Game Theoretical Models on Networks and Discrete Structures

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Preface

In this thesis, we deal with two game theoretical models on networks, the multi-unit trading network model and the marking game. We analyze these models by focusing on discrete structures of them.

After an introduction chapter (Chapter 1), we introduce general terminology of graphs in the beginning of Chapter 2. We then introduce discrete convex analysis, which plays an important role in analyzing the multi-unit trading network model. Moreover, we describe fundamentals of game theory and microeconomics related to this thesis.

In Chapter 3, we consider the multi-unit trading network model. This model is an extension of the single-unit trading network model introduced by Hatfield et al. By extending their model, we can handle multiple units of contracts with the same unit price. In Section 3.1, we give definitions of the model, and then review the previous results shown by Hatfield et al. In the first subsection of Section 3.2, we show that twisted $M^\natural$-concavity and the generalized full substitutes condition are equivalent. This fact is a reason why we use discrete convex analysis to analyze the model. By using discrete convex analysis, we show that there always exists a competitive equilibrium and that the set of competitive equilibrium price vectors forms a lattice under the generalized full substitutes condition. In Section 3.3, we consider the relationship among competitive equilibria, efficient trade vectors, and stable outcomes. In Section 3.4, we study three stability concepts, stability, strong group stability, and chain stability. Most of our results in Sections 3.3 and 3.4 are also proven by using discrete convex analysis.

In Chapter 4, we consider the marking game, which is one of the 2-person zero-sum games. This game is played by Alice and Bob and closely related to graph coloring. In Section 4.1, we give formal definitions of the marking game. We also introduce the game coloring number which is defined as the smallest score such that Alice has a winning strategy in the marking game. The game coloring number gives an upper bound for the game chromatic number which is defined through the chromatic game. In Section 4.2, we estimate an upper bound for the game coloring number of planar graphs with girth at least 4. In Section 4.3, we estimate lower bounds for the game coloring number of planar graphs with girth 4 and 5, respectively.

Contents

Chapter 1

Introduction

In this thesis, we focus on two models which are both related to combinatorial optimization and game theory.

Combinatorial optimization is one of the biggest research topics in applied mathematics. Many people have been studying a lot of combinatorial optimization problems, for example, the maximum flow problem, the graph coloring problem, the maximum matching problem, and so on. The aim is to solve these problems, that is, to find the best solution from the feasible solutions. Combinatorial optimization is also applied in several areas such as operations research, algorithmic theory, auction theory, artificial intelligence, and so on (see [39, 52, 58]).

Since the number of feasible solutions is very huge in general, it is not practical to check all of them. Indeed, there are many combinatorial optimization problems which cannot be solved in polynomial time (unless $P = NP$). However, there are polynomial time algorithms for solving some combinatorial optimization problems (e.g. the maximum flow problem [14], the maximum matching problem [9]). We can efficiently solve some of combinatorial optimization problems.

Solving combinatorial optimization problems efficiently, we often focus on discrete structures like graphs and matroids. Graphs are one of the most popular structures in combinatorics and discrete mathematics. Most of combinatorial optimization problems can be represented in terms of graphs and some properties of graphs are useful to solve them. Matroids are also discrete structures which generalize the notion of linear independence in vector system [56, 70, 71]. They are related to discrete convex analysis, which is a theoretical framework of discrete optimization proposed by Murota [43, 45]. It is based on convex analysis and matroid theory. A lot of combinatorial optimization problems can be analyzed by using discrete convex analysis. In this thesis, graphs and discrete convex analysis play important roles.

On the other hand, game theory is the study of mathematical models in which two or more players interact with each other. In the early 20th century, some people analyzed a game theoretical situation by using mathematics. For instance, Zermelo [73] studied the optimal chess strategy and von Neumann [67] showed the minimax theorem for the 2-person zero-sum game. Then von Neumann and Morgenstern published the book "Theory of Games and Economic Behavior" [68] which built on the basics of game theory. Since then, game theory has been remarkably developed by many people. At present, it is applied in several fields including economics, computer science, biology, and so on [5, 6, 53].

Unlike basic optimization models, there is conflict between players in game theoretical models. This feature makes problems more complicated because we have to consider strategies of each player. However, many positive results are shown. For example, Nash [51] proved that there exists an equilibrium point, called the Nash equilibrium, in non-cooperative n-person games.

In this thesis, we focus on two models although there are many types of games. Both of them are models on networks (graphs) and closely related to combinatorial optimization and game theory. The first model is called the "*multi-unit trading network model*," which is a generalization of the stable matching model. This model is also related to the market game. However, unlike the market game, we deal with indivisible commodities and utility functions of agents are defined on the set of trade vectors in the multi-unit trading network model. We study competitive equilibria and stable outcomes in the multi-unit trading network model by using discrete convex analysis. The second model is the "*marking game*," which is a 2-person zero-sum game on graphs with perfect information. We analyze strategies for players and consider the game coloring number of graphs, which is associated with the game.

1.1 Multi-unit trading network model

We give introduction of the *multi-unit trading network model.* This model is an extension of the two-sided market model. The feature of the model is that we can handle multiple units of contracts which cannot be handled in the previous models. We use discrete convex analysis to analyze the multi-unit trading network model. We first explain background and history of the two-sided market model. Then we describe our motivation and the outline of the results of the model.

The two-sided market model is well known and well studied in the fields of game theory, economics, and operations research. Many significant results of the two-sided market model are known and it has been successfully applied to many real world problems (e.g. college admissions problem [57], hospitals/residents problem [29]).

All variants and extensions of the two-sided market model stem from the stable matching model due to Gale and Shapley [20] and the assignment model due to Shapley and Shubik [61]. These two models differ in that the assignment model allows commodity prices, while the stable matching model does not. At present, there are many extensions and generalizations of these two models, not all of which can be clearly classified as being based on which one. In fact, some models can be viewed as generalizations of both models. Moreover, results on the stable matching model can be interpreted into results on the assignment model, and vice versa. Our model, the multi-unit trading network model, also can be regarded as a common extension of the two models.

We introduce the stable matching model and extensions which are related to our model. The stable matching model was first proposed in 1962 by Gale and Shapley [20]. They introduced the deferred acceptance algorithm by which they proved the existence of stable matchings. Hatfield and Milgrom [29] introduced the concept of contracts, which allows discrete prices to be implicitly considered. They showed that under the condition called the substitutes condition, stable outcomes always exist and that they form a lattice structure. Ostrovsky [55] further generalized the Hatfield-Milgrom model by extending the two-sidedness to the supply chain network, which is represented by an acyclic directed graph. Furthermore, he introduced two new conditions, the same side substitutability (SSS) and cross side complementarity (CSC). He showed the existence of chain stable allocations when these conditions are satisfied.

On the other hand, the assignment model was proposed by Shapley and Shubik [61] in a game theoretical setting. They proved the nonemptiness of the core and showed that the core has a lattice structure. Then Sotomayor [64] extended this result to the multiple-partners assignment model. Kelso and Crawford [35] introduced the two-sided job matching model in which the gross substitutes condition plays an important role. The gross substitutes condition[1] is an often-used condition in economics. Indeed, there exists a core allocation in the two-sided job matching model, if preferences of firms satisfy the gross substitutes condition [35]. Roughly speaking, the gross substitutes condition requires that when the salary of some worker increases, no firms have an incentive to fire the workers whose salaries do not change. There are some conditions which are equivalent to the gross substitutes condition. Gul and Stacchetti [23] introduced the two new conditions called the single improvement condition and the no complementarity condition and proved that these two conditions are equivalent to the gross substitutes condition. Recently, in the same way as Ostrovsky [55] extended the two-sided stable matching model to acyclic directed graphs, Hatfield et al. [27] generalized the assignment model to general networks (graphs). They defined the full substitutes condition, which is a generalization of the SSS and CSC condition, and showed that there exists a competitive equilibrium when preferences of all agents satisfy the full substitutes condition. They also studied the relationship between competitive equilibria and stable outcomes.

[1]The gross substitutes condition also appears in auction theory. Gul and Stacchetti [24] used the gross substitutes condition in an ascending auction. They showed that there exists an auction mechanism which converges to the smallest Walrasian equilibrium prices if the preferences of all bidders satisfy the gross substitutes condition. Sun and Yang [65, 66] extended the gross substitutes condition to the gross substitutability and complementarity (GSC) condition and showed that there exists a Walrasian equilibrium in the auction model under the GSC condition.

The above models consider only single-unit trades, that is, all trades are restricted to one unit of commodity. In the stable matching model, trades can be generalized to multi-units by creating duplicate trades, however, as we have to consider commodity prices in the assignment model, this technique does not work. One of our motivation is to generalize the model of Hatfield et al. [27] (we call this model the *single-unit trading network model*) to multi-units. To accomplish this, we believe that the most useful tool is discrete convex analysis due to Murota [43, 45].

Discrete convex analysis, which is centered on $\mathrm{M}^\natural$ and $\mathrm{L}^\natural$-convexity, has been successfully utilized in various economical models (for more details, see [7, 8, 10, 17, 18, 40, 62]). In particular, $\mathrm{M}^\natural$-concavity has many nice properties. For example, Fujishige and Yang [19] showed the equivalence of the gross substitutes condition and $\mathrm{M}^\natural$-concavity on the unit hypercube. Ikebe and Tamura [33] showed that twisted $\mathrm{M}^\natural$-concave functions (a variant of $\mathrm{M}^\natural$-concave functions) satisfy the SSS and CSC conditions. Recently, Shioura and Yang [63] considered the auction model, and introduced the generalized gross substitutes and complements (GGSC) condition, which is a generalization of the GSC condition introduced by Sun and Yang [65, 66]. They showed that the GGSC condition is equivalent to twisted $\mathrm{M}^\natural$-concavity. Moreover, they proved that under the GGSC condition, a Lyapunov function is a generalized $\mathrm{L}^\natural$-convex function and the set of Walrasian equilibrium price vectors forms a good structure, called a generalized lattice.

In this thesis, we deal with the multi-unit trading network model, which is an extension of the single-unit trading network model due to Hatfield et al. [27]. In our model, valuation functions of each agent are twisted $\mathrm{M}^\natural$-concave functions defined on the set of integer vectors. This allows us to consider multiple units of contracts, which cannot be handled in the single-unit trading network model. We also introduce the generalized full substitutes condition and show that this condition is equivalent to twisted $\mathrm{M}^\natural$-concavity.

Our main results of the multi-unit trading network model concern the existence of competitive equilibria and the structure of competitive equilibrium price vectors. We show that there exists a competitive equilibrium under the generalized full substitutes condition and that the set of competitive equilibrium price vectors forms a lattice. We also consider the relationship among competitive equilibria, stability, and efficiency. Although these concepts are not equivalent, we prove that a competitive equilibrium is stable, and a competitive equilibrium trade vector is efficient. Moreover, we show that we can construct a competitive equilibrium from a stable outcome by changing the price vector under the generalized full substitutes condition. Unlike the single-unit trading network model, there is an obvious gap between competitive equilibria and stable outcomes with respect to price vectors in our model.

Furthermore, we deeply study stability in the model. Although there are many types of stability concepts, we focus on the three stability concepts in this thesis: stability, strong group stability, and chain stability. While these concepts are not equivalent in general, we show that these concepts are equivalent under the generalized full substitutes condition. The greater part of our results are shown by using discrete convex analysis, which gives a novel view of the trading network model.

1.2 Marking game

Games associated with graphs are being considered by many people and there are many types of games (e.g. the hat game [13], the cops and robbers game [3], the Maker-Breaker game [31]). In this thesis, we focus on the *marking game*.

The origin of the marking game is the *map-coloring game*. The map-coloring game was invented by Brams in the study of the Four Color Theorem and first published by Gardner [1, 21]. The map-coloring game is played by two players, Alice and Bob, with Alice playing first. Given a simple planar graph G and a color set C. They take turns coloring uncolored regions (faces) of G with a color in C so that any two adjacent regions are colored with different colors. Alice wins the game if all regions are colored properly, otherwise Bob wins the game. By the rule of the game, whether Alice has a winning strategy for the map-coloring game depends on the structure of a graph and the number of available colors.

As we mentioned above, the map-coloring game was first defined only on planar graphs. Then Bodlaender [2] extended the concept of the game to general graphs. We call the extended game the *chromatic game*. In the chromatic game, players color vertices of a graph instead of regions (vertices and regions of a planar graph are essentially equivalent because we can regard a region of a planar graph as a vertex of its dual graph). The *game chromatic number* of a graph is the smallest number of colors such that Alice has a winning strategy.

To analyze the game chromatic number, Zhu [74] introduced the new concept, the *game coloring number*. The game coloring number is related to the coloring number in graph theory introduced by Erdös and Hajnal [11]. The coloring number was introduced to study graph coloring. Similarly, we use the game coloring number to analyze the game chromatic number.

The game coloring number is defined through the marking game. Although the marking game is similar to the chromatic game, some settings are different. In the marking game, players "mark" vertices of a graph instead of coloring them. The *score* of the marking game is determined when the game ends. The definition of the score of the marking game on graph G is given by

$$s = 1 + \max_{v \in V(G)} b(v),$$

where $b(v)$ is the number of neighbors of v which are marked before v. Alice wins the game if the score of the game is at most k, which is given before starting the game, otherwise Bob wins the game. The game coloring number of a graph is the minimum number k such that Alice has a winning strategy for the marking game.

The game coloring number gives an upper bound for the game chromatic number by their definitions. Since we handle the game coloring number more easily than the game chromatic number, many researchers analyze an upper bound for the game coloring number to estimate that of the game chromatic number. For several classes of graphs, the exact game coloring numbers are determined. We give the game coloring numbers

Table 1.1: Game coloring numbers of graphs (ω is the clique number).

Graph classes	forests	interval graphs	outerplanar graphs	k-trees ($k \geq 2$)
Game coloring number	4	$3\omega - 2$	7	$3k + 2$

Table 1.2: Upper bounds for the game coloring numbers of planar graphs.

Girth	3–4	5	6–7	8 and over
Upper bounds	17	8	6	5

of some classes of graphs in Table 1.1 (these results were proven in [12, 22, 37, 72, 75]). The values in Table 1.1 are exact, that is,

- there exists Alice's strategy such that the score of the game is at most the value in Table 1.1.
- there exists a graph of which the game coloring number is at least the value in Table 1.1.

One big problem is determining the exact value of the game coloring number of planar graphs. At present, the best known upper bound is 17 [76] and the best known lower bound is 11 [72]. From the point of the edge density of planar graphs, we think that 17 is a quite large upper bound. Therefore, some people researched the game coloring number of planar graphs restricted their girth, which belong to between forests and plane triangulations. He et al. [30] proved that the game coloring number of planar graphs with girth at least 5 is at most 8. Kleitman [38] showed that the game coloring number of planar graphs with girth at least 6 is at most 6. Other known upper bounds are given in Table 1.2. Moreover, Borodin et al. [4] showed that the game coloring number of quadrangle-free planar graphs is at most 9.

In this thesis, we deal with the game coloring number of planar graphs with girth 4 and 5. We show the following statements:

- The game coloring number of planar graphs with girth 4 is at most 13.

- The game coloring number of planar graphs with girth 4 is at least 7.
- The game coloring number of planar graphs with girth 5 is at least 6.

Considering Alice's strategy, called the *activation strategy*, is helpful to show the first statement. The activation strategy depends on a total order on the vertex set of a graph. We prove the first statement by giving an appropriate total order on the vertex set. The last two statements are shown by constructing graphs of which the game coloring numbers are above values.

1.3 Summary of the thesis

The rest of this thesis is organized as follows. We first give some preliminaries. In Chapter 2, we give definitions and notations of graphs, and introduce discrete convex analysis, game theory, and microeconomics related to this thesis. In Chapter 3, we deal with the multi-unit trading network model. In Section 3.1, we introduce the model and review the results on the single-unit trading network model. We next give the formal definition of the generalized full substitutes condition in Section 3.2. In this section, we also discuss the existence and lattice structure of competitive equilibria. We then consider the relationship among competitive equilibria, stability, and efficiency in Section 3.3. We study three stability concepts in Section 3.4. In Chapter 4, we discuss the game coloring number of graphs. In Section 4.1, we introduce the marking game and define the game coloring number. We consider the upper bounds for the game coloring number of planar graphs in Section 4.2 and the lower bounds in Section 4.3.

Chapter 2

Preliminaries

Before we take up the main issue, we give some preliminaries. In the beginning of this chapter, we give fundamental notations and define some terminology of graphs. We then introduce the fundamentals of discrete convex analysis, game theory, and microeconomics.

2.1 Basic notations and orders

We denote by $\mathbf{R}$ the set of reals and by $\mathbf{Z}$ the set of integers. We also denote by $\mathbf{R}_+$ the set of nonnegative reals and by $\mathbf{Z}_+$ the set of nonnegative integers.

Let V be a finite set. For vectors $x, y \in \mathbf{R}^V$, the *inner product* of x and y is denoted by $\langle x, y \rangle$, that is,

$$\langle x, y \rangle = \sum_{t \in V} x_t \cdot y_t.$$

A relation $\preceq$ on a set V is called a *partial order* if the following three conditions hold for all $x, y, z \in V$:

- $x \preceq x$ (reflexivity);
- $x \preceq y$ and $y \preceq x \Rightarrow x = y$ (anti-symmetry);
- $x \preceq y$ and $y \preceq z \Rightarrow x \preceq z$ (transitivity).

A partial order $\preceq$ is called a *total order* if $x \preceq y$ or $y \preceq x$ holds for all $x, y \in V$.

2.2 Preliminaries on graphs

In this section, we give some preliminaries on graphs. For more details, see [39, 58].

An *undirected graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a set of 2-element subset of $V(G)$, that is,

$$E(G) \subseteq \{U \subseteq V(G) : |U| = 2\}.$$

The elements of $V(G)$ are called *vertices* and the elements of $E(G)$ are called *edges*.

A *directed graph* is also a pair $G = (V(G), E(G))$, where $V(G)$ is a finite set of vertices and $E(G)$ is defined by

$$E(G) \subseteq \{(u, w) \in V(G) \times V(G) : u \neq w\}.$$

The elements of $E(G)$ are called *directed edges*.

We say that edge $\{u, w\}$ (or directed edge (u, w)) *connects* u and w. Vertices u and w are *adjacent* if there exists an edge (or a directed edge) which connects u and w. For a directed graph, we say that directed edge (u, w) *leaves* u and *enters* w.

The *underlying undirected graph* G' of a directed graph G is the undirected graph which satisfies

$$V(G') = V(G), \quad E(G') = \{\{u, w\} : (u, w) \in E(G)\}.$$

A graph G' is called a *subgraph* of an undirected graph G if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$, and for $\{u, w\} \in E(G')$ it holds that $u, w \in V(G')$.

For a vertex u of an undirected graph G, we define

$$\delta_G(u) = \{e \in E(G) : u \in e\}, \quad N_G(u) = \{v \in V(G) : \{u, v\} \in E\}.$$

The element of $N_G(u)$ is called a *neighbor*. The *degree* of vertex u is defined by

$$d_G(u) = |\delta_G(u)|.$$

The *maximum degree* of G is defined by

$$\Delta(G) = \max\{d_G(v) : v \in V(G)\}.$$

In the directed case, we define

$$\delta_G^+(u) = \{(v,w) \in E(G) : v = u\}, \qquad \delta_G^-(u) = \{(v,w) \in E(G) : w = u\},$$
$$N_G^+(u) = \{v \in V(G) : (u,v) \in E(G)\}, \qquad N_G^-(u) = \{v \in V(G) : (v,u) \in E(G)\}.$$

The element of $N_G^+(u)$ is called an *out-neighbor* and the element of $N_G^-(u)$ is called an *in-neighbor*. The *out-degree* and *in-degree* of vertex u are respectively defined by

$$d_G^+(u) = |\delta_G^+(u)|, \quad d_G^-(u) = |\delta_G^-(u)|.$$

2.2.1 Paths, cycles, and connectivity

A *walk* in a graph G is a sequence

$$P = v_0, e_1, v_1, \ldots, e_k, v_k$$

such that $k \geq 0$, $e_i = \{v_{i-1}, v_i\} \in E(G)$ (or $e_i = (v_{i-1}, v_i) \in E(G)$) for all $i = 1, \ldots, k$, and $e_1, \ldots, e_k$ are all distinct. In this case, P is also called a v_0-v_k walk. If $v_0 = v_k$, P is said to be a *closed walk*. A walk is called *Eulerian* if it is a closed walk and contains every edge of the graph. A walk P is called a v_0-v_k *path* if $v_i \neq v_j$ for all $0 \leq i < j \leq k$. The vertices v_0 and v_k are called the *end points* of P. The *length* of walk P is defined by k. A *cycle* is a walk $v_0, e_1, v_1, \ldots, e_k, v_k$ with $k \geq 1$ which is a closed walk and $v_1, \ldots, v_k$ are all distinct. The length of the shortest cycle in a graph G is called the *girth* of G.

We can consider a path P as a graph whose vertex set and edge set are

$$V(P) = \{v_0, v_1, \ldots, v_k\}, \quad E(P) = \{e_1, \ldots, e_k\},$$

respectively. Similarly, we can consider a cycle as a graph.

A directed graph G is called *acyclic* if it has no cycles. A *chord* of a cycle P in a graph G is an edge $e \in E(G) \setminus E(P)$ which connects two vertices in P.

An undirected graph G is *connected* if there exists a u-w path for all $u, w \in V(G)$. A maximal connected nonempty subgraph is called a *connected component*. For a directed graph, we also define connectivity. A directed graph G is connected if its underlying undirected graph is connected.

At the end of this subsection, we introduce the proposition about Eulerian walks, which is a very famous result in graph theory.

Proposition 2.1. *A connected undirected graph has an Eulerian walk if and only if the degree of each vertex is even. A connected directed graph G has an Eulerian walk if and only if $d_G^+(v) = d_G^-(v)$ for all $v \in V(G)$.*

2.2.2 Classes of graphs

We introduce some classes of graphs. In this subsection, we consider only undirected graphs.

An undirected graph is a *complete graph* if any two vertices are adjacent. We denote the complete graph with n vertices by K_n.

Let G be an undirected graph. A subset M of $E(G)$ is called a *matching* of G if $e \cap f = \emptyset$ for any $e, f \in M$. A matching M is said to be a *perfect matching* if for all $v \in V(G)$, there exists $e \in M$ such that $v \in e$ (then we say that M *covers* v). A subset C of $V(G)$ is called a *stable set* of G if every two distinct vertices in C are not adjacent, and a *clique* if every two distinct vertices in C are adjacent. We define the maximum size of a clique of G, the *clique number* of G, by

$$\omega(G) = \max\{|C| : C \text{ is a clique in } G\}.$$

A *coloring* of an undirected graph G is a partition of $V(G)$ into stable sets. Then each stable set is called a *color*. The *chromatic number* of G is the minimum number of colors in colorings and is defined by $\chi(G)$. If $\chi(G) \leq 2$, G is called a *bipartite* *graph*. A bipartite graph G is called *complete* if $V(G)$ can be partitioned into two sets U and W ($U \cup W = V(G), U \cap W = \emptyset$) such that U and W are colors (stable sets) and $\{u, w\} \in E(G)$ for all $u \in U$ and $w \in W$. We denote the complete bipartite graph with color size m and n by $K_{m,n}$.

A graph is called a *forest* if it has no cycles. A *tree* is a connected forest.

A *chordal graph* is a graph in which every cycle with length at least 4 has a chord. Let

$$\begin{aligned} V(G) &= \{[a_i, b_i] : i = 1, \ldots, n\}, \\ E(G) &= \{\{[a_i, b_i], [a_j, b_j]\} : i \neq j, [a_i, b_i] \cap [a_j, b_j] \neq \emptyset\}, \end{aligned}$$

where $[a_i, b_i]$ is an interval, that is,

$$[a_i, b_i] = \{x \in \mathbf{R} : a_i \leq x \leq b_i\}.$$

Then $G = (V(G), E(G))$ is called an *interval graph*. We next define a k-*tree*. It is defined inductively as follows:

(i) K_k is a k-tree.

(ii) Let G be a k-tree and C be a clique in G with size k. Then a graph which is obtained from G by adding vertex v and the edges between v and each vertex of C is a k-tree.

Interval graphs and k-trees are special cases of chordal graphs. A *partial* k-*tree* is a subgraph of a k-tree.

A graph which can be embedded in the plane $\mathbf{R}^2$ is called a *planar graph*. Such an embedding is called a *planar embedding*. A graph is called an *outerplanar graph* if it can be embedded in the plane $\mathbf{R}^2$ such that each vertex is in the outer boundary. A maximal outerplanar graph is a special case of a 2-tree.

Planar graphs have the following property, which is well known in graph theory.

Proposition 2.2. *For a planar graph $G = (V(G), E(G))$ with $|V(G)| \geq 3$, we have*

$$|E(G)| \leq 3|V(G)| - 6,$$
$$\sum_{v \in V(G)} d_G(v) < 6|V(G)|.$$

2.3 Fundamental results in discrete convex analysis

Discrete convex analysis, which is proposed by Murota [43], is a framework for discrete optimization problems. It is based on convex analysis and matroid theory. In other words, the framework of convex analysis is applied to discrete settings and results in matroid theory is extended to continuous settings in discrete convex analysis.

We first consider continuous settings. We define the *effective domain* of $f : \mathbf{R}^V \to \{\pm\infty\}$ by

$$\mathrm{dom}\, f = \{x : -\infty < f(x) < +\infty\}.$$

The sets of minimizers and maximizers of f are denoted by $\mathrm{argmin}\, f$ and $\mathrm{argmax}\, f$, respectively, that is,

$$\mathrm{argmin}\, f = \{x : f(x) \leq f(y)\ (y \in \mathbf{R}^V)\},$$
$$\mathrm{argmax}\, f = \{x : f(x) \geq f(y)\ (y \in \mathbf{R}^V)\}.$$

The following is the definition of convex functions.

