m_v))$$

for $1\le j\le s$, by definition we have

$$\zeta_{\mathcal{A}}(X,k) = \sum_{\substack{(m_v)\in\mathbb{Z}_{\ge1}^{V_\bullet} \text{ s.t.} \\ \sum_{v\in V_\bullet} m_v=p}} \prod_{i=1}^{r} \frac{1}{L_{e_i}(\mathrm{rt}_X,(m_v))^{k_i}} \times \frac{1}{L_{e'}(\mathrm{rt}_X,(m_v))^{k'}}$$

$$\times \prod_{j=1}^{s} \prod_{e\in E(T_j)} \frac{1}{L_e(\mathrm{rt}_X,(m_v))^{k(e)}} \times \frac{1}{M_1^{l_1}\cdots M_s^{l_s}}.$$

By (1.5), we have

$$\frac{1}{M_1^{l_1}\cdots M_s^{l_s}} = \frac{1}{M_1+\cdots+M_s}\left(\frac{1}{M_1^{l_1-1}M_2^{l_2}\cdots M_s^{l_s}}+\cdots+\frac{1}{M_1^{l_1}\cdots M_{s-1}^{l_{s-1}}M_s^{l_s-1}}\right).$$

Therefore, since $L_{e'}(\mathrm{rt}_X,(m_v)) = M_1+\cdots+M_s$, we obtain

$$\zeta_{\mathcal{A}}(X,k) = \sum_{j=1}^{s} \zeta_{\mathcal{A}}(X,\alpha_j).$$

Here, α_j is an index on X defined by

$$\alpha_j(e) = \begin{cases} l_j-1 & \text{if } e=f_j, \\ k'+1 & \text{if } e=e', \\ k(e) & \text{otherwise,} \end{cases}$$

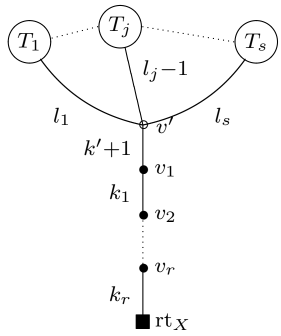
and

$$\zeta_{\mathcal{A}}(X,\alpha_j) = \sum_{\substack{(m_v)\in\mathbb{Z}_{\ge1}^{V_\bullet} \text{ s.t.} \\ \sum_{v\in V_\bullet} m_v=p}} \prod_{i=1}^{r} \frac{1}{L_{e_i}(\mathrm{rt}_X,(m_v))^{k_i}} \times \frac{1}{L_{e'}(\mathrm{rt}_X,(m_v))^{k'+1}}$$

$$\times \prod_{j=1}^{s} \prod_{e\in E(T_j)} \frac{1}{L_e(\mathrm{rt}_X,(m_v))^{k(e)}} \times \frac{1}{M_1^{l_1}\cdots M_j^{l_j-1}\cdots M_s^{l_s}}.$$

To use the induction hypotheses, we need to consider two cases whether (X,α_j) is harvestable or not.

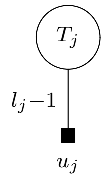
(i) First, consider the case that the pair (X, α_j) is harvestable. This is the case when $l_j > 1$ or the child of v' incident to f_j is in $V_\circ$. In this case, the pair (X, α_j) has the following shape.



Since $S(X, \alpha_j) = S(X, k) - 1 < S(X, k)$, by the induction hypotheses, we obtain

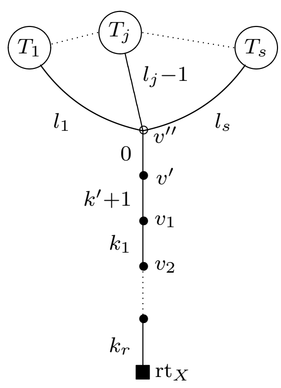
$$\zeta_{\mathcal{A}}(X, \alpha_j) = Z_{\mathcal{A}}\left(\left(\mathop{\text{Ш}}_{\substack{a=1\\a\neq j}}^{s} w_a \mathbin{\text{ш}} w_j'\right) x^{k'+1} z_{k_1}\cdots z_{k_r}\right).$$

Here w_j' is the element of $\mathfrak{H}^1$ corresponding to the following harvestable pair (X_j, β_j).



Note that $w_j = w_j' x$ in this case.

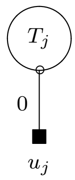
(ii) Next, consider the case that the pair (X, α_j) is not harvestable. This is the case when $l_j = 1$ and the child of v' incident to f_j is in $V_\bullet$ because $\alpha_j(f_j) = l_j - 1 = 0$. By using Proposition 2.2 to contract f_j and insert edges e' with $\alpha_j(e') = 0$, we obtain a harvestable form $(X_{\mathrm{h}}, \alpha_{j,\mathrm{h}})$ of (X, α_j) as follows.



Since $S(X, \alpha_j) = S(X_{\mathrm{h}}, \alpha_{j,\mathrm{h}}) = S(X, k) - 1 < S(X, k)$, by the induction hypotheses, we obtain

$$\begin{aligned}\zeta_{\mathcal{A}}(X, \alpha_j) = \zeta_{\mathcal{A}}(X_{\mathrm{h}}, \alpha_{j,\mathrm{h}}) &= Z_{\mathcal{A}}\left(\left(\mathop{\text{Ш}}_{\substack{a=1\\a\neq j}}^{s} w_a \,\text{ш}\, w'_j\right) x^0 z_{k'+1} z_{k_1} \cdots z_{k_r}\right)\\ &= Z_{\mathcal{A}}\left(\left(\mathop{\text{Ш}}_{\substack{a=1\\a\neq j}}^{s} w_a \,\text{ш}\, w'_j\right) y x^{k'} z_{k_1} \cdots z_{k_r}\right).\end{aligned}$$

Here w'_j is the element of $\mathfrak{H}^1$ corresponding to the following harvestable pair (X'_j, β_j).



Note also that $w_j = w'_j y$ in this case.

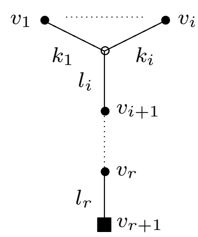
Therefore, by the definition of the shuffle product, we obtain

$$\zeta_{\mathcal{A}}(X, k) = \sum_{j=1}^{s} \zeta_{\mathcal{A}}(X, \alpha_j) = Z_{\mathcal{A}}\left(\left(\mathop{\text{Ш}}_{j=1}^{s} w_j\right) x^{k'} z_{k_1} \cdots z_{k_r}\right).$$

Therefore, $w := (\text{Ш}_{j=1}^{s} w_j) x^{k'} z_{k_1} \cdots z_{k_r}$ is the desired element in $\mathfrak{H}^1$. □

Proof of Theorem 1.1. By Proposition 2.9, for a given pair (X, k) consisting of a 2-colored rooted tree X and an essentially positive index k on X, there exists a harvestable pair $(X_{\mathrm{h}}, k_{\mathrm{h}})$ satisfying (2.1). Since the pair $(X_{\mathrm{h}}, k_{\mathrm{h}})$ is harvestable, by Proposition 2.12, the right hand side of (2.1) can be written explicitly as a $\mathbb{Z}$-linear combination of the usual finite multiple zeta values, so can $\zeta_{\mathcal{A}}(X, k)$. This completes the proof of our main theorem. □

Example 2.13. (i) For $1 \leq i \leq r$, consider the following 2-colored rooted tree X and the essentially positive index k on X.



Set $\mathrm{rt}_X = v_{r+1}$. Since k is essentially positive, k_j $(1 \le j \le i)$ and l_j $(i+1 \le j \le r)$ are positive. Then we have

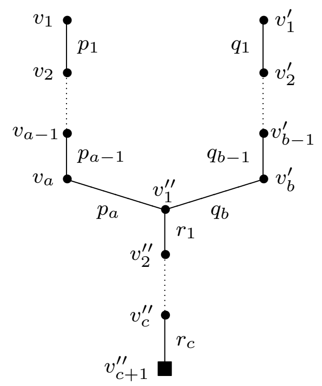
$$\begin{aligned}\zeta_{\mathcal{A}}(X,k) = & \sum_{\substack{m_1,\dots,m_r\ge 1\\ m_1+\cdots+m_r\le p-1}} \frac{1}{m_1^{k_1}\cdots m_i^{k_i}(m_1+\cdots+m_i)^{l_i}\cdots(m_1+\cdots+m_r)^{l_r}}\\ =& Z_{\mathcal{A}}((z_{k_1} ш \cdots ш z_{k_i})x^{l_i}z_{l_{i+1}}\cdots z_{l_r}),\end{aligned}$$

which is Kamano's result [Kam, Theorem 2.1]. In particular, if $i = r$, then we obtain the case of finite multiple zeta values of Mordell-Tornheim type

$$\begin{aligned}\zeta_{\mathcal{A}}(X,k) = & \sum_{\substack{m_1,\dots,m_r\ge 1\\ m_1+\cdots+m_r\le p-1}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}(m_1+\cdots+m_r)^{l_{r+1}}}\\ =& Z_{\mathcal{A}}((z_{k_1} ш \cdots ш z_{k_r})x^{l_{r+1}}),\end{aligned}$$

which is also Kamano's result [Kam, Theorem 1.2].

(ii) Consider the following 2-colored tree X and the essentially positive index k on X.



Here, a, b and c are non-negative integers. Set $\mathrm{rt}_X = v''_{c+1}$. Since k is essentially positive, p_x $(1 \le x \le a)$, q_y $(1 \le y \le b)$ and r_z $(1 \le z \le c)$ are all positive. Then we have

$$\begin{aligned}\zeta_{\mathcal{A}}(X,k) = & \sum_{\substack{0<l_1<\cdots<l_a\\ 0<m_1<\cdots<m_b\\ l_a+m_b<n_1<\cdots<n_c<p}} \frac{1}{l_1^{p_1}\cdots l_a^{p_a}m_1^{q_1}\cdots m_b^{q_b}n_1^{r_1}\cdots n_c^{r_c}}\\ =& Z_{\mathcal{A}}((z_{p_1}\cdots z_{p_a} ш z_{q_1}\cdots z_{q_b})z_{r_1}\cdots z_{r_c}),\end{aligned}$$

which is a finite analogue of a result of Komori, Matsumoto and Tsumura [KMT, Theorem 1].

Remark 2.14. Since the only tool used to prove Proposition 2.12 is the partial fraction decomposition (1.5), the analogous statement of Proposition 2.12 for the MZVs holds. For example, we obtain a result

$$\sum_{\substack{0<l_1<\cdots<l_a \\ 0<m_1<\cdots<m_b \\ l_a+m_b<n_1<\cdots<n_c}} \frac{1}{l_1^{p_1}\cdots l_a^{p_a} m_1^{q_1}\cdots m_b^{q_b} n_1^{r_1}\cdots n_c^{r_c}} = Z((z_{p_1}\cdots z_{p_a} ш z_{q_1}\cdots z_{q_b})z_{r_1}\cdots z_{r_c}) \tag{2.2}$$

of Komori-Matsumoto-Tsumura [KMT, Theorem 1] by using our method. Here,

$$Z : \mathfrak{H}^0 := \mathbb{Q} + y\mathfrak{H}x \to \mathbb{R}\ ;\quad z_{k_1}\cdots z_{k_r} \mapsto \zeta(k_1,\ldots,k_r)$$

is a $\mathbb{Q}$-linear map. The left hand side of (2.2) can be regarded as a special value of the multiple zeta function $\zeta(\mathbf{s};A_r)$ of the root system of type A_r

$$\zeta(\mathbf{s};A_r) := \sum_{m_1,\ldots,m_r\geq 1}\prod_{1\leq i<j\leq r+1}(m_i+\cdots+m_{j-1})^{-s_{ij}},$$

which was first defined by Matsumoto and Tsumura in [MT]. Indeed, we have

$$\begin{aligned}&\sum_{\substack{0<l_1<\cdots<l_a \\ 0<m_1<\cdots<m_b \\ l_a+m_b<n_1<\cdots<n_c}} \frac{1}{l_1^{p_1}\cdots l_a^{p_a} m_1^{q_1}\cdots m_b^{q_b} n_1^{r_1}\cdots n_c^{r_c}} \\ =&Z((z_{p_1}\cdots z_{p_a} ш z_{q_1}\cdots z_{q_b})z_{r_1}\cdots z_{r_c}) \\ =&\zeta((k_{i,j});A_r)\end{aligned}$$

for

$$k_{ij} = \begin{cases} p_{j-1} & i=1, 2\leq j\leq a+1, \\ q_{j-(a+1)} & i=a+1, a+2\leq j\leq a+b+1, \\ r_{j-(a+b+1)} & i=1, a+b+2\leq j\leq a+b+c+1, \\ 0 & \text{otherwise.}\end{cases}$$

2.3. **Applications for shuffle relations among finite multiple zeta values.** In this subsection, using Theorem 1.1 and Proposition 2.4, we give another proof of the shuffle relation among finite multiple zeta values, which was first proved by Kaneko and Zagier in [KZ].

