On the isomorphism classes of Iwasawa modules

September 2015

Kazuaki Murakami

A Thesis for the Degree of Ph. D. in Science

On the isomorphism classes of Iwasawa modules

September 2015

Graduate School of Science and Technology

Keio University

Kazuaki Murakami

Contents

Chapter 1

Introduction

In Iwasawa theory, we study Galois actions on several arithmetic objects like ideal class groups and Galois groups. More precisely, suppose that $\mathbb{Z}_p$ is the ring of p-adic integers for a prime p, and k_∞/k is a Galois extension whose Galois group Γ is isomorphic to $\mathbb{Z}_p$. We call such a Galois extension k_∞/k a $\mathbb{Z}_p$-extension of k. We study $\mathbb{Z}_p$-modules with Γ-action. Suppose that L_∞/k_∞ is the maximal abelian pro-p extension unramified everywhere. We denote by X_{k_∞} the Galois group of L_∞/k_∞. Then X_{k_∞} is a $\mathbb{Z}_p$-module, and Γ acts on X_{k_∞} by conjugation. This X_{k_∞} is called the Iwasawa module for k_∞/k, which is regarded as a $\mathbb{Z}_p[[\Gamma]]$-module, where $\mathbb{Z}_p[[\Gamma]]$ is the completed group ring of Γ over $\mathbb{Z}_p$. Iwasawa proved that X_{k_∞} is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module. Serre pointed out that $\mathbb{Z}_p[[\Gamma]]$ is isomorphic to $\Lambda = \mathbb{Z}_p[[T]]$, where $\mathbb{Z}_p[[T]]$ is the ring of formal power series in one variable over $\mathbb{Z}_p$. Thus X_{k_∞} becomes a finitely generated torsion Λ-module. By the structure theorem of finitely generated torsion Λ-modules, we can classify such modules up to pseudo isomorphism, where a pseudo isomorphism is a morphism with finite kernel and cokernel. Further, we can define the characteristic ideal for a finitely generated torsion Λ-module by the structure theorem. In this thesis, we study the problems whether one can derive more precise information on a Λ-module than its characteristic ideal and whether one can classify Λ-modules up to isomorphism. Out main result is to classify such modules up to isomorphism under several assumptions. We apply our theorems to the Iwasawa modules associated to the cyclotomic $\mathbb{Z}_p$-extensions of imaginary quadratic fields.

In the following, we begin with some historical background of our thesis.

1.1 Ideal class groups

Let k_∞/k be a $\mathbb{Z}_p$-extension of an algebraic number field k. By class field theory, the Iwasawa module X_{k_∞} for k_∞/k is isomorphic to the projective limit of the ideal class groups of algebraic number fields. For the details, see the next Section 1.2. In number theory, the ideal class group of a number field is an important object. First, we introduce a historical overview of the ideal class group. In the 19th century, Kummer introduced the notion of "ideal primfactors" to study Fermat's Last Theorem, which was proved by Andrew Wiles [22]. Kummer's notion was taken up and extended by Dedekind. This led to "Ideal theory". Dedekind defined an ideal as a subset of a set of numbers, composed of algebraic integers that satisfy polynomial equations with integer coefficients. He proved that non-zero ideals of the ring of the integers of a number field can be uniquely decomposed into prime ideals. He also defined ideal class groups. We review the definition of the ideal class group for an algebraic number field k. We denote by $I(k)$ and $P(k)$ the group of fractional ideals and the subgroup of principal fractional ideals, respectively. The ideal class group of k is the quotient group $Cl(k) = I(k)/P(k)$. It is known that $Cl(k)$ is a finite abelian group. We call the order of $Cl(k)$ the class number of k. If $Cl(k)$ is trivial, by the definition of $Cl(k)$, the ring of integers of k is a principal ideal domain, especially a unique factorization domain. Hence the ideal class group measures how close the ring of integers of k is to a principal ideal domain.

1.2 Iwasawa's class number formula

In this section, we briefly introduce a part of Iwasawa theory. Recall that, for a finite Galois extension $k/\mathbb{Q}$, the Galois group $\mathrm{Gal}(k/\mathbb{Q})$ acts naturally on $Cl(k)$. It is important to investigate the structure of $Cl(k)$ including the action of $\mathrm{Gal}(k/\mathbb{Q})$. Especially, in Iwasawa theory, one often studies ideal class groups on which the Galois group of a $\mathbb{Z}_p$-extension acts. We give a typical example (Iwasawa's class number formula [8]) of this idea. We introduce the Iwasawa's class number formula [8, Theorem 11] in the following. Let p be a prime number. Let k_∞/k be a $\mathbb{Z}_p$-extension. For each $n \geq 0$, we denote by k_n the intermediate field of k_∞/k such that k_n is the unique cyclic extension over k of degree p^n. Namely, we have a tower of number fields

$$k_0 \subset k_1 \subset \cdots \subset k_n \subset \cdots \subset k_\infty, \quad k_0 = k, \quad k_\infty = \bigcup_{n=0}^{\infty} k_n.$$

Let A_n be the p-Sylow subgroup of the ideal class group of k_n. We denote the order of A_n by p^{e_n}. Then Iwasawa's class number formula states that there exist non-negative integers λ, μ, and an integer ν such that

$$e_n = \lambda n + \mu p^n + \nu \tag{1.1}$$

for sufficiently large n. A key of his idea is not to treat each k_n independently but to treat the whole $\{k_n\}_n$. Put $\Gamma = \mathrm{Gal}(k_\infty/k)$ satisfying $\Gamma \cong \mathbb{Z}_p$ as a topological group. Iwasawa considered the inverse limit $X_{k_\infty} = \varprojlim_n A_n$, where the inverse limit is taken with respect to the relative norms. We note that $\varprojlim_n A_n$ is isomorphic to $\mathrm{Gal}(L_\infty/k_\infty)$, where $\mathrm{Gal}(L_\infty/k_\infty)$ is the maximal abelian pro-p extension unramified everywhere. The module X_{k_∞} is called the Iwasawa module for k_∞/k. Since the Galois group Γ acts naturally on X_{k_∞}, it becomes a $\mathbb{Z}_p[[\Gamma]]$-module. He proved that X_{k_∞} is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module. The class number formula above is proved by investigating a rough structure of X_{k_∞} as a $\mathbb{Z}_p[[\Gamma]]$-module.

1.3 Iwasawa modules and its properties

Put $\Gamma = \mathrm{Gal}(k_\infty/k)$. If k is a CM-filed, the complex conjugation ρ acts naturally on the Iwasawa module X_{k_∞}. Further if p is odd, then we can decompose X_{k_∞} into $X_{k_\infty} = X_{k_\infty}^+ \oplus X_{k_\infty}^-$, where $X_{k_\infty}^+ = \{x \in X_{k_\infty} \mid x = \rho(x)\}$ and $X_{k_\infty}^- = \{x \in X_{k_\infty} \mid x = -\rho(x)\}$. Consider the following properties (P1) and (P2) for a $\mathbb{Z}_p[[\Gamma]]$-module M:

(P1) The module M is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module.

(P2) The module M has no non-trivial finite $\mathbb{Z}_p[[\Gamma]]$-submodule.

Iwasawa proved that the minus part $X_{k_\infty}^-$ of X_{k_∞} satisfies (P1) and (P2). In Iwasawa theory, there are many $\mathbb{Z}_p[[\Gamma]]$-modules M satisfying (P1) and (P2). We introduce some of them here:

1: Let K be a totally real field and put $k = K(\zeta_p)$, where ζ_p is a primitive p-th root of unity. Let k_∞/k be the cyclotomic $\mathbb{Z}_p$-extension. Let M_∞ be the maximal abelian p-extension unramified outside p and put $M = \mathrm{Gal}(M_\infty/K_\infty)$. Then M satisfies (P1) and (P2) (cf. [7, Theorem 18]).

2: Let M be a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module. Then the adjoint module of M has no non-trivial finite $\mathbb{Z}_p[[\Gamma]]$-submodule (cf. [21, Proposition 15.28]).

1.4 Structure theorem and pseudo-isomorphism classes and some invariants

As in the previous section, we put $\Gamma = \mathrm{Gal}(k_\infty/k)$. Let γ be a fixed topological generator of Γ. Serre ([18]) pointed out the existence of an isomorphism $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$. We put $\Lambda = \mathbb{Z}_p[[T]]$. We introduce the structure theorem for finitely generated torsion Λ-modules (cf. [21, Theorem 13.12]). This theorem was first proved by Iwasawa in terms of the group ring $\mathbb{Z}_p[[\Gamma]]$. If M is a finitely generated torsion Λ-module, there exists a homomorphism

$$M \to \bigoplus_{i=1}^{s} \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(f_j(T)^{n_j}) \tag{1.2}$$

with finite kernel and finite cokernel, where $s, t, m_i, n_j \in \mathbb{Z}_{\geq 0}$, and $f_j(T)$ is an irreducible distinguished polynomial. We note that the decomposition (1.2) is uniquely determined by M. A Λ-module homomorphism with finite kernel and finite cokernel is called a pseudo-homomorphism. We will use the structure theorem to prove our main theorems. We define the characteristic ideal of M by

$$\mathrm{char}(M) = \left(\prod_{i=1}^{s} p^{m_i} \prod_{j=1}^{t} f_j(T)^{n_j} \right)$$

and define the λ-invariant and μ-invariant of M by

$$\lambda(M) = \sum_{j=1}^{t} n_j \deg(f_j(T)), \quad \mu(M) = \sum_{i=1}^{s} m_i,$$

respectively. We also define an equivalence relation $\sim$ for the set of finitely generated torsion Λ-modules as follows. For M_1 and M_2, we write $M_1 \sim M_2$ if there exists a pseudo-isomorphism $M_1 \to M_2$. In classical Iwasawa theory, one studies Iwasawa modules up to pseudo-isomorphism. In this thesis, we consider finitely generated torsion Λ-modules M with $\mu(M) = 0$ and $\lambda(M) \leq 4$.

1.5 Modules up to isomorphism

In this thesis, we study Iwasawa modules up to Λ-isomorphism. Especially, our aim is to generalize Sumida's results (cf. [19], [20]).

Let E be a finite extension over the field $\mathbb{Q}_p$ of p-adic numbers and $\mathcal{O}_E$ the ring of integers of E. Let π be a prime element of $\mathcal{O}_E$. We put $\Lambda_E = \mathcal{O}_E[[T]]$, the ring of power series in one variable over $\mathcal{O}_E$. For a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, Sumida considered finitely generated torsion Λ_E-modules whose characteristic ideals are $(f(T))$, and defined the set $\mathcal{M}^E_{f(T)}$ by

$$\mathcal{M}^E_{f(T)} = \left\{ [M]_E \;\middle|\; \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda_E\text{-module,} \\ \mathrm{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \end{array} \right\},$$

where $[M]_E$ denotes the isomorphism class of M as a Λ_E-module. We denote the Λ_E-isomorphism class of M by $[M]_E$ or simply by $[M]$. He proved in [19] that $\mathcal{M}^E_{f(T)}$ is a finite set if and only if $f(T)$ is separable, where $f(T)$ is said to be separable if $f(T)$ has no multiple roots in the algebraic closure of E. The case of $\deg(f(T)) \leq 3$ was treated in [4], [9], [10], [12], [19], and [20]. Sumida and Koike classified $\mathcal{M}^E_{f(T)}$ in the case of $\deg(f(T)) \leq 2$ ([9, Theorem 2.1] and [19, Proposition 10]). Kurihara also classified $\mathcal{M}^E_{f(T)}$ in the case of $\deg(f(T)) = 2$, using higher Fitting ideals [10, Corollary 9.3].

We review the result of Sumida. He considered

$$f(T) = (T - \alpha)(T - \beta),$$

where α and β are distinct elements of $\pi\mathcal{O}_E$. We put $\mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta)$. Let $[M]_E$ be an element of $\mathcal{M}^E_{f(T)}$. Since M has no non-trivial finite Λ_E-submodule, there exists an injective Λ_E-homomorphism $\varphi : M \hookrightarrow \mathcal{E}$ with finite cokernel. Therefore every class of $\mathcal{M}^E_{f(T)}$ can be represented by a Λ_E-submodule of $\mathcal{E}$.