Definition 2.3. *A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is* convex *if*

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)$$

holds for all $x, y \in \mathbf{R}^V$ and $\lambda \in [0, 1]$.

A set $S \subseteq \mathbf{R}^V$ is called *convex* if for any $x, y \in S$, it holds that $\lambda x + (1-\lambda)y \in S$ for all $\lambda \in [0,1]$. For a finite number of vectors $x_1, \ldots, x_n \in S$,

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \quad \left(\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0 \, (\forall i = 1, \ldots, n) \right)$$

is called a *convex combination* of these vectors. The *convex hull* of S is the smallest convex set that contains S.

We now consider discretizing convex functions. In discrete convex analysis, two classes of convex functions, M/M$^\natural$-convex functions and L/L$^\natural$-convex functions, play an important role. They are conjugate to each other under the Legendre-Fenchel transformation. The letters "M" and "L" stand for the first letter of "matroid" and "lattice," respectively.

The remain of this section is organized as follows. We first introduce the definition of M$^\natural$/L$^\natural$-convex functions. We next introduce conjugacy theorem, and some properties of the sum of two M$^\natural$-concave functions. Moreover, we introduce a twisted M$^\natural$-concave function which is a variant of an M$^\natural$-concave function. We use it in the multi-unit trading network model proposed in Section 3.1. For more information about discrete convex analysis, see [15, 45, 46].

2.3.1 M$^\natural$-convex functions

Originally, Murota [43] proposed M-convex functions. M$^\natural$-convex functions are introduced by Murota and Shioura [47]. M-convex functions and M$^\natural$-convex functions are essentially equivalent although the class of M$^\natural$-convex functions are strictly larger than that of M-convex functions. Therefore, we introduce only M$^\natural$-convex functions in this subsection.

Let V be a finite set. The *characteristic vector* $\chi_X \in \{0,1\}^V$ of a subset $X \subseteq V$ is defined by

$$(\chi_X)_t = \begin{cases} 1 & (t \in X), \\ 0 & (t \in V \setminus X). \end{cases}$$

We use the notation $\chi_0 = \mathbf{0}$ and $\chi_t = \chi_{\{t\}}$, where $\mathbf{0}$ is the all-zero vector (the vector of which all elements are equal to 0). Consider a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$. We define the *positive support* and the *negative support* of a vector $x = (x_t | t \in V) \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(x) = \{t \in V : x_t > 0\}, \quad \mathrm{supp}^-(x) = \{t \in V : x_t < 0\}.$$

We are ready to introduce the definitions of M$^\natural$-convex sets and M$^\natural$-convex functions.

Definition 2.4. *A nonempty set $Q \subseteq \mathbf{Z}^V$ is called an* M$^\natural$-convex set *if for $x, y \in Q$ and $s \in \mathrm{supp}^+(x - y)$ one of the following properties is satisfied:*

(i) $x - \chi_s \in Q$ *and* $y + \chi_s \in Q$*;*

(ii) *there exists $t \in \mathrm{supp}^-(x - y)$ such that $x - \chi_s + \chi_t \in Q$ and $y + \chi_s - \chi_t \in Q$.*

Definition 2.5. *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is an* M$^\natural$-convex function *if for all $x, y \in \mathrm{dom}\, f$ and $s \in \mathrm{supp}^+(x - y)$, there exists $t \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that*

$$f(x) + f(y) \geq f(x - \chi_s + \chi_t) + f(y + \chi_s - \chi_t).$$

A set $S \subseteq \mathbf{Z}^V$ is called *discretely convex* or *hole free* if it holds that

$$S = \bar{S} \cap \mathbf{Z}^V,$$

where $\bar{S}$ is the convex hull of S. An M$^\natural$-convex set is a discretely convex set.

M$^\natural$-convex functions have the following properties (for more detail, see [42, 45]).

Proposition 2.6 (see [45])**.** *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$^\natural$-convex function with a bounded nonempty effective domain. Then the following statements hold.*

(i) $\mathrm{dom}\, f$ *is an M$^\natural$-convex set.*

(ii) $\mathrm{argmin}\, f$ *is an M$^\natural$-convex set.*

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is called an *M$^\natural$-concave* function if $-f$ is M$^\natural$-convex.

We show some examples of M$^\natural$-convex functions.

Example 2.1. Let $Q \subseteq \mathbf{Z}^V$ be an M$^\natural$-convex set. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} 0 & (x \in Q), \\ +\infty & (x \notin Q), \end{cases}$$

is M$^\natural$-convex. We check it easily by using the above definitions.

Example 2.2. Let $\alpha \in \mathbf{R}$ and $p \in \mathbf{R}^V$. A linear function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \alpha + \langle p, x \rangle \quad (x \in \operatorname{dom} f)$$

of which the effective domain is an M$^\natural$-convex set is M$^\natural$-convex (also M$^\natural$-concave).

Example 2.3. Let $\mathcal{T}$ be a *laminar family* of 2^V (i.e. it holds that $X \cap Y = \emptyset$, $X \subseteq Y$, or $Y \subseteq X$ for any $X, Y \in \mathcal{T}$). For any $X \in \mathcal{T}$, we define function $f_X : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$. If f_X is an M$^\natural$-convex function for all $X \in \mathcal{T}$,

$$f(x) = \sum_{X \in \mathcal{T}} f_X(x(X)) \quad (x \in \mathbf{Z}^V)$$

is M$^\natural$-convex, where $x(X) = \sum_{t \in X} x_t$.

Finally, we describe a property of M$^\natural$-convex functions.

Proposition 2.7 ([47])**.** *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M$^\natural$-convex function. For every $x, y \in \operatorname{dom} f$ with $\sum_{t \in V} x_t < \sum_{t \in V} y_t$, there exists $s \in \operatorname{supp}^-(x - y)$ such that*

$$f(x) + f(y) \geq f(x + \chi_s) + f(y - \chi_s).$$

2.3.2 L$^\natural$-convex functions

We give the definition of L$^\natural$-convex functions. L$^\natural$-convex functions are introduced by Fujishige and Murota [16][1].

[1] Analogous to M/M$^\natural$-convex functions, Murota [43] proposed the concept of L-convex functions before they proposed the concept of L$^\natural$-convex functions. L-convex and L$^\natural$-convex functions are essentially equivalent although the class of L$^\natural$-convex functions is strictly larger than that of L-convex functions. Thus, we introduce only L$^\natural$-convex functions.

For vectors $p, q \in \mathbf{R}^V$, we define $p \vee q, p \wedge q \in \mathbf{R}^V$ by

$$(p \vee q)_t = \max\{p_t, q_t\}, \quad (p \wedge q)_t = \min\{p_t, q_t\}.$$

We also define the all-one vector by $\mathbf{1}$. The followings are the definitions of L$^\natural$-convex sets and L$^\natural$-convex functions.

Definition 2.8. *A nonempty set $P \subseteq \mathbf{Z}^V$ is called an* L$^\natural$-convex set *if for $p, q \in P$ it holds that*

$$(p - \alpha\mathbf{1}) \vee q,\, p \wedge (q + \alpha\mathbf{1}) \in P \quad (\forall \alpha \in \mathbf{Z}_+).$$

Definition 2.9. *A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with* $\operatorname{dom} f \neq \emptyset$ *is an* L$^\natural$-convex function *if*

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1}))$$

for all $p, q \in \mathbf{Z}^V$ and $\alpha \in \mathbf{Z}_+$.

Similar to the relationship between M$^\natural$-convex sets and M$^\natural$-convex functions, the following properties are satisfied for L$^\natural$-convex sets and L$^\natural$-convex functions. Moreover, L$^\natural$-convex set is discretely convex.

Proposition 2.10 (see [45])**.** *Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an L$^\natural$-convex function with a bounded nonempty effective domain. Then the following statements hold.*

(i) $\operatorname{dom} g$ *is an L$^\natural$-convex set.*

(ii) $\operatorname{argmin} g$ *is an L$^\natural$-convex set.*

We show some examples of L$^\natural$-convex functions and describe a property of them.

Example 2.4. Let $\alpha \in \mathbf{R}$ and $x \in \mathbf{R}^V$. A linear function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \alpha + \langle p, x \rangle \quad (p \in \operatorname{dom} g)$$

of which the effective domain is an L$^\natural$-convex set is L$^\natural$-convex.

Example 2.5. The function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \max\{p_t : t \in V\}$$

is L$^\natural$-convex.

Proposition 2.11 (see [45])**.** *Let $g_1, g_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be L$^\natural$-convex functions and assume that* $\mathrm{dom}\, g_1 \cap \mathrm{dom}\, g_2 \neq \emptyset$. *Then $g_1 + g_2$ is an L$^\natural$-convex function.*

The sum of L$^\natural$-convex functions is also L$^\natural$-convex, provided its effective domain is nonempty. However, the sum of M$^\natural$-convex functions is not M$^\natural$-convex in general.

2.3.3 M$^\natural$/L$^\natural$-convex functions defined on the set of real vectors

The effective domain of an M$^\natural$-convex (also L$^\natural$-convex) function is a set of integer vectors. We consider M$^\natural$/L$^\natural$-convexity of functions defined on the set of real vectors. Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ be a function. The *epigraph* of f is the set defined by

$$\mathrm{epi}\, f = \{(x, \alpha) : x \in \mathbf{R}^V, \alpha \in \mathbf{R}, \alpha \geq f(x)\}\ (\subseteq \mathbf{R}^V \times \mathbf{R}).$$

A convex function is said to be *polyhedral* if its epigraph can be expressed by as a polyhedron, that is, there exists a matrix A and a vector b such that

$$\mathrm{epi}\, f = \left\{ (x, \alpha) : A \begin{pmatrix} x \\ \alpha \end{pmatrix} \leq b \right\}.$$

We now define M$^\natural$/L$^\natural$-convexity of polyhedral convex functions. These concepts were introduced in [48].

Definition 2.12. *A polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with* $\mathrm{dom}\, f \neq \emptyset$ *is called* M$^\natural$-convex *if for all $x, y \in \mathrm{dom}\, f$ and $s \in \mathrm{supp}^+(x - y)$, there exist $t \in \mathrm{supp}^-(x - y) \cup \{0\}$ and $\alpha_0 \in \mathbf{R}_+ \setminus \{0\}$ such that*

$$f(x) + f(y) \geq f(x - \alpha(\chi_s - \chi_t)) + f(y + \alpha(\chi_s - \chi_t))$$

for all $\alpha \in [0, \alpha_0]$.

Definition 2.13. *A polyhedral convex function* $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\operatorname{dom} g \neq \emptyset$ *is called* $\mathrm{L}^\natural$-convex *if it satisfies*

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1}))$$

for all $p, q \in \mathbf{R}^V$ *and* $\alpha \in \mathbf{R}_+$.

We now define an $\mathrm{M}^\natural/\mathrm{L}^\natural$-convex polyhedron.

Definition 2.14. *A set* $Q \subseteq \mathbf{R}^V$ *is an* $\mathrm{M}^\natural$-convex polyhedron *if for* $x, y \in Q$ *and* $s \in \operatorname{supp}^+(x - y)$, *there exist* $t \in \operatorname{supp}^-(x - y) \cup \{0\}$ *and* $\alpha_0 \in \mathbf{R}_+ \setminus \{0\}$ *such that* $x - \alpha(\chi_s - \chi_t) \in Q$ *and* $y + \alpha(\chi_s - \chi_t) \in Q$ *for all* $\alpha \in [0, \alpha_0]$.

Definition 2.15. *A set* $P \subseteq \mathbf{R}^V$ *is an* $\mathrm{L}^\natural$-convex polyhedron *if for* $p, q \in P$, *it holds that* $(p - \alpha\mathbf{1}) \vee q, p \wedge (q + \alpha\mathbf{1}) \in P$ *for all* $\alpha \in \mathbf{R}_+$.

For functions $f, g : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$, $p \in \mathbf{R}^V$, and $x \in \mathbf{R}^V$, we define

$$f[-p](x) = f(x) - \langle p, x \rangle,$$
$$g[-x](p) = g(p) - \langle p, x \rangle.$$

We describe properties of polyhedral $\mathrm{M}^\natural/\mathrm{L}^\natural$-convex functions.

Proposition 2.16 (see [45]). *Let* $g_1, g_2 : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be polyhedral* $L^\natural$*-convex functions and assume that* $\operatorname{dom} g_1 \cap \operatorname{dom} g_2 \neq \emptyset$. *Then* $g_1 + g_2$ *is a polyhedral* $L^\natural$*-convex function.*

Theorem 2.17 ([48]). *Let* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral* $M^\natural$*-convex function with a bounded nonempty effective domain. Then* $\operatorname{argmin} f[-p]$ *is an* $M^\natural$*-convex polyhedron for all* $p \in \mathbf{R}^V$.

Theorem 2.18 ([48]). *Let* $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral* $L^\natural$*-convex function with a bounded nonempty effective domain. Then* $\operatorname{argmin} g[-x]$ *is an* $L^\natural$*-convex polyhedron for all* $x \in \mathbf{R}^V$.

We next consider extending the effective domain of $\mathrm{M}^\natural/\mathrm{L}^\natural$-convex functions to a set of real vectors. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function defined on a set of integer vectors.

The *convex closure* of f is a function $\bar{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ given by

$$\bar{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x\rangle + \alpha : \langle p, y\rangle + \alpha \leq f(y)\ (\forall y \in \mathbf{Z}^V)\}.$$

If f satisfies

$$\bar{f}(x) = f(x)$$

for all $x \in \mathbf{Z}^V$, we say that f is *convex-extensible* and that $\bar{f}$ is the *convex extension* of f. Also, we say that $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is *concave-extensible* if $-f$ is convex extensible. For a convex-extensible function, the following statement holds.

Proposition 2.19 (see [45])**.** *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with a bounded nonempty effective domain. Then f is convex-extensible if and only if* $\operatorname{argmin} f[-p]$ *is discretely convex for each $p \in \mathbf{R}^V$.*

M$^\natural$/L$^\natural$-convex functions are convex-extensible and have the following property.

Theorem 2.20 ([42, 48, 49])**.** *An M$^\natural$-convex function is convex-extensible. If the convex extension of an M$^\natural$-convex function is polyhedral, it is polyhedral M$^\natural$-convex. Similarly, an L$^\natural$-convex function is convex-extensible. If the convex extension of an L$^\natural$-convex function is polyhedral, it is polyhedral L$^\natural$-convex.*

2.3.4 Conjugacy theorem

We describe the conjugacy theorem in this subsection. The *Legendre-Fenchel transform* of a function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is a function $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) : x \in \mathbf{R}^V\}.$$

For polyhedral M$^\natural$/L$^\natural$-convex functions, the following statement, the conjugacy theorem, holds.

Theorem 2.21 ([48])**.** *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral M$^\natural$-convex function and a polyhedral L$^\natural$-convex function. Then $f^\bullet$ is polyhedral L$^\natural$-convex* *and $g^\bullet$ is polyhedral M$^\natural$-convex. Moreover, $f^{\bullet\bullet} = f$ and $g^{\bullet\bullet} = g$, that is, there is one-to-one correspondence between the class of polyhedral M$^\natural$-convex functions and that of polyhedral L$^\natural$-convex functions.*

We next define the discrete Legendre-Fenchel transform of an integer-valued function whose effective domain is a set of integer vectors and nonempty. For a function $f : \mathbf{Z}^V \to \mathbf{Z}\cup\{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, the discrete Legendre-Fenchel transform $f^\bullet : \mathbf{Z}^V \to \mathbf{Z}\cup\{+\infty\}$ of f is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) : x \in \mathbf{Z}^V\}.$$

Note that $f^\bullet(p)$ is integral for each $p \in \mathbf{Z}^V$ by the definition.

Theorem 2.22 ([43])**.** *Let $f : \mathbf{Z}^V \to \mathbf{Z}\cup\{+\infty\}$ and $g : \mathbf{Z}^V \to \mathbf{Z}\cup\{+\infty\}$ be an integer-valued M$^\natural$-convex function and an integer-valued L$^\natural$-convex function. Then $f^\bullet$ is an integer-valued L$^\natural$-convex function and $g^\bullet$ is an integer-valued M$^\natural$-convex function. Moreover, $f^{\bullet\bullet} = f$ and $g^{\bullet\bullet} = g$, that is, there is one-to-one correspondence between the class of integer-valued M$^\natural$-convex functions and that of integer-valued L$^\natural$-convex functions.*

In this thesis, we use the following theorem in Section 3.2.

Theorem 2.23 ([7])**.** *Let $f : \mathbf{Z}^V \to \mathbf{R}\cup\{+\infty\}$ be an M$^\natural$-convex function with a bounded nonempty effective domain. We define $f^\bullet : \mathbf{R}^V \to \mathbf{R}\cup\{+\infty\}$ by*

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) : x \in \mathbf{Z}^V\}.$$

Then $f^\bullet$ is a polyhedral L$^\natural$-convex function.

2.3.5 Properties of the sum of two M$^\natural$-concave functions

We focus on the sum of two M$^\natural$-concave functions[2]. Note that the sum of two M$^\natural$-concave functions may not be M$^\natural$-concave. However, a maximizer of the sum of two M$^\natural$-concave functions can be characterized as follows.

[2]We deal with M$^\natural$-concave functions instead of M$^\natural$-convex functions after this subsection because we apply the M$^\natural$-concave intersection theorem in Section 3.2.

Theorem 2.24 ([42, 44]). *Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be $M^\natural$-concave functions and $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ be a vector. Then*

$$f_1(x^*) + f_2(x^*) \geq f_1(x) + f_2(x)$$

for all $x \in \mathbf{Z}^V$ if and only if there exists $p^* \in \mathbf{R}^V$ such that

$$f_1[-p^*](x^*) \geq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V),$$
$$f_2[+p^*](x^*) \geq f_2[+p^*](x) \quad (\forall x \in \mathbf{Z}^V).$$

In other words, there exists such a vector p^* with

$$\mathrm{argmax}(f_1 + f_2) = \mathrm{argmax}\, f_1[-p^*] \cap \mathrm{argmax}\, f_2[+p^*].$$

Theorem 2.24 plays an important role in the proof of the existence of competitive equilibria in the multi-unit trading network model (see Section 3.2). We next describe an optimality criterion of the sum of two M$^\natural$-concave functions.

Theorem 2.25 ([50]). *Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be $M^\natural$-concave functions and $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ be a vector. Then we have*

$$f_1(x) + f_2(x) \geq f_1(y) + f_2(y)$$

for all $y \in \mathbf{Z}^V$ if and only if

$$f_1(x) + f_2(x) \geq f_1(x + \chi_Y - \chi_Z) + f_2(x + \chi_Y - \chi_Z)$$

for any $Y, Z \subseteq V$ with $-1 \leq |Y| - |Z| \leq 1$.

Theorem 2.25 leads to the following corollary.

Corollary 2.26. *Let $f_1, f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be $M^\natural$-concave functions and $x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ be a vector. If $x \notin \mathrm{argmax}(f_1 + f_2)$, then there exist $Y, Z \subseteq V$ such that*

$$f_1(x) + f_2(x) < f_1(x + \chi_Y - \chi_Z) + f_2(x + \chi_Y - \chi_Z).$$

2.3.6 Twisted M$^\natural$-concave functions

The class of *twisted M$^\natural$-concave functions* is a variant of that of M$^\natural$-concave functions. Twisted M$^\natural$-concave functions are related to some economical properties (we explain the detail in the following chapter). The concept of twisted M$^\natural$-concave functions was first introduced by Sun and Yang [65] as GM-concave functions. They applied to the single-unit auction model where goods are substitutable or complementary. Their results are extended to the multi-unit auction model by Shioura and Yang [63]. The term "twisted M$^\natural$-concave function" was first used by Ikebe and Tamura [33].

Let W be a subset of V. We define $\mathrm{twist}(x; W) \in \mathbf{Z}^V$ by

$$(\mathrm{twist}(x\,; W))_t = \begin{cases} x_t & (t \notin W), \\ -x_t & (t \in W). \end{cases}$$

We are ready to give the definition of a twisted M$^\natural$-concave function.

Definition 2.27. *Let $W \subseteq V$. Then a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is a W-*twisted M$^\natural$-concave function *if there exists an M$^\natural$-concave function $\hat{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ such that*

$$f(x) = \hat{f}(\mathrm{twist}(x\,; W)) \quad (\forall x \in \mathbf{Z}^V).$$

By the definition and properties of M$^\natural$-concave functions, the following statements hold.

Proposition 2.28. *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be a W-twisted M$^\natural$-concave function. Then the following conditions hold:*

(i) *Let $x, y \in \mathrm{dom}\, f$ and $s \in \mathrm{supp}^+(x-y) \setminus W$. Then there exists $t \in \mathrm{supp}^-(x-y) \setminus W$ such that*

$$f(x) + f(y) \le f(x - \chi_s + \chi_t) + f(y + \chi_s - \chi_t),$$

or there exists $t \in (\mathrm{supp}^+(x-y) \cap W) \cup \{0\}$ *such that*

$$f(x) + f(y) \leq f(x - \chi_s - \chi_t) + f(y + \chi_s + \chi_t).$$

(ii) *Let* $x, y \in \mathrm{dom}\, f$ *and* $s \in \mathrm{supp}^-(x-y) \cap W$. *Then there exists* $t \in \mathrm{supp}^-(x-y) \setminus W$ *such that*

$$f(x) + f(y) \leq f(x + \chi_s + \chi_t) + f(y - \chi_s - \chi_t),$$

or there exists $t \in (\mathrm{supp}^+(x-y) \cap W) \cup \{0\}$ *such that*

$$f(x) + f(y) \leq f(x + \chi_s - \chi_t) + f(y - \chi_s + \chi_t).$$

Proposition 2.29. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *be a* W*-twisted* $\mathrm{M}^\natural$*-concave function. For* $x, y \in \mathrm{dom}\, f$ *with* $\sum_{t \in V} (\mathrm{twist}(x\,; W))_t < \sum_{t \in V} (\mathrm{twist}(y\,; W))_t$, *at least one of the following holds:*

(i) *There exists* $s \in \mathrm{supp}^-(x-y) \setminus W$ *such that*

$$f(x) + f(y) \leq f(x + \chi_s) + f(y - \chi_s).$$

(ii) *There exists* $s \in \mathrm{supp}^+(x-y) \cap W$ *such that*

$$f(x) + f(y) \leq f(x - \chi_s) + f(y + \chi_s).$$

Note that if W is empty, a W-twisted $\mathrm{M}^\natural$-concave function is equal to an $\mathrm{M}^\natural$-concave function. Therefore the class of twisted $\mathrm{M}^\natural$-concave functions contains the class of $\mathrm{M}^\natural$-concave functions. Moreover, the following statement holds.

Proposition 2.30. *Let* f *be a* W*-twisted* $\mathrm{M}^\natural$*-concave function defined on* $\mathbf{Z}^V$. *Then* f *is* $(V \setminus W)$*-twisted* $\mathrm{M}^\natural$*-concave.*

Proof. Since f is W-twisted $\mathrm{M}^\natural$-concave, there exists $\mathrm{M}^\natural$-concave function $\hat{f}$ such that

$$f(x) = \hat{f}(\mathrm{twist}(x\,; W)) \quad (\forall x \in \mathbf{Z}^V).$$

We define function g by

$$g(x) = \hat{f}(-x) \quad (\forall x \in \mathbf{Z}^V).$$

Then we have

$$f(x) = \hat{f}(\mathrm{twist}(x\,; W)) = g(\mathrm{twist}(x\,; V \setminus W)) \quad (\forall x \in \mathbf{Z}^V).$$

By the definition of an M$^\natural$-concave function, g is also M$^\natural$-concave. Therefore f is a $(V \setminus W)$-twisted M$^\natural$-concave function. □

We finally describe some properties of W-twisted M$^\natural$-concave functions. The following statement follows from the property of M$^\natural$-concave functions.

Proposition 2.31 (see [45]). *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be a W-twisted M$^\natural$-concave function. Then the following statements hold:*

(i) *the sum of a W-twisted M$^\natural$-concave function and a linear function is W-twisted M$^\natural$-concave, that is, for $p \in \mathbf{R}^V$, $f[+p]$ is W-twisted M$^\natural$-concave.*

(ii) *for $a, b \in (\mathbf{Z} \cup \{\pm\infty\})^V$ with $a \leq b$, function $f_{[a,b]}$ defined by*

$$f_{[a,b]}(x) = \begin{cases} f(x) & \text{if } a_t \leq x_t \leq b_t \text{ for all } t \in V, \\ -\infty & \text{otherwise,} \end{cases}$$

is W-twisted M$^\natural$-concave.

(iii) *for $A \subseteq V$, we define function $f^A : \mathbf{Z}^A \to \mathbf{R} \cup \{-\infty\}$ by*

$$f^A(y) = \sup\{f(x) : x_t = y_t\,(\forall t \in A), x \in \mathbf{Z}^V\}.$$

If $f^A(y) < +\infty$ for all $y \in \mathbf{Z}^A$, f^A is $(W \cap A)$-twisted M$^\natural$-concave.

2.4 Fundamentals of Game theory

In this section, we describe fundamentals of game theory and microeconomics related to this thesis. We first introduce the concept of a game, then we describe an extensive form of a game. Finally, we introduce the market game and define some economical terminology. For more details on game theory and microeconomics, see [34, 41, 54].

There are many situations in which players interact with each other. Then players' payoffs depend on their strategic actions. In game theory, such a situation is called a *game* and represented by mathematical models. This enables us to analyze situations with strategic actions.

A game consists of three things, players, strategies, and payoffs. We give an example of a game.

Example 2.6 (Matching Pennies)**.** This game is played by two players, 1 and 2. To begin with, each player puts a penny down simultaneously. If the two pennies match (i.e. both pennies are heads up or tails up), Player 1 wins, otherwise (i.e. one penny is heads up and the other is tails up) Player 2 wins. The winner obtains a penny from the other player. The game Matching Pennies can be represented by the following three objects:

(i) Players: Player 1 and Player 2.