Corollary 2.15. ([KZ]) For positive integers $k_1,\ldots,k_r,l_1,\ldots,l_s$ and elements $w := z_{k_1}\cdots z_{k_r}, w' := z_{l_1}\cdots z_{l_s} \in \mathfrak{H}^1$, we have

$$Z_{\mathcal{A}}(w ш w') = (-1)^{l_1+\cdots+l_s} Z_{\mathcal{A}}(z_{k_1}\cdots z_{k_r} z_{l_s}\cdots z_{l_1}).$$

First, we give the proof of the shuffle relation due to Kaneko-Zagier. We introduce a notation. If we are given a power series

$$f(T)=\sum_{n=0}^{\infty}a_nT^n,$$

we denote the coefficient of f at the degree k by

$$\mathrm{Coeff}[f(T);T^k]:=a_k.$$

Proof. Denote indices corresponding to words $w=z_{k_1}\cdots z_{k_r}, w'=z_{l_1}\cdots z_{l_s}$ by $\mathbf{k}=(k_1,\dots,k_r), \mathbf{l}=(l_1,\dots,l_s)$. Note that

$$\sum_{0<n_1<\cdots<n_r<p}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}=\sum_{0<i<p}\mathrm{Coeff}[\mathrm{Li}_{\mathbf{k}}(z);z^i]$$

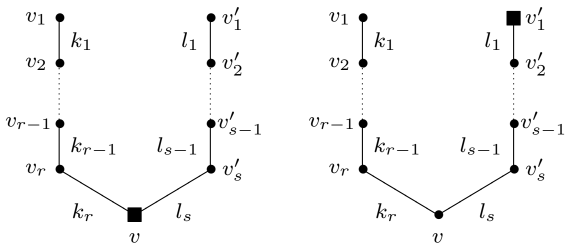
and multiple polylogarithms satisfy the shuffle relation (1.8). Then we obtain

$$\begin{aligned}
&Z_{\mathcal{A}}(w\,ш\,w')\\
=&\sum_{0<n<p}\mathrm{Coeff}[\mathrm{Li}_{\mathbf{k}}(z)\mathrm{Li}_{\mathbf{l}}(z);z^n]\\
=&\sum_{\substack{0<i,j<p\\ i+j<p}}\mathrm{Coeff}[\mathrm{Li}_{\mathbf{k}}(z);z^i]\mathrm{Coeff}[\mathrm{Li}_{\mathbf{l}}(z);z^j]\\
=&\sum_{\substack{0<i,j<p\\ i+j<p}}\left(\sum_{0<m_1<\cdots<m_r<0}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}i^{k_r}}\right)\left(\sum_{0<n_1<\cdots<n_{s-1}<j}\frac{1}{n_1^{l_1}\cdots n_{s-1}^{l_{s-1}}j^{l_s}}\right)\\
=&\sum_{\substack{0<i,j<p\\ i+j<p}}\left(\sum_{0<m_1<\cdots<m_r<0}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}i^{k_r}}\right)\\
&\qquad\times\left(\sum_{0<n_1<\cdots<n_{s-1}<j}\frac{(-1)^{l_1+\cdots+l_s}}{(p-n_1)^{l_1}\cdots(p-n_{s-1})^{l_{s-1}}(p-j)^{l_s}}\right)\\
=&\sum_{\substack{0<i,j<p\\ i+j<p}}\left(\sum_{0<m_1<\cdots<m_r<0}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}i^{k_r}}\right)\left(\sum_{j<n_{s-1}<\cdots<n_1<p}\frac{(-1)^{l_1+\cdots+l_s}}{j^{l_s}n_{s-1}^{l_{s-1}}\cdots n_1^{l_1}}\right)\\
=&\sum_{0<m_1<\cdots<m_{r-1}<i<j<n_2<\cdots<n_s<p}\frac{(-1)^{l_1+\cdots+l_s}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}i^{k_r}j^{l_s}n_{s-1}^{l_{s-1}}\cdots n_1^{l_1}}\\
=&(-1)^{l_1+\cdots+l_s}Z_{\mathcal{A}}(z_{k_1}\cdots z_{k_r}z_{l_s}\cdots z_{l_1}),
\end{aligned}$$

which completes the proof of the shuffle relation. $\square$

Next, we give a proof of the shuffle relation using Theorem 1.1.

Proof. Consider the following two 2-colored rooted trees X, X', whose roots are v and v'_1, and index k on X and X'.



Then, by Proposition 2.4, we have

$$\zeta_{\mathcal{A}}(X, k) = (-1)^{k(P(v,v'_1))}\zeta_{\mathcal{A}}(X', k). \tag{2.3}$$

By Theorem 1.1, the left hand side of (2.3) coincides with

$$Z_{\mathcal{A}}(w ш w').$$

On the other hand, by Proposition 2.12, the right hand side of (2.3) coincides with

$$(-1)^{l_1+\cdots+l_s} Z_{\mathcal{A}}(z_{k_1}\cdots z_{k_r} z_{l_s}\cdots z_{l_1}).$$

Therefore, we obtain the shuffle relation among finite multiple zeta values. $\square$

Remark 2.16. The case $r = s = 1, k_1 = 1$ and $l_1 = k - 1$ for $k > 1$, the Proposition 2.15 says that

$$Z_{\mathcal{A}}(z_1 ш z_{k-1}) = -Z_{\mathcal{A}}(z_{k-1}z_1). \tag{2.4}$$

The right hand side of (2.4) is $-\zeta_{\mathcal{A}}(k-1, 1)$, which is equal to $B_{\mathbf{p}-k}$ by Hoffman's result [Hof2, Theorem 6.1]. Here $B_{\mathbf{p}-k} := (B_{p-k})_p \in \mathcal{A}$ and B_n is the n-th Bernoulli number. On the other hand, since

$$z_1 ш z_{k-1} = z_1 z_{k-1} + \sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=k}} z_{k_1}z_{k_2},$$

the left hand side of (2.4) is equal to

$$\zeta_{\mathcal{A}}(1, k-1) + \sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=k}} \zeta_{\mathcal{A}}(k_1, k_2).$$

Therefore, by [Hof2, Theorem 4.4], we have

$$\sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=k}} \zeta_{\mathcal{A}}(k_1,k_2) = -(\zeta_{\mathcal{A}}(k-1,1)+\zeta_{\mathcal{A}}(1,k-1)) = 0.$$

This equality is equivalent to the sum formula for double finite multiple zeta values [SW1, Theorem 1.4]. Indeed, by [SW1, Theorem 1.4] and [Hof2, Theorem 6.1], we have

$$\sum_{\substack{k_1,k_2\geq 1,k_i\geq 2\\ k_1+k_2=k}} \zeta_{\mathcal{A}}(k_1,k_2) = (-1)^{k+i}B_{p-k} = \begin{cases} -\zeta_{\mathcal{A}}(1,k-1) & \text{if } i=1,\\ -\zeta_{\mathcal{A}}(k-1,1) & \text{if } i=2. \end{cases}$$

Part 3. **On finite multiple polylogarithms**

3. Finite multiple polylogarithms

For non-negative integers a, b, c and $l_1, \dots, l_a, m_1, \dots, m_b, n_1, \dots, n_c$, we set

$$\begin{cases} L_i := l_1 + \cdots + l_i & (0 \le i \le a), \\ M_j := m_1 + \cdots + m_j & (0 \le j \le b), \\ N_k := n_1 + \cdots + n_k & (0 \le k \le c). \end{cases} \tag{3.1}$$

3.1. **A variant of finite multiple zeta values.** In this subsection, we define a variant $\zeta_{\mathcal{A}}^{(i)}$ of finite multiple zeta values. This variant turns out to be a sum of finite multiple zeta values (Proposition 3.4), and will play an important role in the proof of our second main theorem.

Definition 3.1. For an index $\mathbf{k} = (k_1, \dots, k_r)$ and $1 \le i \le r$, we define a variant of finite multiple zeta values as an element in $\mathcal{A}$ by

$$\zeta_{\mathcal{A}}^{(i)}(\mathbf{k}) := \sideset{}{'}\sum_{\substack{0<l_1,\dots,l_r<p \\ (i-1)p<L_r<ip}} \frac{1}{L_1^{k_1} \cdots L_r^{k_r}}.$$

Here, we set $L_j := l_1 + \cdots + l_j$ for $1 \le j \le r$.

Note that $\zeta_{\mathcal{A}}^{(1)}(\mathbf{k})$ coincides with $\zeta_{\mathcal{A}}(\mathbf{k})$ by definition.

Remark 3.2. By setting $l_i' := p - l_i$, we see that $\zeta_{\mathcal{A}}^{(r+1-i)}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\zeta_{\mathcal{A}}^{(i)}(\mathbf{k})$.

To explain that $\zeta_{\mathcal{A}}^{(i)}$ can be expressed as a sum of $\zeta_{\mathcal{A}}$'s, we introduce more notation. For a positive integer r, we set $[r] := \{1, \dots, r\}$. For positive integers r and s, set

$$X_r := \{(l_1, \dots, l_r) \in [p-1]^r \mid (L_1, p) = (L_2, p) = \cdots = (L_r, p) = 1\},$$

$$\Phi_r := \bigsqcup_{s=1}^{r} \Phi_{r,s}, \quad \Phi_{r,s} := \{\phi : [r] \to [s] : \text{surjective} \mid \phi(a) \ne \phi(a+1) \text{ for all } a \in [r-1]\},$$

$$Y_s := \{(A_1, \dots, A_s) \in [p-1]^s \mid 0 < A_1 < \cdots < A_s < p\}$$

and we define an integer $\delta_\phi(i)$ by

$$\delta_\phi(i) := \#\{a \in [i-1] \mid \phi(a) > \phi(a+1)\} \quad (1 \le i \le r) \tag{3.2}$$

for $\phi \in \Phi_r$. Next, for $x = (l_1, \dots, l_r) \in X_r$, there exist

$$s \in [r], \quad \phi = \phi_x \in \Phi_{r,s}, \quad (A_1, \dots, A_s) \in Y_s$$

uniquely satisfying that $L_i = l_1 + \cdots + l_i \equiv A_{\phi(i)} \pmod{p}$ for $i = 1, \dots, r$. Indeed, s is the number of distinct remainders of $L_1, \dots, L_r$ modulo p, $A_1, \dots, A_s$ is those remainders, and ϕ is the map defined by $L_i \equiv A_{\phi(i)} \pmod p$ for $i = 1, \dots, r$. Moreover, using s, ϕ and $(A_1, \dots, A_s)$ above, we define the map $f : X_r \to \bigsqcup_{s=1}^r (\Phi_{r,s} \times Y_s)$ by sending $x \in X_r$ to $(\phi, (A_1, \dots, A_s))$.

