Mathematical Analysis on the Thin Film Approximation for the Flow of a Viscous Incompressible Fluid down an Inclined Plane

February 2017

Hiroki Ueno

A Thesis for the Degree of Ph.D. in Science

Mathematical Analysis on the Thin Film Approximation for the Flow of a Viscous Incompressible Fluid down an Inclined Plane

February 2017

Graduate School of Science and Technology
Keio University

Hiroki Ueno

Contents

Chapter 1

Introduction

1.1 Background

In this dissertation, we consider a two-dimensional motion of a liquid film of a viscous and incompressible fluid flowing down an inclined plane under the influence of the gravity and the surface tension on the interface. The motion can be mathematically formulated as a free boundary problem for the incompressible Navier–Stokes equations.

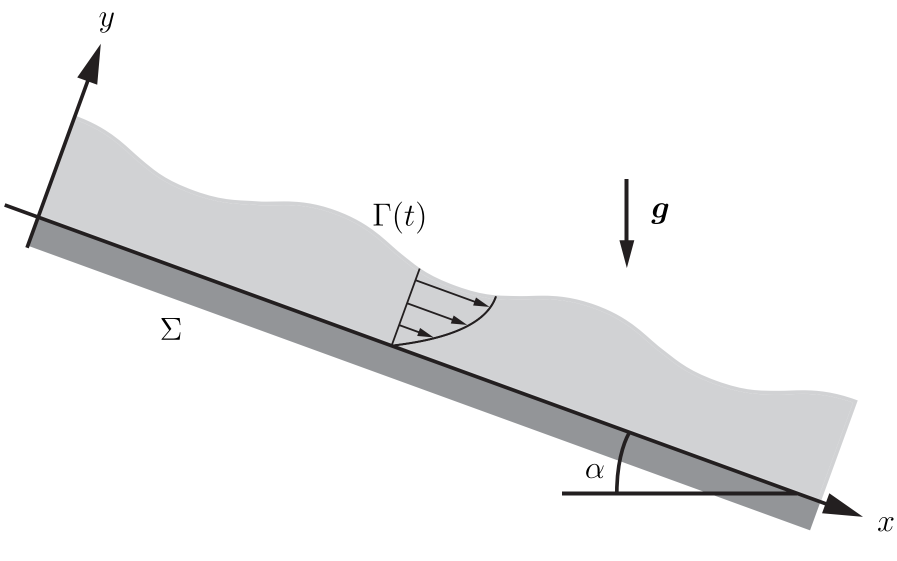

Figure 1.1: Sketch of a thin liquid film flowing down an inclined plane

We assume that the domain $\Omega(t)$ occupied by the liquid at time $t \geq 0$, the liquid surface $\Gamma(t)$, and the rigid plane Σ are of the forms

$$\begin{cases} \Omega(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < h_0 + \eta(x,t)\}, \\ \Gamma(t) = \{(x,y) \in \mathbb{R}^2 \mid y = h_0 + \eta(x,t)\}, \\ \Sigma = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}, \end{cases}$$

where h_0 is the mean thickness of the liquid film and $\eta(x,t)$ is the amplitude of the liquid surface. Here we choose a coordinate system (x,y) so that x axis is down and y axis is normal to the plane. The motion of the liquid is described by the velocity $\boldsymbol{u} = (u, v)^{\mathrm{T}}$ and the pressure p satisfying the Navier–Stokes equations

$$(1.1.1)\qquad \begin{cases} \rho\big(\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) = \nabla\cdot\mathbf{P} + \rho g(\sin\alpha, -\cos\alpha)^{\mathrm{T}} & \text{in} \quad \Omega(t),\ t>0, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in} \quad \Omega(t),\ t>0, \end{cases}$$

where

$$\mathbf{P} = -p\mathbf{I} + 2\mu\mathbf{D}$$

is the stress tensor,

$$\mathbf{D} = \frac{1}{2}\big(\mathrm{D}\boldsymbol{u} + (\mathrm{D}\boldsymbol{u})^{\mathrm{T}}\big)$$

is the deformation tensor, $\mathrm{D}\boldsymbol{u}$ is the Jacobian matrix of $\boldsymbol{u}$, $\mathbf{I}$ is the unit matrix, ρ is a constant density of the liquid, g is the acceleration of the gravity, α is the angle of inclination, and μ is the shear viscosity coefficient. The dynamical and kinematic boundary conditions on the liquid surface are

$$(1.1.2)\qquad \begin{cases} \mathbf{P}\boldsymbol{n} = -p_0\boldsymbol{n} + \sigma H\boldsymbol{n} & \text{on} \quad \Gamma(t),\ t>0, \\ \eta_t + u\eta_x - v = 0 & \text{on} \quad \Gamma(t),\ t>0, \end{cases}$$

where $\boldsymbol{n}$ is the unit outward normal vector to the liquid surface, that is,

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1)^{\mathrm{T}},$$

p_0 is a constant atmospheric pressure, σ is the surface tension coefficient, and H is the mean curvature of the liquid surface, that is,

$$H = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x.$$

The boundary condition on the rigid plane is the non-slip condition

$$(1.1.3)\qquad \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Sigma,\ t>0.$$

(1.1.1)–(1.1.3) have a laminar steady solution of the form

$$(1.1.4)\qquad \eta = 0, \quad u = \frac{\rho g\sin\alpha}{2\mu}(2h_0y - y^2), \quad v = 0, \quad p = p_0 - \rho g\cos\alpha(y - h_0),$$

which is called the Nusselt flat film solution (see [24]). Throughout this dissertation, we assume that the flow is downward l_0-periodic or approaches asymptotically this flat film solution at spacial infinity.

Concerning the instability of this laminar flow, there are vast research literatures from the physical and engineering points of view. The first investigation of the wave motion of thin film including the effect of the surface tension was provided by Kapitza [16]. In particular, he considered the case where the liquid film flows down a vertical wall, that is, the case $\alpha = \frac{\pi}{2}$. Yih [40] first formulated the linear stability problem of the laminar flow of the liquid film flowing down an inclined plane as an eigenvalue problem for the complex phase velocity, more specifically, the Orr-Sommerfeld problem although he neglected the effect of the surface tension. Benjamin [4] took into account the effect of the surface tension and showed that the critical Reynolds number $\mathrm{R}_c^{\mathrm{Benjamin}} = \frac{5}{6}\frac{1}{\tan\alpha}$ by expanding the normal mode solution in powers of y. Later, Yih [41] showed the same condition by expanding the normal mode solution in powers of the aspect ratio of the film which will be denoted by δ in this dissertation. An approach taking into account the nonlinearity was first given by Mei [21] and Benney [5]. While Mei considered the gravity waves, Benney considered the capillary-gravity waves and he recovered Benjamin's and Yih's linear stability theories.

Using the mean thickness of the liquid h_0, the characteristic scale of the streamwise direction l_0, and the typical amplitude of the liquid surface a_0, Benney introduced two non-dimensional parameters

$$\delta = \frac{h_0}{l_0}, \quad \varepsilon = \frac{a_0}{h_0}.$$

It is to be noted that we do not determine a characteristic scale l_0 in x a priori because l_0 is a typical wavelength of a nontrivial wave pattern which arises as a consequence of a destabilization and l_0 itself is an object of scientific interest. While the destabilization appears theoretically as a long wave instability in the case $\delta \to 0$, which corresponds to the case $l_0 \to \infty$, l_0 is often determined experimentally by observing waves generated by an external vibrator. As for more details of the long wave instability, see [31]. Benney derived the following single nonlinear evolution equation

$$\begin{aligned}\eta_t =& A(\eta)\eta_x + \delta\big(B(\eta)\eta_{xx} + \varepsilon C(\eta)\eta_x^2\big)\\ &+ \delta^2\big(D(\eta)\eta_{xxx} + \varepsilon E(\eta)\eta_x\eta_{xx} + \varepsilon^2 F(\eta)\eta_x^3\big)\\ &+ \delta^3\big(G(\eta)\eta_{xxxx} + \varepsilon H(\eta)\eta_x\eta_{xxx} + \varepsilon I(\eta)\eta_{xx}^2 + \varepsilon^2 J(\eta)\eta_x^2\eta_{xx} + \varepsilon^3 K(\eta)\eta_x^4\big)\\ &+ O(\delta^4),\end{aligned} \tag{1.1.5}$$

where

$$\begin{cases} A(\eta) = -2(1+\varepsilon\eta)^2, \\ B(\eta) = -\dfrac{8}{15}\mathrm{R}(1+\varepsilon\eta)^6 + \dfrac{2}{3\tan\alpha}(1+\varepsilon\eta)^3, \\ C(\eta) = -\dfrac{16}{5}R(1+\varepsilon\eta)^5 + \dfrac{2}{\tan\alpha}(1+\varepsilon\eta)^2, \\ D(\eta) = -2(1+\varepsilon\eta)^4 - \dfrac{32}{63}\mathrm{R}^2(1+\varepsilon\eta)^{10} + \dfrac{40}{63}\dfrac{\mathrm{R}}{\tan\alpha}(1+\varepsilon\eta)^7, \\ E(\eta) = -\dfrac{52}{3}(1+\varepsilon\eta)^3 - \dfrac{3632}{315}\mathrm{R}^2(1+\varepsilon\eta)^4 + \dfrac{392}{45}\dfrac{\mathrm{R}}{\tan\alpha}(1+\varepsilon\eta)^6, \\ F(\eta) = -14(1+\varepsilon\eta)^2 - \dfrac{1016}{35}\mathrm{R}^2(1+\varepsilon\eta)^8 + \dfrac{64}{5}\dfrac{\mathrm{R}}{\tan\alpha}(1+\varepsilon\eta)^5, \\ G(0) = -\dfrac{2}{3}\dfrac{\mathrm{W}}{\sin\alpha} - \dfrac{157}{56}\mathrm{R} - \dfrac{8}{45}\dfrac{\mathrm{R}}{\tan^2\alpha} + \dfrac{138904}{155925}\dfrac{\mathrm{R}^2}{\tan\alpha} - \dfrac{1213952}{2027025}\mathrm{R}^3 \end{cases}$$

by using a perturbation expansion of the solution (u, v, p) with respect to δ under the thin film regime $\delta \ll 1$. Here, R is the Reynolds number, W is the Weber Number.

Thereafter, several authors have followed Benney's approach. We note that if $\mathrm{W} = O(1)$, then the effect of the surface tension does not appear up to the term of $O(\delta^3)$ in (1.1.5). Since Benney considered the case $\mathrm{W} = O(1)$ and calculated the terms up to $O(\delta^2)$, the effect of the surface tension was omitted in his stability analysis. Consequently, his results showed that linearly unstable waves grow more rapidly in the nonlinear range. Nakaya [22] computed the terms up to $O(\delta^3)$ and showed that the surface tension has a stabilization effect in the development of the monochromatic waves. On the other hand, Gjevik [13] incorporated the effect of the surface tension into the equation by assuming the condition $\mathrm{W} = O(\delta^{-2})$ and investigated the growth of an initially unstable periodic surface perturbation and its nonlinear interaction with the higher harmonics. Their results imply that the surface tension plays an important role in investigating the stability of surface waves, which have already been pointed out by Kapitza [16]. We remark that the condition $\mathrm{W} = O(\delta^{-2})$ holds for many kinds of fluid such as water and alcohol at normal temperature. Moreover, several authors extended Benney's results to the three-dimensional case. Roskes [27] calculated the terms up to $O(\delta^2)$ and investigated the interactions between two-dimensional and three-dimensional weakly nonlinear waves on the liquid film under the condition $\mathrm{W} = O(1)$, which implies that he did not consider the effect of the surface tension. Atherton and Homsy [2] and Lin and Krishna [19] calculated the terms up to $O(\delta)$ and $O(\delta^2)$, respectively, under the condition $\mathrm{W} = O(\delta^{-2})$, namely, they took the effect of the surface tension in the equation in three-dimensional case. Furthermore, while the case where $\mathrm{R} - \mathrm{R}_c = O(1)$ had been considered, Topper and Kawahara [36] derived approximate equations under the conditions $\mathrm{W} = O(\delta^{-2})$ and $\mathrm{R} - \mathrm{R}_c = O(\delta)$. More details or a list of useful references about a physical aspect of the thin film approximation can be found in [1, 8, 9, 10, 11, 15, 18, 20, 25].

Concerning a mathematical analysis of the problem, Teramoto [33] showed that the initial value problem to the Navier–Stokes equations (1.1.1)–(1.1.3) has a unique solution globally in time under the assumptions that the Reynolds number and the initial data are sufficiently small (see also [34]). Nishida, Teramoto, and Win [23] showed the exponential stability of the Nusselt flat film solution under the assumptions that the angle of inclination is sufficiently small and the flow is downward periodic in addition to the assumptions in [33]. Furthermore, Uecker [37] studied the asymptotic behavior of the solution as $t \to \infty$ in the case of $x \in \mathbb{R}$ and showed that the perturbation of the Nusselt flat film solution decays like the self-similar solution of the Burgers equation under the assumptions that the initial data are sufficiently small and $\mathrm{R} < \mathrm{R}_c$. However, they did not consider the δ scaling because they non-dimensionalized x and y components by using the same unit length h_0.

1.2 Aim of the present study

Under the weekly nonlinear regime, we rewrite (1.1.5) as

$$\begin{aligned}\eta_t + 2(1+\varepsilon\eta)^2\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + C_1\delta^2\eta_{xxx} \\ + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + \frac{2}{3}\frac{\mathrm{W}}{\sin\alpha}\delta^3\eta_{xxxx} = O(\delta^3 + \varepsilon^2\delta + \varepsilon\delta^2),\end{aligned} \tag{1.2.1}$$

where R_c is the critical Reynolds number defined by

$$\mathrm{R}_c = \frac{5}{4}\frac{1}{\tan\alpha}$$

and C_1 and C_2 are the constants defined by

$$\begin{cases} C_1 = -D(0) = 2 + \dfrac{32}{63}\mathrm{R}^2 - \dfrac{40}{63}\dfrac{\mathrm{R}}{\tan\alpha}, \\ C_2 = -C(0) = \dfrac{16}{5}\mathrm{R} - \dfrac{2}{\tan\alpha}. \end{cases}$$

Here, R_c differs from Benjamin's critical Reynolds number $\mathrm{R}_c^{\mathrm{Benjamin}}$ because Benney [5] defined Reynolds number by using the speed of the Nusselt flat film solution on the liquid surface, whereas Benjamin [4] used the average speed of the solution. In what follows, we adopt this constant R_c according to Benney. Many approximate equations are obtained from (1.2.1) by assuming that parameters ε, W, and R have appropriate orders in δ. In the following, we assume $\mathrm{R} < \mathrm{R}_c$ unless we note in particular. Moreover, let us set

$$\eta(x,t) = \zeta(x - 2t, \varepsilon t). \tag{1.2.2}$$

I. Burgers equation

Assuming $\mathrm{W}_1 \le \mathrm{W} \le \delta^{-1}\mathrm{W}_2$ in (1.2.1), we have

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} = O(\delta^2).$$

Plugging (1.2.2) in the above equation and passing to the limit $\varepsilon = \delta \to 0$, we obtain

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} = 0. \tag{1.2.3}$$

II. Burgers equation with a fourth order dissipation term

Assuming $\mathrm{W} = \delta^{-2}\mathrm{W}_2$ in (1.2.1), we have

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta\eta_{xxxx} = O(\delta^2).$$

Plugging (1.2.2) in the above equation and passing to the limit $\varepsilon = \delta \to 0$, we obtain

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\zeta_{xxxx} = 0. \tag{1.2.4}$$

III. Burgers equation with dispersion and nonlinear terms

Assuming $\mathrm{W}_1 \le \mathrm{W} \le \mathrm{W}_2$ in (1.2.1), we have

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x = O(\delta^3).$$

Plugging (1.2.2) in the above equation, assuming $\varepsilon = \delta$, and neglecting the terms of $O(\delta^3)$, we obtain

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx} + \delta\big\{C_1\zeta_{xxx} + C_2\big(\zeta\zeta_{xx} + \zeta_x^2\big) + 2\zeta^2\zeta_x\big\} = 0. \tag{1.2.5}$$

IV. Burgers equation with fourth order dissipation, dispersion, and nonlinear terms

Assuming $\mathrm{W} = \delta^{-1}\mathrm{W}_2$ in (1.2.1), we have

$$\begin{aligned}\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx}& \\ + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta^2\eta_{xxxx} &= O(\delta^3).\end{aligned}$$

Plugging (1.2.2) in the above equation, assuming $\varepsilon = \delta$, and neglecting the terms of $O(\delta^3)$, we obtain

$$\begin{aligned}\zeta_\tau + 4\zeta\zeta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\zeta_{xx}& \\ + \delta\left\{C_1\zeta_{xxx} + C_2\big(\zeta\zeta_{xx} + \zeta_x^2\big) + 2\zeta^2\zeta_x + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\zeta_{xxxx}\right\} &= 0.\end{aligned} \tag{1.2.6}$$

We remark that (1.2.5) and (1.2.6) are higher order approximate equations to the Burgers equation (1.2.3). If $\mathrm{R} > \mathrm{R}_c$, then (1.2.4) is the Kuramoto–Sivashinsky equation (see [17], [29], and [30]). If $\mathrm{R}_c - \mathrm{R} = \delta\tilde{\mathrm{R}} > 0$, then we obtain the δ-independent KdV–Burgers equation (see [14])

$$\zeta_\tau + 4\zeta\zeta_x - \frac{8\tilde{\mathrm{R}}}{15}\zeta_{xx} + C_1\zeta_{xxx} = 0 \tag{1.2.7}$$

by plugging (1.2.2) in (1.2.1) and passing to the limit $\varepsilon = \delta^2 \to 0$ under the assumption $\mathrm{W}_1 \le \mathrm{W} \le \mathrm{W}_2$. Moreover if $\mathrm{R}_c - \mathrm{R} = -\delta\tilde{\mathrm{R}} < 0$, we obtain the δ-independent KdV–Kuramoto–Sivashinsky equation (see [36])

$$\zeta_\tau + 4\zeta\zeta_x + \frac{8\tilde{\mathrm{R}}}{15}\zeta_{xx} + C_1\zeta_{xxx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\zeta_{xxxx} = 0 \tag{1.2.8}$$

by plugging (1.2.2) in (1.2.1) and passing to the limit $\varepsilon = \delta^2 \to 0$ under the assumption $\mathrm{W} = \delta^{-1}\mathrm{W}_2$. Moreover, by assuming $\varepsilon = 1$, that is, the strongly nonlinear case and $\mathrm{W} = \delta^{-2}\widetilde{\mathrm{W}}$ and neglecting the terms of $O(\delta^2)$, we obtain the so-called Benney equation (see [13])

$$\eta_t = \Big[-\frac{2}{3}(1+\eta)^3 + \delta\Big\{ \frac{2}{3\tan\alpha}(1+\eta)^3\eta_x - \frac{8\mathrm{R}}{15}(1+\eta)^6\eta_x - \frac{2\widetilde{\mathrm{W}}}{3\sin\alpha}(1+\eta)^3\eta_{xxx} \Big\} \Big]_x . \tag{1.2.9}$$

Our aim is to give a mathematically rigorous justification of these thin film approximations by establishing error estimates between the solution of the Navier–Stokes equations (1.1.1)–(1.1.3) and those of the approximate equations (1.2.3)–(1.2.6), which will be performed in Chapter 4. More specifically, we will estimate a norm of the difference between the solution η^δ of Navier–Stokes equations and the solution η^{app} of approximate equations (1.2.3)–(1.2.6) and show that a norm goes to 0 as $\delta \to 0$. To our knowledge, this is the first rigorous justification of a thin film approximation in the sense of comparing the solution of the Navier–Stokes equations with those of the approximate equations. We remark that Bresch and Noble [7] justified the shallow water model by proving that remainder terms converge to 0 as $\delta \to 0$ (see also [6]). Moreover, Giacomelli and Otto [12] justified a lubrication approximation in the sense that an equilibrium contact angle is preserved throughout the evolution for a Darcy flow. As for more details of the lubrication approximation, see [25, 26]. Furthermore, Shih and Shen [28] and Sun and Shen [32] justified a thin film approximation for linear equations with analytic initial data.

In order to carry out the justification, the most difficult task is to derive a uniform estimate for the solution of the Navier–Stokes equations with respect to δ in the thin film regime $\delta \ll 1$. In Chapter 3, we derive a uniform estimate for the solution with respect to δ when the Reynolds number, the angle of inclination, and the initial date are sufficiently small under the conditions $O(1) \le \mathrm{W} \le O(\delta^{-2})$, $\alpha = O(1)$, and $x \in \mathbb{T}$ or $\mathbb{R}$. We remark that Bresch and Noble [7] have already derived a uniform estimate for the solution with respect to δ by assuming $\mathrm{W} = O(\delta^{-2})$, $\mathrm{R} = O(\delta)$, $\alpha = O(\sqrt{\delta})$, $x \in \mathbb{T}$, and that initial data are sufficiently small. Their assumptions on R and α are too restrictive when we consider the asymptotic behavior of the solution as $\delta \to 0$. Moreover, they assumed $\varepsilon = \delta$ and excluded the case of $\varepsilon = 1$, so that their uniform estimate cannot be applied to the justification for the Benney equation (1.2.9). Therefore, our results are not included in their works. We note that we cannot just yet justify the Kuramoto–Sivashinsky equation, the δ-independent KdV–Burgers equation (1.2.7), and the KdV–Kuramoto–Sivashinsky equation (1.2.8) because without the assumption $\mathrm{R} \ll \mathrm{R}_c$ we have not yet obtain a uniform estimate in δ for the solution.

1.3 Preliminaries

1.3.1 Notations

We put

$$\Omega = \mathbb{G} \times (0,1), \quad \Gamma = \mathbb{G} \times \{y = 1\},$$

where $\mathbb{G}$ is the flat torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or $\mathbb{R}$. For a Banach space X, we denote by $\|\cdot\|_X$ the norms in X. For $1 \le p \le \infty$, we put

$$\|u\|_{L^p} = \|u\|_{L^p(\Omega)}, \quad \|u\| = \|u\|_{L^2}, \quad |u|_{L^p} = \|u(\cdot, 1)\|_{L^p(\mathbb{G})}, \quad |u|_0 = |u|_{L^2}.$$

We denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. For $s \ge 0$, we denote by $H^s(\Omega)$ and $H^s(\Gamma)$ the L^2 Sobolev spaces of order s on Ω and Γ, respectively. The norms of these spaces are denoted by $\|\cdot\|_s$ and $|\cdot|_s$. For a function $u = u(x,y)$ on Ω, a Fourier multiplier $P(D_x)$ ($D_x = -\mathrm{i}\partial_x$) is defined by

$$(P(D_x)u)(x,y) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} P(n)\hat{u}_n(y)\mathrm{e}^{2\pi\mathrm{i}nx} & \text{in the case } \mathbb{G} = \mathbb{T}, \\ \displaystyle\int_{\mathbb{R}} P(\xi)\hat{u}(\xi,y)\mathrm{e}^{2\pi\mathrm{i}\xi x}\mathrm{d}\xi & \text{in the case } \mathbb{G} = \mathbb{R}, \end{cases}$$

where

$$\hat{u}_n(y) = \int_0^1 u(x,y)\mathrm{e}^{-2\pi\mathrm{i}nx}\,\mathrm{d}x, \quad \hat{u}(\xi,y) = \int_{\mathbb{R}} u(x,y)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\mathrm{d}x$$

are the Fourier coefficient and the Fourier transform in x, respectively. We put

$$\nabla_\delta = (\delta\partial_x, \partial_y)^{\mathrm{T}}, \quad \Delta_\delta = \nabla_\delta \cdot \nabla_\delta.$$

For operators A and B, we denote by

$$[A,B] = AB - BA$$

the commutator. We put

$$\partial_y^{-1} f(x,y) = -\int_y^1 f(x,z)\mathrm{d}z.$$

$f \lesssim g$ means that there exists a non-essential positive constant C such that $f \le Cg$ holds.

1.3.2 Basic inequalities

We will prove the following lemmas in Appendix.

Lemma 1.3.1. (*Korn's inequality*) *There exists a constant K independent of δ such that for any $0 < \delta \le 1$ and $\boldsymbol{u} = (u,v)^{\mathrm{T}}$ satisfying*

$$\begin{cases} u_x + v_y = 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \Sigma, \end{cases}$$

we have

$$\iint_\Omega (\delta^2 u_x^2 + u_y^2 + \delta^4 v_x^2 + \delta^2 v_y^2)\mathrm{d}x\mathrm{d}y \le K \iint_\Omega \big(2\delta^2 u_x^2 + (u_y + \delta^2 v_x)^2 + 2\delta^2 v_y^2\big)\mathrm{d}x\mathrm{d}y.$$

Remark 1.3.2. Teramoto and Tomoeda [35] proved that the best constant of K is 3. Note that in the case of $\delta = 1$, this inequality is well-known.

Lemma 1.3.3. (*Trace theorem*) *For* $0 < \delta \le 1$, *we have*

$$|f|_0^2 + \delta ||D_x|^{\frac{1}{2}} f|_0^2 \lesssim \|f\|^2 + \delta^2 \|f_x\|^2 + \|f_y\|^2.$$

Remark 1.3.4. This trace theorem is also well-known in the case of $\delta = 1$.

Lemma 1.3.5. *If* $f(x, 0) = 0$, *then we have*

$$\|f\|_{L^\infty} \lesssim \|f_y\| + \|f_{xy}\|.$$

Lemma 1.3.6. *For any integer* $k \ge 0$, *we have*

$$\begin{aligned}
\|\partial_x^k (af)\| &\lesssim \|a\|_{L^\infty} \|\partial_x^k f\| + (\|\partial_x^k a\| + \|\partial_x^k a_y\|)(\|f\| + \|f_x\|), \\
\|\partial_x^k (abf)\| &\lesssim \|a\|_{L^\infty} \|b\|_{L^\infty} \|\partial_x^k f\| + \|b\|_{L^\infty} (\|\partial_x^k a\| + \|\partial_x^k a_y\|)(\|f\| + \|f_x\|) \\
&\quad + \|a\|_{L^\infty} (\|\partial_x^k b\| + \|\partial_x^k b_y\|)(\|f\| + \|f_x\|).
\end{aligned}$$

Lemma 1.3.7. *For any integer* $k \ge 1$, *we have*

$$\|[\partial_x^k, a] f\| \lesssim \|a_x\|_{L^\infty} \|\partial_x^{k-1} f\| + (\|\partial_x^k a\| + \|\partial_x^k a_y\|)(\|f\| + \|f_x\|).$$

Chapter 2

Main results

In this chapter, we rewrite the problem in a non-dimensional form, transform the problem in a time dependent domain to a problem in a time independent domain by using an appropriate diffeomorphism, and give our main theorems in this dissertation.

2.1 Reformulation of the problem

2.1.1 Nondimensional form

We seek a stationary solution $(\bar{u}, \bar{v}, \bar{p}, \bar{\eta})$ to the system (1.1.1)–(1.1.3) of the following form

$$\bar{v} = \bar{v}(y), \quad \bar{u} = \bar{u}(y), \quad \bar{p} = \bar{p}(y), \quad \bar{\eta} = 0.$$

Plugging these into (1.1.1)–(1.1.3), we have

$$\begin{cases} \mu \bar{u}_{yy} = \rho \bar{v} \bar{u}_y - \rho g \sin\alpha & \text{in} \quad 0 < y < h_0, \\ \mu \bar{v}_{yy} - \bar{p}_y + \rho \bar{v} \bar{v}_y = \rho g \cos\alpha & \text{in} \quad 0 < y < h_0, \\ \bar{v}_y = 0 & \text{in} \quad 0 < y < h_0, \\ \bar{u}_y = 0 & \text{on} \quad y = h_0, \\ \bar{p} - 2\mu \bar{v}_y = p_0 & \text{on} \quad y = h_0, \\ \bar{u} = \bar{v} = 0 & \text{on} \quad y = 0. \end{cases}$$

Solving the above boundary value problem, we obtain the Nusselt flat film solution (1.1.4).

We proceed to consider fluctuations on a laminar stationary motion. We use prime sign to represent fluctuations, that is,

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad p = \bar{p} + p', \quad \eta = \eta'$$

and rewrite (1.1.1)–(1.1.3) as

$$\begin{cases} \rho(u_t + (\bar{u} + u)u_x + v(\bar{u}_y + u_y)) + p_x = \mu(u_{xx} + u_{yy}) & \text{in} \quad \Omega(t), \quad t > 0, \\ \rho(v_t + (\bar{u} + u)v_x + vv_y) + p_y = \mu(v_{xx} + v_{yy}) & \text{in} \quad \Omega(t), \quad t > 0, \\ u_x + v_y = 0 & \text{in} \quad \Omega(t), \quad t > 0, \end{cases} \tag{2.1.1}$$

$$
(2.1.2) \quad \begin{cases} (p - 2\mu u_x)\eta_x + \mu(\bar{u}_y + u_y + v_x) \\ \qquad = \rho g \cos\alpha (y - h_0)\eta_x - \dfrac{\sigma \eta_x \eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} & \text{on } \Gamma(t), \quad t > 0, \\ -\mu(\bar{u}_y + u_y + v_x)\eta_x - p + 2\mu v_y \\ \qquad = -\rho g \cos\alpha (y - h_0) + \dfrac{\sigma \eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} & \text{on } \Gamma(t), \quad t > 0, \\ \eta_t + (\bar{u} + u)\eta_x - v = 0 & \text{on } \Gamma(t), \quad t > 0, \end{cases}
$$

$$
(2.1.3) \qquad u = v = 0 \qquad \text{on} \quad \Sigma, \quad t > 0
$$

where the prime sign is dropped in the notation.

We proceed to rewrite (2.1.1)–(2.1.3) in a non-dimensional form. We rescale the independent and dependent variables by

$$
\begin{cases} x = l_0 x', \quad y = h_0 y', \quad t = t_0 t', \\ \eta = a_0 \eta', \quad u = \varepsilon U_0 u', \quad v = \varepsilon V_0 v', \quad p = \varepsilon P_0 p', \end{cases}
$$

where

$$
U_0 = \frac{\rho g h_0^2 \sin\alpha}{2\mu}, \quad V_0 = \frac{h_0}{l_0} U_0, \quad t_0 = \frac{l_0}{U_0}, \quad \bar{u}' = 2y' - y'^2, \quad P_0 = \rho g h_0 \sin\alpha.
$$

Putting these into (2.1.1)–(2.1.3) and dropping the prime sign in the notation, we obtain

$$
(2.1.4) \quad \begin{cases} \delta \boldsymbol{u}_t^\delta + \big((\bar{\boldsymbol{u}} + \varepsilon \boldsymbol{u}^\delta) \cdot \nabla_\delta\big)\boldsymbol{u}^\delta + (\boldsymbol{u}^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}} \\ \qquad\qquad + \dfrac{2}{\mathrm{R}} \nabla_\delta p - \dfrac{1}{\mathrm{R}} \Delta_\delta \boldsymbol{u}^\delta = \boldsymbol{0} & \text{in } \Omega_\varepsilon(t), \ t > 0, \\ \nabla_\delta \cdot \boldsymbol{u}^\delta = 0 & \text{in } \Omega_\varepsilon(t), \ t > 0, \end{cases}
$$

$$
(2.1.5) \quad \begin{cases} \big(\boldsymbol{D}_\delta(\varepsilon \boldsymbol{u}^\delta + \bar{\boldsymbol{u}}) - \varepsilon p \boldsymbol{I}\big)\boldsymbol{n}^\delta \\ \qquad = \left(-\dfrac{1}{\tan\alpha} \varepsilon\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha} \dfrac{\varepsilon \eta_{xx}}{(1 + (\varepsilon\delta\eta_x)^2)^{\frac{3}{2}}} \right) \boldsymbol{n}^\delta & \text{on } \Gamma_\varepsilon(t), \ t > 0, \\ \eta_t + \big(1 - (\varepsilon\eta)^2 + \varepsilon u\big)\eta_x - v = 0 & \text{on } \Gamma_\varepsilon(t), \ t > 0, \end{cases}
$$

$$
(2.1.6) \qquad \boldsymbol{u}^\delta = \boldsymbol{0} \quad \text{on} \quad \Sigma, \ t > 0,
$$

where

$$
\boldsymbol{u}^\delta = (u, \delta v)^{\mathrm{T}}, \quad \bar{\boldsymbol{u}} = (\bar{u}, 0)^{\mathrm{T}}, \quad \bar{u} = 2y - y^2,
$$

$$
\boldsymbol{D}_\delta \boldsymbol{f} = \frac{1}{2}\big\{\nabla_\delta(\boldsymbol{f}^{\mathrm{T}}) + \big(\nabla_\delta(\boldsymbol{f}^{\mathrm{T}})\big)^{\mathrm{T}}\big\}, \quad \boldsymbol{n}^\delta = (-\varepsilon\delta\eta_x, 1)^{\mathrm{T}},
$$

and

$$
\mathrm{R} = \frac{\rho U_0 h_0}{\mu}, \quad \mathrm{W} = \frac{\sigma}{\rho g h_0^2}
$$

are the Reynolds number and the Weber number. In this scaling, the liquid domain $\Omega_\varepsilon(t)$ and the liquid surface $\Gamma_\varepsilon(t)$ are of the forms

$$\begin{cases} \Omega_\varepsilon(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < 1 + \varepsilon\eta(x,t)\}, \\ \Gamma_\varepsilon(t) = \{(x,y) \in \mathbb{R}^2 \mid y = 1 + \varepsilon\eta(x,t)\}. \end{cases}$$

2.1.2 Properties of a diffeomorphism

Next, we transform the problem in the moving domain $\Omega_\varepsilon(t)$ to a problem in the fixed domain Ω by using an appropriate diffeomorphism $\Phi : \Omega \to \Omega_\varepsilon(t)$ defined by

$$\Phi(x,y,t) = \big(x, y(1 + \varepsilon\tilde{\eta}(x,y,t))\big), \tag{2.1.7}$$

where $\tilde{\eta}$ is an extension of η to Ω. We need to choose the extension $\tilde{\eta}$ carefully and in this dissertation we adopt the following extension. For $\phi \in H^s(\Gamma)$, we define its extension $\tilde{\phi}$ to Ω by

$$\tilde{\phi}(x,y) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} \frac{\hat{\phi}_n}{1 + (\delta n(1-y)y)^4} \mathrm{e}^{2\pi i n x} & \text{in the case } \mathbb{G} = \mathbb{T}, \\ \displaystyle\int_{\mathbb{R}} \frac{\hat{\phi}(\xi)}{1 + (\delta\xi(1-y)y)^4} \mathrm{e}^{2\pi i \xi x} \mathrm{d}\xi & \text{in the case } \mathbb{G} = \mathbb{R}. \end{cases} \tag{2.1.8}$$

By the definition, it is easy to see that

$$\partial_y^j \tilde{\phi}(x,1) = \partial_y^j \tilde{\phi}(x,0) = 0 \qquad \text{for } j = 1,2,3. \tag{2.1.9}$$

As usual, this extension operator has a regularizing effect so that $\tilde{\phi} \in H^{s+\frac{1}{2}}(\Omega)$. However, if we use such a regularizing property, then we need to pay the cost of a power of δ. Moreover, in this extension, ∂_y corresponds to $\delta\partial_x$. More precisely, we have the following lemma.

Lemma 2.1.1. *Let i and j be non-negative integers such that $j \le 4$. Then, for the extension* (2.1.8) *we have*

$$\|\partial_x^i \partial_y^j \tilde{\phi}\| \lesssim \delta^j |\partial_x^{i+j}\phi|_0, \tag{2.1.10}$$

$$\|\partial_x^i \partial_y^j \tilde{\phi}\|_{L^\infty} \lesssim \delta^j |\partial_x^{i+j}\phi|_1. \tag{2.1.11}$$

If, in addition, $i + j \ge 1$, then

$$\|\partial_x^i \partial_y^j \tilde{\phi}\| \lesssim \delta^{j-\frac{1}{2}} ||D_x|^{i+j-\frac{1}{2}}\phi|_0. \tag{2.1.12}$$

Proof. We first prove (2.1.12) in the case $\mathbb{G} = \mathbb{T}$. Since $\partial_x^i \tilde{\phi} = \widetilde{\partial_x^i \phi}$, it is sufficient to show $\|\partial_y^j \phi\|^2 \lesssim \delta^{2j-1} ||D_x|^{j-\frac{1}{2}}\phi|_0^2$. Moreover, without loss of generality, we can assume $\hat{\phi}_0 = 0$. Therefore, we rewrite (2.1.8) as

$$\tilde{\phi}(x,y) = \sum_{n\neq 0} f(\delta n(1-y)y)\hat{\phi}_n \mathrm{e}^{2\pi i n x},$$

where $f(z) := \frac{1}{1+z^4}$. In view of $|f^{(j)}(z)| \lesssim \frac{|z|^{4-j}}{(1+z^4)^2}$, we easily obtain

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}y^j} f(\delta n(1-y)y)\right| \lesssim \frac{|\delta n|^j}{(1+|\delta n(1-y)y|)^{j+4}} \quad \text{for } j = 0, 1, \ldots, 4.$$

Hence, by Parseval's identity we see that

$$\begin{aligned}
\|\partial_y^j \tilde{\phi}\|^2 &\lesssim \sum_{n\neq 0} |\delta n|^{2j} |\hat{\phi}_n|^2 \int_0^1 \frac{1}{(1+|\delta n(1-y)y|)^{2(j+4)}} \mathrm{d}y \\
&\lesssim \sum_{n\neq 0} |\delta n|^{2j-1} |\hat{\phi}_n|^2 \int_0^{\frac{|\delta n|}{2}} \frac{1}{(1+|z|/2)^{2(j+4)}} \mathrm{d}z \\
&\lesssim \sum_{n\neq 0} |\delta n|^{2j-1} |\hat{\phi}_n|^2 = \delta^{2j-1} ||D_x|^{j-\frac{1}{2}} \phi|_0^2.
\end{aligned}$$

Therefore, (2.1.12) holds.

Moreover, by using

$$\int_0^{\frac{|\delta n|}{2}} \frac{1}{(1+|z|/2)^{2(j+4)}} \mathrm{d}z \le \frac{|\delta n|}{2}$$

in above calculation, we also obtain (2.1.10).