(ii) Strategies of each player: Heads up or tails up.

(iii) Payoffs of each player: $(1, -1)$ (if Player 1 wins), $(-1, 1)$ (if Player 2 wins).

2.4.1 Extensive form of a game

We next describe an extensive form of a game. An extensive form is represented by a graph theoretic tree. We introduce an extensive form by using the following game.

Example 2.7 (Matching Pennies Ver.2)**.** We consider another version of Matching Pennies. At the beginning, Player 1 puts a penny down and covers her penny with her hand. Then Player 2 puts a penny down. If the two pennies match, Player 1 wins, otherwise Player 2 wins.

The *extensive form* of Matching Pennies Ver.2 is represented in Figure 2.1.

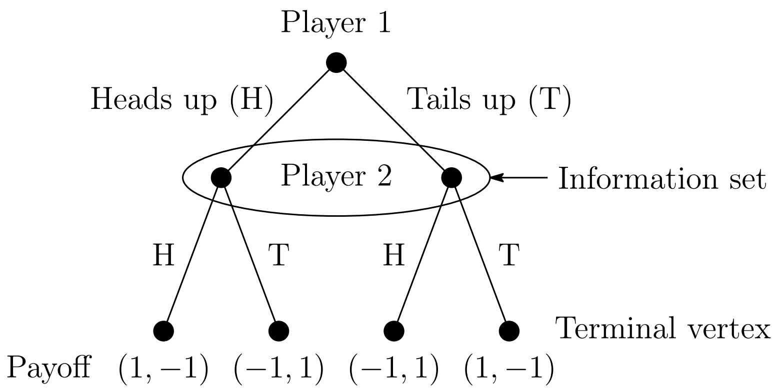

Figure 2.1: The extensive form of Matching Pennies Ver.2.

The game starts at the initial vertex, where Player 1 puts her penny. Player 1 has two strategies. One is putting her penny down heads up. The other is putting it down tails up. Each strategy corresponds to an edge which connects the initial vertex and a decision vertex of Player 2. At a decision vertex of Player 2, she chooses her strategy. After Player 2's move, we reach a terminal vertex and each player obtains a payoff.

In Matching Pennies Ver.2, Player 1 covers her penny with her hand after choosing her strategy. Therefore, Player 2 does not see Player 1's choice until the game ends. This means that Player 2 cannot distinguish the two vertices. We represent this situation by using an *information set*. A player cannot distinguish between vertices which belong to the same information set.

An extensive form game has *perfect information* if all information sets are singletons. A game is said to be a *zero-sum game* if the sum of the payoffs of all players is always zero. By Figure 2.1, Matching Pennies Ver.2 is a zero-sum game.

2.4.2 Market game

We next introduce the market game, which is a cooperative game. A cooperative game is a game where players can make groups, called coalitions.

First, we define a cooperative game by using mathematical terminology. Let $N = \{1, \ldots, n\}$ be the set of players and $v : 2^N \to \mathbf{R}$ be the characteristic function of the game. A *cooperative game* is represented by a pair (N, v). A subset $S \subseteq N$ is called a *coalition* and $v(S)$ represents the value of coalition S. We would like to know whether players have incentives to make coalitions or not, and how to share their aggregate payoffs.

Next we define the market game. Let $N = \{1, \ldots, n\}$ and L, respectively, be the set of agents and goods. Each agent i has preference on bundles $x_i \in \mathbf{R}_+^L$ which is represented by utility function $u_i : \mathbf{R}_+^L \to \mathbf{R}$. In this thesis, each u_i is monotone (i.e. $u_i(x_i) \leq u_i(y_i)$ for $x_i \leq y_i$) and concave. We define the initial bundle of agent i by $w_i \in \mathbf{R}_+^L$. An *allocation* $(x_1, \ldots, x_n) \in \mathbf{R}_+^L \times \cdots \times \mathbf{R}_+^L$ is defined by a tuple of bundles. We define a feasible allocation.

Definition 2.32. *An allocation* $(x_1, \ldots, x_n)$ *is* feasible *if*

$$\sum_{i \in N} x_{il} \leq \sum_{i \in N} w_{il}$$

for all $l \in L$.

We define $v : 2^N \to \mathbf{R}$ by

$$v(S) = \max \left\{ \sum_{i \in S} u_i(x_i) : (x_1, \ldots, x_n) \text{ is a feasible allocation} \right\} \quad (\forall S \subseteq N).$$

Then the pair (N, v) is called the *market game.*

We next define Pareto efficiency of allocations. Pareto efficiency is an important property in economics.

Definition 2.33. *A feasible allocation* $(x_1, \ldots, x_n)$ *is* Pareto efficient *if there is no other feasible allocation* $(x'_1, \ldots, x'_n)$ *such that* $u_i(x'_i) \geq u_i(x_i)$ *for all* $i \in N$ *and* $u_i(x'_i) > u_i(x_i)$ *for some* $i \in N$.

By the definition of a Pareto efficient allocation, it is impossible to increase someone's utility without decreasing utilities of the other agents. Hence a Pareto efficient allocation is one of the "balanced" allocations.

Now we define a competitive equilibrium in the market game. We denote a price vector by $p \in \mathbf{R}_+^L$.

Definition 2.34. *Let* $p^* \in \mathbf{R}^L_+$ *be a price vector. Then* $(p^*, x_1^*, \dots, x_n^*)$*, a tuple of a price vector and an allocation, is said to be a* competitive equilibrium *if the following statements hold:*

(i) *For any* $i \in N$ *and* $x_i \in \mathbf{R}^L_+$*, we have*

$$u_i(x_i^*) - \langle p^*, x_i^* - w_i \rangle \geq u_i(x_i) - \langle p^*, x_i - w_i \rangle.$$

(ii) $\sum_{i \in N} x_i^* = \sum_{i \in N} w_i$.

The first property requires x_i^* to be the best bundle for agent i, when the price of good l is given by p_l^*. The second property means that the total amount of the aggregate supply and demand is equivalent.

We now assume that players would like to form coalition N (i.e. the best strategy for all players is forming coalition N). Thus we focus on how to share $v(N)$. We describe the concept of imputations of the market game.

Definition 2.35. *A vector* $z \in \mathbf{R}^N$ *is said to be an* imputation *of the market game* (N, v) *if the following conditions hold:*

(i) $z_i \geq v(\{i\})$ *for all* $i \in N$.

(ii) $\sum_{i \in N} z_i = v(N)$.

The first condition says that each player obtains more payoff by participating in a coalition than not. The second condition means that aggregate payoff is shared thoroughly. In the market game, the following imputation has a good property.

Theorem 2.36 (see [41, 54])**.** *Let* $(p^*, x_1^*, \dots, x_n^*)$ *be a competitive equilibrium. We define vector* $z \in \mathbf{R}^N$ *by*

$$z_i = u_i(x^*) - \langle p^*, x_i^* - w_i \rangle \quad (i \in N).$$

Then z *is an imputation of the market game. Furthermore, there do not exist* $y \in \mathbf{R}^N$ *and* $S \subseteq N$ *such that*

(i) $v(S) \geq \sum_{i \in S} y_i$.

(ii) $y_i > x_i$ *for all* $i \in S$.

Imputation z has good properties because properties (i) and (ii) say that there is no incentive for all players to leave coalition N.

Fundamental theorems of welfare economics

In microeconomics, Pareto efficiency and competitive equilibria are closely related. Indeed, in the market game, $(x_1^*, \ldots, x_n^*)$ is a Pareto efficient allocation if $(p^*, x_1^*, \ldots, x_n^*)$ is a competitive equilibrium. The following statement is called the first fundamental theorem of welfare economics.

Theorem 2.37 (The first fundamental theorem of welfare economics)**.** *A competitive equilibrium is Pareto efficient.*

We can regard Theorem 2.37 as the formal description of the "invisible hand" which is used by Adam Smith. The converse of Theorem 2.37 does not hold in general. However, the following theorem, the second fundamental theorem of welfare economics, holds[3].

Theorem 2.38 (The second fundamental theorem of welfare economics)**.** *We can achieve a competitive equilibrium from a Pareto efficient allocation by a wealth redistribution (if some properties are satisfied).*

[3]To be exact, some assumptions are necessary to hold the second fundamental theorem of welfare economics. For more information, see [41].

Chapter 3

Multi-unit Trading Network Model

3.1 Definitions and Previous Results

We deal with the multi-unit trading network model. This model is an extension of the single-unit trading network model introduced by Hatfield et al. [27]. The model is related to the market game. In our model, however, each agent has preference on trades instead of bundles. In the first half of this section, we give definitions of the model. We next review the previous results on the single-unit trading network model so that readers can understand our results clearly.

3.1.1 Definitions of the model

The multi-unit trading network model consists of agents and trades. Agents have their own preferences on trades and they are represented by functions. Let I and Ω, respectively, be finite sets of agents and trades. Each agent plays as a buyer or seller in some trades. For trade t, we define the buyer and seller by $b(t)$ and $s(t)$, respectively. The relationship among the agents can be represented by a directed graph, that is, each agent corresponds to a vertex and each trade t corresponds to a directed edge $(s(t), b(t))$[1]. We denote by $\omega_t \in \mathbf{Z}_+$ the number of units of t, and by $p_t \in \mathbf{R}$ the unit price for t. A

[1]In the model, there may exist trades $t_1, t_2 (t_1 \neq t_2)$ such that $b(t_1) = b(t_2)$ and $s(t_1) = s(t_2)$. Then we consider that the corresponding directed graph has multiple directed edges.

trade vector and a price vector are respectively defined by $\omega = (\omega_t \mid t \in \Omega) \in \mathbf{Z}_+^\Omega$ and $p = (p_t \mid t \in \Omega) \in \mathbf{R}^\Omega$. A pair (ω, p) is called an *outcome*.

For agent $i \in I$, we define

$$\Omega_{\to i} = \{t \in \Omega : b(t) = i\}, \quad \Omega_{i\to} = \{t \in \Omega : s(t) = i\}, \quad \Omega_i = \Omega_{\to i} \cup \Omega_{i\to}.$$

By the above definitions, $\Omega_{\to i}$ represents the set of trades in which i is a buyer and $\Omega_{i\to}$ represents the set of trades in which i is a seller. The set of trades which is associated with agent i is denoted by Ω_i.

For a subset $T \subseteq \Omega$, $\mathrm{ag}(T)$ represents the set of agents who are involved in trades in T, that is,

$$\mathrm{ag}(T) = \bigcup_{t \in T} \{b(t), s(t)\}.$$

Each agent $i \in I$ has a *valuation function* $u_i : \mathbf{Z}^\Omega \to \mathbf{R} \cup \{-\infty\}$ which represents her/his preference. Throughout this paper, we assume that

$$u_i(\omega) = u_i(\omega') \quad \text{if } \omega_t = \omega'_t \text{ for all } t \in \Omega_i, \tag{3.1}$$

and

$$\mathbf{0} \in \mathrm{dom}\, u_i \subseteq \{\omega \in \mathbf{Z}^\Omega : 0 \le \omega_t \le M \ (\forall t \in \Omega_i)\} \text{ for some } M \in \mathbf{Z}_+. \tag{3.2}$$

Condition (3.1) requires that the value of u_i depends only on trades associated with herself/himself. Therefore, we can regard the domain of u_i as $\mathbf{Z}^{\Omega_i}$. Condition (3.2) requires that the number of trades is finite. We think that these conditions are natural requirements. We note that $\max\{u_i(\omega) : \omega \in \mathbf{Z}^\Omega\}$ always exists by these conditions.

We next define utility functions. The *utility function* $U : \mathbf{Z}^\Omega \times \mathbf{R}^\Omega \to \mathbf{R} \cup \{-\infty\}$ associated with u_i is defined by

$$U(\omega, p\,; u_i) = u_i(\omega) + \sum_{t \in \Omega_{i\to}} \omega_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega_t \cdot p_t.$$

The second and third terms of U represent the amount of money that agent i obtains and pays. Valuations of each agent are measured in monetary terms. The *indirect utility function* $V : \mathbf{R}^\Omega \to \mathbf{R}$ associated with u_i is defined by

$$V(p\,;u_i) = \max\{U(\omega, p\,;u_i) : \omega \in \mathbf{Z}^\Omega\}.$$

The value of $V(p\,;u_i)$ means the maximum utility of agent i under the price p.

We then introduce the demand correspondence and choice correspondence. For a price vector p, we define the *demand correspondence* $D(p\,;u_i)$ of agent i by

$$D(p\,;u_i) = \mathrm{argmax}\{U(\omega, p\,;u_i) : \omega \in \mathbf{Z}^\Omega\}.$$

For an outcome (ω, p), we define the *choice correspondence* $C(\omega, p\,;u_i)$ of agent i by

$$C(\omega, p\,;u_i) = \mathrm{argmax}\{U(\omega', p\,;u_i) : \omega' \leq \omega,\ \omega' \in \mathbf{Z}^\Omega\}.$$

The demand correspondence $D(p\,;u_i)$ is the set of trade vectors which maximizes utility of agent i under the price p. On the other hand, the choice correspondence $C(\omega, p\,;u_i)$ is the set of trade vectors which maximizes utility of agent i under the price p when available trades are given by ω. Since the model is determined by Ω and u_i $(i \in I)$, we represent the model by the pair $(\Omega, \{u_i\}_{i\in I})$, and call it an *economy*.

We next give definitions of stable outcomes and competitive equilibria. These concepts play an important role in the model. For this, we define individual rationality and blocking sets.

Definition 3.1. *An outcome* (ω, p) *is said to be* individually rational *if* $\omega \in C(\omega, p\,;u_i)$ *for all* $i \in I$.

Individual rationality means that no agent has an incentive to cancel some trades when taking ω under the price p.

Definition 3.2. *A pair* (z, p') *is said to be a* blocking set *of* (ω, p) *in economy* $(\Omega, \{u_i\}_{i\in I})$ *if the following conditions hold:*

(i) $z \in \mathbf{Z}^\Omega_+ \setminus \{\mathbf{0}\}$.

(ii) $p' \in \mathbf{R}^\Omega$ *and* $p'_t = 0$ *for all* $t \in \Omega$ *with* $t \notin \mathrm{supp}^+(z)$.

(iii) *for all* $i \in \mathrm{ag}(\mathrm{supp}^+(z))$ *and* $\omega' \in C(\omega + z, p + p' \,; u_i)$, *we have* $\omega'_t = (\omega + z)_t$ *for all* $t \in \mathrm{supp}^+(z) \cap \Omega_i$.

Since the definition of blocking sets is complicated, we give an intuition of a blocking set. We assume that (z, p') is a blocking set of (ω, p) and consider the case where the number of available units of trade t increases by z_t and the unit price for t changes from p_t to $p_t + p'_t$ for all $t \in \mathrm{supp}^+(z)$. Then each agent $i \in \mathrm{ag}(\mathrm{supp}^+(z))$ strictly prefers taking $(\omega + z)_t$ trades for all $t \in \mathrm{supp}^+(z) \cap \Omega_i$. In other words, all agents in $\mathrm{ag}(\mathrm{supp}^+(z))$ take all additional units of trades. Therefore we regard (ω, p) as blocked by the pair (z, p'). By the definition of blocking sets, it holds that

$$U(\omega', p + p' \,; u_i) > U(\omega, p + p' \,; u_i)$$

for all $\omega' \in C(\omega + z, p + p' \,; u_i)$ when (z, p') is a blocking set of (ω, p). We are ready to define stable outcomes and competitive equilibria in economy $(\Omega, \{u_i\}_{i \in I})$. Both are one of the balanced situations.

Definition 3.3. *An outcome* (ω, p) *is* stable *in economy* $(\Omega, \{u_i\}_{i \in I})$ *if the following conditions hold:*

(i) (ω, p) *is individually rational.*

(ii) (ω, p) *does not have any blocking sets.*

Definition 3.4. *An outcome* (ω, p) *is said to be a* competitive equilibrium *in economy* $(\Omega, \{u_i\}_{i \in I})$ *if* $\omega \in D(p \,; u_i)$ *for all* $i \in I$. *Then the vectors* ω *and* p *are called a* competitive equilibrium trade vector *and a* competitive equilibrium price vector, *respectively.*

By Definition 3.4, if (ω, p) is a competitive equilibrium, taking ω is the best choice for all agents under the price p. In other words, no player has incentive to increase (decrease) the number of trades without changing prices. This corresponds to property (i) in Definition 2.34. Moreover, the total amount of the aggregate supply and demand is equivalent for any competitive equilibrium, which corresponds to property (ii) in Defi-

Table 3.1: The valuations of agents in Example 3.1.

(t_1, t_2)	$u_a(\cdot)$	$u_b(\cdot)$	$u_c(\cdot)$
$(0, 0)$	0	0	0
$(1, 0)$	3	-1	0
$(0, 1)$	4	0	-1
$(1, 1)$	2	-1	-1

nition 2.34. Thus Definition 3.4 can be regarded as an adaptation of Definition 2.34 to the multi-unit trading network model.

We give an example to explain stable outcomes and competitive equilibria.

Example 3.1. There are three agents, a, b, and c and two types of trades, t_1 and t_2 such that $b(t_1) = a$, $s(t_1) = b$, $b(t_2) = a$, and $s(t_2) = c$. Valuation functions of each agent are given in Table 3.1. We consider outcome $(\omega, p) = ((1, 0), (2, 0))$. Although this outcome is individually rational, it has a blocking set $(z, p') = ((0, 1), (0, 2))$. In this case, there exists a competitive equilibrium. In fact, $((0, 1), (1, 1))$ is a competitive equilibrium because $(0, 1) \in D((1, 1)\,; u_i)$ for all $i \in \{a, b, c\}$.

Finally, we give the definition of efficient trade vectors.

Definition 3.5. *A trade vector* ω *is* efficient *on* $\{u_i\}_{i\in I}$ *if*

$$\sum_{i\in I} u_i(\omega) \geq \sum_{i\in I} u_i(\omega') \quad (\forall \omega' \in \mathbf{Z}^\Omega).$$

Here, "efficient" is equal to Pareto efficient. In fact, for an efficient trade vector ω, there is no trade ω' such that $u_i(\omega') \geq u_i(\omega)$ for all $i \in I$ and $u_i(\omega') > u_i(\omega)$ for some $i \in I$. Furthermore, if there exists $\tau \in \mathbf{Z}^\Omega$ such that $\sum_{i\in I} u_i(\tau) > \sum_{i\in I} u_i(\omega)$, (ω, p) is not stable for all $p \in \mathbf{R}^\Omega$ (see Section 3.3).

A big difference between the multi-unit trading network model and the single-unit trading network model is about valuation functions. They are defined on $\mathbf{Z}^\Omega$ in the multi-unit trading network model whereas they are defined on the power set of Ω in the single-unit trading network model. In the multi-unit trading network model, a vector whose elements are 0 or 1 can be regarded as a subset of trades. In fact, if $\operatorname{dom} u_i \subseteq \{0,1\}^\Omega$ for all $i \in I$, the multi-unit trading network model represents the single-unit trading network model in [27] and the definitions of stable outcomes and competitive equilibria in [27] coincide with the above definitions in the multi-unit trading network model.

3.1.2 Review of the previous works

As mentioned in the last paragraph in Subsection 3.1.1, if $\operatorname{dom} u_i \subseteq \{0,1\}^\Omega$ for all $i \in I$, we regard the multi-unit trading network model as the single-unit trading network model. In this subsection, we describe the results concerning the single-unit trading network model which were shown by Hatfield et al. [27]. We first explain the definitions of the full substitutes condition and the laws of aggregate supply and demand.

Full substitutes condition and laws of aggregate supply and demand

The formal definition of the full substitutes condition is given by as follows.

Definition 3.6 ([27])**.** *A valuation function* $u_i : \{0,1\}^\Omega \to \mathbf{R} \cup \{-\infty\}$ *of agent* i *satisfies the* full substitutes condition *if the following conditions hold:*

(i) *For any* $p \in \mathbf{R}^\Omega$, $t \in \Omega_{i\to}$, $\omega \in D(p\,; u_i)$, *and* $\delta > 0$, *there exists* $\tau \in D(p + \delta\chi_t\,; u_i)$ *such that*

$$\omega_s \geq \tau_s \quad (\forall s \in \Omega_{i\to} \setminus \{t\}), \qquad \omega_s \leq \tau_s \quad (\forall s \in \Omega_{\to i}).$$

(ii) *For any* $p \in \mathbf{R}^\Omega$, $t \in \Omega_{\to i}$, $\omega \in D(p\,; u_i)$, *and* $\delta > 0$, *there exists* $\tau \in D(p - \delta\chi_t\,; u_i)$ *such that*

$$\omega_s \geq \tau_s \quad (\forall s \in \Omega_{\to i} \setminus \{t\}), \qquad \omega_s \leq \tau_s \quad (\forall s \in \Omega_{i\to}).$$

The first condition in Definition 3.6 says that when the price of $t \in \Omega_{i\to}$ increases, the following statements hold:

- agent i's supply of trades other than t does not increase.
- agent i's demand does not decrease.

We consider the situation where the price of $t \in \Omega_{i\rightarrow}$ increases to obtain an intuition. Then agent i has no incentives to decrease the amount of trade t. The first statement requires that agent i never has incentives to increase the amount of trades in $\Omega_{i\rightarrow} \setminus \{t\}$. This means that agent i views trades in $\Omega_{i\rightarrow}$ as substitutes. Moreover, by the second statement, agent i may increase the amount of trades in $\Omega_{\rightarrow i}$ in this situation. The second condition in Definition 3.6 is symmetric to the first condition. It requires that when the price of $t \in \Omega_{\rightarrow i}$ decreases, the following statements hold:

- agent i's supply does not decrease.
- agent i's demand of trades other than t does not increase.

In summary, agent i views trades in which she/he is a buyer (seller) as substitutes, and a trade in $\Omega_{i\rightarrow}$ and one in $\Omega_{\rightarrow i}$ as complements if u_i satisfies the full substitutes condition.

The full substitutes condition contains the gross substitutes condition. Indeed, if $\Omega_{i\rightarrow} = \emptyset$, the full substitutes condition coincides with the gross substitutes condition (we can regard hiring a worker i with salary p as buying from i with price p). We next give the definition of the laws of aggregate supply and demand.

Definition 3.7 ([27]). *A valuation function $u_i : \{0,1\}^{\Omega} \to \mathbf{R} \cup \{-\infty\}$ of agent i satisfies the* law of aggregate supply *if for every $p \in \mathbf{R}^{\Omega}$, $t \in \Omega_{i\rightarrow}$, $\omega \in D(p\,; u_i)$, and $\delta > 0$, there exists $\tau \in D(p + \delta\chi_t\,; u_i)$ such that*

$$\sum_{s\in\Omega_{i\rightarrow}} \tau_s - \sum_{s\in\Omega_{i\rightarrow}} \omega_s \geq \sum_{s\in\Omega_{\rightarrow i}} \tau_s - \sum_{s\in\Omega_{\rightarrow i}} \omega_s.$$

Similarly, u_i satisfies the law of aggregate demand *if for every $p \in \mathbf{R}^{\Omega}$, $t \in \Omega_{\rightarrow i}$, $\omega \in D(p\,; u_i)$, and $\delta > 0$, there exists $\tau \in D(p - \delta\chi_t\,; u_i)$ such that*

$$\sum_{s\in\Omega_{\rightarrow i}} \tau_s - \sum_{s\in\Omega_{\rightarrow i}} \omega_s \geq \sum_{s\in\Omega_{i\rightarrow}} \tau_s - \sum_{s\in\Omega_{i\rightarrow}} \omega_s.$$

The laws of aggregate supply and demand are conditions about the amount of increase of trades. Two inequalities in Definition 3.7 represent the relationship between the amount of trades in $\Omega_{i\rightarrow}$ and $\Omega_{\rightarrow i}$. The law of aggregate supply means that when the price of $t \in \Omega_{i\rightarrow}$ increases, the amount of increase of trades in $\Omega_{\rightarrow i}$ does not exceed that of increasing trades in $\Omega_{i\rightarrow}$. Similarly, the law of aggregate demand means that when the price of $t \in \Omega_{\rightarrow i}$ decreases, the amount of increase of trades in $\Omega_{i\rightarrow}$ does not exceed that of increasing trades in $\Omega_{\rightarrow i}$.

The laws of aggregate supply and demand were first introduced by Hatfield and Kominers [26]. Hatfield et al. [27] applied these concepts to the single-unit trading network model. For the above conditions, the following statement holds.

Theorem 3.8 ([27]). *If u_i satisfies the full substitutes condition it satisfies the laws of aggregate supply and demand.*

Previous results

The full substitutes condition plays an important role in the single-unit trading network model. Under the full substitutes condition, there exists a competitive equilibrium.

Theorem 3.9 ([27]). *Suppose that every $u_i : \{0,1\}^\Omega \to \mathbf{R} \cup \{-\infty\}$ satisfies the full substitutes condition. Then there exists a competitive equilibrium in economy $(\Omega, \{u_i\}_{i\in I})$.*

The full substitutes condition is necessary to guarantee the existence of competitive equilibria. If a preference of only one agent does not satisfy the full substitutes condition, we cannot guarantee the existence of competitive equilibria. We next explain the relationship between competitive equilibria and efficiency of trade vectors.