Now we fix a notation to express such submodules in $\mathcal{E}$. First, by using the canonical isomorphism $\Lambda_E/(T-\alpha) \cong \mathcal{O}_E \quad (f(T) \longmapsto f(\alpha))$, we define an isomorphism $\iota : \mathcal{E} \longrightarrow \mathcal{O}_E^{\oplus 2}$ by $(f_1(T), f_2(T)) \longmapsto (f_1(\alpha), f_2(\beta))$. We identify $\mathcal{E}$ with $\mathcal{O}_E^{\oplus 2}$ via ι. Thus an element in $\mathcal{E}$ is expressed as $(a_1, a_2) \in \mathcal{O}_E^{\oplus 2}$. Since the rank of M over $\mathcal{O}_E$ is equal to 2, we can write M in the form

$$M = \langle (a_1, a_2), (b_1, b_2) \rangle_{\mathcal{O}_E} \ \subset \ \mathcal{E},$$

where $\langle * \rangle_{\mathcal{O}_E}$ is the $\mathcal{O}_E$-submodule generated by $*$. Further, using this notation, we can express the action of $T \in \Lambda_E$ by

$$T(a_1, a_2) = (\alpha a_1, \beta a_2).$$

In this case, Sumida proved that

$$\mathcal{M}^E_{f(T)} = \{[\langle (1,1), (0, \pi^k) \rangle_{\mathcal{O}_E}] \mid 0 \leq k \leq \mathrm{ord}_E(\beta - \alpha)\},$$

where ord_E is the normalized additive valuation on E such that $\mathrm{ord}_E(\pi) = 1$ (see Proposition 3.1.4).

1.6 Main Theorem for $\lambda = 3$

In this thesis, we classify Λ_E-modules in the case of $\lambda = 3$ and that of $\lambda = 4$ with $\mu = 0$ (namely, Λ_E-modules which are free over $\mathcal{O}_E$ of rank 3 or 4). Here, we state our results in the case of $\lambda = 3$. In this case, we consider

$$f(T) = (T-\alpha)(T-\beta)(T-\gamma), \tag{1.3}$$

where α, β, and γ are distinct elements of the maximal ideal of $\mathcal{O}_E$. We put $\mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma)$. We note that $\mathcal{E}$ is an integral closure of $\Lambda_E/(T-\alpha)(T-\beta)(T-\gamma)$. Using the structure theorem of Λ_E-modules (1.2), we regard a Λ_E-module M satisfying $[M] \in \mathcal{M}^E_{f(T)}$ as a Λ_E-submodule of $\mathcal{E}$. We first prove that for each isomorphism class $\mathfrak{C} \in \mathcal{M}^E_{f(T)}$, we can take a submodule

$$M(m,n,x) := \langle (1,1,1), (0,\pi^m,x), (0,0,\pi^n) \rangle_{\mathcal{O}_E} \tag{1.4}$$

of $\mathcal{E}$ with $[M(m,n,x)] = \mathfrak{C}$. Here m and n are non-negative integers and x is an element of $\mathcal{O}_E$. The non-negative integers m and n are determined only by $[M(m,n,x)]$ (Corollary 4.2.2). Our first main theorem is as follows.

Theorem 1 (Theorem 4.1.5)**.** *There is a bijection* Φ :

$$\begin{array}{ccc} \mathcal{M}^E_{f(T)} & \longrightarrow & Z/\sim \\ \Psi & & \Psi \\ \Big[M(m,n,x)\Big] & \longmapsto & \Big[\overline{(m,n,x)}\Big]. \end{array}$$

The definitions of the set Z *and the relation* $\sim$ *will be given in Chapter 4.*

We briefly explain the definition of the set Z here. First, we define a certain equivalence relation $\sim'$ on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$ and define $Z' = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E)/\sim'$. Let Z be a subset of Z' satisfying certain conditions. An element of Z' is written as $\overline{(m,n,x)}$. We also define an equivalence relation $\sim$ on Z and consider $Z/\sim$. An element of $Z/\sim$ is written as $\Big[\overline{(m,n,x)}\Big]$. By Theorem 1, we have the following corollary, which explicitly gives a necessary and sufficient condition for the two Λ_E-modules $M(m,n,x)$ and $M(m,n,x')$ to be isomorphic.

Corollary 1 (Corollary 4.1.7)**.** *Let* $[M(m,n,x)]$ *and* $[M(m,n,x')]$ *be elements of* $\mathcal{M}^E_{f(T)}$*. Suppose that* $\mathrm{ord}_E(x) < n$ *or* $x = 0$ *and that* $\mathrm{ord}_E(x') < n$ *or* $x' = 0$*, where* ord_E *is the normalized additive valuation on* E *such that* $\mathrm{ord}_E(\pi) = 1$*. Then the following statements are equivalent:*

(i) *We have* $M(m,n,x) \cong M(m,n,x')$ *as* Λ_E*-modules.*

(ii) *We have* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and one of* (I′)*,* (II′)*, and* (III′) *holds, where* (I′)*,* (II′)*, and* (III′) *are*

$$\begin{array}{ll} (\mathrm{I}') & m \neq 0,\ x' \neq 0,\ \text{and} \\ & \min\left\{ \mathrm{ord}_E\left(\dfrac{\pi^n}{x'}\right), \mathrm{ord}_E(\pi^m - x') \right\} \leq \mathrm{ord}_E\left(\dfrac{x}{x'} - 1\right), \\ (\mathrm{II}') & x' = 0,\ \text{and} \\ (\mathrm{III}') & m = 0 \text{ and } \mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x'). \end{array}$$

1.7 Main Theorem for $\lambda = 4$

In this section, we state our second main theorem in the case of $\lambda = 4$. More precisely, we treat the case in which

$$f(T) = (T-\alpha)(T-\beta)(T-\gamma)(T-\delta),$$

where α, β, γ, and δ are distinct elements of the maximal ideal of $\mathcal{O}_E$. In the same way as in the case of deg $(f(T)) = 3$, for each isomorphism class $\mathfrak{C} \in \mathcal{M}^E_{f(T)}$, we can take a submodule

$$M(\ell, m, n; x, y, z) := \langle (1,1,1,1), (0, \pi^\ell, x, y), (0,0,\pi^m, z), (0,0,0,\pi^n) \rangle_{\mathcal{O}_E}$$

of $\Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \oplus \Lambda_E/(T-\delta)$ with $[M(\ell, m, n; x, y, z)] = \mathfrak{C}$, where ℓ, m, n are non-negative integers and x, y, z are elements of $\mathcal{O}_E$. We can prove that ℓ, m, and n are determined by $\mathfrak{C}$ (see Proposition 5.1.2). In Chapter 5, we define the notion of "admissibility" (see Definition 5.1.5). Let $(\ell, m, n; x, y, z)$ be a 6-tuple with $\ell, m, n \in \mathbb{Z}_{\geq 0}$ and $x, y, z \in \mathcal{O}_E$ satisfying the conditions (a), (b), $\ldots$, and (f) in Lemma 5.1.1 of Chapter 5. We prove that there is an admissible 6-tuple $(\ell, m, n; x, y, z)$ such that $[M] = [M(\ell, m, n; x, y, z)]$ for each $[M] \in \mathcal{M}^E_{f(T)}$ (see Proposition 5.1.6 (2)). By the definition of admissibility of $(\ell, m, n; x, y, z)$, we have $[M(\ell, m, n; x, y, z)] \in \mathcal{M}^E_{f(T)}$ if $(\ell, m, n; x, y, z)$ is admissible (see Proposition 5.1.6 (1)).

The following is our second main theorem, which gives a necessary and sufficient condition for the two Λ_E-modules $M(\ell, m, n; x, y, z)$ and $M(\ell, m, n; x', y', z')$ to be isomorphic:

Theorem 2 (Theorem 5.3.1)**.** *Let $(\ell, m, n;\ x, y, z)$ and $(\ell, m, n;\ x', y', z')$ be admissible* 6*-tuples. Suppose that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$*, where* ord_E *is the normalized additive valuation on E such that* $\mathrm{ord}_E(\pi) = 1$*. Suppose also that* $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$ *if* $\ell = 0$*. Then the following statements are equivalent:*

(i) *We have $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ as Λ_E-modules.*

(ii) *One of* (I), (II), $\ldots$, *and* (XII) *holds for $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$, where the statements* (I), (II), $\ldots$, *and* (XII) *are described in Chapter* 5.

We note that our assumptions $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, and $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$ are necessary conditions for the two modules to be isomorphic (see Proposition 5.3.2, Lemma 5.3.3).

The classification in the case of $\lambda = 4$ is essentially different from that of $\lambda = 3$. Although in the case of $\lambda = 3$, we need only one element $x \in \mathcal{O}_E$ to study $M(m, n, x)$, we have to investigate three elements x, y, and $z \in \mathcal{O}_E$ to study $M(\ell, m, n; x, y, z)$ in the case of $\lambda = 4$. For a 6-tuple $(\ell, m, n; x, y, z)$, the valuation $\mathrm{ord}_E(y)$ is not uniquely determined by the class $[M(\ell, m, n; x, y, z)]$ (cf. Proposition 5.3.2).

1.8 Applications to Iwasawa theory

Finally, we apply our theorems to Iwasawa theory in Chapter 7. We briefly explain our application below. Let k be a finite, imaginary, abelian extension of $\mathbb{Q}$ and k_∞/k the cyclotomic $\mathbb{Z}_p$-extension. We denote by X_{k_∞} the Iwasawa module for k_∞/k. As we stated in Section 1.3, the minus part $X_{k_\infty}^-$ of X_{k_∞} is a finitely generated torsion Λ-module and has no non-trivial Λ-submodule (properties (P1) and (P2)). Let $f(T)$ be a generator of the characteristic ideal $\mathrm{char}(X_{k_\infty}^-)$. Iwasawa conjectured that $\mu(X_{k_\infty}^-) = 0$ for the cyclotomic $\mathbb{Z}_p$-extension for any k. When k is a finite abelian extension of $\mathbb{Q}$, this was proven by Ferrero and Washington [3]. Therefore if $f(T)$ is a separable polynomial, then we have

$$[X_{k_\infty}^-] \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}.$$

Then we can apply our theorems to the class $[X_{k_\infty}^-]$ for a finite imaginary abelian extension k over $\mathbb{Q}$. For a positive integer n, we put $\Gamma_n = \Gamma^{p^n}$. For a Λ-module M, we define

$$M_{\Gamma_n} = M/((1+T)^{p^n} - 1)M.$$

Let A_n^- be the minus part of A_n. We assume that exactly one prime of k is ramified in k_∞/k and this prime is totally ramified. Then we have

$$(X_{k_\infty}^-)_{\Gamma_n} \cong {A_n}^- \tag{1.5}$$

as $\mathbb{Z}_p[\mathrm{Gal}(k_n/k)]$-modules. By this isomorphism, we can determine the structure of A_n^- for non-negative integer n as a $\mathbb{Z}_p[\mathrm{Gal}(k_n/k)]$-module if we determine the isomorphism class of $X_{k_\infty}^-$.

Let us give an example. Suppose that $p = 3$ and $k = \mathbb{Q}(\sqrt{-9069})$. Since k is an imaginary quadratic field, we have $X_{k_\infty} = X_{k_\infty}^-$. In this case, we can check that $f(T)$ is separable. Using Theorem 1, we have

$$X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,1,1),$$

where $E/\mathbb{Q}_p$ is the minimal splitting field of $f(T)$ and $M(0,1,1)$ is defined by (1.4). This implies that

$$\begin{aligned} \sharp M(0,1,1)_{\Gamma_n} &= p^{6n+4}, \\ M(0,1,1)_{\Gamma_n} &\cong \mathcal{O}_E/(\pi^{2n+2}) \oplus \mathcal{O}_E/(\pi^{2n}) \oplus \mathcal{O}_E/(\pi^{2n+2}). \end{aligned}$$

By (1.5), we can determine the structure of $A_n \otimes \mathcal{O}_E$ for all $n \geq 0$. In particular, we get

$$\begin{aligned} \sharp A_n &= p^{3n+2}, \\ A_n &\cong \mathbb{Z}/(p^{n+1}) \oplus \mathbb{Z}/(p^n) \oplus \mathbb{Z}/(p^{n+1}) \end{aligned}$$

for all $n \geq 0$. In this way, we get more precise information than Iwasawa's class number formula (1.1). We note that only knowing $f(T)$ does not give the information above. For details about the computations above, see Chapter 6 and Chapter 7.

1.9 Overview

The outline of this thesis is as follows. In Chapter 2, we briefly review some properties of Λ_E and prove the structure theorem for finitely generated torsion Λ_E-modules. In Chapter 3, we state some known results about the isomorphism classes of Λ_E-modules. In Chapter 4, we prove Theorem 1. In Chapter 5, we introduce the notion of admissibility of a 6-tuple $(\ell, m, n; x, y, z)$ and give a proof of Theorem 2. As an application, in Corollary 5.3.16 we determine the number of the elements of $\mathcal{M}_{f(T)}^E$ when $E = \mathbb{Q}_p$ and $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\delta) =$

$\mathrm{ord}_p(\delta - \alpha) = \mathrm{ord}_p(\beta - \delta) = \mathrm{ord}_p(\alpha - \gamma)$=1. Here we write ord_p for $\mathrm{ord}_{\mathbb{Q}_p}$. In Chapter 6, we introduce the notion of the higher Fitting ideals and study the relationships between Λ_E-modules and their higher Fitting ideals. In Chapter 7, we determine the isomorphism classes of Iwasawa modules associated to the cyclotomic $\mathbb{Z}_p$-extension of imaginary quadratic fields for $p = 3, 5$.