Lemma 3.3. f is a bijection.

Proof. We construct the inverse map of f. For any $\phi \in \Phi_{r,s}, (A_1, \dots, A_s) \in Y_s$ and $i \in [r]$, we define an integer l_i as follows:

$$l_i := \begin{cases} A_{\phi(1)} & (i = 1), \\ (A_{\phi(i)} + \delta_\phi(i)p) - (A_{\phi(i-1)} + \delta_\phi(i-1)p) & (2 \le i \le r). \end{cases} \tag{3.3}$$

Since

$$\delta_\phi(i) - \delta_\phi(i-1) = \begin{cases} 0 & \text{if } \phi(i-1) < \phi(i), \\ 1 & \text{if } \phi(i-1) > \phi(i), \end{cases} \tag{3.4}$$

we have $0 < l_i < p$ for all i. Indeed, if $i = 1$, by $l_1 = A_{\phi(1)}$ and $0 < A_{\phi(1)} < p$, we have $0 < l_1 < p$. If $2 \le i \le p$ and $\phi(i) > \phi(i-1)$, by (3.4) and $0 < A_{\phi(i-1)} < A_{\phi(i)} < p$, we have $0 < l_i = A_{\phi(i)} - A_{\phi(i-1)} < p$. If $\phi(i) < \phi(i-1)$, by (3.4) and $0 < A_{\phi(i)} < A_{\phi(i-1)} < p$,we have $0 < l_i = A_{\phi(i)} - A_{\phi(i-1)} + p < p$. Moreover, since

$$\begin{aligned} l_1 + \dots + l_i =& A_{\phi(1)} + (A_{\phi(2)} + \delta_\phi(2)p) - A_{\phi(1)} \\ & + (A_{\phi(3)} + \delta_\phi(3)p) - (A_{\phi(2)} + \delta_\phi(2)p) \\ & + \dots + (A_{\phi(i)} + \delta_\phi(i)p) - (A_{\phi(i-1)} + \delta_\phi(i-1)p) \\ =& A_{\phi(i)} + \delta_\phi(i)p, \end{aligned} \tag{3.5}$$

the remainder of $l_1 + \dots + l_i$ modulo p is $A_{\phi(i)}$ and $l_1 + \dots + l_i$ is prime to p. Thus we obtain a map $g : \bigsqcup_{s=1}^r (\Phi_{r,s} \times Y_s) \to X_r$ by $g(\phi, (A_1, \dots, A_s)) = (l_1, \dots, l_r)$.

We prove that g is the inverse of f. First, we prove $g \circ f = \mathrm{id}$. For any $(l_1, \dots, l_r) \in X_r$, set

$$(\phi, (A_1, \dots, A_s)) := f(l_1, \dots, l_r) \in \Phi_{r,s} \times Y_s$$

and

$$(l'_1, \dots, l'_r) := g(\phi, (A_1, \dots, A_s)) \in X_r.$$

By the definition of g, we have $l'_1 + \dots + l'_i \equiv A_{\phi(i)} \pmod p$ for $i = 1, \dots, r$. We prove $l'_i = l_i$ for $i = 1, \dots, r$ by induction on i. For $i = 1$, by the definition of g, we have $l'_1 = A_\phi(1)$. Since $0 < l_1, A_{\phi(1)} < p$, we obtain $l'_1 = A_{\phi(1)} = l_1$. Suppose that $l'_1 = l_1, \dots, l'_{i-1} = l_{i-1}$. By (3.5), we see that $l'_1 + \dots + l'_i \equiv A_{\phi(1)} \equiv l_1 + \dots + l_i \pmod p$. By the induction hypothesis and $0 < l_i, l'_i < p$, we have $l'_i = l_i$.

Next, we prove $f \circ g = \mathrm{id}$. For any $(\phi, (A_1, \ldots, A_s)) \in \Phi_{r,s} \times Y_s$, set

$$(l_1, \ldots, l_r) := g(\phi, (A_1, \ldots, A_s)) \in X_r$$

and

$$(\phi', (A'_1, \ldots, A'_s)) := f(l_1, \ldots, l_r) \in \Phi_{r,s} \times Y_s.$$

By the definition of g, we have $A'_{\phi(i)} \equiv l_1 + \cdots + l_i = A_{\phi(i)} \pmod{p}$. Then we have $(A'_1, \ldots, A'_s) = (A_1, \ldots, A_s)$. Therefore, by the definition of g, we have $\phi' = \phi$. $\square$

Next we define two maps

$$\alpha : X_r \to [r], \quad \beta : \Phi_r \to [r]$$

as follows:

$\alpha(l_1, \ldots, l_r)$ is defined to be the unique integer n satisfying $(n-1)p < l_1 + \cdots + l_r < np$,

$$\beta(\phi) := \delta_\phi(r) + 1.$$

Using α and β, for $1 \le i \le r$, we set

$$X_r^i := \alpha^{-1}(i), \quad \Phi_r^i := \beta^{-1}(i),$$

and $X_\phi := \{x \in X_r \mid \phi_x = \phi\}$ for $\phi \in \Phi_r$. Then we have

$$X_r^i = \bigsqcup_{\phi \in \Phi_r^i} X_\phi \tag{3.6}$$

for $1 \le i \le r$. Further, for $\phi \in \Phi_{r,s}^i := \Phi_r^i \cap \Phi_{r,s}$, the composition

$$X_\phi \xrightarrow{f} \{\phi\} \times Y_s \xrightarrow{\mathrm{pr}_2} Y_s \tag{3.7}$$

is a bijection.

The following is the main result in this subsection.

Proposition 3.4. For $1 \le i \le r$ and an index $\mathbf{k} = (k_1, \ldots, k_r)$ in I_r, we have

$$\zeta_{\mathcal{A}}^{(i)}(\mathbf{k}) = \sum_{\phi \in \Phi_r^i} \zeta_{\mathcal{A}} \left(\sum_{\phi(j)=1} k_j, \ldots, \sum_{\phi(j)=s} k_j \right).$$

Proof. By the definition of X_r^i, (3.6) and (3.7), we obtain

$$\begin{aligned}
\zeta_{\mathcal{A}}^{(i)}(k_1,\ldots,k_r) &= \sum_{(l_1,\ldots,l_r)\in X_r^i} \frac{1}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} \quad (\text{definition of } X_r^i)\\
&= \sum_{\phi\in\Phi_r^i}\sum_{(l_1,\ldots,l_r)\in X_\phi} \frac{1}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} \quad (\text{by } (3.6))\\
&= \sum_{\phi\in\Phi_r^i}\sum_{0<A_1<\cdots<A_s<p} \frac{1}{A_{\phi(1)}^{k_1}\cdots A_{\phi(r)}^{k_r}} \quad (\text{by } (3.7))\\
&= \sum_{\phi\in\Phi_r^i}\sum_{0<A_1<\cdots<A_s<p} \frac{1}{A_1^{\sum_{\phi(j)=1}k_j}\cdots A_s^{\sum_{\phi(j)=s}k_j}}\\
&= \sum_{\phi\in\Phi_r^i}\zeta_{\mathcal{A}}\left(\sum_{\phi(j)=1}k_j,\ldots,\sum_{\phi(j)=s}k_j\right).
\end{aligned}$$

□

Remark 3.5. By Proposition 3.4, we see that $\zeta_{\mathcal{A}}^{(i)}(\mathbf{k})$ is a sum of finite multiple zeta values of weight $\mathrm{wt}(\mathbf{k})$. If we denote the reverse index $\overline{\mathbf{k}} := (k_r,\ldots,k_1)$ of $\mathbf{k}=(k_1,\ldots,k_r)$, using Remark 3.2 and $(-1)^{\mathrm{wt}(\mathbf{k})}\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}(\overline{\mathbf{k}})$, we see that $\zeta_{\mathcal{A}}^{(r+1-i)}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\zeta_{\mathcal{A}}^{(i)}(\mathbf{k}) = \zeta_{\mathcal{A}}^{(i)}(\overline{\mathbf{k}})$.

Example 3.6. We present examples of Proposition 3.4 for $r=3,4$. Note that, since $\zeta_{\mathcal{A}}^{(1)}(\mathbf{k}) = \zeta_{\mathcal{A}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}^{(r+1-i)}(\mathbf{k}) = \zeta_{\mathcal{A}}^{(i)}(\overline{\mathbf{k}})$ (Remark 3.5), we may assume $2\le i\le \frac{r+1}{2}$. We write the terms of the right hand side in lexicographic order.

(i) For $r=3$,

$$\begin{aligned}
\zeta_{\mathcal{A}}^{(2)}(k_1,k_2,k_3) =&\zeta_{\mathcal{A}}(k_1,k_3,k_2)+\zeta_{\mathcal{A}}(k_2,k_1,k_3)+\zeta_{\mathcal{A}}(k_2,k_3,k_1)\\
&+\zeta_{\mathcal{A}}(k_3,k_1,k_2)+\zeta_{\mathcal{A}}(k_1+k_3,k_2)+\zeta_{\mathcal{A}}(k_2,k_1+k_3).
\end{aligned}$$

(ii) For $r=4$,

$$\begin{aligned}
&\zeta_{\mathcal{A}}^{(2)}(k_1,k_2,k_3,k_4)\\
=&\zeta_{\mathcal{A}}(k_1,k_2,k_4,k_3)+\zeta_{\mathcal{A}}(k_1,k_3,k_2,k_4)+\zeta_{\mathcal{A}}(k_1,k_3,k_4,k_2)+\zeta_{\mathcal{A}}(k_1,k_4,k_2,k_3)\\
&+\zeta_{\mathcal{A}}(k_2,k_1,k_3,k_4)+\zeta_{\mathcal{A}}(k_2,k_3,k_1,k_4)+\zeta_{\mathcal{A}}(k_2,k_3,k_4,k_1)+\zeta_{\mathcal{A}}(k_3,k_1,k_2,k_4)\\
&+\zeta_{\mathcal{A}}(k_3,k_1,k_4,k_2)+\zeta_{\mathcal{A}}(k_3,k_4,k_1,k_2)+\zeta_{\mathcal{A}}(k_4,k_1,k_2,k_3)+\zeta_{\mathcal{A}}(k_1,k_2+k_4,k_3)\\
&+\zeta_{\mathcal{A}}(k_1,k_3,k_2+k_4)+\zeta_{\mathcal{A}}(k_1+k_3,k_2,k_4)+\zeta_{\mathcal{A}}(k_1+k_3,k_4,k_2)+\zeta_{\mathcal{A}}(k_1+k_4,k_2,k_3)\\
&+\zeta_{\mathcal{A}}(k_2,k_1+k_3,k_4)+\zeta_{\mathcal{A}}(k_2,k_3,k_1+k_4)+\zeta_{\mathcal{A}}(k_3,k_1+k_4,k_2)+\zeta_{\mathcal{A}}(k_3,k_1,k_2+k_4)\\
&+\zeta_{\mathcal{A}}(k_1+k_3,k_2+k_4).
\end{aligned}$$

3.2. **Second main result.** In this subsection, we define FMPs $\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)$ of type $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ and prove our second main theorem using a method inspired by Komori, Matsumoto and Tsumura [KMT]. $\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)$ is a generalization of both FMPs and products of two FMPs.

Definition 3.7. For indices $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_a), \boldsymbol{\mu}=(\mu_1,\dots,\mu_b)$ and $\boldsymbol{\nu}=(\nu_1,\dots,\nu_c)$ $(a,b,c\in\mathbb{Z}_{\geq 0})$, we define a FMP of type $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ by

$$\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T):=\sideset{}{'}\sum_{\substack{0<l_1,\dots,l_a<p\\0<m_1,\dots,m_b<p\\0<n_1,\dots,n_c<p}}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a}L_x^{\lambda_x}\prod_{y=1}^{b}M_y^{\mu_y}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}}\in\mathcal{B}.$$

Here we also set $L_x:=l_1+\cdots+l_x$ for $1\leq x\leq a$, $M_y:=m_1+\cdots+m_y$ for $1\leq y\leq b$ and $N_z:=n_1+\cdots+n_z$ for $1\leq z\leq c$.