Theorem 3.10 ([27]). *Let (ω, p) be a competitive equilibrium in $(\Omega, \{u_i\}_{i\in I})$. Then ω is efficient on $\{u_i\}_{i\in I}$, that is,*

$$\sum_{i\in I} u_i(\omega) \geq \sum_{i\in I} u_i(\omega') \quad (\forall \omega' \in \{0,1\}^\Omega).$$

Theorem 3.11 ([27]). *Suppose that every u_i satisfies the full substitutes condition and consider the economy $(\Omega, \{u_i\}_{i\in I})$. Then for any competitive equilibrium (ω, p) and efficient trade vector ω' on $\{u_i\}_{i\in I}$, (ω', p) is also a competitive equilibrium. Moreover, the set of competitive equilibrium price vectors forms a lattice, that is, if p and q are competitive equilibrium price vectors, $p \wedge q$ and $p \vee q$ are also competitive equilibrium price vectors.*

In other words, Theorem 3.10 and the first part of Theorem 3.11 say that the fundamental theorems of welfare economics hold under the full substitutes condition.

Finally, we describe the relationship between stable outcomes and competitive equilibria in the single-unit trading network model.

Theorem 3.12 ([27]). *If an outcome (ω, p) is a competitive equilibrium in $(\Omega, \{u_i\}_{i\in I})$, then it is stable.*

Theorem 3.13 ([27]). *Suppose that every u_i satisfies the full substitutes condition and (ω, p) is stable in $(\Omega, \{u_i\}_{i\in I})$. Then there exists $q \in \mathbf{R}^\Omega$ such that (ω, q) is a competitive equilibrium and $q_t = p_t$ for all $t \in \mathrm{supp}^+(\omega)$.*

Theorem 3.12 holds without assumptions that preferences of all agents satisfy the full substitutes condition. This means that competitive equilibria are always stricter properties than stable outcomes. However, in the single-unit trading network model, these properties are essentially equivalent under the full substitutes condition. Theorem 3.13 says that we can construct a competitive equilibrium from a stable outcome by changing prices of trades which are not taken.

3.2 Existence and Lattice Structure of Competitive Equilibria

We now consider the existence of competitive equilibria. We first introduce the generalized full substitutes condition, which plays a central role in the multi-unit trading network model. The important fact is that the generalized full substitutes condition is equivalent to twisted M$^\natural$-concavity. By using this fact, we show the existence of competitive equilibria under the generalized full substitutes condition. Moreover, we prove that the set of competitive equilibrium price vector is an L$^\natural$-convex polyhedron. The results in this section can be found in [32].

3.2.1 Generalized full substitutes condition

The full substitutes condition is defined on a valuation function whose effective domain is contained in $\{0,1\}^{\Omega}$. In this subsection, we consider extending the full substitutes condition to the multi-unit trading network model, that is, we define the analogous condition on a function whose effective domain is a set of integer vectors.

Definition 3.14. *A valuation function* $u_i : \mathbf{Z}^{\Omega} \to \mathbf{R} \cup \{-\infty\}$ *of agent* i *satisfies the* generalized full substitutes condition *if the following conditions hold:*

(i) $D(p\,; u_i)$ *is a discretely convex set for all* $p \in \mathbf{R}^{\Omega}$.

(ii) *For any* $p \in \mathbf{R}^{\Omega}$, $t \in \Omega_{i\to}$, $\omega \in D(p\,; u_i)$, *and* $\delta > 0$, *there exists* $\tau \in D(p + \delta\chi_t\,; u_i)$ *such that*

$$\omega_s \geq \tau_s \quad (\forall s \in \Omega_{i\to} \setminus \{t\}), \qquad \omega_s \leq \tau_s \quad (\forall s \in \Omega_{\to i}), \tag{3.3}$$

$$\sum_{s\in\Omega_{i\to}} \tau_s - \sum_{s\in\Omega_{i\to}} \omega_s \geq \sum_{s\in\Omega_{\to i}} \tau_s - \sum_{s\in\Omega_{\to i}} \omega_s. \tag{3.4}$$

(iii) *For any* $p \in \mathbf{R}^{\Omega}$, $t \in \Omega_{\to i}$, $\omega \in D(p\,; u_i)$, *and* $\delta > 0$, *there exists* $\tau \in D(p - \delta\chi_t\,; u_i)$ *such that*

$$\omega_s \geq \tau_s \quad (\forall s \in \Omega_{\to i} \setminus \{t\}), \qquad \omega_s \leq \tau_s \quad (\forall s \in \Omega_{i\to}), \tag{3.5}$$

$$\sum_{s\in\Omega_{\to i}} \tau_s - \sum_{s\in\Omega_{\to i}} \omega_s \geq \sum_{s\in\Omega_{i\to}} \tau_s - \sum_{s\in\Omega_{i\to}} \omega_s. \tag{3.6}$$

The first condition is essentially equivalent[2] to the condition that u_i is concave-extensible by Proposition 2.19. We see from Definition 3.14 that the generalized full substitutes condition is a natural combination of the full substitutes condition and the laws of aggregate supply and demand introduced in Subsection 3.1.2.

[2] The word "essentially" means that there is a possibly that this statement does not hold because $\mathrm{dom}\, u_i$ is not bounded in our setting (note that u_i is defined on not $\mathbf{Z}^{\Omega_i}$ but $\mathbf{Z}^{\Omega}$). However, when we regard u_i as a function defined on $\mathbf{Z}^{\Omega_i}$, it is concave-extensible.

Obviously, the generalized full substitutes condition implies the full substitutes condition. On the other hand, the converse does not hold in general, while it turns out that the converse holds true for a valuation function u_i with $\mathrm{dom}\, u_i \subseteq \{0,1\}^{\Omega}$.

The generalized full substitutes condition is closely related to twisted M$^\natural$-concavity of a valuation function. This is not a surprising fact because Fujishige and Yang [19] showed that a valuation function satisfies the gross substitutes condition, which is included in the generalized full substitutes condition, if and only if it is M$^\natural$-concave on the unit hypercube. Their results were extended by several researchers. For instance, Ikebe and Tamura [33] showed that if $\mathrm{dom}\, u_i \subseteq \{0,1\}^{\Omega}$, an $\Omega_{i\to}$-twisted M$^\natural$-concave function satisfies the full substitutes condition. Recently, Shioura and Yang [63] showed that twisted M$^\natural$-concavity of a valuation function is equivalent to the generalized gross substitutes and complements (GGSC) condition, which is used in auction models (for more detail, see [62]). In this thesis, we show the following statement.

Theorem 3.15. *A valuation function $u_i : \mathbf{Z}^{\Omega} \to \mathbf{R} \cup \{-\infty\}$ of agent i satisfies the generalized full substitutes condition if and only if it is an $\Omega_{i\to}$-twisted M$^\natural$-concave function.*

Recall that a valuation function u_i satisfies condition (3.1). Therefore, if u_i is an $\Omega_{i\to}$-twisted M$^\natural$-concave function it is also an A-twisted M$^\natural$-concave function where $\Omega_{i\to} \subseteq A \subseteq \Omega \setminus \Omega_{\to i}$. Unless otherwise noted, a valuation function of agent i is simply said to be a twisted M$^\natural$-concave function if it is an $\Omega_{i\to}$-twisted M$^\natural$-concave function.

As we mentioned above, Shioura and Yang [63] proved that the GGSC condition is equivalent to twisted M$^\natural$-concavity. In the proof, they used the following property, (GGS$\pm$). Let $\hat{u}_i : \mathbf{Z}^{\Omega_i} \to \mathbf{R} \cup \{-\infty\}$ be a function whose effective domain is bounded.

(GGS$\pm$)

(i) For all $p \in \mathbf{R}^{\Omega_i}$, $\tilde{D}(p\,;\hat{u}_i)$ is a discretely convex set where

$$\tilde{D}(p\,;\hat{u}_i) = \mathrm{argmax}\left\{\hat{u}_i(\omega) - \sum_{s\in\Omega_i} p_s \cdot \omega_s : \omega \in \mathbf{Z}^{\Omega_i}\right\}.$$

(ii) Let $p \in \mathbf{R}^{\Omega_i}$, $\delta > 0$, and $\omega \in \tilde{D}(p\,; \hat{u}_i)$.

a) For all $t \in \Omega_{i\to}$, there exists $\tau \in \tilde{D}(p + \delta\chi_t\,; \hat{u}_i)$ such that

$$\tau_s \geq \omega_s \ (\forall s \in \Omega_i \setminus \{t\}), \qquad \sum_{s\in\Omega_i} \tau_s \leq \sum_{s\in\Omega_i} \omega_s.$$

b) For all $t \in \Omega_{\to i}$, there exists $\tau \in \tilde{D}(p - \delta\chi_t\,; \hat{u}_i)$ such that

$$\tau_s \leq \omega_s \ (\forall s \in \Omega_i \setminus \{t\}), \qquad \sum_{s\in\Omega_i} \tau_s \geq \sum_{s\in\Omega_i} \omega_s.$$

The condition (GGS±) is equivalent to M$^\natural$-concavity.

Theorem 3.16 ([63])**.** *A function $\hat{u}_i : \mathbf{Z}^{\Omega_i} \to \mathbf{R} \cup \{-\infty\}$ whose effective domain is bounded satisfies (GGS±) condition if and only if it is an M$^\natural$-concave function.*

By condition (3.1), we can regard a valuation function in our model as a function defined on $\mathbf{Z}^{\Omega_i}$. Then the effective domain of a valuation function is bounded by condition (3.2). By the definition of an $\Omega_{i\to}$-twisted M$^\natural$-concave function and Theorem 3.16, the following statement holds.

Corollary 3.17. *Let $u_i : \mathbf{Z}^{\Omega_i} \to \mathbf{R} \cup \{-\infty\}$ be a function whose effective domain is bounded. Then u_i is an $\Omega_{i\to}$-twisted M$^\natural$-concave function if and only if the following conditions hold.*

(i) *For all $p \in \mathbf{R}^{\Omega_i}$, $D(p\,; u_i)$ is a discretely convex set where*

$$D(p\,; u_i) = \operatorname{argmax}\left\{ u_i(\omega) + \sum_{s\in\Omega_{i\to}} p_s \cdot \omega_s - \sum_{s\in\Omega_{\to i}} p_s \cdot \omega_s : \omega \in \mathbf{Z}^{\Omega_i} \right\}.$$

(ii) *Let $p \in \mathbf{R}^{\Omega_i}$, $\delta > 0$, and $\omega \in D(p\,; u_i)$.*

a) *For all $t \in \Omega_{i\to}$, there exists $\tau \in D(p + \delta\chi_t\,; u_i)$ such that*

$$\omega_s \geq \tau_s \quad (\forall s \in \Omega_{i\to} \setminus \{t\}), \qquad \omega_s \leq \tau_s \quad (\forall s \in \Omega_{\to i}),$$
$$\sum_{s\in\Omega_{i\to}} \tau_s - \sum_{s\in\Omega_{i\to}} \omega_s \geq \sum_{s\in\Omega_{\to i}} \tau_s - \sum_{s\in\Omega_{\to i}} \omega_s.$$

b) *For all* $t \in \Omega_{\to i}$*, there exists* $\tau \in D(p - \delta\chi_t\,; u_i)$ *such that*

$$\omega_s \geq \tau_s \quad (\forall s \in \Omega_{\to i} \setminus \{t\}), \qquad \omega_s \leq \tau_s \quad (\forall s \in \Omega_{i\to}),$$
$$\sum_{s\in\Omega_{\to i}} \tau_s - \sum_{s\in\Omega_{\to i}} \omega_s \geq \sum_{s\in\Omega_{i\to}} \tau_s - \sum_{s\in\Omega_{i\to}} \omega_s.$$

By Corollary 3.17 and condition (3.1), we verify that Theorem 3.15 is satisfied.

3.2.2 Existence of competitive equilibria

We consider competitive equilibria in the multi-unit trading network model under the generalized full substitutes condition, which is equivalent to twisted M$^\natural$-concavity of a valuation function. Since the concept of twisted M$^\natural$-concave functions is a variant of M$^\natural$-concave functions, the existing results for M$^\natural$-concave functions in discrete convex analysis can be applied to show the existence of competitive equilibria.

Theorem 3.18. *Suppose that every* u_i *satisfies the generalized full substitutes condition, that is,* u_i *is twisted M*$^\natural$*-concave for all* $i \in I$*. Then there exists a competitive equilibrium in economy* $(\Omega, \{u_i\}_{i\in I})$*.*

Proof. We divide each trade $t \in \Omega$ into two trades t_b and t_s, and define

$$\tilde{\Omega} = \bigcup_{t\in\Omega}\{t_b, t_s\}, \qquad \tilde{\Omega}_i = \{t_b : t \in \Omega_{\to i}\} \cup \{t_s : t \in \Omega_{i\to}\} \quad (i \in I).$$

Since there are exactly one buyer and one seller for each trade $t \in \Omega$, it holds that

$$\tilde{\Omega}_j \cap \tilde{\Omega}_k = \emptyset \quad (j \neq k). \tag{3.7}$$

We now construct two M$^\natural$-concave functions u and f defined on $\mathbf{Z}^{\tilde{\Omega}}$. For each $i \in I$, we define a function $\tilde{u}_i : \mathbf{Z}^{\tilde{\Omega}_i} \to \mathbf{R} \cup \{-\infty\}$ by

$$\tilde{u}_i(\tilde{\omega}) = u_i(T_i(\tilde{\omega})) \quad (\forall \tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}_i}), \tag{3.8}$$

where u_i is a valuation function defined on $\mathbf{Z}^{\Omega}$ and $T_i(\tilde{\omega}) \in \mathbf{Z}^{\Omega}$ is defined by

$$(T_i(\tilde{\omega}))_t = \begin{cases} \tilde{\omega}_{t_b} & \text{if } t \in \Omega_{\to i}, \\ -\tilde{\omega}_{t_s} & \text{if } t \in \Omega_{i\to}, \\ \text{arbitrary} & \text{if } t \notin \Omega_i. \end{cases}$$

Although $T_i(\tilde{\omega})$ is not unique for each $\tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}_i}$, the value of $u_i(\tilde{\omega})$ is unique. This is because the value of u_i depends only on the number of trades in Ω_i. By the definition of $\tilde{u}_i$, it is an M$^\natural$-concave function. Let $u : \mathbf{Z}^{\tilde{\Omega}} \to \mathbf{R} \cup \{-\infty\}$ be defined by

$$u(\tilde{\omega}) = \sum_{i \in I} \tilde{u}_i(\tilde{\omega}^{\tilde{\Omega}_i}) \quad (\tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}}), \tag{3.9}$$

where $\tilde{\omega}^{\tilde{\Omega}_i}$ is the restriction of $\tilde{\omega}$ on $\tilde{\Omega}_i$. Namely, u is the sum of all $\tilde{u}_i$ which implies that u is also an M$^\natural$-concave function (note that (3.7) holds). Moreover, $\operatorname{dom} u$ is bounded by condition (3.2). We next define $f : \mathbf{Z}^{\tilde{\Omega}} \to \mathbf{R} \cup \{-\infty\}$ by

$$f(\tilde{\omega}) = \begin{cases} 0 & \text{if } \tilde{\omega}_{t_b} + \tilde{\omega}_{t_s} = 0 \text{ for all } t \in \Omega, \\ -\infty & \text{otherwise.} \end{cases} \tag{3.10}$$

It is easy to check that $\operatorname{dom} f$ is an M$^\natural$-convex set. Hence f is an M$^\natural$-concave function.

Let $\tilde{\omega}^*$ be a maximizer of $u + f$. Such a maximizer exists because $\operatorname{dom} u$ is bounded and $\operatorname{dom} u \cap \operatorname{dom} f \neq \emptyset$ (note that $\mathbf{0} \in \operatorname{dom} u \cap \operatorname{dom} f$). We apply Theorem 2.24 to u and f. Then there exists $\tilde{p}^* \in \mathbf{R}^{\tilde{\Omega}}$ such that

$$u(\tilde{\omega}^*) - \sum_{k \in \tilde{\Omega}} \tilde{\omega}_k^* \cdot \tilde{p}_k^* \geq u(\tilde{\omega}) - \sum_{k \in \tilde{\Omega}} \tilde{\omega}_k \cdot \tilde{p}_k^* \quad (\forall \tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}}), \tag{3.11}$$

$$f(\tilde{\omega}^*) + \sum_{k \in \tilde{\Omega}} \tilde{\omega}_k^* \cdot \tilde{p}_k^* \geq f(\tilde{\omega}) + \sum_{k \in \tilde{\Omega}} \tilde{\omega}_k \cdot \tilde{p}_k^* \quad (\forall \tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}}). \tag{3.12}$$

By (3.12), it holds that $\tilde{\omega}^* \in \operatorname{dom} f$, which means $\tilde{\omega}^*_{t_b} + \tilde{\omega}^*_{t_s} = 0$ for all $t \in \Omega$. We claim that $\tilde{p}^*_{t_b} = \tilde{p}^*_{t_s}$ for all $t \in \Omega$. If $\tilde{p}^*_{t_b} > \tilde{p}^*_{t_s}$ for some $t \in \Omega$, we have

$$\begin{aligned} f(\tilde{\omega}^* + \chi_{t_b} - \chi_{t_s}) + \sum_{k \in \tilde{\Omega}} (\tilde{\omega}^* + \chi_{t_b} - \chi_{t_s})_k \cdot \tilde{p}^*_k &= f(\tilde{\omega}^*) + \sum_{k \in \tilde{\Omega}} \tilde{\omega}^*_k \cdot \tilde{p}^*_k + \tilde{p}^*_{t_b} - \tilde{p}^*_{t_s} \\ &> f(\tilde{\omega}^*) + \sum_{k \in \tilde{\Omega}} \tilde{\omega}^*_k \cdot \tilde{p}^*_k. \end{aligned}$$

This contradicts (3.12). We also obtain a contradiction from the assumption that $\tilde{p}^*_{t_b} < \tilde{p}^*_{t_s}$. Therefore $\tilde{\omega}^*_{t_b} + \tilde{\omega}^*_{t_s} = 0$ and $\tilde{p}^*_{t_b} = \tilde{p}^*_{t_s}$ for all $t \in \Omega$.

We define $\omega^* \in \mathbf{Z}^\Omega$ and $p^* \in \mathbf{R}^\Omega$ by

$$\omega^*_t = \tilde{\omega}^*_{t_b} \, (= -\tilde{\omega}^*_{t_s}), \quad p^*_t = \tilde{p}^*_{t_b} \, (= \tilde{p}^*_{t_s}) \quad (\forall t \in \Omega).$$

We finally show that (ω^*, p^*) is a competitive equilibrium in $(\Omega, \{u_i\}_{i \in I})$, that is, $\omega^* \in D(p^* \, ; u_i)$ for all $i \in I$. By (3.7), we obtain

$$\sum_{i \in I} \tilde{u}_i(\tilde{\omega}^{\tilde{\Omega}_i}) - \sum_{k \in \tilde{\Omega}} \tilde{\omega}_k \cdot \tilde{p}^*_k = \sum_{i \in I} \left(\tilde{u}_i(\tilde{\omega}^{\tilde{\Omega}_i}) - \sum_{k \in \tilde{\Omega}_i} \tilde{\omega}_k \cdot \tilde{p}^*_k \right).$$

This fact and (3.11) imply that

$$\tilde{u}_i(\tilde{\omega}^{*\tilde{\Omega}_i}) - \sum_{k \in \tilde{\Omega}_i} \tilde{\omega}^*_k \cdot \tilde{p}^*_k \geq \tilde{u}_i(\tilde{\omega}^{\tilde{\Omega}_i}) - \sum_{k \in \tilde{\Omega}_i} \tilde{\omega}_k \cdot \tilde{p}^*_k \quad (\forall i \in I, \tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}}). \tag{3.13}$$

Furthermore, for all $i \in I$ and $\tilde{\omega} \in \mathbf{Z}^{\tilde{\Omega}}$ we have

$$\begin{aligned} \tilde{u}_i(\tilde{\omega}^{\tilde{\Omega}_i}) - \sum_{k \in \tilde{\Omega}_i} \tilde{\omega}_k \cdot \tilde{p}^*_k &= u_i(T_i(\tilde{\omega}^{\tilde{\Omega}_i})) + \sum_{t \in \Omega_{i \to}} (T_i(\tilde{\omega}^{\tilde{\Omega}_i}))_t \cdot p^*_t - \sum_{t \in \Omega_{\to i}} (T_i(\tilde{\omega}^{\tilde{\Omega}_i}))_t \cdot p^*_t \\ &= U(T_i(\tilde{\omega}^{\tilde{\Omega}_i}), p^* \, ; u_i). \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$U(\omega^*, p^* \, ; u_i) \geq U(\omega, p^* \, ; u_i) \quad (\forall i \in I, \omega \in \mathbf{Z}^\Omega).$$

Thus $\omega^* \in D(p^* \, ; u_i)$ for all $i \in I$, that is, (ω^*, p^*) is a competitive equilibrium. $\square$

3.2.3 Lattice structure of the set of competitive equilibrium price vectors

In this subsection, we consider the structure of the set of competitive equilibrium price vectors under the generalized full substitutes condition. Similar to the single-unit trading network model, the set of competitive equilibrium price vectors forms a lattice in our model.

Theorem 3.19. *Suppose that every u_i satisfies the generalized full substitutes condition and consider economy $(\Omega, \{u_i\}_{i \in I})$. Then the set of competitive equilibrium price vectors forms a lattice.*

Before we prove Theorem 3.19, we provide the following proposition. We often use this proposition to prove our results.

Proposition 3.20. *For all $p \in \mathbf{R}^{\Omega}$, we have*

$$\sum_{i \in I} U(\omega, p\,; u_i) = \sum_{i \in I} u_i(\omega).$$

Proof. Since each trade $t \in \Omega$ is associated with exactly one buyer and one seller, we have

$$\begin{aligned}\sum_{i \in I} U(\omega, p\,; u_i) &= \sum_{i \in I} \left(u_i(\omega) + \sum_{t \in \Omega_{i \to}} \omega_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega_t \cdot p_t \right) \\ &= \sum_{i \in I} u_i(\omega) + \sum_{i \in I} \left(\sum_{t \in \Omega_{i \to}} \omega_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega_t \cdot p_t \right) = \sum_{i \in I} u_i(\omega).\end{aligned}$$

□

We now prove Theorem 3.19. We roughly describe the idea of the proof. We first show that the sum of indirect utility functions $\sum_{i \in I} V(\cdot\,; u_i)$ is a polyhedral L$^\natural$-convex function (Proposition 3.21). Then we prove that p is a competitive equilibrium price vector if and only if p is a minimizer of $\sum_{i \in I} V(\cdot\,; u_i)$ (Proposition 3.22). By Theorem 2.18, we verify that the set of competitive equilibrium price vectors forms a lattice because it is an L$^\natural$-convex polyhedron.

Proposition 3.21. *Suppose that u_i is twisted M$^\natural$-concave for all $i \in I$. Then $\sum_{i \in I} V(\cdot\,; u_i)$ is a polyhedral L$^\natural$-convex function.*

Proof. By Proposition 2.16, the sum of polyhedral L$^\natural$-convex functions is also polyhedral L$^\natural$-convex. It suffices to show that each indirect utility function $V(\cdot\,; u_i)$ is polyhedral L$^\natural$-convex. By Proposition 2.30, u_i is $\Omega_{\to i}$-twisted M$^\natural$-concave (where u_i is defined on $\mathbf{Z}^{\Omega_i}$). Thus there exists an M$^\natural$-concave function $v_i : \mathbf{Z}^{\Omega_i} \to \mathbf{R} \cup \{-\infty\}$ such that

$$u_i(\omega) = v_i(\mathrm{twist}(\omega\,; \Omega_{\to i})).$$

We consider the Legendre-Fenchel transform of $-v_i$. It is represented by

$$\begin{aligned}
(-v_i)^\bullet(p) &= \sup\left\{\langle p, \omega\rangle - (-v_i(\omega)) : \omega \in \mathbf{Z}^{\Omega_i}\right\} \\
&= \sup\left\{\langle p, \omega\rangle + u_i(\mathrm{twist}(\omega\,; \Omega_{\to i})) : \omega \in \mathbf{Z}^{\Omega_i}\right\} \\
&= \sup\left\{\langle p, \mathrm{twist}(\omega\,; \Omega_{\to i})\rangle + u_i(\omega) : \omega \in \mathbf{Z}^{\Omega_i}\right\} \\
&= \sup\left\{u_i(\omega) + \sum_{t \in \Omega_{i\to}} p_t \cdot \omega_t - \sum_{t \in \Omega_{\to i}} p_t \cdot \omega_t : \omega \in \mathbf{Z}^{\Omega_i}\right\} \\
&= \max\left\{u_i(\omega) + \sum_{t \in \Omega_{i\to}} p_t \cdot \omega_t - \sum_{t \in \Omega_{\to i}} p_t \cdot \omega_t : \omega \in \mathbf{Z}^{\Omega_i}\right\},
\end{aligned}$$

where the last equality follows from the fact that $\mathrm{dom}\, u_i$ is bounded. On the other hand, for $p \in \mathbf{R}^\Omega$, we have

$$\begin{aligned}
V(p\,; u_i) &= \left\{U(\omega, p\,; u_i) : \omega \in \mathbf{Z}^\Omega\right\} \\
&= \max\left\{u_i(\omega) + \sum_{t \in \Omega_{i\to}} p_t \cdot \omega_t - \sum_{t \in \Omega_{\to i}} p_t \cdot \omega_t : \omega \in \mathbf{Z}^{\Omega_i}\right\} = (-v_i)^\bullet(p^{\Omega_i}).
\end{aligned}$$

By Theorem 2.23, $V(\cdot\,; u_i)$ is a polyhedral L$^\natural$-convex function. □

We next show that the set of minimizers of $V(\cdot\,; u_i)$ coincides with the set of competitive equilibrium price vectors.