Chapter 2

Preliminary

In this chapter, we prove the structure theorem (1.2) in Chapter 1. Let p be a prime and E a finite extension over the field $\mathbb{Q}_p$ of p-adic numbers. We put $\Lambda = \mathcal{O}[[T]]$, where $\mathcal{O}$ is the ring of integers of E. We denote a prime element of E by π.

2.1 Structure theorem

First, we review some properties of the ring Λ. The following is so-called division lemma.

Lemma 2.1.1 ([21], Proposition 7.2)**.** *Let $f(T) = \sum_{n=1}^{\infty} a_n T^n$ be an element of $\mathcal{O}[[T]]$. Assume that there exists an integer $s \geq 0$ such that*

$$a_0, a_1, \ldots, a_{s-1} \in (\pi), \text{ and } a_s \in \mathcal{O}^{\times}.$$

Then for every power series $g(T) \in \Lambda$, there exist $q(T) \in \Lambda$ and $r(T) \in \mathcal{O}[T]$ such that

$$g(T) = q(T)f(T) + r(T), \quad \deg(r(T)) \leq s - 1.$$

Definition 2.1.2 (Distinguished polynomial)**.** Let $f(T)$ be a polynomial over $\mathcal{O}$. We call $f(T)$ a distinguished polynomial if it is of the form

$$f(T) = T^n + a_{n-1}T^{n-1} + a_{n-2}T^{n-2} + \cdots + a_1 T + a_0$$

with coefficients $a_0, \ldots, a_{n-1}$ contained in the maximal ideal of $\mathcal{O}$.

Proposition 2.1.3 ([21], Theorem 7.3, p-adic Weierstrass Preparation Theorem)**.** *Let $f(T) \in \Lambda$ be non-zero element. Then $f(T)$ is uniquely written as*

$$f(T) = \pi^n P(T)U(T),$$

where $P(T)$ is a distinguished polynomial, $U(T)$ is a unit of Λ, and n is a non-negative integer.

Proposition 2.1.4. *The prime ideals of Λ are*

$$(0),\ (\pi),\ (f(T)),\ \text{and}\ (\pi, T),$$

where $f(T) \in \mathcal{O}[T]$ is an irreducible distinguished polynomial.

Proof. It is obvious that (π, T) is the maximal ideal. Let $f(T) \in \mathcal{O}[T]$ be an irreducible distinguished polynomial. Since π and $f(T)$ are irreducible elements of Λ, (π) and $(f(T))$ are prime ideals. Conversely, we suppose that $\mathfrak{p}$ is a prime ideal. Then there exists an irreducible element $h(T) \in \mathfrak{p}$. We assume that $\mathfrak{p} \neq (h)$. We apply the following

Lemma 2.1.5. *Suppose that $f(T)$ and $g(T) \in \Lambda$ are relatively prime. Then the ideal (f, g) is of finite index in Λ.*

The lemma above can be proved by using Lemma 2.1.1. We put $M = \Lambda/\mathfrak{p}$. By Lemma 2.1.5, M is finite. Then $T^n M = \pi^n M = 0$ for some $n \geq 0$. Hence we have $T^n, \pi^n \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, we have $(\pi, T) \subset \mathfrak{p}$. This implies $\mathfrak{p} = (\pi, T)$. Thus we get the conclusion. $\square$

We define the notion of pseudo-nulls and pseudo-isomorphisms.

Definition 2.1.6 (pseudo-null)**.** Let R be a noetherian integrally closed domain. A finitely generated R-module M is called pseudo-null if $M_{\mathfrak{p}} = 0$ for all prime ideal $\mathfrak{p}$ satisfying $\mathrm{ht}(\mathfrak{p}) \leq 1$, where $\mathrm{ht}(\mathfrak{p})$ is the height of $\mathfrak{p}$.

Definition 2.1.7 (pseudo-isomorphism)**.** Let R be a noetherian integrally closed domain. Let $f : M \to N$ be a homomorphism between finitely generated R-modules. We call f pseudo-isomorphism if $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are pseudo-null.

By the definition of a pseudo-isomorphism, we get the following

Proposition 2.1.8. *Let $f : M \to N$ be a homomorphism between finitely generated R-modules. Then the following statements are equivalent:*
(i) *The map f is a pseudo-isomorphism.*
(ii) *The induced map $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism for every prime ideal $\mathfrak{p}$ satisfying $\mathrm{ht}(\mathfrak{p}) \leq 1$.*

In the case of $R = \Lambda$, we can prove that a pseudo-null module is a finite module. To show the fact, we prepare the following

Lemma 2.1.9. *Let M be a finitely generated Λ-module. Then the following statements are equivalent:*
(i) *There exist relatively prime elements $f(T)$ and $g(T) \in \Lambda$ such that $f(T)M = g(T)M = 0$.*
(ii) *The module M is finite.*

Proof. First, we prove (ii) $\Rightarrow$ (i). Since $\pi^n M = T^n M = 0$ for some $n \geq 0$, we get (ii). Next, we prove (i) $\Rightarrow$ (ii). Since M is a finitely generated Λ-module, we have a surjective map $(\Lambda/(f(T), g(T)))^{\oplus r} \to M$ for some positive integer r. By Lemma 2.1.5, $\Lambda/(f(T), g(T))$ is finite. Thus we get (ii). □

Proposition 2.1.10. *Let M be a finitely generated Λ-module. Then the following statements are equivalent:*
(i) *The module M is finite.*
(ii) *The module M is pseudo-null.*

Proof. First, we suppose (i). Since M is finite, M is a torsion Λ-module. Hence we have $M_{(0)} = 0$. Further, using Lemma 2.1.9, we have relatively prime elements $f(T), g(T) \in \Lambda$ such that $f(T)M = g(T)M = 0$. Thus we have $f(T) \notin \mathfrak{p}$ or $g(T) \notin \mathfrak{p}$ for every $\mathfrak{p} \in P^1(\Lambda)$, where $P^1(\Lambda) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal with } \mathrm{ht}(\mathfrak{p}) = 1\}$. This implies that $M_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in P^1(\Lambda)$. Therefore we get (ii).

Next, we suppose (ii). In this case, we note that $\mathrm{Ann}_{\Lambda}(M) \neq 0$ and there is no $\mathfrak{p} \in P^1(\Lambda)$ such that $\mathrm{Ann}_{\Lambda}(M) \subset \mathfrak{p}$. Hence we have $\sqrt{\mathrm{Ann}_{\Lambda}(M)} = (\pi, T)$. This implies that $\pi^n, T^n \in \mathrm{Ann}_{\Lambda}(M)$ for some $n \geq 0$. Using Lemma 2.1.9, we get (i). □

Theorem 2.1.11 (Structure theorem for torsion Λ-modules)**.** *Let M be a finitely generated torsion Λ-module. Then there exists a pseudo-isomorphism*

$$M \to \bigoplus_{i=1}^{s} \Lambda/(\pi^{m_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(f_j(T)^{n_j}),$$

where ℓ, s, t, m_i, and n_j are integers and $f_j(T)$ is an irreducible distinguished polynomial.

Proof. First, we use the following

Lemma 2.1.12. *Let M be a finitely generated module. Then*

$$\{\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0 \text{ for all } \mathfrak{p} \in P^1(\Lambda)\}$$

is a finite set.

Proof. We assume that $M_{\mathfrak{p}} \neq 0$ for a prime $\mathfrak{p} \in P^1(\Lambda)$. This is equivalent to saying that $sM \neq 0$ for all $s \in \Lambda \backslash \mathfrak{p}$. This implies that $\mathrm{Ann}_{\Lambda}(M) \subset \mathfrak{p}$. Since $\mathrm{Ann}_{\Lambda}(M) \neq 0$, $\mathfrak{p}$ is one of the prime factors of $\mathrm{Ann}_{\Lambda}(M)$. Thus we get the conclusion. $\square$

Using Lemma 2.1.12, we put

$$\{\mathfrak{p} \mid M_{\mathfrak{p}} \neq 0, \text{ for all } \mathfrak{p} \in P^1(\Lambda)\} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_h\}$$

for some positive integer h. We also put

$$S = \bigcap_{i=1}^{h} (\Lambda \backslash \mathfrak{p}_i).$$

Then the set S becomes a multiplicatively closed set and $S^{-1}\Lambda$ is a principal ideal domain. Indeed, the maximal ideals of $S^{-1}\Lambda$ are $\mathfrak{p}_1 S^{-1}\Lambda$, $\mathfrak{p}_2 S^{-1}\Lambda$, $\ldots$, $\mathfrak{p}_h S^{-1}\Lambda$. By the structure theorem for finitely generated modules over a principal ideal domain, we have

$$S^{-1}M \cong \bigoplus_{i} S^{-1}\Lambda/{\mathfrak{p}_i}^{n_i} S^{-1}\Lambda \cong S^{-1}\left(\bigoplus_{i} \Lambda/{\mathfrak{p}_i}^{n_i}\right)$$

for some non-negative integers n_i. Thus there exists an isomorphism

$$\phi : S^{-1}M \to S^{-1}\left(\bigoplus_i \Lambda/\mathfrak{p}_i{}^{n_i}\right).$$

We use the following

Proposition 2.1.13 ([1], Chapter II, §2, no 7, Proposition 19)**.** *Let S be a multiplicatively closed set of Λ. Assume that M and N are finitely generated torsion Λ-modules. Then we have*

$$S^{-1}(\mathrm{Hom}_\Lambda(M, N)) \cong \mathrm{Hom}_{S^{-1}\Lambda}(S^{-1}M, S^{-1}N).$$

By this proposition, there exists $s \in S$ such that $s\phi : M \to \bigoplus_i \Lambda/\mathfrak{p}_i{}^{n_i}$ is a pseudo-isomorphism. Thus we get the conclusion. □

Chapter 3

Known results about isomorphism classes

In this chapter, we introduce some known results about isomorphism classes of modules. Especially, we review the results of Sumida, Koike, Kurihara, and Franks.

3.1 Sumida's and Koike's results

Let E be a finite extension over the field $\mathbb{Q}_p$ of p-adic numbers. Let $\mathcal{O}_E$, π_E, and ord_E be the ring of integers in E, a prime element, and the normalized additive valuation on E such that $\mathrm{ord}_E(\pi_E) = 1$, respectively. We put $\Lambda_E := \mathcal{O}_E[[T]]$, the ring of power series over $\mathcal{O}_E$.

Let M be a finitely generated torsion Λ_E-module. By the structure theorem 2.1.11, there is a Λ_E-homomorphism

$$\varphi : M \longrightarrow \left(\bigoplus_i \Lambda_E/({\pi_E}^{m_i}) \right) \oplus \left(\bigoplus_j \Lambda_E/(f_j(T)^{n_j}) \right)$$

with finite kernel and finite cokernel, where m_i, n_j are non-negative integers and $f_j(T) \in \mathcal{O}_E[T]$ is a distinguished irreducible polynomial. We put

$$\mathrm{char}(M) = \left(\prod_i {\pi_E}^{m_i} \prod_j f_j(T)^{n_j} \right)$$

which is an ideal in Λ_E. We denote the Λ_E-isomorphism class of M by $[M]_E$ or simply by $[M]$.