Remark 3.8. By the definition of $\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)$, we have

$$\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)=\mathrm{li}(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\nu};T)$$

for any indices $\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}$ and

$$\mathrm{li}(\boldsymbol{\lambda},\emptyset,\emptyset;T)=\mathrm{li}(\emptyset,\boldsymbol{\lambda},\emptyset;T)=\mathrm{li}(\emptyset,\emptyset,\boldsymbol{\lambda};T)=\mathrm{li}_{\boldsymbol{\lambda}}(T),$$
$$\mathrm{li}(\boldsymbol{\lambda},\emptyset,\boldsymbol{\mu};T)=\mathrm{li}(\emptyset,\boldsymbol{\lambda},\boldsymbol{\mu};T)=\mathrm{li}_{\boldsymbol{\lambda}\bullet\boldsymbol{\mu}}(T),\quad \mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\emptyset;T)=\mathrm{li}_{\boldsymbol{\lambda}}(T)\mathrm{li}_{\boldsymbol{\mu}}(T).$$

for any indices $\boldsymbol{\lambda},\boldsymbol{\mu}$.

To state our second main theorem, we introduce the $\mathcal{Z}_{\mathcal{A}}[T^p]$-submodule

$$\mathcal{R}:=\sum_{\mathbf{k}:\mathrm{index}}\mathcal{Z}_{\mathcal{A}}[T^p]\mathrm{li}_{\mathbf{k}}(T)$$

of $\mathcal{B}$, and $\mathbb{Q}$-subspaces $\mathcal{R}\supset\mathcal{R}_a\supset\mathcal{R}_{a,b}$ as follows.

Definition 3.9. For a non-negative integer a, we define a $\mathbb{Q}$-subspace

$$\mathcal{R}_a:=\langle\zeta_{\mathcal{A}}(\mathbf{k})\cdot(T^p)^n\cdot\mathrm{li}_{\mathbf{k}'}(T)\mid n\in\mathbb{Z}_{\geq0},\mathrm{wt}(\mathbf{k})+\mathrm{wt}(\mathbf{k}')=a\rangle_{\mathbb{Q}}$$

of $\mathcal{R}$. Then we have $\mathcal{R}=\sum_{a=0}^{\infty}\mathcal{R}_a$. Moreover, for non-negative integers a and $b\in\{0,\dots,a\}$, we define a $\mathbb{Q}$-subspace

$$\mathcal{R}_{a,b}:=\langle\zeta_{\mathcal{A}}(\mathbf{k})\cdot(T^p)^n\cdot\mathrm{li}_{\mathbf{k}'}(T)\mid n\in\mathbb{Z}_{\geq0},\mathrm{wt}(\mathbf{k})+\mathrm{wt}(\mathbf{k}')=a,\mathrm{wt}(\mathbf{k}')\leq b\rangle_{\mathbb{Q}}$$

of $\mathcal{R}_a$. Then we have an increasing filtration

$$\mathcal{R}_{a,0}\subset\mathcal{R}_{a,1}\subset\cdots\subset\mathcal{R}_{a,a-1}\subset\mathcal{R}_{a,a}=\mathcal{R}_a$$

of $\mathbb{Q}$-vector spaces.

Remark 3.10. Let $\mathcal{P}$ be the $\mathbb{Q}$-vector space generated by the multiple polylogarithms:

$$\mathcal{P} := \sum_{\mathbf{k}\,:\,\text{index}} \mathbb{Q}\mathrm{Li}_{\mathbf{k}}(T).$$

Then the shuffle relation (1.8) implies that $\mathcal{P}$ forms a $\mathbb{Q}$-algebra. On the other hand, a simple computation using (3.13) shows that

$$\mathrm{li}_1(T)\mathrm{li}_{(1,1)}(T) = 3\mathrm{li}_{(1,1,1)}(T) + \zeta_{\mathcal{A}}(2,1)T^p + \zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p}$$

(See Example 3.24 (i) below). This suggests that the direct analogue $\sum_{\mathbf{k}:\text{index}} \mathbb{Q}\mathrm{li}_{\mathbf{k}}(T)$ of $\mathcal{P}$ is inadequate if we want a subalgebra of $\mathcal{B}$, and that it is more natural to take $\mathcal{Z}_{\mathcal{A}}[T^p]$ as the coefficient ring. Such a consideration motivates the above definition of $\mathcal{R}$, and in fact, as we prove below (Corollary 3.14), $\mathcal{R}$ forms a $\mathcal{Z}_{\mathcal{A}}[T^p]$-subalgebra of $\mathcal{B}$.

The second main theorem of this thesis is the following.

Theorem 3.11 ([OY, Theorem 1.3])**.** *For indices $\mathbf{k} = (k_1, \dots, k_r), \mathbf{k}' = (k'_1, \dots, k'_{r'})$ with $k := \mathrm{wt}(\mathbf{k})$ and $k' := \mathrm{wt}(\mathbf{k}')$, we have*

$$\mathrm{li}_{\mathbf{k}}(T)\mathrm{li}_{\mathbf{k}'}(T) \equiv \mathrm{li}_{\mathbf{k}\,ш\,\mathbf{k}'}(T) \pmod{\mathcal{R}_{k+k',k+k'-1}}.$$

Moreover, $\mathrm{li}_{\mathbf{k}}(T)\mathrm{li}_{\mathbf{k}'}(T) - \mathrm{li}_{\mathbf{k}\,ш\,\mathbf{k}'}(T) \in \mathcal{R}_{k+k',k+k'-1}$ can be calculated explicitly from $\mathrm{li}_{\mathbf{k}}(T)$ and $\mathrm{li}_{\mathbf{k}'}(T)$.

To prove Theorem 3.11, we first show the following proposition.

Proposition 3.12. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_a)$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_b)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_c)$ be indices ($a, b, c \in \mathbb{Z}_{\geq 0}$). Assume that $\boldsymbol{\lambda}, \boldsymbol{\mu} \neq \emptyset$. Then we have the following equality.

$$\begin{aligned}(3.8)\quad &\mathrm{li}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) \\ &\quad = f(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) + f(\boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\nu}; T) - g(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) - g(\boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\nu}; T) + h(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T)\end{aligned}$$

Here,

$$f(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) = \sum_{\tau=0}^{\mu_b - 1} \binom{\lambda_a - 1 + \tau}{\tau} \mathrm{li}((\lambda_1, \dots, \lambda_{a-1}), (\mu_1, \dots, \mu_{b-1}, \mu_b - \tau), (\lambda_a + \tau) \bullet \boldsymbol{\nu}; T),$$

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) = \binom{\lambda_a + \mu_b - 1}{\lambda_a} \left(\sum_{j=1}^{a-1} \zeta_{\mathcal{A}}^{(j)}(\lambda_1, \dots, \lambda_{a-1})(T^p)^j \right) \mathrm{li}_{(\mu_1, \dots, \mu_{b-1}) \bullet (\lambda_a + \mu_b) \bullet \boldsymbol{\nu}}(T),$$

and

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) = (-1)^{\mathrm{wt}(\boldsymbol{\mu})} \left(\sum_{i=1}^{a+b-1} \zeta_{\mathcal{A}}^{(i)}(\boldsymbol{\lambda} \star \boldsymbol{\mu})(T^p)^i \right) \mathrm{li}_{\boldsymbol{\nu}}(T).$$

Here, $\boldsymbol{\lambda} \bullet \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \star \boldsymbol{\mu}$ are indices defined by

$$\boldsymbol{\lambda} \bullet \boldsymbol{\mu} := (\lambda_1, \ldots, \lambda_a, \mu_1, \ldots, \mu_b),\ \boldsymbol{\lambda} \star \boldsymbol{\mu} := (\lambda_1, \ldots, \lambda_{a-1}, \lambda_a + \mu_b, \mu_{b-1}, \ldots, \mu_1).$$

The next lemma plays the most important role in the proof of Proposition 3.12.

Lemma 3.13 ([Wei] equation(2))**.** For indeterminates X and Y and positive integers α and β, we have the following partial fraction decomposition:

$$\frac{1}{X^\alpha Y^\beta} = \sum_{\tau=0}^{\beta-1} \binom{\alpha-1+\tau}{\tau} \frac{1}{(X+Y)^{\alpha+\tau} Y^{\beta-\tau}} + \sum_{\tau=0}^{\alpha-1} \binom{\beta-1+\tau}{\tau} \frac{1}{(X+Y)^{\beta+\tau} X^{\alpha-\tau}}.$$

Proof of Proposition 3.12. First, we separate $\mathrm{li}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T)$ into two parts according to whether $L_a + M_b$ is prime to p or not:

$$\mathrm{li}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}; T) = \left({\sum_{\substack{0<l_\bullet, m_\bullet, n_\bullet<p \\ p \nmid L_a + M_b}}}' + {\sum_{\substack{0<l_\bullet, m_\bullet, n_\bullet<p \\ p \mid L_a + M_b}}}' \right) \frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a} L_x^{\lambda_x} \prod_{y=1}^{b} M_y^{\mu_y} \prod_{z=1}^{c} (L_a + M_b + N_z)^{\nu_z}}. \tag{3.9}$$

Here we abbreviate ${\sum}'_{\substack{0<l_1,\ldots,l_a<p \\ 0<m_1,\ldots,m_b<p \\ 0<n_1,\ldots,n_c<p}}$ to ${\sum}'_{0<l_\bullet, m_\bullet, n_\bullet<p}$. The second term in (3.9) is calculated as

$$\begin{aligned}
&\sum_{i=1}^{a+b-1} {\sum_{\substack{0<l_\bullet, m_\bullet, n_\bullet<p \\ L_a+M_b=ip}}}' \frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a} L_x^{\lambda_x} \prod_{y=1}^{b} M_y^{\mu_y} \prod_{z=1}^{c} (L_a + M_b + N_z)^{\nu_z}} \\
=&(-1)^{\mathrm{wt}(\boldsymbol{\mu})} \sum_{i=1}^{a+b-1} (T^p)^i \\
&\times {\sum_{\substack{0<l_\bullet, m_\bullet, n_\bullet<p \\ (i-1)p<L_a+m_b+\cdots+m_2<ip}}}' \frac{T^{N_c}}{\prod_{x=1}^{a-1} L_x^{\lambda_x} L_a^{\lambda_a+\mu_b} \prod_{y=1}^{b-1} (L_a + m_b + \cdots + m_{y+1})^{\mu_y} \prod_{z=1}^{c} N_z^{\nu_z}},
\end{aligned}$$

by using the congruences $L_a + m_b + \cdots + m_{y+1} \equiv -M_y \pmod p$ $(1 \le y \le b-1)$. The reason why the sum $m_b + \cdots + m_{y+1}$ has its indices written backwards is that they correspond to l-terms in increasing index order, that is, if we put $l_{a+1} := m_b, \ldots, l_{a+b-1} := m_2$, we obtain

$$
\begin{aligned}
&(-1)^{\mathrm{wt}(\boldsymbol{\mu})}\sum_{i=1}^{a+b-1}(T^p)^i \\
&\times \sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\(i-1)p<L_a+m_b+\cdots+m_2<ip}} \frac{T^{N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}L_a^{\lambda_a+\mu_b}\prod_{y=1}^{b-1}(L_a+m_b+\cdots+m_{y+1})^{\mu_y}\prod_{z=1}^{c}N_z^{\nu_z}} \\
=&(-1)^{\mathrm{wt}(\boldsymbol{\mu})}\sum_{i=1}^{a+b-1}(T^p)^i \sideset{}{'}\sum_{0<l_1,\dots,l_{a+b-1}<p}\frac{T^{N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}L_a^{\lambda_a+\mu_b}\prod_{y=1}^{b-1}L_{a+y}^{\mu_{b-y}}}\sideset{}{'}\sum_{0<n_1,\dots,n_c<p}\frac{T^{N_c}}{N_1^{\nu_1}\cdots N_c^{\nu_c}} \\
=&(-1)^{\mathrm{wt}(\boldsymbol{\mu})}\left(\sum_{i=1}^{a+b-1}\zeta_{\mathcal{A}}^{(i)}(\boldsymbol{\lambda}\star\boldsymbol{\mu})(T^i)^p\right)\mathrm{li}_{\boldsymbol{\nu}}(T) \\
=&h(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T).
\end{aligned}
$$

Therefore, we see that the second term in (3.9) coincides with $h(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)$.