As for (2.1.11), by Schwarz' inequality and Parseval's identity, we get

$$\begin{aligned}
|\partial_x^i \partial_y^j \tilde{\phi}(x,y)| &\le \delta^j \sum_{n\neq 0} |n^{i+j} \hat{\phi}_n| \\
&\le \delta^j \left(\sum_{n\neq 0} \frac{1}{|n|^2}\right)^{\frac{1}{2}} \left(\sum_{n\neq 0} |n^{i+j+1} \hat{\phi}_n|^2\right)^{\frac{1}{2}} \lesssim \delta^j |\partial_x^{i+j+1} \phi|_0,
\end{aligned}$$

which implies the desired inequality. □

The solenoidal condition on the velocity field is destroyed in general by the transformation. To keep the condition, following Beale [3], we also change the dependent variables and introduce new unknown functions (u', v', p') defined in Ω by

$$u' = J(u \circ \Phi), \quad v' = v \circ \Phi - y\varepsilon \tilde{\eta}_x (u \circ \Phi), \quad p' = p \circ \Phi,$$

where

$$J = 1 + \varepsilon (y\tilde{\eta})_y$$

is the Jacobian of the diffeomorphism Φ. Putting

$$a_1 = -yJ^{-1}\varepsilon\delta\tilde{\eta}_x, \quad b_1 = J^{-1} - 1,$$

$$A_1 = \begin{pmatrix} 1+b_1 & 0 \\ -a_1 & 1 \end{pmatrix} = N_1 + I, \quad \boldsymbol{u}'^{\delta} = \begin{pmatrix} u' \\ \delta v' \end{pmatrix},$$

we have

$$\boldsymbol{u}^{\delta} \circ \Phi = A_1 \boldsymbol{u}'^{\delta}. \tag{2.1.13}$$

Here N_1 is the nonlinear part of A_1. We note that b_1 is the term which is hard to handle because it contains the term without δ in the coefficient. Then, the second equation in (2.1.5) is transformed to

$$\eta_t + \eta_x - v' = h_3, \tag{2.1.14}$$

where

$$h_3 = \varepsilon^2 \eta^2 \eta_x.$$

We easily obtain that

$$\begin{cases} (\nabla_\delta \phi) \circ \Phi = A_2 \nabla_\delta (\phi \circ \Phi), & (2.1.15) \\ (\Delta_\delta \phi) \circ \Phi = \delta^2 (\phi \circ \Phi)_{xx} + (1 + b_2)(\phi \circ \Phi)_{yy} + P_\delta(\tilde{\eta}, D)(\phi \circ \Phi), & (2.1.16) \\ \delta(\phi_t \circ \Phi) = \delta(\phi \circ \Phi)_t - y J^{-1} \varepsilon \delta \tilde{\eta}_t (\phi \circ \Phi)_y, & (2.1.17) \end{cases}$$

where

$$A_2 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 + b_1 \end{pmatrix} = N_2 + I,$$

N_2 is the nonlinear part of A_2,

$$b_2 = a_1^2 + 2b_1 + b_1^2,$$

and $P_\delta(\tilde{\eta}, D)$ is a second order differential operator defined by

$$P_\delta(\tilde{\eta}, D) f = 2\delta a_1 f_{xy} + \left\{ \delta a_{1x} + a_1 a_{1y} + (1 + b_1) b_{1y} \right\} f_y.$$

We confirm that solenoidal condition holds. Using integration by parts and (2.1.15), for all test function ϕ we see that

$$\begin{aligned} 0 &= \iint_{\Omega(t)} (\nabla_\delta \cdot \boldsymbol{u}^\delta) \phi \, \mathrm{d}x \mathrm{d}y = - \iint_{\Omega(t)} \boldsymbol{u}^\delta \cdot \nabla_\delta \phi \, \mathrm{d}x \mathrm{d}y \\ &= - \iint_{\Omega} (\boldsymbol{u}^\delta \circ \Phi) \cdot A_2 \nabla_\delta (\phi \circ \Phi) J \, \mathrm{d}x \mathrm{d}y \\ &= \iint_{\Omega} \{ \nabla_\delta \cdot J A_2^{\mathrm{T}} (\boldsymbol{u}^\delta \circ \Phi) \} (\phi \circ \Phi) \, \mathrm{d}x \mathrm{d}y. \end{aligned}$$

Therefore, thanks to fundamental lemma of calculus of variations we have

$$\nabla_\delta \cdot J A_2^{\mathrm{T}} (\boldsymbol{u}^\delta \circ \Phi) = 0.$$

In view of

$$J A_2^{\mathrm{T}} = A_1^{-1},$$

and (2.1.13), we have

$$\nabla_\delta \cdot \boldsymbol{u}'^\delta = 0. \tag{2.1.18}$$

2.1.3 Transformation of the system

We begin to transform the equations in (2.1.4). By (2.1.13) and (2.1.17), we obtain

$$\delta \boldsymbol{u}_t^\delta \circ \Phi = \delta A_1 \boldsymbol{u}_t'^\delta + \boldsymbol{f}_1, \tag{2.1.19}$$

where

$$\boldsymbol{f}_1 = \delta A_{1t} \boldsymbol{u}'^\delta - y J^{-1} \varepsilon \delta \tilde{\eta}_t (A_1 \boldsymbol{u}'^\delta)_y.$$

By (2.1.13) and (2.1.15), we obtain

$$\big\{\big((\bar{\boldsymbol{u}} + \varepsilon \boldsymbol{u}^\delta) \cdot \nabla_\delta\big)\boldsymbol{u}^\delta + (\boldsymbol{u}^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}}\big\} \circ \Phi = (\bar{\boldsymbol{u}} \cdot \nabla_\delta)\boldsymbol{u}'^\delta + (\boldsymbol{u}'^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}} + \boldsymbol{f}_2, \tag{2.1.20}$$

where

$$\begin{aligned} \boldsymbol{f}_2 &= (\bar{\boldsymbol{u}} \cdot \nabla_\delta) N_1 \boldsymbol{u}'^\delta + (\bar{\boldsymbol{u}} \cdot N_2 \nabla_\delta) A_1 \boldsymbol{u}'^\delta + \big((\boldsymbol{V} + \varepsilon A_1 \boldsymbol{u}'^\delta) \cdot A_2 \nabla_\delta\big) A_1 \boldsymbol{u}'^\delta \\ &\quad + (\boldsymbol{u}'^\delta \cdot N_2 \nabla_\delta)\bar{\boldsymbol{u}} + \big((N_1 \boldsymbol{u}'^\delta) \cdot (A_2 \nabla_\delta)\big)\bar{\boldsymbol{u}} + \big((A_1 \boldsymbol{u}'^\delta) \cdot (A_2 \nabla_\delta)\big)\boldsymbol{V}, \\ \boldsymbol{V} &= \begin{pmatrix} 2\varepsilon y \tilde{\eta} - 2\varepsilon y^2 \tilde{\eta} - (\varepsilon y \tilde{\eta})^2 \\ 0 \end{pmatrix}. \end{aligned}$$

By (2.1.15), we have

$$(\nabla_\delta p) \circ \Phi = A_2 \nabla_\delta p'. \tag{2.1.21}$$

By (2.1.13) and (2.1.16), we obtain

$$(\Delta_\delta \boldsymbol{u}^\delta) \circ \Phi = A_1 \big(\delta^2 \boldsymbol{u}_{xx}'^\delta + (I + A_3)\boldsymbol{u}_{yy}'^\delta\big) + \boldsymbol{f}_3, \tag{2.1.22}$$

where

$$\begin{aligned} A_3 &= \begin{pmatrix} b_2 & 0 \\ 0 & 0 \end{pmatrix}, \\ \boldsymbol{f}_3 &= [\delta^2 \partial_x^2, A_1]\boldsymbol{u}'^\delta + (1 + b_2)\big[\partial_y^2, A_1\big]\boldsymbol{u}'^\delta + P_\delta(\tilde{\eta}, D)\big(A_1 \boldsymbol{u}'^\delta\big) + A_1 \begin{pmatrix} 0 \\ \delta b_2 v_{yy}' \end{pmatrix}. \end{aligned}$$

Thus combining (2.1.19)–(2.1.22), we transform the first equation in (2.1.4) to

$$\delta \boldsymbol{u}_t'^\delta + (\bar{\boldsymbol{u}} \cdot \nabla_\delta)\boldsymbol{u}'^\delta + (\boldsymbol{u}'^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}} + \frac{2}{\mathrm{R}}(I + A_4)\nabla_\delta p' - \frac{1}{\mathrm{R}}\big(\delta^2 \boldsymbol{u}_{xx}'^\delta + (I + A_3)\boldsymbol{u}_{yy}'^\delta\big) = \boldsymbol{f}, \tag{2.1.23}$$

where

$$\boldsymbol{f} = -N\big\{(\bar{\boldsymbol{u}} \cdot \nabla_\delta)\boldsymbol{u}'^\delta + (\boldsymbol{u}'^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}}\big\} + A_1^{-1}\Big(-\boldsymbol{f}_1 - \boldsymbol{f}_2 + \frac{1}{\mathrm{R}}\boldsymbol{f}_3\Big), \tag{2.1.24}$$

$$A_4 = A_1^{-1} A_2 - I = \begin{pmatrix} (y\varepsilon\tilde{\eta})_y & -y\varepsilon\delta\tilde{\eta}_x \\ -y\varepsilon\delta\tilde{\eta}_x & J^{-1}\big((y\varepsilon\delta\tilde{\eta}_x)^2 - (y\varepsilon\tilde{\eta})_y\big) \end{pmatrix}, \tag{2.1.25}$$

and N is the nonlinear part of A_1^{-1}. We remark that $\boldsymbol{f}$ is a collection of nonlinear terms, which does not contain $\boldsymbol{u}_t'^\delta$, u_{yy}', $\nabla_\delta p'$, nor any function of $\tilde{\eta}$ only.

Next, we transform the boundary conditions. By (2.1.13) and (2.1.15), we see that

$$\{(\boldsymbol{D}_\delta(\varepsilon \boldsymbol{u}^\delta + \bar{\boldsymbol{u}}) - \varepsilon p I)\boldsymbol{n}^\delta\} \circ \Phi = \begin{pmatrix} \frac{1}{2}\varepsilon(\delta^2 v'_x + u'_y - 2\eta) \\ \varepsilon\delta v'_y \end{pmatrix} - \varepsilon p' \boldsymbol{n}^\delta + \boldsymbol{h} \quad \text{on} \quad \Gamma,$$

where

$$\begin{aligned} \boldsymbol{h} = &- \begin{pmatrix} \varepsilon^2\delta^2\eta_x u'_x \\ \frac{1}{2}\varepsilon^2\delta\eta_x(\delta^2 v'_x + u'_y - 2\eta) \end{pmatrix} \\ &+ \frac{\varepsilon}{2}\{\nabla_\delta(N_1\boldsymbol{u}'^\delta)^{\mathrm{T}} + (\nabla_\delta(N_1\boldsymbol{u}'^\delta)^{\mathrm{T}})^{\mathrm{T}} + N_2\nabla_\delta(A_1\boldsymbol{u}'^\delta)^{\mathrm{T}} + (N_2\nabla_\delta(A_1\boldsymbol{u}'^\delta)^{\mathrm{T}})^{\mathrm{T}}\}\boldsymbol{n}^\delta. \end{aligned}$$

Taking the inner product of a tangential vector $\boldsymbol{t}^\delta = (1, \varepsilon\delta\eta_x)^{\mathrm{T}}$ with the first equation in (2.1.5), we obtain

$$\delta^2 v'_x + u'_y - 2\eta = h_4 \quad \text{on} \quad \Gamma, \tag{2.1.26}$$

where

$$h_4 = -\frac{2}{\varepsilon}(\varepsilon^2\delta^2\eta_x v'_y + \boldsymbol{h}\cdot\boldsymbol{t}^\delta).$$

On the other hand, taking the inner product of a normal vector $\boldsymbol{n}^\delta$ with the first equation in (2.1.5), we obtain

$$p' - \delta v'_y - \frac{1}{\tan\alpha}\eta + \frac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = h_2 \quad \text{on} \quad \Gamma, \tag{2.1.27}$$

where

$$\begin{aligned} h_2 = &\frac{1}{\varepsilon}\Big\{ - \frac{(\varepsilon\delta\eta_x)^2}{1+(\varepsilon\delta\eta_x)^2}\varepsilon\delta v'_y + \frac{1}{1+(\varepsilon\delta\eta_x)^2}\Big(-\frac{1}{2}\varepsilon^2\delta\eta_x(\delta^2 v'_x + u'_y - 2\eta) + \boldsymbol{h}\cdot\boldsymbol{n}^\delta\Big)\Big\} \\ &+ \frac{\delta^2\mathrm{W}}{\sin\alpha}\Big(1 - \frac{1}{(1+(\varepsilon\delta\eta_x)^2)^{\frac{3}{2}}}\Big)\eta_{xx} \\ =&: h_{2,1} + \delta^2\mathrm{W} h_{2,2} \end{aligned} \tag{2.1.28}$$

and h_2 does not contain p' nor any function of η only. Note that the term $\delta^2\mathrm{W}h_{2,2}$ is the only nonlinear term which contains W. Here, by a straightforward calculation we see that

$$\boldsymbol{h}\cdot\boldsymbol{t}^\delta = \varepsilon(b_4 u'_y + h_5),$$

where

$$\begin{aligned} b_4 &= -\frac{1}{2}(\varepsilon\delta\eta_x)^2 + \left\{ \begin{pmatrix} a_1(1+b_1) & \frac{1}{2}(-a_1^2 + b_1(2+b_1)) \\ \frac{1}{2}(-a_1^2 + b_1(2+b_1)) & -a_1(1+b_1) \end{pmatrix} \boldsymbol{n}^\delta \right\} \cdot \boldsymbol{t}^\delta, \\ h_5 &= -\varepsilon\delta^2\eta_x u'_x - \frac{1}{2}(\varepsilon\delta\eta_x)^2(\delta^2 v'_x - 2\eta) \\ &\quad + \left\{ \begin{pmatrix} \delta(b_1 u')_x & \frac{1}{2}\{\delta(-a_1 u')_x - a_1 a_{1y} u' + \delta a_1 v'_y\} \\ \frac{1}{2}\{\delta(-a_1 u')_x - a_1 a_{1y} u' + \delta a_1 v'_y\} & -a_{1y}(1+b_1)u' + \delta b_1 v'_y \end{pmatrix} \boldsymbol{n}^\delta \right\} \cdot \boldsymbol{t}^\delta, \end{aligned}$$

and h_5 does not contain u'_y. Thus we can rewrite (2.1.26) as

$$\delta^2 v'_x + u'_y - (2 + b_3)\eta = h_1 \quad \text{on} \quad \Gamma, \tag{2.1.29}$$

where

$$b_3 = -\frac{4b_4}{1 + 2b_4}, \tag{2.1.30}$$

$$h_1 = \frac{2b_4}{1 + 2b_4}\delta^2 v'_x - \frac{2}{1 + 2b_4}(\varepsilon\delta^2\eta_x v'_y + h_5). \tag{2.1.31}$$

Note that h_1 does not contain u'_y, p', nor any function of η only.

Summarizing (2.1.14), (2.1.18), (2.1.23), (2.1.27), and (2.1.29) and dropping the prime sign in the notation, we have

$$\begin{cases} \delta \boldsymbol{u}^\delta_t + (\bar{\boldsymbol{u}} \cdot \nabla_\delta)\boldsymbol{u}^\delta + (\boldsymbol{u}^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}} \\ \qquad + \dfrac{2}{\mathrm{R}}(I + A_4)\nabla_\delta p - \dfrac{1}{\mathrm{R}}\{\delta^2 \boldsymbol{u}^\delta_{xx} + (I + A_3)\boldsymbol{u}^\delta_{yy}\} = \boldsymbol{f} & \text{in} \quad \Omega,\ t > 0, \\ u_x + v_y = 0 & \text{in} \quad \Omega,\ t > 0, \end{cases} \tag{2.1.32}$$

$$\begin{cases} \delta^2 v_x + u_y - (2 + b_3)\eta = h_1 & \text{on} \quad \Gamma,\ t > 0, \\ p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = h_2 & \text{on} \quad \Gamma,\ t > 0, \\ \eta_t + \eta_x - v = h_3 & \text{on} \quad \Gamma,\ t > 0, \end{cases} \tag{2.1.33}$$

$$u = v = 0 \quad \text{on} \quad \Sigma,\ t > 0. \tag{2.1.34}$$

In the following, we will consider the initial value problem to (2.1.32)–(2.1.34) under the initial conditions

$$\eta|_{t=0} = \eta_0 \quad \text{on} \quad \Gamma, \quad (u, v)^{\mathrm{T}}|_{t=0} = (u_0, v_0)^{\mathrm{T}} \quad \text{in} \quad \Omega. \tag{2.1.35}$$

We denote b_3 and h_1 determined from the initial data by $b_3^{(0)}$ and $h_1^{(0)}$, respectively.

2.2 Main results

2.2.1 Uniform estimate

For simplicity, we set

$$\begin{aligned} E_m^{(0)} =& |(1 + \delta|D_x|)^2\eta_0|_m + \|(1 + |D_x|)^m(u_0, \delta v_0)^{\mathrm{T}}\| + \|(1 + |D_x|)^m D_\delta(u_0, \delta v_0)^{\mathrm{T}}\| \\ &+ \|(1 + |D_x|)^m D_\delta^2(u_0, \delta v_0)^{\mathrm{T}}\| + \delta^2\mathrm{W}|(1 + \delta|D_x|)\eta_{0x}|_{m+1} + \sqrt{\delta^2\mathrm{W}}\|(1 + |D_x|)^m \delta v_{0xy}\|. \end{aligned}$$

We state one of our main results in this dissertation.

Theorem 2.2.1. (H. Ueno, A. Shiraishi, and T. Iguchi [39]) *There exist small positive constants* R_0 *and* α_0 *such that the following statement holds: Let* m *be an integer satisfying*

$m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, *and* $0 < \alpha \leq \alpha_0$. *There exist positive constants* c_0 *and* T *such that if the initial data* (η_0, u_0, v_0) *and the parameters* δ, ε, R, *and* W *satisfy the compatibility conditions*

$$\begin{cases} u_{0x} + v_{0y} = 0 & \text{in } \Omega, \\ u_{0y} + \delta^2 v_{0x} - (2 + b_3^{(0)})\eta_0 = h_1^{(0)} & \text{on } \Gamma, \\ u_0 = v_0 = 0 & \text{on } \Sigma, \end{cases}$$

and

$$\begin{cases} E_2^{(0)} \leq c_0, \quad E_m^{(0)} < \infty, \\ 0 < \delta, \varepsilon \leq 1, \quad \mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0, \quad \mathrm{W}_1 \leq \mathrm{W} \leq \delta^{-2}\mathrm{W}_2, \end{cases}$$

then the initial value problem (2.1.32)–(2.1.35) *has a unique solution* (η, u, v, p) *on the time interval* $[0, T/\varepsilon]$ *and the solution satisfies the estimate*

$$\begin{aligned}
&|(1+\delta|D_x|)^2\eta(t)|_m^2 + \delta^2|\eta_t(t)|_m^2 + \delta^2\mathrm{W}\{|(1+\delta|D_x|)^2\eta_x(t)|_m^2 + \delta^2|\eta_{tx}(t)|_m^2\} \\
&\quad + \|(1+|D_x|)^m(1+\delta|D_x|)^2\boldsymbol{u}^\delta(t)\|^2 + \|(1+|D_x|)^m\boldsymbol{u}_y^\delta(t)\|^2 + \delta^2\|(1+|D_x|)^m\boldsymbol{u}_t^\delta(t)\|^2 \\
&+ \int_0^t \{\delta|\eta_x(\tau)|_m^2 + \delta|(1+\delta|D_x|)^{\frac{5}{2}}\eta_t(\tau)|_m^2 \\
&\qquad + (\delta^2\mathrm{W})\delta|\eta_{xx}(\tau)|_m^2 + (\delta^2\mathrm{W})^2\{\delta|\eta_{xxx}(\tau)|_m^2 + \delta^2||D_x|^{\frac{7}{2}}\eta(\tau)|_m^2\} \\
&\qquad + \delta\|(1+|D_x|)^m\boldsymbol{u}_x^\delta(\tau)\|^2 + \delta\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta\boldsymbol{u}_x^\delta(\tau)\|^2 + \delta\|(1+|D_x|)^m\nabla_\delta\boldsymbol{u}_t^\delta(\tau)\|^2 \\
&\qquad + \delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)\boldsymbol{u}_{yy}^\delta(\tau)\|^2 + \delta\|(1+|D_x|)^m\partial_y^{-1}p_x(\tau)\|^2 \\
&\qquad + \delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta p(\tau)\|^2 + \delta\|(1+|D_x|)^{m-1}\nabla_\delta p_t(\tau)\|^2\}\mathrm{d}\tau \leq C
\end{aligned}$$

for $0 \leq t \leq T/\varepsilon$ *with a constant* $C = C(\mathrm{R}_1, \mathrm{W}_1, \mathrm{W}_2, \alpha, M)$ *independent of* δ, ε, R, *and* W, *where* M *is an upper bound of* $E_m^{(0)}$. *Moreover, the following uniform estimate holds.*

$$\begin{aligned}
|\eta(t)|_m + \|(1+|D_x|)^{m-1}u(t)\|_1 + \|\partial_x^m u_y(t)\| \\
+ \|(1+|D_x|)^{m-2}v(t)\|_1 + \|\partial_x^{m-1}v_{yy}(t)\| \leq C
\end{aligned} \tag{2.2.1}$$

for $0 \leq t \leq T/\varepsilon$. *If, in addition,* $0 \leq \varepsilon \lesssim \delta$, *then the solution can be extended for all* $t \geq 0$ *and the above estimates hold for* $t \geq 0$.

Remark 2.2.2. In the case $\varepsilon \simeq 1$, this theorem gives a uniform boundedness of the solution only for a short time interval $[0, T]$. However, this is essential and we cannot extend this uniform estimate for all $t \geq 0$ in general, because by (1.2.1) we see that the limiting equation for η as $\delta \to 0$ becomes a nonlinear hyperbolic conservation law of the form

$$\eta_t + 2(1+\varepsilon\eta)^2\eta_x = 0,$$

whose solution will have a singularity in finite time in general.

Remark 2.2.3. In the case where $\mathbb{G} = \mathbb{T}, \varepsilon \lesssim \delta$, and $\int_0^1 \eta_0(x)\mathrm{d}x = 0$, we also obtain the following exponential decay in time property of the solution.

$$
\begin{aligned}
(2.2.2) \qquad & |(1+\delta|D_x|)^2\eta(t)|_m^2 + \delta^2|\eta_t(t)|_m^2 + \delta^2\mathrm{W}\{|(1+\delta|D_x|)^2\eta_x(t)|_m^2 + \delta^2|\eta_{tx}(t)|_m^2\} \\
& + \|(1+|D_x|)^m(1+\delta|D_x|)^2\boldsymbol{u}^\delta(t)\|^2 + \|(1+|D_x|)^m\boldsymbol{u}_y^\delta(t)\|^2 \\
& + \delta^2\|(1+|D_x|)^m\boldsymbol{u}_t^\delta(t)\|^2 \leq C\mathrm{e}^{-c\delta t}.
\end{aligned}
$$

Remark 2.2.4. In order to derive a uniform estimate in R, the constant C in the above estimate is required to depend on a lower bound R_1 of R for a technical reason. However, for a justification of the thin film approximation this restriction matters little because we are interested in the case where R is close enough to $\mathrm{R_c}$.

2.2.2 Error estimate

Before we state another main result, we set the following assumption for later use.

Assumption 2.2.5. *Let* $\mathrm{R}_0, \mathrm{R}_1, \alpha_0, \mathrm{W}_1, c_0$, *and* M *be positive constants and* $m \geq 2$ *be an integer.*

(1) Conditions for parameters

Parameters $\mathrm{R}, \alpha, \mathrm{W}, \delta$, *and* ε *satisfy*

$$
\mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0, \quad 0 < \alpha \leq \alpha_0, \quad \mathrm{W}_1 \leq \mathrm{W}, \quad 0 < \varepsilon = \delta \leq 1.
$$

(2) Smallness of initial data

Initial data (η_0, u_0, v_0) *and parameters* W *and* δ *satisfy*

$$
\begin{aligned}
& |(1+\delta|D_x|)^2\eta_0|_2 + \|(1+|D_x|)^2(u_0, \delta v_0)^{\mathrm{T}}\| + \|(1+|D_x|)^2 D_\delta(u_0, \delta v_0)^{\mathrm{T}}\| \\
& \quad + \|(1+|D_x|)^2 D_\delta^2(u_0, \delta v_0)^{\mathrm{T}}\| + \delta^2\mathrm{W}|(1+\delta|D_x|)\eta_{0x}|_3 + \sqrt{\delta^2\mathrm{W}}\|(1+|D_x|)^2\delta v_{0xy}\| \leq c_0.
\end{aligned}
$$

(3) Regularity of initial data

Initial data (η_0, u_0, v_0) *satisfies*

$$
\|(1+|D_x|)^{m+1}(u_0, v_0)^{\mathrm{T}}\|_{H^2(\Omega)} + |\eta_0|_{m+4} \leq M.
$$

(4) Compatibility conditions

Initial data (η_0, u_0, v_0) *and parameters* δ *and* ε *satisfy*

$$
\begin{cases}
u_{0x} + v_{0y} = 0 & \text{in } \Omega, \\
u_{0y} + \delta^2 v_{0x} - 2(1+\varepsilon\eta_0)^2\eta_0 = \delta^3 h_1^{(0)} & \text{on } \Gamma, \\
u_0 = v_0 = 0 & \text{on } \Sigma.
\end{cases}
$$

Remark 2.2.6. Under the assumption that there exist small positive constants R_0, α_0, and c_0 such that Assumption 2.2.5 is fulfilled, Theorem 2.2.1 holds.

Moreover, we define the norm of a difference between the solution $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ of the Navier–Stokes equations (2.1.32)–(2.1.35) and the approximate solution $(\zeta^{\rm app}, u^{\rm app}, v^{\rm app}, p^{\rm app})$ as

$$
\begin{aligned}
(2.2.3)\quad \mathcal{D}(t;\zeta^{\rm app}, u^{\rm app}, v^{\rm app}, p^{\rm app}) =& |\eta^\delta(t) - \zeta^{\rm app}(\cdot - 2t, \varepsilon t)|_m^2 + \|(1+|D_x|)^m (u^\delta - u^{\rm app})(t)\|^2 \\
&+ \|(1+|D_x|)^{m-1}(v^\delta - v^{\rm app})(t)\|^2 \\
&+ \|(1+|D_x|)^{m-1}(p^\delta - p^{\rm app})(t)\|^2.
\end{aligned}
$$

Let $\zeta^I, \zeta^{II}, \zeta^{III}$, and ζ^{IV} be the solution of (1.2.3)–(1.2.6) under the initial condition $\zeta|_{\tau=0} = \eta_0$, respectively.

Now we are ready to state our main results in this dissertation. Note that the definitions of the approximate solutions $u^I, v^I, p^I, u^{II}, \dots$ appeared in the following statement will be given in Section 4.3 (see (4.3.1) and (4.3.24)–(4.3.26)).

Theorem 2.2.7. (H. Ueno and T. Iguchi [38]) *Let us assume $\mathbb{G} = \mathbb{T}$. There exist small positive constants R_0 and α_0 such that the following statement holds: Let m be an integer satisfying $m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, and $0 < \alpha \leq \alpha_0$. There exists small positive constant c_0 such that if the initial data (η_0, u_0, v_0) and the parameters δ, ε, R, and W satisfy Assumption 2.2.5, then we have the following estimates.*

I. Burgers equation

If the parameters δ and W and the initial data η_0 and u_0 satisfy

$$(2.2.4)\qquad \mathrm{W}_1 \leq \mathrm{W} \leq \delta^{-1}\mathrm{W}_2, \quad |\eta_0|_{m+7} + \delta^{-1}\|(1+|D|_x)^{m+1} u_{0yy}\| \leq M < \infty,$$

then the following error estimate holds.

$$(2.2.5)\qquad \mathcal{D}(t;\zeta^I, u^I, v^I, p^I) \leq C\delta^2 \mathrm{e}^{-c\varepsilon t}.$$

II. Burgers equation with a fourth order dissipation term

If the parameters δ and W and the initial data η_0 and u_0 satisfy

$$(2.2.6)\qquad \mathrm{W} = \delta^{-2}\mathrm{W}_2, \quad |\eta_0|_{m+12} + \delta^{-1}\|(1+|D|_x)^{m+1} u_{0yy}\| \leq M < \infty,$$

then the following error estimate holds.

$$(2.2.7)\qquad \mathcal{D}(t;\zeta^{II}, u^{II}, v^{II}, p^{II}) \leq C\delta^2 \mathrm{e}^{-c\varepsilon t}.$$

III. Burgers equation with dispersion and nonlinear terms

If the parameters δ and W and the initial data η_0 and u_0 satisfy

$$(2.2.8)\qquad \mathrm{W}_1 \leq \mathrm{W} \leq \mathrm{W}_2, \quad |\eta_0|_{m+13} + \delta^{-2}\|(1+|D|_x)^{m+1}(u_{0yy} - u_{yy}^{III}|_{t=0})\| \leq M < \infty,$$

then the following error estimate holds.

$$(2.2.9)\qquad \mathcal{D}(t;\zeta^{III}, u^{III}, v^{III}, p^{III}) \leq C\delta^4 \mathrm{e}^{-c\varepsilon t}.$$

IV. Burgers equation with fourth order dissipation, dispersion, and nonlinear terms

If the parameters δ and W and the initial data η_0 and u_0 satisfy

$$\mathrm{W} = \delta^{-1}\mathrm{W}_2, \quad |\eta_0|_{m+17} + \delta^{-2}\|(1+|D|_x)^{m+1}(u_{0yy} - u_{yy}^{IV}|_{t=0})\| \le M < \infty, \tag{2.2.10}$$

then the following error estimate holds.

$$\mathcal{D}(t; \zeta^{IV}, u^{IV}, v^{IV}, p^{IV}) \le C\delta^4 \mathrm{e}^{-c\varepsilon t}. \tag{2.2.11}$$

Here, positive constants C and c depend on $\mathrm{R}_1, \mathrm{W}_1, \mathrm{W}_2, \alpha$, and M but are independent of δ, ε, R, and W.

Remark 2.2.8. *It follows from the above error estimates that*

$$\begin{cases} |\eta^\delta(t) - \zeta^I(\cdot - 2t, \varepsilon t)|_m^2 \le C\delta^2 \mathrm{e}^{-c\varepsilon t}, \\ |\eta^\delta(t) - \zeta^{II}(\cdot - 2t, \varepsilon t)|_m^2 \le C\delta^2 \mathrm{e}^{-c\varepsilon t}, \\ |\eta^\delta(t) - \zeta^{III}(\cdot - 2t, \varepsilon t)|_m^2 \le C\delta^4 \mathrm{e}^{-c\varepsilon t}, \\ |\eta^\delta(t) - \zeta^{IV}(\cdot - 2t, \varepsilon t)|_m^2 \le C\delta^4 \mathrm{e}^{-c\varepsilon t}. \end{cases}$$

Remark 2.2.9. The assumptions for u_{0yy} in (2.2.4) and (2.2.6) represent the restriction on the initial profile of the velocity. Moreover, the assumptions for u_{0yy} in (2.2.8) and (2.2.10) mean that the initial profile of the velocity has to be equal to that of the approximate solution up to $O(\delta^2)$.

Remark 2.2.10. We see formally that the order of error terms in (1.2.3) is of $O(\delta)$, which implies that the error estimates (2.2.5) and (2.2.7) are natural. In a similar way, we see that the error estimates (2.2.9) and (2.2.11) are natural.

Remark 2.2.11. By introducing the slow time scale $\tau = \varepsilon t$, the norm decays exponentially and uniformly in τ.

Chapter 3

Uniform estimate for the solution of the Navier–Stokes equations

In this chapter, we will show Theorem 2.2.1. We remark that an outline of the proof is same as [23]. The plan of this chapter is as follows. In Section 3.1, we derive energy estimates to (2.1.32)–(2.1.34). Only by following [23], we cannot obtain a uniform estimate in δ because it is difficult to control lower order terms just by using energies derived in [23]. Hence, we introduce an essentially new energy function in order to control lower order terms which is one of difficulties to obtain a uniform boundedness of the solution in δ. Therefore, Section 3.1 is a key section in this chapter and thus this dissertation. In Section 3.2, we give estimates for the pressure. In order to obtain a uniform estimate in δ, we need to carefully estimate the pressure, while in [23] there was no need to use such an estimate. In Section 3.3, we estimate carefully nonlinear terms appeared in the right-hand side of the energy inequality so that we can get a uniform estimate in δ. Finally, combining the estimates obtained in the last three sections, we derive a uniform estimate for the solution in Section 3.4.

3.1 Energy estimates

3.1.1 Basic energy estimates

The following proposition is a slight modification of the energy estimate obtained in [23].

Proposition 3.1.1. *There exists a positive constant* R_0 *such that if* $0 < \mathrm{R} \le \mathrm{R}_0$*, then the solution* (η, u, v, p) *of* (2.1.32)–(2.1.34) *satisfies*

$$\begin{aligned}(3.1.1)\qquad &\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{u}^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\} + \frac{1}{4KR}\|\nabla_\delta \boldsymbol{u}^\delta\|^2 \\ &\le \frac{4K}{\mathrm{R}}(|\eta|_0^2 + |b_3\eta|_0^2) + \frac{1}{\mathrm{R}}(h_1, u)_\Gamma - \frac{2}{\mathrm{R}}(h_2, \delta v)_\Gamma \\ &\quad + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta h_3\right)_\Gamma + (\boldsymbol{F}_1, \boldsymbol{u}^\delta)_\Omega,\end{aligned}$$

where K is the constant in Korn's inequality and

$$\boldsymbol{F}_1 = \boldsymbol{f} - \frac{2}{\mathrm{R}} A_4 \nabla_\delta p + \frac{1}{\mathrm{R}} \begin{pmatrix} b_2 u_{yy} \\ 0 \end{pmatrix}. \tag{3.1.2}$$

Proof. Note that Lemma 1.3.1 implies

$$\|\nabla_\delta \boldsymbol{u}^\delta\|^2 \le K |||\boldsymbol{u}^\delta|||^2, \tag{3.1.3}$$

where

$$|||\boldsymbol{u}^\delta|||^2 = 2\|\delta u_x\|^2 + \|u_y + \delta^2 v_x\|^2 + 2\|\delta v_y\|^2.$$

Taking the inner product of $\boldsymbol{u}^\delta$ with the first equation in (2.1.32), we have

$$\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\delta\|^2 + (u, \bar{u}_y \delta v)_\Omega + \frac{1}{\mathrm{R}}(2\nabla_\delta p - \Delta_\delta \boldsymbol{u}^\delta, \boldsymbol{u}^\delta)_\Omega = (\boldsymbol{F}_1, \boldsymbol{u}^\delta)_\Omega. \tag{3.1.4}$$

Using the second equation in (2.1.32) and integration by parts in x and y, we see that

$$\begin{aligned}
&(2\nabla_\delta p - \Delta_\delta \boldsymbol{u}^\delta, \boldsymbol{u}^\delta)_\Omega \\
&= 2(p, \delta v)_\Gamma - (2\delta^2 u_{xx} + \delta^2 v_{xy} + u_{yy}, u)_\Omega - (\delta^3 v_{xx} + 2\delta v_{yy} + \delta u_{xy}, \delta v)_\Omega \\
&= 2(p, \delta v)_\Gamma + 2\|\delta u_x\|^2 + (\delta^2 v_x + u_y, u_y)_\Omega - (\delta^2 v_x + u_y, u)_\Gamma \\
&\quad + 2\|\delta v_y\|^2 - 2(\delta v_y, \delta v)_\Gamma + (\delta^2 v_x + u_y, \delta^2 v_x)_\Omega \\
&= |||\boldsymbol{u}^\delta|||^2 + 2(p - \delta v_y, \delta v)_\Gamma - (\delta^2 v_x + u_y, u)_\Gamma.
\end{aligned}$$

By (2.1.33) and integration by parts in x, the boundary terms in the right-hand side of the above equality are calculated as

$$\begin{aligned}
2(p - \delta v_y, \delta v)_\Gamma &= 2(\frac{1}{\tan\alpha}\eta - \frac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx}, \delta(\eta_t + \eta_x - h_3))_\Gamma + 2(h_2, \delta v)_\Gamma \\
&= \delta\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2 \mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right\} + 2(h_2, \delta v)_\Gamma \\
&\quad - 2(\frac{1}{\tan\alpha}\eta - \frac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx}, \delta h_3)_\Gamma
\end{aligned} \tag{3.1.5}$$

and

$$-(\delta^2 v_x + u_y, u)_\Gamma = -((2 + b_3)\eta, u)_\Gamma - (h_1, u)_\Gamma.$$

Moreover, by the Cauchy–Schwarz and Poincaré's inequalities we see that

$$|(u, \bar{u}_y \delta v)_\Omega| \le 2\|u\|\|\delta v\| \le \|\boldsymbol{u}^\delta\|^2 \le \|\boldsymbol{u}_y^\delta\|^2 \le \|\nabla_\delta \boldsymbol{u}^\delta\|^2$$

and that

$$\frac{2}{\mathrm{R}}|(\eta, u)_\Gamma| \le \frac{2}{\mathrm{R}}|\eta|_0\|u_y\| \le \frac{1}{4K R}\|u_y\|^2 + \frac{4K}{R}|\eta|_0^2.$$

Here, we used the inequality

$$|u(\cdot, 1)|_0 = |u(\cdot, 1) - u(\cdot, 0)|_0 \le \|u_y\|$$

thanks to the boundary condition (2.1.34). In the following, we use frequently this type of inequality without any comment. Thus we can rewrite (3.1.4) as

$$
\begin{aligned}
&\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{u}^\delta\|^2+\frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2+\frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\}+\frac{1}{2K\mathrm{R}}\|\nabla_\delta\boldsymbol{u}^\delta\|^2\\
&\le\|\nabla_\delta\boldsymbol{u}^\delta\|^2+\frac{4K}{\mathrm{R}}(|\eta|_0^2+|b_3\eta|_0^2)+\frac{1}{\mathrm{R}}(h_1,u)_\Gamma-\frac{2}{\mathrm{R}}(h_2,\delta v)_\Gamma\\
&\quad+\frac{2}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta-\frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx},\delta h_3)_\Gamma+(\boldsymbol{F}_1,\boldsymbol{u}^\delta)_\Omega,
\end{aligned}
$$

where we used Korn's inequality (3.1.3). Therefore, taking R_0 sufficiently small so that

$$4K\mathrm{R}_0\le 1,$$

for $0<\mathrm{R}\le\mathrm{R}_0$ we obtain the desired energy estimate. □

Note that we can take the tangential and time derivatives of the boundary conditions. Applying $\delta\partial_x$, $\delta^2\partial_x^2$, and $\delta\partial_t$ to (2.1.32)–(2.1.34) and using the above proposition, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\delta^2\|\boldsymbol{u}_x^\delta\|^2+\frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_x|_0^2+\frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{xx}|_0^2\right)\right\}+\frac{1}{4K\mathrm{R}}\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2\\
&\le\frac{4K}{\mathrm{R}}(\delta|\eta_x|_0^2+\delta|(b_3\eta)_x|_0^2)+\frac{1}{\mathrm{R}}\delta(h_{1x},u_x)_\Gamma-\frac{2}{\mathrm{R}}\delta(h_{2x},\delta v_x)_\Gamma\\
&\quad+\frac{2}{\mathrm{R}}\delta(\frac{1}{\tan\alpha}\eta_x-\frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xxx},\delta h_{3x})_\Gamma+\delta(\boldsymbol{F}_{1x},\boldsymbol{u}_x^\delta)_\Omega,
\end{aligned}
\tag{3.1.6}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\delta^4\|\boldsymbol{u}_{xx}^\delta\|^2+\frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^4|\eta_{xx}|_0^2+\frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^4|\eta_{xxx}|_0^2\right)\right\}+\frac{1}{4K\mathrm{R}}\delta^3\|\nabla_\delta\boldsymbol{u}_{xx}^\delta\|^2\\
&\le\frac{4K}{\mathrm{R}}(\delta^3|\eta_{xx}|_0^2+\delta^3|(b_3\eta)_{xx}|_0^2)+\frac{1}{\mathrm{R}}\delta^3(h_{1xx},u_{xx})_\Gamma-\frac{2}{\mathrm{R}}\delta^3(h_{2xx},\delta v_{xx})_\Gamma\\
&\quad+\frac{2}{\mathrm{R}}\delta^3(\frac{1}{\tan\alpha}\eta_{xx}-\frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xxxx},\delta h_{3xx})_\Gamma+\delta^3(\boldsymbol{F}_{1xx},\boldsymbol{u}_{xx}^\delta)_\Omega,
\end{aligned}
\tag{3.1.7}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\delta^2\|\boldsymbol{u}_t^\delta\|^2+\frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_t|_0^2+\frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{tx}|_0^2\right)\right\}+\frac{1}{4K\mathrm{R}}\delta\|\nabla_\delta\boldsymbol{u}_t^\delta\|^2\\
&\le\frac{4K}{\mathrm{R}}(\delta|\eta_t|_0^2+\delta|(b_3\eta)_t|_0^2)+\frac{1}{\mathrm{R}}\delta(h_{1t},u_t)_\Gamma-\frac{2}{\mathrm{R}}\delta(h_{2t},\delta v_t)_\Gamma\\
&\quad+\frac{2}{\mathrm{R}}\delta(\frac{1}{\tan\alpha}\eta_t-\frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{txx},\delta h_{3t})_\Gamma\\
&\quad+\delta(\boldsymbol{f}_t,\boldsymbol{u}_t^\delta)_\Omega-\frac{2}{\mathrm{R}}\delta((A_4\nabla_\delta p)_t,\boldsymbol{u}_t^\delta)_\Omega+\frac{1}{\mathrm{R}}\delta((b_2u_{yy})_t,u_t)_\Omega.
\end{aligned}
\tag{3.1.8}
$$

For later use, we will compute $-\frac{2}{\mathrm{R}}\delta(\partial_x^k\big(A_4\nabla_\delta p\big)_t,\partial_x^k\boldsymbol{u}_t^\delta)_\Omega$ for a nonnegative integer k. Applying $\delta\partial_t$ to the first equation in (2.1.32), we have

$$\delta^2\boldsymbol{u}_{tt}^\delta=-\frac{2}{\mathrm{R}}\delta(I+A_4)\nabla_\delta p_t-\frac{2}{\mathrm{R}}\delta A_{4t}\nabla_\delta p+\delta\boldsymbol{F}_{3t},\tag{3.1.9}$$

where

$$\boldsymbol{F}_3 = -(\bar{\boldsymbol{u}} \cdot \nabla_\delta)\boldsymbol{u}^\delta - (\boldsymbol{u}^\delta \cdot \nabla_\delta)\bar{\boldsymbol{u}} + \frac{1}{\mathrm{R}}\big(\delta^2 \boldsymbol{u}^\delta_{xx} + (I + A_3)\boldsymbol{u}^\delta_{yy}\big) + \boldsymbol{f}. \tag{3.1.10}$$

Moreover, we can rewrite (2.1.32) as

$$\frac{2}{\mathrm{R}} A_4 \nabla_\delta p = -\delta A_5 \boldsymbol{u}^\delta_t + A_5 \boldsymbol{F}_3, \tag{3.1.11}$$

where

$$A_5 = A_4 (I + A_4)^{-1}.$$

Note that A_5 is a symmetric matrix due to the symmetry of A_4 (see (2.1.25)). Applying $\delta\partial_x^k\partial_t$ to the above equation, we have

$$\frac{2}{\mathrm{R}}\delta\partial_x^k(A_4\nabla_\delta p)_t = -\delta^2 A_5 \partial_x^k \boldsymbol{u}^\delta_{tt} - \delta^2 \partial_x^k (A_{5t}\boldsymbol{u}^\delta_t) - \delta^2[\partial_x^k, A_5]\boldsymbol{u}^\delta_{tt} + \delta\partial_x^k(A_5\boldsymbol{F}_3)_t.$$

This together with (3.1.9) yields

$$\begin{aligned} -\frac{2}{\mathrm{R}}\delta(\partial_x^k(A_4\nabla_\delta p)_t, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2(A_5\partial_x^k\boldsymbol{u}^\delta_t, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega \\ &\quad + \delta(\partial_x^k\{\frac{1}{2}\delta A_{5t}\boldsymbol{u}^\delta_t - (A_5\boldsymbol{F}_3)_t\}, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega + \delta(\boldsymbol{G}_k, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega, \end{aligned} \tag{3.1.12}$$

where

$$\boldsymbol{G}_k = [\partial_x^k, A_5]\bigg\{ -\frac{2}{\mathrm{R}}(I + A_4)\nabla_\delta p_t - \frac{2}{\mathrm{R}}A_{4t}\nabla_\delta p + \boldsymbol{F}_{3t}\bigg\} + \frac{1}{2}\delta[\partial_x^k, A_{5t}]\boldsymbol{u}^\delta_t. \tag{3.1.13}$$

In particular, in the case of $k = 0$, we have

$$-\frac{2}{\mathrm{R}}\delta((A_4\nabla_\delta p)_t, \boldsymbol{u}^\delta_t)_\Omega = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2(A_5\boldsymbol{u}^\delta_t, \boldsymbol{u}^\delta_t)_\Omega + \delta(\frac{1}{2}\delta A_{5t}\boldsymbol{u}^\delta_t - (A_5\boldsymbol{F}_3)_t, \boldsymbol{u}^\delta_t)_\Omega.$$

By substituting this into (3.1.8), we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg\{\delta^2((I - A_5)\boldsymbol{u}^\delta_t, \boldsymbol{u}^\delta_t)_\Omega + \frac{2}{\mathrm{R}}\bigg(\frac{1}{\tan\alpha}\delta^2|\eta_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{tx}|_0^2\bigg)\bigg\} + \frac{1}{4K\mathrm{R}}\delta\|\nabla_\delta\boldsymbol{u}^\delta_t\|^2 \\ &\quad \le \frac{4K}{\mathrm{R}}(\delta|\eta_t|_0^2 + \delta|(b_3\eta)_t|_0^2) + \frac{1}{\mathrm{R}}\delta(h_{1t}, u_t)_\Gamma - \frac{2}{\mathrm{R}}\delta(h_{2t}, \delta v_t)_\Gamma \\ &\qquad + \frac{2}{\mathrm{R}}\delta(\frac{1}{\tan\alpha}\eta_t - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{txx}, \delta h_{3t})_\Gamma + \delta(\boldsymbol{F}_2, \boldsymbol{u}^\delta_t)_\Omega, \end{aligned} \tag{3.1.14}$$

where

$$\boldsymbol{F}_2 = \boldsymbol{f}_t + \frac{1}{\mathrm{R}}\begin{pmatrix}(b_2u_{yy})_t \\ 0\end{pmatrix} + \frac{1}{2}\delta A_{5t}\boldsymbol{u}^\delta_t - (A_5\boldsymbol{F}_3)_t. \tag{3.1.15}$$

Note that $I - A_5$ is positive definite for small solutions.