For a distinguished polynomial $f(T) \in \mathcal{O}_E[T]$, we consider finitely generated torsion Λ_E-modules whose characteristic ideals are $(f(T))$, and define the set $\mathcal{M}^E_{f(T)}$ by

$$\mathcal{M}^E_{f(T)} = \left\{ [M]_E \;\middle|\; \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda_E\text{-module,} \\ \mathrm{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \end{array} \right\}. \tag{3.1}$$

Sumida proved the following

Proposition 3.1.1 ([19], Theorem 2)**.** *Let $f(T)$ and $\mathcal{M}^E_{f(T)}$ be the same as above. Then $\mathcal{M}^E_{f(T)}$ is finite if and only if $f(T)$ is separable.*

Let $\overline{E}$ be a splitting field of $f(T)$. Sumida and Koike considered

$$f(T) = (T - \alpha)(T - \beta),$$

where α and β are elements of $\overline{E}$. They classified all the elements of $\mathcal{M}^E_{f(T)}$ in [9] and [19]. Let us introduce their results in the following. There are three cases to consider.

$$\begin{cases} \text{(i)} & \text{The polynomial } f(T) \text{ is separable and reducible over } E. \\ \text{(ii)} & \text{The polynomial } f(T) \text{ is irreducible over } E. \\ \text{(iii)} & \text{The polynomial } f(T) \text{ is inseparable.} \end{cases}$$

First, we consider the case (i). Let $f(T)$ be a separable and reducible polynomial. In other words, we assume that

$$f(T) = (T - \alpha)(T - \beta),$$

where α and β are distinct elements of $\pi_E\mathcal{O}_E$. Let $[M]_E$ be an element of $\mathcal{M}^E_{f(T)}$. Since M has no non-trivial finite Λ_E-submodule, there exists an injective Λ_E-homomorphism

$$\varphi : M \hookrightarrow \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)$$

with finite cokernel. We fix the notation to express such submodules in $\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)$. By using the canonical isomorphism $\Lambda_E/(T-\alpha)\cong\mathcal{O}_E\ (f(T)\longmapsto f(\alpha))$, we define an isomorphism

$$\iota:\mathcal{E}=\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta)\longrightarrow\mathcal{O}_E^{\oplus 2}$$

by $(f_1(T),f_2(T))\longmapsto(f_1(\alpha),f_2(\beta))$. We identify $\mathcal{E}$ with $\mathcal{O}_E^{\oplus 2}$ via ι. Thus an element in $\mathcal{E}$ is expressed as $(a_1,a_2)\in\mathcal{O}_E^{\oplus 2}$. Since the rank of M is equal to two, we can write M of the form

$$M=\langle(a,b),(c,d)\rangle_{\mathcal{O}_E}\ \subset\ \Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta),$$

where $\langle *\rangle_{\mathcal{O}_E}$ is the $\mathcal{O}_E$-submodule generated by $*$. Further, using this notation, we can express the action of T by

$$T(a,b)=(\alpha a,\beta b).$$

Remark 3.1.2. The module $M=\langle(a,b),(c,d)\rangle_{\mathcal{O}_E}$ is an $\mathcal{O}_E$-module. A necessary and sufficient condition for M to be a Λ_E-module is the following

Lemma 3.1.3. *We assume that* $\mathrm{ord}_E(a)\le\mathrm{ord}_E(c)$*. Then an* $\mathcal{O}_E$*-module* $\langle(a,b),(c,d)\rangle_{\mathcal{O}_E}$ *is a* Λ_E*-module if and only if* $\mathrm{ord}_E(d-a^{-1}bc)-\mathrm{ord}_E(b)\le\mathrm{ord}_E(\beta-\alpha)$.

Then Sumida proved the following

Proposition 3.1.4 ([19], Proposition 10)**.** *Let $f(T)$ be the same polynomial as above. Then we have*

$$\mathcal{M}^E_{f(T)}=\{[M(m)]_E\mid 0\le m\le\mathrm{ord}_E(\beta-\alpha)\},$$

where

$$M(m)=\langle(1,1),(0,\pi_E^m)\rangle_{\mathcal{O}_E}\subset\Lambda_E/(T-\alpha)\oplus\Lambda_E/(T-\beta).$$

Further, we have

$$M(m)\cong M(m')\Leftrightarrow m=m'.$$

Next, we consider the case (ii). Let $f(T)$ be an irreducible polynomial. We put

$$f(T)=T^2+c_1T+c_0\in\mathcal{O}_E[T].$$

By the same method as in the case (i), there exists an injective Λ_E-homomorphism

$$\varphi : M \hookrightarrow \Lambda_E/(f(T)).$$

Since the rank of M is equal to two, we can write M of the form

$$M = \langle aT + b, cT + d\rangle_{\mathcal{O}_E} \ \subset \ \Lambda_E/(f(T)),$$

where a, b, c, and d are elements of $\mathcal{O}_E$. Further, using this notation, we can express the action of T by

$$T(aT + b, cT + d) = ((b - ac_1)T - ac_0, (d - cc_1)T - cc_0).$$

Remark 3.1.5. The module $M = \langle aT + b, cT + d\rangle_{\mathcal{O}_E}$ is an $\mathcal{O}_E$-module. A necessary and sufficient condition for M to be a Λ_E-module is the following:

Lemma 3.1.6. *We assume that* $\mathrm{ord}_E(a) \leq \mathrm{ord}_E(c)$. *Then an* $\mathcal{O}_E$*-module* $\langle aT + b, cT + d\rangle_{\mathcal{O}_E}$ *is a* Λ_E*-module if and only if*

$$\begin{cases} \mathrm{ord}_E(a) \leq \mathrm{ord}_E(b) \text{ and} \\ \mathrm{ord}_E(a) \leq \mathrm{ord}_E(d - a^{-1}bc) \leq \mathrm{ord}_E(a) + \mathrm{ord}_E(f(-\frac{b}{a})). \end{cases}$$

Then Koike proved the following

Theorem 3.1.7 ([9], Theorem 2.1)**.** *Let $f(T)$ be the same polynomial as above. Then we have*

$$\mathcal{M}_f^E(T) = \left\{ [N]_E \;\middle|\; N = \left\langle T + \frac{c_1}{2}, \pi_E^x \right\rangle_{\mathcal{O}_E}, 0 \leq x \leq \frac{1}{2}\mathrm{ord}_E(c_1^2 - 4c_0) \right\}.$$

Finally, we consider the case (iii). Let $f(T) \in \mathcal{O}_E[T]$ be an inseparable polynomial. In other words, we suppose that

$$f(T) = T^2 + c_1T + c_0 = (T - \alpha)^2 \in \mathcal{O}_E[T].$$

Then there exists an injective Λ_E-homomorphism

$$\varphi : M \hookrightarrow \mathcal{E},$$

where we put $\mathcal{E} = \Lambda_E/(T - \alpha)$ or $\Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \alpha)$. In the case where $\mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha)$, it is easy to see that $N \cong \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha)$. In the case where $\mathcal{E} = \Lambda_E/(T - \alpha)^2$, Koike proved the following

Theorem 3.1.8 ([9], Theorem 2.1). *Let $f(T)$ be the same polynomial as above. Then we have*

$$\mathcal{M}_f^E(T) = \left\{ [N] \,\middle|\, N = N_\infty \text{ or } N = \left\langle T + \frac{c_1}{2}, \pi_E^x \right\rangle_{\mathcal{O}_E} \;\; (0 \le x < \infty) \right\},$$

where $N_\infty = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\alpha)$.

3.2 Kurihara's results

Kurihara determined the isomorphism classes of modules, using higher Fitting ideals. We give the definition of Fitting ideals in Chapter 6.

Lemma 3.2.1 ([10], Lemma 9.1). *Put $f(T) = (T-\alpha)(T-\beta) \in \mathcal{O}_E[T]$. Let $[M]$ be an element of $\mathcal{M}_{f(T)}^E$.*
(1) Suppose that α and β belong to $\mathcal{O}_E$. Then we have an exact sequence of Λ-modules

$$0 \to \Lambda_E^2 \xrightarrow{\varphi} \Lambda_E^2 \to M \to 0$$

such that the matrix A_φ corresponding to the Λ-homomorphism φ is of the form

$$A_\varphi = \begin{pmatrix} T-\alpha & \pi_E^i \\ 0 & T-\beta \end{pmatrix}$$

for some i with $0 < i \le \mathrm{ord}_E(\beta - \alpha)$. Here if $\alpha = \beta$, $i = \infty$ is allowed. Further, the isomorphism class of M is determined by the value i.

(2) Suppose that $f(T)$ is irreducible. We define

$$a = \frac{\alpha + \beta}{2}.$$

Then we have an exact sequence of Λ-modules

$$0 \to \Lambda_E^2 \xrightarrow{\varphi} \Lambda_E^2 \to M \to 0$$

such that the matrix A_φ corresponding to the Λ-homomorphism φ is of the form

$$A_\varphi = \begin{pmatrix} T-a & \pi_E^i \\ c & T-a \end{pmatrix}$$

for some i such that $0 < i \leq \mathrm{ord}_E(\beta - \alpha)$ and for some $c \in \mathcal{O}_E$ with $\mathrm{ord}_E(c) \geq i$. Further, the isomorphism class of M is determined by the value i.

Remark 3.2.2. 1. This lemma says that the isomorphism class $[M] \in \mathcal{M}^E_{f(T)}$ is determined by the 1-st Fitting ideal $\mathrm{Fitt}_{1,\Lambda_E}(M)$ of M. Indeed, we have $\mathrm{Fitt}_{1,\Lambda_E}(M) = (\pi_E^i)$ in this lemma.

2. In general (in the case of $\mathrm{rank}_{\mathcal{O}_E}(M) \geq 3$), the Fitting ideals $\mathrm{Fitt}_{i,\Lambda_E}(M)$ ($i \geq 0$) do not determine the isomorphism class of M. We will state the relationships between Λ_E-modules and their higher Fitting ideals in Chapter 6.

3.3 Franks's results

Chase Franks [4] studies the Λ_E-isomorphism classes. He gave an algorithm to determine whether two Λ_E-modules are isomorphic or not for any separable polynomial $f(T)$ of degree $\lambda \geq 0$. He determined all the elements of $\mathcal{M}^E_{f(T)}$ for a separable distinguished polynomial $f(T)$ with $\deg(f(T)) = 4$ satisfying some conditions [4, Section 5.3]. This algorithm is proceeded by checking whether some matrices he defined belong to $GL_\lambda(\mathcal{O}_E)$, where $\lambda = \deg(f(T))$ and $GL_\lambda(\mathcal{O}_E)$ is the group of $\lambda \times \lambda$ matrices over $\mathcal{O}_E$ that are invertible.

We introduce his results in the case of $\lambda = 4$ shortly. We suppose that

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma)(T - \delta),$$

where α, β, γ, and δ are distinct elements of the maximal ideal of $\mathcal{O}_E$. Let π be a prime element of $\mathcal{O}_E$. For each isomorphism class $\mathfrak{C} \in \mathcal{M}^E_{f(T)}$, we can take a submodule

$$M(\ell, m, n; x, y, z) := \langle (1,1,1,1), (0, \pi_E^\ell, x, y), (0, 0, \pi_E^m, z), (0,0,0,\pi_E^n) \rangle_{\mathcal{O}_E}$$

of $\Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \oplus \Lambda_E/(T-\delta)$ with $[M(\ell, m, n; x, y, z)] = \mathfrak{C}$. Franks considered a map

$$\varphi_{1,2} : (\mathcal{O}_E^\times)^4 \longrightarrow \mathrm{GL}_4(E)$$

for Λ_E-modules $M_1 = M(\ell_1, m_1, n_1; x_1, y_1, z_1)$ and $M_2 = M(\ell_2, m_2, n_2; x_2, y_2, z_2)$. This map is defined by $\varphi_{1,2}(u_1, u_2, u_3, u_4) = G_2^{-1}\mathrm{diag}(u_1, u_2, u_3, u_4)G_1$, where

$\mathrm{diag}(u_1, u_2, u_3, u_4)$ is the diagonal matrix with u_1, u_2, u_3, and $u_4 \in \mathcal{O}_E^\times$ along its diagonal and

$$G_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pi_E^{\ell_i} & 0 & 0 \\ 1 & x_i & \pi_E^{m_i} & 0 \\ 1 & y_i & z_i & \pi_E^{n_i} \end{pmatrix}$$

for $i = 1, 2$. Franks proved the following

Theorem 3.3.1 ([4], Theorem 2.1.2)**.** *Let M_1 and M_2 be as above. Then $M_1 \cong M_2$ as Λ_E-modules if and only if* $\mathrm{im}(\varphi_{1,2}) \cap \mathrm{GL}_4(\mathcal{O}_E) \neq \emptyset$.

In order to check this condition $\mathrm{im}(\varphi_{1,2}) \cap \mathrm{GL}_4(\mathcal{O}_E) \neq \emptyset$, he took some finite set $S \subset (\mathcal{O}_E^\times)^4$ and reduced this condition to $\varphi_{1,2}(S) \cap \mathrm{GL}_4(\mathcal{O}_E) \neq \emptyset$. It is known that $\sharp S \leq p^{\ell+m+n}$ in the case of $E = \mathbb{Q}_p$, where $\sharp S$ denotes the number of elements of S. Further he reduced $\sharp S$ which have to be checked (cf. [4, Theorem 5.2.1]). Consequently, he gave an algorithm [4, Section 5.3] which is proceeded by checking the condition above for all elements in S. For the details about his algorithm, see Section 5 in [4].

Chapter 4

Proof of Theorem 1

In this chapter, we give a proof of Theorem 1. This is the generalization of Proposition 3.1.4 in Chapter 3. Roughly speaking, Theorem 1 states that there is an one to one correspondence between $\mathcal{M}^E_{f(T)}$ and the equivalence classes of $Z/\sim$, where the set Z and the relation $\sim$ will be defined in Section 4.1.

4.1 Some results

As in Chapter 3, let E be a finite extension over the field $\mathbb{Q}_p$ of p-adic numbers. Let $\mathcal{O}_E$, π, and ord_E be the ring of integers in E, a prime element, and the normalized additive valuation on E such that $\mathrm{ord}_E(\pi) = 1$, respectively. We put $\Lambda_E := \mathcal{O}_E[[T]]$, the ring of power series over $\mathcal{O}_E$.