Next, using Lemma 3.13 for $(X,Y)=(L_a,M_b)$ and $(\alpha,\beta)=(\lambda_a,\mu_b)$, we calculate the first term in (3.9) as follows:

(3.10)
$$
\sum_{\tau=0}^{\mu_b-1}\binom{\lambda_a-1+\tau}{\tau}\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\p\nmid L_a}}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}\prod_{y=1}^{b-1}M_y^{\mu_y}M_b^{\mu_b-\tau}(L_a+M_b)^{\lambda_a+\tau}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}}
$$

(3.11)
$$
+\sum_{\tau=0}^{\lambda_a-1}\binom{\mu_b-1+\tau}{\tau}\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\p\nmid M_b}}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}L_a^{\lambda_a-\tau}\prod_{y=1}^{b-1}M_y^{\mu_y}(L_a+M_b)^{\mu_b+\tau}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}}.
$$

Here, (3.10) is rewritten as follows:

$$
\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\p\nmid L_a,}}=\sideset{}{'}\sum_{0<l_\bullet,m_\bullet,n_\bullet<p}-\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\p\mid L_a}}.
$$

The first sum in the right hand side coincides with $\mathrm{li}((\lambda_1,\dots,\lambda_{a-1}),(\mu_1,\dots,\mu_{b-1},\mu_b-\tau),(\lambda_a+\tau)\bullet\boldsymbol{\nu};T)$. Therefore, we obtain

$$\sum_{\tau=0}^{\mu_b-1}\binom{\lambda_a-1+\tau}{\tau}\sideset{}{'}\sum_{0<l_\bullet,m_\bullet,n_\bullet<p}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}\prod_{y=1}^{b-1}M_y^{\mu_y}M_b^{\mu_b-\tau}(L_a+M_b)^{\lambda_a+\tau}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}}$$
$$=f(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T).$$

The second can be calculated as follows:

$$\sum_{j=1}^{a-1}\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\ L_a=jp}}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}\prod_{y=1}^{b-1}M_y^{\mu_y}M_b^{\mu_b-\tau}(L_a+M_b)^{\lambda_a+\tau}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}} \tag{3.12}$$
$$=\sum_{j=1}^{a-1}(T^p)^j\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\ (j-1)p<L_{a-1}<jp}}\frac{T^{M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}\prod_{y=1}^{b-1}M_y^{\mu_y}M_b^{\lambda_a+\mu_b}\prod_{z=1}^{c}(M_b+N_z)^{\nu_z}}$$
$$=\left(\sum_{j=1}^{a-1}\zeta_{\mathcal{A}}^{(j)}(\lambda_1,\dots,\lambda_{a-1})(T^p)^j\right)\mathrm{li}_{(\mu_1,\dots,\mu_{b-1})\bullet(\lambda_a+\mu_b)\bullet\boldsymbol{\nu}}(T).$$

Moreover, recall that $(1-x)^{-m}=\sum_{\tau=0}^{\infty}\binom{m-1+\tau}{\tau}x^\tau$ for $m\in\mathbb{Z}_{\geq 0}$. Looking at the coefficients of $x^{\mu_b}-1$ in the product of $(1-x)^{-\lambda_a}$ and $(1-x)^{-1}$, we have

$$\sum_{\tau=0}^{\mu_b-1}\binom{\lambda_a-1+\tau}{\tau}=\binom{\lambda_a+\mu_b-1}{\mu_b-1}=\binom{\lambda_a+\mu_b-1}{\lambda_a}. \tag{3.13}$$

Therefore, by (3.12) and (3.13), we obtain

$$\sum_{\tau=0}^{\mu_b-1}\binom{\lambda_a+\mu_b-1}{\lambda_a}\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\ p\mid L_a}}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}\prod_{y=1}^{b-1}M_y^{\mu_y}M_b^{\mu_b-\tau}(L_a+M_b)^{\lambda_a+\tau}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}}$$
$$=g(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T).$$

By the same calculation for (3.11), we obtain

$$\sum_{\tau=0}^{\lambda_a-1}\binom{\lambda_a+\mu_b-1}{\mu_b}\sideset{}{'}\sum_{\substack{0<l_\bullet,m_\bullet,n_\bullet<p\\ p\nmid L_a}}\frac{T^{L_a+M_b+N_c}}{\prod_{x=1}^{a-1}L_x^{\lambda_x}L_a^{\lambda_a-\tau}(L_a+M_b)^{\mu_b+\tau}\prod_{y=1}^{b-1}M_y^{\mu_y}\prod_{z=1}^{c}(L_a+M_b+N_z)^{\nu_z}}$$
$$=f(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\nu};T)-g(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\nu};T).$$

Therefore, we obtain the desired formula. □

Proof of Theorem 3.11. By Proposition 3.4, we see that the variant $\zeta_{\mathcal{A}}^{(i)}$ of finite multiple zeta value is contained in $\mathcal{Z}_{\mathcal{A}}$. Further, note that all terms in (3.8) have total weight $w:=\mathrm{wt}(\boldsymbol{\lambda})+\mathrm{wt}(\boldsymbol{\mu})+\mathrm{wt}(\boldsymbol{\nu})$, and all the terms in the 3rd, 4th and 5th sum on the right hand side in (3.8) belong to $\mathcal{R}_{w,w-1}$. Hence we have

$$\begin{aligned}(3.14)\qquad &\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)\\ &\equiv\sum_{\tau=0}^{\mu_b-1}\binom{\lambda_a-1+\tau}{\tau}\mathrm{li}((\lambda_1,\dots,\lambda_{a-1}),(\mu_1,\dots,\mu_{b-1},\mu_b-\tau),(\lambda_a+\tau)\bullet\boldsymbol{\nu};T)\\ &\quad+\sum_{\tau=0}^{\lambda_a-1}\binom{\mu_b-1+\tau}{\tau}\mathrm{li}((\lambda_1,\dots,\lambda_{a-1},\lambda_a-\tau),(\mu_1,\dots,\mu_{b-1}),(\mu_b+\tau)\bullet\boldsymbol{\nu};T)\\ (3.15)\qquad &=f(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)+f(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\nu};T)\end{aligned}$$

in $\mathcal{R}_w/\mathcal{R}_{w,w-1}$. Therefore, the main theorem is obtained from the same argument in [KMT]. Namely, consider the $\mathbb{Q}$-linear map

$$Z_{\mathcal{R}}:\mathfrak{H}\longrightarrow\mathcal{R};\quad z_{\mathbf{k}}\longmapsto\mathrm{li}_{\mathbf{k}}(T).$$

Then by equation (17) in [KMT], we have

$$\begin{aligned}(3.16)\qquad &Z_{\mathcal{R}}((z_{\boldsymbol{\lambda}}\,\text{ш}\,z_{\boldsymbol{\mu}})z_{\boldsymbol{\nu}})\\ &=\sum_{\tau=0}^{\mu_b-1}\binom{\lambda_a-1+\tau}{\tau}Z_{\mathcal{R}}((z_{\lambda_1}\cdots z_{\lambda_{a-1}}\,\text{ш}\,z_{\mu_1}\cdots z_{\mu_{b-1}}z_{\mu_b-\tau})z_{\lambda_a+\tau}z_{\boldsymbol{\nu}})\\ &\quad+\sum_{\tau=0}^{\lambda_a-1}\binom{\mu_b-1+\tau}{\tau}Z_{\mathcal{R}}((z_{\lambda_1}\cdots z_{\lambda_{a-1}}z_{\lambda_a-\tau}\,\text{ш}\,z_{\mu_1}\cdots z_{\mu_{b-1}})z_{\mu_b+\tau}z_{\boldsymbol{\nu}}).\end{aligned}$$

Starting from $Z_{\mathcal{R}}(z_{\boldsymbol{\nu}})=\mathrm{li}_{\boldsymbol{\nu}}(T)=\mathrm{li}(\emptyset,\emptyset,\boldsymbol{\nu};T)$, we can prove the congruence $Z_{\mathcal{R}}((z_{\boldsymbol{\lambda}}\,\text{ш}\,z_{\boldsymbol{\mu}})z_{\boldsymbol{\nu}})\equiv\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu};T)$ in $\mathcal{R}_w/\mathcal{R}_{w,w-1}$ by induction on $\mathrm{dep}(\boldsymbol{\lambda})+\mathrm{dep}(\boldsymbol{\mu})$, by using (3.14) and (3.16). In particular, we obtain

$$\mathrm{li}_{\boldsymbol{\lambda}}(T)\mathrm{li}_{\boldsymbol{\mu}}(T)=\mathrm{li}(\boldsymbol{\lambda},\boldsymbol{\mu},\emptyset;T)\equiv Z_{\mathcal{R}}(z_{\boldsymbol{\lambda}}\,\text{ш}\,z_{\boldsymbol{\mu}})=\mathrm{li}_{\boldsymbol{\lambda}\,\text{ш}\,\boldsymbol{\mu}}(T)\pmod{\mathcal{R}_{w,w-1}}.$$

Moreover, since each step in the induction is explicit by (3.8), $\mathrm{li}_{\mathbf{k}}(T)\mathrm{li}_{\mathbf{k}'}(T) - \mathrm{li}_{\mathbf{k}Ш\mathbf{k}'}(T)$ can be calculated explicitly from $\mathrm{li}_{\mathbf{k}}(T)$ and $\mathrm{li}_{\mathbf{k}'}(T)$. This completes the proof of Theorem 3.11. $\square$

The following is a corollary of Proposition 3.12.

Corollary 3.14. $\mathcal{R}$ forms a $\mathcal{Z}_{\mathcal{A}}[T^p]$-subalgebra of $\mathcal{B}$.

3.3. **Known results on finite multiple zeta values.** In this subsection, we give the proofs of the facts on finite multiple zeta values. These facts will be used in the next subsection.

Proposition 3.15 ([Hof2, Theorem 4.3], [Zha, Lemma 2.2])**.** For a positive integer k, we have $\zeta_{\mathcal{A}}(k) = 0$.

Proof. This proposition follows from the fact that

$$\sum_{n=1}^{p-1} \frac{1}{n^k} \equiv 0 \bmod p$$

for all primes p satisfying $p-1 \nmid k$. $\square$

For the proof of the following proposition about finite double zeta values, we set

$$\zeta^{\star}_{\mathcal{A}}(k_1, \ldots, k_r) := \sum_{1 \le m_1 \le \cdots \le m_r \le p-1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

as an element in $\mathcal{A}$ and we call this finite multiple zeta star values (FMZSVs) of weight $k_1 + \cdots + k_r$ and depth r. Note that $\zeta^{\star}_{\mathcal{A}}(\mathbf{k})$ can be written as a sum of finite multiple zeta values and finite multiple zeta values can be written as an alternative sum of finite multiple zeta star values. That is, it can be seen easily that

$$\zeta^{\star}_{\mathcal{A}}(k_1, \ldots, k_r) = \sum_{\mathbf{k}'} \zeta_{\mathcal{A}}(\mathbf{k}'), \quad \zeta_{\mathcal{A}}(k_1, \ldots, k_r) = \sum_{\mathbf{k}'} (-1)^{\sigma(\mathbf{k}')} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}'), \tag{3.17}$$

where $\mathbf{k}'$ runs over all indices of the forms

$$\mathbf{k}' = (k_1 \square k_2 \square \cdots \square k_r)$$

in which square will be filled by comma , or plus $+$ and $\sigma(\mathbf{k}')$ denotes the number of $+$ used in $\mathbf{k}'$. For example, by Proposition 3.15, we have $\zeta^{\star}_{\mathcal{A}}(k_1, k_2) = \zeta_{\mathcal{A}}(k_1, k_2) + \zeta_{\mathcal{A}}(k_1 + k_2) = \zeta_{\mathcal{A}}(k_1, k_2)$.