3.1.2 Modified energy estimate

The lowest order energy obtained in (3.1.1) is not appropriate in order to get the uniform estimate in δ, which is our goal in this chapter. We thereby need to modify the lowest energy estimate. Now it follows from the first and second equations in (2.1.32) that

$$\delta^2 v_t + \bar{u}\delta^2 v_x + \frac{2}{\mathrm{R}} p_y - \frac{1}{\mathrm{R}}\delta(\delta^2 v_x + u_y)_x - \frac{2}{\mathrm{R}}\delta v_{yy} = f_1,$$

where

$$f_1 = \left(\boldsymbol{f} - \frac{2}{\mathrm{R}} A_4 \nabla_\delta p \right) \cdot \boldsymbol{e}_2 \tag{3.1.16}$$

and $\boldsymbol{e}_2 = (0,1)^{\mathrm{T}}$. Taking the inner product of δv with the above equation, we obtain

$$\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2\|v\|^2 - \frac{2}{\mathrm{R}}(p, \delta v_y)_\Omega + \frac{1}{\mathrm{R}}(\delta^2 v_x + u_y, \delta^2 v_x)_\Omega + \frac{2}{\mathrm{R}}\delta^2\|v_y\|^2 + \frac{2}{\mathrm{R}}(p - \delta v_y, \delta v)_\Gamma = (f_1, \delta v)_\Omega.$$

Thus using the second equation in (2.1.32) and integration by parts in x, we have

$$\begin{aligned} &\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2\|v\|^2 + \frac{2}{\mathrm{R}}(p - \delta v_y, \delta v)_\Gamma + \frac{1}{\mathrm{R}}\delta^4\|v_x\|^2 + \frac{2}{\mathrm{R}}\delta^2\|v_y\|^2 \\ &= \frac{2}{\mathrm{R}}(\delta p_x, u)_\Omega + \frac{1}{\mathrm{R}}(\delta u_{xy}, \delta v)_\Omega + (f_1, \delta v)_\Omega. \end{aligned} \tag{3.1.17}$$

Lemma 3.1.2. *The following inequality holds.*

$$\begin{aligned} &\frac{2}{\mathrm{R}}(\delta p_x, u)_\Omega + \frac{1}{3\mathrm{R}}\left(\frac{1}{\tan^2\alpha}\delta^2|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta^2|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta^2|\eta_{xxx}|_0^2 \right) + \frac{1}{\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 \\ &\quad \le I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{cases} I_1 = -\dfrac{2}{\mathrm{R}}(\delta\partial_y^{-1}p_x, (2+b_3)\eta)_\Omega, \\ I_2 = -\dfrac{2}{\mathrm{R}}(\delta\partial_y^{-1}p_x, -\delta^2 v_x(\cdot,1) + h_1 + \partial_y^{-1}(u_{yy} - 2\delta p_x))_\Omega, \\ I_3 = \dfrac{1}{\mathrm{R}}(2\delta^4|u_{xx}|_0^2 + 2\delta^2|h_{2x}|_0^2 + 3\delta^2\|\partial_y^{-2}p_{xy}\|^2). \end{cases}$$

Proof. By the first equation in (2.1.33) and (2.1.34), we see that

$$\begin{aligned} \frac{2}{\mathrm{R}}(\delta p_x, u)_\Omega &= -\frac{2}{\mathrm{R}}(\partial_y^{-1}\delta p_x, u_y)_\Omega = -\frac{2}{\mathrm{R}}(\partial_y^{-1}\delta p_x, u_y(\cdot,1) + \partial_y^{-1}u_{yy})_\Omega \\ &= -\frac{2}{\mathrm{R}}(\partial_y^{-1}\delta p_x, (2+b_3)\eta \\ &\qquad - \delta^2 v_x(\cdot,1) + h_1 + 2\partial_y^{-1}\delta p_x + \partial_y^{-1}(u_{yy} - 2\delta p_x))_\Omega \\ &= -\frac{4}{\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 + I_1 + I_2. \end{aligned} \tag{3.1.18}$$

On the other hand, it follows from the second equations in (2.1.32) and (2.1.33) that

$$\begin{aligned} p(x,y) &= p(x,1) + (\partial_y^{-1}p_y)(x,y) \\ &= -\delta u_x(x,1) + \frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} + h_2 + (\partial_y^{-1}p_y)(x,y). \end{aligned} \tag{3.1.19}$$

Thus applying $\delta \mathrm{R}^{-\frac{1}{2}}\partial_y^{-1}\partial_x$ to the above equation, we obtain

$$\begin{aligned}
&\frac{y-1}{\mathrm{R}^{\frac{1}{2}}}\left(\frac{1}{\tan\alpha}\delta\eta_x - \frac{\delta^2 \mathrm{W}}{\sin\alpha}\delta\eta_{xxx}\right) \\
&= \frac{\delta}{\mathrm{R}^{\frac{1}{2}}}(\partial_y^{-1}p_x)(x,y) + \frac{y-1}{\mathrm{R}^{\frac{1}{2}}}(\delta^2 u_{xx}(x,1) - \delta h_{2x}) - \frac{\delta}{\mathrm{R}^{\frac{1}{2}}}(\partial_y^{-2}p_{xy})(x,y).
\end{aligned}$$

Squaring both sides of the above equation and integrating the resulting equality on Ω, we have

$$\frac{1}{3\mathrm{R}}\left(\frac{1}{\tan^2\alpha}\delta^2|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta^2|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta^2|\eta_{xxx}|_0^2\right) \le \frac{3}{\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 + I_3,$$

where we used integration by parts in x. This and (3.1.18) lead to the desired inequality. □

This lemma together with (3.1.5) and (3.1.17) implies that

$$\begin{aligned}
(3.1.20)\quad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\delta^2\|v\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\} + \frac{1}{\mathrm{R}}(\delta^3\|v_x\|^2 + 2\delta\|v_y\|^2 + \delta\|\partial_y^{-1}p_x\|^2) \\
&+ \frac{1}{3\mathrm{R}}\left(\frac{1}{\tan^2\alpha}\delta|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta|\eta_{xxx}|_0^2\right) \\
&\le -\frac{2}{\mathrm{R}}(h_2, v)_\Gamma + \frac{1}{\mathrm{R}}\delta(u_{xy}, v)_\Omega + (f_1, v)_\Omega + \frac{2}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, h_3)_\Gamma \\
&+ \delta^{-1}(I_1 + I_2 + I_3).
\end{aligned}$$

The first three terms in the right-hand side are estimated as

$$-\frac{2}{\mathrm{R}}(h_2, v)_\Gamma + \frac{1}{\mathrm{R}}\delta(u_{xy}, v)_\Omega + (f_1, v)_\Omega \le \frac{1}{\mathrm{R}}\delta\|v_y\|^2 + \frac{1}{\mathrm{R}}(2\delta^{-1}|h_2|_0^2 + \delta\|u_{xy}\|^2) + \mathrm{R}\delta^{-1}\|f_1\|^2$$

and the first term in the right-hand side can be absorbed in the left-hand side of (3.1.20). We proceed to estimate I_1, I_2, and I_3. By (3.1.19) and integration by parts in x, I_1 is rewritten as

$$\begin{aligned}
(3.1.21)\qquad I_1 &= -\frac{2}{\mathrm{R}}(\delta\partial_y^{-1}\left(-\delta u_x(\cdot,1) + \frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} + h_2 + \partial_y^{-1}p_y\right)_x, (2+b_3)\eta)_\Omega \\
&= I_4 + I_5,
\end{aligned}$$

where

$$\begin{aligned}
I_4 &= \frac{2}{\mathrm{R}}((y-1)(-\delta u_x(\cdot,1) + h_2) + \partial_y^{-2}p_y, \delta\big((2+b_3)\eta\big)_x)_\Omega, \\
(3.1.22)\qquad I_5 &= -\frac{1}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta(b_3\eta)_x)_\Gamma.
\end{aligned}$$

Here we used identities $(\eta, \eta_x)_\Gamma = (\eta_{xx}, \eta_x)_\Gamma = 0$. We estimate I_2, I_3, and I_4 as follows.

Lemma 3.1.3. *There exists a positive constant C independent of δ, R, W, and α such that the following estimates hold.*

$$
\begin{aligned}
|I_2| &\le \frac{1}{2\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 + C\Big\{\frac{1}{\mathrm{R}}(\delta^4\|v_{xy}\|^2 + |h_1|_0^2 + \delta^4\|u_{xx}\|^2) \\
&\qquad + \mathrm{R}(\delta^2\|u_{ty}\|^2 + \delta^2\|u_x\|^2 + \delta^2\|v_y\|^2 + \|f_2\|^2)\Big\}, \\
|I_3| &\le C\Big\{\frac{1}{\mathrm{R}}(\delta^4\|u_{xxy}\|^2 + \delta^2|h_{2x}|_0^2 + \delta^8\|v_{xxx}\|^2 + \delta^4\|v_{xyy}\|^2) \\
&\qquad + \mathrm{R}(\delta^6\|v_{tx}\|^2 + \delta^6\|v_{xx}\|^2 + \delta^2\|f_{1x}\|^2)\Big\}, \\
|I_4| &\le \frac{1}{6\mathrm{R}\tan^2\alpha}(\delta^2|\eta_x|_0^2 + \delta^2|(b_3\eta)_x|_0^2) \\
&\quad + C\Big\{\frac{\tan^2\alpha}{\mathrm{R}}(\delta^2\|u_{xy}\|^2 + \delta^6\|v_{xx}\|^2 + \delta^2\|v_{yy}\|^2 + |h_2|_0^2) \\
&\qquad + R\tan^2\alpha(\delta^4\|v_{ty}\|^2 + \delta^4\|v_x\|^2 + \|f_1\|^2)\Big\},
\end{aligned}
$$

where

$$
f_2 = -\frac{b_2}{1+b_2}\left(\delta u_t + \bar{u}\delta u_x + \bar{u}_y\delta v - \frac{1}{\mathrm{R}}\delta^2 u_{xx}\right) - \frac{2b_2}{\mathrm{R}(1+b_2)}\delta p_x - \frac{1}{1+b_2}f_3, \tag{3.1.23}
$$

$f_3 = (\boldsymbol{f} - \frac{2}{\mathrm{R}}A_4\nabla_\delta p)\cdot\boldsymbol{e}_1$, *and* $\boldsymbol{e}_1 = (1,0)^{\mathrm{T}}$.

Proof. We can easily estimate I_3 and I_4 by using the second component of the first equation in (2.1.32) so as to eliminate p_y. As for I_2, by the first component of the first equation in (2.1.32), we have

$$
\frac{1}{\mathrm{R}}\left(u_{yy} - \frac{2}{1+b_2}\delta p_x\right) = \frac{1}{1+b_2}\left(\delta u_t + \bar{u}\delta u_x + \bar{u}_y\delta v - \frac{1}{\mathrm{R}}\delta^2 u_{xx}\right) - \frac{1}{1+b_2}f_3.
$$

Substituting the above equation into I_2, we easily obtain the desired estimate. □

Combining (3.1.20), (3.1.21), and Lemma 3.1.3, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\delta^2\|v\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\} + \frac{1}{\mathrm{R}}\left(\delta\|\boldsymbol{u}_x^\delta\|^2 + \frac{1}{2}\delta\|\partial_y^{-1}p_x\|^2\right) \\
&\quad + \frac{1}{3\mathrm{R}}\left(\frac{1}{2\tan^2\alpha}\delta|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta|\eta_{xxx}|_0^2\right) \\
&\le C_1\Big\{\frac{1}{\mathrm{R}}\big((1+\tan^2\alpha)\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + \delta^3\|\nabla_\delta\boldsymbol{u}_{xx}^\delta\|^2 \\
&\qquad + \delta^{-1}|h_1|_0^2 + (1+\tan^2\alpha)\delta^{-1}|h_2|_0^2 + \delta|h_{2x}|_0^2\big) \\
&\quad + \mathrm{R}\big(\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + (1+\tan^2\alpha)\delta\|\nabla_\delta\boldsymbol{u}_t^\delta\|^2 \\
&\qquad + (1+\tan^2\alpha)\delta^{-1}\|f_1\|^2 + \delta^{-1}\|f_2\|^2 + \delta\|f_{1x}\|^2\big)\Big\} \\
&\quad + \frac{2\delta^2\mathrm{W}}{\mathrm{R}\sin\alpha}\delta^{-1}|(\eta_{xx},\delta h_3)_\Gamma| + \frac{1}{6\mathrm{R}\tan^2\alpha}\delta|(b_3\eta)_x|_0^2 + \delta^{-1}I_5,
\end{aligned} \tag{3.1.24}
$$

where we used the second equation in (2.1.32) and $(\eta, h_3)_\Gamma = (\eta, \varepsilon^2\eta^2\eta_x)_\Gamma = 0$. Here the constant C_1 does not depend on δ, R, W, nor α. This is the modified energy estimate. In the left-hand side, we have a new term $\delta\|\partial_y^{-1}p_x\|^2$, which plays an important role in this chapter.

3.1.3 Energy estimate

In view of the energy estimates obtained in this section, we define an energy function E_0, a dissipation function F_0, and a collection of the nonlinear terms N_0 by

$$
\begin{aligned}
E_0(\eta, \boldsymbol{u}^\delta) &= \delta^2\|v\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right) \\
&\quad + \beta_1\left\{\delta^2\|\boldsymbol{u}_x^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_x|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{xx}|_0^2\right)\right\} \\
&\quad + \beta_2\left\{\delta^4\|\boldsymbol{u}_{xx}^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^4|\eta_{xx}|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^4|\eta_{xxx}|_0^2\right)\right\} \\
&\quad + \beta_3\left\{\delta^2((I - A_5)\boldsymbol{u}_t^\delta, \boldsymbol{u}_t^\delta)_\Omega + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{tx}|_0^2\right)\right\},
\end{aligned} \tag{3.1.25}
$$

$$
\begin{aligned}
F_0(\eta, \boldsymbol{u}^\delta, p) &= \frac{1}{2\mathrm{R}}\left(\delta\|\boldsymbol{u}_x^\delta\|^2 + \frac{1}{2}\delta\|\partial_y^{-1}p_x\|^2\right) \\
&\quad + \frac{1}{6\mathrm{R}}\left(\frac{1}{2\tan^2\alpha}\delta|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta|\eta_{xxx}|_0^2\right) \\
&\quad + \frac{1}{8K\mathrm{R}}(\beta_1\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + \beta_2\delta^3\|\nabla_\delta\boldsymbol{u}_{xx}^\delta\|^2 + \beta_3\delta\|\nabla_\delta\boldsymbol{u}_t^\delta\|^2),
\end{aligned} \tag{3.1.26}
$$

$$
\begin{aligned}
N_0(Z) &= \delta^{-1}|h_1|_0^2 + \delta^{-1}|h_2|_0^2 + \delta|h_{1x}|_0^2 + \delta|h_{2x}|_0^2 \\
&\quad + \delta|h_3|_0^2 + \delta^3|h_{3t}|_0^2 + \delta^3|h_{3x}|_0^2 + \delta^5|h_{3xx}|_0^2 \\
&\quad + \delta^2||D_x|^{\frac{1}{2}}h_{1x}|_0^2 + \delta^2||D_x|^{\frac{1}{2}}h_{2x}|_0^2 + \delta|(h_{1t}, u_t)_\Gamma| + \delta|(h_{2t}, \delta v_t)_\Gamma| \\
&\quad + \delta|(b_3\eta)_x|_0^2 + \delta^3|(b_3\eta)_{xx}|_0^2 + \delta|(b_3\eta)_t|_0^2 + |(\eta, (b_3\eta)_x)_\Gamma| \\
&\quad + \delta^2\mathrm{W}\{\delta^{-1}|(\eta_{xx}, \delta h_3 + \delta(b_3\eta)_x)_\Gamma| + \delta^3|(\eta_{xxxx}, \delta h_{3xx})_\Gamma| + \delta|(\eta_{xxt}, \delta h_{3t})_\Gamma|\} \\
&\quad + \delta^{-1}\|f_1\|^2 + \delta^{-1}\|f_2\|^2 + \delta\|f_{1x}\|^2 \\
&\quad + \delta|(\boldsymbol{F}_{1x}, \boldsymbol{u}_x^\delta)_\Omega| + \delta^3|(\boldsymbol{F}_{1xx}, \boldsymbol{u}_{xx}^\delta)_\Omega| + \delta|(\boldsymbol{F}_2, \boldsymbol{u}_t^\delta)_\Omega|,
\end{aligned} \tag{3.1.27}
$$

where $Z = (\eta, \boldsymbol{u}^\delta, h_1, h_2, h_3, b_3\eta, f_1, f_2, \boldsymbol{F}_1, \boldsymbol{F}_2)$ and we will determine the constants β_1, β_2, and β_3 later. Note that the terms $|(\eta, (b_3\eta)_x)_\Gamma|$ and $(\delta^2\mathrm{W})\delta^{-1}|(\eta_{xx}, \delta(b_3\eta)_x)_\Gamma|$ come from I_5. Summarizing our energy estimates, we obtain the following proposition.

Proposition 3.1.4. *Let* W_1 *be a positive constant. There exists a positive constant* α_0 *such that if* $0 < \mathrm{R}_1 \le \mathrm{R} \le \mathrm{R}_0$, $\mathrm{W}_1 \le \mathrm{W}$, *and* $0 < \alpha \le \alpha_0$, *then the solution* (η, u, v, p) *of* (2.1.32)–(2.1.34) *satisfies*

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_0 + F_0 \le C_2 N_0,
$$

where R_0 *is the constant in Proposition* 3.1.1 *and the constant* $C_2(\mathrm{R}_1, \mathrm{W}_1, \alpha)$ *is independent of* δ, R, *and* W.

Proof. Multiplying (3.1.6), (3.1.7), and (3.1.14) by β_1, β_2, and β_3, respectively, and adding these and (3.1.24), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0 + 2F_0 \le L + C(N + N_0),$$

where

$$\begin{aligned}
L &= \frac{4K}{\mathrm{R}}\big((\beta_1 + 3\beta_3)\delta|\eta_x|_0^2 + \beta_2\delta^3|\eta_{xx}|_0^2\big) + \left\{C_1\left(\frac{1+\tan^2\alpha}{\mathrm{R}} + \mathrm{R}\right) + \frac{12K}{\mathrm{R}}\beta_3\right\}\delta\|\nabla_\delta \boldsymbol{u}_x^\delta\|^2 \\
&\quad + \frac{C_1}{\mathrm{R}}\delta^3\|\nabla_\delta \boldsymbol{u}_{xx}^\delta\|^2 + C_1\mathrm{R}(1+\tan^2\alpha)\delta\|\nabla_\delta \boldsymbol{u}_t^\delta\|^2, \\
N &= \delta|(h_{1x}, u_x)_\Gamma| + \delta|(h_{2x}, \delta v_x)_\Gamma| + \delta|(\eta_x, \delta h_{3x})_\Gamma| + |((\delta^2\mathrm{W})\delta^{1/2}\eta_{xxx}, \delta^{3/2}h_{3x})_\Gamma| \\
&\quad + \delta^3|(h_{1xx}, u_{xx})_\Gamma| + \delta^3|(h_{2xx}, \delta v_{xx})_\Gamma| + \delta^3|(\eta_{xx}, \delta h_{3xx})_\Gamma| + \delta|(\eta_t, \delta h_{3t})_\Gamma| + \delta^{-1}|I_5|,
\end{aligned}$$

and C is a positive constant which is depend on $\mathrm{R}_1, \mathrm{W}_1, \alpha, \beta_1, \beta_2$, and β_3. Here we used

$$|\eta_t|_0 \le |\eta_x|_0 + \|u_{xy}\| + |h_3|_0,$$

which comes from the second equation in (2.1.32), the third equation in (2.1.33), and Poincaré's inequality. Moreover, it is easy to see that for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that $N \le \epsilon F_0 + C_\epsilon N_0$. Therefore, if we take $(\beta_1, \beta_2, \beta_3)$ so that

$$\begin{cases}
\dfrac{4K}{\mathrm{R}}(\beta_1 + 3\beta_3) < \dfrac{1}{12\mathrm{R}\tan^2\alpha}, \quad \dfrac{4K}{\mathrm{R}}\beta_2 < \dfrac{\mathrm{W}}{3\mathrm{R}\tan\alpha\sin\alpha}, \\
C_1\left(\dfrac{1+\tan^2\alpha}{\mathrm{R}} + \mathrm{R}\right) + \dfrac{12K}{\mathrm{R}}\beta_3 < \dfrac{\beta_1}{8K\mathrm{R}}, \quad \dfrac{C_1}{\mathrm{R}} < \dfrac{\beta_2}{8K\mathrm{R}}, \\
C_1\mathrm{R}(1+\tan^2\alpha) < \dfrac{\beta_3}{8K\mathrm{R}},
\end{cases} \tag{3.1.28}$$

and if we choose $\epsilon > 0$ sufficiently small, then we obtain $L + CN \le F_0 + C_\epsilon N_0$. Here taking $(\beta_1, \beta_2, \beta_3)$ as

$$\beta_2 := 16KC_1, \quad \beta_3 := 16KC_1\mathrm{R}_0^2(1+\tan^2\alpha), \quad \beta_1 := 16K\big\{C_1(1+\tan^2\alpha + \mathrm{R}_0^2) + 12K\beta_3\big\},$$

we see that (3.1.28) is equivalent to

$$48K(\beta_1 + 3\beta_3)\tan^2\alpha < 1, \quad 12K\beta_2\tan\alpha\sin\alpha < \mathrm{W}_1.$$

Thus there exists a small constant α_0 which depends on W_1 such that (3.1.28) is fulfilled and we obtain the desired energy inequality. □

Hereafter, m is an integer satisfying $m \ge 2$. We define a higher order energy and a dissipation functions E_m and F_m and a collection of the nonlinear terms N_m by

$$E_m = \sum_{k=0}^{m} E_0(\partial_x^k\eta, \partial_x^k\boldsymbol{u}^\delta), \quad F_m = \sum_{k=0}^{m} F_0(\partial_x^k\eta, \partial_x^k\boldsymbol{u}^\delta, \partial_x^k p), \tag{3.1.29}$$

$$N_m = \sum_{k=0}^{m} N_0(\partial_x^k Z) + \sum_{k=1}^{m} \big(\delta|(\boldsymbol{G}_k, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega| + |(\partial_x^k \eta, \partial_x^k h_3)_\Gamma|\big). \tag{3.1.30}$$

Here, we note that $\delta|(\boldsymbol{G}_k, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega|$ is the term appearing in (3.1.12) and that $(\eta, h_3)_\Gamma = 0$. Under an appropriate assumption of the solution, we have the following equivalence uniformly in δ.

$$\begin{aligned}
E_m &\simeq |(1+\delta|D_x|)^2\eta|_m^2 + \delta^2|\eta_t|_m^2 + \delta^2\mathrm{W}\big\{|(1+\delta|D_x|)^2\eta_x|_m^2 + \delta^2|\eta_{tx}|_m^2\big\} \\
&\quad + \delta^2\|(1+|D_x|)^m v\|^2 + \delta^2\|(1+|D_x|)^m(1+\delta|D_x|)\boldsymbol{u}_x^\delta\|^2 + \delta^2\|(1+|D_x|)^m \boldsymbol{u}_t^\delta\|^2 \\
&\simeq |\eta|_m^2 + \delta^2\big\{|(\eta_x,\eta_t)|_m^2 + \|(1+|D_x|)^m(v,u_x,u_t)\|^2\big\} \\
&\quad + \delta^4\big\{|(\eta_{xx},\eta_{tx})|_m^2 + \|(1+|D_x|)^m(v_x,u_{xx},v_t)\|^2\big\} + \delta^6\|(1+|D_x|)^m v_{xx}\|^2 \\
&\quad + \delta^2\mathrm{W}\big\{|\eta_x|_m^2 + \delta^2|(\eta_{xx},\eta_{tx})|_m^2 + \delta^4|\eta_{xxx}|_m^2\big\},
\end{aligned}$$

$$\begin{aligned}
F_m &\simeq \delta|\eta_x|_m^2 + (\delta^2\mathrm{W})\delta|\eta_{xx}|_m^2 + (\delta^2\mathrm{W})^2\delta|\eta_{xxx}|_m^3 + \delta\|(1+|D_x|)^m\partial_y^{-1}p_x\|^2 \\
&\quad + \delta\|(1+|D_x|)^m\boldsymbol{u}_x^\delta\|^2 + \delta\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + \delta\|(1+|D_x|)^m\nabla_\delta\boldsymbol{u}_t^\delta\|^2 \\
&\simeq \delta\big\{|\eta_x|_m^2 + \|(1+|D_x|)^m(v_y,u_x,u_{xy},u_{ty},\partial_y^{-1}p_x)\|^2\big\} \\
&\quad + \delta^3\|(1+|D_x|)^m(v_x,v_{xy},v_{ty},u_{xx},u_{xxy},u_{tx})\|^2 \\
&\quad + \delta^5\|(1+|D_x|)^m(v_{xx},v_{xxy},v_{tx},u_{xxx})\|^2 + \delta^7\|(1+|D_x|)^m v_{xxx}\|^2 \\
&\quad + (\delta^2\mathrm{W})\delta|\eta_{xx}|_m^2 + (\delta^2\mathrm{W})^2\delta|\eta_{xxx}|_m^2.
\end{aligned}$$

Applying ∂_x^k to (2.1.32)–(2.1.34), using Proposition 3.1.4, and adding the resulting inequalities for $0 \le k \le m$, we obtain a higher order energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m + F_m \le C_2 N_m. \tag{3.1.31}$$

3.2 Estimate for the pressure

We will use an elliptic estimate for the pressure p. First, we derive an equation for p. Applying $\nabla_\delta\cdot$ to the first equation in (2.1.4) and using the second equation in (2.1.4), we have

$$\begin{aligned}
\frac{2}{\mathrm{R}}\Delta_\delta p &= -\{\varepsilon(\delta u_x)^2 + 2\delta^2 v_x(\varepsilon u_y + \bar{u}_y) + \varepsilon(\delta v_y)^2\} \\
&= -\varepsilon^{-1}\mathrm{tr}\big(\nabla_\delta(\varepsilon\boldsymbol{u}^\delta + \bar{\boldsymbol{u}})^{\mathrm{T}}\big)^2 =: f.
\end{aligned}$$

We transform this by the diffeomorphism Φ introduced by (2.1.7) and obtain

$$\nabla_\delta \cdot A_6 \nabla_\delta p' = \frac{1}{2}\mathrm{R}J(f\circ\Phi) =: g, \tag{3.2.1}$$

where $p' = p\circ\Phi$ and $A_6 = JA_2^{\mathrm{T}}A_2$. On the other hand, by the definition of f and (2.1.13), we have

$$f\circ\Phi = -\varepsilon^{-1}\mathrm{tr}\big((A_2\nabla_\delta)(\varepsilon A_1\boldsymbol{u}'^\delta + \bar{\boldsymbol{u}}')^{\mathrm{T}}\big)^2,$$

where $\boldsymbol{u}'^{\delta}$ is defined by (2.1.13) and $\bar{\boldsymbol{u}}' = (\bar{u}', 0)^{\mathrm{T}} := \bar{\boldsymbol{u}} \circ \Phi$. Here we see that

$$(A_2\nabla_\delta)(\varepsilon A_1\boldsymbol{u}'^{\delta} + \bar{\boldsymbol{u}}')^{\mathrm{T}} = \begin{pmatrix} \delta\partial_x + a_1\partial_y \\ J^{-1}\partial_y \end{pmatrix}(\varepsilon J^{-1}u' + \bar{u}', -\varepsilon a_1 u' + \varepsilon\delta v') = \varepsilon F_1 u'_y + F_2,$$

where

$$F_1 := \begin{pmatrix} a_1 J^{-1} & -a_1^2 \\ J^{-2} & -a_1 J^{-1} \end{pmatrix},$$

$$F_2 := \begin{pmatrix} \delta(\varepsilon J^{-1}u')_x + \varepsilon a_1 (J^{-1})_y u' & \varepsilon\delta(-a_1 u' + \delta v')_x - \varepsilon a_1 a_{1y} u' + \varepsilon\delta a_1 v'_y \\ \varepsilon J^{-1}(J^{-1})_y u' + J^{-1}\bar{u}'_y & -\varepsilon J^{-1} a_{1y} u' + \varepsilon\delta J^{-1} v'_y \end{pmatrix}.$$

Here, in the above calculation, we used the identity $\delta\bar{u}'_x + a_1\bar{u}'_y = 0$. It follows from $F_1^2 = O$ that

$$g = -\frac{1}{2}\mathrm{R}J\{\mathrm{tr}(F_1F_2 + F_2F_1)u'_y + \varepsilon^{-1}\mathrm{tr}(F_2^2)\}, \tag{3.2.2}$$

where F_1 and F_2 do not contain u'_y.

Next, as for the boundary condition on Γ, by the second equation in (2.1.33), we obtain

$$p' = -\delta u'_x + \frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} + h_2 =: \phi' \qquad \text{on} \quad \Gamma. \tag{3.2.3}$$

Moreover, as for the boundary condition on Σ, taking the trace of the second component of the first equation in (2.1.4) on Σ, we obtain $(p + \frac{1}{2}u_x)_y = 0$ on Σ. In view of (2.1.13) and (2.1.15), this is transformed into

$$J^{-1}\left\{p' + \frac{\delta}{2}(J^{-1}u')_x + \frac{1}{2}a_1(J^{-1}u')_y\right\}_y = 0 \qquad \text{on} \quad \Sigma.$$

Recalling $a_1 = -yJ^{-1}\varepsilon\delta\tilde{\eta}_x$, $J = 1 + \varepsilon(y\tilde{\eta})_y$, and (2.1.9), we have $a_1|_{y=0} = 0$, $a_{1yy}|_{y=0} = 0$, and $J_y|_{y=0} = 0$, so that we obtain $(a_1(J^{-1}u')_y)_y|_{y=0} = (J^{-1}a_{1y}u')_y|_{y=0}$. Therefore we have

$$(p' + g_0)_y = 0 \qquad \text{on} \quad \Sigma, \tag{3.2.4}$$

where

$$g_0 = \frac{1}{2}\{\delta(J^{-1}u')_x + J^{-1}a_{1y}u'\}. \tag{3.2.5}$$

Summarizing (3.2.1), (3.2.3), and (3.2.4), we have

$$\begin{cases} \nabla_\delta \cdot A_6\nabla_\delta p = g & \text{in} \quad \Omega, \\ p = \phi & \text{on} \quad \Gamma, \\ (p + g_0)_y = 0 & \text{on} \quad \Sigma. \end{cases} \tag{3.2.6}$$

Here we dropped the prime sign in the notation.

We proceed to derive an elliptic estimate for p. To this end, we will consider the following boundary value problem

$$\begin{cases} \Delta_\delta q = g + \nabla_\delta \cdot \boldsymbol{g} & \text{in} \quad \Omega, \\ q = \psi_1 & \text{on} \quad \Gamma, \\ q_y = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.2.7}$$

and show the following lemma.

Lemma 3.2.1. *For any $g, \boldsymbol{g} \in L^2(\Omega)$ and $\psi_1 \in H^{\frac{1}{2}}(\Gamma)$, there exists a unique solution $q \in H^1(\Omega)$ of (3.2.7) satisfying*

$$\|\nabla_\delta q\|^2 \lesssim \|g\|^2 + \|\boldsymbol{g}\|^2 + \delta ||D_x|^{\frac{1}{2}}\psi_1|_0^2.$$

Proof. First, we will construct a solution of the following equation

$$\Delta_\delta q_1 = g + \nabla_\delta \cdot \boldsymbol{g} \qquad \text{in} \quad \Omega. \tag{3.2.8}$$

We extend g and $g_1 := \boldsymbol{g} \cdot \boldsymbol{e}_1$ as even and 4-periodic functions in y satisfying $\int_0^4 g(x,y)\mathrm{d}y = \int_0^4 g_1(x,y)\mathrm{d}y = 0$ and $g_2 := \boldsymbol{g} \cdot \boldsymbol{e}_2$ as an odd and 4-periodic function. By these extension and Fourier series expansion in x and y, we can construct a solution of (3.2.8) satisfying

$$q_{1y}(x,0) = 0, \tag{3.2.9}$$

$$\|q_1\|^2 + \|\nabla_\delta q_1\|^2 \lesssim \|g\|^2 + \|\boldsymbol{g}\|^2. \tag{3.2.10}$$

Next, let us seek the solution of (3.2.7) in the form $q = q_1 + q_2$, where q_2 should be the solution of the following boundary value problem

$$\begin{cases} \Delta_\delta q_2 = 0 & \text{in} \quad \Omega, \\ q_2 = \psi_2 & \text{on} \quad \Gamma, \\ q_{2y} = 0 & \text{on} \quad \Sigma, \end{cases}$$

where $\psi_2 = \psi_1 - q_1|_{y=1}$ and we used (3.2.9). By Fourier series expansion in x, we easily construct a solution of the above problem satisfying

$$\|\nabla_\delta q_2\|^2 \lesssim \delta ||D_x|^{\frac{1}{2}}\psi_2|_0^2. \tag{3.2.11}$$

Here, Lemma 1.3.3 yields $\delta ||D_x|^{\frac{1}{2}} q_1|_0^2 \lesssim \|q_1\|^2 + \|\nabla_\delta q_1\|^2$, which together with (3.2.10) and (3.2.11) implies the desired estimate. The uniqueness of the solution is well-known, so that the proof is complete. □

Now, we rewrite (3.2.6) as

$$\begin{cases} \Delta_\delta q = g + \nabla_\delta \cdot (\nabla_\delta g_0 - N_6 \nabla_\delta p) & \text{in} \quad \Omega, \\ q = \phi + g_0 & \text{on} \quad \Gamma, \\ q_y = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.2.12}$$

where $q = p + g_0$ and N_6 is a nonlinear part of A_6, that is, $A_6 = I + N_6$. Applying Lemma 3.2.1 to the above boundary value problem, we have

$$\|\nabla_\delta p\|^2 \lesssim \|g\|^2 + \|g_0\|^2 + \|\nabla_\delta g_0\|^2 + \|N_6 \nabla_\delta p\|^2 + \delta ||D_x|^{\frac{1}{2}}\phi|_0^2, \tag{3.2.13}$$

where we used $\delta ||D_x|^{\frac{1}{2}} g_0|_0^2 \lesssim \|g_0\|^2 + \|\nabla_\delta g_0\|^2$ which comes from Lemma 1.3.3. Differentiating (3.2.12) in x and t, likewise we deduce

$$\begin{cases} \delta\|\nabla_\delta p_x\|^2 \lesssim \delta\|g_x\|^2 + \delta\|g_{0x}\|^2 + \delta\|\nabla_\delta g_{0x}\|^2 + \delta\|(N_6\nabla_\delta p)_x\|^2 + \delta^2 ||D_x|^{\frac{1}{2}}\phi_x|_0^2, \\ \delta\|\nabla_\delta p_t\|^2 \lesssim \delta\|g_t\|^2 + \delta\|g_{0t}\|^2 + \delta\|\nabla_\delta g_{0t}\|^2 + \delta\|(N_6\nabla_\delta p)_t\|^2 + \delta^2 ||D_x|^{\frac{1}{2}}\phi_t|_0^2. \end{cases} \tag{3.2.14}$$

Here, for the same reason as the modification of the lowest order energy, we need to modify (3.2.13), that is, we estimate $\delta^{-1}\|\nabla_\delta p\|^2$ in a different way. As for $\delta^{-1}\|p_y\|^2$, by using the second component of the first equation in (2.1.32), we see that

$$\delta^{-1}\|p_y\|^2 \lesssim F_0 + \delta^{-1}\|f_1\|^2, \tag{3.2.15}$$

where f_1 is defined by (3.1.16). To estimate $\delta\|p_x\|^2$ in terms of the dissipation function F_0, we use the term $\delta\|\partial_y^{-1}p_x\|^2$ in the following way. We compute

$$\begin{aligned}
\delta\|p_x\|^2 &= \delta \iint_\Omega p_x(x,y)\Big(\frac{\partial}{\partial y}\int_0^y p_x(x,z)\mathrm{d}z\Big)\mathrm{d}x\mathrm{d}y \\
&= -\delta \iint_\Omega p_{xy}(x,y)\Big(\int_0^y p_x(x,z)\mathrm{d}z\Big)\mathrm{d}x\mathrm{d}y + \delta\int_0^1 p_x(x,1)\Big(\int_0^1 p_x(x,z)\mathrm{d}z\Big)\mathrm{d}x \\
&\le \delta\|p_{xy}\|(\|p_x\| + \|\partial_y^{-1}p_x\|) + \delta|p_x|_0\|p_x\|,
\end{aligned}$$

so that we have

$$\delta\|p_x\|^2 \lesssim \delta\|p_{xy}\|^2 + \delta\|\partial_y^{-1}p_x\|^2 + \delta|p_x|_0^2. \tag{3.2.16}$$

Here, it follows from the second equation in (2.1.33) that $\delta|p_x|_0^2 \lesssim F_0 + \delta|h_{2x}|_0^2$. This together with (3.2.15) and (3.2.16) yields

$$\delta^{-1}\|\nabla_\delta p\|^2 \lesssim F_0 + \delta\|p_{xy}\|^2 + \delta|h_{2x}|_0^2 + \delta^{-1}\|f_1\|^2.$$

This is the modified estimate of $\delta^{-1}\|\nabla_\delta p\|^2$.