In this chapter, we consider

$$f(T) = (T-\alpha)(T-\beta)(T-\gamma) \in \mathcal{O}_E[T],$$

where α, β, and γ are distinct elements of $\pi\mathcal{O}_E$. Let $[M]_E$ be an element of $\mathcal{M}^E_{f(T)}$. Since M has no non-trivial finite Λ_E-submodule, there exists an injective Λ_E-homomorphism

$$\varphi : M \hookrightarrow \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) =: \mathcal{E} \tag{4.1}$$

with finite cokernel. We write $\mathcal{E}$ for the right-hand side. The fact above implies that every class of $\mathcal{M}^E_{f(T)}$ can be represented by a Λ_E-submodule of $\mathcal{E}$. Let M be an $\mathcal{O}_E$-submodule of $\mathcal{E}$ with $\mathrm{rank}_{\mathcal{O}_E}(M) = 3$ of the form

$$M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{\mathcal{O}_E} \ \subset \ \mathcal{E}.$$

We put

$$\begin{aligned} s &= \min\{i \in \mathbb{Z}_{\geq 0} |\ {}^{\exists}a, b \in \mathcal{O}_E \text{ s.t. } (\pi^i, a, b) \in M\}, \\ t &= \min\{i \in \mathbb{Z}_{\geq 0} |\ {}^{\exists}c \in \mathcal{O}_E \text{ s.t. } (0, \pi^i, c) \in M\}, \text{ and} \\ u &= \min\{i \in \mathbb{Z}_{\geq 0} |\ (0, 0, \pi^i) \in M\}. \end{aligned}$$

Then we have

$$M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}.$$

Suppose that $(a_1, a_2, a_3) \in M$. Since $\mathrm{ord}_E(a_1) \geq s$, there exists $x \in \mathcal{O}_E$ such that $a_1 = x\pi^s$. Hence $(a_1, a_2, a_3) - x(\pi^s, a, b) = (0, a_2 - xa, a_3 - xb) \in M$. Since $\mathrm{ord}_E(a_2 - xa) \geq t$, there exists $y \in \mathcal{O}_E$ such that $a_2 - xa = y\pi^t$. By the same method as above, we get $(0, 0, a_3 - xb - yc) \in M$. Finally, there exists $z \in \mathcal{O}_E$ such that $a_3 - xb - yc = z\pi^u$. Then we have $(a_1, a_2, a_3) = x(\pi^s, a, b) + y(0, \pi^t, c) + z(0, 0, \pi^u)$.

The following lemma gives a necessary and sufficient condition for an $\mathcal{O}_E$-module M to be a Λ_E-submodule.

Lemma 4.1.1. *Put $M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$. Then the following two statements are equivalent:*
(i) *The $\mathcal{O}_E$-module M is a Λ_E-submodule.*
(ii) *Integers $a, b, c, s, t,$ and u satisfy*

$$\begin{cases} t & \leq \mathrm{ord}_E(\beta - \alpha) + \mathrm{ord}_E(a), \\ u & \leq \mathrm{ord}_E\{(\gamma - \alpha)b - (\beta - \alpha)\pi^{-t}ac\}, \text{ and} \\ u & \leq \mathrm{ord}_E(\gamma - \beta) + \mathrm{ord}_E(c). \end{cases}$$

Proof. We first suppose that M is a Λ_E-submodule. Hence M satisfies $TM \subset M$

and we have

$$\begin{aligned}
T(\pi^s, a, b) &= (\alpha\pi^s, \beta a, \gamma b) \\
&= \alpha(\pi^s, a, b) + (\beta - \alpha)\pi^{-t} a(0, \pi^t, c) \\
&\quad + \{(\gamma - \alpha)b - (\beta - \alpha)\pi^{-t} ac\}\pi^{-u}(0, 0, \pi^u), \\
T(0, \pi^t, c) &= (0, \beta\pi^t, \gamma c) \\
&= \beta(0, \pi^t, c) + (\gamma - \beta)c\pi^{-u}(0, 0, \pi^u).
\end{aligned}$$

Since these coefficients belong to $\mathcal{O}_E$, we get (ii). Conversely, if an $\mathcal{O}_E$-module M satisfies (ii), M is naturally regarded as an $\mathcal{O}_E[T]$-module by the action as above. We show that M becomes a Λ_E-module. For a positive integer n, we put $v_n = \sum_{k=0}^{n} d_k T^k \in \mathcal{O}_E[T]$ and $v = \sum_{n=0}^{\infty} d_n T^n \in \mathcal{O}_E[[T]]$. Then we have

$$\begin{aligned}
v_n(\pi^s, a, b) &= \left(\pi^s \sum_{k=0}^{n} d_k\alpha^k, a\sum_{k=0}^{n} d_k\beta^k, b\sum_{k=0}^{n} d_k\gamma^k\right) \\
&= \sum_{k=0}^{n} d_k\alpha^k(\pi^s, a, b) + a\left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t}(0, \pi^t, c) + \\
&\quad \left\{b\left(\sum_{k=0}^{n} d_k\gamma^k - \sum_{k=0}^{n} d_k\alpha^k\right) - \right. \\
&\quad \left.\left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t}ac\right\}\pi^{-u}(0, 0, \pi^u).
\end{aligned}$$

Since M is an $\mathcal{O}_E[T]$-module, we have $v_n(\pi^s, a, b) \in M$. Thus we obtain

$$a\left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t} \in \mathcal{O}_E$$

and

$$\left\{b\left(\sum_{k=0}^{n} d_k\gamma^k - \sum_{k=0}^{n} d_k\alpha^k\right) - \left(\sum_{k=0}^{n} d_k\beta^k - \sum_{k=0}^{n} d_k\alpha^k\right)\pi^{-t}ac\right\}\pi^{-u} \in \mathcal{O}_E.$$

Since $d_k\alpha^k, d_k\beta^k$, and $d_k\gamma^k \to 0$ $(k \to \infty)$, $\sum_{k=0}^{\infty} d_k\alpha^k, \sum_{k=0}^{\infty} d_k\beta^k$, and $\sum_{k=0}^{\infty} d_k\gamma^k$ converge in $\mathcal{O}_E$. Thus we have $v(\pi^s, a, b) \in M$. For $(0, \pi^t, c)$ and $(0, 0, \pi^u)$, we can define the action of the elements of Λ_E by the same method as above. $\square$

We use the following lemma to fix a representative of the Λ_E-isomorphism class of M.

Lemma 4.1.2 ([20], Lemma 1)**.** *Let* $M = \langle (a_1, a_2, a_3),\ (b_1, b_2, b_3),\ (c_1, c_2, c_3)\rangle_{\mathcal{O}_E}$ *be a* Λ_E*-submodule of* $\mathcal{E}$*. Suppose that* u_1, u_2*, and* u_3 *are non-zero elements of* $\mathcal{O}_E$*. Then we have*

$$M \cong \langle (u_1 a_1, u_2 a_2, u_3 a_3), (u_1 b_1, u_2 b_2, u_3 b_3), (u_1 c_1, u_2 c_2, u_3 c_3)\rangle_{\mathcal{O}_E}$$

as Λ_E*-modules.*

Proof. The injective homomorphism

$$\varphi : \mathcal{E} \to \mathcal{E}, \quad (x_1, x_2, x_3) \mapsto (u_1 x_1, u_2 x_2, u_3 x_3)$$

induces a Λ_E-isomorphism $M \to \varphi(M)$. We have thus proved the lemma. $\square$

We take M to be a Λ_E-submodule of $\mathcal{E}$ with finite index. Then we can write

$$M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$$

as we explained in the beginning of this section. By Lemma 4.1.2, there exist non-negative integers m, n, and $x \in \mathcal{O}_E$ such that there is an isomorphism

$$M \ \cong \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n)\rangle_{\mathcal{O}_E}$$

as Λ_E-modules. In fact, by Lemma 4.1.2, M is isomorphic to $M' = \langle (1, a, b),$ $(0, \pi^t, c),\ (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$. In the case of $\mathrm{ord}_E(a) \le t$, by Lemma 4.1.2, M is isomorphic to $\langle (1, 1, b),\ (0, a^{-1}\pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$. On the other hand, in the case of $\mathrm{ord}_E(a) > t$, since $M' = \langle (1, a + \pi^t, b + c), (0, \pi^t, c), (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$, we can proceed by the same method as in the case of $\mathrm{ord}_E(a) \le t$. Therefore M is isomorphic to $M'' = \langle (1, 1, b),\ (0, a'\pi^t, c),\ (0, 0, \pi^u)\rangle_{\mathcal{O}_E}$ for some $a' \in E$. By applying the same method as above, M'' is isomorphic to $\langle (1, 1, 1), (0, \pi^m, x),\ (0, 0, \pi^n)\rangle_{\mathcal{O}_E}$ for some non-negative integers m, n, and $x \in \mathcal{O}_E$.

We define $M(m, n, x)$ by

$$M(m, n, x) := \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n)\rangle_{\mathcal{O}_E} \ \subset \ \mathcal{E}.$$

Proposition 4.1.3. *Let $f(T) \in \mathcal{O}_E[T]$ be a distinguished polynomial. Then we have*

$$\mathcal{M}^E_{f(T)} = \left\{ [M(m,n,x)]_E \;\middle|\; m, n, x \text{ satisfy } (*) \right\},$$

where $[M(m,n,x)]_E$ is the Λ_E-isomorphism class of $M(m,n,x)$ and $(*)$ is as follows:

$$(*) \begin{cases} \text{(A)} \quad 0 \le m \le \mathrm{ord}_E(\beta - \alpha), \\ \text{(B)} \quad 0 \le n \le \mathrm{ord}_E(\gamma - \beta) + \mathrm{ord}_E(x), \text{ and} \\ \text{(C)} \quad n \le \mathrm{ord}_E\{(\gamma - \alpha) - (\beta - \alpha)\pi^{-m}x\}. \end{cases}$$

Proof. Let M be a Λ_E-module such that $[M]_E \in \mathcal{M}^E_{f(T)}$. Then we see that $[M]_E = [M(m,n,x)]_E$ for some m, n, and x satisfying $(*)$ by Lemma 4.1.1. We will show the converse. We suppose that m, n, and x satisfy $(*)$. By Lemma 4.1.1, $M(m,n,x)$ becomes a finitely generated Λ_E-module. Since $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$ annihilates $M(m,n,x)$, it is a torsion Λ_E-module. Moreover, by the definition of $M(m,n,x)$, it is a free $\mathcal{O}_E$-module. Finally, we show that $\mathrm{char}(M(m,n,x)) = (f(T))$. The Λ_E-module $M(m,n,x)$ is a submodule of $\mathcal{E}$ with finite index. In fact, since $\mathrm{rank}_{\mathcal{O}_E}(\mathcal{E}) = \mathrm{rank}_{\mathcal{O}_E}(M(m,n,x)) = 3$, $\mathcal{E}/M(m,n,x)$ is finite. This implies that $\mathrm{char}(M(m,n,x)) = \mathrm{char}(\mathcal{E})$. Thus we get $[M(m,n,x)]_E \in \mathcal{M}^E_{f(T)}$. $\square$

Remark 4.1.4. (i) If $x \equiv x' \bmod \pi^n$, we have $M(m,n,x) = M(m,n,x')$ since $(0, \pi^m, x) = (0, \pi^m, x') + a(0, 0, \pi^n)$ for some $a \in \mathcal{O}_E$. In particular, if $\mathrm{ord}_E(x) \ge n$, we have $M(m,n,x) = M(m,n,0)$. This means that we may assume that $x = 0$ or $\mathrm{ord}_E(x) < n$.
(ii) We have

$$\frac{(\gamma - \alpha)(\gamma - \beta)}{\pi^n} = \frac{(\gamma - \beta)x}{\pi^n} \cdot \frac{\beta - \alpha}{\pi^m} + (\gamma - \beta) \cdot \frac{(\gamma - \alpha) - (\beta - \alpha)\pi^{-m}x}{\pi^n}.$$

Therefore if $(*)$ holds, we get

$$0 \le n \le \mathrm{ord}_E(\gamma - \alpha) + \mathrm{ord}_E(\gamma - \beta).$$

Let $M(m,n,x)$ and $M(m',n',x')$ be elements of $\mathcal{M}^E_{f(T)}$. We will investigate a relation among m, m', n, n', x, and x' when $M(m,n,x)$ is isomorphic to $M(m',n',x')$ as Λ_E-modules. We note that we may assume that $x = 0$ or $\mathrm{ord}_E(x) < n$ by Remark 6.1.4 (i).