Proposition 3.22. *Suppose that there exists a competitive equilibrium. Then p is a competitive equilibrium price vector if and only if p is a minimizer of $\sum_{i \in I} V(\cdot\,; u_i)$.*

Proof. Let (ω^*, p^*) be a competitive equilibrium and $p \in \mathbf{R}^{\Omega}$. By the definition of $V(\cdot\,; u_i)$, we have

$$V(p^*\,; u_i) = U(\omega^*, p^*\,; u_i) \qquad (\forall i \in I), \tag{3.15}$$

$$V(p\,; u_i) \geq U(\omega^*, p\,; u_i) \qquad (\forall i \in I). \tag{3.16}$$

By (3.15) and (3.16), we have

$$\begin{aligned} \sum_{i \in I} V(p\,; u_i) &\geq \sum_{i \in I} U(\omega^*, p\,; u_i) \\ &= \sum_{i \in I} u_i(\omega^*) \\ &= \sum_{i \in I} U(\omega^*, p^*\,; u_i) = \sum_{i \in I} V(p^*\,; u_i), \end{aligned} \tag{3.17}$$

where the first and second equalities follow from Proposition 3.20. This shows that p^* is a minimizer of $\sum_{i \in I} V(\cdot\,; u_i)$.

We then assume that p is a minimizer of $\sum_{i \in I} V(\cdot\,; u_i)$. Since p^* is also a minimizer, we have

$$\sum_{i \in I} V(p\,; u_i) = \sum_{i \in I} V(p^*\,; u_i).$$

Hence (3.16) holds with equality, which implies that $\omega^* \in D(p\,; u_i)$ for all $i \in I$, that is, p is a competitive equilibrium price vector. □

By Proposition 2.18, the set of competitive equilibrium price vectors is an $\mathrm{L}^\natural$-convex polyhedron. Thus it forms a lattice. This concludes the proof of Theorem 3.19.

3.3 Competitive equilibria, efficiency, and stability

In this section, we consider the connection among competitive equilibria, efficiency, and stability in the multi-unit trading network model. We generalize most of the results in Subsection 3.1.2 concerning the single-unit trading network model, while Theorem 3.13 in Subsection 3.1.2 does not admit a generalization. The results in this section are also based on [32].

3.3.1 Competitive equilibria and efficiency

We first show the relationship between competitive equilibria and the efficiency of trade vectors.

Theorem 3.23. *Let (ω, p) be a competitive equilibrium in economy $(\Omega, \{u_i\}_{i\in I})$. Then ω is efficient on $\{u_i\}_{i\in I}$.*

Proof. By the definition of competitive equilibria, we have

$$\begin{aligned}
u_i(\omega) + \sum_{t\in\Omega_{i\to}} \omega_t \cdot p_t - \sum_{t\in\Omega_{\to i}} \omega_t \cdot p_t &= U(\omega, p\,; u_i) \\
&\geq U(\omega', p\,; u_i) \\
&= u_i(\omega') + \sum_{t\in\Omega_{i\to}} \omega'_t \cdot p_t - \sum_{t\in\Omega_{\to i}} \omega'_t \cdot p_t
\end{aligned}$$

for all $i \in I$ and $\omega' \in \mathbf{Z}^\Omega$. Since $\{\Omega_{i\to} : i \in I\}$ and $\{\Omega_{\to i} : i \in I\}$ are partitions of Ω, by summing these inequalities over all $i \in I$ we have

$$\sum_{i\in I} u_i(\omega) \geq \sum_{i\in I} u_i(\omega') \quad (\forall \omega' \in \mathbf{Z}^\Omega).$$

Therefore ω is efficient on $\{u_i\}_{i\in I}$. $\square$

Theorem 3.24. *Let (ω, p) be a competitive equilibrium in economy $(\Omega, \{u_i\}_{i\in I})$. For any efficient trade vector ω' on $\{u_i\}_{i\in I}$, (ω', p) is also a competitive equilibrium.*

Proof. By the efficiency of ω' and Proposition 3.20, it holds that

$$\begin{aligned}
\sum_{i\in I} U(\omega', p\,; u_i) &= \sum_{i\in I} u_i(\omega') \\
&\geq \sum_{i\in I} u_i(\omega) = \sum_{i\in I} U(\omega, p\,; u_i).
\end{aligned}$$

On the other hand,

$$U(\omega, p\,; u_i) \geq U(\omega', p\,; u_i)$$

holds for all $i \in I$ because (ω, p) is a competitive equilibrium. From these inequalities, we obtain

$$U(\omega, p\,; u_i) = U(\omega', p\,; u_i) \quad (\forall i \in I).$$

Therefore $\omega' \in D(p\,; u_i)$ for all $i \in I$, that is, (ω', p) is a competitive equilibrium. $\square$

Theorem 3.23 says that a competitive equilibrium trade vector is always efficient on $\{u_i\}_{i \in I}$. This theorem can be regarded as the first fundamental theorem of welfare economics. Theorem 3.24 says that the pair of an efficient trade vector and a competitive equilibrium price vector is a competitive equilibrium. We can regard Theorem 3.24 as the second fundamental theorem of welfare economics[3]. Note that no assumptions are required for preferences of agents in these two theorems (however, there may not exist a competitive equilibrium).

3.3.2 Competitive equilibria and stable outcomes

We next consider the connection between competitive equilibria and stable outcomes. We show that a competitive equilibrium is always stable.

Theorem 3.25. *If an outcome (ω, p) is a competitive equilibrium in economy $(\Omega, \{u_i\}_{i \in I})$, then it is stable.*

Proof. Suppose that (ω, p) is not stable. We prove that (ω, p) is not a competitive equilibrium, that is, $\omega \notin D(p\,; u_j)$ holds for some $j \in I$. This fact leads to Theorem 3.25.

We first consider the case where (ω, p) is not individually rational. Then, there exists $j \in I$ such that $\omega \notin C(\omega, p\,; u_j)$. Thus it holds that

$$U(\omega', p\,; u_j) > U(\omega, p\,; u_j) \qquad (\forall \omega' \in C(\omega, p\,; u_j)),$$

which implies $\omega \notin D(p\,; u_j)$.

[3]The fundamental theorems of welfare economics are well known in microeconomics. For more information, see [41].

We next consider the case where (ω, p) is individually rational and there exists a blocking set (z, p') of (ω, p). We denote $J = \mathrm{ag}(\mathrm{supp}^+(z))$. By the definition of blocking sets, we have

$$\omega^i_t = (\omega + z)_t \quad (t \in \mathrm{supp}^+(z) \cap \Omega_i)$$

for $i \in J$ and $\omega^i \in C(\omega + z, p + p'\,; u_i)$. This means that $\omega \notin C(\omega + z, p + p'\,; u_i)$ for all $i \in J$, implying that

$$U(\omega, p + p'\,; u_i) < U(\omega^i, p + p'\,; u_i) \quad (\forall i \in J).$$

By summing these inequalities over all $i \in J$, we obtain

$$\sum_{i\in J} U(\omega, p + p'\,; u_i) < \sum_{i\in J} U(\omega^i, p + p'\,; u_i). \tag{3.18}$$

As shown below, it holds that

$$\sum_{i\in J} U(\omega, p + p'\,; u_i) = \sum_{i\in J} U(\omega, p\,; u_i), \tag{3.19}$$

$$\sum_{i\in J} U(\omega^i, p + p'\,; u_i) = \sum_{i\in J} U(\omega^i, p; u_i). \tag{3.20}$$

By (3.18), (3.19), and (3.20), we have

$$\sum_{i\in J} U(\omega, p\,; u_i) < \sum_{i\in J} U(\omega^i, p\,; u_i). \tag{3.21}$$

By (3.21), there exists $j \in J$ such that $U(\omega^j, p\,; u_j) > U(\omega, p\,; u_j)$. Hence, $\omega \notin D(p\,; u_j)$ holds and (ω, p) is not a competitive equilibrium.

We need to prove the equations (3.19) and (3.20) for the correctness of the proof. We first prove the equation (3.19). By Proposition 3.20, we have

$$\sum_{i\in I} U(\omega, p\,; u_i) = \sum_{i\in I} u_i(\omega) = \sum_{i\in I} U(\omega, p + p'\,; u_i). \tag{3.22}$$

Since $p'_t = 0$ for all $t \in \Omega \setminus \mathrm{supp}^+(z)$, it holds that

$$\sum_{i \in I \setminus J} U(\omega, p\,; u_i) = \sum_{i \in I \setminus J} U(\omega, p + p'\,; u_i). \tag{3.23}$$

The equation (3.19) follows from (3.22) and (3.23).

We finally show that the equation (3.20) holds. For all $t \in \mathrm{supp}^+(z) \cap \Omega_i$ and $i \in J$, we have $\omega^i_t = (\omega + z)_t$. Moreover, $p'_t = 0$ for all $t \in \Omega \setminus \mathrm{supp}^+(z)$. Therefore, it holds that

$$\begin{aligned}
&\sum_{i \in J} \left(\sum_{t \in \Omega_{i\to}} \omega^i_t \cdot p'_t - \sum_{t \in \Omega_{\to i}} \omega^i_t \cdot p'_t \right) \\
&\qquad = \sum_{i \in J} \left(\sum_{t \in \Omega_{i\to}} (\omega + z)_t \cdot p'_t - \sum_{t \in \Omega_{\to i}} (\omega + z)_t \cdot p'_t \right) = 0.
\end{aligned}$$

Hence, we can obtain (3.20) as follows:

$$\begin{aligned}
\sum_{i \in J} U(\omega^i, p\,; u_i) &= \sum_{i \in J} \left(u_i(\omega^i) + \sum_{t \in \Omega_{i\to}} \omega^i_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega^i_t \cdot p_t \right) \\
&= \sum_{i \in J} \left(u_i(\omega^i) + \sum_{t \in \Omega_{i\to}} \omega^i_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega^i_t \cdot p_t \right) \\
&\qquad\qquad + \sum_{i \in J} \left(\sum_{t \in \Omega_{i\to}} \omega^i_t \cdot p'_t - \sum_{t \in \Omega_{\to i}} \omega^i_t \cdot p'_t \right) \\
&= \sum_{i \in J} \left(u_i(\omega^i) + \sum_{t \in \Omega_{i\to}} \omega^i_t \cdot (p + p')_t - \sum_{t \in \Omega_{\to i}} \omega^i_t \cdot (p + p')_t \right) \\
&= \sum_{i \in J} U(\omega^i, p + p'\,; u_i).
\end{aligned}$$

□

The converse of Theorem 3.25 does not hold even if preferences of all agents satisfy the generalized full substitutes condition, as shown in the following example.

Example 3.2. Consider an economy with two agents, a and b, and only one trade ω. Agent a is the buyer and agent b is the seller. Valuation functions of each agent are given in Table 3.2.

Table 3.2: The valuations and utilities of agents in Example 3.2.

ω	$u_a(\omega)$	$u_b(\omega)$	$U(\omega, 2\,; u_a)$	$U(\omega, 2\,; u_b)$
0	0	0	0	0
1	4	-1	2	1
2	7	-2	3	2
3	0	-3	-6	3
4	$-\infty$	$-\infty$	$-\infty$	$-\infty$

We can easily check that u_a and u_b are twisted M$^\natural$-concave functions. Thus preferences of all agents satisfy the generalized full substitutes condition. In this example, we can verify that $(\omega, p) = (2, 2)$ is a stable outcome. However, $(2, 2)$ is not a competitive equilibrium because it holds that

$$2 \notin D(2\,; u_b) = \{3\}.$$

Furthermore, Example 3.2 shows that the following statement, a generalization of Theorem 3.13, does not hold in the multi-unit trading network model:

> **Statement A.** Suppose that every u_i satisfies the generalized full substitutes condition. If an outcome (ω, p) is stable in economy $(\Omega, \{u_i\}_{i\in I})$, then there exists $q \in \mathbf{R}^\Omega$ such that (ω, q) is a competitive equilibrium and $q_t = p_t$ for all $t \in \mathrm{supp}^+(\omega)$.

This shows an obvious gap between competitive equilibria and stable outcomes with respect to price vectors in the multi-unit trading network model. In fact, stable outcomes are strictly wider concept than competitive equilibria. On the other hand, stability is closely related to the efficiency of a trade vector.

Theorem 3.26. *Suppose that every u_i satisfies the generalized full substitutes condition. If an outcome (ω, p) is stable in economy $(\Omega, \{u_i\}_{i\in I})$, then ω is efficient on $\{u_i\}_{i\in I}$.*

We prove Theorem 3.26 in the next subsection. By Theorems 3.24 and 3.26, we have the following weaker statement than Statement A.

Table 3.3: The valuations of agents in Example 3.3.

(t_1, t_2)	$u_a(\cdot)$	$u_b(\cdot)$	$u_c(\cdot)$
$(0,0)$	0	0	0
$(1,0)$	4	-1	0
$(0,1)$	4	0	-1
$(1,1)$	0	-1	-1

Corollary 3.27. *Suppose that every u_i satisfies the generalized full substitutes condition. If an outcome (ω, p) is stable in economy $(\Omega, \{u_i\}_{i\in I})$, then there exists $q \in \mathbf{R}^{\Omega}$ such that (ω, q) is a competitive equilibrium in $(\Omega, \{u_i\}_{i\in I})$.*

Corollary 3.27 says that we can construct a competitive equilibrium from a stable outcome by changing prices of trades. Unlike the single-unit trading network model, competitive equilibria and stable outcomes are not essentially equivalent in the multi-unit trading network model.

By Theorem 3.26, a price vector of a stable outcome is efficient. We can easily check that the converse of Theorem 3.26 does not hold. Indeed, if p is sufficiently large, (ω, p) does not satisfy the individual rationality. In addition, even if ω is efficient on $\{u_i\}_{i\in I}$ and (ω, p) is individually rational, (ω, p) may not be stable (see Example 3.3).

Example 3.3. Let a, b, and c be three agents, and t_1 and t_2 be two types of trades such that $b(t_1) = a$, $s(t_1) = b$, $b(t_2) = a$, and $s(t_2) = c$. Valuation functions of each agent are given in Table 3.3. Then $(\omega, p) = ((1,0),(3,0))$ has a blocking set $(z, p') = ((0,1),(0,2))$ while $((1,0),(3,0))$ is individually rational and $(t_1, t_2) = (1,0)$ is efficient on $\{u_i\}_{i\in I}$.

3.3.3 Proof of Theorem 3.26

We give the proof of Theorem 3.26. Suppose that (ω, p) is individually rational. We show that (ω, p) has a blocking set if ω is not efficient on $\{u_i\}_{i\in I}$. We first define a new valuation function. For a nonempty $A \subseteq \Omega$ we define the valuation function

$u_i^A(\cdot\,;(\omega,p)) : \mathbf{Z}^A \to \mathbf{R} \cup \{-\infty\}$ of agent i by

$$u_i^A(\tau\,;(\omega,p)) = \max\Biggl\{u_i(\omega') + \sum_{t\in\Omega_{i\to}\setminus A} \omega'_t \cdot p_t - \sum_{t\in\Omega_{\to i}\setminus A} \omega'_t \cdot p_t : \\ \omega'_t \le \omega_t\,(t \in \Omega \setminus A),\, \omega'_t = \tau_t\,(t\in A),\, \omega' \in \mathbf{Z}^\Omega\Biggr\}.$$

For convenience, we define $\max\{-\infty\} = -\infty$. By Proposition 2.31, $u_i^A(\cdot\,;(\omega,p))$ is $(\Omega_{i\to} \cap A)$-twisted M$^\natural$-concave (note that we can replace "max" with "sup" by conditions (3.1) and (3.2)). Function $u_i^A(\cdot\,;(\omega,p))$ is helpful in investigating the relationship between the efficiency and stability. In fact, the following lemma holds.

Lemma 3.28. *Suppose that every u_i is twisted M$^\natural$-concave and (ω,p) is individually rational in economy $(\Omega,\{u_i\}_{i\in I})$. Then ω^A is not efficient on $\{u_i^A(\cdot\,;(\omega,p))\}_{i\in I}$ for some $A \subseteq \Omega$ if and only if there exists a blocking set of (ω,p). Furthermore, if ω^A is not efficient on $\{u_i^A(\cdot\,;(\omega,p))\}_{i\in I}$, then there exists a blocking set (z,p') of (ω,p) such that $z \in \{0,1\}^\Omega$ and $\mathrm{supp}^+(z) \subseteq A$.*

If ω is not efficient on $\{u_i\}_{i\in I}$, ω^Ω is not efficient on $\{u_i^\Omega(\cdot\,;(\omega,p))\}_{i\in I}$ (note that $\omega^\Omega = \omega$ and $u_i^\Omega(\cdot\,;(\omega,p)) = u_i$). Therefore, Theorem 3.26 follows from Lemma 3.28. The proof of Lemma 3.28 is given in the last part of this subsection.

Before proving Lemma 3.28, we discuss the inefficiency of trade vectors and some properties of $u_i^A(\cdot\,;(\omega,p))$. Some propositions and lemmas in this subsection are used in the proofs of the next section.

Inefficiency of trade vectors

We discuss the inefficiency of trade vectors. We show that if ω is not efficient on $\{u_i\}_{i\in I}$, there exists ω' such that $\sum_{i\in I} u_i(\omega) < \sum_{i\in I} u_i(\omega')$ and $\max_{t\in\Omega} |\omega'_t - \omega_t| \le 1$.

Lemma 3.29. *Suppose that u_i is twisted M$^\natural$-concave for all $i \in I$ and ω is not efficient on $\{u_i\}_{i\in I}$. Then there exist $A, B \subseteq \Omega$ such that $A \cap B = \emptyset$, and*

$$\sum_{i\in I} u_i(\omega) < \sum_{i\in I} u_i(\omega + \chi_A - \chi_B).$$

Proof. Let u and f be functions defined by (3.9) and (3.10), respectively. Recall that these functions are M$^\natural$-concave. It is easy to check that

$$\sum_{i \in I} u_i(\omega') = u(\tilde{T}(\omega')) + f(\tilde{T}(\omega')) \quad (\forall \omega' \in \mathbf{Z}^{\Omega}), \tag{3.24}$$

where $\tilde{T}(\omega') \in \mathbf{Z}^{\tilde{\Omega}}$ is defined by

$$(\tilde{T}(\omega'))_{t_b} = \omega'_t, \quad (\tilde{T}(\omega'))_{t_s} = -\omega'_t \quad (\forall t \in \Omega). \tag{3.25}$$

Since ω is not efficient on $\{u_i\}_{i \in I}$, $\tilde{T}(\omega) \notin \operatorname{argmax}\{u + f\}$. By Corollary 2.26, there exist $U, W \subseteq \tilde{\Omega}$ such that

$$u(\tilde{T}(\omega)) + f(\tilde{T}(\omega)) < u(\tilde{T}(\omega) + \chi_U - \chi_W) + f(\tilde{T}(\omega) + \chi_U - \chi_W). \tag{3.26}$$

This inequality means $f(\tilde{T}(\omega) + \chi_U - \chi_W) > -\infty$. Therefore for each $t \in \Omega$ we have

$$t_b \in U \Leftrightarrow t_s \in W, \qquad t_b \in W \Leftrightarrow t_s \in U. \tag{3.27}$$

By (3.24), (3.25), and (3.27), we have

$$u(\tilde{T}(\omega) + \chi_U - \chi_W) + f(\tilde{T}(\omega) + \chi_U - \chi_W) = \sum_{i \in I} u_i(\omega + \chi_A - \chi_B),$$

where

$$\begin{aligned} A &= \{t \in \Omega : t_b \in U\} \, (= \{t \in \Omega : t_s \in W\}), \\ B &= \{t \in \Omega : t_s \in U\} \, (= \{t \in \Omega : t_b \in W\}). \end{aligned}$$

Therefore, it holds by (3.26) that

$$\begin{aligned} \sum_{i \in I} u_i(\omega) &= u(\tilde{T}(\omega)) + f(\tilde{T}(\omega)) \\ &< u(\tilde{T}(\omega) + \chi_U - \chi_W) + f(\tilde{T}(\omega) + \chi_U - \chi_W) \\ &= \sum_{i \in I} u_i(\omega + \chi_A - \chi_B). \end{aligned}$$

□

Properties of u_i^A

We next describe some properties of $u_i^A(\cdot\,;(\omega,p))$ used in the proof of Lemma 3.28. Let $\omega^A \in \mathbf{Z}^A$ and $p^A \in \mathbf{R}^A$ be restrictions of ω and p on A, respectively. Propositions we introduce below state the relationship between two kinds of valuation functions, u_i and $u_i^A(\cdot\,;(\omega,p))$. We define the utility function associated with $u_i^A(\cdot\,;(\omega,p))$ by $U(\cdot,\cdot\,;u_i^A(\cdot\,;(\omega,p)))$, that is,

$$U(\tau,q\,;u_i^A(\cdot\,;(\omega,p))) = u_i^A(\tau\,;(\omega,p)) + \sum_{t\in\Omega_{i\to}\cap A} \tau_t \cdot q_t - \sum_{t\in\Omega_{\to i}\cap A} \tau_t \cdot q_t. \tag{3.28}$$

Proposition 3.30. *For $\tau \in \mathbf{Z}^A$ and $q \in \mathbf{R}^A$, it holds that*

$$\begin{aligned} &U(\tau,q\,;u_i^A(\cdot\,;(\omega,p))) \\ &\quad = \max\left\{U(\omega',q'\,;u_i) : \omega'_t \le \omega_t\,(t\in\Omega\setminus A),\ \omega'_t = \tau_t\,(t\in A),\ \omega'\in\mathbf{Z}^\Omega\right\}, \end{aligned} \tag{3.29}$$

where $q' \in \mathbf{R}^\Omega$ is the vector such that $q'_t = q_t\,(\forall t\in A)$ and $q'_t = p_t\,(\forall t\in\Omega\setminus A)$. Moreover, the following statements hold:

(i) *Suppose that (z,p') is a blocking set of (ω,p) in $(\Omega,\{u_i\}_{i\in I})$ and define $A = \mathrm{supp}^+(z)$. Then, for $i\in\mathrm{ag}(A)$ and $\tau \le \omega^A + z^A$ with $\tau \ne \omega^A + z^A$, we have*

$$U(\tau,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p))) < U(\omega^A+z^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p))).$$

(ii) *Suppose that (ω,p) is individually rational in economy $(\Omega,\{u_i\}_{i\in I})$. Then for each $A\subseteq\Omega$, we have*

$$U(\omega^A,p^A\,;u_i^A(\cdot\,;(\omega,p))) = U(\omega,p\,;u_i), \tag{3.30}$$

$$\sum_{i\in I} u_i^A(\omega^A\,;(\omega,p)) = \sum_{i\in I} u_i(\omega). \tag{3.31}$$

(iii) *$U(\cdot,\cdot\,;u_i^A(\cdot\,;(\omega,p)))$ is a constant function if $\Omega_i\cap A=\emptyset$, that is, the value of $U(\tau,q\,;u_i^A(\cdot\,;(\omega,p)))$ does not depend on τ and q.*

Proof. The equation (3.29) follows immediately from the definitions of a utility function and $u_i^A(\cdot\,;(\omega,p))$. Indeed, we have

$$
\begin{aligned}
U(\tau,q\,;u_i^A(\cdot\,;(\omega,p))) &= u_i^A(\tau\,;(\omega,p)) + \sum_{t\in\Omega_{i\to}\cap A}\tau_t\cdot q_t - \sum_{t\in\Omega_{\to i}\cap A}\tau_t\cdot q_t \\
&= \max\Biggl\{u_i(\omega'') + \sum_{t\in\Omega_{i\to}\setminus A}\omega''_t\cdot p_t - \sum_{t\in\Omega_{\to i}\setminus A}\omega''_t\cdot p_t \\
&\qquad\qquad : \omega''_t \le \omega_t\,(t\in\Omega\setminus A),\ \omega''_t = \tau_t\,(t\in A),\ \omega''\in\mathbf{Z}^\Omega\Biggr\} \\
&\qquad\qquad + \sum_{t\in\Omega_{i\to}\cap A}\tau_t\cdot p_t - \sum_{t\in\Omega_{\to i}\cap A}\tau_t\cdot p_t \\
&= \max\Biggl\{u_i(\omega'') + \sum_{t\in\Omega_{i\to}\setminus A}\omega''_t\cdot p_t - \sum_{t\in\Omega_{\to i}\setminus A}\omega''_t\cdot p_t \\
&\qquad + \sum_{t\in\Omega_{i\to}\cap A}\tau_t\cdot q_t - \sum_{t\in\Omega_{\to i}\cap A}\tau_t\cdot q_t \\
&\qquad\qquad : \omega''_t \le \omega_t\,(t\in\Omega\setminus A),\ \omega''_t = \tau_t\,(t\in A),\ \omega''\in\mathbf{Z}^\Omega\Biggr\} \\
&= \max\Bigl\{U(\omega',q'\,;u_i) : \omega'_t \le \omega_t\,(t\in\Omega\setminus A), \\
&\qquad\qquad \omega'_t = \tau_t\,(t\in A),\ \omega'\in\mathbf{Z}^\Omega\Bigr\}.
\end{aligned}
$$

We then prove the claim (i). Since (z,p') is a blocking set of (ω,p), we have

$$\omega^*_t = (\omega+z)_t \quad (\forall t\in\mathrm{supp}^+(z)\cap\Omega_i) \tag{3.32}$$

for each $\omega^*\in C(\omega+z,p+p'\,;u_i)$. By (3.29) and (3.32), it holds that

$$
\begin{aligned}
U(\tau,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p))) &< U(\omega^*,p+p'\,;u_i), \\
U(\omega^A+z^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p))) &= U(\omega^*,p+p'\,;u_i).
\end{aligned}
$$

Hence we obtain the claim (i).