Proposition 3.16 ([Hof2, Theorem 6.1], [Zha, Theorem 3.1])**.** For positive integers k_1, k_2, we have

$$\zeta_{\mathcal{A}}(k_1, k_2) = (-1)^{k_2} \binom{k_1 + k_2}{k_1} \frac{B_{p-(k_1+k_2)}}{k_1 + k_2}.$$

Proof. We see that

$$\begin{aligned}
\zeta^{\star}_{\mathcal{A}}(k_1,k_2) &= \sum_{m_2=1}^{p-1}\frac{1}{m_2^{k_2}}\sum_{m_1=1}^{m_2}\frac{1}{m_1^{k_1}}\\
&\equiv \sum_{m_2=1}^{p-1}\frac{1}{m_2^{k_2}}\sum_{m_1=1}^{m_2} m_1^{p-1-k_1}\\
&= \sum_{m_2=1}^{p-1}\frac{1}{m_2^{k_2}}\frac{1}{p-k_1}\sum_{j=0}^{p-1-k_1}\binom{p-k_1}{j}B_j m_2^{p-k_1-j}\\
&=\frac{1}{p-k_1}\sum_{j=0}^{p-1-k_1}\binom{p-k_1}{j}B_j\sum_{m_2=1}^{p-1} m_2^{p-k_1-k_2-j}.
\end{aligned}$$

Note that the first congruence comes from the Fermat's little theorem and the second equality comes from the Faulharber's formula. Further, if $j \neq p-k_1-k_2$, then we have

$$\sum_{m_2=1}^{p-1} m_2^{p-k_1-k_2-j} \equiv 0.$$

Therefore, we obtain

$$\sum_{0<m_1\le m_2<p}\frac{1}{m_1^{k_1}m_2^{k_2}} \equiv \frac{1}{p-k_1}\binom{p-k_1}{p-k_1-k_2}B_{p-k_1-k_2}(p-1).$$

Furthermore, we see that

$$\begin{aligned}
\frac{p-1}{p-k_1}\binom{p-k_1}{p-k_1-k_2} &\equiv \frac{1}{k_1}\binom{p-k_1}{k_2}\\
&\equiv \frac{1}{k_1}\frac{(p-k_1)(p-k_1-1)\cdots(p-k_1-k_2+1)}{k_2!}\\
&\equiv \frac{1}{k_1}\frac{(-k_1)(-k_1-1)\cdots(-k_1-k_2+1)}{k_2!}\\
&=(-1)^{k_2}\frac{(k_1+1)\cdots(k_1+k_2-1)}{k_2!}\\
&=\frac{(-1)^{k_2}}{k_1+k_2}\binom{k_1+k_2}{k_2},
\end{aligned}$$

which leads us to the desired formula. □

Proposition 3.17 ([Hof2, Theorem 4.5], [Zha, Lemma 3.3])**.** If $\mathbf{k} = (k_1,\dots,k_r)$ is an index, then we have $\zeta_{\mathcal{A}}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\zeta_{\mathcal{A}}(\overline{\mathbf{k}})$.

Proof. By transforming $n_i \to p - n_i$ for each i, we have

$$\begin{aligned}
\zeta_{\mathcal{A}}(\mathbf{k}) &= \sum_{0<n_1<\cdots<n_r<p} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \\
&= \sum_{0<p-n_1<\cdots<p-n_r<p} \frac{1}{(p-n_1)^{k_1}\cdots(p-n_r)^{k_r}} \\
&= (-1)^{k_1+\cdots+k_r} \sum_{0<n_r<\cdots<n_1<p} \frac{1}{n_r^{k_r}\cdots n_1^{k_1}} \\
&= (-1)^{\mathrm{wt}(\mathbf{k})}\zeta_{\mathcal{A}}(\overline{\mathbf{k}}),
\end{aligned}$$

which is the desired formula. □

We end this section by proving that finite multiple zeta values with weight 4 are all 0.

Proposition 3.18. If $\mathbf{k}$ is an index with weight 4, then we have $\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}^{\star}(\mathbf{k}) = 0$.

To prove this proposition, we need two propositions. The first proposition says that the summation of finite multiple zeta (star) values over the symmetric group is zero.

Proposition 3.19 ([Hof2, Theorem 4.4])**.** If $\mathbf{k} = (k_1, \ldots, k_r)$ is a non empty index, then we have

$$\sum_{\sigma\in\mathfrak{S}_r} \zeta_{\mathcal{A}}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = \sum_{\sigma\in\mathfrak{S}_r} \zeta_{\mathcal{A}}^{\star}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = 0.$$

In particular, if $\mathbf{k} = \{k\}^r$ for positive integers k, r, we see that $\zeta_{\mathcal{A}}(\{k\}^r) = \zeta_{\mathcal{A}}^{\star}(\{k\}^r) = 0$.

The second proposition is the duality theorem for FMZSVs. To state this duality theorem, we need to define the Hoffman dual of indices.

Definition 3.20. The Hoffman dual of an index $\mathbf{k}$ of weight k is the index $\mathbf{k}^{\vee}$ uniquely determined by

$$A(\mathbf{k}) \sqcup A(\mathbf{k}^{\vee}) = \{1, 2, \ldots, k-1\}.$$

Here, $A(\mathbf{k}) := \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_{r-1}\}$ if $\mathbf{k} = (k_1, \ldots, k_r)$. For example, we see that

$$\{r\}^{\vee} = \{1\}^r, \quad (k_1, k_2)^{\vee} = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}), \quad (k_1, \{1\}^{k_2-1})^{\vee} = (\{1\}^{k_1-1}, k_2)$$

for positive integers k_1, k_2, r.

Proposition 3.21 ([Hof2, Theorem 4.6])**.** For an index $\mathbf{k}$, we have $\zeta_{\mathcal{A}}^{\star}(\mathbf{k}) = -\zeta_{\mathcal{A}}^{\star}(\mathbf{k}^{\vee})$.

Proof of Proposition 3.18. If $\mathrm{dep}(\mathbf{k})=1$, $\zeta_{\mathcal{A}}(\mathbf{k})=\zeta_{\mathcal{A}}^{\star}(\mathbf{k})=0$ by Proposition 3.15. If $\mathrm{dep}(\mathbf{k})=4$, $\zeta_{\mathcal{A}}(\mathbf{k})=\zeta_{\mathcal{A}}^{\star}(\mathbf{k})=0$ by Proposition 3.19. If $\mathrm{dep}(\mathbf{k})=2$, $\zeta_{\mathcal{A}}(\mathbf{k})=\zeta_{\mathcal{A}}^{\star}(\mathbf{k})=0$ by Proposition 3.16. If $\mathrm{dep}(\mathbf{k})=3$, $\zeta_{\mathcal{A}}^{\star}(\mathbf{k})=-\zeta_{\mathcal{A}}^{\star}(\mathbf{k}^{\vee})=0$ by Proposition 3.21 and $\mathrm{dep}(\mathbf{k}^{\vee})=2$. This leads to $\zeta_{\mathcal{A}}(\mathbf{k})=0$ for an index $\mathbf{k}$ of depth 3. □

3.4. **Examples of products of finite multiple polylogarithms.** In this subsection, using Proposition 3.12, we give examples of the products of finite multiple polylogarithms whose total depth is smaller than or equal to 5.

First, we express the product of two finite polylogarithms by a sum of finite multiple polylogarithms. In this case, thanks to Proposition 3.15, the variant $\zeta_{\mathcal{A}}^{(i)}$ of finite multiple zeta values do not appear. Therefore, this case coincides with the product of two classical polylogarithms.

Proposition 3.22. For positive integers k and l, we have

$$\mathrm{li}_k(T)\mathrm{li}_l(T)=\mathrm{li}_{(k)\text{ш}(l)}(T).$$

Next, we express the product of a finite polylogarithm and a finite double polylogarithm. In this case, the terms of the variant $\zeta_{\mathcal{A}}^{(i)}$ of finite multiple zeta values appear in general.

Proposition 3.23. For positive integers k, l_1 and l_2, we have

$$\mathrm{li}_k(T)\mathrm{li}_{(l_1,l_2)}(T)-\mathrm{li}_{(k)\text{ш}(l_1,l_2)}(T)=(-1)^{l_1+l_2}(\zeta_{\mathcal{A}}(k+l_2,l_1)T^p+\zeta_{\mathcal{A}}^{(2)}(k+l_2,l_1)T^{2p}).$$

Note that we see that $\mathrm{li}_k(T)\mathrm{li}_{(l_1,l_2)}(T)=\mathrm{li}_{(k)\text{ш}(l_1,l_2)}(T)$ if $k+l_1+l_2$ is even since $\zeta_{\mathcal{A}}(k+l_2,l_1)=\zeta_{\mathcal{A}}(l_1,k+l_2)=0$ by Proposition 3.16.

Example 3.24. We describe all the products of a finite polylogarithm and a finite double polylogarithm whose total weight is 5 as follows.

(i) $\mathrm{li}_1(T)\mathrm{li}_{(1,1)}(T)-\mathrm{li}_{(1)\text{ш}(1,1)}(T)=\zeta_{\mathcal{A}}(2,1)T^p+\zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p}=\zeta_{\mathcal{A}}(2,1)T^p(1-T^p)$.
(ii) $\mathrm{li}_1(T)\mathrm{li}_{(1,2)}(T)-\mathrm{li}_{(1)\text{ш}(1,2)}(T)=0$.
(iii) $\mathrm{li}_1(T)\mathrm{li}_{(2,1)}(T)-\mathrm{li}_{(1)\text{ш}(2,1)}(T)=0$.
(iv) $\mathrm{li}_2(T)\mathrm{li}_{(1,1)}(T)-\mathrm{li}_{(2)\text{ш}(1,1)}(T)=0$.
(v) $\mathrm{li}_3(T)\mathrm{li}_{(1,1)}(T)-\mathrm{li}_{(3)\text{ш}(1,1)}(T)=\zeta_{\mathcal{A}}(4,1)T^p(1-T^p)$.
(vi) $\mathrm{li}_2(T)\mathrm{li}_{(2,1)}(T)-\mathrm{li}_{(2)\text{ш}(2,1)}(T)=-\zeta_{\mathcal{A}}(3,2)T^p(1-T^p)$.
(vii) $\mathrm{li}_2(T)\mathrm{li}_{(1,2)}(T)-\mathrm{li}_{(2)\text{ш}(1,2)}(T)=-\zeta_{\mathcal{A}}(4,1)T^p(1-T^p)$.
(viii) $\mathrm{li}_1(T)\mathrm{li}_{(3,1)}(T)-\mathrm{li}_{(1)\text{ш}(3,1)}(T)=\zeta_{\mathcal{A}}(2,3)T^p(1-T^p)$.
(ix) $\mathrm{li}_1(T)\mathrm{li}_{(2,2)}(T)-\mathrm{li}_{(1)\text{ш}(2,2)}(T)=\zeta_{\mathcal{A}}(3,2)T^p(1-T^p)$.
(x) $\mathrm{li}_1(T)\mathrm{li}_{(1,3)}(T)-\mathrm{li}_{(1)\text{ш}(1,3)}(T)=\zeta_{\mathcal{A}}(4,1)T^p(1-T^p)$.