First of all, we prepare some notations. For (m,n,x) and $(m',n',x') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$, we define

$$(m,n,x) \sim' (m',n',x') \Longleftrightarrow m = m',\ n = n' \text{ and } x \equiv x' \mod \pi^n\mathcal{O}_E.$$

We put $Z' := (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E)/\sim'$ and introduce a set

$$Z := \left\{ \overline{(m,n,x)} \in Z' \;\middle|\; m, n, x \text{ satisfy } (*) \right\}, \tag{4.2}$$

where $(*)$ is the inequalities (A), (B), and (C) in Proposition 4.1.3 and $\overline{(m,n,x)}$ is the equivalence class of (m,n,x). The class $\overline{(m,n,x)}$ is determined by m, n, and x mod $\pi^n\mathcal{O}_E$. We note that the condition $(*)$ does not depend on the choice of a representative of (m,n,x).

For an element $x \in \mathcal{O}_E$ and $z = \overline{x} \in \mathcal{O}_E/\pi^n\mathcal{O}_E$, we define $\mathrm{ord}_E(z) = \mathrm{ord}_E(x \mod \pi^n)$ as follows:

$$\mathrm{ord}_E(z) := \begin{cases} \mathrm{ord}_E(x) & \text{if } \overline{x} \neq 0, \\ \infty & \text{if } \overline{x} = 0. \end{cases}$$

For $\overline{(m,n,x)}$ and $\overline{(m',n',x')} \in Z$, we put $k = \mathrm{ord}_E(x \mod \pi^n)$ and $\ell = \mathrm{ord}_E(x' - \pi^m)$. We define $\overline{(m,n,x)} \sim \overline{(m',n',x')}$ as follows.

(I) Suppose $m \neq 0$.

(a) When $\ell + k \geq n$, we define

$$\overline{(m,n,x)} \sim \overline{(m',n',x')} \Longleftrightarrow m = m',\ n = n' \text{ and } \overline{x} = \overline{x'} \text{ in } \mathcal{O}_E/\pi^n\mathcal{O}_E.$$

(b) When $\ell + k < n$, we define

$$\begin{aligned}\overline{(m,n,x)} \sim \overline{(m',n',x')} \quad \Longleftrightarrow \quad & m = m',\ n = n' \text{ and} \\ & \overline{x} = \varepsilon\overline{x'} \text{ in } \mathcal{O}_E/\pi^n\mathcal{O}_E \text{ for some } \varepsilon \in 1 + \pi^\ell\mathcal{O}_E.\end{aligned}$$

(II) Suppose $m = 0$. We define

$$\begin{aligned}\overline{(m,n,x)} \sim \overline{(m',n',x')} \iff \ & m = m' = 0,\ n = n', \\ & \mathrm{ord}_E(x \mod \pi^n) = \mathrm{ord}_E(x' \mod \pi^n) \text{ and} \\ & \overline{1-x} = \varepsilon\overline{(1-x')} \text{ in } \mathcal{O}_E/\pi^n\mathcal{O}_E \text{ for some } \varepsilon \in \mathcal{O}_E^\times.\end{aligned}$$

Here, for $s \leq 0$, we define $1 + \pi^s\mathcal{O}_E = \mathcal{O}_E^\times$. We can prove that $\sim$ is an equivalence relation. The following is our first main theorem, whose proof will be given in Section 4.2.

Theorem 4.1.5. *There is a bijection* Φ :

$$\begin{array}{ccc} \mathcal{M}^E_{f(T)} & \longrightarrow & Z/\sim \\ \Cup & & \Cup \\ \Big[M(m,n,x)\Big]_E & \longmapsto & \Big[\overline{(m,n,x)}\Big], \end{array}$$

where $\mathcal{M}^E_{f(T)}$ *is defined by* (3.1) *in Chapter* 3*,* Z *is defined by* (4.2) *after Remark 4.1.4, and* $\sim$ *is the equivalence relation of* Z *defined above. The symbol* $\Big[M(m,n,x)\Big]_E$ *is the class of* $M(m,n,x)$ *and* $\Big[\overline{(m,n,x)}\Big]$ *is the class of* $\overline{(m,n,x)}$*.*

Remark 4.1.6. When $\overline{(m,n,x)} \sim \overline{(m',n',x')}$ and $\ell + k \leq n$, we have $\ell = \mathrm{ord}_E(x' - \pi^m) = \mathrm{ord}_E(x - \pi^m)$.

Using Theorem 1, we get the following

Corollary 4.1.7. *Let* $[M(m,n,x)]$ *and* $[M(m,n,x')]$ *be elements of* $\mathcal{M}^E_{f(T)}$*. Suppose that* $\mathrm{ord}_E(x) < n$ *or* $x = 0$ *and that* $\mathrm{ord}_E(x') < n$ *or* $x' = 0$*. Then the following statements are equivalent:*

(i) *We have* $M(m,n,x) \cong M(m,n,x')$ *as* Λ_E*-modules.*

(ii) *We have* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and one of* (I′)*,* (II′)*, and* (III′) *holds, where* (I′)*,* (II′)*, and* (III′) *are*

$$\begin{array}{ll} (\mathrm{I}') & m \neq 0,\ x' \neq 0,\ \text{and} \\ & \min\left\{\mathrm{ord}_E\left(\dfrac{\pi^n}{x'}\right), \mathrm{ord}_E(\pi^m - x')\right\} \leq \mathrm{ord}_E\left(\dfrac{x}{x'} - 1\right), \\ (\mathrm{II}') & x' = 0, \\ (\mathrm{III}') & m = 0 \text{ and } \mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x'). \end{array}$$

Sumida [20] determined all the elements of $\mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ for $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ and $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\alpha) = 1$. We can also obtain the same result from Theorem 1.

Corollary 4.1.8 ([20], Theorem 1)**.** *Put* $E = \mathbb{Q}_p$ *and* $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ *with* $\alpha, \beta,$ *and* $\gamma \in \mathbb{Z}_p$*. Assume that* $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\alpha) = 1$*. Then we have* $\sharp\mathcal{M}_{f(T)} = 7$ *and*

$$\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \{(0,0,0),(0,1,0),(1,0,0),(0,1,1),(1,2,up),(1,1,0),(0,1,2)\},$$

where $u = \dfrac{\gamma-\alpha}{\beta-\alpha}$ *and* (m,n,x) *means* $[M(m,n,x)]_{\mathbb{Q}_p}$*.*

Proof. We prove this corollary by using Theorem 1. For fixed integers m and n we put

$$Z(m,n) = \{\text{ the equivalence class of } \overline{(m,n,x)} \text{ in } Z/\sim \;\big|\; \overline{(m,n,x)} \in Z\}.$$

Then by definition we have

$$Z/\sim \;\; = \coprod_m \coprod_n Z(m,n).$$

We determine all the elements of $Z(m,n)$ for each m and n in order to determine all the elements of $\mathcal{M}_{f(T)}$.

We first assume $\left[\overline{(m,n,x)}\right] \in Z/\sim$, where $\left[\overline{(m,n,x)}\right]$ is the equivalence class of $\overline{(m,n,x)}$. Then by Proposition 4.1.3, $M(m,n,x)$ is a Λ_E-module satisfying (A), (B), and (C). By the inequality (A), we have $0 \le m \le 1$. Now we investigate $\coprod_n Z(m,n)$ for $m = 0,1$.

(I) Suppose $m = 0$. In this case, by the inequalities (B) and (C), we have $0 \le n \le 1$. When $n \ge 2$, we get $\mathrm{ord}_p(x) = 0$ by (C). This contradicts (B). When $n = 0$, we have $\overline{(0,0,x)} = \overline{(0,0,0)}$. Therefore we get $Z(0,0) = \left\{\left[\overline{(0,0,0)}\right]\right\}$. When $n = 1$, we have

$$Z(0,1) = \left\{\left[\overline{(0,1,0)}\right], \left[\overline{(0,1,1)}\right], \left[\overline{(0,1,2)}\right]\right\}.$$

By the definition of the equivalence relation, we have $\overline{(0,1,x)} \sim \overline{(0,1,x')}$ if and only if

$$\mathrm{ord}_p(x \mod p) = \mathrm{ord}_p(x' \mod p) \;\text{ and }\; \overline{1-x} = \varepsilon\overline{(1-x')}$$

for some $\varepsilon \in \mathbb{Z}_p^\times$.
By the definition of $\mathrm{ord}_p(x \mod p)$, we have

$$\mathrm{ord}_p(x \mod p) = \begin{cases} 0 & x \notin p\mathbb{Z}_p, \\ \infty & x \in p\mathbb{Z}_p. \end{cases}$$

We investigate the case of $\mathrm{ord}_p(x \mod \pi) = 0$. Suppose $x = 1$. Then we have

$$\begin{aligned}
\left[\overline{(0,1,1)}\right] &= \{\overline{(0,1,x)} \mid \overline{(0,1,1)} \sim \overline{(0,1,x)}\} \\
&= \{\overline{(0,1,x)} \mid \mathrm{ord}_p(x) = 0,\ \overline{0} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\
&= \{\overline{(0,1,x)} \mid x \equiv 1 \mod p\} \\
&= \{\overline{(0,1,1)}\}.
\end{aligned}$$

If $x = 2$, then we have

$$\begin{aligned}
\left[\overline{(0,1,2)}\right] &= \{\overline{(0,1,x)} \mid \mathrm{ord}_p(x) = 0,\ \overline{-1} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\
&= \{\overline{(0,1,x)} \mid x \not\equiv 0, 1\} \\
&= \{\overline{(0,1,2)}, \ldots, \overline{(0,1,p-1)}\}.
\end{aligned}$$

Therefore we get $Z(0,1) = \left\{\left[\overline{(0,1,0)}\right], \left[\overline{(0,1,1)}\right], \left[\overline{(0,1,2)}\right]\right\}$.

(II) Suppose $m = 1$. By Remark 4.1.4 (ii), we have $0 \le n \le 2$. When $n = 0$, we have $Z(1,0) = \left\{\left[\overline{(1,0,0)}\right]\right\}$. When $n = 1$, we have $Z(1,1) = \left\{\left[\overline{(1,1,0)}\right]\right\}$. In fact, we suppose $\left[\overline{(1,1,x)}\right] \in Z(1,1)$. Then we have $\overline{x} = 0$ by (C). When $n = 2$, we have $Z(1,2) = \left\{\left[\overline{(1,2,up)}\right]\right\}$. Indeed, we suppose $\left[\overline{(1,2,x)}\right] \in Z(1,2)$. For some $v \in \mathbb{Z}_p^\times$, we have

$$\begin{aligned}
x &= \left(1 - \frac{vp^2}{\gamma - \alpha}\right)\frac{\gamma - \alpha}{\beta - \alpha}p \\
&\equiv \frac{\gamma - \alpha}{\beta - \alpha}p \qquad \mod p^2
\end{aligned}$$

by (C). Thus

$$\begin{aligned}
Z/\sim \ = \Big\{&\left[\overline{(0,0,0)}\right], \left[\overline{(0,1,0)}\right], \left[\overline{(1,0,0)}\right], \left[\overline{(0,1,1)}\right], \left[\overline{(1,2,up)}\right], \\
&\left[\overline{(1,1,0)}\right], \left[\overline{(0,1,2)}\right]\Big\}.
\end{aligned}$$

We complete the proof by Theorem 1. $\square$

Corollary 4.1.9. *Put $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ and $E = \mathbb{Q}_p$. Assume that $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\alpha) = 2$. Then we have $\sharp\mathcal{M}_{f(T)} = p+18$ and*

$$\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \left\{ \begin{array}{l} (0,0,0),(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1), \\ (0,2,2),(0,2,p),(0,2,p+1),(1,0,0),(1,1,0), \\ (1,1,1),(1,2,0),(1,2,p),\cdots,(1,2,(p-1)p),(1,3,up), \\ (2,0,0),(2,1,0),(2,2,0),(2,3,up^2),(2,4,up^2) \end{array} \right\},$$

where $u = \dfrac{\gamma-\alpha}{\beta-\alpha}$ and (m,n,x) means $[M(m,n,x)]_{\mathbb{Q}_p}$.

Proof. We use the same notation as in Corollary 4.1.8. By definition, we have

$$Z/\sim \;\; = \;\; \coprod_m \coprod_n Z(m,n).$$

We determine all the elements of $Z(m,n)$ for each m and n in order to determine all the elements of $\mathcal{M}_{f(T)}^{\mathbb{Q}_p}$.

We first assume that $\left[\overline{(m,n,x)}\right] \in Z/\sim$, where $\left[\overline{(m,n,x)}\right]$ is the equivalence class of $\overline{(m,n,x)}$. Then $M(m,n,x)$ becomes a Λ_E-module satisfying (A), (B), and (C). By the inequality (A), we have $0 \le m \le 2$. Now we investigate $\coprod_n Z(m,n)$ for each m.