We show the claim (ii). Suppose that (ω, p) is individually rational in $(\Omega, \{u_i\}_{i\in I})$, that is,

$$\omega \in C(\omega, p\,; u_i) = \mathrm{argmax}\{U(\omega', p\,; u_i) : \omega' \leq \omega,\ \omega' \in \mathbf{Z}^{\Omega}\} \quad (\forall i \in I).$$

Since

$$\omega \in \{\omega' \in \mathbf{Z}^{\Omega} : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A),\ \omega'_t = \omega_t\,(t \in A)\} \subseteq \{\omega' \in \mathbf{Z}^{\Omega} : \omega' \leq \omega\},$$

it holds that

$$\omega \in \mathrm{argmax}\{U(\omega', p\,; u_i) : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A), \omega'_t = \omega_t\,(t \in A),\ \omega' \in \mathbf{Z}^{\Omega}\} \quad (\forall i \in I).$$

Hence we have

$$\begin{aligned} U(\omega^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) &= \max\Big\{U(\omega', p\,; u_i) : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A), \\ &\qquad\qquad \omega'_t = \omega_t\,(t \in A),\ \omega' \in \mathbf{Z}^{\Omega}\Big\} \\ &= U(\omega, p\,; u_i). \end{aligned} \tag{3.33}$$

By (3.33) and Proposition 3.20, we obtain

$$\sum_{i\in I} u_i^A(\omega^A\,; (\omega, p)) = \sum_{i\in I} U(\omega^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) = \sum_{i\in I} U(\omega, p\,; u_i) = \sum_{i\in I} u_i(\omega).$$

We finally consider the case where $\Omega_i \cap A = \emptyset$. By (3.28), we have

$$U(\tau, q\,; u_i^A(\cdot\,; (\omega, p))) = u_i^A(\tau\,; (\omega, p)) \quad (\forall q \in \mathbf{R}^A).$$

By the definition of $u_i^A(\cdot\,; (\omega, p))$ and the fact that the value of u_i depends only on trades in Ω_i, the claim (iii) holds. □

Proposition 3.31. *Let A and A' be subsets of Ω with $A \supseteq A'$. Then for all $\tau \in \mathbf{Z}^\Omega$ with $\tau_t \leq \omega_t\,(\forall t \in \Omega \setminus A')$ and $q \in \mathbf{R}^\Omega$ with $q_t = p_t\,(\forall t \in \Omega \setminus A')$, it holds that*

$$U(\tau, q\,; u_i) \leq U(\tau^A, q^A\,; u_i^A(\cdot\,; (\omega, p))) \leq U(\tau^{A'}, q^{A'}\,; u_i^{A'}(\cdot\,; (\omega, p))), \tag{3.34}$$

$$\sum_{i \in I} u_i(\tau) \leq \sum_{i \in I} u_i^A(\tau^A\,; (\omega, p)) \leq \sum_{i \in I} u_i^{A'}(\tau^{A'}\,; (\omega, p)). \tag{3.35}$$

Proof. The equation (3.34) follows from the equation (3.29) in Proposition 3.30 and

$$\begin{aligned}\tau \in \{&\omega' \in \mathbf{Z}^\Omega : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A),\, \omega'_t = \tau_t\,(t \in A)\} \\ &\subseteq \{\omega' \in \mathbf{Z}^\Omega : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A'),\, \omega'_t = \tau_t\,(t \in A')\}\end{aligned}$$

for all $A \supseteq A'$. The equation (3.35) follows from (3.34) and Proposition 3.20. □

The next statement shows a useful inequality for the value $\sum_{i \in I} u_i^A(\cdot\,; (\omega, p))$ under the individual rationality.

Proposition 3.32. *Suppose that (ω, p) is individually rational in economy $(\Omega, \{u_i\}_{i \in I})$. For $\tau \leq \omega^A$, we have*

$$\sum_{i \in I} u_i^A(\tau\,; (\omega, p)) \leq \sum_{i \in I} u_i^A(\omega^A\,; (\omega, p)).$$

Proof. By Proposition 3.20 and the equation (3.29) in Proposition 3.30, we have

$$\begin{aligned}\sum_{i \in I} u_i^A(\tau\,; (\omega, p)) &= \sum_{i \in I} U(\tau, p^A\,; u_i^A(\cdot\,; (\omega, p))) \\ &= \sum_{i \in I} \max \big\{U(\omega', p\,; u_i) : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A), \\ &\qquad\qquad \omega'_t = \tau_t\,(t \in A),\, \omega' \in \mathbf{Z}^\Omega\big\} \\ &\leq \sum_{i \in I} \max \big\{U(\omega', p\,; u_i) : \omega' \leq \omega,\, \omega' \in \mathbf{Z}^\Omega\big\},\end{aligned}$$

where the last inequality follows from

$$\{\omega' \in \mathbf{Z}^\Omega : \omega'_t \leq \omega_t\,(t \in \Omega \setminus A),\, \omega'_t = \tau_t\,(t \in A)\} \subseteq \{\omega' \in \mathbf{Z}^\Omega : \omega' \leq \omega\}.$$

By the individual rationality, Proposition 3.20, and (3.35) in Proposition 3.31, we have

$$\begin{aligned}\sum_{i\in I}\max\left\{U(\omega',p\,;u_i):\omega'\leq\omega,\,\omega\in\mathbf{Z}^{\Omega}\right\}&=\sum_{i\in I}U(\omega,p\,;u_i)\\&=\sum_{i\in I}u_i(\omega)\\&\leq\sum_{i\in I}u_i^A(\omega^A\,;(\omega,p)).\end{aligned}$$

Hence, we have

$$\sum_{i\in I}u_i^A(\tau\,;(\omega,p))\leq\sum_{i\in I}u_i^A(\omega^A\,;(\omega,p)).$$

□

Proof of Lemma 3.28

We now prove Lemma 3.28. We first show the "if" part of Lemma 3.28.

Proposition 3.33. *If (ω,p) has a blocking set, there exists $A\subseteq\Omega$ such that ω^A is not efficient on $\{u_i^A(\cdot\,;(\omega,p))\}_{i\in I}$.*

Proof. Let (z,p') be a blocking set of (ω,p). We denote $A=\operatorname{supp}^+(z)$. By the claim (i) in Proposition 3.30, we have

$$U(\omega^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p)))<U(\omega^A+z^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p)))\tag{3.36}$$

for all $i\in\operatorname{ag}(A)$. Since $\Omega_i\cap A=\emptyset$ for each $i\in I\setminus\operatorname{ag}(A)$, by the claim (iii) in Proposition 3.30, we have

$$U(\omega^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p)))=U(\omega^A+z^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p)))\tag{3.37}$$

for all $i\in I\setminus\operatorname{ag}(A)$. By (3.36), (3.37), and Proposition 3.20, we obtain

$$\begin{aligned}\sum_{i\in I}u_i^A(\omega^A\,;(\omega,p))&=\sum_{i\in I}U(\omega^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p)))\\&<\sum_{i\in I}U(\omega^A+z^A,(p+p')^A\,;u_i^A(\cdot\,;(\omega,p)))\\&=\sum_{i\in I}u_i^A(\omega^A+z^A\,;(\omega,p)).\end{aligned}$$

Hence, ω^A is not efficient on $\{u_i^A(\cdot\,;(\omega,p))\}_{i\in I}$. $\square$

Proposition 3.33 holds even if the preferences of agents do not satisfy the generalized full substitutes condition. We then prove the "only if" part of Lemma 3.28.

Proposition 3.34. *Suppose that every u_i is twisted $M^\natural$-concave and (ω,p) is individually rational in economy $(\Omega,\{u_i\}_{i\in I})$. If ω^A is not efficient on $\{u_i^A(\cdot\,;(\omega,p))\}_{i\in I}$ for some $A\subseteq\Omega$, there exists a blocking set (z,p') of (ω,p) with $z\in\{0,1\}^\Omega$ and* $\mathrm{supp}^+(z)\subseteq A$.

Proof. For $A\subseteq\Omega$, we define

$$\mu_i^A(\tau)\equiv {u_i^A}_{[\mathbf{0},\omega^A+\mathbf{1}]}(\tau\,;(\omega,p))=\begin{cases} u_i^A(\tau\,;(\omega,p)) & \text{if } \tau\in[\mathbf{0},\omega^A+\mathbf{1}],\\ -\infty & \text{otherwise.}\end{cases}$$

We consider the economy $(A,\{\mu_i^A\}_{i\in I})$. Note that μ_i^A is $(\Omega_{i\to}\cap A)$-twisted M$^\natural$-concave by Proposition 2.31. Therefore we can apply Lemma 3.29 to μ_i^A, that is, there exist $U,W\subseteq A$ such that $U\cap W=\emptyset$ and

$$\begin{aligned}\sum_{i\in I}\mu_i^A(\omega^A)&=\sum_{i\in I}u_i^A(\omega^A\,;(\omega,p))\\ &<\sum_{i\in I}u_i^A(\omega^A+\chi_U-\chi_W\,;(\omega,p))=\sum_{i\in I}\mu_i^A(\omega^A+\chi_U-\chi_W).\end{aligned}$$

This implies that ω^A is not efficient on $\{\mu_i^A\}_{i\in I}$. Let A^* be a minimal subset of Ω such that ω^{A^*} is not efficient on $\{\mu_i^{A^*}\}_{i\in I}$. Note that $A^*\neq\emptyset$. By $(\Omega_{i\to}\cap A^*)$-twisted M$^\natural$-concavity of $\mu_i^{A^*}$ and Theorem 3.18, there exists a competitive equilibrium (τ^*,p^*) in economy $(A^*,\{\mu_i^{A^*}\}_{i\in I})$.

We now describe the outline of the rest of the proof. We first show that τ^* is uniquely given by $\tau^*=\omega^{A^*}+\mathbf{1}$. Then we introduce a new function $\bar u_i$, which is a perturbation of $\mu_i^{A^*}$, and show that there exists $q\in\mathbf{R}^{A^*}$ such that (τ^*,q) is a competitive equilibrium in $(A^*,\{\bar u_i\}_{i\in I})$. Finally, we construct a blocking set of (ω,p) by using τ^* and q.

We claim that $\tau^*=\omega^{A^*}+\mathbf{1}$. By Theorem 3.23, τ^* is efficient on $\{\mu_i^{A^*}\}_{i\in I}$. Since ω^{A^*} is not efficient on $\{\mu_i^{A^*}\}_{i\in I}$ by assumption, it holds that

$$\sum_{i\in I}\mu_i^{A^*}(\omega^{A^*})<\sum_{i\in I}\mu_i^{A^*}(\tau^*). \tag{3.38}$$

Let $A' = \{t \in A^* : \tau^*_t = \omega_t + 1\}$. Then by Proposition 3.32, $A' \neq \emptyset$ holds. Since $\tau^*_t \leq \omega_t$ for all $t \in A^* \setminus A'$, by Proposition 3.31, we have

$$\sum_{i \in I} \mu_i^{A^*}(\tau^*) \leq \sum_{i \in I} \mu_i^{A'}(\tau^{*A'}). \tag{3.39}$$

Furthermore, by the equation (3.31) in Proposition 3.30, we have

$$\sum_{i \in I} \mu_i^{A^*}(\omega^{A^*}) = \sum_{i \in I} u_i(\omega) = \sum_{i \in I} \mu_i^{A'}(\omega^{A'}). \tag{3.40}$$

From (3.38), (3.39), and (3.40), it follows that

$$\sum_{i \in I} \mu_i^{A'}(\omega^{A'}) < \sum_{i \in I} \mu_i^{A'}(\tau^{*A'}).$$

That is, $\omega^{A'}$ is not efficient on $\{\mu_i^{A'}\}_{i \in I}$. Due to the minimality of A^*, we have $A' = A^*$. Thus $\tau^* = \omega^{A^*} + \mathbf{1}$.

We then define the new valuation function $\bar{u}_i$ as

$$\bar{u}_i(\tau) = \mu_i^{A^*}(\tau) - \delta \sum_{t \in A^* \cap \Omega_i} \tau_t \quad (i \in I),$$

where δ is a sufficiently small positive number. Since τ^* is the unique competitive equilibrium trade vector in the economy $(A^*, \{\mu_i^{A^*}\}_{i \in I})$, τ^* is also the unique efficient trade vector by Theorem 3.24. We assume that δ is sufficiently small so that τ^* is the unique efficient trade vector in economy $(A^*, \{\bar{u}_i\}_{i \in I})$. By Proposition 2.31, each $\bar{u}_i$ is $(\Omega_{i\to} \cap A^*)$-twisted M$^\natural$-concave, which means that there exists a competitive equilibrium $(\bar{\omega}^*, q)$ in economy $(A^*, \{\bar{u}_i\}_{i \in I})$ by Theorem 3.18. Then it follows from Theorem 3.23 that $\bar{\omega}^* = \tau^*$.

Finally, we claim that (z, p') given by

$$z_t = \begin{cases} 1 & \text{if } t \in A^*, \\ 0 & \text{otherwise,} \end{cases} \qquad p'_t = \begin{cases} q_t - p_t & \text{if } t \in A^*, \\ 0 & \text{otherwise,} \end{cases}$$

is a blocking set of (ω, p). It suffices to show that for all $j \in \mathrm{ag}(A^*)$ and $\phi \in C(\omega + z, p + p'\,; u_j)$, we have $\phi_t = \omega_t + 1 = \tau^*_t$ for all $t \in A^* \cap \Omega_j$.

In this setting, $\phi_t \leq \omega_t$ and $p'_t = 0$ hold for all $t \in \Omega \setminus A^*$. By using Proposition 3.31, we have

$$U(\phi, p + p' ; u_j) \leq U(\phi^{A^*}, (p + p')^{A^*} ; \mu_j^{A^*}) = U(\phi^{A^*}, q ; \mu_j^{A^*}). \tag{3.41}$$

Moreover, by the equation (3.29) in Proposition 3.30, there exists $\psi \in \mathbf{Z}^\Omega$ such that $\psi \leq \omega + z$ and

$$U(\tau^*, q ; \mu_j^{A^*}) = U(\psi, p + p' ; u_j). \tag{3.42}$$

Since (τ^*, q) is a competitive equilibrium in $(A^*, \{\bar{u}_i\}_{i\in I})$, we have

$$U(\tau^*, q ; \bar{u}_j) \geq U(\phi^{A^*}, q ; \bar{u}_j).$$

This can be rewritten as

$$U(\tau^*, q ; \mu_j^{A^*}) - \delta \sum_{t\in A^*\cap\Omega_j} \tau_t^* \geq U(\phi^{A^*}, q ; \mu_j^{A^*}) - \delta \sum_{t\in A^*\cap\Omega_j} \phi_t,$$

which, together with (3.41) and (3.42), implies that

$$\delta \sum_{t\in A^*\cap\Omega_j} (\phi_t - \tau_t^*) \geq U(\phi, p + p' ; u_j) - U(\psi, p + p' ; u_j) \geq 0,$$

where the last inequality follows from $\phi \in C(\omega+z, p+p' ; u_j)$. Since $\phi^{A^*} \leq \omega^{A^*} + z^{A^*} = \tau^*$, we have $\phi_t = \tau_t^*$ for all $t \in A^* \cap \Omega_j$. Hence, (z, p') is a blocking set of (ω, p) and we complete the proof of Proposition 3.34. □

Lemma 3.28 follows from Propositions 3.33 and 3.34. This implies that Theorem 3.26 holds.

3.4 Relationship among three stability concepts

We study stability concepts in the multi-unit trading network model. Although we define stable outcomes in Subsection 3.1.1, another stability concepts in trading networks were also being considered by several researchers.

In this section, we focus on stability, strong group stability, and chain stability in the multi-unit trading network model. While these stability concepts are different in general, they are equivalent in the case where preferences of all agents satisfy the generalized full substitutes condition. The results in this section is found in [32].

3.4.1 Strong group stability

We first consider the concept of strong group stability. Strong group stability was originally introduced by Hatfield et al. [27] in the single-unit trading network model, as a combination of strong stability by Hatfield and Kominers [25] and group stability by Roth and Sotomayor [57].

Before giving the definition of strong group stability, we make preparation. For $i \in I$, an outcome (ω, p), $z \in \mathbf{Z}^{\Omega}_{+} \setminus \{\mathbf{0}\}$, and $p' \in \mathbf{R}^{\Omega}$, we define

$$\begin{aligned}\Delta(z, p' ; u_i, (\omega, p)) = \max\left\{U(\omega', p ; u_i) : \omega' \leq \omega + z,\, \omega'_t = (\omega + z)_t\, (t \in \mathrm{supp}^+(z))\right\} \\ + \sum_{t \in \Omega_{i\to}} z_t \cdot (p'_t - p_t) - \sum_{t \in \Omega_{\to i}} z_t \cdot (p'_t - p_t) - U(\omega, p ; u_i).\end{aligned}$$

The value $\Delta(z, p' ; u_i, (\omega, p))$ can be understood as follows: when an agent i takes z_t more trades with price p'_t for all $t \in \mathrm{supp}^+(z)$ (and may cancel some trades which are not in $\mathrm{supp}^+(z)$) she/he can increase her/his utility by up to $\Delta(z, p' ; u_i, (\omega, p))$. The value $\Delta(z, p' ; u_i, (\omega, p))$ can be represented by an alternative formulation:

$$\begin{aligned}\Delta(z, p' ; u_i, (\omega, p)) = \max\Bigg\{u_i(\omega') + \sum_{t \in \Omega_{i\to}} \omega'_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega'_t \cdot p_t \\ : \omega' \leq \omega + z,\, \omega'_t = (\omega + z)_t\, (t \in \mathrm{supp}^+(z))\Bigg\} \\ + \sum_{t \in \Omega_{i\to}} z_t \cdot (p'_t - p_t) - \sum_{t \in \Omega_{\to i}} z_t \cdot (p'_t - p_t) - U(\omega, p ; u_i).\end{aligned}$$

Definition 3.35. *Let (ω, p) be an outcome. A pair (z, p') is a* strong group blocking set *of (ω, p) in economy $(\Omega, \{u_i\}_{i \in I})$ if the following conditions hold:*

(i) $z \in \mathbf{Z}^{\Omega}_{+} \setminus \{\mathbf{0}\}$.

(ii) $p' \in \mathbf{R}^{\Omega}$ *and* $p'_t = p_t$ *for all* $t \in \Omega$ *with* $t \notin \mathrm{supp}^+(z)$.

(iii) *for all* $i \in \mathrm{ag}(\mathrm{supp}^+(z))$, $\Delta(z, p'\,; u_i, (\omega, p)) > 0$.

Definition 3.36. *An outcome* (ω, p) *is* strong group stable *in* $(\Omega, \{u_i\}_{i\in I})$ *if the following conditions hold:*

(i) (ω, p) *is individually rational.*

(ii) (ω, p) *does not have any strong group blocking sets.*

The strong group stability defined above coincides with the original strong group stability in [27] when applied to the single-unit trading network model. Strong group stability is a more restricted condition than stability.

Proposition 3.37. *Let* (ω, p) *be an outcome.*

(i) *If* (ω, p) *has a blocking set, then it also has a strong group blocking set.*

(ii) *If* (ω, p) *is strong group stable, then it is stable.*

Proof. We prove the claim (i). The claim (ii) immediately follows from the claim (i). Let (z, p') be a blocking set of (ω, p) and define $J = \mathrm{ag}(\mathrm{supp}^+(z))$. Then for $i \in J$ and $\omega' \in C(\omega + z, p + p'\,; u_i)$, we have $\omega'_t = (\omega + z)_t$ for all $t \in \mathrm{supp}^+(z) \cap \Omega_i$. Thus, it holds that

$$U(\omega', p + p'\,; u_i) > U(\omega, p + p'\,; u_i).$$

By the definition of a utility function, we have

$$\begin{aligned}
&U(\omega', p + p'\,; u_i) - U(\omega, p + p'\,; u_i) \\
&\quad = u_i(\omega') + \sum_{t\in\Omega_{i\to}} \omega'_t \cdot (p_t + p'_t) - \sum_{t\in\Omega_{\to i}} \omega'_t \cdot (p_t + p'_t) \\
&\qquad\qquad - \left(u_i(\omega) + \sum_{t\in\Omega_{i\to}} \omega_t \cdot (p_t + p'_t) - \sum_{t\in\Omega_{\to i}} \omega_t \cdot (p_t + p'_t) \right) \\
&\quad = u_i(\omega') - u_i(\omega) + \sum_{t\in\Omega_{i\to}} (\omega'_t - \omega_t) \cdot p_t + \sum_{t\in\Omega_{i\to}} z_t \cdot p'_t \\
&\qquad\qquad - \sum_{t\in\Omega_{\to i}} (\omega'_t - \omega_t) \cdot p_t - \sum_{t\in\Omega_{\to i}} z_t \cdot p'_t
\end{aligned} \tag{3.43}$$

for all $i \in J$. On the other hand, we obtain

$$\begin{aligned}\Delta(z, p+p'; u_i, (\omega, p)) \geq u_i(\omega') + \sum_{t \in \Omega_{i\to}} \omega'_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega'_t \cdot p_t + \sum_{t \in \Omega_{i\to}} z_t \cdot p'_t - \sum_{t \in \Omega_{\to i}} z_t \cdot p'_t \\ - \left(u_i(\omega) + \sum_{t \in \Omega_{i\to}} \omega_t \cdot p_t - \sum_{t \in \Omega_{\to i}} \omega_t \cdot p_t \right) \\ = u_i(\omega') - u_i(\omega) + \sum_{t \in \Omega_{i\to}} (\omega'_t - \omega_t) \cdot p_t + \sum_{t \in \Omega_{i\to}} z_t \cdot p'_t \\ - \sum_{t \in \Omega_{\to i}} (\omega'_t - \omega_t) \cdot p_t - \sum_{t \in \Omega_{\to i}} z_t \cdot p'_t \qquad (3.44)\end{aligned}$$

for all $i \in J$. By (3.43) and (3.44), we have

$$\Delta(z, p+p'; u_i, (\omega, p)) \geq U(\omega', p+p'; u_i) - U(\omega, p+p'; u_i) > 0 \quad (\forall i \in J).$$

This means that $(z, p+p')$ is a strong group blocking set of (ω, p). □

By Proposition 3.37, strong group stable outcomes are always stable. Although strong group stability and stability are not equivalent in general, the following statement holds.

Proposition 3.38 ([27])**.** *For the single-unit trading network model, strong group stability is equivalent to stability if preferences of all agents satisfy the full substitutes condition.*

We can extend Proposition 3.38 to the multi-unit trading network model. In fact, stability leads to strong group stability under the generalized full substitutes condition.

Theorem 3.39. *Suppose that every u_i satisfies the generalized full substitutes condition. Let (ω, p) be an outcome.*

(i) *Suppose that (ω, p) is individually rational. If (ω, p) has a strong group blocking set, then it also has a blocking set.*

(ii) *If (ω, p) is stable, then it is strong group stable.*

Proof. We prove the claim (i). The claim (ii) immediately follows from the claim (i). Let (z, p') be a strong group blocking set of (ω, p) and define $A = \mathrm{supp}^+(z)$ and $J = \mathrm{ag}(A)$. By the definition of $\Delta(z, p'; u_i, (\omega, p))$, the equation (3.29), and the claim (ii) in Proposition 3.30, we have

$$\begin{aligned}\Delta(z, p'; u_i, (\omega, p)) &= U(\omega^A + z^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) + \sum_{t \in \Omega_{i\to}} z_t \cdot (p'_t - p_t) \\ &\qquad - \sum_{t \in \Omega_{\to i}} z_t \cdot (p'_t - p_t) - U(\omega, p\,; u_i) \\ &= U(\omega^A + z^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) - U(\omega^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) \\ &\qquad + \sum_{t \in \Omega_{i\to}} z_t \cdot (p'_t - p_t) - \sum_{t \in \Omega_{\to i}} z_t \cdot (p'_t - p_t).\end{aligned}$$

Since $\Delta(z, p'; u_i, (\omega, p)) > 0$, we obtain

$$\begin{aligned}&U(\omega^A + z^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) \\ &\quad > U(\omega^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) + \sum_{t \in \Omega_{\to i}} z_t \cdot (p'_t - p_t) - \sum_{t \in \Omega_{i\to}} z_t \cdot (p'_t - p_t).\end{aligned} \tag{3.45}$$

Moreover, it holds by the definition of J that

$$\sum_{i \in J} \left(\sum_{t \in \Omega_{\to i}} z_t \cdot (p'_t - p_t) - \sum_{t \in \Omega_{i\to}} z_t \cdot (p'_t - p_t) \right) = 0. \tag{3.46}$$

It follows from (3.45) and (3.46) that

$$\sum_{i \in J} U(\omega^A + z^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) > \sum_{i \in J} U(\omega^A, p^A\,; u_i^A(\cdot\,; (\omega, p))). \tag{3.47}$$

On the other hand, $\Omega_i \cap A = \emptyset$ holds for all $i \in I \setminus J$. Thus, by the claim (iii) in Proposition 3.30 we have

$$U(\omega^A + z^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) = U(\omega^A, p^A\,; u_i^A(\cdot\,; (\omega, p))) \quad (\forall i \in I \setminus J). \tag{3.48}$$

Table 3.4: The valuations and utilities of agents in Example 3.4.

ω	$u_a(\omega)$	$u_b(\omega)$	$U(\omega, 3\,;u_a)$	$U(\omega, 3\,;u_b)$	$U(\omega, 1.5\,;u_a)$	$U(\omega, 1.5\,;u_b)$
0	0	0	0	0	0	0
1	4	-1	1	2	2.5	0.5
2	8	-2	2	4	5	1
3	10	-3	1	6	5.5	1.5
4	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$

By (3.47) and (3.48), we obtain

$$\begin{aligned}\sum_{i\in I} u_i^A(\omega^A + z^A\,;(\omega,p)) &= \sum_{i\in I} U(\omega^A + z^A, p^A\,; u_i^A(\cdot\,;(\omega,p)))\\ &> \sum_{i\in I} U(\omega^A, p^A\,; u_i^A(\cdot\,;(\omega,p)))\\ &= \sum_{i\in I} u_i^A(\omega^A\,;(\omega,p)).\end{aligned}$$

This inequality implies that ω^A is not efficient on $\{u_i^A(\cdot\,;(\omega,p))\}_{i\in I}$. By Lemma 3.28, (ω,p) has a blocking set. $\square$

From Proposition 3.37 and Theorem 3.39, strong group stability and stability are equivalent under the generalized full substitutes condition.