Further, we express the product of two finite double polylogarithms.

Proposition 3.25. For positive integers k_1, k_2, l_1, l_2, we have

$$\begin{aligned}
&\mathrm{li}_{(k_1,k_2)}(T)\mathrm{li}_{(l_1,l_2)}(T) - \mathrm{li}_{(k_1,k_2)ш(l_1,l_2)}(T)\\
=&(-1)^{l_1+l_2}(\zeta_{\mathcal{A}}(k_1,k_2+l_2,l_1)T^p + \zeta_{\mathcal{A}}^{(2)}(k_1,k_2+l_2,l_1)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(k_1,k_2+l_2,l_1)T^{3p})\\
&+\sum_{\tau=0}^{l_2-1}\binom{k_2-1+\tau}{\tau}(-1)^{l_1+l_2-\tau}(\zeta_{\mathcal{A}}(k_1+l_2-\tau,l_1)T^p + \zeta_{\mathcal{A}}^{(2)}(k_1+l_2-\tau,l_1)T^{2p})\mathrm{li}_{k_2+\tau}(T)\\
&+\sum_{\tau=0}^{k_2-1}\binom{l_2-1+\tau}{\tau}(-1)^{l_1}(\zeta_{\mathcal{A}}(k_1,k_2-\tau+l_1)T^p + \zeta_{\mathcal{A}}^{(2)}(k_1,k_2-\tau+l_1)T^{2p})\mathrm{li}_{l_2+\tau}(T).
\end{aligned}$$

Example 3.26. (i) We have

$$\begin{aligned}
&\mathrm{li}_{(1,1)}(T)^2 - \mathrm{li}_{(1,1)ш(1,1)}(T)\\
=&(\zeta_{\mathcal{A}}(1,2,1)T^p + \zeta_{\mathcal{A}}^{(2)}(1,2,1)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(1,2,1)T^{3p})\\
&+(\zeta_{\mathcal{A}}(2,1)T^p + \zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p})\mathrm{li}_1(T) - (\zeta_{\mathcal{A}}(1,2)T^p + \zeta_{\mathcal{A}}^{(2)}(1,2)T^{2p})\mathrm{li}_1(T)\\
=&2\zeta_{\mathcal{A}}(2,1)T^p(1-T^p)\mathrm{li}_1(T).
\end{aligned}$$

Note that the second equality is due to that $\zeta_{\mathcal{A}}(1,2,1), \zeta_{\mathcal{A}}^{(2)}(1,2,1)$ and $\zeta_{\mathcal{A}}^{(3)}(1,2,1)$ are sums of finite multiple zeta values of weight 4 by Example 3.6 (i) and therefore they are all zero since finite multiple zeta values with weight 4 vanish by Proposition 3.18, $\zeta_{\mathcal{A}}^{(2)}(2,1) = -\zeta_{\mathcal{A}}(2,1)$ by Remark 3.2, $\zeta_{\mathcal{A}}(1,2) = -\zeta_{\mathcal{A}}(2,1)$ by Proposition 3.17 and $\zeta_{\mathcal{A}}^{(2)}(1,2) = -\zeta_{\mathcal{A}}(1,2) = \zeta_{\mathcal{A}}(2,1)$ by Remark 3.2 and Proposition 3.17.

(ii) We have

$$\begin{aligned}
&\mathrm{li}_{(1,1)}(T)\mathrm{li}_{(1,2)}(T) - \mathrm{li}_{(1,1)ш(1,2)}(T)\\
=&-(\zeta_{\mathcal{A}}(1,3,1)T^p + \zeta_{\mathcal{A}}^{(2)}(1,3,1)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(1,3,1)T^{3p})\\
&-(\zeta_{\mathcal{A}}(3,1)T^p + \zeta_{\mathcal{A}}^{(2)}(3,1)T^{2p}))\mathrm{li}_1(T) + (\zeta_{\mathcal{A}}(2,1)T^p + \zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p})\mathrm{li}_2(T)\\
&-(\zeta_{\mathcal{A}}(1,2)T^p + \zeta_{\mathcal{A}}^{(2)}(1,2)T^{2p})\mathrm{li}_2(T)\\
=&2\zeta_{\mathcal{A}}(2,1)T^p(1-T^p)\mathrm{li}_2(T).
\end{aligned}$$

Also note that the second equality comes from $\zeta_{\mathcal{A}}^{(2)}(1,3,1) = 2(\zeta_{\mathcal{A}}(1,1,3) + \zeta_{\mathcal{A}}(3,1,1)) + (\zeta_{\mathcal{A}}(2,3) + \zeta_{\mathcal{A}}(3,2))$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(1,3,1) = \zeta_{\mathcal{A}}(1,1,3) + \zeta_{\mathcal{A}}(3,1,1) = \zeta_{\mathcal{A}}(2,3) + \zeta_{\mathcal{A}}(3,2) = 0$ by Proposition 3.17, $\zeta^{(3)}(1,3,1) = -\zeta_{\mathcal{A}}(1,3,1)$ and this is 0 by Remark 3.2 and Proposition 3.17, $\zeta_{\mathcal{A}}(3,1) = \zeta_{\mathcal{A}}^{(2)}(3,1) = 0$

by Remark 3.2 and Proposition 3.18, $\zeta_{\mathcal{A}}^{(2)}(2,1) = -\zeta_{\mathcal{A}}(2,1)$ by Remark 3.2, $\zeta_{\mathcal{A}}(1,2) = -\zeta_{\mathcal{A}}(2,1), \zeta^{(2)}(1,2) = -\zeta_{\mathcal{A}}(1,2) = \zeta_{\mathcal{A}}(2,1)$ by Proposition 3.17 and Remark 3.2.

(iii) We have

$$\begin{aligned}
&\mathrm{li}_{(1,1)}(T)\mathrm{li}_{(2,1)}(T) - \mathrm{li}_{(1,1)\sqcup\!\sqcup(2,1)}(T)\\
=&-(\zeta_{\mathcal{A}}(1,2,2)T^p + \zeta_{\mathcal{A}}^{(2)}(1,2,2)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(1,2,2)T^{3p})\\
&-(\zeta_{\mathcal{A}}(2,2)T^p + \zeta_{\mathcal{A}}^{(2)}(2,2)T^{2p})\mathrm{li}_1(T) - (\zeta_{\mathcal{A}}(1,3)T^p + \zeta_{\mathcal{A}}^{(2)}(1,3)T^{2p})\mathrm{li}_1(T)\\
=&-\zeta_{\mathcal{A}}(1,2,2)T^p(1-T^p)(1+T^p).
\end{aligned}$$

The second equality also comes from $\zeta_{\mathcal{A}}^{(2)}(1,2,2) = 2\zeta_{\mathcal{A}}(2,1,2) + (\zeta_{\mathcal{A}}(1,2,2) + \zeta_{\mathcal{A}}(2,2,1)) + (\zeta_{\mathcal{A}}(3,2) + \zeta_{\mathcal{A}}(2,3))$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(2,1,2) = \zeta_{\mathcal{A}}(1,2,2) + \zeta_{\mathcal{A}}(2,2,1) = \zeta_{\mathcal{A}}(3,2) + \zeta_{\mathcal{A}}(2,3) = 0$ by Proposition 3.17, $\zeta_{\mathcal{A}}^{(3)}(1,2,2) = -\zeta_{\mathcal{A}}(1,2,2)$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(2,2) = \zeta_{\mathcal{A}}^{(2)}(2,2) = 0$ and $\zeta_{\mathcal{A}}(1,3) = \zeta_{\mathcal{A}}^{(2)}(1,3) = 0$ by Proposition 3.18 and Remark 3.2.

At the end of this subsection, we give products of a finite polylogarithm and a finite triple polylogarithm.

Proposition 3.27. For positive integers k, l_1, l_2, l_3, we have

$$\begin{aligned}
&\mathrm{li}_k(T)\mathrm{li}_{(l_1,l_2,l_3)}(T) - \mathrm{li}_{(k)\sqcup\!\sqcup(l_1,l_2,l_3)}(T)\\
=&(-1)^{l_1+l_2+l_3}(\zeta_{\mathcal{A}}(k+l_3,l_2,l_1)T^p + \zeta_{\mathcal{A}}^{(2)}(k+l_3,l_2,l_1)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(k+l_3,l_2,l_1)T^{3p})\\
&+\sum_{\tau=0}^{k-1}\binom{l_3-1+\tau}{\tau}(-1)^{l_1+l_2}(\zeta_{\mathcal{A}}(k_1-\tau+l_2,l_1)T^p + \zeta_{\mathcal{A}}^{(2)}(k_1-\tau+l_2,l_1)T^{2p})\mathrm{li}_{l_3+\tau}(T)\\
&-\binom{k+l_3-1}{l_3}(\zeta_{\mathcal{A}}(l_1,l_2)T^p + \zeta_{\mathcal{A}}^{(2)}(l_1,l_2)T^{2p})\mathrm{li}_{k+l_3}(T).
\end{aligned}$$

Example 3.28. (i) We have

$$\begin{aligned}
&\mathrm{li}_1(T)\mathrm{li}_{(1,1,1)}(T) - \mathrm{li}_{(1)\sqcup\!\sqcup(1,1,1)}(T)\\
=&-(\zeta_{\mathcal{A}}(2,1,1)T^p + \zeta_{\mathcal{A}}^{(2)}(2,1,1)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(2,1,1)T^{3p})\\
&+(\zeta_{\mathcal{A}}(2,1)T^p + \zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p})\mathrm{li}_1(T) - (\zeta_{\mathcal{A}}(1,1)T^p + \zeta_{\mathcal{A}}^{(2)}(1,1)T^{2p})\mathrm{li}_2(T)\\
=&\zeta_{\mathcal{A}}(2,1)T^p(1-T^p).
\end{aligned}$$

The second equality comes from the facts that $\zeta_{\mathcal{A}}(2,1,1), \zeta_{\mathcal{A}}^{(2)}(2,1,1), \zeta_{\mathcal{A}}^{(3)}(2,1,1)$ are all zero since they are sums of finite multiple zeta values of weight 4 by

Example 3.6 (i) and finite multiple zeta values of weight 4 are zero by Proposition 3.18, $\zeta_{\mathcal{A}}^{(2)}(2,1)=-\zeta_{\mathcal{A}}(2,1)$ by Remark 3.2, $\zeta_{\mathcal{A}}^{(2)}(1,1)=\zeta_{\mathcal{A}}(1,1)=0$ by Remark 3.2 and Proposition 3.16.

(ii) We have

$$\begin{aligned}&\mathrm{li}_1(T)\mathrm{li}_{(1,1,2)}(T)-\mathrm{li}_{(1)ш(1,1,2)}(T)\\=&\zeta_{\mathcal{A}}(3,1,1)T^p+\zeta_{\mathcal{A}}^{(2)}(3,1,1)T^{2p}+\zeta_{\mathcal{A}}^{(3)}(3,1,1)T^{3p}\\&+(\zeta_{\mathcal{A}}(2,1)T^p+\zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p})\mathrm{li}_2(T)-(\zeta_{\mathcal{A}}(1,1)T^p+\zeta_{\mathcal{A}}^{(2)}(1,1)T^{2p})\mathrm{li}_3(T)\\=&\zeta_{\mathcal{A}}(2,1)T^p(1-T^p)+\zeta_{\mathcal{A}}(3,1,1)T^p(1-T^p)(1+T^p).\end{aligned}$$

The second equality is due to $\zeta_{\mathcal{A}}^{(2)}(3,1,1)=2\zeta_{\mathcal{A}}(1,3,1)+(\zeta_{\mathcal{A}}(3,1,1)+\zeta_{\mathcal{A}}(1,1,3))+(\zeta_{\mathcal{A}}(4,1)+\zeta_{\mathcal{A}}(1,4))$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(1,3,1)=\zeta_{\mathcal{A}}(3,1,1)+\zeta_{\mathcal{A}}(1,1,3)=\zeta_{\mathcal{A}}(1,4)+\zeta_{\mathcal{A}}(4,1)=0$ by Proposition 3.17, $\zeta_{\mathcal{A}}^{(3)}(3,1,1)=-\zeta_{\mathcal{A}}(3,1,1),\zeta_{\mathcal{A}}^{(2)}(2,1)=-\zeta_{\mathcal{A}}(2,1),\zeta_{\mathcal{A}}^{(2)}(1,1)=\zeta_{\mathcal{A}}(1,1)$ by Remark 3.2, $\zeta_{\mathcal{A}}(1,1)=0$ by Proposition 3.16.