(I) Suppose $m = 0$. In this case, by the inequalities (B) and (C), we have $0 \le n \le 2$. In fact, if $\mathrm{ord}_p(x) \ge 1$, we get $n \le 2$ by (C) and if $\mathrm{ord}_p(x) = 0$, then we get $n \le 2$ by (B). When $n = 0$, we have $\overline{(0,0,x)} = \overline{(0,0,0)}$ and $Z(0,0) = \left\{\left[\overline{(0,0,0)}\right]\right\}$. When $n = 1$, we have $Z(0,1) = \left\{\left[\overline{(0,1,0)}\right], \left[\overline{(0,1,1)}\right], \left[\overline{(0,1,2)}\right]\right\}$ by the same method as in the proof of Corollary 4.1.8. When $n = 2$, we have

$$Z(0,2) = \left\{\left[\overline{(0,2,0)}\right], \left[\overline{(0,2,1)}\right], \left[\overline{(0,2,2)}\right], \left[\overline{(0,2,p)}\right], \left[\overline{(0,2,p+1)}\right]\right\}. \qquad (4.3)$$

In fact, if $\left[\overline{(0,2,x)}\right] \in Z(0,2)$, then we have $\overline{x} = \overline{0}$ or $\mathrm{ord}_p(\overline{x}) \le 1$. We first investigate the case of $\mathrm{ord}_p(x) = 0$. Then, $\overline{(0,2,x)} \sim \overline{(0,2,x')}$ if and only if

$$0 = \mathrm{ord}_p(x) = \mathrm{ord}_p(x') \;\text{ and }\; \overline{1-x} = \varepsilon\overline{(1-x')} \;\text{ for some } \varepsilon \in \mathbb{Z}_p^\times.$$

By the same method as above, we get

$$\begin{aligned}
\left[\overline{(0,2,1)}\right] &= \{\overline{(0,2,1)}\}, \\
\left[\overline{(0,2,2)}\right] &= \{\overline{(0,2,x)} \mid \overline{x} \neq \overline{0}, \overline{1}\}, \text{ and} \\
\left[\overline{(0,2,p+1)}\right] &= \{\overline{(0,2,x)} \mid \mathrm{ord}_p(x) = 0,\ \overline{-p} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\
&= \{\overline{(0,2,1+x_1p)} \mid 1 \leq x_1 < p\}.
\end{aligned}$$

Next, we investigate the case of $\mathrm{ord}_p(x) = 1$. We suppose $x = p$. Then we have

$$\begin{aligned}
\left[\overline{(0,2,p)}\right] &= \{\overline{(0,2,x)} \mid \mathrm{ord}_p(x) = 1,\ \overline{1-p} = \varepsilon\overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\
&= \{\overline{(0,2,x_1p)} \mid 1 \leq x_1 < p\}.
\end{aligned}$$

Thus we get (4.3).

(II) Suppose $m = 1$. By the inequalities (B) and (C), we have $0 \leq n \leq 3$. If $\mathrm{ord}_p(x) \leq 1$, we have $n \leq 3$ by (B). If $\mathrm{ord}_p(x) > 1$, we have $n \leq 2$ by (C). When $n = 0$, we have $Z(1,0) = \left\{\left[\overline{(1,0,0)}\right]\right\}$. When $n = 1$, we have $Z(1,1) = \left\{\left[\overline{(1,1,0)}\right], \left[\overline{(1,1,1)}\right]\right\}$. If $\left[\overline{(1,1,x)}\right] \in Z(1,1)$, then we have $\overline{x} = 0$ or $\mathrm{ord}_p(\overline{x}) = 0$. We suppose $\mathrm{ord}_p(\overline{x}) = 0$. We have $\ell = \mathrm{ord}_p(x-p) = 0$. This is the case where $\ell + k < n$. By the definition of the equivalence relation, $\overline{(1,1,x)} \sim \overline{(1,1,x')}$ if and only if

$$\overline{x} = \varepsilon\overline{x'} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times.$$

Here we note that $\ell = \mathrm{ord}_E(x'-p) = 0$. Then we have

$$\begin{aligned}
\left[\overline{(1,1,x)}\right] &= \{\overline{(1,1,x')} \mid \overline{x} = \varepsilon\overline{x'} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\} \\
&= \{\overline{(1,1,x')} \mid \overline{x'} \neq \overline{0}\}.
\end{aligned}$$

Therefore we get $Z(1,1) = \left\{\left[\overline{(1,1,0)}\right], \left[\overline{(1,1,1)}\right]\right\}$. When $n = 2$, we have $Z(1,2) = \left\{\left[\overline{(1,2,x)}\right] \,\middle|\, x = 0, p, 2p, \ldots, (p-1)p\right\}$. In fact, we suppose $\left[\overline{(1,2,x)}\right] \in Z(1,2)$. By the inequality (C), we have

$$2 \leq \mathrm{ord}_p\{(\gamma-\alpha) - (\beta-\alpha)p^{-1}x\}.$$

If $\mathrm{ord}_p(x) = 0$, the order of the right-hand side is 1. This is a contradiction. Thus we may assume $1 \le \mathrm{ord}_p(x)$. If $\mathrm{ord}_p(x) \ge 2$, we get $\left[\overline{(1,2,x)}\right] = \{\overline{(1,2,0)}\}$. We suppose $\mathrm{ord}_p(x) = 1$. Then $\overline{(1,2,x)} \sim \overline{(1,2,x')}$ if and only if

$$\overline{x} = \overline{x'}.$$

Here we note that this is the case where $\ell + k \ge n$ since $\ell = \mathrm{ord}_p(x' - p) \ge 1$. For each $x = \varepsilon p$, where $1 \le \varepsilon < p$, we have

$$\left[\overline{(1,2,x)}\right] = \{\overline{(1,2,x)}\}.$$

Thus we get $Z(1,2) = \left\{ \left[\overline{(1,2,x)}\right] \;\middle|\; x = 0, p, 2p, \dots, (p-1)p \right\}$. When $n = 3$, we have $Z(1,3) = \left\{ \left[\overline{(1,3,up)}\right] \right\}$. In fact, we suppose $\left[\overline{(1,3,x)}\right] \in Z(1,3)$. By the same method as in the case of $n = 2$, we get $\mathrm{ord}_p(x) = 1$ and $\overline{(1,3,x)} \sim \overline{(1,3,up)}$ if and only if

$$\overline{x} = \varepsilon \overline{up} \qquad \text{for some } \varepsilon \in 1 + p\mathbb{Z}_p.$$

Here we note that this is the case where $\ell + k < n$ since $\ell = \mathrm{ord}_E(up - p) = 1$. Moreover, by (C) we have

$$x = \left(1 - \frac{vp^3}{\gamma - \alpha}\right) \frac{\gamma - \alpha}{\beta - \alpha} p \qquad \text{for some } v \in \mathbb{Z}_p^\times.$$

Since $1 - \dfrac{vp^3}{\gamma - \alpha} \in 1 + p\mathbb{Z}_p$, we have

$$\left[\overline{(1,3,up)}\right] = \{\overline{(1,3,x)} \mid \overline{x} = \varepsilon \overline{up} \qquad \text{for some } \varepsilon \in 1 + p\mathbb{Z}_p\},$$

where $u = \dfrac{\gamma - \alpha}{\beta - \alpha}$. Thus we get $Z(1,3) = \left\{ \left[\overline{(1,3,up)}\right] \right\}$.

(III) Suppose $m = 2$. By the same method as (I) and (II), we get

$$\begin{aligned}
Z(2,0) &= \left\{ \left[\overline{(2,0,0)}\right] \right\}, Z(2,1) = \left\{ \left[\overline{(2,1,0)}\right] \right\}, \\
Z(2,2) &= \left\{ \left[\overline{(2,2,0)}\right] \right\}, Z(2,3) = \left\{ \left[\overline{(2,3,up^2)}\right] \right\}, \\
\text{and } Z(2,4) &= \left\{ \left[\overline{(2,4,up^2)}\right] \right\}.
\end{aligned}$$

Thus we complete the proof. $\square$

4.2 Proof of Theorem 1

For any $\xi \in \Lambda_E$, we define a map $\Pi_\xi = \Pi_\xi^M : M \longrightarrow M$ by $\Pi_\xi(y) = \xi y$.

Lemma 4.2.1. *Put $q = \sharp(\mathcal{O}_E/(\pi))$ and $M = M(m,n,x)$. Then we have*

$$\begin{aligned}
\sharp(\mathrm{Ker}(\Pi^N_{(T-\alpha)})/\mathrm{Im}(\Pi^N_{(T-\beta)})) &= q^{\{\mathrm{ord}_E(\alpha-\beta)-m\}} \text{ and} \\
\sharp(\mathrm{Ker}(\Pi^M_{(T-\gamma)})/\mathrm{Im}(\Pi^M_{(T-\alpha)(T-\beta)})) &= q^{\{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)-n\}},
\end{aligned}$$

where $N = \mathrm{Im}(\Pi_{(T-\gamma)})$.

Proof. We first compute $\mathrm{Ker}(\Pi_{(T-\gamma)})$. For $y \in M = M(m,n,x)$, there exist λ_1, λ_2, and $\lambda_3 \in \mathcal{O}_E$ such that

$$\begin{aligned}
y &= \lambda_1(1,1,1) + \lambda_2(0,\pi^m,x) + \lambda_3(0,0,\pi^n) \\
&= (\lambda_1, \lambda_1 + \lambda_2\pi^m, \lambda_1 + \lambda_2 x + \lambda_3\pi^n).
\end{aligned}$$

Thus we have $\Pi_{(T-\gamma)}(y) = ((\alpha-\gamma)\lambda_1, (\beta-\gamma)(\lambda_1+\lambda_2\pi^m), 0)$. If $y \in \mathrm{Ker}(\Pi_{(T-\gamma)})$, we get $\lambda_1 = 0$ and $\lambda_1 + \lambda_2\pi^m = 0$, since α, β and γ are distinct elements of $\mathcal{O}_E$. Therefore $y = (0,0,\lambda_3\pi^n)$ and $\mathrm{Ker}(\Pi_{(T-\gamma)}) = (0,0,\pi^n\mathcal{O}_E)$. On the other hand, by $y = (\lambda_1, \lambda_1+\lambda_2\pi^m, \lambda_1+\lambda_2 x+\lambda_3\pi^n)$, we have

$$\begin{aligned}
\Pi_{(T-\alpha)(T-\beta)}(y) &= \Pi_{(T-\alpha)}((\alpha-\beta)\lambda_1, 0, (\gamma-\beta)(\lambda_1+\lambda_2 x+\lambda_3\pi^n)) \\
&= (0,0,(\gamma-\alpha)(\gamma-\beta)(\lambda_1+\lambda_2 x+\lambda_3\pi^n)).
\end{aligned}$$

Thus we have $\mathrm{Im}(\Pi_{(T-\alpha)(T-\beta)}) = (0,0,\pi^{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)}\mathcal{O}_E)$ and

$$\begin{aligned}
\sharp(\mathrm{Ker}(\Pi_{(T-\gamma)})/\mathrm{Im}(\Pi_{(T-\alpha)(T-\beta)})) &= \sharp(\pi^n\mathcal{O}_E/\pi^{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)}\mathcal{O}_E) \\
&= q^{\{\mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta)-n\}}.
\end{aligned}$$

Next, we put $N = \mathrm{Im}(\Pi_{(T-\gamma)})$. We have

$$\begin{aligned}
\mathrm{Ker}(\Pi^N_{(T-\alpha)}) &= (\pi^{\mathrm{ord}_E(\alpha-\gamma)+m}\mathcal{O}_E, 0, 0) \text{ and} \\
\mathrm{Im}(\Pi^N_{(T-\beta)}) &= (\pi^{\mathrm{ord}_E(\alpha-\gamma)+\mathrm{ord}_E(\alpha-\beta)}\mathcal{O}_E, 0, 0).
\end{aligned}$$

Therefore we get

$$\sharp(\mathrm{Ker}(\Pi^N_{(T-\alpha)})/\mathrm{Im}(\Pi^N_{(T-\beta)})) = q^{\{\mathrm{ord}_E(\alpha-\beta)-m\}}. \quad \square$$

Corollary 4.2.2. *Let $[M(m,n,x)]_E$ and $[M(m',n',x')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. If $[M(m,n,x)]_E = [M(m',n',x')]_E$, then we have $m = m'$ and $n = n'$.*

Proof. We put $M = M(m,n,x)$ and $M' = M(m',n',x')$. Since $M \cong M'$, we have $N = \mathrm{Im}(\Pi^M_{(T-\gamma)}) \cong \mathrm{Im}(\Pi^{M'}_{(T-\gamma)}) = N'$. Therefore we have

$$\mathrm{Ker}(\Pi^N_{(T-\alpha)})/\mathrm{Im}(\Pi^N_{(T-\beta)}) \cong \mathrm{Ker}(\Pi^{N'}_{(T-\alpha)})/\mathrm{Im}(\Pi^{N'}_{(T-\beta)}).$$

This implies $m = m'$ by Lemma 4.2.1. We get $n = n'$ by the same method. $\square$

By using the canonical isomorphism $\Lambda_E/(T-\alpha) \cong \mathcal{O}_E \ \ (f(T) \longmapsto f(\alpha))$, we define an isomorphism

$$\iota : \mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \longrightarrow \mathcal{O}_E^{\oplus 3}$$

by $(f_1(T), f_2(T), f_3(T)) \longmapsto (f_1(\alpha), f_2(\beta), f_3(\gamma))$. Then ι induces an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} E^{\oplus 3}$$

such that $(f_1(T),\ f_2(T), f_3(T)) \otimes y \longmapsto (f_1(\alpha)y, f_2(\beta)y, f_3(\gamma)y)$.