Example 3.4. Let ω be a trade and a and b be two agents. Agent a is the buyer and agent b is the seller. Valuation functions of each agent are given in Table 3.4.

We can easily check that u_a and u_b satisfy the generalized full substitutes condition. We consider outcome $(\omega,p) = (2,3)$. It is not stable because it has a blocking set $(z,p') = (1,-1.5)$. This means that $\{3\} = C(3,1.5\,;u_a)$ and $\{3\} = C(3,1.5\,;u_b)$. It is doubtful that agent b prefers choosing $(3,1.5)$ to choosing $(2,3)$ because it holds that

$$4 = U(2,3\,;u_b) > U(3,1.5\,;u_b) = 1.5.$$

However, Theorem 3.39 guarantees that $(2,3)$ has a strong group blocking set (for instance, $(1,1.5)$ is a strong group blocking set of $(2,3)$). This implies that both a and b can increase their utilities by taking new trades.

3.4.2 Chain stability

We consider another stability concept, chain stability. This concept was first introduced by Ostrovsky [55] in the supply chain network, then extended to the single-unit trading network model by Hatfield et al. [28]. The definition of chain stability in the multi-unit trading network model is similar to that in [28]. We first give the definition of a chain.

Definition 3.40. *A trade vector $z \in \{0,1\}^{\Omega}$ is a* chain *if there exists an ordering $t_1, \ldots, t_k$ of elements in $\mathrm{supp}^+(z)$ such that $s(t_{l+1}) = b(t_l)$ for all $l = 1, \ldots, k-1$, where $k = |\mathrm{supp}^+(z)|$. A blocking set (z, p') is called a blocking chain if z is a chain.*

Note that a chain is a 0-1 vector. Chain stable outcomes are defined as follows.

Definition 3.41. *An outcome (ω, p) is* chain stable *in $(\Omega, \{u_i\}_{i \in I})$ if the following conditions hold:*

(i) *(ω, p) is individually rational.*

(ii) *(ω, p) does not have any blocking chains.*

Trivially, chain stability is a weaker condition than stability since a blocking chain is a blocking set. However, under the generalized full substitutes condition, chain stability is equivalent to stability.

Theorem 3.42. *Suppose that every u_i satisfies the generalized full substitutes condition. Let (ω, p) be an outcome.*

(i) *Suppose that (ω, p) is individually rational. If (ω, p) has a blocking set, then it has a blocking chain.*

(ii) *(ω, p) is stable if and only if it is chain stable.*

The proof of Theorem 3.42 is given in Subsection 3.4.3. For a blocking chain $z \in \{0,1\}^{\Omega}$, there may exist $j \in I$ such that $|\mathrm{supp}^+(z) \cap \Omega_{j\to}| > 1$ or $|\mathrm{supp}^+(z) \cap \Omega_{\to j}| > 1$. This condition cannot be reduced. In fact, there is a case where (ω, p) is not stable when there is no blocking chain (z, p') such that $|\mathrm{supp}^+(z) \cap \Omega_{j\to}| \leq 1$ and $|\mathrm{supp}^+(z) \cap \Omega_{\to j}| \leq 1$ for all $j \in I$ (see Example 3.5).

Table 3.5: The valuations of agents in Example 3.5.

(t_1, t_2, t_3)	$u_a(\cdot)$	$u_b(\cdot)$	$u_c(\cdot)$
$(0,0,0)$	0	0	0
$(0,0,1)$	-2	0	-2
$(0,1,0)$	0	0	-4
$(0,1,1)$	-2	0	1

(t_1, t_2, t_3)	$u_a(\cdot)$	$u_b(\cdot)$	$u_c(\cdot)$
$(1,0,0)$	1	-2	0
$(1,0,1)$	3	-2	-2
$(1,1,0)$	1	-2	-4
$(1,1,1)$	3	-2	1

Example 3.5. Let a, b, and c be three agents, and t_1, t_2, and t_3 be three types of trades such that $b(t_1) = a$, $s(t_1) = b$, $b(t_2) = a$, $s(t_2) = c$, $b(t_3) = c$, and $s(t_3) = a$. Valuation functions of each agent are given in Table 3.5. Then valuation functions of each agent satisfy the generalized full substitutes condition. We consider outcome $(\omega, p) = ((0,0,0),(0,0,0))$. It is not stable because ω is not efficient on $\{u_i\}_{i\in\{a,b,c\}}$. Since (ω, p) is individually rational, there exists a blocking set (z, p') of (ω, p). By Table 3.5, we have

$$\sum_{i\in\{a,b,c\}} u_i((0,0,0)) > \sum_{i\in\{a,b,c\}} u_i(\omega) \quad (\forall \omega \in \{0,1\}^3 \setminus \{(1,1,1)\}).$$

This implies that $z = (1,1,1)$ must hold. Then it follows that $|\mathrm{supp}^+(z) \cap \Omega_{\to a}| > 1$.

By Proposition 3.37 and Theorems 3.39 and 3.42, we obtain the equivalence among the three stability concepts under the generalized full substitutes condition.

Corollary 3.43. *Suppose that every u_i satisfies the generalized full substitutes condition. Then stability, strong group stability, and chain stability are equivalent.*

3.4.3 Proof of Theorem 3.42

Since the claim (ii) follows from the claim (i), we prove the claim (i). Suppose that (ω, p) is individually rational. Let (z, p') be a blocking set of (ω, p) such that $z \in \{0,1\}^\Omega$ and $\mathrm{supp}^+(z)$ is minimal among all blocking sets of (ω, p). Such a blocking set exists because of Lemma 3.28. We claim that z is a chain.

We denote $A = \mathrm{supp}^+(z)$. We consider the directed graph $G = (\mathrm{ag}(A), E(A))$ where

$$E(A) = \{(s(t), b(t)) : t \in A\}.$$

The graph G must be a connected directed graph. If G is not connected, that is, there are more than one connected components in G, we can construct a blocking set of (z', q) such that $\mathrm{supp}^+(z') \subsetneq A$. In fact, by condition (3.1), for a connected component $(\mathrm{ag}(A'), E(A'))$ of G, (z', q) is a blocking set where

$$z'_t = \begin{cases} z_t & (t \in A'), \\ 0 & (t \notin A'), \end{cases} \qquad q_t = \begin{cases} p'_t & (t \in A'), \\ 0 & (t \notin A'). \end{cases}$$

This contradicts the minimality of A.

We show that there is a walk in G such that all directed edges in G are contained in the walk, which means that z is a chain. We first consider the case where

$$|\Omega_{\to i} \cap A| = |\Omega_{i\to} \cap A| \quad (\forall i \in \mathrm{ag}(A)).$$

Then we have $d_G^+(i) = d_G^-(i)$ for all $i \in \mathrm{ag}(A)$. By Proposition 2.1, $G = (\mathrm{ag}(A), E(A))$ has an Eulerian walk, implying that z is a chain.

We next consider the case where there exists $k \in \mathrm{ag}(A)$ with

$$|\Omega_{\to k} \cap A| < |\Omega_{k\to} \cap A|. \tag{3.49}$$

For each $i \in I$, we define the function $U_i : \mathbf{Z}^A \to \mathbf{R} \cup \{-\infty\}$ by

$$U_i(\tau) = U(\tau, (p+p')^A ; u_i^A(\cdot\,; (\omega, p))) \quad (i \in I).$$

By the definition of U_i and condition (3.1), we can regard the domain of U_i as $\mathbf{Z}^{\Omega_i \cap A}$. By Proposition 2.31, U_i is $(\Omega_{i\to} \cap A)$-twisted M$^\natural$-concave. By showing that z is a chain, we use the following algorithm, FIND_CHAIN. This algorithm is valid (prove later) and always returns a chain z^*.

FIND_CHAIN

Step 1. Take $k \in \mathrm{ag}(A)$ with (3.49) and $t \in \Omega_{k\to} \cap A$ with $U_k(\omega^A) < U_k(\omega^A + \chi_t)$. Set $j := b(t)$ and $\hat{z} := \chi_t$ ($\hat{z} \in \{0,1\}^A$).

Step 2. **While** $U_j(\omega^A) \geq U_j(\omega^A + \hat{z})$ **do** {

Take $t \in \mathrm{supp}^+(z^A - \hat{z}) \cap \Omega_{j\to}$ so that $U_j(\omega^A) < U_j(\omega^A + \hat{z} + \chi_t)$ holds, and set $j := b(t)$ and $\hat{z} := \hat{z} + \chi_t$.

}.

Step 3. Set $z^*_t = \hat{z}_t$ for each $t \in A$ and $z^*_t = 0$ for each $t \notin A$, and return z^*.

FIND_CHAIN terminates in a finite number of iterations because $\mathrm{supp}^+(z^A - \hat{z})$ is strictly reduced in each update of $\hat{z}$. Moreover, it is easy to check that z^* is a chain if FIND_CHAIN works correctly. We claim that Step 1 works.

Proposition 3.44. *For $k \in \mathrm{ag}(A)$ satisfying* (3.49)*, there exists $t \in \Omega_{k\to} \cap A$ such that*

$$U_k(\omega^A) < U_k(\omega^A + \chi_t).$$

Proof. Recall that U_k is $(\Omega_{k\to} \cap A)$-twisted M$^\natural$-concave. By (3.49), we have

$$\sum_{s \in \Omega_k \cap A} (\mathrm{twist}(\omega^A + z^A ; \Omega_{k\to} \cap A))_s < \sum_{s \in \Omega_k \cap A} (\mathrm{twist}(\omega^A ; \Omega_{k\to} \cap A))_s.$$

Moreover, it follows that

$$\mathrm{supp}^-(z^A) \setminus (\Omega_{k\to} \cap A) = \emptyset, \quad \mathrm{supp}^+(z^A) \cap (\Omega_{k\to} \cap A) = \Omega_{k\to} \cap A.$$

By Proposition 2.29, there exists $t \in \Omega_{k\to} \cap A$ such that

$$U_k(\omega^A + z^A) + U_k(\omega^A) \leq U_k(\omega^A + z^A - \chi_t) + U_k(\omega^A + \chi_t). \tag{3.50}$$

By the claim (i) in Proposition 3.30, it holds that

$$U_k(\omega^A + z^A) > U_k(\omega^A + z^A - \chi_t). \tag{3.51}$$

Therefore, by (3.50) and (3.51), we have

$$U_k(\omega^A) < U_k(\omega^A + \chi_t).$$

□

We next show that Step 2 works.

Proposition 3.45. *FIND_ CHAIN preserves the following properties:*

(a) *at the beginning of each while-loop of Step 2, if* $U_j(\omega^A) \geq U_j(\omega^A + \hat{z})$ *there exists* $t \in \mathrm{supp}^+(z^A - \hat{z}) \cap \Omega_{j\rightarrow}$ *such that* $U_j(\omega^A) < U_j(\omega^A + \hat{z} + \chi_t)$.

(b) *at the end of Step 1 and each while-loop, we have*

$$U_i(\omega^A) = U_i(\omega^A + \hat{z}) \quad (\forall i \in I \setminus \mathrm{ag}(\mathrm{supp}^+(\hat{z}))), \tag{3.52}$$

$$U_i(\omega^A) < U_i(\omega^A + \hat{z}) \quad (\forall i \in \mathrm{ag}(\mathrm{supp}^+(\hat{z})) \setminus \{j\}). \tag{3.53}$$

Proof. Since $\mathrm{supp}^+(\hat{z}) \cap \Omega_i = \emptyset$ for all $i \in I \setminus \mathrm{ag}(\mathrm{supp}^+(\hat{z}))$, (3.52) always holds by the claim (iii) in Proposition 3.30. At the end of Step 1, (3.53) holds because of the definition of k and $\mathrm{ag}(\mathrm{supp}^+(\hat{z})) \setminus \{j\} = \{k\}$. If property (a) is preserved, we obtain (3.53). Therefore it suffices to show property (a).

We assume that $U_j(\omega^A) \geq U_j(\omega^A + \hat{z})$ and (3.53) hold before the last update of $\hat{z}$ and j. Let $t' \in \Omega_{\rightarrow j} \cap A$ be the trade which is added to $\hat{z}$ in the last update of $\hat{z}$; such t' always exists. From the above assumption, it follows that

$$U_j(\omega^A) \leq U_j(\omega^A + \hat{z} - \chi_{t'}). \tag{3.54}$$

By Proposition 2.28, there exists $t \in (\mathrm{supp}^+(z - \hat{z}) \cap (\Omega_{j\rightarrow} \cap A)) \cup \{0\}$ such that

$$U_j(\omega^A + z^A) + U_j(\omega^A + \hat{z} - \chi_{t'}) \leq U_j(\omega^A + z^A - \chi_{t'} - \chi_t) + U_j(\omega^A + \hat{z} + \chi_t). \tag{3.55}$$

On the other hand, by the claim (i) in Proposition 3.30, we have

$$U_j(\omega^A + z^A) > U_j(\omega^A + z^A - \chi_{t'} - \chi_t). \tag{3.56}$$

By (3.54), (3.55), and (3.56), we have

$$U_j(\omega^A) \leq U_j(\omega^A + \hat{z} - \chi_{t'}) < U_j(\omega^A + \hat{z} + \chi_t).$$

Since $U_j(\omega^A) \geq U_j(\omega^A + \hat{z})$, $t \neq 0$ must hold. □

We now prove $z^* = z$. At Step 3, it holds by Proposition 3.45 that

$$U_i(\omega^A) = U_i((\omega + z^*)^A) \quad (\forall i \in I \setminus \mathrm{ag}(\mathrm{supp}^+(z^*))), \tag{3.57}$$

$$U_i(\omega^A) < U_i((\omega + z^*)^A) \quad (\forall i \in \mathrm{ag}(\mathrm{supp}^+(z^*))). \tag{3.58}$$

We define $A^* = \mathrm{supp}^+(z^*)$. By (3.57) and (3.58), we have

$$\sum_{i \in I} U_i(\omega^A) < \sum_{i \in I} U_i((\omega + z^*)^A). \tag{3.59}$$

It follows from Proposition 3.20 and the claim (ii) in Proposition 3.30 that

$$\sum_{i \in I} U_i(\omega^A) = \sum_{i \in I} u_i^A(\omega^A\, ; (\omega, p)) = \sum_{i \in I} u_i^{A^*}(\omega^{A^*}\, ; (\omega, p)). \tag{3.60}$$

Moreover, by Propositions 3.20 and 3.31, it holds that

$$\sum_{i \in I} U_i((\omega + z^*)^A) = \sum_{i \in I} u_i^A((\omega + z^*)^A\, ; (\omega, p)) \le \sum_{i \in I} u_i^{A^*}((\omega + z^*)^{A^*}\, ; (\omega, p)). \tag{3.61}$$

By (3.59), (3.60) and (3.61), we have

$$\sum_{i \in I} u_i^{A^*}(\omega^{A^*}\, ; (\omega, p)) < \sum_{i \in I} u_i^{A^*}((\omega + z^*)^{A^*}\, ; (\omega, p)),$$

which implies that ω^{A^*} is not efficient on $\{u_i^{A^*}(\cdot\, ; (\omega, p))\}_{i \in I}$. By Lemma 3.28, there exists a blocking set $(\tilde{z}, \tilde{p})$ of (ω, p) such that $\tilde{z} \in \{0, 1\}^{\Omega}$ and $\mathrm{supp}^+(\tilde{z}) \subseteq A^*$. By the minimality of A, we have $\mathrm{supp}^+(\tilde{z}) = A^* = A$. Hence, $z^* = z$. We can similarly show that z is a chain when $|\Omega_{\to k} \cap A| > |\Omega_{k \to} \cap A|$ for some $k \in \mathrm{ag}(A)$.

Chapter 4

Marking Game

4.1 Preliminaries

We study the game coloring number of graphs. The game coloring number is defined through the marking game and it is often used to analyze the game chromatic number, which is defined through the chromatic game. In this section, we give the formal definitions of the game chromatic number and the game coloring number.

4.1.1 Game chromatic number

We first define the chromatic game and the game chromatic number. The *chromatic game* is played by two players, Alice and Bob.

Given an undirected graph $G = (V(G), E(G))$ and a color set C. Alice and Bob take turns coloring vertices of G with a color in C (at the beginning, Alice colors a vertex of G). In each turn, she/he colors only one uncolored vertex of G so that any two adjacent vertices are colored with different colors. If all vertices are colored properly, Alice is the winner of the chromatic game. Otherwise, Bob is the winner (then there exists an uncolored vertex u such that u has a neighbor colored with c for any $c \in C$).

The *game chromatic number* of G, defined by $\chi_g(G)$, is the smallest number of colors such that Alice has a winning strategy in the chromatic game. The game chromatic number was formally introduced by Bodlaender [2].

For a class of graphs $\mathcal{H}$, we also define the game chromatic number of the class. We denote the game chromatic number of $\mathcal{H}$ by

$$\chi_g(\mathcal{H}) = \max\left\{\chi_g(G) : G \in \mathcal{H}\right\}.$$

4.1.2 Game coloring number

We next define the marking game and the game coloring number. The *marking game* is also played by two players, Alice and Bob. The marking game is similar to the chromatic game. However, players just "mark" vertices of a graph in the marking game instead of coloring them.

Given an undirected graph $G = (V(G), E(G))$ and $k \in \mathbf{Z}_+$ which is called an upper limit on the score. Alice and Bob take turns marking vertices of G until all vertices are marked (at the beginning, Alice marks a vertex of G). In each turn, she/he marks only one unmarked vertex of G. When all vertices are marked, we calculate the *score* of the game. The score s of the marking game is defined by

$$s = 1 + \max_{v \in V(G)} b(v),$$

where $b(v)$ is the number of neighbors of v which are marked before v. If the score s is at most k, Alice is the winner of the game. Otherwise, Bob is the winner.

The *game coloring number* of G, defined by $\mathrm{col}_g(G)$, is the smallest number k such that Alice has a winning strategy in the marking game. The game coloring number was formally introduced by Zhu [74]. He used it as a tool to study the game chromatic number. We define the game coloring number of a class of graphs $\mathcal{H}$ by

$$\mathrm{col}_g(\mathcal{H}) = \max\left\{\mathrm{col}_g(G) : G \in \mathcal{H}\right\}.$$

We define payoffs of each player in the marking game by

$$(\text{Alice's payoff}) = \begin{cases} 1 & \text{if Alice wins,} \\ -1 & \text{if Bob wins,} \end{cases} \qquad (\text{Bob's payoff}) = \begin{cases} 1 & \text{if Bob wins,} \\ -1 & \text{if Alice wins.} \end{cases}$$

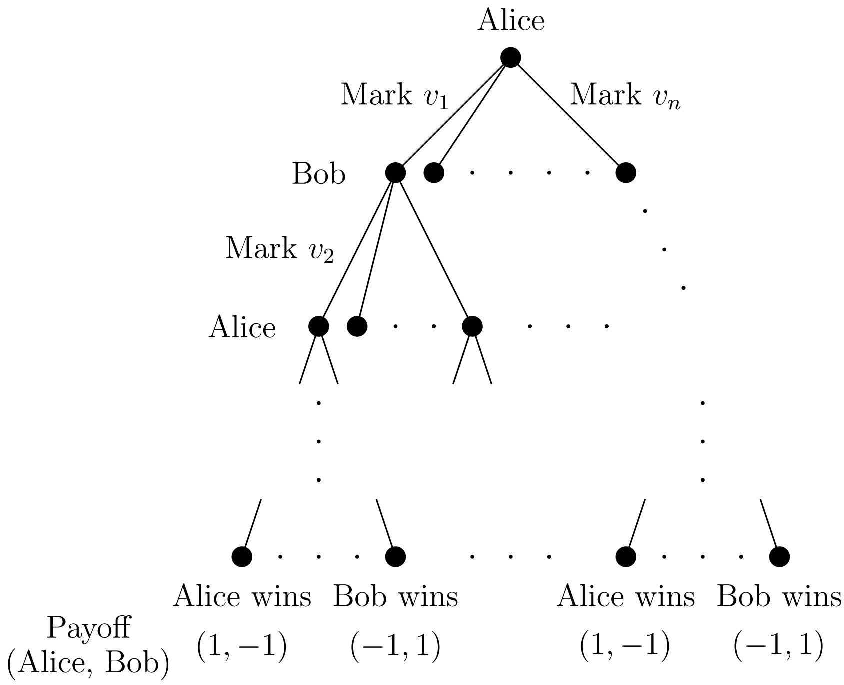

Figure 4.1: Extensive form of the marking game.

Then the marking game is represented by an extensive form. We give the extensive form of the marking game on G in Figure 4.1 (where $V(G) = \{v_1, \ldots, v_n\}$). By the rule of the game, both players see all vertices during the game. This means that all information sets of the marking game are singletons. Moreover, since the sum of the payoffs is always zero, the marking game is a 2-person zero-sum game with perfect information.

4.1.3 Previous results

In this subsection, we describe previous results of the game chromatic number and the game coloring number. We first explain the relationship between the game chromatic number and the game coloring number. It is easy to show the following proposition.

Proposition 4.1. *For any undirected graph G, we have*

$$\chi_g(G) \leq \mathrm{col}_g(G).$$

By Proposition 4.1, we can estimate an upper bound for the game chromatic number to analyze the game coloring number. We can handle the game coloring number more easily than the game chromatic number for several reasons. One of them is the monotonicity of the game coloring number.

Proposition 4.2 ([72]). *Let H be a subgraph of G. Then*

$$\mathrm{col}_g(H) \leq \mathrm{col}_g(G).$$

Proof. Obviously, the statement holds if $V(H) = V(G)$. Indeed, if Alice takes an optimal strategy for G in the game on H, the score is at most $\mathrm{col}_g(G)$. We show that $\mathrm{col}_g(G-x) \leq \mathrm{col}_g(G)$ for all $x \in V(G)$ where

$$\begin{aligned} V(G-x) &= V(G) \setminus \{x\}, \\ E(G-x) &= \{\{u,w\} : \{u,w\} \in E(G), u, w \in V(G) \setminus \{x\}\}. \end{aligned}$$

We define a vertex which Alice marks next when she uses an optimal strategy for G by $S(W)$, where $W \subseteq V(G)$ is the set of marked vertices. We assume that Alice and Bob play the game on $G - x$. We construct the strategy S'.

Strategy S'

Alice first sets $W := \emptyset$. She maintains W until the game finishes. She marks a vertex by using the following strategy. Let $b \in V(G-x)$ be the vertex which is marked by Bob in his last turn.

1. Set $W := W \cup \{b\}$ (in her first turn, this step is skipped).
2. If she can mark $S(W)$, she marks it and sets $W := W \cup S(W)$.
3. If $S(W) = x$, she sets $W := W \cup \{x, y_1\}$, where y_1 is a vertex whose degree is minimum in $V(G-x) \setminus W$. Then she marks $S(W)$.
4. If $S(W)$ is already marked, that is, Bob marks y_i in his last turn, she sets $W := W \cup \{y_{i+1}\}$, where y_{i+1} is a vertex whose degree is minimum in $V(G-x) \setminus W$. Then she marks $S(W)$.

By the definition of S', $b(v) \leq \mathrm{col}_g(G) - 1$ holds for each vertex $v \in V(G - x) \setminus \{y_1, \dots\}$. Moreover, we have

$$d_G(y_i) \leq \mathrm{col}_g(G) - 1$$

for each y_i. Otherwise, $b(u) \geq \mathrm{col}_g(G)$ holds for the vertex u which is marked at the end of the game. This means that the score is at least $\mathrm{col}_g(G) + 1$, a contradiction. Therefore by using S', the score is at most $\mathrm{col}_g(G)$ which means $\mathrm{col}_g(G - x) \leq \mathrm{col}_g(G)$ for all $x \in V(G)$. Applying this inequality inductively, we have

$$\mathrm{col}_g(H) \leq \mathrm{col}_g(G)$$

for any subgraph H of G. $\square$

By Proposition 4.2, when we study the game coloring number of a class of graphs $\mathcal{H}$, we focus on the game coloring number of maximal graphs in $\mathcal{H}$. Unlike the game coloring number, the game chromatic number does not have the monotonicity. This is one reason why handling the game chromatic number is difficult. In fact, for a complete bipartite graph $K_{t,t}$ and a perfect matching M of $K_{t,t}$, it holds that

$$\chi_g(K_{t,t}) = 3, \quad \chi_g(K_{t,t} - M) = t,$$

where $K_{t,t} - M = (V(K_{t,t}), E(K_{t,t}) \setminus M)$.

We next explain previous results of the game coloring number. For several classes of graphs, the exact values of the game coloring numbers are known.

Theorem 4.3 ([12, 22, 37, 72, 75])**.** *The exact values of the game coloring numbers of forests, interval graphs, outerplanar graphs, and partial k-trees are as follows:*

- $\mathrm{col}_g(\mathcal{F}) = 4$ ($\mathcal{F}$*: the family of forests*).
- $\mathrm{col}_g(\mathcal{I}_k) = 3k - 2$ ($\mathcal{I}_k$*: the family of interval graphs with clique number k*).
- $\mathrm{col}_g(\mathcal{Q}) = 7$ ($\mathcal{Q}$*: the family of outerplanar graphs*).
- $\mathrm{col}_g(\mathcal{PT}_k) = 3k + 2$ ($\mathcal{PT}_k$*: the family of partial k-trees* ($k \geq 2$)).

We next describe previous results of the game coloring number of planar graphs, which was shown by several researchers.

Theorem 4.4 ([30, 38, 69, 72, 76])**.** *Let $\mathcal{P}_k$ be the family of planar graphs with girth at least k. Then the following statements hold:*

(i) $11 \leq \mathrm{col}_g(\mathcal{P}_3) \leq 17$.

(ii) $\mathrm{col}_g(\mathcal{P}_5) \leq 8$.