(iii) We have

$$\begin{aligned}&\mathrm{li}_1(T)\mathrm{li}_{(1,2,1)}(T)-\mathrm{li}_{(1)ш(1,2,1)}(T)\\=&(\zeta_{\mathcal{A}}(2,2,1)T^p+\zeta_{\mathcal{A}}^{(2)}(2,2,1)T^{2p}+\zeta_{\mathcal{A}}^{(3)}(2,2,1)T^{3p})\\&+(\zeta_{\mathcal{A}}(3,1)T^p+\zeta_{\mathcal{A}}^{(2)}(3,1)T^{2p})\mathrm{li}_1(T)-(\zeta_{\mathcal{A}}(1,2)T^p+\zeta_{\mathcal{A}}^{(2)}(1,2)T^{2p})\mathrm{li}_2(T)\\=&-\zeta_{\mathcal{A}}(1,2)T^p(1-T^p)\mathrm{li}_2(T).\end{aligned}$$

The second equality is due to $\zeta_{\mathcal{A}}^{(2)}(2,2,1)=2\zeta_{\mathcal{A}}(2,1,2)+(\zeta_{\mathcal{A}}(2,2,1)+\zeta_{\mathcal{A}}(1,2,2))+(\zeta_{\mathcal{A}}(3,2)+\zeta_{\mathcal{A}}(2,3))$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(2,1,2)=\zeta_{\mathcal{A}}(2,2,1)+\zeta_{\mathcal{A}}(1,2,2)=\zeta_{\mathcal{A}}(3,2)+\zeta_{\mathcal{A}}(2,3)=0$ by Proposition 3.17, $\zeta_{\mathcal{A}}^{(2)}(3,1)=\zeta_{\mathcal{A}}(3,1)=0$ by Remark 3.2 and Proposition 3.18, $\zeta_{\mathcal{A}}^{(2)}(1,2)=-\zeta_{\mathcal{A}}(1,2)$ by Remark 3.2.

(iv) We have

$$\begin{aligned}&\mathrm{li}_1(T)\mathrm{li}_{(2,1,1)}(T)-\mathrm{li}_{(1)ш(2,1,1)}(T)\\=&(\zeta_{\mathcal{A}}(2,1,2)T^p+\zeta_{\mathcal{A}}^{(2)}(2,1,2)T^{2p}+\zeta_{\mathcal{A}}^{(3)}(2,1,2)T^{3p})\\&-(\zeta_{\mathcal{A}}(2,2)T^p+\zeta_{\mathcal{A}}^{(2)}(2,2)T^{2p})\mathrm{li}_1(T)-(\zeta_{\mathcal{A}}(2,1)T^p+\zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p})\mathrm{li}_2(T)\\=&\zeta_{\mathcal{A}}(2,1)T^p(1-T^p)\mathrm{li}_2(T).\end{aligned}$$

The second equality is due to $\zeta_{\mathcal{A}}(2,1,2)=0$ by Proposition 3.17, $\zeta_{\mathcal{A}}^{(2)}(2,1,2)=2(\zeta_{\mathcal{A}}(2,2,1)+\zeta_{\mathcal{A}}(1,2,2))+(\zeta_{\mathcal{A}}(3,2)+\zeta_{\mathcal{A}}(2,3))$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(2,2,1)+\zeta_{\mathcal{A}}(1,2,2)=\zeta_{\mathcal{A}}(3,2)+\zeta_{\mathcal{A}}(2,3)=0$ by Proposition 3.17, $\zeta_{\mathcal{A}}^{(3)}(2,1,2)=-\zeta_{\mathcal{A}}(2,1,2)$

and this is 0 by Remark 3.2 and Proposition 3.17, $\zeta_{\mathcal{A}}(2,2) = \zeta_{\mathcal{A}}^{(2)}(2,2) = 0$ by Remark 3.2 and Proposition 3.18.

(v) We have

$$
\begin{aligned}
&\mathrm{li}_2(T)\mathrm{li}_{(1,1,1)}(T) - \mathrm{li}_{(2)ш(1,1,1)}(T) \\
= &-\left(\zeta_{\mathcal{A}}(3,1,1)T^p + \zeta_{\mathcal{A}}^{(2)}(3,1,1)T^{2p} + \zeta_{\mathcal{A}}^{(3)}(3,1,1)T^{3p}\right) \\
&+ (\zeta_{\mathcal{A}}(3,1)T^p + \zeta_{\mathcal{A}}^{(2)}(3,1)T^{2p})\mathrm{li}_1(T) + (\zeta_{\mathcal{A}}(2,1)T^p + \zeta_{\mathcal{A}}^{(2)}(2,1)T^{2p})\mathrm{li}_2(T) \\
&- 2(\zeta_{\mathcal{A}}(1,1)T^p + \zeta_{\mathcal{A}}^{(2)}(1,1)T^{2p})\mathrm{li}_3(T) \\
= &- \zeta_{\mathcal{A}}(3,1,1)T^p(1-T^p)(1+T^p) + \zeta_{\mathcal{A}}(2,1)T^p(1-T^p)\mathrm{li}_2(T).
\end{aligned}
$$

The second equality is due to $\zeta_{\mathcal{A}}^{(2)}(3,1,1) = 2\zeta_{\mathcal{A}}(1,3,1)+(\zeta_{\mathcal{A}}(3,1,1)+\zeta_{\mathcal{A}}(1,1,3))+(\zeta_{\mathcal{A}}(4,1)+\zeta_{\mathcal{A}}(1,4))$ by Example 3.6 (i), $\zeta_{\mathcal{A}}(1,3,1) = \zeta_{\mathcal{A}}(3,1,1)+\zeta_{\mathcal{A}}(1,1,3) = \zeta_{\mathcal{A}}(4,1)+\zeta_{\mathcal{A}}(1,4) = 0$ by Proposition 3.17, $\zeta_{\mathcal{A}}^{(3)}(3,1,1) = -\zeta_{\mathcal{A}}(3,1,1), \zeta_{\mathcal{A}}^{(2)}(2,1) = -\zeta_{\mathcal{A}}(2,1)$ by Remark 3.2, $\zeta_{\mathcal{A}}^{(2)}(3,1) = \zeta_{\mathcal{A}}(3,1) = 0$ by Remark 3.2 and Proposition 3.18, $\zeta_{\mathcal{A}}^{(2)}(1,1) = \zeta_{\mathcal{A}}(1,1) = 0$ by Remark 3.2 and Proposition 3.16.

3.5. **An algebraic interpretation.** In this section, we will give an algebraic interpretation of our second main theorem (Theorem 3.11). Note that by Corollary 3.14, we have

$$\mathcal{R}_{a_1,b_1} \cdot \mathcal{R}_{a_2,b_2} \subset \mathcal{R}_{a_1+a_2,b_1+b_2} \tag{3.18}$$

for all a_1, a_2, b_1, b_2 with $a_1 \geq b_1 \geq 0$ and $a_2 \geq b_2 \geq 0$.

Definition 3.29. For non-negative integers a and $b \in \{0,\dots,a\}$, we define $\mathbb{Q}$-vector spaces

$$\overline{\mathcal{R}_{a,b}} := \mathcal{R}_{a,b}/\mathcal{R}_{a,b-1}$$

and set

$$\overline{\mathcal{R}} := \bigoplus_{a\geq b\geq 0} \overline{\mathcal{R}_{a,b}}.$$

Note that $\overline{\mathcal{R}}$ has a natural $\mathbb{Q}$-algebraic structure induced from (3.18).

Definition 3.30. We define a $\mathbb{Q}$-vector space $\mathcal{S}$ by

$$\mathcal{S} := \bigoplus_{\mathbf{k},\mathbf{k}':\text{indices}} \mathbb{Q}u_{\mathbf{k}}v_{\mathbf{k}'}.$$

Here $u_{\mathbf{k}}$ and $v_{\mathbf{k}'}$ are indeterminates associated to $\mathbf{k}$ and $\mathbf{k}'$. We define the product on $\mathcal{S}$ by the usual product of $\mathbb{Q}$, the stuffle product for $u_{\mathbf{k}}$ and the shuffle product for $v_{\mathbf{k}'}$, i.e., for $a, b \in \mathbb{Q}$ and indices $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$,

$$(au_{\mathbf{k}_1}v_{\mathbf{k}'_1}) \cdot (bu_{\mathbf{k}_2}v_{\mathbf{k}'_2}) := ab(u_{\mathbf{k}_1} * u_{\mathbf{k}_2})(v_{\mathbf{k}'_1} ш v_{\mathbf{k}'_2}).$$

Then $(\mathcal{S}, \cdot)$ is a $\mathbb{Q}$-algebra and we call $\mathcal{S}$ the shuffle-stuffle algebra over $\mathbb{Q}$.

An algebraic interpretation of our main theorem is as follows.

Corollary 3.31. The $\mathbb{Q}$-linear homomorphism

$$\varphi : \mathcal{S} \to \overline{\mathcal{R}}\ ; \ \ u_{\mathbf{k}}v_{\mathbf{k}'} \mapsto \zeta_{\mathcal{A}}(\mathbf{k})\mathrm{li}_{\mathbf{k}'}(T)$$

is a $\mathbb{Q}$-algebra homomorphism.

Proof. Note that finite multiple zeta values satisfy the stuffle relation. For any elements $\zeta_{\mathcal{A}}(\mathbf{k}_1)(T^p)^{n_1}\mathrm{li}_{\mathbf{k}'_1}(T) \in \overline{\mathcal{R}_{a_1,b_1}}$ and $\zeta_{\mathcal{A}}(\mathbf{k}_2)(T^p)^{n_2}\mathrm{li}_{\mathbf{k}'_2}(T)$ in $\overline{\mathcal{R}_{a_2,b_2}}$ ($n_1, n_2, a_1, a_2, b_1, b_2 \in \mathbb{Z}_{\geq 0}$ with $a_1 \geq b_1 \geq 0$ and $a_2 \geq b_2 \geq 0$), we have the following equality by Theorem 3.11 and the stuffle relation of finite multiple zeta values:

$$\begin{aligned}\zeta_{\mathcal{A}}(\mathbf{k}_1)(T^p)^{n_1}\mathrm{li}_{\mathbf{k}'_1}(T) \cdot \zeta_{\mathcal{A}}(\mathbf{k}_2)(T^p)^{n_2}\mathrm{li}_{\mathbf{k}'_2}(T) =&\zeta_{\mathcal{A}}(\mathbf{k}_1)\zeta_{\mathcal{A}}(\mathbf{k}_2)(T^p)^{n_1}(T^p)^{n_2}\mathrm{li}_{\mathbf{k}'_1}\mathrm{li}_{\mathbf{k}'_2}(T)\\ =&\zeta_{\mathcal{A}}(\mathbf{k}_1 * \mathbf{k}_2)(T^p)^{n_1+n_2}\mathrm{li}_{\mathbf{k}'_1 ш \mathbf{k}'_2}(T)\\ \in&\overline{\mathcal{R}_{a_1+a_2,b_1+b_2}}.\end{aligned}$$

This completes the proof. $\square$

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