By differentiating (3.2.12) with respect to x and applying the above argument and (3.2.14), we obtain the following lemma.

Lemma 3.2.2. *For $0 \le k \le m$ and $1 \le l \le m$, we have*

$$\delta^{-1}\|\nabla_\delta\partial_x^k p\|^2 \lesssim F_m + \delta\|\partial_x^k p_{xy}\|^2 + \delta|\partial_x^k h_{2x}|_0^2 + \delta^{-1}\|\partial_x^k f_1\|^2, \tag{3.2.17}$$

$$\begin{aligned}
\delta\|\nabla_\delta\partial_x^k p_x\|^2 &\lesssim \delta\|\partial_x^k g_x\|^2 + \delta\|\partial_x^k g_{0x}\|^2 + \delta\|\nabla_\delta\partial_x^k g_{0x}\|^2 \\
&\quad + \delta\|\partial_x^k(N_6\nabla_\delta p)_x\|^2 + \delta^2||D_x|^{k+\frac{1}{2}}\phi_x|_0^2,
\end{aligned} \tag{3.2.18}$$

$$\begin{aligned}
\delta\|\nabla_\delta\partial_x^{l-1}p_t\|^2 &\lesssim \delta\|\partial_x^{l-1}g_t\|^2 + \delta\|\partial_x^{l-1}g_{0t}\|^2 + \delta\|\nabla_\delta\partial_x^{l-1}g_{0t}\|^2 \\
&\quad + \delta\|\partial_x^{l-1}(N_6\nabla_\delta p)_t\|^2 + \delta^2||D_x|^{l-\frac{1}{2}}\phi_t|_0^2.
\end{aligned} \tag{3.2.19}$$

3.3 Estimate for nonlinear terms

We modify the energy and the dissipation functions E_m and F_m defined by (3.1.29) as

$$\tilde{E}_m = E_m + \|(1+|D_x|)^m u\|^2 + \|(1+|D_x|)^m u_y\|^2, \tag{3.3.1}$$

$$\begin{aligned}
\tilde{F}_m &= F_m + \delta|(1+\delta|D_x|)^{\frac{5}{2}}\eta_t|_m^2 + (\delta^2\mathrm{W})^2\delta^2||D_x|^{\frac{7}{2}}\eta|_m^2 \\
&\quad + \delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta p\|^2 + \delta\|(1+|D_x|)^{m-1}\nabla_\delta p_t\|^2.
\end{aligned} \tag{3.3.2}$$

We also introduce another energy function D_m by

(3.3.3)
$$D_m = |(1+\delta|D_x|)^2\eta|_m^2 + \|(1+|D_x|)^m \boldsymbol{u}^\delta\|^2 + \|(1+|D_x|)^m D_\delta \boldsymbol{u}^\delta\|^2 + \|(1+|D_x|)^m D_\delta^2 \boldsymbol{u}^\delta\|^2 + (\delta^2 \mathrm{W})^2|(1+\delta|D_x|)\eta_x|_{m+1}^2 + (\delta^2\mathrm{W})\delta^2\|(1+|D_x|)^m v_{xy}\|^2,$$

which does not include any time derivatives. Moreover, we have the following equivalence uniformly in δ.

$$\begin{aligned} D_m \simeq\ & |\eta|_m^2 + \|(1+|D_x|)^m(u, u_y, u_{yy})\|^2 \\ & + \delta^2\{|\eta_x|_m^2 + \|(1+|D_x|)^m(v, v_y, u_x, u_{xy}, v_{yy})\|^2\} \\ & + \delta^4\{|\eta_{xx}|_m^2 + \|(1+|D_x|)^m(v_x, v_{xy}, u_{xx})\|^2\} \\ & + \delta^6\{|\eta_{xxx}|_m^2 + \|(1+|D_x|)^m v_{xx}\|^2\} \\ & + \delta^2\mathrm{W}\{|\eta_x|_m^2 + \delta^2|\eta_{xx}|_m^2 + \delta^4|\eta_{xxx}|_m^2 + \delta^2\|(1+|D_x|)^m v_{xy}\|^2\} \\ & + (\delta^2\mathrm{W})^2\{|\eta_{xx}|_m^2 + \delta^2|\eta_{xxx}|_m^2\}. \end{aligned}$$

Since the proof of nonlinear estimates derived in this section is particularly long, we give a guiding principle of the proof. A goal of Section 3.3 is to estimate the nonlinear terms in terms of $\tilde{E}_2\tilde{F}_m, \tilde{F}_2\tilde{E}_m$, and D_2D_m. As for $\tilde{E}_2\tilde{F}_m$, by using a smallness of the energy this term can be absorbed in the right-hand side of the energy inequality (3.1.31). As for $\tilde{F}_2\tilde{E}_m$, using a boundedness of $\int_0^t \tilde{F}_2(\tau)\mathrm{d}\tau$ and a standard Gronwall's inequality we can estimate this term. As for D_2D_m, we use this estimate in order to estimate an initial energy $E(0)$. Here, what we should be careful is that if we use the Sobolev embedding theorem in Ω, that is, $\|u\|_{L^\infty} \lesssim \|u\|_{H^2}$ and Poincaré's inequality for η, that is, $|\eta|_{L^\infty} \lesssim |\eta_x|_0$, we cannot obtain uniform estimates in δ. Therefore, we have to estimate nonlinear terms carefully.

Throughout this section, we assume that

$$\tilde{E}_2(t) \le c_1 \quad \text{for} \quad t \in [0, T/\varepsilon], \tag{3.3.4}$$

where T and c_1 will be determine later. We also assume that (η, u, v, p) is a solution of (2.1.32)–(2.1.34), $0 < \delta, \varepsilon \le 1$, $\mathrm{W}_1 \le \mathrm{W} \le \delta^{-2}\mathrm{W}_2$, k and l are integers satisfying $0 \le k \le m$ and $1 \le l \le m$.

3.3.1 Notations

We put
$$D_\delta^i f = \{(\delta\partial_x)^{i_1}\partial_y^{i_2} f \mid i_1 + i_2 = i\}.$$

We denote smooth functions of $\boldsymbol{f}$ by the same symbol $\Phi = \Phi(\boldsymbol{f})$ and Φ_0 is such a function satisfying $\Phi_0(\mathbf{0}) = 0$. We denote a function Φ_0 depending also on $y \in [0,1]$ by $\Phi_0(\boldsymbol{f}; y)$, that is, $\Phi_0(\mathbf{0}; y) \equiv 0$.

3.3.2 Auxiliary lemmas

We prepare several lemmas to proceed nonlinear estimates.

Lemma 3.3.1. *The following estimates hold.*

$$\|\tilde\eta\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2\}, \quad \|D_\delta^i\tilde\eta\|_{L^\infty}^2 \lesssim \min\{\delta^2\tilde E_2, \delta^2 D_2, \delta\tilde F_2\} \quad \textit{for} \quad 1\le i\le 4, \tag{3.3.5}$$

$$\delta^2 \mathrm{W}\|D_\delta^i\tilde\eta_x\|_{L^\infty}^2 + \delta^4\mathrm{W}\|D_\delta^i\tilde\eta_{xx}\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2, \delta\tilde F_2\} \quad \textit{for} \quad i=0,1, \tag{3.3.6}$$

$$\begin{cases} \|\partial_x^i \boldsymbol{u}^\delta\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2\}, \quad \delta\|\partial_x^i\boldsymbol{u}_x^\delta\|_{L^\infty}^2 + \delta\|\partial_x^i\boldsymbol{u}_t^\delta\|_{L^\infty}^2 \lesssim \tilde F_2 \quad \textit{for} \quad i=0,1, \\ \delta^4\|v_{xx}\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2\}, \end{cases} \tag{3.3.7}$$

$$\begin{cases} \|\tilde\eta_t\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2\}, \quad \delta\|\tilde\eta_t\|_{L^\infty}^2 \lesssim \tilde F_2, \\ \|D_\delta^i\tilde\eta_t\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2, \delta\tilde F_2\} \quad \textit{for} \quad i=1,2, \qquad \|D_\delta^3\tilde\eta_t\|_{L^\infty}^2 \lesssim \delta\tilde F_2, \end{cases} \tag{3.3.8}$$

$$\delta\|D_\delta^i\tilde\eta_{tt}\|_{L^\infty}^2 \lesssim \tilde F_2, \quad \textit{for} \quad i=0,1. \tag{3.3.9}$$

In particular, we have

$$\begin{cases} \|(\tilde\eta, \tilde\eta_t, D_\delta\tilde\eta_t, D_\delta^2\tilde\eta_t, \boldsymbol{u}^\delta, \boldsymbol{u}_x^\delta)\|_{L^\infty}^2 \lesssim \min\{\tilde E_2, D_2\}, \\ \|(D_\delta\tilde\eta, D_\delta^2\tilde\eta, D_\delta^3\tilde\eta)\|_{L^\infty}^2 \lesssim \delta\min\{\tilde E_2, D_2\}. \end{cases} \tag{3.3.10}$$

Remark 3.3.2. Using (3.3.5) and taking c_1 sufficiently small, we see that $J = 1+\varepsilon(y\tilde\eta)_y$ and $I - A_5$ are positive definite.

Proof. By (2.1.11) in Lemma 2.1.1, we have

$$\begin{aligned} &\|\tilde\eta\|_{L^\infty}^2 \lesssim |\eta|_1^2 \lesssim \min\{\tilde E_2, D_2\} \\ &\|D_\delta^i\tilde\eta\|_{L^\infty}^2 \lesssim \delta^{2i}|\partial_x^i\eta|_1^2 \lesssim \min\{\delta^2\tilde E_2, \delta^2 D_2, \delta\tilde F_2\} \quad \text{for} \quad 1\le i\le 4. \end{aligned}$$

Thus (3.3.5) holds. Similarly, we obtain (3.3.6). (3.3.7) is obtained from Lemma 1.3.5 and the second equation in (2.1.32). By the second equation in (2.1.32), we have $|v|_1 \lesssim \|(1+|D_x|)v_y\| = \|(1+|D_x|)u_x\|$. In view of the assumption (3.3.4), we have

$$|\partial_x^j h_3|_0 \lesssim \varepsilon^2|\eta|_{L^\infty}^2|\partial_x^{j+1}\eta|_0 \lesssim |\partial_x^{j+1}\eta|_0 \quad \text{for} \quad j\ge 0.$$

Therefore, by (2.1.11) in Lemma 2.1.1 and the third equation in (2.1.33), we see that

$$\begin{aligned} &\|\tilde\eta_t\|_{L^\infty} \lesssim |\eta_t|_1 \lesssim |v|_1 + |\eta_x|_1 + |h_3|_1 \lesssim \|(1+|D_x|)u_x\| + |\eta_x|_1, \\ &\|D_\delta^i\tilde\eta_t\|_{L^\infty} \lesssim \delta^i|\partial_x^i\eta_t|_1 \lesssim \delta^i(|\partial_x^i v|_1 + |\partial_x^{i+1}\eta|_1 + |\partial_x^i h_3|_1) \lesssim \delta^i(\|(1+|D_x|)\partial_x^{i+1}u\| + |\partial_x^{i+1}\eta|_1). \end{aligned}$$

These estimates give (3.3.8). Similarly, we see that

$$\begin{aligned} &\|\tilde\eta_{tt}\|_{L^\infty} \lesssim |\eta_{tt}|_1 \lesssim |v_t|_1 + |\eta_{tx}|_1 + |h_{3t}|_1 \lesssim \|(1+|D_x|)u_{tx}\| + |\eta_t|_2, \\ &\|D_\delta\tilde\eta_{tt}\|_{L^\infty} \lesssim \delta|\partial_x\eta_{tt}|_1 \lesssim \delta(|\partial_x v_t|_1 + |\partial_x^2\eta_t|_1 + |\partial_x h_{3t}|_1) \lesssim \delta(\|(1+|D_x|)\partial_x u_{tx}\| + |\eta_{tx}|_2 + |\eta_x|_2). \end{aligned}$$

Here, we used $|\eta_t|_{L^\infty} \le \|\tilde\eta_t\|_{L^\infty} \lesssim \sqrt{\tilde E_2}$ and the assumption (3.3.4). Thus (3.3.9) holds. The proof is complete. □

Lemma 3.3.3. *The following estimates hold.*

$$\|\partial_x^k\tilde{\eta}\|^2 \lesssim \min\{\tilde{E}_m, D_m\}, \qquad \|\partial_x^k D_\delta^i\tilde{\eta}\|^2 \lesssim \min\{\tilde{E}_m, D_m, \delta\tilde{F}_m\} \quad \textit{for} \quad i=1,2,3, \tag{3.3.11}$$

$$\delta^2 \mathrm{W}\|\partial_x^k D_\delta^i\tilde{\eta}_x\|^2 \lesssim \min\{\tilde{E}_m, D_m\} \quad \textit{for} \quad i=1,2 \tag{3.3.12}$$

$$\|\partial_x^k D_\delta^4\tilde{\eta}\|^2 \lesssim \delta\tilde{F}_m, \tag{3.3.13}$$

$$\delta^2\|\partial_x^k D_\delta^i\tilde{\eta}_t\|^2 \lesssim \min\{\tilde{E}_m, D_m, \delta\tilde{F}_m\} \quad \textit{for} \quad i=0,1,2, \qquad \delta\|\partial_x^k D_\delta^3\tilde{\eta}_t\|^2 \lesssim \tilde{F}_m, \tag{3.3.14}$$

$$\delta^3\|\partial_x^k D_\delta^i\tilde{\eta}_{tt}\|^2 \lesssim \tilde{F}_m \quad \textit{for} \quad i=0,1. \tag{3.3.15}$$

Proof. By (2.1.10) and (2.1.12) in Lemma 2.1.1, we have

$$\begin{aligned}
&\|\partial_x^k D_\delta^i\tilde{\eta}\| \lesssim \delta^i|\partial_x^{k+i}\eta|_0 \quad \text{for} \quad i \geq 0, \\
&\|\partial_x^k D_\delta^4\tilde{\eta}\| \lesssim \delta^{\frac{7}{2}}||D_x|^{k+\frac{7}{2}}\eta|_0,
\end{aligned}$$

which give (3.3.11) and (3.3.13), respectively. Similarly, we obtain (3.3.12). By (2.1.10) in Lemma 2.1.1 and a similar argument in the proof of Lemma 3.3.1, we see that

$$\begin{aligned}
\|\partial_x^k D_\delta^i\tilde{\eta}_t\| &\lesssim \delta^i|\partial_x^{k+i}\eta_t|_0 \lesssim \delta^i(\|\partial_x^{k+i+1}u\| + |\partial_x^{k+i+1}\eta|_0) \\
&\lesssim \delta^i\big(\|(1+|D_x|)^m\partial_x^{i+1}u\| + |\partial_x^{i+1}\eta|_m\big).
\end{aligned}$$

By (2.1.12) in Lemma 2.1.1, Lemma 1.3.3, Poincaré's inequality, and the estimate

$$||D_x|^{k+\frac{5}{2}}h_3|_0 \lesssim |(\eta,\eta_x)|_{L^\infty}^2||D_x|^{k+\frac{7}{2}}\eta|_0 \lesssim ||D_x|^{k+\frac{7}{2}}\eta|_0, \tag{3.3.16}$$

we see that

$$\begin{aligned}
\|\partial_x^k D_\delta^3\tilde{\eta}_t\| &\lesssim \delta^{\frac{5}{2}}||D_x|^{\frac{1}{2}}\partial_x^{k+2}\eta_t|_0 \leq \delta^{\frac{5}{2}}(||D_x|^{\frac{1}{2}}\partial_x^{k+2}v|_0 + ||D_x|^{\frac{1}{2}}\partial_x^{k+2}\eta_x|_0 + ||D_x|^{\frac{1}{2}}\partial_x^{k+2}h_3|_0) \\
&\lesssim \delta^3\|\partial_x^k v_{xxx}\| + \delta^2\|\partial_x^k v_{xxy}\| + \delta^{\frac{5}{2}}||D_x|^{k+\frac{7}{2}}\eta|_0 \\
&\lesssim \delta^3\|(1+|D_x|)^m v_{xxx}\| + \delta^2\|(1+|D_x|)^m v_{xxy}\| + \delta^{\frac{5}{2}}||D_x|^{k+\frac{7}{2}}\eta|_0.
\end{aligned}$$

These estimates give (3.3.14). It is easy to see that

$$|\partial_x^j h_{3t}|_0 \lesssim \varepsilon^2(|\eta|_{L^\infty}^2|\partial_x^{j+1}\eta_t|_0 + |\eta|_{L^\infty}|\eta_t|_{L^\infty}|\partial_x^{j+1}\eta|_0) \lesssim |\partial_x^{j+1}\eta_t|_0 + |\partial_x^{j+1}\eta|_0 \quad \text{for} \quad j \geq 0.$$

Therefore, by (2.1.10) in Lemma 2.1.1 and the third equation in (2.1.33), we see that

$$\begin{aligned}
\|\partial_x^k\tilde{\eta}_{tt}\| &\lesssim |\partial_x^k\eta_{tt}|_0 \lesssim \|(1+|D_x|)^m u_{tx}\|_0 + |\eta_{tx}|_m + |h_{3t}|_m \\
&\lesssim \|(1+|D_x|)^m u_{tx}\|_0 + |\eta_{tx}|_m + |\eta_x|_m.
\end{aligned}$$

Similarly, by (2.1.12) in Lemma 2.1.1 we obtain

$$\begin{aligned}
\|\partial_x^k D_\delta\tilde{\eta}_{tt}\| &\lesssim \delta^{\frac{1}{2}}||D_x|^{\frac{1}{2}}\partial_x^k\eta_{tt}|_0 \lesssim \delta^{\frac{1}{2}}(||D_x|^{\frac{1}{2}}\partial_x^k v_t|_0 + ||D_x|^{\frac{1}{2}}\partial_x^{k+1}\eta_t|_0 + ||D_x|^{\frac{1}{2}}\partial_x^k h_{3t}|_0) \\
&\lesssim \delta\|\partial_x^k v_{tx}\| + \|\partial_x^k v_{ty}\| + \delta|\partial_x^k\eta_{txx}|_0 + |\partial_x^k\eta_{tx}|_0 + \delta|\partial_x^{k+1}h_{3t}|_0 + |\partial_x^k h_{3t}|_0 \\
&\lesssim \delta\|(1+|D_x|)^m v_{tx}\| + \|(1+|D_x|)^m v_{ty}\| + \delta|\eta_{txx}|_m + |(\eta_{xx},\eta_{tx},\eta_x)|_m.
\end{aligned}$$

These estimates give (3.3.15). The proof is complete. □

Lemma 3.3.4. *The following estimates hold.*

$$(3.3.17)\quad \delta^{2i+1}||D_x|^{i+\frac{1}{2}}\eta|_m^2 \lesssim \min\{\tilde{E}_m, D_m\}, \quad \delta^{2i+2}||D_x|^{i+\frac{1}{2}}\eta_x|_m^2 \lesssim \tilde{F}_m \quad \textit{for} \quad i=0,1,2,$$

$$(3.3.18)\quad \delta^3\mathrm{W}||D_x|^{\frac{1}{2}}\eta_x|_m^2 \lesssim \min\{\tilde{E}_m, D_m\}, \quad \delta^{2i+4}\mathrm{W}||D_x|^{i+\frac{1}{2}}\eta_{xx}|_m^2 \lesssim \tilde{F}_m \quad \textit{for} \quad i=0,1,$$

$$(3.3.19)\quad \delta|\boldsymbol{u}^\delta|_{m+\frac{1}{2}}^2 \lesssim \min\{\tilde{E}_m, D_m\}, \quad \delta^3|\boldsymbol{u}_x^\delta|_{m+\frac{1}{2}}^2 \lesssim \min\{D_m, \delta\tilde{F}_m\}, \quad \delta^5|\boldsymbol{u}_{xx}^\delta|_{m+\frac{1}{2}}^2 \lesssim \delta\tilde{F}_m,$$

$$(3.3.20)\quad |\boldsymbol{u}^\delta|_m^2 \lesssim \min\{\tilde{E}_m, D_m\}, \quad \delta^{2i}|\partial_x^i\boldsymbol{u}^\delta|_m^2 \lesssim \delta\tilde{F}_m \quad \textit{for} \quad i=1,2,$$

$$(3.3.21)\quad \delta^2|\boldsymbol{u}_t^\delta|_{m+\frac{1}{2}}^2 \lesssim \tilde{F}_m.$$

Proof. By an interpolation inequality, we have $\delta^{2i+1}||D_x|^{i+\frac{1}{2}}\eta|_m^2 \lesssim \delta^{2i}|\partial_x^i\eta|_m^2 + \delta^{2i+2}|\partial_x^i\eta_x|_m^2$, which gives the first estimate in (3.3.17). Similarly, we can show the second estimate in (3.3.17) for $i=0,1$, and the case $i=2$ follows directly from the definition of $\tilde{F}_m$. Likewise, we obtain (3.3.18). By Lemma 1.3.3 and Poincaré's inequality, we see that

$$\begin{aligned}\delta^{2i+1}|\partial_x^i\boldsymbol{u}^\delta|_{m+\frac{1}{2}}^2 &\lesssim \delta^{2i+1}|\partial_x^i\boldsymbol{u}^\delta|_0^2 + \delta^{2i+1}||D_x|^{\frac{1}{2}}\partial_x^{m+i}\boldsymbol{u}^\delta|_0^2 \\ &\lesssim \delta^{2i+1}\|\partial_x^i\boldsymbol{u}_y^\delta\|^2 + \delta^{2(i+1)}\|\partial_x^{m+i}\boldsymbol{u}_x^\delta\|^2 + \delta^{2i}\|\partial_x^{m+i}\boldsymbol{u}_y^\delta\|^2\end{aligned}$$

for $i \geq 0$, which leads to (3.3.19). Similarly, we can show (3.3.21). Poincaré's inequality and the second equation in (2.1.32) yield (3.3.20). The proof is complete. □

In view of Lemmas 3.3.1 and 3.3.3 and the inequality $\|\partial_x^k\Phi_0(\boldsymbol{f};y)\| \leq C(\|\boldsymbol{f}\|_{L^\infty})\|\partial_x^k\boldsymbol{f}\|$, we obtain the following lemma.

Lemma 3.3.5. *For $j=0,1$, the following estimates hold.*

$$(3.3.22)\quad \|\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, D_\delta^3\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, \delta D_\delta^2\tilde{\eta}_t, \boldsymbol{u}^\delta, \delta\boldsymbol{u}_x^\delta, \delta^3 v_{xx}; y)\|_{L^\infty}^2 \lesssim \min\{\tilde{E}_2, D_2\},$$

$$(3.3.23)\quad \|\partial_x^k\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, D_\delta^3\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, \delta D_\delta^2\tilde{\eta}_t, \boldsymbol{u}^\delta, \delta\boldsymbol{u}_x^\delta, \delta^3 v_{xx}; y)\|^2 \lesssim \min\{\tilde{E}_m, D_m\},$$

$$(3.3.24)\quad \|\partial_x^k\partial_y^j\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, \boldsymbol{u}^\delta, \delta^2 v_x; y)\|^2 \lesssim \min\{\tilde{E}_m, D_m\}$$

$$(3.3.25)\quad \delta\|\partial_x^l\partial_y^j\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, D_\delta^3\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, \delta D_\delta^2\tilde{\eta}_t, \boldsymbol{u}^\delta, \delta\boldsymbol{u}_x^\delta; y)\|^2 \lesssim \tilde{F}_m.$$

Remark 3.3.6. As for (3.3.25), if Φ_0 does not contain $\tilde{\eta}$ and u, then δ appearing in the coefficient of the term $\|\partial_x^l\partial_y^j\Phi_0\|^2$ is unnecessary and we can replace l with k.

This lemma together with Lemma 1.3.3 gives the following lemma.

Lemma 3.3.7. *The following estimates hold.*

$$(3.3.26)\quad |\Phi_0(\eta, \delta\eta_x, \delta^2\eta_{xx}, \boldsymbol{u}^\delta|_\Gamma, \delta^2 v_x|_\Gamma)|_{L^\infty}^2 \lesssim \min\{\tilde{E}_2, D_2\},$$

$$(3.3.27)\quad \delta|\Phi_0(\eta, \delta\eta_x, \delta^2\eta_{xx}, \boldsymbol{u}^\delta|_\Gamma, \delta^2 v_x|_\Gamma)|_{m+\frac{1}{2}}^2 \lesssim \min\{\tilde{E}_m, D_m\},$$

$$(3.3.28)\quad |\Phi_0(\eta, \delta\eta_x, \delta^2\eta_{xx}, \boldsymbol{u}^\delta|_\Gamma, \delta^2 v_x|_\Gamma)|_m^2 \lesssim \min\{\tilde{E}_m, D_m\}.$$

By (3.3.6) in Lemma 3.3.1, (3.3.12) in Lemma 3.3.3 and Lemma 1.3.3, we obtain the following lemma.

Lemma 3.3.8. *The following estimates hold.*

$$\mathrm{W}|\Phi_0(\delta\eta_x, \delta^2\eta_{xx})|_{L^\infty}^2 \lesssim \min\{\tilde{E}_2, D_2\}, \tag{3.3.29}$$

$$\delta\mathrm{W}|\Phi_0(\delta\eta_x, \delta^2\eta_{xx})|_{m+\frac{1}{2}}^2 \lesssim \min\{\tilde{E}_m, D_m\}, \tag{3.3.30}$$

$$\mathrm{W}|\Phi_0(\delta\eta_x, \delta^2\eta_{xx})|_m^2 \lesssim \min\{\tilde{E}_m, D_m\}. \tag{3.3.31}$$

We set

$$(w_1, \dots, w_7) := (D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, D_\delta^3\tilde{\eta}, \delta D_\delta^2\tilde{\eta}_t, \delta\boldsymbol{u}_x^\delta). \tag{3.3.32}$$

Lemma 3.3.9. *For $j = 0, 1$, the following estimates hold.*

$$\delta^{-1}\|w_\lambda\|_{L^\infty}^2 \lesssim \min\{\delta\tilde{E}_2, \tilde{F}_2\} \quad \textit{for} \quad 1 \le \lambda \le 7, \tag{3.3.33}$$

$$\delta^{-1}\|\partial_x^k\partial_y^j w_\lambda\|^2 \lesssim \tilde{F}_m \quad \textit{for} \quad 1 \le \lambda \le 7, \tag{3.3.34}$$

$$\delta^{-2}\|\partial_x^{l-1}\partial_y^j w_\lambda\|^2 \lesssim \tilde{E}_m \quad \textit{for} \quad 1 \le \lambda \le 4. \tag{3.3.35}$$

Proof. (3.3.33) and (3.3.34) follow from Lemmas 3.3.1 and 3.3.5, respectively. In the same way as the proof of Lemma 3.3.5, we can show (3.3.35). □

3.3.3 Estimate for nonlinear terms in boundary conditions

We begin to estimate the nonlinear terms. First, we will estimate h_1, h_2, h_3, and $b_3\eta$. By the explicit form of h_1 defined by (2.1.31), h_1 is consist of terms in the form

$$\begin{cases} \Phi_0(\varepsilon\eta, \varepsilon\delta\eta_x, \varepsilon\boldsymbol{u}^\delta|_\Gamma)\delta^i\partial_x^i\eta & \text{for} \quad i = 1, 2, \\ \Phi_0(\varepsilon\eta, \varepsilon\delta\eta_x)\delta\boldsymbol{u}_x^\delta|_\Gamma. \end{cases} \tag{3.3.36}$$

Although $h_{2,1}$ contains $\Phi(\varepsilon\eta, \varepsilon\delta\eta_x)\varepsilon\delta\eta_x u_y$ in addition to the above terms (see (2.1.28)), by using the boundary condition $u_y = -\delta^2 v_x + (2 + b_3)\eta + h_1$ on Γ, we can reduce the estimate of $h_{2,1}$ to that of h_1. Moreover, we note that $\delta^2\mathrm{W}h_{2,2}$ is of the form $\delta^2\mathrm{W}\Phi_0(\varepsilon^2\delta^2\eta_x^2)\eta_{xx}$.

Lemma 3.3.10. *For any $\epsilon > 0$ there exists a positive constant C_ϵ such that we have*

$$\delta^{-1}|(h_1, h_2)|_m^2 + \delta|(h_{1x}, h_{2x})|_m^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m, \tag{3.3.37}$$

$$\delta^2|(h_{1x}, h_{2x})|_{m+\frac{1}{2}}^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m, \tag{3.3.38}$$

$$\delta^{-1}|(h_1, h_2)|_{m-\frac{1}{2}}^2 \lesssim \tilde{E}_2\tilde{E}_m, \tag{3.3.39}$$

$$\delta|h_2|_{m+\frac{1}{2}}^2 \lesssim D_2 D_m, \tag{3.3.40}$$

$$\delta|(\partial_x^k h_{1t}, \partial_x^k u_t)_\Gamma| + \delta|(\partial_x^k h_{2t}, \delta\partial_x^k v_t)_\Gamma| \le \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m), \tag{3.3.41}$$

$$\begin{cases} \delta|h_3|_m^2 + \delta^3|(h_{3x}, h_{3t})|_m^2 + \delta^5|h_{3xx}|_m^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m, \\ \delta^6\mathrm{W}|(\partial_x^k\eta_{xxxx}, \partial_x^k h_{3xx})_\Gamma| \le \epsilon\tilde{F}_m + C_\epsilon\tilde{E}_2\tilde{F}_m, \end{cases} \tag{3.3.42}$$

$$\begin{cases} \delta|(b_3\eta)_x|_m^2 + \delta^3|(b_3\eta)_{xx}|_m^2 + \delta|(b_3\eta)_t|_m^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m, \\ |(\partial_x^k\eta, \partial_x^k(b_3\eta)_x)_\Gamma| + |(\partial_x^k\eta, \partial_x^k h_3)_\Gamma| + \delta^2\mathrm{W}|(\partial_x^k\eta_{xx}, \partial_x^k h_3 + \partial_x^k(b_3\eta)_x)_\Gamma| \\ \qquad +\delta^4\mathrm{W}|(\partial_x^k\eta_{txx}, \partial_x^k h_{3t})_\Gamma| \le \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m + \varepsilon\sqrt{\tilde{E}_2}\tilde{E}_m). \end{cases} \tag{3.3.43}$$

Moreover, if $\varepsilon \lesssim \delta$, then we have

$$
\begin{aligned}
(3.3.44) \qquad & |(\partial_x^k \eta, \partial_x^k (b_3\eta)_x)_\Gamma| + |(\partial_x^k \eta, \partial_x^k h_3)_\Gamma| + \delta^2 \mathrm{W} |(\partial_x^k \eta_{xx}, \partial_x^k h_3 + \partial_x^k (b_3\eta)_x)_\Gamma| \\
& + \delta^4 \mathrm{W} |(\partial_x^k \eta_{txx}, \partial_x^k h_{3t})_\Gamma| \leq \epsilon \tilde{F}_m + C_\epsilon (\tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m).
\end{aligned}
$$

Remark 3.3.11. Concerning the terms in the left-hand side of (3.3.43), in the case where ε is not dominated by δ, we cannot estimate these terms by using $\tilde{F}_m$ because the power of δ of these terms is not enough. These are the only terms which prevent from deriving a uniform estimate for the solution for all time.

Proof. Since ε is the nonlinear parameter, that is, ε measures the nonlinearity, it is sufficient to show the estimates in the case $\varepsilon = 1$ except the last estimate (3.3.44). Therefore, we will assume that $\varepsilon = 1$ in the following.

As for (3.3.37), it suffices to estimate

$$
\begin{cases}
J_1 := \delta^{2i-1} |\Phi_0^1 \partial_x^i \eta|_m^2 & \text{for} \quad i = 1, 2, 3, \\
J_2 := \delta^{2i-1} |\Phi_0^1 \partial_x^i \boldsymbol{u}^\delta|_m^2 & \text{for} \quad i = 1, 2, \\
J_3 := \delta^{2i+3} \mathrm{W}^2 |\Phi_0^2 \partial_x^i \eta_{xx}|_m^2 & \text{for} \quad i = 0, 1,
\end{cases}
$$

where $\Phi_0^1 = \Phi_0(\eta, \delta\eta_x, \delta^2\eta_{xx}, \boldsymbol{u}^\delta|_\Gamma, \delta^2 v_x|_\Gamma)$ and $\Phi_0^2 = \Phi_0(\delta\eta_x, \delta^2\eta_{xx})$. Note that we included the term $\delta^2 v_x|_\Gamma$ in Φ_0^1 for later use, although we can drop it. In the following we use the inequality

$$
(3.3.45) \qquad |fg|_s \lesssim |f|_{L^\infty} |g|_s + |g|_{L^\infty} |f|_s.
$$

By (3.3.45), (3.3.5) in Lemma 3.3.1, and (3.3.26) and (3.3.28) in Lemma 3.3.7, we obtain $J_1 \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. By (3.3.45), the second inequality in (3.3.7) in Lemma 3.3.1, (3.3.26) and (3.3.28) in Lemma 3.3.7, and the second inequality in (3.3.20) in Lemma 3.3.4, we obtain $J_2 \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. By (3.3.45), (3.3.6) in Lemma 3.3.1, and (3.3.29) and (3.3.31) in Lemma 3.3.8, we obtain $J_3 \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. Thus (3.3.37) holds.

As for (3.3.38), it suffices to estimate

$$
\begin{cases}
J_4 := \delta^{2i} |\Phi_0^1 \partial_x^i \eta|_{m+\frac{1}{2}}^2 & \text{for} \quad i = 1, 2, 3, \\
J_5 := \delta^{2i} |\Phi_0^1 \partial_x^i \boldsymbol{u}^\delta|_{m+\frac{1}{2}}^2 & \text{for} \quad i = 1, 2, \\
J_6 := \delta^{2i+4} \mathrm{W}^2 |\Phi_0^2 \partial_x^i \eta_{xx}|_{m+\frac{1}{2}}^2 & \text{for} \quad i = 0, 1.
\end{cases}
$$

By (3.3.45), (3.3.5) in Lemma 3.3.1, the second inequality in (3.3.17) in Lemma 3.3.4, and (3.3.26) and (3.3.27) in Lemma 3.3.7, we obtain $J_4 \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. By (3.3.45), the second inequality in (3.3.7) in Lemma 3.3.1, the second and third inequalities in (3.3.19) in Lemma 3.3.4, and (3.3.26) and (3.3.27) in Lemma 3.3.7 we obtain $J_5 \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. By (3.3.45), (3.3.6) in Lemma 3.3.1, the second inequality in (3.3.18) in Lemma 3.3.4, and (3.3.29) and (3.3.30) in Lemma 3.3.8, we obtain $J_6 \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. Thus (3.3.38) holds.

As for (3.3.39), it suffices to estimate

$$\begin{cases} J_7 := \delta^{2i+1}|\Phi_0^1\partial_x^i\eta_x|_{m-\frac{1}{2}}^2 \quad \text{for} \quad i=0,1, \\ J_8 := \delta|\Phi_0^1\boldsymbol{u}_x^\delta|_{m-\frac{1}{2}}^2, \\ J_9 := \delta^3\mathrm{W}^2|\Phi_0^2\eta_{xx}|_{m-\frac{1}{2}}^2. \end{cases}$$

By (3.3.45), (3.3.5) in Lemma 3.3.1, the first inequality in (3.3.17) in Lemma 3.3.4, and (3.3.26) and (3.3.28) in Lemma 3.3.7, we obtain $J_7 \lesssim \tilde{E}_2\tilde{E}_m$. By (3.3.45), (3.3.7) in Lemma 3.3.1, the first inequality in (3.3.19) in Lemma 3.3.4, and (3.3.26) and (3.3.28) in Lemma 3.3.7, we obtain $J_8 \lesssim \tilde{E}_2\tilde{E}_m$. By (3.3.45), (3.3.6) in Lemma 3.3.1, the first inequality in (3.3.18) in Lemma 3.3.4, and (3.3.29) and (3.3.31) in Lemma 3.3.8, we obtain $J_9 \lesssim \tilde{E}_2\tilde{E}_m$. Thus (3.3.39) holds.

As for (3.3.40), it suffices to estimate

$$\begin{cases} J_{10} := \delta^{2i+1}|\Phi_0^1\partial_x^i\eta|_{m+\frac{1}{2}}^2 \quad \text{for} \quad i=1,2, \\ J_{11} := \delta^3|\Phi_0^1\boldsymbol{u}_x^\delta|_{m+\frac{1}{2}}^2, \\ J_{12} := \delta^5\mathrm{W}^2|\Phi_0^2\eta_{xx}|_{m+\frac{1}{2}}^2. \end{cases}$$

By (3.3.45), (3.3.5) in Lemma 3.3.1, the first inequality in (3.3.17) in Lemma 3.3.4, and (3.3.26) and (3.3.27) in Lemma 3.3.7, we obtain $J_{10} \lesssim D_2D_m$. By (3.3.45), (3.3.7) in Lemma 3.3.1, the first inequality in (3.3.19) in Lemma 3.3.4, and (3.3.26) and (3.3.27) in Lemma 3.3.7, we obtain $J_{11} \lesssim D_2D_m$. By (3.3.45), (3.3.6) in Lemma 3.3.1, the first inequality in (3.3.18) in Lemma 3.3.4, and (3.3.29) and (3.3.30) in Lemma 3.3.8, we obtain $J_{12} \lesssim D_2D_m$. Thus (3.3.40) holds.

We proceed to estimate (3.3.41). By the third equation in (2.1.33), we can reduce the estimates of the terms which contain η_t except the terms which accompany W to those of J_1, J_2, and J_4. Thus it suffices to estimate

$$\begin{cases} J_{13} := \delta^9\mathrm{W}^2|\Phi^3\eta_x\eta_{xx}\eta_{tx}|_m^2, \\ J_{14} := \delta^9\mathrm{W}^2|\Phi^3\eta_x^2\eta_{txx}|_m^2, \\ J_{15} := \delta|\Phi_0^1\boldsymbol{u}_t^\delta|_m^2, \\ J_{16} := \delta^2|(\partial_x^k(\Phi_0^4\boldsymbol{u}_{tx}^\delta), \partial_x^k\boldsymbol{u}_t^\delta)_\Gamma|, \end{cases}$$

where $\Phi^3 = \Phi(\delta\eta_x)$, $\Phi_0^4 = \Phi_0(\eta, \delta\eta_x)$ and we used $h_{2,2} = \Phi_0(\delta^2\eta_x^2)\eta_{xx} = \Phi(\delta\eta_x)\delta^2\eta_x^2\eta_{xx}$. Taking into account that $\delta^9\mathrm{W}^2 \lesssim \delta^5$, by the third equation in (2.1.33), we can reduce the estimates of J_{13} and J_{14} to those of J_1, J_2, and J_4. By (3.3.45), the second inequality in (3.3.7) in Lemma 3.3.1, (3.3.26) and (3.3.28) in Lemma 3.3.7, and $\delta|\boldsymbol{u}_t^\delta|_m^2 \lesssim \delta\|(1+|D_x|)^m\boldsymbol{u}_{ty}^\delta\|^2 \lesssim$

$\tilde F_m$, we obtain $J_{15} \lesssim \tilde E_2\tilde F_m + \tilde F_2\tilde E_m$. By Lemma 1.3.3, we see that

$$
\begin{aligned}
J_{16} &= \delta^2|(\partial_x^k\{(\Phi_0^4\boldsymbol{u}_t^\delta)_x - \Phi_{0x}^4\boldsymbol{u}_t^\delta\}, \partial_x^k\boldsymbol{u}_t^\delta)_\Gamma| \\
&\le \delta^2||D_x|^{\frac12}\partial_x^k(\Phi_0^4\boldsymbol{u}_t^\delta)|_0||D_x|^{\frac12}\partial_x^k\boldsymbol{u}_t^\delta|_0 + \delta^2|(\partial_x^k(\Phi_{0x}^4\boldsymbol{u}_t^\delta), \partial_x^k\boldsymbol{u}_t^\delta)_\Gamma| \\
&\le \epsilon(\delta\|\partial_x^k\boldsymbol{u}_t^\delta\|^2 + \delta^3\|\partial_x^k\boldsymbol{u}_{tx}^\delta\|^2 + \delta\|\partial_x^k\boldsymbol{u}_{ty}^\delta\|^2) + C_\epsilon\big(\delta^2|\Phi_0^4\boldsymbol{u}_t^\delta|_{m+\frac12}^2 + \delta^3|\Phi_{0x}^4\boldsymbol{u}_t^\delta|_m^2\big).
\end{aligned}
$$

Here, we can reduce the estimate of $\delta^3|\Phi_{0x}^4\boldsymbol{u}_t^\delta|_m^2$ to that of J_8. By (3.3.45), the second inequality in (3.3.7) in Lemma 3.3.1, (3.3.21) in Lemma 3.3.4, and (3.3.26) and (3.3.27) in Lemma 3.3.7, we obtain $\delta^2|\Phi_0^4\boldsymbol{u}_t^\delta|_{m+\frac12}^2 \lesssim \tilde E_2\tilde F_m + \tilde F_2\tilde E_m$. We thereby deduce $J_{16} \le \epsilon\tilde F_m + C_\epsilon(\tilde E_2\tilde F_m + \tilde F_2\tilde E_m)$. Thus (3.3.41) holds.