(iii) $\mathrm{col}_g(\mathcal{P}_6) \leq 6$.

(iv) $\mathrm{col}_g(\mathcal{P}_8) \leq 5$.

In this thesis, we study the game coloring number of planar graphs with a given girth. In particular, we study the game coloring number of planar graphs with girth 4 and 5.

4.2 Upper bounds for the game coloring number

We study upper bounds for the game coloring number of planar graphs with girth 4. We consider Alice's strategy, called the activation strategy, to analyze an upper bound for the game coloring number. Therefore we first explain the activation strategy introduced by Kierstead [36]. Then we show that the game coloring number of planar graphs with girth 4 is at most 13.

4.2.1 Activation strategy

We explain the *activation strategy* introduced by Kierstead [36]. This strategy is helpful to analyze upper bounds for the game coloring number.

For an undirected graph $G = (V(G), E(G))$, we define the set of total orders on $V(G)$ by $\Pi(G)$. The *orientation* of G with respect to $L \in \Pi(G)$ is a directed graph $G_L = (V(G), E_L)$ where

$$E_L = \{(u, v) : \{u, v\} \in E(G), u \succeq_L v\}.$$

For a vertex $u \in V(G_L)$, we define

$$V^+_{G_L}(u) = \{v \in V(G_L) : v \preceq_L u\}, \qquad V^-_{G_L}(u) = \{v \in V(G_L) : v \succeq_L u\}.$$

Furthermore, let

$$N^+_{G_L}[u] = N^+_{G_L}(u) \cup \{u\}, \qquad N^-_{G_L}[u] = N^-_{G_L}(u) \cup \{u\},$$
$$V^+_{G_L}[u] = V^+_{G_L}(u) \cup \{u\}, \qquad V^-_{G_L}[u] = V^-_{G_L}(u) \cup \{u\}.$$

We now explain the activation strategy $S(L, G)$ with respect to $L \in \Pi(G)$. For every turn, Alice marks a vertex according to the following rule.

Activation strategy $S(L, G)$

Alice first sets $A := \emptyset$. She maintains A until the game ends. In her first turn, she marks the minimum vertex v with respect to L and updates $A := \{v\}$. Otherwise, she decides which vertex to mark by using the following algorithm. Let U be the set of unmarked vertices and $b \in V(G)$ be the vertex which is marked by Bob in his last turn.

Step 1. Set $A := A \setminus \{b\}$ and $x := b$.

Step 2. **While** $x \notin A$ **do**:

$A := A \cup \{x\}, s(x) := \min_L \left(N^+_{G_L}[x] \cap (U \cup \{b\})\right), x := s(x)$.

Step 3. **If** $x \neq b$ **then** mark x.

Else { $y := \min_L U$.

If $y \notin A$ **then** $A := A \cup \{y\}$.

Mark y.}

A vertex u is said to be *activated* if $u \in A$. Intuitively, the above algorithm works as follows. Alice first activates b. Then she moves to $s(x)$ which is the minimum vertex with respect to L among neighbors of x. If $s(x)$ is not activated, she repeats this operation until she finds an activated vertex. When she finds an activated vertex, she marks it. Since the activation strategy is a little complicated, we give an example below.

Example 4.1. We explain the activation strategy by using the graph G in the left side of Figure 4.2. This example was introduced in [59]. We define partial order L on $V(G)$

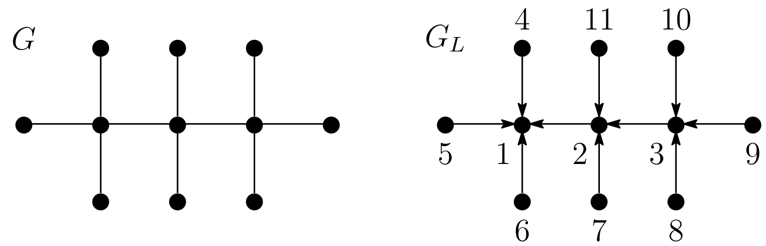

Figure 4.2: Graph G and the orientation of G with respect to L.

as in the right side of Figure 4.2. In other words, we define L such that $u \preceq_L v$ if and only if $u \leq v$ for $u, v \in V(G)$. We consider the situation where Alice takes the activation strategy $S(L, G)$.

(i) At the beginning of the game, Alice marks 1, the minimum vertex with respect to L (see Figure 4.3).

(ii) We assume that Bob marks 10 after Alice marks 1. At this time, she first activates 10 and calculate $s(10)$. Since $s(10) = 3$ and $3 \notin A$, she activates 3. In the same way, she activates 2 because $s(3) = 2$. After activating 2, she moves to Step 3, and marks 2 (see Figure 4.4). In Figure 4.4, black squares represent marked vertices and underlined numbers represent activated vertices (the same applies in Figure 4.5 and 4.6).

(iii) We assume that Bob marks 9 after (ii). Then, Alice first activates 9 and calculate $s(9)$. Since $s(9) = 3$ and $3 \in A$, she marks 3 (see Figure 4.5).

(iv) We assume that Bob marks 8 after (iii). Alice activates 8 and calculate $s(8)$. Since $s(8) = 8$ holds and 8 is already marked, she marks 4, the minimum unmarked vertex with respect to L (see Figure 4.6).

By using the activation strategy, we can estimate upper bounds for the game coloring number. We first define the *matching number*.

Definition 4.5. *Let $A, B \subseteq V(G)$. We say that a matching M of G is a* matching from A to B *if M covers all vertices in A and each edge in M connects a vertex in A and a vertex in $B \setminus A$.*

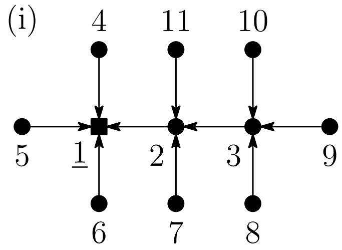

Figure 4.3: Beginning of the game.

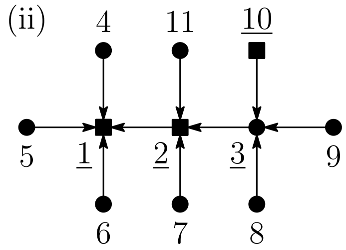

Figure 4.4: Alice's second turn.

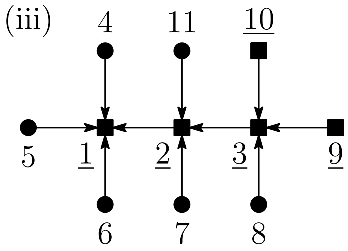

Figure 4.5: Alice's third turn.

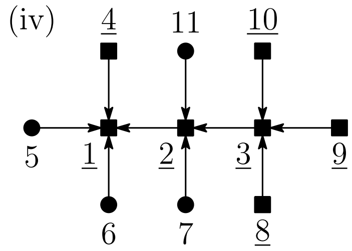

Figure 4.6: Alice's fourth turn.

Definition 4.6. *Let $L \in \Pi(G)$ be a total order on $V(G)$. For $u \in V(G)$, we consider $Z \subseteq N^-_{G_L}[u]$ which satisfies the following condition.*

(*) *There exists a partition $Z = X \cup Y$ such that there exist a matching from X to $V^+_{G_L}(u)$ and a matching from Y to $V^+_{G_L}[u]$.*

The matching number $m(u, L, G)$ *of G with respect to L is defined to be the maximum size of Z satisfying the condition* (*).

For graph G, $L \in \Pi(G)$, and $u \in V(G)$, we define

$$r(u, L, G) = d^+_{G_L}(u) + m(u, L, G).$$

Moreover, we define

$$r(L, G) = \max_{u \in V(G)} r(u, L, G), \quad r(G) = \min_{L \in \Pi(G)} r(L, G).$$

Then the following theorem holds.

Theorem 4.7 ([36])**.** *Let G be an undirected graph and $L \in \Pi(G)$. If Alice uses the activation strategy with respect to L, the score will be at most $1 + r(L, G)$. In particular, $\mathrm{col}_g(G) \leq 1 + r(G)$.*

Proof. Assume that Alice uses the activation strategy $S(L, G)$. We consider the situation just before Alice marks a vertex v. We define the set of activated vertices by A. Then all marked vertices and v are activated by the definition of the activation strategy. We show that

$$|N^-_{G_L}(u) \cap A| \leq m(u, L, G)$$

for any unmarked vertex u. This fact leads $\mathrm{col}_g(G) \leq 1 + r(G)$.

We define

$$P = \{x \in A : x \text{ is activated before } s(x)\},$$
$$Q = \{x \in A : x \text{ is activated after } s(x)\},$$

where $s(x)$ is defined by $S(L, G)$ (note that $s(x)$ is already calculated for all $x \in A$). Then the following proposition holds.

Proposition 4.8. *For $x, y \in P$ with $x \neq y$, we have $s(x) \neq s(y)$. Similarly, for $x, y \in Q$ with $x \neq y$, we have $s(x) \neq s(y)$.*

Proof. Let $x, y \in P$ with $x \neq y$. We assume that x is activated before y. Then $s(x)$ is activated just after x. This implies that $y = s(x)$ or $s(x)$ is activated before y. Since $y \in P$, $s(x)$ is activated before $s(y)$. Therefore $s(x) \neq s(y)$. We next prove the latter statement. Let $x, y \in Q$ with $x \neq y$. We assume that x is activated before y. Then $s(x)$ is marked just after x is activated because $x \in Q$. Therefore $s(x)$ is already marked before y is activated. This means that $s(x) \neq s(y)$. □

We define

$$P' = N^-_{G_L}(u) \cap P, \quad Q' = N^-_{G_L}(u) \cap Q.$$

We have $s(x) \in V^+_{G_L}[u]$ and $s(x) \neq x$ for all $x \in N^-_{G_L}(u) \cap A$ because $s(x) \preceq_L u$ holds. This means that $P' \cup Q' = N^-_{G_L}(u) \cap A$. Furthermore,

$$S_{P'} = \{\{x, s(x)\} : x \in P'\}, \quad S_{Q'} = \{\{x, s(x)\} : x \in Q'\}$$

are matchings from $N^-_{G_L}(u)$ to $V^+_{G_L}[u]$ by Proposition 4.8. We now construct sets X and Y by using P' and Q'. We consider two cases.

Case 1. There do not exist $x \in P'$ and $y \in Q'$ such that $s(x) = u = s(y)$.
In this case, at least one of $S_{P'}$ and $S_{Q'}$ does not cover u. If $S_{P'}$ does not cover u, we set $X = P'$ and $Y = Q'$, otherwise $X = Q'$ and $Y = P'$.

Case 2. There exist $x \in P'$ and $y \in Q'$ such that $s(x) = u = s(y)$.
Since x is activated before y and u is unmarked, we have $s(u) \neq u$. If $u \in P$ we set

$$X = (P' \setminus \{x\}) \cup \{u\}, \quad Y = Q',$$

otherwise we set

$$X = (Q' \setminus \{y\}) \cup \{u\}, \quad Y = P'.$$

We denote S_X and S_Y by

$$S_X = \{\{x, s(x)\} : x \in X\}, \quad S_Y = \{\{y, s(y)\} : y \in Y\}.$$

In both cases, S_X is a matching from $N^-_{G_L}[u]$ to $V^+_{G_L}(u)$ and S_Y is a matching from $N^-_{G_L}(u)$ to $V^+_{G_L}[u]$. Hence, we have

$$|N^-_{G_L} \cap A| = |P'| + |Q'| \leq m(u, L, G).$$

□

4.2.2 Planar graphs with girth 4

By using Theorem 4.7, we show that the game coloring number of planar graphs with girth at least 4 is at most 13. The following result is based on [60].

Theorem 4.9. *If G is a planar graph with girth at least 4, then $\mathrm{col}_g(G) \leq 13$. Hence $\mathrm{col}_g(\mathcal{P}_4) \leq 13$, where $\mathcal{P}_4$ is the family of planar graphs with girth at least 4.*

Proof. It suffices to show that there exists a total order L such that $r(L,G) \leq 12$. We fix a planar embedding of G. We choose a vertex of G according to the following rule for every turn. We repeat this operation until all vertices are chosen. Then, we construct L so that $u \preceq_L v$ if and only if v is chosen earlier than u.

We now explain the rule of choosing a vertex. Let $C, U \subseteq V(G)$ be the set of chosen vertices and unchosen vertices of G, respectively. If C is empty, we choose a vertex whose degree is at most 5 (such a vertex exists by Proposition 2.2). We consider the case where C is not empty. We now construct a new graph from G. First, we delete all edges connecting vertices in C and all vertices $x \in C$ such that $|N_G(x) \cap U| \leq 3$. Then we add edges between any two nonadjacent vertices in U that are adjacent to the same deleted vertex x. We define this graph by H. Then H is a planar graph.

We define

$$
\begin{aligned}
S &= \{\{x,y\} \in E(G) : x,y \in U\},\\
S' &= \{\{x,y\} \in E(H) \setminus E(G) : x,y \in U\},\\
A &= \{\{x,y\} \in E(H) : x \in C,\ y \in U,\ 4 \leq d_H(x) \leq 5\},\\
B &= \{\{x,y\} \in E(H) : x \in C,\ y \in U,\ d_H(x) \geq 6\}.
\end{aligned}
$$

By the construction of H and the definitions of S, S', A, and B, we have

$$E(H) = S \cup S' \cup A \cup B. \tag{4.1}$$

We next define the charge of a vertex v by $c(v)$. At the beginning, we set $c(v) = d_H(v)$ for all $v \in V(H)$. We consider transferring the charges among vertices of H. For each edge $\{x,y\} \in A\,(x \in C, y \in U)$, y gives $1/2$ to x. Let the new charge of a vertex v be $c'(v)$. Note that the total charge is unchanged. Therefore, it holds by Proposition 2.2 that

$$\sum_{v \in V(H)} d_H(v) = \sum_{v \in V(H)} c(v) = \sum_{v \in V(H)} c'(v) < 6|V(H)|. \tag{4.2}$$

Furthermore, by the construction of H, we have $c'(v) \geq 6$ for all $v \in C \cap V(H)$. Hence by (4.2) and the half integrality of c', there exists a vertex $u \in U$ such that $c'(u) \leq 5.5$. We choose such u.

Let

$$\sigma = |\{y \in U : \{u,y\} \in S\}|, \qquad \sigma' = |\{y \in U : \{u,y\} \in S'\}|,$$
$$\alpha = |\{x \in C : \{u,x\} \in A\}|, \qquad \beta = |\{x \in C : \{u,x\} \in B\}|.$$

Then by (4.1), we have

$$c'(u) = \sigma + \sigma' + \frac{1}{2}\alpha + \beta \leq 5.5. \tag{4.3}$$

We claim that $r(u, L, G) \leq \sigma + 2\sigma' + \alpha + \beta + \delta + 1$, where

$$\delta = \begin{cases} 0 & \text{if } \sigma = 0, \\ 1 & \text{if } \sigma \geq 1. \end{cases}$$

It suffices to show that $m(u, L, G) \leq 2\sigma' + \alpha + \beta + \delta + 1$ because $d^+_{G_L}(u) = \sigma$ holds. We consider a set $Z \subseteq N^-_{G_L}[u]$ with $|Z| = m(u, L, G)$ such that there exists a partition $Z = X \cup Y$ and there exist matchings M from X to $V^+_{G_L}(u)$ and N from Y to $V^+_{G_L}[u]$. Since the girth of G is at least 4, there is no edge connecting a vertex in $N^-_{G_L}(u)$ and one in $N^+_{G_L}(u)$. This implies that the number of edges of a matching from $\{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \leq 3\}$ to $V^+_{G_L}(u)$ is at most σ'. Therefore we have

$$|X \setminus \{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \geq 4\}| \leq \sigma' + \delta, \tag{4.4}$$
$$|Y \setminus \{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \geq 4\}| \leq \sigma' + 1. \tag{4.5}$$

Moreover, it holds that

$$|\{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \geq 4\} \cap Z| \leq \alpha + \beta. \tag{4.6}$$

By (4.4), (4.5), and (4.6), we have

$$
\begin{aligned}
m(u, L, G) = |Z| &= |X| + |Y| \\
&= |X \setminus \{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \geq 4\}| \\
&\qquad + |Y \setminus \{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \geq 4\}| \\
&\qquad\quad + |\{v \in N^-_{G_L}(u) : |N^+_{G_L}(v) \cap V^+_{G_L}[u]| \geq 4\} \cap Z| \\
&\leq \sigma' + \delta + \sigma' + 1 + \alpha + \beta \\
&\leq 2\sigma' + \alpha + \beta + \delta + 1.
\end{aligned}
$$

Hence $r(u, L, G) \leq \sigma + 2\sigma' + \alpha + \beta + \delta + 1$. We finally show $r(u, L, G) \leq 12$.

Case 1. $\sigma = 0$.

In this case, it follows that $\delta = 0$. By (4.3), we have

$$c'(u) = \sigma' + \frac{1}{2}\alpha + \beta \leq 5.5.$$

Since σ', α, and β are nonnegative integers, we obtain

$$\sigma + 2\sigma' + \alpha + \beta + \delta + 1 = 2\sigma' + \alpha + \beta + 1 \leq 12.$$

Case 2. $\sigma \geq 1$.

In this case, it follows that $\delta = 1$. Let $\sigma^* = \sigma - 1$. Then we have

$$c'(u) = \sigma + \sigma' + \frac{1}{2}\alpha + \beta = \sigma^* + \sigma' + \frac{1}{2}\alpha + \beta + 1 \leq 5.5,$$

implying that

$$\sigma^* + \sigma' + \frac{1}{2}\alpha + \beta \leq 4.5.$$

Since $\sigma^*, \sigma', \alpha$, and β are nonnegative integers it holds that

$$
\begin{aligned}
\sigma + 2\sigma' + \alpha + \beta + \delta + 1 &= \sigma^* + 1 + 2\sigma' + \alpha + \beta + 1 + 1 \\
&= \sigma^* + 2\sigma' + \alpha + \beta + 3 \\
&\leq 12.
\end{aligned}
$$

In both cases, $r(u, L, G) \leq 12$ is satisfied. Hence $\mathrm{col}_g(G) \leq 13$. $\square$

4.3 Lower bounds for the game coloring number

We focus on lower bounds for the game coloring number of planar graphs with girth 4 and 5. The results in this section is also based on [60].

4.3.1 Planar graphs with girth 4

We consider the game coloring number of planar graphs with girth 4. We prove the following theorem.

Theorem 4.10. *There exists a planar graph G with girth 4 such that* $\mathrm{col}_g(G) \geq 7$.

Proof. We consider the graph $G = (A \cup B \cup C, E_A \cup E_B \cup E_C)$ where

$$\begin{aligned}
A &= \{a_{i,j} : i, j \in \{1, 2, \dots, 15\}\}, \\
B &= \{b_{i,j} : i \in \{2, 4, \dots, 14\}, j \in \{1, 2, \dots, 14\}\}, \\
C &= \{c_{i,j} : i, j \in \{1, 2, \dots, 15\}\}, \\
E_A &= \{\{a_{i,j}, a_{i,j+1}\} : a_{i,j}, a_{i,j+1} \in A\} \cup \{\{a_{i,j}, a_{i+1,j}\} : a_{i,j}, a_{i+1,j} \in A\}, \\
E_B &= \{\{b_{i,j}, a_{i+1,j}\} : b_{i,j} \in B, a_{i+1,j} \in A\} \cup \{\{b_{i,j}, a_{i,j+1}\} : b_{i,j} \in B, a_{i,j+1} \in A\}, \\
E_C &= \{\{a_{ij}, c_{ij}\} : i, j \in \{1, 2, \dots, 15\}\}.
\end{aligned}$$

We give a sketch of G in Figure 4.7 (we omit the vertices of C and the edges of E_C). Then G is a planar graph and its girth is 4. We show that $\mathrm{col}_g(G) \geq 7$, that is, Bob has a strategy such that the score is at least 7.

We define

$$B' = B \cup \{a_{i,j} \in A : i \in \{1, 15\} \text{ or } j \in \{1, 15\}\}.$$

Bob marks the vertices of B' as soon as possible. By the definition of B', we have

$$154 = |B'| < |A \setminus B'| = 169. \tag{4.7}$$

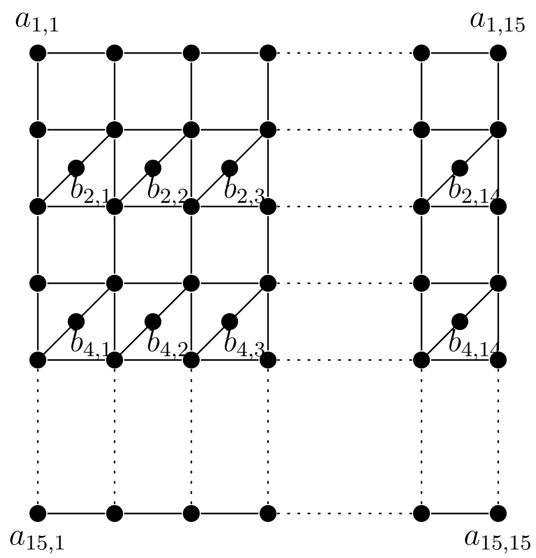

Figure 4.7: The graph G which satisfies $\mathrm{col}_g(G) \geq 7$.

It follows from (4.7) that all the vertices of B' can be marked before all the vertices of $A \setminus B'$ are marked. We consider Bob's turn immediately after all the vertices of B' are marked. Then there is at least 14 unmarked vertices of $A \setminus B'$. Let $a_{i_1,j_1}, \ldots, a_{i_k,j_k}$ be such vertices of $A \setminus B'$ where k is the number of unmarked vertices of $A \setminus B'$. Bob has a strategy such that $c_{i_1,j_1}, \ldots, c_{i_k,j_k}$ are marked before $a_{i_1,j_1}, \ldots, a_{i_k,j_k}$ are marked. Therefore there exists a vertex $v \in A \setminus B'$ which satisfies $1 + b(v) \geq 7$ (note that v has 4 neighbors in A, 1 neighbor in B and 1 neighbor in C). Hence $\mathrm{col}_g(G) \geq 7$. $\square$

4.3.2 Planar graphs with girth 5

We focus on planar graphs with girth 5.

Theorem 4.11. *There exists a planar graph G with girth 5 such that $\mathrm{col}_g(G) \geq 6$.*

Proof. Let H_1 be the cycle of length 6 (we say that H_1 is a *regular hexagon*). We construct H_n $(n \geq 2)$ from H_{n-1} by adding $6(n-1)$ regular hexagons around H_{n-1} (see Figure 4.8).

We label the vertices of a regular hexagon f by $v_1^f, \ldots v_6^f$ in the clockwise order from the upper left. For each regular hexagon f of H_n, we add a vertex a_f and add edges $\{a_f, v_1^f\}$ and $\{a_f, v_4^f\}$. Furthermore, we add a vertex b_v for each $v \in V(H_n)$, and add an

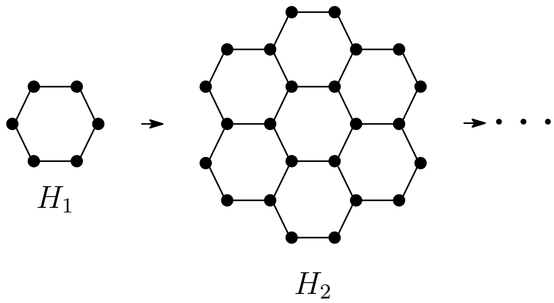

Figure 4.8: The construction of graph H_n.

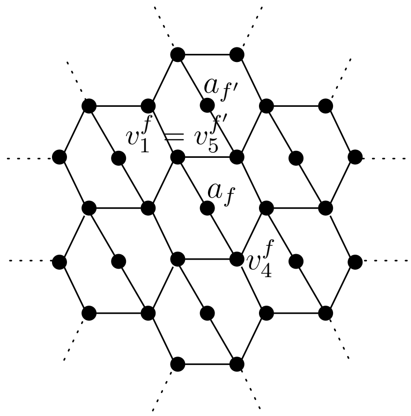

Figure 4.9: The graph G_n which satisfies $\mathrm{col}_g(G_n) \geq 6$.

edge between b_v and v. The resulting graph is defined by G_n and it is given in Figure 4.9 (we omit b_v and an edge which connects $b(v)$ and v for all $v \in V(H_n)$). Then it follows that G_n is a planar graph and its girth is 5. We show that $\mathrm{col}_g(G_n) \geq 6$ if $n \geq 7$.

We define

$$A = \{a_f : f \text{ is a regular hexagon of } H_n\}, \qquad B = \{b_v : v \in V(H_n)\},$$
$$V_I = \{v \in V(H_n) : v \text{ is not in the outer boundary}\}, \qquad V_O = V(H_n) \setminus V_I.$$

Then we have

$$|A| = 3n^2 - 3n + 1, \quad |V_I| = 6n^2 - 12n + 6, \quad |V_O| = 12n - 6.$$

Therefore it follows that

$$|A| + |V_O| + 1 < |V_I| \quad (\forall n \geq 7). \tag{4.8}$$

Bob marks the vertices of A and V_O as soon as possible. By (4.8), Bob has a strategy such that all the vertices of A and V_O are marked before all the vertices of V_I are marked. Each vertex of V_I has 3 neighbors in $V(H_n)$, 1 neighbor in A, and 1 neighbor in B. In the same way as the proof of Theorem 4.10, we verify that there exists a vertex $v \in V_I$ which satisfies $1 + b(v) \geq 6$. Hence $\mathrm{col}_g(G_n) \geq 6$. □

By Theorems 4.4, 4.9, 4.10, and 4.11, we obtain the following statement.

Corollary 4.12. *Let $\mathcal{P}_k$ be the family of planar graphs with girth at least k. Then the following statements hold:*

(i) $7 \leq \mathrm{col}_g(\mathcal{P}_4) \leq 13$.

(ii) $6 \leq \mathrm{col}_g(\mathcal{P}_5) \leq 8$.

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