As for (3.3.42), since $h_3 = \eta^2\eta_x$ is contained in the first term in (3.3.36), we have already checked that the first inequality holds. As for the second inequality, we have

$$
\delta^6\mathrm{W}|(\partial_x^k\eta_{xxxx}, \partial_x^k h_{3xx})_\Gamma| \le \epsilon\delta^6\mathrm{W}||D_x|^{k+\frac72}\eta|_0^2 + C_\epsilon\delta^6\mathrm{W}||D_x|^{k+\frac52}h_3|_0^2.
$$

Here, (3.3.16) leads to $\delta^6\mathrm{W}||D_x|^{k+\frac52}h_3|_0^2 \lesssim \tilde E_2\tilde F_m$. Therefore, we get the second inequality.

As for (3.3.43), taking into account that we can write b_3 as Φ_0^4 (see (2.1.30)), we obtain the first inequality in the same reason as the last estimate. Concerning the term $|(\partial_x^k\eta, \partial_x^k(b_3\eta)_x)_\Gamma|$ in the second inequality, there exist rational functions $b_{3,1}$ and $b_{3,2}$ such that $b_3\eta = b_{3,1}(\eta) + b_{3,2}(\eta, \delta\eta_x)\delta\eta_x$ and $b_{3,2}(\mathbf{0}) = 0$. Since the term $b_{3,2}(\eta, \delta\eta_x)\delta\eta_x$ can be treated in the same way as before, it suffices to estimate

$$
J_{17} := |(\partial_x^k\eta, \partial_x^{k+1}b_{3,1}(\eta))_\Gamma|.
$$

Here we can assume that $k \ge 1$ because we have $(\eta, b_{3,1}(\eta)_x)_\Gamma = 0$ in the case $k = 0$. We see that $J_{17} \le |(\partial_x^k\eta, b'_{3,1}(\eta)\partial_x^{k+1}\eta)_\Gamma| + |(\partial_x^k\eta, [\partial_x^k, b'_{3,1}(\eta)]\eta_x)_\Gamma)|$, where by integration by parts we have $|(\partial_x^k\eta, b'_{3,1}(\eta)\partial_x^{k+1}\eta)_\Gamma| = \frac12|(\partial_x^k\eta, b''_{3,1}(\eta)\eta_x\partial_x^k\eta)_\Gamma| \lesssim \sqrt{\tilde E_2}|\eta_x|_{m-1}^2$. In view of

$$
|[\partial_x^k, b'_{3,1}(\eta)]\eta_x|_0 \le C(|\eta|_{L^\infty})(1 + |\eta_x|_{L^\infty})^{k-1}|\eta_x|_{L^\infty}|\partial_x^{k-1}\eta_x|_0,
$$

we also have $|(\partial_x^k\eta, [\partial_x^k, b'_{3,1}(\eta)]\eta_x)_\Gamma| \lesssim \sqrt{\tilde E_2}|\eta_x|_{m-1}^2$. Therefore, $J_{17} \lesssim \sqrt{\tilde E_2}\min\{\tilde E_m, \delta^{-1}\tilde F_m\}$, so that we obtain

$$
|(\partial_x^k\eta, \partial_x^k(b_3\eta)_x)_\Gamma| \le \epsilon\tilde F_m + C_\epsilon(\tilde E_2\tilde F_m + \tilde F_2\tilde E_m) + C\sqrt{\tilde E_2}\min\{\tilde E_m, \delta^{-1}\tilde F_m\}.
$$

As for the the term $\delta^4\mathrm{W}|(\partial_x^k\eta_{txx}, \partial_x^k h_{3t})_\Gamma|$, integration by parts in x leads to

$$
\begin{aligned}
\delta^4\mathrm{W}|(\partial_x^k\eta_{txx}, \partial_x^k h_{3t})_\Gamma| &\le \delta^4\mathrm{W}|(\partial_x^k\eta_{txx}, \partial_x^k(\eta^2\eta_{tx}))_\Gamma| + \delta^4\mathrm{W}|(\partial_x^k\eta_{txx}, \partial_x^k(2\eta\eta_t\eta_x))_\Gamma| \\
&\le \delta^4\mathrm{W}|(\partial_x^k\eta_{tx}, \eta\eta_x\partial_x^k\eta_{tx})_\Gamma| + \delta^4\mathrm{W}|(\partial_x^k\eta_{tx}, ([\partial_x^k, \eta^2]\eta_{tx})_x)_\Gamma| \\
&\quad + \delta^4\mathrm{W}|(\partial_x^k\eta_{tx}, \partial_x^k(2\eta\eta_t\eta_x)_x)_\Gamma| \\
&=: J_{18} + J_{19} + J_{20}.
\end{aligned}
$$

Here, it follows from the third equation in (2.1.33) that $\delta^4\mathrm{W}|\partial_x^k\eta_{tx}|_0^2 \lesssim \delta^2\big(|\partial_x^k\eta_{xx}|_0^2 + |\partial_x^k v_x|_0^2 + |\partial_x^k h_{3x}|_0^2\big) \lesssim \delta^{-1}\tilde F_m$ and $\delta^4\mathrm{W}|\partial_x^k\eta_{tx}|_0^2 \lesssim \tilde E_m$ so that we have

$$
\delta^4\mathrm{W}|\partial_x^k\eta_{tx}|_0^2 \lesssim \min\{\tilde E_m, \delta^{-1}\tilde F_m\}. \tag{3.3.46}
$$

By (3.3.5) in Lemma 3.3.1 and (3.3.46), we have $J_{18} \lesssim \tilde{E}_2 \min\{\tilde{E}_m, \delta^{-1}\tilde{F}_m\}$. By the estimate

$$|([\partial_x^k, \eta^2]\eta_{tx})_x|_0 \lesssim |\eta|_{L^\infty}|\eta_x|_{L^\infty}|\partial_x^k \eta_{tx}| + |\eta|_{L^\infty}|\eta_{tx}|_{L^\infty}|\partial_x^k \eta_x| + |\eta_x|_{L^\infty}|\eta_{tx}|_{L^\infty}|\partial_x^k \eta|,$$

(3.3.5), (3.3.6), and (3.3.8) in Lemma 3.3.1, and (3.3.46), we easily obtain $J_{19} \leq \epsilon\tilde{F}_m + C_\epsilon \tilde{F}_2\tilde{E}_m + C\tilde{E}_2 \min\{\tilde{E}_m, \delta^{-1}\tilde{F}_m\}$. By (3.3.45) and (3.3.5), (3.3.6), and (3.3.8) in Lemma 3.3.1, and (3.3.46), we have $J_{20} \leq \epsilon\tilde{F}_m + C_\epsilon\tilde{E}_2\tilde{F}_m + C\tilde{E}_2 \min\{\tilde{E}_m, \delta^{-1}\tilde{F}_m\}$. Therefore, we get the third inequality. Thus far, we have assumed that $\varepsilon = 1$. Now, for general $\varepsilon \in (0,1]$ it follows easily from the above estimate that

$$|(\partial_x^k\eta, \partial_x^k(b_3\eta)_x)_\Gamma| \leq \epsilon\tilde{F}_m + \varepsilon^2 C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m) + C\sqrt{\tilde{E}_2}\min\{\varepsilon\tilde{E}_m, \varepsilon\delta^{-1}\tilde{F}_m\}.$$

The term $|(\partial_x^k\eta, \partial_x^k h_3)_\Gamma|$ is of the form J_{17}, so that it also satisfies the above estimate. Moreover, by taking into account that $\delta\sqrt{\mathrm{W}}|\eta_x|_{L^\infty} \lesssim \sqrt{\tilde{E}_2}$ and $\delta^2\mathrm{W}|\eta_x|_{m-1}^2 \lesssim |\eta_x|_{m-1}^2 \lesssim \min\{\tilde{E}_m, \delta^{-1}\tilde{F}_m\}$, the term $\delta^2\mathrm{W}|(\partial_x^k\eta_{xx}, \partial_x^k h_3 + \partial_x^k(b_3\eta)_x)_\Gamma|$ also satisfies the above estimate. Similarly, we obtain

$$\delta^4\mathrm{W}|(\partial_x^k\eta_{txx}, \partial_x^k h_{3t})_\Gamma| \leq \epsilon\tilde{F}_m + \varepsilon^2 C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m) + C\tilde{E}_2\min\{\varepsilon\tilde{E}_m, \varepsilon\delta^{-1}\tilde{F}_m\}.$$

Therefore, the second inequalities in (3.3.43) and (3.3.44) hold. The proof is complete. □

3.3.4 Estimate for nonlinear terms in equations

Next, we will estimate f_1, f_2, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{G}_k$. By the explicit form of $\boldsymbol{f}$ (see (2.1.24)), we see that this is consist of terms in the form

$$\begin{cases} \Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, \boldsymbol{u}^\delta; y)D_\delta^3\tilde{\eta}, \\ \Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}; y)\delta^i\partial_x^i\partial_y^j u & \text{for} \quad (i,j) = (2,0), (1,1), \\ \Phi(\tilde{\eta}, D_\delta\tilde{\eta}, \boldsymbol{u}^\delta, y)w_\lambda u_y & \text{for} \quad 1 \leq \lambda \leq 3, \\ \Phi(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \boldsymbol{u}^\delta, y)w_\lambda\boldsymbol{u}^\delta & \text{for} \quad 1 \leq \lambda \leq 4, \\ \Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \delta\tilde{\eta}_t, \boldsymbol{u}^\delta; y)\delta\boldsymbol{u}_x^\delta, \end{cases}$$

where w_λ is defined by (3.3.32). Thus by the explicit forms of f_1 and f_2 (see (3.1.16) and (3.1.23)), we see that these contain the above terms, $\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}; y)\nabla_\delta p$, and $\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}; y)\delta u_t$ (see also (2.1.25)). In addition to these terms, $\boldsymbol{F}_1$ contains also $\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}; y)u_{yy}$ (see (3.1.2)).

Lemma 3.3.12. *For any $\epsilon > 0$ there exists a positive constant C_ϵ such that the following* *estimates hold.*

$$(3.3.47)\qquad \delta^{-1}\|\partial_x^k f_1\|^2 + \delta^{-1}\|\partial_x^k f_2\|^2 + \delta\|\partial_x^k f_{1x}\|^2 + \delta|(\partial_x^k \boldsymbol{F}_{1x}, \partial_x^k \boldsymbol{u}_x^\delta)_\Omega| + \delta^3|(\partial_x^k \boldsymbol{F}_{1xx}, \partial_x^k \boldsymbol{u}_{xx}^\delta)_\Omega| \le \epsilon \tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m),$$

$$(3.3.48)\qquad \delta^{-2}\|\partial_x^{l-1}\boldsymbol{f}\|^2 \lesssim \tilde{E}_2\tilde{E}_m,$$

$$(3.3.49)\qquad |(\partial_x^l\{A_4\nabla_\delta p + (b_2u_{yy},0)^{\mathrm{T}}\}, \partial_x^l \boldsymbol{u}^\delta)_\Omega| \le (\epsilon + C_\epsilon\tilde{E}_2)\tilde{E}_m,$$

$$(3.3.50)\qquad \|\partial_x^k \boldsymbol{f}\|^2 \lesssim D_2D_m,$$

$$(3.3.51)\qquad \delta|(\partial_x^k \boldsymbol{F}_2, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega| \le \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m),$$

$$(3.3.52)\qquad \delta|(\boldsymbol{G}_k, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega| \le \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m).$$

Proof. As for (3.3.47), the definition of $\boldsymbol{F}_1$ and integration by parts in x imply

$$\delta^3|(\partial_x^k \boldsymbol{F}_{1xx}, \partial_x^k \boldsymbol{u}_{xx}^\delta)_\Omega| \le \epsilon\delta^5\|\partial_x^k \boldsymbol{u}_{xxx}^\delta\|^2 + C_\epsilon\delta\|\partial_x^k(\boldsymbol{f} - \Phi_0(\tilde{\eta}, D_\delta\tilde{\eta})\nabla_\delta p)_x\|^2 + \delta^3|(\partial_x^k(\Phi_0(\tilde{\eta}, D_\delta\tilde{\eta})u_{yy})_{xx}, \partial_x^k \boldsymbol{u}_{xx}^\delta)_\Omega|.$$

Taking this into account, it suffices to estimate

$$\begin{cases} K_1 := \delta^{-1}\|\partial_x^k(\Phi_0^5 D_\delta^i\tilde{\eta})\|^2 & \text{for} \quad 1 \le i \le 4, \\ K_2 := \delta^{-1}\|\partial_x^k(\Phi_0^5\delta\tilde{\eta}_t)\|^2, & \\ K_3 := \delta^{2i-1}\|\partial_x^k(\Phi_0^5\partial_x^i u_y)\|^2 & \text{for} \quad i = 1,2, \\ K_4 := \delta^{-1}\|\partial_x^k(\Phi^5 w_\lambda\partial_y^j u)\|^2 & \text{for} \quad 1 \le \lambda \le 7,\ j = 0,1, \\ K_5 := \delta^{2i-1}\|\partial_x^k(\Phi_0^6\partial_x^i \boldsymbol{u}^\delta)\|^2 & \text{for} \quad i = 1,2,3 \\ K_6 := \delta^{2i-1}|(\partial_x^{k+i}(\Phi_0^7 u_{yy}), \partial_x^{k+i}\boldsymbol{u}^\delta)_\Omega| & \text{for} \quad i = 1,2, \\ K_7 := \delta^{2i-1}\|\partial_x^{k+i}(\Phi_0^7\nabla_\delta p)\|^2 & \text{for} \quad i = 0,1, \end{cases}$$

where

$$\Phi^5 = \Phi(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, \boldsymbol{u}^\delta; y),$$
$$\Phi^6 = \Phi(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, D_\delta^3\tilde{\eta}, \delta\tilde{\eta}_t, \delta D_\delta\tilde{\eta}_t, \delta D_\delta^2\tilde{\eta}_t, \boldsymbol{u}^\delta, \delta\boldsymbol{u}_x^\delta; y),$$
$$\Phi^7 = \Phi(\tilde{\eta}, D_\delta\tilde{\eta}; y).$$

In the following we will use the well-known inequality

$$(3.3.53)\qquad \|\partial_x^k(fg)\| \lesssim \|f\|_{L^\infty}\|\partial_x^k g\| + \|g\|_{L^\infty}\|\partial_x^k f\|.$$

By this, (3.3.5) in Lemma 3.3.1, (3.3.11) and (3.3.13) in Lemma 3.3.3, and (3.3.22) and (3.3.23) in Lemma 3.3.5, we obtain $K_1 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. Similarly, we get $K_2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. By Lemma 1.3.6, we have

$$K_3 \lesssim \|\Phi_0^5\|_{L^\infty}^2\delta^{2i-1}\|\partial_x^{k+i}u_y\|^2 + (\|\partial_x^k\Phi_0^5\|^2 + \|\partial_x^k\Phi_{0y}^5\|^2)\delta^{2i-1}(\|\partial_x^i u_y\|^2 + \|\partial_x^i u_{xy}\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_3 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{E}_m\tilde{F}_2$. By Lemma 1.3.6, we have

$$\begin{aligned}
K_4 &\lesssim \|\Phi^5\|_{L^\infty}^2 \delta^{-1}\|w_\lambda\|_{L^\infty}^2 \|\partial_x^k\partial_y^j u\|^2 \\
&\quad + \delta^{-1}\|w_\lambda\|_{L^\infty}^2 (\|\partial_x^k\Phi^5\|^2 + \|\partial_x^k\Phi_y^5\|^2)(\|\partial_y^j u\|^2 + \|\partial_y^j u_x\|^2) \\
&\quad + \|\Phi^5\|_{L^\infty}^2 \delta^{-1}(\|\partial_x^k w_\lambda\|^2 + \|\partial_x^k w_{\lambda y}\|^2)(\|\partial_y^j u\|^2 + \|\partial_y^j u_x\|^2),
\end{aligned}$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 and (3.3.33) and (3.3.34) in Lemma 3.3.9 gives $K_4 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. As for K_5, it suffices to consider the case of $k \geq 1$ since we can easily treat the case of $k = 0$. By Lemma 1.3.6, we have

$$K_5 \lesssim \|\Phi_0^6\|_{L^\infty}^2 \delta^{2i-1}\|\partial_x^{k+i}\boldsymbol{u}^\delta\|^2 + \delta(\|\partial_x^k\Phi_0^6\|^2 + \|\partial_x^k\Phi_{0y}^6\|^2)\delta^{2(i-1)}(\|\partial_x^i\boldsymbol{u}^\delta\|^2 + \|\partial_x^i\boldsymbol{u}_x^\delta\|^2),$$

which together with (3.3.22) and (3.3.25) in Lemma 3.3.5 gives $K_5 \lesssim \tilde{E}_2\tilde{F}_m$. As for K_6, we will consider the case $i = 2$ only, because the case where $i = 1$ can be treated in a similar but easier way. Using integration by parts in x and y and Lemma 1.3.3, we have

$$\begin{aligned}
K_6 &= \delta^3|(\partial_x^k\{(\Phi_0^7 u_y)_{xxy} - (\Phi_{0y}^7 u_y)_{xx}\}, \partial_x^k\boldsymbol{u}_{xx}^\delta)_\Omega| \\
&\leq \delta^3|(\partial_x^k(\Phi_0^7 u_y)_{xx}, \partial_x^k\boldsymbol{u}_{xxy}^\delta)_\Omega| + \delta^3||D_x|^{\frac12}\partial_x^k(\Phi_0^7 u_y)_x|_0||D_x|^{\frac12}\partial_x^k\boldsymbol{u}_{xx}^\delta|_0 \\
&\quad + \delta^3|(\partial_x^k(\Phi_{0y}^7 u_y)_x, \partial_x^k\boldsymbol{u}_{xxx}^\delta)_\Omega| \\
&\leq \epsilon\tilde{F}_m + C_\epsilon\big(\delta^3\|\partial_x^k(\Phi_0^7 u_y)_{xx}\|^2 + \delta\|\partial_x^k(\Phi_{0y}^7 u_y)_x\|^2 + \delta^2||D_x|^{\frac12}\partial_x^k(\Phi_0^7 u_y)_x|_0^2\big).
\end{aligned}$$

Here we can reduce the estimate of $\delta^3\|\partial_x^k(\Phi_0^7 u_y)_{xx}\|^2 + \delta\|\partial_x^k(\Phi_{0y}^7 u_y)_x\|^2$ to those of K_3 and K_4. Furthermore, using the first equation in (2.1.33) to eliminate $u_y|_\Gamma$, we can reduce the estimate of $\delta^2||D_x|^{\frac12}\partial_x^k(\Phi_0^7 u_y)_x|_0^2$ to those of J_4 and J_5. Thus combining these estimates, we obtain $K_6 \leq \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m)$. By Lemma 1.3.6, we have

$$K_7 \lesssim \|\Phi_0^7\|_{L^\infty}^2 \delta^{2i-1}\|\nabla_\delta\partial_x^{k+i}p\|^2 + \delta^{2i}(\|\partial_x^{k+i}\Phi_0^7\|^2 + \|\partial_x^{k+i}\Phi_{0y}^7\|^2)\delta^{-1}(\|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_7 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{E}_m\tilde{F}_2$. Thus (3.3.47) holds.

As for (3.3.48), it suffices to estimate

$$\begin{cases}
K_8 := \delta^{-2}\|\partial_x^{l-1}(\Phi_0^5 D_\delta^3\tilde{\eta})\|^2, \\
K_9 := \|\partial_x^{l-1}(\Phi_0^5 u_{xy})\|^2, \\
K_{10} := \delta^{-2}\|\partial_x^{l-1}(\Phi^5 w_\lambda\partial_y^j\boldsymbol{u}^\delta)\|^2 \quad \text{for} \quad 1 \leq \lambda \leq 4,\ j = 0, 1, \\
K_{11} := \delta^{2i}\|\partial_x^{l-1}(\Phi_0^5\partial_x^i\boldsymbol{u}_x^\delta)\|^2 \qquad \text{for} \quad i = 0, 1.
\end{cases}$$

By (2.1.10) in Lemma 2.1.1, we have $\delta^{-2}\|\partial_x^{l-1}D_\delta^3\tilde{\eta}\|^2 \lesssim \delta^4|\partial_x^l\eta_{xx}|_0^2$. Therefore, by (3.3.53), (3.3.10) in Lemma 3.3.1, and (3.3.22) and (3.3.23) in Lemma 3.3.5, we obtain $K_8 \lesssim \tilde{E}_2\tilde{E}_m$. By Lemma 1.3.6, we have

$$K_9 \lesssim \|\Phi_0^5\|_{L^\infty}^2\|\partial_x^l\boldsymbol{u}_y^\delta\|^2 + (\|\partial_x^{l-1}\Phi_0^5\|^2 + \|\partial_x^{l-1}\Phi_{0y}^5\|^2)(\|u_{xy}\|^2 + \|u_{xxy}\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_9 \lesssim \tilde{E}_2\tilde{E}_m$. By Lemma 1.3.6, we have

$$\begin{aligned}K_{10} &\lesssim \|\Phi^5\|_{L^\infty}^2\delta^{-2}\|w_\lambda\|_{L^\infty}^2\|\partial_x^{l-1}\partial_y^j\boldsymbol{u}^\delta\|^2\\ &\quad+\delta^{-2}\|w_\lambda\|_{L^\infty}^2(\|\partial_x^{l-1}\Phi^5\|^2+\|\partial_x^{l-1}\Phi_y^5\|^2)(\|\partial_y^j\boldsymbol{u}^\delta\|^2+\|\partial_y^j\boldsymbol{u}_x^\delta\|^2)\\ &\quad+\|\Phi^5\|_{L^\infty}^2\delta^{-2}(\|\partial_x^{l-1}w_\lambda\|^2+\|\partial_x^{l-1}w_{\lambda y}\|^2)(\|\partial_y^j\boldsymbol{u}^\delta\|^2+\|\partial_y^j\boldsymbol{u}_x^\delta\|^2),\end{aligned}$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 and (3.3.33) and (3.3.35) in Lemma 3.3.9 gives $K_{10} \lesssim \tilde{E}_2\tilde{E}_m$. By Lemma 1.3.6, we have

$$K_{11} \lesssim \|\Phi_0^5\|_{L^\infty}^2\delta^{2i}\|\partial_x^{l+i}\boldsymbol{u}^\delta\|^2+(\|\partial_x^{l-1}\Phi_0^5\|^2+\|\partial_x^{l-1}\Phi_{0y}^5\|^2)\delta^{2i}(\|\partial_x^i\boldsymbol{u}_x^\delta\|^2+\|\partial_x^i\boldsymbol{u}_{xx}^\delta\|),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_{11} \lesssim \tilde{E}_2\tilde{E}_m$. Thus (3.3.48) holds.

We proceed to estimate (3.3.49). With the aid of (3.1.11), we can express $A_4\nabla_\delta p$ in terms of the product of Φ_0^7 and derivatives of $\boldsymbol{u}^\delta$ in addition to $\Phi_0^7\boldsymbol{f}$. Taking this into account and using (3.3.48), it suffices to estimate

$$\begin{cases}K_{12} := \delta^2\|\partial_x^l(\Phi_0^7\boldsymbol{u}_t^\delta)\|^2,\\ K_{13} := |(\partial_x^l(\Phi_0^7\boldsymbol{u}_{yy}^\delta),\partial_x^l\boldsymbol{u}^\delta)_\Omega|.\end{cases}$$

By Lemma 1.3.6, we have

$$K_{12} \lesssim \|\Phi_0^7\|_{L^\infty}^2\delta^2\|\partial_x^l\boldsymbol{u}_t^\delta\|^2+(\|\partial_x^l\Phi_0^7\|^2+\|\partial_x^l\Phi_{0y}^7\|^2)\delta^2(\|\boldsymbol{u}_t^\delta\|^2+\|\boldsymbol{u}_{tx}^\delta\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_{12} \lesssim \tilde{E}_2\tilde{E}_m$. Integration by parts in y implies

$$\begin{aligned}K_{13} &= |(\partial_x^l\{(\Phi_0^7\boldsymbol{u}_y^\delta)_y-\Phi_{0y}^7\boldsymbol{u}_y^\delta\},\partial_x^l\boldsymbol{u}^\delta)_\Omega|\\ &\le \|\partial_x^l(\Phi_0^7\boldsymbol{u}_y^\delta)\|\|\partial_x^l\boldsymbol{u}_y^\delta\|+|(\partial_x^l(\Phi_0^7\boldsymbol{u}_y^\delta),\partial_x^l\boldsymbol{u}^\delta)_\Gamma|+\|\partial_x^l(\Phi_{0y}^7\boldsymbol{u}_y^\delta)\|\|\partial_x^k\boldsymbol{u}^\delta\|\\ &\le \epsilon\tilde{E}_m+C_\epsilon\big(\|\partial_x^l(\Phi_0^7\boldsymbol{u}_y^\delta)\|^2+\|\partial_x^l(\Phi_{0y}^7\boldsymbol{u}_y^\delta)\|^2\big)+|(\partial_x^l(\Phi_0^7\boldsymbol{u}_y^\delta),\partial_x^l\boldsymbol{u}^\delta)_\Gamma|.\end{aligned}$$

Here the estimates of $\|\partial_x^l(\Phi_0^7\boldsymbol{u}_y^\delta)\|^2$ and $\|\partial_x^l(\Phi_{0y}^7\boldsymbol{u}_y^\delta)\|^2$ are reduced to that of $\|\partial_x^l(\Phi_0^5\boldsymbol{u}_y^\delta)\|^2$. By Lemma 1.3.6, we have

$$\|\partial_x^l(\Phi_0^5\boldsymbol{u}_y^\delta)\|^2 \lesssim \|\Phi_0^5\|_{L^\infty}^2\|\partial_x^l\boldsymbol{u}_y^\delta\|^2+(\|\partial_x^l\Phi_0^5\|^2+\|\partial_x^l\Phi_{0y}^5\|^2)(\|\boldsymbol{u}_y^\delta\|^2+\|\boldsymbol{u}_{xy}^\delta\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $\|\partial_x^l(\Phi_0^5\boldsymbol{u}_y^\delta)\|^2 \lesssim \tilde{E}_2\tilde{E}_m$. Concerning the boundary integral, by the first equation in (2.1.33) and the second equation in (2.1.32), we can replace u_y and δv_y by $h_1+(2+b_3)\eta-\delta^2 v_x$ and by $-\delta u_x$, respectively, so that we obtain

$$|(\partial_x^l(\Phi_0^7\boldsymbol{u}_y^\delta),\partial_x^l\boldsymbol{u}^\delta)_\Gamma| \lesssim |\Phi_0^7(2+b_3)\eta|_m|\boldsymbol{u}^\delta|_m+(|\Phi_0^7h_1|_{m-\frac{1}{2}}+\delta|\Phi_0^7\boldsymbol{u}_x^\delta|_{m-\frac{1}{2}})|\boldsymbol{u}^\delta|_{m+\frac{1}{2}}.$$

These terms can be treated by the estimate of J_8 and (3.3.19) and (3.3.20) in Lemma 3.3.4. Therefore, we obtain $K_{13} \le (\epsilon+C_\epsilon\tilde{E}_2)\tilde{E}_m$. Thus (3.3.49) holds.

As for (3.3.50), it suffices to estimate

$$\begin{cases} K_{14} := \|\partial_x^k(\Phi_0^5 D_\delta^3 \tilde{\eta})\|^2, \\ K_{15} := \delta^{2i}\|\partial_x^k(\Phi_0^5 \partial_x^i \partial_y^j \boldsymbol{u}^\delta)\|^2 \quad \text{for} \quad 0 \le i+j \le 2,\ j \ne 2. \end{cases}$$

By (3.3.53), (3.3.10) in Lemma 3.3.1, (3.3.11) in Lemma 3.3.3, and (3.3.22) and (3.3.23) in Lemma 3.3.5, we obtain $K_{14} \lesssim D_2 D_m$. By Lemma 1.3.6, we have

$$K_{15} \lesssim \|\Phi_0^5\|_{L^\infty}^2 \delta^{2i}\|\partial_x^{k+i}\partial_y^j \boldsymbol{u}^\delta\|^2 + (\|\partial_x^k \Phi_0^5\|^2 + \|\partial_x^k \Phi_{0y}^5\|^2)\delta^{2i}(\|\partial_x^i \partial_y^j \boldsymbol{u}^\delta\|^2 + \|\partial_x^i \partial_y^j \boldsymbol{u}_x^\delta\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_{15} \lesssim D_2 D_m$. Thus (3.3.50) holds.

As for (3.3.51), by the definition of $\boldsymbol{F}_2$ (see (3.1.15)) and using the third equation in (2.1.33), it suffices to estimate

$$\begin{cases} K_{16} := \delta\|\partial_x^k(\Phi_0^5 D_\delta^i \tilde{\eta}_t)\|^2 & \text{for} \quad i = 1,2,3, \\ K_{17} := \delta\|\partial_x^k(\Phi_0^5 u_{ty})\|^2, \\ K_{18} := \delta\|\partial_x^k(\Phi^5 \boldsymbol{u}_t^\delta u_y)\|^2, \\ K_{19} := \delta^{2i+1}\|\partial_x^k(\Phi_0^6 \partial_x^i \boldsymbol{u}_t^\delta)\|^2 & \text{for} \quad i = 0,1, \\ K_{20} := \delta^3\|\partial_x^k(\Phi^5 D_\delta^i \tilde{\eta}_{tt} \partial_y^j u)\|^2 & \text{for} \quad (i,j) = (0,0),(1,0),(0,1), \\ K_{21} := \delta^3\|\partial_x^k(\Phi^5 D_\delta^i \tilde{\eta}_{tt} \boldsymbol{u}_x^\delta)\|^2, \\ K_{22} := \delta^{i+2}|(\partial_x^k(\Phi_0^7 \partial_x^{i+1}\partial_y^j \boldsymbol{u}_t^\delta), \partial_x^k \boldsymbol{u}_t^\delta)_\Omega| & \text{for} \quad (i,j) = (1,0),(0,1), \\ K_{23} := \delta|(\partial_x^k(\Phi_0^7 u_{yy})_t, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega|. \end{cases}$$

Here we did not list the terms which we have already estimated as $K_1, K_2, \dots, K_5$. By (3.3.53), (3.3.8) in Lemma 3.3.1, (3.3.14) in Lemma 3.3.3, and (3.3.22) and (3.3.23) in Lemma 3.3.5, we obtain $K_{16} \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. By Lemma 1.3.6, we have

$$K_{17} \lesssim \|\Phi_0^5\|_{L^\infty}^2 \delta\|\partial_x^k u_{ty}\|^2 + (\|\partial_x^k \Phi_0^5\|^2 + \|\partial_x^k \Phi_{0y}^5\|^2)\delta(\|u_{ty}\|^2 + \|u_{txy}\|^2),$$

which together with (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_{17} \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{E}_m \tilde{F}_2$. By Lemma 1.3.6, we have

$$\begin{aligned} K_{18} \lesssim{}& \|\Phi^5\|_{L^\infty}^2 \delta\|\boldsymbol{u}_t^\delta\|_{L^\infty}^2 \|\partial_x^k u_y\|^2 + \delta\|\boldsymbol{u}_t^\delta\|_{L^\infty}^2(\|\partial_x^k \Phi^5\|^2 + \|\partial_x^k \Phi_y^5\|^2)(\|u_y\|^2 + \|u_{xy}\|^2) \\ &+ \|\Phi^5\|_{L^\infty}^2 \delta(\|\partial_x^k \boldsymbol{u}_t^\delta\|^2 + \|\partial_x^k \boldsymbol{u}_{ty}^\delta\|^2)(\|u_y\|^2 + \|u_{xy}\|^2), \end{aligned}$$

which together with the second inequality in (3.3.7) in Lemma 3.3.1 and (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_{18} \lesssim \tilde{E}_m \tilde{F}_2 + \tilde{E}_2 \tilde{F}_m$. By (3.3.53), the second inequality in (3.3.7) in Lemma 3.3.1 and (3.3.22) and (3.3.23) in Lemma 3.3.5, we obtain $K_{19} \lesssim \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m$. As for K_{20}, we will consider the case $(i,j) = (0,1)$ only, because the other cases can be treated more easily. By Lemma 1.3.6, we have

$$\begin{aligned} K_{20} \lesssim{}& \|\Phi^5\|_{L^\infty}^2 \delta^3\|\tilde{\eta}_{tt}\|_{L^\infty}^2 \|\partial_x^k u_y\|^2 + \delta^3\|\tilde{\eta}_{tt}\|_{L^\infty}^2(\|\partial_x^k \Phi^5\|^2 + \|\partial_x^k \Phi^5\|^2)(\|u_y\|^2 + \|u_{xy}\|^2) \\ &+ \|\Phi^5\|_{L^\infty}^2 \delta^3(\|\partial_x^k \tilde{\eta}_{tt}\|^2 + \|\partial_x^k \tilde{\eta}_{tty}\|^2)(\|u_y\|^2 + \|u_{xy}\|^2), \end{aligned}$$

which together with (3.3.9) in Lemma 3.3.1 and (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $K_{20} \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. Similarly, we obtain $K_{21} \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. As for K_{22}, integration by parts in x yields

$$\begin{aligned}K_{22} &= \delta^{i+2}|(\partial_x^k\{(\Phi_0^7\partial_x^i\partial_y^j\boldsymbol{u}_t^\delta)_x - \Phi_{0x}^7\partial_x^i\partial_y^j\boldsymbol{u}_t^\delta\}, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega| \\ &\le \delta^{i+2}|(\partial_x^k(\Phi_0^7\partial_x^i\partial_y^j\boldsymbol{u}_t^\delta), \partial_x^k\boldsymbol{u}_{tx}^\delta)_\Omega| + \delta^{i+2}|(\partial_x^k(\Phi_{0x}^7\partial_x^i\partial_y^j\boldsymbol{u}_t^\delta), \partial_x^k\boldsymbol{u}_t^\delta)_\Omega| \\ &\le \epsilon\tilde{F}_m + C_\epsilon\big(\delta^{2i+1}\|\partial_x^k(\Phi_0^7\partial_x^i\partial_y^j\boldsymbol{u}_t^\delta)\|^2 + \delta^{2i+3}\|\partial_x^k(\Phi_{0x}^7\partial_x^i\partial_y^j\boldsymbol{u}_t^\delta)\|^2\big).\end{aligned}$$

Since the estimate of the right-hand side of the above inequality is reduced to those of K_{17} and K_{19}, we obtain $K_{22} \le \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m)$. As for K_{23}, integration by parts in y yields

$$\begin{aligned}K_{23} &= \delta|(\partial_x^k\{(\Phi_0^7u_y)_y - \Phi_{0y}^7u_y\}_t, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega| \\ &\le \delta|(\partial_x^k(\Phi_0^7u_y)_t, \partial_x^k\boldsymbol{u}_{ty}^\delta)_\Omega| + \delta|(\partial_x^k(\Phi_0^7u_y)_t, \partial_x^k\boldsymbol{u}_t^\delta)_\Gamma| + \delta|(\partial_x^k(\Phi_{0y}^7u_y)_t, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega| \\ &\le \epsilon\tilde{F}_m + C_\epsilon\big(\delta\|\partial_x^k(\Phi_0^7u_y)_t\|^2 + \delta\|\partial_x^k(\Phi_{0y}^7u_y)_t\|^2\big) + \delta|(\partial_x^k(\Phi_0^7u_y)_t, \partial_x^k\boldsymbol{u}_t^\delta)_\Gamma|.\end{aligned}$$

Here we can reduce the estimate of $\delta\|\partial_x^k(\Phi_0^7u_y)_t\|^2 + \delta\|\partial_x^k(\Phi_{0y}^7u_y)_t\|^2$ to those of K_4 and K_{17}. Moreover, by the first equation in (2.1.33), we can estimate the term $\delta|(\partial_x^k(\Phi_0^7u_y)_t, \partial_x^k\boldsymbol{u}_t^\delta)_\Gamma|$ in the same way as the proof of (3.3.41) in Lemma 3.3.10. We thereby obtain $K_{23} \le \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m)$. Thus (3.3.51) holds.

As for (3.3.52), by the definition of $\boldsymbol{G}_k$ (see (3.1.13)) we see that

$$\begin{aligned}\delta|(\boldsymbol{G}_k, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega| &\le \epsilon\tilde{F}_m + C_\epsilon\delta\|[\partial_x^k, A_5]\{(I + A_4)\nabla_\delta p_t + A_{4t}\nabla_\delta p\}\|^2 \\ &\quad + C_\epsilon\delta\|[\partial_x^k, A_{5t}]\boldsymbol{u}_t^\delta\|^2 + \delta|([\partial_x^k, A_5]\boldsymbol{F}_{3t}, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega| \\ &=: \epsilon\tilde{F}_m + K_{24} + K_{25} + K_{26}.\end{aligned}$$

Here we can assume $k \ge 1$. By the fact that A_4 and A_5 are of the form Φ_0^7 (see (2.1.25) and (3.1.11)), Lemma 1.3.7, (3.3.5) in Lemma 3.3.1, and (3.3.22) and (3.3.24) in Lemma 3.3.5, we obtain

$$\begin{aligned}K_{24} \le C_\epsilon\{&\tilde{E}_2(\delta\|\nabla_\delta\partial_x^{k-1}p_t\|^2 + \|\nabla_\delta\partial_x^{k-1}p\|^2) \\ &+ \tilde{E}_m(\delta\|\nabla_\delta p_t\|^2 + \delta\|\nabla_\delta p_{tx}\|^2 + \|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2)\},\end{aligned}$$

which gives $K_{24} \le C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{E}_m\tilde{F}_2)$. The estimate for K_{25} is reduced to that of K_{19}. Taking into account the explicit form of $\boldsymbol{F}_3$ (see (3.1.10)), we can estimate K_{26} in the same way as the proof of (3.3.51). Therefore the proof is complete. □

By Lemmas 3.3.10 and 3.3.12, for the nonlinear term N_m defined by (3.1.30), we obtain the following proposition.

Proposition 3.3.13. *For any $\epsilon > 0$ there exists a positive constants C_ϵ such that the following estimate holds.*

$$N_m \le \epsilon\tilde{F}_m + C_\epsilon\big(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m + \varepsilon\sqrt{\tilde{E}_2\tilde{E}_m}\big).$$

Moreover, if $\varepsilon \lesssim \delta$, then we have

$$N_m \le \epsilon \tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m).$$

Finally, we estimate the terms appearing in the right-hand side of (3.2.18) and (3.2.19) in Lemma 3.2.2. By the explicit form of g (see (3.2.2)), this consists of the terms in the form

$$\begin{cases} \Phi(\tilde{\eta}, D_\delta\tilde{\eta}, y)\delta \boldsymbol{u}_x^\delta u_y, \\ \Phi(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \boldsymbol{u}^\delta, y) w_\lambda \partial_y^j \boldsymbol{u}^\delta \qquad \text{for} \quad \lambda = 1,2, \ j = 0,1, \\ \Phi_0(\tilde{\eta}, D_\delta\tilde{\eta}, D_\delta^2\tilde{\eta}, \boldsymbol{u}^\delta, \delta\boldsymbol{u}_x^\delta; y)\delta\boldsymbol{u}_x^\delta. \end{cases}$$

Lemma 3.3.14. *The following estimates hold.*

$$\begin{aligned} &\delta\|\partial_x^k g_x\|^2 + \delta\|\partial_x^k g_{0x}\|^2 + \delta\|\nabla_\delta\partial_x^k g_{0x}\|^2 \\ &\quad + \delta\|\partial_x^k(N_6\nabla_\delta p)_x\|^2 + \delta^2||D_x|^{k+\frac{1}{2}}\phi_x|_0^2 \lesssim \tilde{F}_m + \tilde{F}_2\tilde{E}_m, \end{aligned} \tag{3.3.54}$$

$$\begin{aligned} &\|\partial_x^k g\|^2 + \|\partial_x^k g_0\|^2 + \|\nabla_\delta\partial_x^k g_0\|^2 + \|\partial_x^k(N_6\nabla_\delta p)\|^2 + \delta||D_x|^{k+\frac{1}{2}}\phi|_0^2 \\ &\quad \lesssim (1+D_2)D_m + \tilde{E}_2\|\nabla_\delta\partial_x^k p\|^2 + \min\{\tilde{E}_m, D_m\}(\|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2), \end{aligned} \tag{3.3.55}$$

$$\begin{aligned} &\delta\|\partial_x^{l-1} g_t\|^2 + \delta\|\partial_x^{l-1} g_{0t}\|^2 + \delta\|\nabla_\delta\partial_x^{l-1} g_{0t}\|^2 \\ &\quad + \delta\|\partial_x^{l-1}(N_6\nabla_\delta p)_t\|^2 + \delta^2||D_x|^{l-\frac{1}{2}}\phi_t|_0^2 \lesssim \tilde{F}_m + \tilde{F}_2\tilde{E}_m. \end{aligned} \tag{3.3.56}$$

Proof. By Lemma 1.3.6, we have

$$\begin{aligned} \delta^3\|\partial_x^k(\Phi^5\boldsymbol{u}_{xx}^\delta u_y)\|^2 &\lesssim \|\Phi^5\|_{L^\infty}^2\delta^3\|\boldsymbol{u}_{xx}^\delta\|_{L^\infty}^2\|\partial_x^k u_y\|^2 \\ &\quad + \delta^3\|\boldsymbol{u}_{xx}^\delta\|_{L^\infty}^2(\|\partial_x^k\Phi^5\|^2 + \|\partial_x^k\Phi_y^5\|^2)(\|u_y\|^2 + \|u_{xy}\|^2) \\ &\quad + \|\Phi^5\|_{L^\infty}^2\delta^3(\|\partial_x^k\boldsymbol{u}_{xx}^\delta\|^2 + \|\partial_x^k\boldsymbol{u}_{xxy}^\delta\|^2)(\|u_y\|^2 + \|u_{xy}\|^2), \end{aligned}$$

which together with the second inequality in (3.3.7) in Lemma 3.3.1 and (3.3.22) and (3.3.24) in Lemma 3.3.5, we obtain

$$\delta^3\|\partial_x^k(\Phi^5\boldsymbol{u}_{xx}^\delta u_y)\|^2 \lesssim \tilde{F}_2\tilde{E}_m + \tilde{F}_m\tilde{E}_2.$$

It follows from (3.3.10) in Lemma 3.3.1 that $\|\boldsymbol{u}_x^\delta\|_{L^\infty}^2\delta^3\|\partial_x^k u_{xy}\|^2 \lesssim \tilde{E}_2\tilde{F}_m$. Therefore, in the same way as the above estimate, we obtain

$$\delta^3\|\partial_x^k(\Phi^5\boldsymbol{u}_x^\delta u_{xy})\|^2 \lesssim \tilde{F}_2\tilde{E}_m + \tilde{F}_m\tilde{E}_2.$$

These together with the estimates of K_3, K_4, and K_5 yield $\delta\|\partial_x^k g_x\|^2 \lesssim \tilde{F}_2\tilde{E}_m + \tilde{E}_2\tilde{F}_m$. It follows from the explicit form of g_0 (see (3.2.5)) that $\delta\|\partial_x^k g_{0x}\|^2 + \delta\|\nabla_\delta\partial_x^k g_{0x}\|^2 \lesssim \tilde{F}_m + \tilde{F}_2\tilde{E}_m$, where we used the estimates for $K_1, K_2, \ldots, K_5$. By Lemma 1.3.6, we have

$$\begin{aligned} \delta\|\partial_x^{k+1}(N_6\nabla_\delta p)\|^2 &\lesssim \|N_6\|_{L^\infty}^2\delta\|\partial_x^{k+1}\nabla_\delta p\|^2 \\ &\quad + \delta^2(\|\partial_x^{k+1}N_6\|^2 + \|\partial_x^{k+1}N_{6y}\|)\delta^{-1}(\|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2). \end{aligned}$$

Since N_6 is the nonlinear part of A_6, which is defined by (3.2.1), we see that N_6 is of the form Φ_0^7. Thus by (3.3.22) and (3.3.24) in Lemma 3.3.5, we obtain

$$\delta\|\partial_x^{k+1}(N_6\nabla_\delta p)\|^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{E}_m\tilde{F}_2. \tag{3.3.57}$$

The definition of ϕ (see (3.2.3)), Lemma 1.3.3, and (3.3.38) in Lemma 3.3.10 imply $\delta^2||D_x|^{k+\frac{1}{2}}\phi_x|_0^2 \lesssim \tilde{F}_m + \tilde{F}_2\tilde{E}_m$. Combining the above estimates, we obtain (3.3.54).

By Lemma 1.3.6, (3.3.10) in Lemma 3.3.1 and (3.3.22), and (3.3.24) in Lemma 3.3.5, we obtain

$$\begin{aligned}\delta^2\|\partial_x^k(\Phi^5\boldsymbol{u}_x^\delta u_y)\|^2 &\lesssim \|\Phi^5\|_{L^\infty}^2\delta^2\|\boldsymbol{u}_x^\delta\|_{L^\infty}^2\|\partial_x^k u_y\|^2 \\ &\quad + \delta^2\|\boldsymbol{u}_x^\delta\|_{L^\infty}^2(\|\partial_x^k\Phi^5\|^2 + \|\partial_x^k\Phi_y^5\|^2)(\|u_y\|^2 + \|u_{xy}\|^2) \\ &\quad + \|\Phi^5\|_{L^\infty}^2\delta^2(\|\partial_x^k\boldsymbol{u}_x^\delta\|^2 + \|\partial_x^k\boldsymbol{u}_{xy}^\delta\|^2)(\|u_y\|^2 + \|u_{xy}\|^2) \\ &\lesssim D_2(1 + D_m).\end{aligned}$$

By (3.3.53), (3.3.10) in Lemma 3.3.1, and (3.3.22) and (3.3.23) in Lemma 3.3.5, we get $\delta^4\|\partial_x^k\big(\Phi^5(\boldsymbol{u}_x^\delta)^2\big)\|^2 \lesssim D_2(1 + D_m)$. These together with the estimate of K_{15} yield $\|\partial_x^k g\|^2 \lesssim D_2(1 + D_m)$. By the estimate for K_{15}, we obtain $\|\partial_x^k g_0\|^2 + \|\nabla_\delta\partial_x^k g_0\|^2 \lesssim (1 + D_2)D_m$. In the same way as the proof of (3.3.57), we obtain

$$\|\partial_x^k(N_6\nabla_\delta p)\|^2 \lesssim \tilde{E}_2\|\nabla_\delta\partial_x^k p\|^2 + \min\{\tilde{E}_m, D_m\}(\|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2).$$

Lemma 1.3.3 and (3.3.40) in Lemma 3.3.10 lead to $\delta||D_x|^{k+\frac{1}{2}}\phi|_0^2 \lesssim (1 + D_2)D_m$. Combining the above estimates implies (3.3.55).

By Lemma 1.3.6, we have

$$\begin{aligned}\delta^3\|\partial_x^{l-1}(\Phi^5\boldsymbol{u}_{tx}^\delta u_y)\|^2 &\lesssim \|\Phi^5\|_{L^\infty}^2\delta^3\|\boldsymbol{u}_{tx}^\delta\|_{L^\infty}^2\|\partial_x^{l-1}u_y\|^2 \\ &\quad + \delta^3\|\boldsymbol{u}_{tx}^\delta\|_{L^\infty}^2(\|\partial_x^{l-1}\Phi^5\|^2 + \|\partial_x^{l-1}\Phi_y^5\|^2)(\|u_y\|^2 + \|u_{xy}\|^2) \\ &\quad + \|\Phi^5\|_{L^\infty}^2(\|u_y\|^2 + \|u_{xy}\|^2)\delta^3(\|\partial_x^l\boldsymbol{u}_t^\delta\|^2 + \|\partial_x^l\boldsymbol{u}_{ty}^\delta\|),\end{aligned}$$

which together with the second inequality in (3.3.7) in Lemma 3.3.1 and (3.3.22) and (3.3.24) in Lemma 3.3.5 gives $\delta^3\|\partial_x^{l-1}(\Phi^5\boldsymbol{u}_{tx}^\delta u_y)\|^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. In a similar way, we get $\delta^3\|\partial_x^{l-1}(\Phi^5\boldsymbol{u}_t^\delta u_{xy})\|^2 \lesssim \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m$. Thus, in the same way as the proof of (3.3.54), we obtain (3.3.56). The proof is complete. □

3.4 Uniform estimate

Summarizing the estimates in the last sections, we will prove the following proposition.

Proposition 3.4.1. *Let m be an integer satisfying $m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, and $0 < \alpha \leq \alpha_0$, where R_0 and α_0 are constants in Propositions 3.1.1 and 3.1.4. There exist positive constants c_1, C_5, C_6, and C_7 such that if the solution (η, u, v, p) of (2.1.32)–(2.1.34) and the parameters δ, ε, R, and W satisfy*

$$\tilde{E}_2(t) \leq c_1, \quad 0 < \delta, \varepsilon \leq 1, \quad \mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0, \quad \mathrm{W}_1 \leq \mathrm{W} \leq \delta^{-2}\mathrm{W}_2,$$

then we have

$$(3.4.1)\quad \tilde{E}_2(t) \le C_7 E_2(0)\mathrm{e}^{C_6\varepsilon t}, \quad \tilde{E}_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau \le C_7 E_m(0)\exp(C_5 E_2(0)\mathrm{e}^{C_6\varepsilon t} + C_5\varepsilon t).$$

Moreover, if $\varepsilon \lesssim \delta$*, then we have*

$$\tilde{E}_2(t) \le C_7 E_2(0), \quad \tilde{E}_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau \le C_7 E_m(0)\exp(C_5 E_2(0)).$$

In order to prove the above proposition, we prepare the following lemma.

Lemma 3.4.2. *Under the same assumptions of Proposition* 3.4.1*, for any integer* k *satisfying* $0 \le k \le m$*, the following estimates hold.*

$$(3.4.2)\qquad \tilde{E}_m \lesssim E_m,$$

$$(3.4.3)\qquad \tilde{F}_m \lesssim F_m + \tilde{F}_2\tilde{E}_m,$$

$$(3.4.4)\qquad \|(1+|D_x|)^m \nabla_\delta p\|^2 \lesssim (1+D_2)^2 D_m.$$

Proof. As for (3.4.2), by the definition of $\tilde{E}_m$ (see (3.3.1)) and Poincaré's inequality, it suffices to show that for any $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$(3.4.5)\qquad \|\partial_x^k u_y\|^2 \le \epsilon\tilde{E}_m + C_\epsilon(E_m + \tilde{E}_2\tilde{E}_m).$$

Applying ∂_x^k to (2.1.32)–(2.1.34) and using the argument in the proof of Proposition 3.1.1, we obtain

$$\begin{aligned}\frac{1}{4K}\|\nabla_\delta\partial_x^k \boldsymbol{u}^\delta\|^2 \le& -\Big\{\mathrm{R}\delta(\partial_x^k\boldsymbol{u}^\delta, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega + 2\Big(\frac{1}{\tan\alpha}\delta(\partial_x^k\eta, \partial_x^k\eta_t)_\Gamma + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta(\partial_x^k\eta_x, \partial_x^k\eta_{tx})_\Gamma\Big)\Big\}\\ &+ 4K\big(|\partial_x^k\eta|_0^2 + |\partial_x^k(b_3\eta)|_0^2\big) + (\partial_x^k h_1, \partial_x^k u)_\Gamma - 2(\partial_x^k h_2, \delta\partial_x^k v)_\Gamma\\ &+ 2(\frac{1}{\tan\alpha}\partial_x^k\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\partial_x^k\eta_{xx}, \delta\partial_x^k h_3)_\Gamma\\ &+ \mathrm{R}(\partial_x^k\boldsymbol{f}, \partial_x^k\boldsymbol{u}^\delta)_\Omega + (\partial_x^k\{-2A_4\nabla_\delta p + (b_2u_{yy}, 0)^{\mathrm{T}}\}, \partial_x^k\boldsymbol{u}^\delta)_\Omega.\end{aligned}$$

Here we consider the case $k \ge 1$ only, because the case $k = 0$ can be treated more easily. Then, by Lemma 1.3.3 we obtain

$$\begin{aligned}\|\nabla_\delta\partial_x^k\boldsymbol{u}^\delta\|^2 \lesssim& E_m + |b_3\eta|_m^2 + \delta^{-1}|(h_1,h_2)|_{m-\frac{1}{2}}^2 + \delta^2|h_3|_m^2 + \delta^{-2}\|\partial_x^{k-1}\boldsymbol{f}\|^2\\ &+ |(\partial_x^k\{-2A_4\nabla_\delta p + (b_2u_{yy},0)^{\mathrm{T}}\}, \partial_x^k\boldsymbol{u}^\delta)_\Omega|.\end{aligned}$$

It is easy to see that $|b_3\eta|_m^2 + \delta^2|h_3|_m^2 \lesssim E_m$. Combining these, (3.3.39) in Lemma 3.3.10, and (3.3.48) and (3.3.49) in Lemma 3.3.12, we obtain (3.4.5). Then, taking ϵ and c_1 sufficiently small we get (3.4.2).

As for (3.4.3), in view of the definition of $\tilde{F}_m$ (see (3.3.2)), it suffices to show

$$\begin{aligned}(3.4.6)\qquad &\delta^{-1}\|\nabla_\delta\partial_x^k p\|^2 + \delta\|\nabla_\delta\partial_x^k p_x\|^2 + \delta\|\nabla_\delta\partial_x^{l-1}p_t\|^2\\ &+ \delta^6||D_x|^{k+\frac{7}{2}}\eta|_0^2 + \delta|(1+\delta|D_x|)^{\frac{5}{2}}\partial_x^k\eta_t|_0^2 \lesssim F_m + \tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m.\end{aligned}$$

Combining Lemma 3.2.2, (3.3.37) in Lemma 3.3.10, (3.3.47) in Lemma 3.3.12, and (3.3.54) and (3.3.56) in Lemma 3.3.14, we obtain

$$\delta^{-1}\|\nabla_\delta \partial_x^k p\|^2 + \delta\|\nabla_\delta \partial_x^k p_x\|^2 + \delta\|\nabla_\delta \partial_x^{l-1} p_t\|^2 \lesssim F_m + \tilde{E}_2 \tilde{F}_m + \tilde{F}_2 \tilde{E}_m. \tag{3.4.7}$$

We proceed to estimate $(\delta^2 \mathrm{W})^2 \delta^2 ||D_x|^{k+\frac{7}{2}}\eta|_0^2$. Applying $-\delta|D_x|^{k+\frac{3}{2}}$ to the second equation in (2.1.33) and taking the inner product of $(\delta^2\mathrm{W})\delta|D_x|^{k+\frac{7}{2}}\eta$ with the resulting equality, we have

$$\begin{aligned}&\Big(\frac{1}{\tan\alpha}\delta|D_x|^{k+\frac{3}{2}}\eta + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta|D_x|^{k+\frac{7}{2}}\eta, (\delta^2\mathrm{W})\delta|D_x|^{k+\frac{7}{2}}\eta\Big)_\Gamma \\ &\quad = (\delta|D_x|^{k+\frac{3}{2}}(p - \delta v_y - h_2), (\delta^2\mathrm{W})\delta|D_x|^{k+\frac{7}{2}}\eta)_\Gamma,\end{aligned}$$

which together with Lemma 1.3.3 and the second equation in (2.1.32) leads to

$$\begin{aligned}&(\delta^2\mathrm{W})^2\delta^2||D_x|^{k+\frac{7}{2}}\eta|_0^2 \\ &\lesssim \delta^2||D_x|^{\frac{1}{2}}\partial_x^k(p_x + \delta u_{xx} - h_{2x})|_0^2 \\ &\lesssim \delta\|\partial_x^k p_x\|^2 + \delta\|\partial_x^k \nabla_\delta p_x\|^2 + \delta^3\|\partial_x^k u_{xx}\|^2 + \delta^3\|\partial_x^k \nabla_\delta u_{xx}\| + \delta^2||D_x|^{k+\frac{1}{2}} h_{2x}|_0^2.\end{aligned}$$

Combining this, (3.3.38) in Lemma 3.3.10, and (3.4.7), we obtain the estimate for $(\delta^2\mathrm{W})^2\delta^2||D_x|^{k+\frac{7}{2}}\eta|_0^2$. Finally, the estimate for $\delta|(1+\delta|D_x|)^{\frac{5}{2}}\partial_x^k \eta_t|_0^2$ follows easily from the third equation in (2.1.33) and the estimate for $\delta^6||D_x|^{k+\frac{7}{2}}\eta|_0^2$. Thus, we obtain (3.4.6). Then, taking c_1 sufficiently small we get (3.4.3).

As for (3.4.4), using (3.2.13) and (3.3.55) in Lemma 3.3.14 and taking c_1 sufficiently small, we have

$$\|\nabla_\delta \partial_x^k p\|^2 \lesssim (1 + D_2)D_m + \min\{\tilde{E}_m, D_m\}(\|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2).$$

Considering the case $m = 2$ and $k = 0, 1$ in the above inequality and taking c_1 sufficiently small yield $\|\nabla_\delta p\|^2 + \|\nabla_\delta p_x\|^2 \lesssim (1 + D_2)D_2$, which together with the above estimates gives (3.4.4). The proof is complete. □

Proof of Proposition 3.4.1. Combining (3.1.31), Proposition 3.3.13, and (3.4.2) and (3.4.3) in Lemma 3.4.2 and taking ϵ and c_1 sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t) + \tilde{F}_m(t) \le C_5(\tilde{F}_2(t) + \varepsilon)E_m(t) \tag{3.4.8}$$

for a positive constant C_5 independent of δ. Note that if $\varepsilon \lesssim \delta$, then we can drop the term $C_5\varepsilon E_m(t)$ from the above inequality. Now, let us consider the case where $m = 2$. By taking c_1 sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) + \tilde{F}_2(t) \le C_6\varepsilon E_2(t)$$

for a positive constant C_6 independent of δ. Thus, Gronwall's inequality yields

$$E_2(t) + \int_0^t \exp\big(C_6\varepsilon(t-\tau)\big)\tilde{F}_2(\tau)\mathrm{d}\tau \le E_2(0)\mathrm{e}^{C_6\varepsilon t}. \tag{3.4.9}$$

In particular, we have $\int_0^t \tilde{F}_2(\tau)\mathrm{d}\tau \leq E_2(0)\mathrm{e}^{C_6\varepsilon t}$. By this, (3.4.8), and Gronwall's inequality, we see that

$$\begin{aligned} E_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau &\leq E_m(0)\exp\left(C_5\int_0^t (\tilde{F}_2(\tau)+\varepsilon)\mathrm{d}\tau\right) \\ &\leq E_m(0)\exp\left(C_5\tilde{E}_2(0)\mathrm{e}^{C_6\varepsilon t} + C_5\varepsilon t\right). \end{aligned}$$

This together with (3.4.9) and (3.4.2) in Lemma 3.4.2 gives the desired estimates in Proposition 3.4.1. The proof is complete. □

Proof of Theorem 2.2.1. Since the existence theorem of the solution locally in time is now classical, for example see [33, 23], it is sufficient to give a priori estimate of the solution. The first equation in (2.1.32) leads to

$$\delta^2\|\partial_x^k \boldsymbol{u}_t^\delta\|^2 \lesssim \|\partial_x^k \boldsymbol{u}^\delta\|^2 + \|\nabla_\delta \partial_x^k \boldsymbol{u}^\delta\|^2 + \|\Delta_\delta \partial_x^k \boldsymbol{u}^\delta\|^2 + \|\nabla_\delta \partial_x^k p\|^2 + \|\partial_x^k \boldsymbol{f}\|^2.$$

Thus, by (3.3.50) in Lemma 3.3.12 and (3.4.4) in Lemma 3.4.2, we have $\delta^2\|\partial_x^k \boldsymbol{u}_t^\delta\|^2 \lesssim (1 + D_2)^2 D_m$. By this, the third equation in (2.1.33), and the definitions of E_m and D_m (see (3.1.29) and (3.3.3)), we obtain

$$E_m(0) \leq C_8\big(1 + D_2(0)\big)^2 D_m(0) \tag{3.4.10}$$

for a positive constant C_8 independent of δ. Thus considering the case of $m = 2$ in the above inequality, taking $D_2(0)$ and T sufficiently small so that $2C_7C_8\big(1+D_2(0)\big)^2 D_2(0) \leq c_1$ and $\mathrm{e}^{C_6 T} \leq 2$, and using the first inequality in (3.4.1) in Proposition 3.4.1, we see that the solution satisfies

$$\tilde{E}_2(t) \leq c_1 \qquad \text{for} \quad 0 \leq t \leq T/\varepsilon.$$

Thus, using the second inequality in (3.4.1) in Proposition 3.4.1 together with (3.4.10), we obtain

$$\tilde{E}_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau \leq C, \tag{3.4.11}$$

where the constant C depends on R_1, W_1, W_2, α, and M but not on δ, ε, R, nor W. By the first equation in (2.1.32), we easily obtain $\delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)u_{yy}\|^2 \lesssim \tilde{F}_m$. Therefore, we obtain the desired estimate in Theorem 2.2.1. In view of the explicit form of $\tilde{E}_m$, using the second equation in (2.1.32) and Poincaré's inequality, we easily obtain (2.2.1). Moreover, in the case where $\mathbb{G} = \mathbb{T}, \varepsilon \lesssim \delta$, and $\int_0^1 \eta_0(x)\mathrm{d}x = 0$, it follows from Poincaré's inequality that $\delta E_m(t) \lesssim F_m(t)$, which yields (2.2.2). The proof is complete. □

Chapter 4

Mathematical justification for a thin film approximation

In this chapter, we will show Theorem 2.2.7. The plan of this chapter is as follows. In Section 4.1, we construct an approximate solution of the Navier–Stokes equations by using Benney's method. We first fix $\eta = \eta(x,t)$ arbitrarily and let (u,v,p) be a solution of (2.1.4)–(2.1.6) except the kinematic boundary condition

$$\eta_t + \bigl(1 - (\varepsilon\eta)^2 + \varepsilon u\bigr)\eta_x - v = 0.$$

Expanding the solution with respect to the small parameter δ as

$$\begin{cases} u = u_{(0)} + \delta u_{(1)} + \delta^2 u_{(2)} + \cdots , \\ v = v_{(0)} + \delta v_{(1)} + \delta^2 v_{(2)} + \cdots , \\ p = p_{(0)} + \delta p_{(1)} + \delta^2 p_{(2)} + \cdots \end{cases}$$

and substituting these into (2.1.4)–(2.1.6) except the kinematic boundary condition, we obtain ordinary differential equations in y together with boundary conditions for each order of δ. Solving the boundary value problems, we determine coefficients in the above expansion. Then, neglecting higher order terms in δ, we obtain an approximate solution of the Navier–Stokes equations for the arbitrary function η. We note that the approximate solution is just a polynomial in y whose coefficients depend on η and its derivatives. Substituting the approximate solution into the above kinematic boundary condition, we can recover the approximate equation for η given in Section 1.2. In Section 4.2, we derive an energy estimate for a difference between the solution of the Navier–Stokes equations and the approximate solution constructed in Section 4.1. Since the approximate solution satisfies the Navier–Stokes equations approximately, the difference satisfies linearized Navier–Stokes equations with non-homogeneous terms. Therefore, we apply the energy estimate for the solution of Navier–Stokes equations obtained in Section 3.1 to the difference. In Chapter 2, this energy estimate was the most important and essential step in order to derive the uniform estimate in δ for the solution of the Navier–Stokes equations. This energy structure allows us to derive the desired error estimates and hence Section 4.2 is the main part in this chapter. Finally, in Section 4.3 we complete error estimates. That is, we specify the arbitrary function η as the solution of each approximate equation and estimate nonlinear terms appearing in the right-hand side of the energy inequality in terms of energy functions, where we use essentially Theorem 2.2.1, that is, the uniform estimate for the solution of the Navier–Stokes equations. We remark that calculations performed in nonlinear estimates are technical because we need to carefully treat the dependence of δ in the estimates.

4.1 Approximate solution of the Navier–Stokes equations

In this section, following Benney's perturbation method [5] we will construct an approximate solution of the Navier–Stokes equations. Hereafter, we assume $\varepsilon = \delta$. By a straightforward calculation and $\tilde{\eta} = \eta + O(\delta^4)$, we can rewrite (2.1.32)–(2.1.34) as follows.

$$(4.1.1)\quad \begin{cases} \delta(u_t + \bar{u}u_x + \bar{u}_y v) + \dfrac{2}{\mathrm{R}}\delta p_x - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx} + u_{yy}) \\ \qquad\qquad = -\delta\dfrac{2}{\mathrm{R}}\eta u_{yy} + \delta^2 f_1^{(2)} + \delta^3 f_1^{(3)} \quad \text{in} \quad \Omega,\ t > 0, \\ \delta^2(v_t + \bar{u}v_x) + \dfrac{2}{\mathrm{R}}p_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx} + v_{yy}) \\ \qquad\qquad = \delta\dfrac{2}{\mathrm{R}}\eta p_y + \delta^2 f_2^{(2)} + \delta^3 f_2^{(3)} \quad \text{in} \quad \Omega,\ t > 0, \\ u_x + v_y = 0 \quad \text{in} \quad \Omega,\ t > 0, \end{cases}$$

$$(4.1.2)\quad \begin{cases} \delta^2 v_x + u_y - 2(1+\delta\eta)^2\eta = \delta^3 h_1^{(3)} \quad \text{on} \quad \Gamma,\ t > 0, \\ p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = \delta^2 h_2^{(2)} + \delta^3 h_2^{(3)} \quad \text{on} \quad \Gamma,\ t > 0, \end{cases}$$

$$(4.1.3)\quad u = v = 0 \quad \text{on} \quad \Sigma,\ t > 0,$$

$$(4.1.4)\quad \eta_t + \eta_x - v = \delta^2 h_3^{(2)} \quad \text{on} \quad \Gamma,\ t > 0,$$

where

$$(4.1.5)\quad \begin{cases} f_1^{(2)} = \dfrac{1}{\mathrm{R}}\big(3\eta^2 u_{yy} - 2\eta p_x + 2y\eta_x p_y\big) + \eta_t u + y\eta_t u_y + y^2\eta_x u \\ \qquad\quad + 2y(y-1)\eta u_x - y^2(y-2)\eta_x u_y - uu_x - vu_y + 2(2y-1)\eta v, \\ f_2^{(2)} = \dfrac{1}{\mathrm{R}}\big(-2\eta^2 p_y + 2\eta_x u_y + 2\eta u_{xy}\big), \\ h_2^{(2)} = 2\eta\eta_x + \eta_x u + \eta u_x, \end{cases}$$

$f_1^{(3)}, f_2^{(3)}, h_1^{(3)}, h_2^{(3)}$, and $h_3^{(2)}$ are functions of $O(1)$.

We proceed to construct an approximate solution of the Navier–Stokes equations following Benney [5]. Let $\eta = \eta(x,t)$ be an arbitrary function. For any $\delta \in (0,1]$, let (u,v,p) be a solution of (4.1.1)–(4.1.3) and we expand (u,v,p) as

$$\begin{cases} u = u_{(0)} + \delta u_{(1)} + \delta^2 u_{(2)} + \cdots , \\ v = v_{(0)} + \delta v_{(1)} + \delta^2 v_{(2)} + \cdots , \\ p = p_{(0)} + \delta p_{(1)} + \delta^2 p_{(2)} + \cdots \end{cases} \tag{4.1.6}$$

and substitute this into (4.1.1)–(4.1.3), we obtain a sequence of equations for each order of δ. By assuming $\mathrm{W} = O(1)$, the $O(1)$, $O(\delta)$, and $O(\delta^2)$ problems are as follows.

$$\begin{cases} u_{(0)yy} = 0, \quad p_{(0)y} = 0, \quad u_{(0)x} + v_{(0)y} = 0 & \text{in } \Omega, \\ u_{(0)y} = 2\eta, \quad p_{(0)} = \dfrac{1}{\tan\alpha}\eta & \text{on } \Gamma, \\ u_{(0)} = v_{(0)} = 0 & \text{on } \Sigma, \end{cases} \tag{4.1.7}$$

$$\begin{cases} u_{(1)yy} = \mathrm{R}(u_{(0)t} + (2y-y^2)u_{(0)x} + 2(1-y)v_{(0)}) + 2p_{(0)x} + 2\eta u_{(0)yy} & \text{in } \Omega, \\ 2p_{(1)y} = v_{(0)yy} + 2\eta p_{(0)y}, \quad u_{(1)x} + v_{(1)y} = 0 & \text{in } \Omega, \\ u_{(1)y} = 4\eta^2, \quad p_{(1)} = -u_{(0)x} & \text{on } \Gamma, \\ u_{(1)} = v_{(1)} = 0 & \text{on } \Sigma, \end{cases} \tag{4.1.8}$$

$$\begin{cases} u_{(2)yy} = \mathrm{R}(u_{(1)t} + (2y-y^2)u_{(1)x} + 2(1-y)v_{(1)}) & \\ \qquad\qquad + 2p_{(1)x} + 2\eta u_{(1)yy} - u_{(0)xx} - \mathrm{R}f_1^{(2)}(\eta, u_{(0)}, v_{(0)}, p_{(0)}) & \text{in } \Omega, \\ 2p_{(2)y} = v_{(1)yy} + 2\eta p_{(1)y} & \\ \qquad\qquad - \mathrm{R}\big(v_{(0)t} + (2y-y^2)v_{(0)x}\big) + \mathrm{R}f_2^{(2)}(\eta, u_{(0)}, v_{(0)}, p_{(0)}) & \text{in } \Omega, \\ u_{(2)x} + v_{(2)y} = 0 & \text{in } \Omega, \\ u_{(2)y} = -v_{(0)x} + 2\eta^3, \quad p_{(2)} = -u_{(1)x} + h_2^{(2)}(\eta, u_{(0)}) - \dfrac{\mathrm{W}}{\sin\alpha}\eta_{xx} & \text{on } \Gamma, \\ u_{(2)} = v_{(2)} = 0 & \text{on } \Sigma. \end{cases} \tag{4.1.9}$$

Solving the above boundary value problem for the ordinary differential equations, we have

$$\begin{cases} u_{(0)} = 2y\eta, \\ v_{(0)} = -y^2\eta_x, \\ p_{(0)} = \dfrac{1}{\tan\alpha}\eta, \end{cases} \tag{4.1.10}$$

$$
(4.1.11)\qquad \begin{cases} u_{(1)} = \left(\dfrac{1}{3}y^3 - y\right) \mathrm{R}\eta_t + \left\{(y^2-2y)\dfrac{1}{\tan\alpha} + \left(\dfrac{1}{6}y^4 - \dfrac{2}{3}y\right)\mathrm{R}\right\}\eta_x + 4y\eta^2, \\ v_{(1)} = \left(-\dfrac{1}{12}y^4 + \dfrac{1}{2}y^2\right)\mathrm{R}\eta_{xt} \\ \qquad + \left\{\big(-\dfrac{1}{3}y^3 + y^2\big)\dfrac{1}{\tan\alpha} + \left(-\dfrac{1}{30}y^5 + \dfrac{1}{3}y^2\right)\mathrm{R}\right\}\eta_{xx} - 4y^2\eta\eta_x, \\ p_{(1)} = -(1+y)\eta_x, \end{cases}
$$

$$
(4.1.12)\qquad \begin{cases} u_{(2)} = \left(\dfrac{1}{60}y^5 - \dfrac{1}{6}y^3 + \dfrac{5}{12}y\right)\mathrm{R}^2\eta_{tt} \\ \qquad +\Big\{\left(\dfrac{1}{12}y^4 - \dfrac{1}{3}y^3 + \dfrac{2}{3}y\right)\dfrac{\mathrm{R}}{\tan\alpha} \\ \qquad\quad + \left(-\dfrac{1}{252}y^7 + \dfrac{1}{45}y^6 - \dfrac{1}{12}y^4 - \dfrac{1}{9}y^3 + \dfrac{101}{180}y\right)\mathrm{R}^2\Big\}\eta_{xt} \\ \qquad +\Big\{\left(-\dfrac{2}{3}y^3 - y^2 + 5y\right) + \left(-\dfrac{1}{90}y^6 + \dfrac{1}{15}y^5 - \dfrac{1}{6}y^4 + \dfrac{2}{5}y\right)\dfrac{\mathrm{R}}{\tan\alpha} \\ \qquad\quad + \left(-\dfrac{1}{560}y^8 + \dfrac{2}{315}y^7 - \dfrac{1}{18}y^4 + \dfrac{121}{630}y\right)\mathrm{R}^2\Big\}\eta_{xx} \\ \qquad +2y\eta^3 + \mathrm{R}\left(\dfrac{4}{3}y^3 - 4y\right)\eta\eta_t + \left\{\mathrm{R}(y^4-4y) + (3y^2-6y)\dfrac{1}{\tan\alpha}\right\}\eta\eta_x, \\ v_{(2)} = \left(-\dfrac{1}{360}y^6 + \dfrac{1}{24}y^4 - \dfrac{5}{24}y^2\right)\mathrm{R}^2\eta_{xtt} \\ \qquad +\Big\{\left(-\dfrac{1}{60}y^5 + \dfrac{1}{12}y^4 - \dfrac{1}{3}y^2\right)\dfrac{\mathrm{R}}{\tan\alpha} \\ \qquad\quad + \left(\dfrac{1}{2016}y^8 - \dfrac{1}{315}y^7 + \dfrac{1}{60}y^5 + \dfrac{1}{36}y^4 - \dfrac{101}{360}y^2\right)\mathrm{R}^2\Big\}\eta_{xxt} \\ \qquad +\Big\{\left(\dfrac{1}{6}y^4 + \dfrac{1}{3}y^3 - \dfrac{5}{2}y^2\right) + \left(\dfrac{1}{630}y^7 - \dfrac{1}{90}y^6 + \dfrac{1}{30}y^5 - \dfrac{1}{5}y^2\right)\dfrac{\mathrm{R}}{\tan\alpha} \\ \qquad\quad + \left(\dfrac{1}{5040}y^9 - \dfrac{1}{1260}y^8 + \dfrac{1}{90}y^5 - \dfrac{121}{1260}y^2\right)\mathrm{R}^2\Big\}\eta_{xxx} \\ \qquad -3y^2\eta^2\eta_x + \mathrm{R}\left(-\dfrac{1}{3}y^4 + 2y^2\right)(\eta_x\eta_t + \eta\eta_{tx}) \\ \qquad +\left\{\mathrm{R}\big(-\dfrac{1}{5}y^5 + 2y^2\big) + (-y^3+3y^2)\dfrac{1}{\tan\alpha}\right\}(\eta_x^2 + \eta\eta_{xx}), \\ p_{(2)} = \left(\dfrac{1}{2}y + \dfrac{1}{6}\right)\mathrm{R}\eta_{xt} \\ \qquad +\Big\{-\dfrac{\mathrm{W}}{\sin\alpha} + \left(-\dfrac{1}{2}y^2 + y + \dfrac{1}{2}\right)\dfrac{1}{\tan\alpha} \\ \qquad\quad + \left(-\dfrac{1}{10}y^5 + \dfrac{1}{6}y^4 + \dfrac{1}{3}y + \dfrac{1}{10}\right)\mathrm{R}\Big\}\eta_{xx} + \big\{\mathrm{R}(4y-4) - 5y + 3\big\}\eta\eta_x. \end{cases}
$$

Note that $u_{(0)}, v_{(0)}, p_{(0)}, \dots$ are just polynomials in y whose coefficients depend on η. Then, neglecting higher order terms in δ, we obtain the following approximate solution of the Navier–Stokes equations for an arbitrary function η.

$$\begin{cases} u^{\mathrm{app}}(y;\eta) = u_{(0)} + \delta u_{(1)} + \delta^2 u_{(2)}, \\ v^{\mathrm{app}}(y;\eta) = v_{(0)} + \delta v_{(1)} + \delta^2 v_{(2)}, \\ p^{\mathrm{app}}(y;\eta) = p_{(0)} + \delta p_{(1)} + \delta^2 p_{(2)}. \end{cases} \tag{4.1.13}$$

In order to make the approximate solution satisfy the kinematic boundary condition (4.1.4), η is required to satisfy the following equation.

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x = O(\delta^3),$$

where

$$\begin{cases} C_1 = 2 + \dfrac{32}{63}\mathrm{R}^2 - \dfrac{40}{63}\dfrac{\mathrm{R}}{\tan\alpha}, \\ C_2 = \dfrac{16}{5}\mathrm{R} - \dfrac{2}{\tan\alpha} \end{cases} \tag{4.1.14}$$

and the above equation is the approximate equation given in Section 1.2. Here, (4.1.14) is the explicit form of the coefficients appearing in (1.2.1).

Thus far we have assumed $\mathrm{W} = O(1)$. Taking into account that W is contained only in the second equation in (4.1.2) and modifying the $O(\delta)$ problem under the assumption $\mathrm{W} = O(\delta^{-1})$, we see that (u_0^I, v_0^I, p_0^I) and (u_1^I, v_1^I, p_1^I), which are defined by

$$\begin{cases} u_0^I(y;\eta) := u_{(0)}, \quad v_0^I(y;\eta) := v_{(0)}, \quad p_0^I(y;\eta) := p_{(0)}, \\ u_1^I(y;\eta) := u_{(1)}, \quad v_1^I(y;\eta) := v_{(1)}, \quad p_1^I(y;\eta) := p_{(1)} - \frac{\delta \mathrm{W}}{\sin\alpha}\eta_{xx}, \end{cases} \tag{4.1.15}$$

are the solutions of the problem. Putting $v = v_0^I + \delta v_1^I$ and substituting this into (4.1.4), we obtain the approximate equation

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} = O(\delta^2).$$

Similarly, modifying the $O(1)$ and $O(\delta)$ problems under the assumption $\mathrm{W} = O(\delta^{-2})$ and putting

$$\begin{cases} u_0^{II} := u_{(0)}, \quad v_0^{II} := v_{(0)}, \quad p_0^{II} := p_{(0)} - \dfrac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \\ u_1^{II} := u_{(1)} - \dfrac{\delta^2\mathrm{W}}{\sin\alpha}(y^2 - 2y)\eta_{xxx}, \\ v_1^{II} := v_{(1)} + \dfrac{\delta^2\mathrm{W}}{\sin\alpha}\big(\tfrac{1}{3}y^3 - y^2\big)\eta_{xxxx}, \\ p_1^{II} := p_{(1)}, \end{cases} \tag{4.1.16}$$

we obtain the approximate equation

$$\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta\eta_{xxxx} = O(\delta^2).$$

Moreover, putting

$$(4.1.17)\qquad \begin{cases} u_0^{IV} := u_{(0)}, \quad v_0^{IV} := v_{(0)}, \quad p_0^{IV} := p_{(0)}, \\ u_1^{IV} := u_{(1)}, \quad v_1^{IV} := v_{(1)}, \quad p_1^{IV} := p_{(1)} - \dfrac{\delta\mathrm{W}}{\sin\alpha}\eta_{xx}, \\ u_2^{IV} := u_{(2)} - \dfrac{\delta\mathrm{W}}{\sin\alpha}(y^2 - 2y)\eta_{xxx}, \\ v_2^{IV} := v_{(2)} + \dfrac{\delta\mathrm{W}}{\sin\alpha}\big(\dfrac{1}{3}y^3 - y^2\big)\eta_{xxxx}, \\ p_2^{IV} := p_{(2)} + \dfrac{\mathrm{W}}{\sin\alpha}\eta_{xx} \end{cases}$$

and $v = v_0^{IV} + \delta v_1^{IV} + \delta^2 v_2^{IV}$ and substituting this into (4.1.4), we obtain the approximate equation

$$\begin{aligned} &\eta_t + 2\eta_x + 4\varepsilon\eta\eta_x - \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx} \\ &\quad + C_1\delta^2\eta_{xxx} + C_2\varepsilon\delta(\eta\eta_{xx} + \eta_x^2) + 2\varepsilon^2\eta^2\eta_x + \frac{2}{3}\frac{\mathrm{W}_2}{\sin\alpha}\delta^2\eta_{xxxx} = O(\delta^3) \end{aligned}$$

under the assumption $\mathrm{W} = O(\delta^{-1})$.

4.2 Energy estimate

In this section, we will derive an energy estimate, which is most important step in order to obtain error estimates. Using the arbitrary function η and the approximate solution $(u^{\mathrm{app}}, v^{\mathrm{app}}, p^{\mathrm{app}}) = (u^{\mathrm{app}}(y;\eta), v^{\mathrm{app}}(y;\eta), p^{\mathrm{app}}(y;\eta))$, we define $\psi_1, \psi_2, \phi_1, \phi_2$, and ϕ_3 by the following equalities.

$$(4.2.1)\qquad \begin{cases} \psi_1(y;\eta) := \dfrac{1}{\delta^3}\Big\{\delta(u_t^{\mathrm{app}} + \bar{u}u_x^{\mathrm{app}} + \bar{u}_y v^{\mathrm{app}}) + \dfrac{2}{\mathrm{R}}\delta p_x^{\mathrm{app}} \\ \qquad\qquad - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx}^{\mathrm{app}} + u_{yy}^{\mathrm{app}}) - \delta f_1^{(1)}(\eta, u^{\mathrm{app}}, v^{\mathrm{app}}, p^{\mathrm{app}})\Big\}, \\ \psi_2(y;\eta) := \dfrac{1}{\delta^3}\Big\{\delta^2(v_t^{\mathrm{app}} + \bar{u}v_x^{\mathrm{app}}) + \dfrac{2}{\mathrm{R}}p_y^{\mathrm{app}} \\ \qquad\qquad - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx}^{\mathrm{app}} + v_{yy}^{\mathrm{app}}) - \delta f_2^{(1)}(\eta, u^{\mathrm{app}}, p^{\mathrm{app}})\Big\}, \\ \phi_1(\eta) := \dfrac{1}{\delta^3}\{\delta^2 v_x^{\mathrm{app}} + u_y^{\mathrm{app}} - 2(1+\delta\eta)^2\eta\}|_{y=1}, \\ \phi_2(\eta) := \dfrac{1}{\delta^3}\Big\{p^{\mathrm{app}} - \delta v_y^{\mathrm{app}} - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} - \delta^2 h_2^{(2)}(\eta, u^{\mathrm{app}})\Big\}\Big|_{y=1}, \\ \phi_3(\eta) := \dfrac{1}{\delta^3}\{\eta_t + \eta_x - v^{\mathrm{app}} - \delta^2 h_3(\eta)\}|_{y=1}, \end{cases}$$

where

$$f_1^{(1)} = -\frac{2}{\mathrm{R}}\eta u_{yy}^{\mathrm{app}} + \delta f_1^{(2)}, \quad f_2^{(1)} = \frac{2}{\mathrm{R}}\eta p_y^{\mathrm{app}} + \delta f_2^{(2)}. \tag{4.2.2}$$

Here, $\psi_1, \psi_2, \phi_1, \phi_2$, and ϕ_3 measure how much $(\eta, u^{\mathrm{app}}, v^{\mathrm{app}}, p^{\mathrm{app}})$ fails to be the solution of the Navier–Stokes equations and in the next section we will give explicit forms of these functions (see (4.3.3)). Then, by (4.2.1) and the definition of the approximate solution constructed in Section 4.1, it satisfies the following equations.

$$\begin{cases} \delta(u_t^{\mathrm{app}} + \bar{u}u_x^{\mathrm{app}} + \bar{u}_y v^{\mathrm{app}}) + \dfrac{2}{\mathrm{R}}\delta p_x^{\mathrm{app}} - \dfrac{1}{\mathrm{R}}(\delta^2 u_{xx}^{\mathrm{app}} + u_{yy}^{\mathrm{app}}) \\ \qquad = \delta f_1^{(1)}(\eta, u^{\mathrm{app}}, v^{\mathrm{app}}, p^{\mathrm{app}}) + \delta^3\psi_1(y;\eta) \quad \text{in} \quad \Omega,\ t>0, \\ \delta^2(v_t^{\mathrm{app}} + \bar{u}v_x^{\mathrm{app}}) + \dfrac{2}{\mathrm{R}}p_y^{\mathrm{app}} - \dfrac{1}{\mathrm{R}}\delta(\delta^2 v_{xx}^{\mathrm{app}} + v_{yy}^{\mathrm{app}}) \\ \qquad = \delta f_2^{(1)}(\eta, u^{\mathrm{app}}, p^{\mathrm{app}}) + \delta^3\psi_2(y;\eta) \quad \text{in} \quad \Omega,\ t>0, \\ u_x^{\mathrm{app}} + v_y^{\mathrm{app}} = 0 \quad \text{in} \quad \Omega,\ t>0, \end{cases} \tag{4.2.3}$$

$$\begin{cases} \delta^2 v_x^{\mathrm{app}} + u_y^{\mathrm{app}} - 2(1+\delta\eta)^2\eta = \delta^3\phi_1(\eta) \quad \text{on} \quad \Gamma,\ t>0, \\ p^{\mathrm{app}} - \delta v_y^{\mathrm{app}} - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} = \delta^2 h_2^{(2)}(\eta, u^{\mathrm{app}}) + \delta^3\phi_2(\eta) \quad \text{on} \quad \Gamma,\ t>0, \\ \eta_t + \eta_x - v^{\mathrm{app}} = \delta^2 h_3(\eta) + \delta^3\phi_3(\eta) \quad \text{on} \quad \Gamma,\ t>0, \end{cases} \tag{4.2.4}$$

$$u^{\mathrm{app}} = v^{\mathrm{app}} = 0 \quad \text{on} \quad \Sigma,\ t>0. \tag{4.2.5}$$

In other words, the approximate solution satisfies the Navier–Stokes equations approximately with reminder terms $\psi_1, \psi_2, \phi_1, \phi_2$, and ϕ_3. Let $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ be the solution of (4.1.1)–(4.1.4) and we set

$$H := \eta^\delta - \eta, \quad U := u^\delta - u^{\mathrm{app}}, \quad V := v^\delta - v^{\mathrm{app}}, \quad P := p^\delta - p^{\mathrm{app}}.$$

Taking the difference between (4.1.1)–(4.1.4) and (4.2.3)–(4.2.5), we have

$$\begin{cases} \delta(U_t + \bar{u}U_x + \bar{u}_y V) + \dfrac{2}{\mathrm{R}}\delta P_x - \dfrac{1}{\mathrm{R}}(\delta^2 U_{xx} + U_{yy}) \\ \qquad = F_1 + \delta^3 f_1^{(3)}(\eta^\delta, u^\delta, v^\delta, p^\delta) - \delta^3\psi_1(y;\eta) \quad \text{in} \quad \Omega,\ t>0, \\ \delta^2(V_t + \bar{u}V_x) + \dfrac{2}{\mathrm{R}}P_y - \dfrac{1}{\mathrm{R}}\delta(\delta^2 V_{xx} + V_{yy}) \\ \qquad = F_2 + \delta^3 f_2^{(3)}(\eta^\delta, u^\delta, v^\delta, p^\delta) - \delta^3\psi_2(y;\eta) \quad \text{in} \quad \Omega,\ t>0, \\ U_x + V_y = 0 \quad \text{in} \quad \Omega,\ t>0, \end{cases} \tag{4.2.6}$$

$$
(4.2.7) \qquad \begin{cases} \delta^2 V_x + U_y - \big(2 + b(\eta^\delta, \eta)\big) H = \delta^3 h_1^{(3)}(\eta^\delta, u^\delta, v^\delta) - \delta^3 \phi_1(\eta) & \text{on} \quad \Gamma, \ t > 0, \\ P - \delta V_y - \dfrac{1}{\tan\alpha} H + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha} H_{xx} & \\ \qquad\qquad = G_2 + \delta^3 h_2^{(3)}(\eta^\delta, u^\delta, v^\delta) - \delta^3 \phi_2(\eta) & \text{on} \quad \Gamma, \ t > 0, \\ H_t + H_x - V = G_3 - \delta^3 \phi_3(\eta) & \text{on} \quad \Gamma, \ t > 0, \end{cases}
$$

$$
(4.2.8) \qquad U = V = 0 \quad \text{on} \quad \Sigma, \ t > 0,
$$

where

$$
(4.2.9) \qquad \begin{cases} F_1 = \delta\big(f_1^{(1)}(\eta^\delta, u^\delta, v^\delta, p^\delta) - f_1^{(1)}(\eta, u^{\mathrm{app}}, v^{\mathrm{app}}, p^{\mathrm{app}})\big), \\ F_2 = \delta\big(f_2^{(1)}(\eta^\delta, u^\delta, p^\delta) - f_2^{(1)}(\eta, u^{\mathrm{app}}, p^{\mathrm{app}})\big), \\ b = 2\delta\big(\delta(\eta^\delta)^2 + (2 + \delta\eta)\eta^\delta + \delta\eta^2 + 2\eta\big), \\ G_2 = \delta^2\big(h_2^{(2)}(\eta^\delta, u^\delta, v^\delta) - h_2^{(2)}(\eta, u^{\mathrm{app}}, v^{\mathrm{app}})\big), \\ G_3 = \delta^2\big(h_3^{(2)}(\eta^\delta) - h_3^{(2)}(\eta)\big). \end{cases}
$$

Note that (4.2.6)–(4.2.8) are linearized Navier–Stokes equations with non-homogeneous terms. For convenience, we set

$$
\boldsymbol{U} := (U, \delta V)^{\mathrm{T}}, \quad \boldsymbol{F} := (F_1, F_2)^{\mathrm{T}}, \quad \boldsymbol{f}^{(3)} := (f_1^{(3)}, f_2^{(3)})^{\mathrm{T}}, \quad \boldsymbol{\psi} := (\psi_1, \psi_2)^{\mathrm{T}}.
$$

We proceed to derive an energy estimate to (4.2.6)–(4.2.8) following Section 3.1. In view of the energies obtained in Section 3.1 (see (3.1.6)–(3.1.8) and (3.1.24)), we put

$$
\begin{aligned}
\mathscr{E}_0(H, \boldsymbol{U}) := {} & \delta^2 \|V\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|H|_0^2 + \frac{\delta^2 \mathrm{W}}{\sin\alpha}|H_x|_0^2\right) \\
& + \beta_1\left\{\delta^2\|\boldsymbol{U}_x\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|H_x|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|H_{xx}|_0^2\right)\right\} \\
& + \beta_2\left\{\delta^4\|\boldsymbol{U}_{xx}\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^4|H_{xx}|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^4|H_{xxx}|_0^2\right)\right\} \\
& + \beta_3\left\{\delta^2\|\boldsymbol{U}_t\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|H_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|H_{tx}|_0^2\right)\right\}, \\
\mathscr{F}_0(H, \boldsymbol{U}, P) := {} & \delta\|\boldsymbol{U}_x\|^2 + \delta\|\partial_y^{-1}P_x\|^2 + \delta|H_x|_0^2 + \delta^3\mathrm{W}|H_{xx}|_0^2 + \delta^5\mathrm{W}^2|H_{xxx}|_0^2 \\
& + \delta\|\nabla_\delta \boldsymbol{U}_x\|^2 + \delta^3\|\nabla_\delta\boldsymbol{U}_{xx}\|^2 + \delta\|\nabla_\delta\boldsymbol{U}_t\|^2.
\end{aligned}
$$

Here, β_1, β_2, and β_3 are appropriate positive constants (see (3.1.28)). Integrating by parts and using the third equation in (4.2.7) and Poincaré's inequality, we see that for any $\epsilon > 0$

there exists a positive constant C_ϵ such that

$$\begin{aligned}
&\delta^3|(\{\boldsymbol{F}+\delta^3\boldsymbol{f}^{(3)}-\delta^3\boldsymbol{\psi}\}_{xx},\boldsymbol{U}_{xx})_\Omega| \le \epsilon\delta^5\|\boldsymbol{U}_{xxx}\|^2 + C_\epsilon\delta(\|\boldsymbol{F}_x\|^2+\delta^6\|\boldsymbol{f}_x^{(3)}\|^2+\delta^6\|\boldsymbol{\psi}_x\|^2),\\
&|(H,(bH)_x)_\Gamma| \le \epsilon\delta|H_x|_0^2 + C_\epsilon\delta^{-1}|(bH)_x|_0^2,\\
&\delta^2\mathrm{W}|(H_{xx},(bH)_x)_\Gamma| \le \epsilon\delta^3\mathrm{W}|H_{xx}|_0^2 + C_\epsilon\delta\mathrm{W}|(bH)_x|_0^2,\\
&\delta^2\mathrm{W}|(H_{xx},G_3-\delta^3\phi_3)_\Gamma| \le \epsilon\delta^3\mathrm{W}|H_{xx}|_0^2 + C_\epsilon\delta\mathrm{W}(|G_3|_0^2+\delta^6|\phi_3|_0^2),\\
&\delta^6\mathrm{W}|(H_{xxxx},\delta^3\phi_{3xx})_\Gamma| \le \epsilon\delta^5\mathrm{W}^2|H_{xxx}|_0^2 + C_\epsilon\delta^{13}|\phi_{3xxx}|_0^2,\\
&\delta^4\mathrm{W}|(H_{xxt},G_{3t}-\delta^3\phi_{3t})_\Gamma| \le \epsilon(\delta^5\mathrm{W}^2|H_{xxx}|_0^2+\delta^5\|U_{xxx}\|_0^2)\\
&\qquad\qquad + C_\epsilon(1+\mathrm{W}^2)\delta^3(|G_{3t}|_0^2+\delta^6|\phi_{3t}|_0^2)+\delta^5(|G_{3xx}|_0^2+\delta^6|\phi_{3xx}|_0^2).
\end{aligned}$$

Here, we used the inequality $|V(\cdot,1)|_0 = |V(\cdot,1)-V(\cdot,0)|_0 \le \|V_y\| = \|U_x\|$ thanks to the third equation in (4.2.6) and the second equation in (4.2.8). In the following, we use frequently this type of inequality without any comment. Taking into account the above inequality and (3.1.27), we need to estimate the following quantities.

$$\begin{aligned}
(4.2.10)\quad \mathscr{N}_0^1(Z_1) := {}& (\delta\mathrm{W}+\delta^{-1})|(bH)_x|_0^2 + \delta^3|(bH)_{xx}|_0^2 + \delta|(bH)_t|_0^2\\
&+\delta^{-1}|G_2|_0^2+\delta|G_{2x}|_0^2+\delta^2||D_x|^{\frac12}G_{2x}|_0^2+\delta|(G_{2t},\delta V_t)_\Gamma|\\
&+\delta\mathrm{W}|G_3|_0^2+\delta^3|G_{3x}|_0^2+\delta^5|G_{3xx}|_0^2+\delta^3\mathrm{W}^2|G_{3t}|_0^2+\delta^6\mathrm{W}|(H_{xxxx},G_{3xx})_\Gamma|\\
&+\delta^{-1}\|\boldsymbol{F}\|^2+\delta\|\boldsymbol{F}_x\|^2+\delta|(\boldsymbol{F}_t,\boldsymbol{U}_t)_\Omega|,\\
(4.2.11)\quad \mathscr{N}_0^2(Z_2) := {}& \delta^5|h_1^{(3)}|_0^2+\delta^7|h_{1x}^{(3)}|_0^2+\delta^8||D_x|^{\frac12}h_{1x}^{(3)}|_0^2+\delta^4|(h_{1t}^{(3)},U_t)_\Gamma|\\
&+\delta^5|h_2^{(3)}|_0^2+\delta^7|h_{2x}^{(3)}|_0^2+\delta^8||D_x|^{\frac12}h_{2x}^{(3)}|_0^2+\delta^4|(h_{2t}^{(3)},\delta V_t)_\Gamma|\\
&+\delta^5\|\boldsymbol{f}^{(3)}\|^2+\delta^7\|\boldsymbol{f}_x^{(3)}\|^2+\delta^4|(\boldsymbol{f}_t^{(3)},\boldsymbol{U}_t)_\Omega|,\\
(4.2.12)\quad \mathscr{N}_0^3(Z_3) := {}& \delta^5|\phi_1|_0^2+\delta^7|\phi_{1x}|_0^2+\delta^8||D_x|^{\frac12}\phi_{1x}|_0^2+\delta^7|\phi_{1t}|_0^2+\delta^5|\phi_2|_0^2+\delta^7|\phi_{2x}|_0^2\\
&+\delta^8||D_x|^{\frac12}\phi_{2x}|_0^2+\delta^7|\phi_{2t}|_0^2+\delta^7\mathrm{W}|\phi_3|_0^2+\delta^9|\phi_{3x}|_0^2+\delta^{11}|\phi_{3xx}|_0^2\\
&+\delta^{13}|\phi_{3xxx}|_0^2+\delta^9\mathrm{W}^2|\phi_{3t}|_0^2+\delta^5\|\boldsymbol{\psi}\|^2+\delta^7\|\boldsymbol{\psi}_x\|^2+\delta^7\|\boldsymbol{\psi}_t\|^2,
\end{aligned}$$

where

$$Z_1=(H,\boldsymbol{U},bH,G_2,G_3,\boldsymbol{F}),\quad Z_2=(\boldsymbol{U},h_1^{(3)},h_2^{(3)},h_3^{(2)},\boldsymbol{f}^{(3)}),\quad Z_3=(\phi_1,\phi_2,\phi_3,\boldsymbol{\psi}).$$

For an integer $m\ge 2$, we set

$$(4.2.13)\qquad \mathscr{E}_m(H,\boldsymbol{U}) := \sum_{k=0}^m \mathscr{E}_0(\partial_x^k H,\partial_x^k\boldsymbol{U}),\quad \mathscr{F}_m(H,\boldsymbol{U},P) := \sum_{k=0}^m \mathscr{F}_0(\partial_x^k H,\partial_x^k\boldsymbol{U},\partial_x^k P),$$

$$(4.2.14)\qquad \mathscr{N}_m^1(H,\boldsymbol{U},P;\eta) := \sum_{k=0}^m \bigl\{\mathscr{N}_0^1(\partial_x^k Z_1)+|(\partial_x^k H,\partial_x^k G_3)_\Gamma|\bigr\},$$

$$(4.2.15)\qquad \mathscr{N}_m^2(\boldsymbol{U}) := \sum_{k=0}^m \mathscr{N}_0^2(\partial_x^k Z_2),$$

$$(4.2.16)\qquad \mathscr{N}_m^3(H;\eta) := \sum_{k=0}^m \bigl\{\mathscr{N}_0^3(\partial_x^k Z_3)+|(\partial_x^k H,\delta^3\partial_x^k\phi_3)_\Gamma|\bigr\}.$$

Here, the terms $\sum_{k=0}^m |(\partial_x^k H, \partial_x^k G_3)_\Gamma|$ and $\sum_{k=0}^m |(\partial_x^k H, \delta^3\partial_x^k \phi_3)_\Gamma|$ come from (3.1.30). Applying ∂_x^k to (4.2.6)–(4.2.8), using Proposition 3.1.4, and adding the resulting inequalities for $0 \le k \le m$, we obtain the following lemma.

Lemma 4.2.1. *There exist a small positive constants* R_0 *and* α_0 *such that if* $0 < \mathrm{R}_1 \le \mathrm{R} \le \mathrm{R}_0$, $\mathrm{W}_1 \le \mathrm{W}$, *and* $0 < \alpha \le \alpha_0$, *then the solution* (H, U, V, P) *of* (4.2.6)–(4.2.8) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m + \mathscr{F}_m \le C(\mathscr{N}_m^1 + \mathscr{N}_m^2 + \mathscr{N}_m^3), \tag{4.2.17}$$

where the constant C *is independent of* δ, R, *and* W.

For later use, we modify the energy and dissipation functions $\mathscr{E}_m$ and $\mathscr{F}_m$ as

$$\tilde{\mathscr{E}}_m(H, \boldsymbol{U}) := \mathscr{E}_m(H, \boldsymbol{U}) + \|(1 + |D_x|)^m U\|^2 + \|(1 + |D_x|)^m U_y\|^2, \tag{4.2.18}$$

$$\begin{aligned}\tilde{\mathscr{F}}_m(H, \boldsymbol{U}, P) := {}& \mathscr{F}_m(H, \boldsymbol{U}, P) + \delta|(1 + \delta|D_x|)^{\frac{5}{2}} H_t|_m^2 + (\delta^2\mathrm{W})^2\delta^2||D_x|^{\frac{7}{2}} H|_m^2 \\ & + \delta^{-1}\|(1 + |D_x|)^m(1 + \delta|D_x|)(\nabla_\delta P, U_{yy})\|^2 \\ & + \delta\|(1 + |D_x|)^{m-1}\nabla_\delta P_t\|^2\end{aligned} \tag{4.2.19}$$

and note that $\tilde{E}_m = \tilde{\mathscr{E}}_m(\eta^\delta, \boldsymbol{u}^\delta)$ and $\tilde{F}_m = \tilde{\mathscr{F}}_m(\eta^\delta, \boldsymbol{u}^\delta, p^\delta)$. We also introduce another energy function $\mathscr{D}_m$ by

$$\begin{aligned}\mathscr{D}_m(H, \boldsymbol{U}) := {}& |(1 + \delta|D_x|)^2 H|_m^2 + \delta^2\|(1 + |D_x|)^m V\|^2 + \delta^2\|(1 + |D_x|)^m \boldsymbol{U}_x\|^2 \\ & + \|(1 + |D_x|)^m D_\delta^2 \boldsymbol{U}\|^2 + (\delta^2\mathrm{W})^2|(1 + \delta|D_x|)H_x|_{m+1}^2 \\ & + \sqrt{\delta^2\mathrm{W}}\|(1 + |D_x|)^m \delta V_{xy}\|^2,\end{aligned} \tag{4.2.20}$$

which does not include any time derivatives. By using Theorem 2.2.1 and Proposition 3.4.1, the following uniform estimate holds.

Proposition 4.2.2. *There exist a small positive constants* R_0 *and* α_0 *such that the following statement holds: Let* m *be an integer satisfying* $m \ge 2$, $0 < \mathrm{R}_1 \le \mathrm{R}_0$, $0 < \mathrm{W}_1 \le \mathrm{W}_2$, *and* $0 < \alpha \le \alpha_0$. *There exists small positive constant* c_0 *such that if the initial data* (η_0, u_0, v_0) *and the parameters* δ, ε, R, *and* W *satisfy Assumption 2.2.5 and* $\mathrm{W} \le \delta^{-2}\mathrm{W}_2$, *then the solution* $(\eta^\delta, u^\delta, v^\delta, p^\delta)$ *of* (2.1.32)–(2.1.35) *satisfies*

$$\tilde{E}_2(t) \le c_0, \quad \sup_{t \ge 0} \tilde{E}_{m+1}(t) + \int_0^\infty \tilde{F}_{m+1}(t)\mathrm{d}t \le C, \quad \tilde{E}_{m+1}(t) \le C\mathrm{e}^{-c\delta t}.$$

Here, positive constants C *and* c *depend on* $\mathrm{R}_1, \mathrm{W}_1, \mathrm{W}_2, \alpha$, *and* M *but are independent of* δ, ε, R, *and* W.

Moreover, we easily obtain the following lemma.,

Lemma 4.2.3. *Let* $\mathbb{G} = \mathbb{T}$, $\alpha > 0$, $0 < \mathrm{R}_1 \le \mathrm{R} < \mathrm{R}_c$. *There exists a small positive constant* c_1 *such that if* $s \ge 2$ *and* $|\eta_0|_2^2 \le c_1$, *then the problems* (1.2.3)–(1.2.6) *under the initial* *condition* $\zeta|_{\tau=0} = \eta_0$ *have unique solutions* ζ^I, ζ^{II}, ζ^{III}, *and* ζ^{IV}, *respectively, which satisfy*

$$\sup_{\tau\geq 0} |\zeta^I(\tau)|_s^2 + \int_0^\infty |\zeta_x^I(\tau)|_s^2 \mathrm{d}\tau \leq C|\eta_0|_s^2, \quad |\zeta^I(\tau)|_s^2 \leq C|\eta_0|_s^2 \mathrm{e}^{-c\delta t},$$
$$\sup_{\tau\geq 0} |\zeta^{II}(\tau)|_s^2 + \int_0^\infty \big(|\zeta_x^{II}(\tau)|_s^2 + |\zeta_{xx}^{II}(\tau)|_s^2\big) \mathrm{d}\tau \leq C|\eta_0|_s^2, \quad |\zeta^{II}(\tau)|_s^2 \leq C|\eta_0|_s^2 \mathrm{e}^{-c\delta t},$$
$$\sup_{\tau\geq 0} |\zeta^{III}(\tau)|_s^2 + \int_0^\infty |\zeta_x^{III}(\tau)|_s^2 \mathrm{d}\tau \leq C|\eta_0|_s^2, \quad |\zeta^{III}(\tau)|_s^2 \leq C|\eta_0|_s^2 \mathrm{e}^{-c\delta t},$$
$$\sup_{\tau\geq 0} |\zeta^{IV}(\tau)|_s^2 + \int_0^\infty \big(|\zeta_x^{IV}(\tau)|_s^2 + \delta|\zeta_{xx}^{IV}(\tau)|_s^2\big) \mathrm{d}\tau \leq C|\eta_0|_s^2, \quad |\zeta^{IV}(\tau)|_s^2 \leq C|\eta_0|_s^2 \mathrm{e}^{-c\delta t}.$$

Here, $\mathrm{R}_c = \frac{5}{4}\frac{1}{\tan\alpha}$ *is the critical Reynolds number and positive constants* C *and* c *are independent of* δ *and* R.

4.3 Error estimate

We will show (2.2.9) under Assumption 2.2.5 and (2.2.8) by combining the energy estimate obtained in Section 4.2 and nonlinear estimates which will be performed in this section. We can show the other claims in Theorem 2.2.7 in the same way as the proof of (2.2.9) and we will comment about the discrepancy at the end of this section. Now, we specify the arbitrary function η as the solution of the approximate equation. Let ζ^{III} be the solution of (1.2.5) under the initial condition $\zeta^{III}|_{\tau=0} = \eta_0$ and we put $\eta^{III}(x,t) := \zeta^{III}(x-2t, \varepsilon t)$ and

$$\begin{cases} u^{III}(x,y,t) := u^{\mathrm{app}}(y;\eta^{III}(x,t)), \\ v^{III}(x,y,t) := v^{\mathrm{app}}(y;\eta^{III}(x,t)), \\ p^{III}(x,y,t) := p^{\mathrm{app}}(y;\eta^{III}(x,t)), \end{cases} \tag{4.3.1}$$

where $(u^{\mathrm{app}}, v^{\mathrm{app}}, p^{\mathrm{app}})$ was defined by (4.1.13). Then, we have

$$\begin{aligned} \eta_t^{III} = &- 2\eta_x^{III} + \frac{8}{15}(\mathrm{R}_c - \mathrm{R})\delta\eta_{xx}^{III} - C_1\delta^2\eta_{xxx}^{III} \\ &- 4\delta\eta^{III}\eta_x^{III} - \delta^2\big\{C_2\big(\eta^{III}\eta_{xx}^{III} + (\eta_x^{III})^2\big) + 2(\eta^{III})^2\eta_x^{III}\big\}. \end{aligned} \tag{4.3.2}$$

Using the approximate solution (4.3.1), we define ψ_1, ψ_2, ϕ_1, ϕ_2, and ϕ_3 by (4.2.1). By using the equality (4.3.2) to eliminate the t derivatives of η^{III}, we can rewrite these terms as follows.

$$\begin{cases} \psi_1(y;\eta^{III}) = \mathcal{C}_1(y)\partial_x^3\eta^{III} + \mathcal{C}_2(y)\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_7(y)\delta^6\partial_x^9\eta^{III} + N_1^{III}, \\ \psi_2(y;\eta^{III}) = \mathcal{C}_8(y)\partial_x^3\eta^{III} + \mathcal{C}_9(y)\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_{15}(y)\delta^7\partial_x^{10}\eta^{III} + N_2^{III}, \\ \phi_1(\eta^{III}) = \mathcal{C}_{16}\partial_x^3\eta^{III} + \mathcal{C}_{17}\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_{21}\delta^5\partial_x^8\eta^{III} + N_3^{III}, \\ \phi_2(\eta^{III}) = \mathcal{C}_{22}\partial_x^3\eta^{III} + \mathcal{C}_{23}\delta\partial_x^4\eta^{III} + \cdots + \mathcal{C}_{26}\delta^4\partial_x^7\eta^{III} + N_4^{III}, \\ \phi_3(\eta^{III}) = \mathcal{C}_{27}\partial_x^4\eta^{III} + \mathcal{C}_{28}\delta\partial_x^5\eta^{III} + \cdots + \mathcal{C}_{30}\delta^3\partial_x^7\eta^{III} + N_5^{III}, \end{cases} \tag{4.3.3}$$

where $\mathcal{C}_1,\dots,\mathcal{C}_{15}$ are polynomials in y, $\mathcal{C}_{16},\dots,\mathcal{C}_{30}$ are constants, and $N_1^{III},\dots,N_5^{III}$ are collections of the nonlinear terms of the form

$$\frac{1}{\delta^3}\Phi_0(\delta\eta^{III},\delta^2\partial_x\eta^{III},\dots,\delta^5\partial_x^4\eta^{III};y)\Phi_0(\delta^2\partial_x\eta^{III},\dots,\delta^{10}\partial_x^9\eta^{III};y). \tag{4.3.4}$$

Let $(\eta^\delta,u^\delta,v^\delta,p^\delta)$ be the solution of (2.1.32)–(2.1.34) and we set $H^{III}:=\eta^\delta-\eta^{III}$, $\boldsymbol{U}^{III}:=(u^\delta-u^{III},\delta(v^\delta-v^{III}))^{\mathrm{T}}$, $\tilde{\mathscr{E}}_m^{III}:=\tilde{\mathscr{E}}_m(H^{III},\boldsymbol{U}^{III})$, and so on. In the following, we use same notations in Subsection 3.3.1. We prepare several lemmas to proceed the error estimate. In particular, we estimate nonlinear terms defined by (4.2.14)–(4.2.16) in terms of energy functions.

Lemma 4.3.1. *Under the same assumption as Proposition* 4.2.2*, for any $\epsilon>0$ there exists a positive constant C_ϵ such that we have*

$$\mathscr{N}_m^2(\boldsymbol{U}^{III})(t)\le\epsilon\tilde{\mathscr{F}}_m(t)+C_\epsilon\delta^4\tilde{E}_m(t)\tilde{F}_{m+1}(t), \tag{4.3.5}$$

where $\mathscr{N}_m^2$ is the collection of nonlinear terms defined by (4.2.15)*.*

Proof. By the explicit forms of $\boldsymbol{f}^{(3)}$, $h_1^{(3)}$, and $h_2^{(3)}$ (see (4.1.5) and Subsection 2.1.3), we can obtain the desired estimate in the same but more easier way as proving Lemmas 3.3.10 and 3.3.12. □

Lemma 4.3.2. *Under the same assumption as Proposition* 4.2.2*, for any $\epsilon>0$ there exists a positive constant C_ϵ such that we have*

$$\mathscr{N}_m^3(H;\eta^{III})(t)\le\epsilon\tilde{\mathscr{F}}_m(t)+C_\epsilon\delta^5|\eta_x^{III}(t)|_{m+12}^2,$$

where $\mathscr{N}_m^3$ is the collection of nonlinear terms defined by (4.2.16)*.*

Proof. By the well-known inequalities

$$\begin{cases}\|\partial_x^k(fg)\|\lesssim\|f\|_{L^\infty}\|\partial_x^k g\|+\|g\|_{L^\infty}\|\partial_x^k f\|,\\ \|\partial_x^k\Phi_0(\boldsymbol{f};y)\|\le C(\|\boldsymbol{f}\|_{L^\infty})\|\partial_x^k\boldsymbol{f}\|\end{cases}$$

and (4.3.2)–(4.3.4) lead to

$$\sum_{k=0}^m\mathscr{N}_0^3(\partial_x^k Z_3)\lesssim\bigl(1+|\eta^{III}|_{m+12}^2\bigr)\delta^5|\eta_x^{III}|_{m+12}^2.$$

Moreover, by Poincaré's inequality and (4.3.4), we see that

$$|(\partial_x^k H,\delta^3\partial_x^k\phi_3)_\Gamma|\le\epsilon\delta|\partial_x^k H_x|_0^2+C_\epsilon\delta^5|\partial_x^k\phi_3|_0^2\le\epsilon\tilde{\mathscr{F}}_m+C_\epsilon\bigl(1+|\eta^{III}|_{m+12}^2\bigr)\delta^5|\eta_x^{III}|_{m+12}^2.$$

These together with Lemma 4.2.3 imply the desired inequality. □

Lemma 4.3.3. *Under the same assumption as Proposition* 4.2.2, *for any* $\epsilon > 0$ *there exists a positive constant* C_ϵ *such that we have*

$$\begin{aligned}\mathscr{N}_m^1(H^{III}, \boldsymbol{U}^{III}, P^{III}; \eta^{III})(t) \le & (C_\epsilon \tilde{E}_2(t) + \epsilon)\tilde{\mathscr{F}}_m^{III}(t) + C_\epsilon \big\{ \tilde{E}_m(t)\tilde{\mathscr{F}}_2^{III}(t) \\ & + \delta^4 \tilde{E}_m(t)\tilde{F}_{m+1}(t) + \delta^5 |\eta_x^{III}(t)|_{m+12}^2 \\ & + (\tilde{F}_m(t) + \delta|\eta_x^{III}(t)|_{m+12}^2)\tilde{\mathscr{E}}_m^{III}(t)\big\},\end{aligned} \tag{4.3.6}$$

where $\mathscr{N}_m^1$ *is the collection of nonlinear terms defined by* (4.2.14).

Proof. In this proof, we omit the symbol III appeared in a superscript of solutions for simplicity. By (4.1.5), (4.2.2), and (4.2.9), we see that $\boldsymbol{F}$ is consist of terms of the form

$$\begin{cases} \delta\Phi_0(\eta^\delta, \delta\eta_x^\delta; y)(\nabla_\delta U_y, \nabla_\delta P) + \delta^2(\eta^\delta)^2(U_{yy}, P_y), \\ \delta\Phi_0(\eta^\delta, \delta\eta_x^\delta, u^\delta; y)(\delta V, \delta U_x), \\ \delta\Phi_0(\delta\eta_x^\delta, \delta\eta_t^\delta, \delta v^\delta; y)(U, U_y), \\ \delta\Phi_0(\eta, \boldsymbol{u}, \nabla_\delta \boldsymbol{u}, \nabla_\delta u_y, \nabla_\delta p; y)(\delta H_x, \delta H_t, U, \delta V), \\ \delta^2\eta^\delta(u_{yy} + p_y)H \end{cases}$$

and that $G_2 = \delta^2\{\eta^\delta(2H_x + U_x) + \eta_x^\delta U + (2\eta_x + u_x)H + uH_x\}$, $G_3 = \delta^2\{(\eta^\delta)^2 H_x + (\eta^\delta + \eta)\eta_x H\}$, and $bH = 2\delta\big(\delta(\eta^\delta)^2 + (2+\delta\eta)\eta^\delta + \delta\eta^2 + 2\eta\big)H$. Note that using (4.3.1) and (4.3.2), we can express the approximate solutions $\boldsymbol{u}, \nabla_\delta \boldsymbol{u}, u_{yy}$, and $\nabla_\delta p$ in terms of η and its x derivatives. In view of these, by putting

$$\begin{cases} \Phi^1 = \Phi(\eta^\delta, \delta\eta_x^\delta, \delta\eta_t^\delta, \delta^2\eta_{xx}^\delta, \delta^2\eta_{tx}^\delta, \boldsymbol{u}^\delta, y), \\ \Phi^2 = \Phi(\delta\eta_x^\delta, \delta\eta_t^\delta, \delta^2\eta_{xx}^\delta, \delta^2\eta_{tx}^\delta, \delta^2\eta_{tt}^\delta, \delta v^\delta, \delta\boldsymbol{u}_x^\delta, \delta\boldsymbol{u}_t^\delta, y), \\ \Phi^3 = \Phi(\eta^\delta, \delta\eta_x^\delta, y), \\ \Phi^4 = \Phi(\eta, \delta\eta_x, \dots, \delta^{10}\partial_x^{10}\eta, y), \end{cases}$$

$$\begin{cases} Q_1 := (\delta H_x, \delta H_t, \delta^2 H_{xx}, \delta^2 H_{tx}, \delta^3 H_{xxx}, \delta V, \delta\boldsymbol{U}_x, \delta\boldsymbol{U}_t, \delta\nabla_\delta U_x, \delta\nabla_\delta U_t, \nabla_\delta U_y, \nabla_\delta U_{xy}, \\ \qquad \nabla_\delta P, \nabla_\delta P_x, \delta U_x|_\Gamma, \delta U_t|_\Gamma, \delta^2 U_{xx}|_\Gamma, \delta^{5/2}|D_x|^{5/2}U|_\Gamma), \\ Q_2 := (H, \delta H_x, \delta H_t, \delta^2 H_{xx}, \delta^2 H_{tx}, \delta^3 H_{xxx}, \boldsymbol{U}, \nabla_\delta \boldsymbol{U}, \delta\boldsymbol{U}_t, U|_\Gamma), \end{cases}$$

it suffices to estimate

$$\begin{cases} I_1 = \delta\|\partial_x^k(\Phi_0^1 Q_1)\|^2, \\ I_2 = \delta\|\partial_x^k(\Phi_0^2 Q_2)\|^2, \\ I_3 = \delta^3|(\partial_x^k(\eta^\delta \boldsymbol{U}_{tx}), \partial_x^k V_t)_\Gamma|, \\ I_4 = \delta^2|(\partial_x^k(\Phi_0^3 \nabla_\delta U_{ty}), \partial_x^k \boldsymbol{U}_t)_\Omega|, \\ I_5 = \delta^2|(\partial_x^k(\Phi_0^3 \nabla_\delta P_t), \partial_x^k \boldsymbol{U}_t)_\Omega|, \\ I_6 = \delta\|\partial_x^k(\Phi^1\Phi_0^4 Q_2)\|^2, \\ I_7 = \delta^4|(\partial_x^k(\Phi_0^4 H_{tt}), \partial_x^k V_t)_\Gamma|, \\ I_8 = \delta^6 \mathrm{W}|(\partial_x^k H_{xxxx}, \partial_x^k G_{3xx})_\Gamma| \end{cases}$$

for $0 \le k \le m$.

By Proposition 4.2.2 and $\|(u,v)\|_{L^\infty} \lesssim \|(u_y,v_y)\| + \|(u_{xy},v_{xy})\|$ thanks to the boundary condition $u|_{y=0} = v|_{y=0} = 0$, we obtain

$$\|\Phi_0^1\|_{L^\infty}^2 \lesssim \tilde{E}_2, \quad \|\partial_x^k \Phi_0^1\|^2 + \|\partial_x^k \Phi_{0y}^1\|^2 \lesssim \tilde{E}_m, \tag{4.3.7}$$

$$\|\Phi_0^2\|_{L^\infty}^2 \lesssim \tilde{F}_2, \quad \|\partial_x^k \Phi_0^2\|^2 + \|\partial_x^k \Phi_{0y}^2\|^2 \lesssim \tilde{F}_m. \tag{4.3.8}$$

In the same way as the proof of Lemma 4.3.2, we have

$$\delta\|\Phi_0^4\|_{L^\infty}^2 \lesssim \delta|\eta_x|_{m+12}^2, \quad \delta(\|\partial_x^k \Phi_0^4\|^2 + \|\partial_x^k \Phi_{0y}^4\|^2) \lesssim \delta|\eta_x|_{m+12}^2, \quad |\Phi_0^4|_{m-\frac{1}{2}}^2 \lesssim |\eta|_{m+12}^2. \tag{4.3.9}$$

On the other hand, it is easy to see that

$$\|Q_1\|^2 + \|Q_{1x}\|^2 \lesssim \tilde{\mathscr{F}}_2, \quad \|\partial_x^k Q_1\|^2 \lesssim \tilde{\mathscr{F}}_m, \tag{4.3.10}$$

$$\|Q_2\|^2 + \|Q_{2x}\|^2 \lesssim \tilde{\mathscr{E}}_2, \quad \|\partial_x^k Q_2\|^2 \lesssim \tilde{\mathscr{E}}_m, \tag{4.3.11}$$

where we used the trace theorem Lemma 1.3.3 to estimate the term $\delta^5 ||D_x|^{\frac{5}{2}} U|_0^2$.

As for I_1, by (4.3.7), (4.3.10), and Lemma 1.3.6, we have $I_1 \lesssim \tilde{E}_2 \tilde{\mathscr{F}}_m + \tilde{E}_m \tilde{\mathscr{F}}_2$. As for I_2, by (4.3.8), (4.3.11), and Lemma 1.3.6, we have $I_2 \lesssim \tilde{F}_m \tilde{\mathscr{E}}_m$. As for I_3, by integration by parts, we have $I_3 \lesssim C_\epsilon \delta^3 |\eta^\delta U_{tx}|_{m-\frac{1}{2}}^2 + \epsilon\delta^3 |V_t|_{m+\frac{1}{2}}^2 \le C_\epsilon(\tilde{E}_2 \tilde{\mathscr{F}}_m + \tilde{E}_m \tilde{\mathscr{F}}_2) + \epsilon \tilde{\mathscr{F}}_m$. As for I_4, by integration by parts in y, we have

$$\begin{aligned}
I_4 &\le C_\epsilon \delta^2 \big(\|\partial_x^k(\Phi_0^3 \nabla_\delta U_t)\|^2 + \|\partial_x^k(\Phi_{0y}^3 \nabla_\delta U_t)\|^2 \big) \\
&\quad + \delta^3 |(\partial_x^k(\Phi_0^3 U_{tx}), \partial_x^k \boldsymbol{U}_t)_\Gamma| + \delta^2 |(\partial_x^k(\Phi_0^3 U_{ty}), \partial_x^k \boldsymbol{U}_t)_\Gamma| + \epsilon\delta \|\partial_x^k \boldsymbol{U}_{ty}\|^2 \\
&\le I_{4,1} + I_{4,2} + I_{4,3} + \epsilon \tilde{\mathscr{F}}_m,
\end{aligned}$$

where

$$\begin{cases}
I_{4,1} = C_\epsilon \delta^2 \big(\|\partial_x^k(\Phi_0^3 \nabla_\delta U_t)\|^2 + \|\partial_x^k(\Phi_{0y}^3 \nabla_\delta U_t)\|^2 \big), \\
I_{4,2} = \delta^3 |(\partial_x^k(\Phi_0^3 U_{tx}), \partial_x^k \boldsymbol{U}_t)_\Gamma|, \\
I_{4,3} = \delta^2 |(\partial_x^k(\Phi_0^3 U_{ty}), \partial_x^k \boldsymbol{U}_t)_\Gamma|.
\end{cases}$$

The estimates for $I_{4,1}$ and $I_{4,2}$ are reduced to the estimates for I_1 and I_3, respectively. Thus, taking into account that we can eliminate the term $U_y|_\Gamma$ in $I_{4,3}$ by the first equation in (4.2.7), this together with the estimates for I_2, I_3, $\delta^3 h_1$, and $\delta^3 \phi_1$ yields $I_4 \le \epsilon \tilde{\mathscr{F}}_m + C_\epsilon \big\{ \tilde{E}_2 \tilde{\mathscr{F}}_m + \tilde{E}_m (\tilde{\mathscr{F}}_2 + \delta^4 \tilde{F}_{m+1} + |\eta|_{m+12}^2 \delta^5 |\eta_x|_{m+12}^2) \big\}$. As for I_5, it suffices to show the case of $k \ge 1$ because we can treat easily the case of $k = 0$. Integrating by parts in x, (4.3.7), and Lemma 1.3.6, we have $I_5 \le \epsilon\delta^3 \|\partial_x^k \boldsymbol{U}_{tx}\|^2 + C_\epsilon \delta \|\partial_x^{k-1}(\Phi_0^3 \nabla_\delta P_t)\|^2 \le \epsilon \tilde{\mathscr{F}}_m + C_\epsilon \big(\tilde{E}_2 \tilde{\mathscr{F}}_m + \tilde{E}_m \tilde{\mathscr{F}}_2 \big)$. As for I_6, by (4.3.7), (4.3.9), (4.3.11), and Lemma 1.3.6, we have

$$\begin{aligned}
I_6 &\lesssim \delta \big\{ \|\Phi_0^4\|_{L^\infty}^2 (\|\partial_x^k \Phi^1\|^2 + \|\partial_x^k \Phi_y^1\|^2)(\|Q_2\|^2 + \|Q_{2x}\|^2) \\
&\quad + \|\Phi^1\|_{L^\infty}^2 (\|\partial_x^k \Phi_0^4\|^2 + \|\partial_x^k \Phi_{0y}^4\|^2)(\|Q_2\|^2 + \|Q_{2x}\|^2) + \|\Phi^1\|_{L^\infty}^2 \|\Phi_0^4\|_{L^\infty}^2 \|\partial_x^k Q_2\|^2 \big\} \\
&\lesssim (\tilde{E}_m + |\eta|_{m+12}^2) \delta |\eta_x|_{m+12}^2 \tilde{\mathscr{E}}_m.
\end{aligned}$$

As for I_7, it suffices to show the case of $k \geq 1$ because we can treat easily the case of $k = 0$. By the third equation in (4.2.7), integration by parts, and the trace theorem, we have

$$\begin{aligned}
I_7 &\leq C_\epsilon \delta^4 ||D_x|^{\frac{1}{2}} \partial_x^{k-1} (\Phi_0^4 V_t)|_0^2 + C_\epsilon \delta^5 |\partial_x^k (\Phi_0^4 H_{xt} + \Phi_0^4 G_{3t})|_0^2 + C_\epsilon \delta^5 |\delta^3 \partial_x^k \phi_{3t}|_0^2 \\
&\quad + \epsilon \big(\delta^4 ||D_x|^{\frac{1}{2}} \partial_x^k V_t|_0^2 + \delta^3 |\partial_x^k V_t|_0^2 \big) \\
&\leq I_{7,1} + I_{7,2} + I_{7,3} + \epsilon \tilde{\mathscr{F}}_m,
\end{aligned}$$

where

$$\begin{cases}
I_{7,1} = C_\epsilon \delta^4 ||D_x|^{\frac{1}{2}} \partial_x^{k-1} (\Phi_0^4 V_t)|_0^2, \\
I_{7,2} = C_\epsilon \delta^5 |\partial_x^k (\Phi_0^4 H_{xt} + \Phi_0^4 G_{3t})|_0^2, \\
I_{7,3} = C_\epsilon \delta^5 |\delta^3 \partial_x^k \phi_{3t}|_0^2.
\end{cases}$$

By Lemma 1.3.3 , the second equation in (4.2.6), and (4.3.9), we have

$$\begin{aligned}
I_{7,1} &\lesssim |\Phi_0^4|_{m-\frac{1}{2}}^2 \delta^3 |V_t|_{L^\infty}^2 + \delta |\Phi_0^4|_{L^\infty}^2 \delta^3 ||D_x|^{\frac{1}{2}} \partial_x^{k-1} V_t|_0^2 \\
&\lesssim |\Phi_0^4|_{m-\frac{1}{2}}^2 \delta^3 \|U_{txx}\|^2 + \delta |\Phi_0^4|_{L^\infty}^2 (\delta^2 \|\partial_x^k U_t\|^2 + \delta^4 \|\partial_x^k V_t\|^2) \\
&\lesssim |\eta|_{m+12}^2 \tilde{\mathscr{F}}_2 + \delta |\eta_x|_{m+12}^2 \tilde{\mathscr{E}}_m.
\end{aligned}$$

Recalling the explicit form of G_3, we see that the estimate of $I_{7,2}$ is reduced to I_6. Taking into account that we have already estimated $I_{7,3}$ in the proof of Lemma 4.3.2, we obtain $I_7 \leq C_\epsilon \{ |\eta|_{m+12}^2 \tilde{\mathscr{F}}_2 + (\tilde{E}_m + |\eta|_{m+12}^2) \delta |\eta_x|_{m+12}^2 \tilde{\mathscr{E}}_m + |\eta|_{m+12}^2 \delta^5 |\eta_x|_{m+12}^2 \} + \epsilon \tilde{\mathscr{F}}_m$. As for I_8, integration by parts, (4.3.7), and (4.3.9) lead to

$$\begin{aligned}
\delta^6 \mathrm{W} |(\partial_x^k H_{xxxx}, \partial_x^k G_{3xx})_\Gamma| &\leq \epsilon (\delta^2 \mathrm{W})^2 \delta^2 ||D_x|^{\frac{7}{2}} H|_m^2 + C_\epsilon \delta^6 ||D_x|^{\frac{5}{2}} G_3|_m^2 \\
&\leq \epsilon \tilde{\mathscr{F}}_m + C_\epsilon \big\{ \delta^2 (\tilde{F}_m + \delta |\eta_x|_{m+12}^2) \tilde{\mathscr{E}}_2 + \tilde{E}_2 \tilde{\mathscr{F}}_m \big\}.
\end{aligned}$$

Therefore, by the boundedness of the terms $\tilde{E}_m$ and $|\eta|_{m+12}^2$ which comes from Proposition 4.2.2 and Lemma 4.2.3, the proof is complete. □

Lemma 4.3.4. *Under the same assumption as Proposition* 4.2.2, *we have*

(4.3.12) $$\tilde{\mathscr{E}}_m^{III}(t) \lesssim \mathscr{E}_m^{III}(t) + \delta^4 (\tilde{E}_{m+1}(t) + |\eta^{III}(t)|_{m+12}^2),$$

(4.3.13) $$\begin{aligned}\tilde{\mathscr{F}}_m^{III}(t) &\lesssim \mathscr{F}_m^{III}(t) + (\tilde{F}_m(t) + \delta |\eta_x^{III}(t)|_{m+12}^2) \tilde{\mathscr{E}}_m^{III}(t) \\ &\quad + \delta^4 \tilde{E}_m(t) \tilde{F}_{m+1}(t) + \delta^5 |\eta_x^{III}(t)|_{m+12}^2,\end{aligned}$$

(4.3.14) $$\mathscr{E}_m^{III}(t) \lesssim \mathscr{D}_m^{III}(t) + \delta^4.$$

Proof. In view of the discrepancy of non-homogeneous terms in the equations, modifying the proof of (3.4.2) in Lemma 3.4.2, we obtain (4.3.12). Taking into account that we can eliminate U_{yy} in $\mathscr{F}_m^{III}$ by using the first equation in (4.2.6), modifying the proof of (3.4.3) in Lemma 3.4.2, it is not difficult to check that (4.3.13) holds. Moreover, modifying the proof of (3.4.10), we obtain (4.3.14). □

Lemma 4.3.5. *Under the same assumption as Proposition* 4.2.2*, we have*

$$\mathscr{D}_m^{III}(0) \lesssim \delta^4.$$

Remark 4.3.6. This lemma together with (4.3.14) yields

$$\mathscr{E}_m^{III}(0) \lesssim \delta^4. \tag{4.3.15}$$

Proof. By the second and third equations in the compatibility conditions in Assumption 2.2.5, we see that

$$\begin{aligned} u_0(x,y) &= y u_{0y}(x,1) - \int_0^y \int_z^1 u_{0yy}(x,w)\mathrm{d}w\mathrm{d}z \\ &= \big(2y\eta_0 + 4y\delta\eta_0^2 + 2y\delta^2\eta_0^3\big) + \delta y\big(-\delta v_{0x} + \delta^2 h_1^{(0)}\big) \\ &\quad - \int_0^y \int_z^1 u_{0yy}(x,w)\mathrm{d}w\mathrm{d}z. \end{aligned} \tag{4.3.16}$$

It follows from (2.2.8) and $\|(1+|D_x|)^{m+1}u_{yy}^{III}|_{t=0}\| \lesssim \delta$ (see the explicit form of u^{III}, that is, (4.1.10)–(4.1.13) and (4.3.1)) that $\|(1+|D_x|)^{m+1}u_{0yy}\| \lesssim \delta$. Thus, by (4.3.16), the explicit form of u^{III}, (2.2.8), and the uniform estimate for $\delta^2|h_1^{(0)}|_{m+1}$ (see the proof of Lemma 4.3.1), we obtain $\|(1+|D_x|)^{m+1}U|_{t=0}\| \lesssim \delta$. Combining this and the first equation in the compatibility conditions leads to $\|(1+|D_x|)^{m}V|_{t=0}\| \lesssim \delta$. Therefore, in view of the definition of $\mathscr{D}_m$ (see (4.2.20)), using these and $H|_{t=0} = 0$, we obtain the desired estimate. □

Proof of (2.2.9) *in Theorem* 2.2.7. By Proposition 4.2.2, Lemmas 4.2.1, 4.3.1–4.3.3, and (4.3.12) and (4.3.13) in Lemma 4.3.4, if c_0 and ϵ are sufficiently small, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_m^{III}(t) + \tilde{\mathscr{F}}_m^{III}(t) \le C_1\big(\varphi_1(t)\mathscr{E}_m^{III}(t) + \tilde{E}_m(t)\tilde{\mathscr{F}}_2^{III}(t) + \delta^4\varphi_2(t)\big), \tag{4.3.17}$$

where

$$\varphi_1(t) = \tilde{F}_m(t) + \delta|\eta_x^{III}(t)|_{m+12}^2, \quad \varphi_2(t) = \tilde{E}_m(t)\tilde{F}_{m+1}(t) + \delta|\eta_x^{III}(t)|_{m+12}^2. \tag{4.3.18}$$

By considering the case of $m = 2$ in (4.3.17) and using Gronwall's inequality and Proposition 4.2.2, if c_0 is sufficiently small, then we have $\mathscr{E}_2^{III}(t) + \int_0^t \tilde{\mathscr{F}}_2^{III}(s)\mathrm{d}s \le \varphi_3(t)$, where

$$\varphi_3(t) = \mathscr{E}_2^{III}(0)\exp\left(C_1\int_0^t \varphi_1(s)\mathrm{d}s\right) + C_1\int_0^t \delta^4\varphi_2(s)\exp\left(C_1\int_s^t \varphi_1(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s, \tag{4.3.19}$$

which leads to

$$\int_0^t \tilde{\mathscr{F}}_2^{III}(s)\mathrm{d}s \le \varphi_3(t). \tag{4.3.20}$$

Note that by Proposition 4.2.2 and Lemma 4.2.3, we have the exponential decay estimate for $\tilde{E}_{m+1}(t)$ and $|\eta^{III}(t)|_{m+13}^2$. This together with (4.3.17), Gronwall's inequality, and $\delta\mathscr{E}_m^{III} \lesssim$

$\tilde{\mathscr{F}}_m^{III}$ which comes from $|H|_0 \lesssim |H_x|_0$ and $\|V\| \lesssim \|V_y\| = \|U_x\|$ (see (4.2.13) and (4.2.19)) yields

$$\mathscr{E}_m^{III}(t) \le \left\{ \mathscr{E}_m^{III}(0) \exp\left(C_1 \int_0^t \varphi_1(s) \mathrm{d}s \right) + \varphi_4(t) \right\} \mathrm{e}^{-c\delta t},$$

where

$$\varphi_4(t) = C_1 \int_0^t \left(\tilde{\mathscr{F}}_2^{III}(s) + \delta^4 \tilde{F}_{m+1}(s) \right) \exp\left(C_1 \int_s^t \varphi_1(\sigma) \mathrm{d}\sigma \right) \mathrm{d}s. \tag{4.3.21}$$

Combining the above inequality and (4.3.12) and (4.3.14) in Lemma 4.3.4, we obtain

$$\tilde{\mathscr{E}}_m^{III}(t) \le C_2 \big(\delta^4 + \mathscr{D}_m^{III}(0) + \varphi_4(t) \big) \mathrm{e}^{-c\delta t}. \tag{4.3.22}$$

Here, recalling the definition $\eta^{III}(x,t) = \zeta^{III}(x-2t, \varepsilon t)$ and the assumption $\varepsilon = \delta$ and using Lemma 4.2.3, we have $\int_0^\infty \delta |\eta_x^{III}(t)|_s^2 \mathrm{d}t = \frac{1}{\varepsilon} \int_0^\infty \delta |\zeta_x^{III}(\tau)|_s^2 \mathrm{d}\tau \lesssim |\eta_0|_s$. By this, the integrability of $\tilde{F}_{m+1}$ which comes from Proposition 4.2.2, and (4.3.15), we have $\varphi_3(t) \lesssim \delta^4$ (see (4.3.18) and (4.3.19)). This together with (4.3.20) leads to $\varphi_4(t) \lesssim \delta^4$ (see (4.3.21)). Combining this, (4.3.22), and Lemma 4.3.5, we have

$$\tilde{\mathscr{E}}_m^{III}(t) \le C_3 \delta^4 \mathrm{e}^{-c\varepsilon t}, \tag{4.3.23}$$

which implies $\mathcal{D}(t; \zeta^{III}, u^{III}, v^{III}, p^{III}) \lesssim \delta^4 \mathrm{e}^{-c\varepsilon t}$ (see (2.2.3) and (4.2.18)). Here, we used $\|V\| \lesssim \|V_y\| = \|U_x\|$. Moreover, by taking into account the equality $P(x,y,t) = P(x,1,t) - \int_y^1 P_y(x,z,t)\mathrm{d}z$ and using the second equation in (4.2.6), the second equation in (4.2.7), and the uniform estimate (4.3.23), we easily obtain $\|(1+|D_x|)^m (p^\delta - p^{III})(t)\|^2 \lesssim \delta^4 \mathrm{e}^{-c\varepsilon t}$. Note that in the case of $O(\delta^{-1}) \le \mathrm{W} \le O(\delta^{-2})$ we can estimate the term $\frac{\delta^2 \mathrm{W}}{\sin\alpha} \partial_x^m H_{xx}$ which comes from the second equation in (4.2.7) by $\tilde{\mathscr{E}}_{m+1}^{III}$. Therefore, the proof of (2.2.9) in Theorem 2.2.7 is complete. $\square$

We proceed to prove (2.2.5), (2.2.7), and (2.2.11). Let ζ^I, ζ^{II}, and ζ^{IV} be the solution for (1.2.3), (1.2.4), and (1.2.6), respectively under the initial condition $\zeta^I|_{\tau=0} = \zeta^{II}|_{\tau=0} = \zeta^{IV}|_{\tau=0} = \eta_0$. We put $\eta^I(x,t) := \zeta^I(x-2t, \varepsilon t)$, $\eta^{II}(x,t) := \zeta^{II}(x-2t, \varepsilon t)$, $\eta^{IV}(x,t) := \zeta^{IV}(x-2t, \varepsilon t)$ and

$$\begin{cases} u^I(x,y,t) := u_0^I(y; \eta^I(x,t)) + \delta u_1^I(y; \eta^I(x,t)), \\ v^I(x,y,t) := u_0^I(y; \eta^I(x,t)) + \delta v_1^I(y; \eta^I(x,t)), \\ p^I(x,y,t) := p_0^I(y; \eta^I(x,t)) + \delta p_1^I(y; \eta^I(x,t)), \end{cases} \tag{4.3.24}$$

$$\begin{cases} u^{II}(x,y,t) := u_0^{II}(y; \eta^{II}(x,t)) + \delta u_1^{II}(y; \eta^{II}(x,t)), \\ v^{II}(x,y,t) := u_0^{II}(y; \eta^{II}(x,t)) + \delta v_1^{II}(y; \eta^{II}(x,t)), \\ p^{II}(x,y,t) := p_0^{II}(y; \eta^{II}(x,t)) + \delta p_1^{II}(y; \eta^{II}(x,t)), \end{cases} \tag{4.3.25}$$

$$
(4.3.26) \qquad \begin{cases} u^{IV}(x,y,t) := u_0^{IV}(y;\eta^{IV}(x,t)) + \delta u_1^{IV}(y;\eta^{IV}(x,t)) + \delta^2 u_2^{IV}(y;\eta^{IV}(x,t)), \\ v^{IV}(x,y,t) := u_0^{IV}(y;\eta^{IV}(x,t)) + \delta v_1^{IV}(y;\eta^{IV}(x,t)) + \delta^2 v_2^{IV}(y;\eta^{IV}(x,t)), \\ p^{IV}(x,y,t) := p_0^{IV}(y;\eta^{IV}(x,t)) + \delta p_1^{IV}(y;\eta^{IV}(x,t)) + \delta^2 p_2^{IV}(y;\eta^{IV}(x,t)), \end{cases}
$$

where $u_0^I, v_0^I, p_0^I, \ldots$ were defined by (4.1.15)–(4.1.17). In view of this, by applying the same argument as showing (2.2.9), it is not difficult to check that (2.2.5), (2.2.7), and (2.2.11) hold. Therefore, the proof of Theorem 2.2.7 is complete. $\square$

Acknowledgement

First and foremost, my deepest gratitude goes to my advisor Professor Tatsuo Iguchi. He gave me not only an excellent academic training but also good lessons for the rest of my life, especially the importance of having principles. I am here because of his guidance and no words can express my appreciation for him.

I would like to thank Professor Atusi Tani, Professor Shun Shimomura, and Professor Toshihiko Sugiura for the fruitful discussions and instructive advices on this dissertation. In particular, Professor Tani taught me the interests of mathematical analysis during my first year of undergraduate and this dissertation would not exist without him.

I would also like thank Professor Masahiro Takayama for meaningful discussion.

Moreover, I am deeply grateful to Doctor Masumi Kawasaki for his great encouragement, who is a chief of the mathematical department at Kaijo junior and senior high school.

Finally, I would like to thank my parents Daisuke and Harumi Ueno for their generous support. I am happy and proud to be your son.

Bibliography

[1] S. V. Alekseenko, V. E. Nakoryakov, and B. G. Pokusaev, *Wave flow of liquid films*, Begell House, 1994.

[2] R. W. Atherton and G. M. Homsy, On the derivation of evolution equations for interfacial waves, *Chem. Eng. Comm.*, **2** (1976), 57–77.

[3] J. T. Beale, Large-time regularity of viscous surface waves, *Arch. Rational Mech. Anal.*, **84** (1984), 307–352.

[4] T. B. Benjamin, Wave formation in laminar flow down an inclined plane, *J. Fluid Mech.*, **2** (1957), 554–574.

[5] D. J. Benney, Long waves on liquid film, *J. Math. Phys.*, **45** (1966), 150–155.

[6] D. Bresch, *Shallow-water equations and related topics*, Handbook of differential equations: evolutionary equations, 5, 1–104, Elsevier/North-Holland, 2009.

[7] D. Bresch and P. Noble, Mathematical justification of a shallow water model, *Methods Appl. Anal.*, **14** (2007), 87–117.

[8] H. - C. Chang and E. A. Demekhin, *Complex wave dynamics on thin films*, Studies in Interface Science, 14, Elsevier Science B.V., 2002.

[9] R. V. Craster and O. K. Matar, Dynamics and stability of thin liquid films, *Rev. Mod. Phys.*, **81** (2009), 1131–1198.

[10] A. E. Dukler, Characterization, effects, and modeling of the wavy gas-liquid interface, *Progress in Heat and Mass Transfer*, **6** (1972), 207–234.

[11] G. D. Fulford, The flow of liquids in thin films, *J. Adv. Chem. Engng.*, **5** (1964), 151–236.

[12] L. Giacomelli and F. Otto, Rigorous lubrication approximation, *Interfaces Free Bound.*, **5** (2003), 483—529.

[13] B. Gjevik, Occurrence of finite-amplitude surface waves on falling liquid films, *Phys. Fluids*, **13** (1970), 1918–1925.

[14] R. S. Johnson, Shallow water waves on a viscous fluid—The undular bore, *Phys. Fluids*, **15** (1972), 1693–1699.

[15] S. Kalliadasis, C. Ruyer-Quil, B. Scheid, and M. G. Velarde, *Falling Liquid film*, Applied Mathematical Sciences, 176, Springer, 2012.

[16] P. L. Kapitza, Wave flow in thin layers of a viscous fluid, *Zh. Eksp. Teor. Fiz.*, **19** (1948), 105–120 (*Collected Works of P. L. Kapitza*, Pergamon, 1965).

[17] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Progr. Theor. Phys.*, **55** (1976), 356–369.

[18] S. P. Lin, Film waves, *Waves on Fluid Interfaces*, (1983), 261–289.

[19] S. P. Lin and M. V. G. Krishna, Stability of a liquid film with respect to initially finite three-dimensional disturbances, *Phys. Fluids*, **20** (1977), 2005–2011.

[20] S. P. Lin and C. Y. Wang, Modeling wavy film flows, *Encyclopedia of fluid mechanics*, **1** (1985), 931–951.

[21] C. C. Mei, Nonlinear gravity waves in a thin sheet of viscous fluid, *J. Math. Phys.*, **45** (1966), 266–288.

[22] C. Nakaya, Long waves on a thin fluid layer flowing down an inclined plane, *Phys. Fluids*, **18** (1975), 1407–1412.

[23] T. Nishida, Y. Teramoto, and H. A. Win, Navier–Stokes flow down an inclined plane: downward periodic motion, *J. Math. Kyoto Univ.*, **33** (1993), 787–801.

[24] W. Nusselt, Die oberflachenkondensation des wasserdampfes, *Z. Ver. Dt. Ing.*, **60** (1916), 541–546.

[25] A. Oron, S. H. Davis, and S. G. Bankoff, Long-scale evolution of thin liquid films, *Rev. Mod. Phys.*, **69** (1997), 931–980.

[26] O. Reynolds, On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil, *Proc. R. Soc. Lond.*, **40** (1886), 191–203.

[27] G. J. Roskes, Three-dimensional long waves on a liquid film, *Phys. Fluids*, **13** (1970), 1440–1445.

[28] S. M. Shih and M. C. Shen, Uniform asymptotic approximation for viscous fluid flow down an inclined plane, *SIAM J. Math. Anal.*, **6** (1975), 560–582.

[29] G. I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames–I. Derivation of basic equations, *Acta Astronautica*, **4** (1977), 1177–1206.

[30] G. I. Sivashinsky and D. M. Michelson, On irregular wavy flow of a liquid film down a vertical plane, *Progr. Theor. Phys.*, **63** (1980), 2112–2114.

[31] M. K. Smith, The mechanism for the long–wave instability in thin liquid films, *J. Fluid Mech.*, **217** (1990), 469–485.

[32] S. M. Sun and M. C. Shen, Justification of the linear long–wave approximation to viscous fluid flow down an inclined plane, *Quart. Appl. Math.*, **52** (1994), 759–775.

[33] Y. Teramoto, On the Navier–Stokes flow down an inclined plane, *J. Math. Kyoto Univ.*, **32** (1992), 593–619.

[34] Y. Teramoto, The initial value problem for viscous incompressible flow down an inclined plane, *Hiroshima Math. J.*, **15** (1985), 619–643.

[35] Y. Teramoto and K. Tomoeda, Optimal Korn's inequality for solenoidal vector fields on a periodic slab, *Proc. Japan Acad. Ser. A Math. Sci.*, **88** (2012), 168–172.

[36] J. Topper and T. Kawahara, Approximate equations for long nonlinear waves on a viscous fluid, *J. Phys. Soc. Japan*, **44** (1978), 663–666.

[37] H. Uecker, Self-similar decay of spatially localized perturbations of the Nusselt solution for the inclined film problem, *Arch. Rational Mech. Anal.*, **184** (2007), 401–447.

[38] H. Ueno and T. Iguchi, A mathematical justification of a thin film approximation for the flow down an inclined plane, *J. Math. Anal. Appl.*, **444** (2016), 804–824.

[39] H. Ueno, A. Shiraishi, and T. Iguchi, Uniform estimates for the flow of a viscous incompressible fluid down an inclined plane in the thin film regime, *J. Math. Anal. Appl.*, **436** (2016), 248–287.

[40] C. S. Yih, Stability of parallel laminar flow with a free surface, *Proceedings of the Second U. S. National Congress of Applied Mechanics*, Ann Arbor, 1954, pp. 623–628. *American Society of Mechanical Engineers*, New York, 1955.

[41] C. S. Yih, Stability of liquid flow down an inclined plane, *Phys. Fluids*, **6** (1963), 321–334.

Appendix A

Proofs of lemmas

A.1 Proof of Lemma 1.3.1

Put $f := u_y + \delta^2 v_x$. Then it holds that

$$\begin{cases} u_x + v_y = 0 & \text{in } \Omega, \\ u_y + \delta^2 v_x = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \Sigma. \end{cases}$$

Taking Fourier series expansion with respect to x, we have

$$\begin{cases} in\hat{u}_n + \hat{v}_n' = 0, \\ \hat{u}_n' + \mathrm{i}n\delta^2\hat{v}_n = \hat{f}_n, \\ \hat{u}_n(0) = \hat{v}_n(0) = 0, \end{cases}$$

which can be written in the following matrix form

$$\hat{\boldsymbol{u}}_n' = A\hat{\boldsymbol{u}}_n + \hat{\boldsymbol{f}}_n,$$

where

$$\hat{\boldsymbol{u}} = \begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix}, \quad \hat{\boldsymbol{f}}_n = \begin{pmatrix} \hat{f}_n \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\mathrm{i}n\delta^2 \\ -\mathrm{i}n & 0 \end{pmatrix}.$$

The solution of this initial value problem is given by

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \int_0^y \mathrm{e}^{A(y-z)} \begin{pmatrix} \hat{f}_n(z) \\ 0 \end{pmatrix} \mathrm{d}z.$$

Since

$$\mathrm{e}^{At} = \begin{pmatrix} \cos(n\delta t) & -\mathrm{i}\delta\sin(n\delta t) \\ -\frac{\mathrm{i}}{\delta}\sin(n\delta t) & \cos(n\delta t) \end{pmatrix},$$

we have

$$\hat{\boldsymbol{u}}_n(y) = \int_0^y \hat{f}_n(z) \begin{pmatrix} \cos\left(n\delta(y-z)\right) \\ -\frac{\mathrm{i}}{\delta}\sin\left(n\delta(y-z)\right) \end{pmatrix} \mathrm{d}z. \tag{A.1.1}$$

Differentiating (A.1.1) with respect to y, we have

$$\hat{\boldsymbol{u}}_n'(y) = \begin{pmatrix} \hat{f}_n(y) \\ 0 \end{pmatrix} - \int_0^y \hat{f}_n(z) \begin{pmatrix} n\delta \sin(n\delta(y-z)) \\ -in\cos(n\delta(y-z)) \end{pmatrix} \mathrm{d}z. \tag{A.1.2}$$

On the other hand, by Parseval's identity, inequality (1.3.1) is equivalent to

$$\sum_{n\in\mathbb{Z}} \int_0^1 (|\hat{u}_n'|^2 + n^2\delta^4|\hat{v}_n|^2)\mathrm{d}y \le K \sum_{n\in\mathbb{Z}} \int_0^1 (n^2\delta^2|\hat{u}_n|^2 + \delta^2|\hat{v}_n'|^2 + |\hat{f}_n|^2)\mathrm{d}y.$$

Substituting (A.1.1) and (A.1.2) for the above equality, we see that (1.3.1) is equivalent to

$$\sum_{n\in\mathbb{Z}} \int_0^1 \left\{ \left| \hat{f}_n(y) - n\delta \int_0^y \sin(n\delta(y-z))\hat{f}_n(z)\mathrm{d}z \right|^2 + \left| n\delta \int_0^y \sin(n\delta(y-z))\hat{f}_n(z)\mathrm{d}z \right|^2 \right\} \mathrm{d}y$$

$$\le K \sum_{n\in\mathbb{Z}} \int_0^1 \left\{ \left| n\delta \int_0^y \cos(n\delta(y-z))\hat{f}_n(z)\mathrm{d}z \right|^2 + |\hat{f}_n(y)|^2 \right\} \mathrm{d}y.$$

Therefore, it is sufficient to prove the inequality

$$\begin{aligned} &\int_0^1 (n\delta)^2 \left| \int_0^y \sin(n\delta(y-z))\, \hat{f}_n(z)\mathrm{d}z \right|^2 \mathrm{d}y \\ &\qquad \le K \int_0^1 \left\{ (n\delta)^2 \left| \int_0^y \cos(n\delta(y-z))\, \hat{f}_n(z)\mathrm{d}z \right|^2 + \left| \hat{f}_n(y) \right|^2 \right\} \mathrm{d}y \end{aligned} \tag{A.1.3}$$

for $n = 1, 2, 3, \cdots$, where K is a positive constant independent of δ and n. By extending $\hat{f}_n$ defined on $(0,1)$ by zero to $\mathbb{R}$ and putting

$$a := \frac{\pi}{2n\delta}, \quad g(y,z) := \cos(n\delta(y-z))\hat{f}_n(z),$$

the left hand side of (A.1.3) is evaluated as

$$\begin{aligned} &\int_0^1 (n\delta)^2 \left| \int_0^y \cos\left\{ n\delta\left(\left(y - \frac{\pi}{2n\delta} \right) - z \right) \right\} \hat{f}_n(z)dz \right|^2 \mathrm{d}y \\ &= \int_{-a}^{1-a} (n\delta)^2 \left| \int_0^y g(y,z)\mathrm{d}z + \int_y^{y+a} g(y,z)\mathrm{d}z \right|^2 \mathrm{d}y \\ &\le 2\int_{-a}^{1-a} \left\{ (n\delta)^2 \left| \int_0^y g(y,z)\mathrm{d}z \right|^2 \right\} \mathrm{d}y + 2(n\delta)^2 \int_{-a}^{1-a} \left| \int_y^{y+a} g(y,z)\mathrm{d}z \right|^2 \mathrm{d}y \\ &\le 2\int_0^1 \left\{ (n\delta)^2 \left| \int_0^y g(y,z)\mathrm{d}z \right|^2 \right\} \mathrm{d}y + \pi n\delta \int_{-a}^{1-a} \int_y^{y+a} |\hat{f}(z)|^2 \mathrm{d}z\mathrm{d}y, \end{aligned} \tag{A.1.4}$$

where we used Schwarz' inequality in the last line. The first term of (A.1.4) is bounded by the right hand side of (A.1.3) for $K = 2$. Since

$$\begin{aligned} \int_{-a}^{1-a} \int_y^{y+a} |\hat{f}(z)|^2 dzdy &\le \int_0^1 \int_{z-a}^z |\hat{f}(z)|^2 \mathrm{d}y\mathrm{d}z \\ &= a \int_0^1 |\hat{f}(z)|^2 \mathrm{d}y\mathrm{d}z, \end{aligned}$$

the inequality (A.1.3) is satisfied if we take $K \ge \frac{\pi^2}{2}$. $\square$

A.2 Proof of Lemma 1.3.3

.

First, we consider the case that $f(x,y)$ is 1-periodic in x and it is defined for all $y \in \mathbb{R}$. It is well-known that

$$f(x,y) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_n(\eta) \mathrm{e}^{2\pi \mathrm{i}(nx+\eta y)} \mathrm{d}\eta, \quad \text{where } \hat{f}_n(\eta) = \int_0^1 \int_{\mathbb{R}} f(x,y) e^{-2\pi \mathrm{i}(nx+\eta y)} \mathrm{d}y\mathrm{d}x.$$

Put $\phi(x) := f(x,0)$, then we see that

$$\hat{\phi}_n = \int_{\mathbb{R}} \hat{f}_n(\eta)\mathrm{d}\eta,$$

so that

$$|\hat{\phi}_n| \le \left(\int_{\mathbb{R}} (1+|n\delta|+|\eta|)^{-2} \mathrm{d}\eta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1+|n\delta|+|\eta|)^{2} |\hat{f}_n(\eta)|^2 \mathrm{d}\eta \right)^{\frac{1}{2}}.$$

In view of

$$\int_{\mathbb{R}} (1+|n\delta|+|\eta|)^{-2} \mathrm{d}\eta = \frac{2}{1+|n\delta|},$$

we have

$$\sum_{n\in\mathbb{Z}} (1+|n\delta|)|\hat{\phi}_n|^2 \lesssim \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}} (1+|n\delta|+|\eta|)^{2} |\hat{f}_n(\eta)|^2 d\eta \right).$$

Thanks to Parseval's identity, it is equivalent to

$$|\phi|_0^2 + \delta ||D_x|^{\frac{1}{2}}\phi|^2 \lesssim \|f\|^2 + \delta^2\|f_x\|^2 + \|f_y\|^2.$$

Next, we consider the case that $f(x,y)$ is 1-periodic in x and is defined for all $y \ge 0$. We extend $f(x,y)$ to $\mathbb{R}$ as

$$F(x,y) := \begin{cases} f(x,y) & \text{for } y \ge 0, \\ f(x,-y) & \text{for } y < 0. \end{cases}$$

Using the result of previous case, we have

$$|f|_0^2 + \delta ||D_x|^{\frac{1}{2}} f|_0^2 \lesssim \|f\|^2 + \delta^2\|f_x\|^2 + \|f_y\|^2.$$

Finally, we consider the case that $F(x,y)$ is 1-periodic in x and is defined for $0 \le y \le 1$. We introduce cutoff function $\lambda(y) \in C_0^\infty[0,\infty)$ such that

$$\begin{cases} \lambda(1) = 1, \\ \lambda(y) = 0 \quad \text{for } 1 \ge y, \end{cases}$$

and define $\tilde{F}(x,y) \in H^1(\mathbb{T}\times[0,\infty))$ as

$$\tilde{F}(x,y) = \begin{cases} \lambda(y) f(x,y) & \text{for } 0 \le y \le 1, \\ 0 & \text{for } y \ge 1. \end{cases}$$

By the result of previous case, we obtain

$$\begin{aligned}
|\tilde{F}|^2 + \delta||D_x|^{\frac{1}{2}}\tilde{F}|^2 &\lesssim \|\tilde{F}\|^2 + \delta^2\|\tilde{F}_x\|^2 + \|\tilde{F}_y\|^2 \\
&\lesssim \|F\|^2 + \delta^2\|F_x\|^2 + \|\lambda' F + \lambda F_y\|^2 \\
&\lesssim \|F\|^2 + \delta^2\|F_x\|^2 + \|F_y\|^2,
\end{aligned}$$

which gives a desired estimate. □

A.3 Proofs of Lemmas 1.3.5–1.3.7

Proof of Lemma 1.3.5. By the Sobolev embedding theorem, we see that

$$\begin{aligned}
|f(x,y)|^2 = |f(x,y) - f(x,0)|^2 &= \left|\int_0^y f_y(x,y)\mathrm{d}y\right|^2 \\
&\leq \int_0^1 |f_y(x,y)|^2\mathrm{d}y \lesssim \int_0^1 \big(\|f_y(\cdot,y)\|_{L^2(\mathbb{G})}^2 + \|f_{xy}(\cdot,y)\|_{L^2(\mathbb{G})}^2\big)\mathrm{d}y,
\end{aligned}$$

which is the desired inequality. □

Proof of Lemma 1.3.6. By the well-known inequality

$$\|\partial_x^k(af)(\cdot,y)\|_{L^2(\mathbb{G})} \lesssim \|a(\cdot,y)\|_{L^\infty(\mathbb{G})}\|\partial_x^k f(\cdot,y)\|_{L^2(\mathbb{G})} + \|f(\cdot,y)\|_{L^\infty(\mathbb{G})}\|\partial_x^k a(\cdot,y)\|_{L^2(\mathbb{G})},$$

and the Sobolev embedding theorem, we see that

$$\begin{aligned}
\|\partial_x^k(af)\|^2 &\lesssim \int_0^1 \big(\|a(\cdot,y)\|_{L^\infty(\mathbb{G})}^2\|\partial_x^k f(\cdot,y)\|_{L^2(\mathbb{G})}^2 + \|f(\cdot,y)\|_{L^\infty(\mathbb{G})}^2\|\partial_x^k a(\cdot,y)\|_{L^2(\mathbb{G})}^2\big)\mathrm{d}y \\
&\lesssim \|a\|_{L^\infty}^2\|\partial_x^k f\|^2 + \sup_{y\in(0,1)}\|\partial_x^k a(\cdot,y)\|_{L^2(\mathbb{G})}^2 \int_0^1 \|f(\cdot,y)\|_{L^\infty(\mathbb{G})}^2\mathrm{d}y \\
&\lesssim \|a\|_{L^\infty}^2\|\partial_x^k f\|^2 + (\|\partial_x^k a\|^2 + \|\partial_x^k a_y\|^2)(\|f\|^2 + \|f_x\|^2).
\end{aligned}$$

We can prove the second inequality in a similar way. □

Proof of Lemma 1.3.7. In view of the well-known inequality

$$\|[\partial_x^k, a]f(\cdot,y)\|_{L^2(\mathbb{G})} \lesssim \|a(\cdot,y)\|_{L^\infty(\mathbb{G})}\|\partial_x^{k-1} f(\cdot,y)\|_{L^2(\mathbb{G})} + \|f(\cdot,y)\|_{L^\infty(\mathbb{G})}\|\partial_x^k a(\cdot,y)\|_{L^2(\mathbb{G})},$$

the desired inequality follows in a similar way as the proof of Lemma 1.3.6. □