Proposition 4.2.3. *Let $[M(m,n,x)]_E$ and $[M(m,n,x')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. Put $M = M(m,n,x)$ and $M' = M(m,n,x')$. Let $g : M \longrightarrow M'$ be a Λ_E-isomorphism. Define an E-linear map F_A by the following commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ {\scriptstyle \varphi\otimes 1}\downarrow & & \downarrow{\scriptstyle \varphi'\otimes 1} \\ \mathcal{E}\otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E}\otimes_{\mathcal{O}_E} E \\ {\scriptstyle \iota\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota\otimes 1} \\ E^{\oplus 3} & \xrightarrow[F_A]{} & E^{\oplus 3}. \end{array}$$

In the diagram, φ and φ' *are natural inclusions defined by (4.1). When we take the standard basis of $E^{\oplus 3}$, F_A corresponds to a diagonal matrix*

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

for some $a_1, a_2,$ and $a_3 \in \mathcal{O}_E^\times$.

Proof. Consider the map $\Pi_T : M \longrightarrow M$. Then Π_T induces a map $F_B : E^{\oplus 3} \longrightarrow E^{\oplus 3}$ and the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Pi_T} & M \\ {\scriptstyle \varphi\otimes 1}\downarrow & & \downarrow{\scriptstyle \varphi\otimes 1} \\ \mathcal{E}\otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E}\otimes_{\mathcal{O}_E} E \\ {\scriptstyle \iota\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota\otimes 1} \\ E^{\oplus 3} & \xrightarrow[F_B]{} & E^{\oplus 3}. \end{array}$$

Thus we get

$$(\natural) \qquad F_B \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x) = (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx)$$

for $x \in M$. Let A be the matrix corresponding to F_A. By the diagram above, we get

$$(\sharp) \qquad F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx) = (\iota \otimes 1) \circ (\varphi' \otimes 1)(g(Tx)).$$

By $(\natural)$ and the diagrams, the left-hand side of $(\sharp)$ is

$$F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx) = F_A \circ F_B \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x).$$

The right-hand side of $(\sharp)$ is

$$\begin{aligned} (\iota \otimes 1) \circ (\varphi' \otimes 1)(Tg(x)) &= F_B \circ (\iota \otimes 1) \circ (\varphi' \otimes 1)(g(x)) \\ &= F_B \circ F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x). \end{aligned}$$

Since this holds for every $x \in M$, we have $F_A \circ F_B = F_B \circ F_A$. If we take the standard basis of $E^{\oplus 3}$, then F_B corresponds to the matrix

$$B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Therefore we have

$$A\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} A.$$

Since α, β, and γ are distinct elements, we get

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

with a_1, a_2, and $a_3 \in E$. Since $g((1,1,1)) = (a_1, a_2, a_3) \in M'$, we have a_1, a_2, and $a_3 \in \mathcal{O}_E$. Furthermore, by the same argument for g^{-1}, we have a_1^{-1}, a_2^{-1}, and $a_3^{-1} \in \mathcal{O}_E$. Hence we get a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$. $\square$

By the commutativity of the diagram, we obtain the following

Corollary 4.2.4. *Suppose that M, F_A, ι, φ and φ' are the same as in Proposition 4.2.3. Then we have*

$$\langle (F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M) \rangle_{\mathcal{O}_E} = \langle (\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) \rangle_{\mathcal{O}_E}. \quad \square$$

Proposition 4.2.5. *Let $[M(m,n,x)]_E$ and $[M(m,n,x')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. Then the following statements are equivalent:*

(i) *We have $M(m,n,x) \cong M(m,n,x')$ as Λ_E-modules.*

(ii) *There exist a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying*

$$\mathrm{ord}_E(a_2 - a_1) \geq m, \tag{4.4}$$

$$\mathrm{ord}_E(a_3 x - a_2 x') \geq n, \text{ and} \tag{4.5}$$

$$\mathrm{ord}_E\{a_3 - a_1 - (a_2 - a_1)\pi^{-m} x'\} \geq n. \tag{4.6}$$

Proof. We put $M = M(m,n,x)$ and $M' = M(m,n,x')$. We first prove that (i) implies (ii). If M is isomorphic to M' as Λ_E-modules, there exists a Λ_E-isomorphism $g : M \xrightarrow{\sim} M'$. By Proposition 4.2.3, there exists a diagonal matrix

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

with a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ which corresponds to g. We have

$$\begin{aligned}F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M) &= F_A(M(m,n,x)) \\ &= \langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3 x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E}\end{aligned}$$

and

$$\begin{aligned}(\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) &= (\iota \otimes 1) \circ (\varphi' \otimes 1)(M') \\ &= \langle (1,1,1), (0, \pi^m, x'), (0,0,\pi^n) \rangle_{\mathcal{O}_E}.\end{aligned}$$

By Corollary 4.2.4, we get

$$\langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3 x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E} = \langle (1,1,1), (0, \pi^m, x'), (0,0,\pi^n) \rangle_{\mathcal{O}_E}.$$

Since the left-hand side is contained in the right-hand side, we have

$$\begin{aligned}(a_1, a_2, a_3) &= a_1(1,1,1) + (a_2 - a_1)\pi^{-m}(0, \pi^m, x') \\ &\quad + \{a_3 - a_1 - (a_2 - a_1)\pi^{-m}x'\}\pi^{-n}(0,0,\pi^n), \\ (0, a_2\pi^m, a_3 x) &= a_2(0, \pi^m, x') + (a_3 x - a_2 x')\pi^{-n}(0,0,\pi^n).\end{aligned}$$

Since these coefficients should belong to $\mathcal{O}_E$, we have (4.4), (4.5), and (4.6). It is easy to prove that (ii) implies (i). □

We can simplify the inequalities (4.4), (4.5), and (4.6). The following is easy to see.

Lemma 4.2.6. *The following conditions are equivalent:*

(i) *There exist a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying* (4.4), (4.5), *and* (4.6).

(ii) *There exist a_1 and $a_2 \in \mathcal{O}_E^\times$ satisfying*

$$\operatorname{ord}_E(a_2 - a_1) \geq m, \tag{4.7}$$

$$\operatorname{ord}_E(x - a_2 x') \geq n, \text{ and} \tag{4.8}$$

$$\operatorname{ord}_E\{1 - a_1 - (a_2 - a_1)\pi^{-m}x'\} \geq n. \tag{4.9}$$

Corollary 4.2.7. *Let $[M(m,n,x)]_E$ and $[M(m,n,x')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. Assume that $\operatorname{ord}_E(x) < n$. If $[M(m,n,x)]_E = [M(m,n,x')]_E$, then we have $\operatorname{ord}_E(x) = \operatorname{ord}_E(x')$.*

Proof. If $\mathrm{ord}_E(x) < \mathrm{ord}_E(x')$, we have $n \leq \mathrm{ord}_E(a_3x - a_2x') = \mathrm{ord}_E(x)$ by the inequality (4.5). This contradicts the assumption $\mathrm{ord}_E(x) < n$. If we assume $\mathrm{ord}_E(x) > \mathrm{ord}_E(x')$, we get the same contradiction. Therefore we obtain $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. □

To prove Theorem 1, we prepare a lemma and some propositions.

Proposition 4.2.8. *The following statements are equivalent:*

(i) *We have* $M(m, n, x) \cong M(m, n, 0)$ *as* Λ_E*-modules.*

(ii) *We have* $\overline{(m, n, x)} \sim \overline{(m, n, 0)}$.

Proof. We show that (i) implies (ii). If $\mathrm{ord}_E(x) < n$, we have $\mathrm{ord}_E(x) = \mathrm{ord}_E(0)$ by Corollary 4.2.7, which is a contradiction. Hence we have $\mathrm{ord}_E(x) \geq n$ and $M(m, n, x) = M(m, n, 0)$. Then we have $\overline{(m, n, x)} = \overline{(m, n, 0)}$ by Remark 4.1.4 (i). □

Put $M = M(m, n, x)$ and $M' = M(m, n, x')$. Now we suppose that $x' \neq 0$ and the existence of $a_1, a_2 \in \mathcal{O}_E^\times$ satisfying (4.7), (4.8), and (4.9). By Proposition 4.2.5 and Lemma 4.2.6, M is isomorphic to M'. From the inequalities (4.7) and (4.8), there are $s, v \in \mathcal{O}_E$ such that $a_2 - a_1 = \pi^m s$ and $x - a_2x' = \pi^n v$. Thus we have

$$a_1 = \frac{x}{x'} - \frac{\pi^n}{x'}v - \pi^m s, \tag{4.10}$$

$$a_2 = \pi^m s + a_1 = \frac{x}{x'} - \frac{\pi^n}{x'}v. \tag{4.11}$$

By the inequality (4.9), we get

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x \tag{4.12}$$

for some $w \in \mathcal{O}_E$.

Lemma 4.2.9. *Suppose that* $m, n \neq 0$, *and* $\mathrm{ord}_E(x) < n$. *The following two statements are equivalent:*

(i) *There exist* $a_1, a_2 \in \mathcal{O}_E^\times$ *satisfying* (4.7), (4.8), *and* (4.9).

(ii) *We have* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ and *there exist* s, v, *and* $w \in \mathcal{O}_E$ *satisfying* (4.12).

Proof. We have already proved that (i) implies (ii). We prove that (ii) implies (i). We put a_1 and a_2 by the equalities (4.10) and (4.11). Since $m, n \neq 0$ and

$\operatorname{ord}_E(x) = \operatorname{ord}_E(x') < n$, we have $a_1, a_2 \in \mathcal{O}_E^\times$. Then we have

$$a_2 - a_1 = \pi^m s, \quad x - a_2 x' = \pi^n v$$

and

$$1 - a_1 - (a_2 - a_1)\pi^{-m} x' = \pi^n w.$$

Therefore we get (4.7), (4.8), and (4.9). $\square$

Proposition 4.2.10. *Suppose that $m, n \neq 0$, and $\operatorname{ord}_E(x) < n$. Then the following statements are equivalent:*

(i) *We have $M(m,n,x) \cong M(m,n,x')$ as Λ_E-modules.*

(ii) *We have $\overline{(m,n,x)} \sim \overline{(m,n,x')}$.*

Proof. We first suppose that $M(m,n,x)$ is isomorphic to $M(m,n,x')$ as Λ_E-modules. Put $k = \operatorname{ord}_E(x)$ and $\ell = \operatorname{ord}_E(x' - \pi^m)$. By Lemma 4.2.9, we have $\operatorname{ord}_E(x) = \operatorname{ord}_E(x') = k$ and there exist s, v, and $w \in \mathcal{O}_E$ such that

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x.$$

We put $\varepsilon = x x'^{-1} \in \mathcal{O}_E^\times$. Dividing the equality by x', we have

$$(x' - \pi^m)s - \frac{\pi^n}{x'} v + \pi^n w = 1 - \varepsilon.$$

Thus we have

$$\begin{aligned} \operatorname{ord}_E(1-\varepsilon) &\geq \min\left\{ \operatorname{ord}_E((x' - \pi^m)s), \operatorname{ord}_E\left(-\frac{\pi^n}{x'} v\right), \operatorname{ord}_E(\pi^n w) \right\} \\ &\geq \min\{\ell, n-k, n\} = \min\{\ell, n-k\}. \end{aligned}$$

In the case where $\ell \geq n-k$, we have $\operatorname{ord}_E(1-\varepsilon) \geq n-k$. Thus we get $\overline{x} = \overline{\varepsilon x'} = \overline{x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$. Therefore we have $\overline{(m,n,x)} \sim \overline{(m,n,x')}$. In the case where $\ell < n-k$, we have $\operatorname{ord}_E(1-\varepsilon) \geq \ell$ and $\overline{x} = \overline{\varepsilon x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$. Therefore we get $\overline{(m,n,x)} \sim \overline{(m,n,x')}$. Conversely we assume that $\overline{(m,n,x)} \sim \overline{(m,n,x')}$. In the case where $\ell \geq n-k$, we have $\overline{x} = \overline{x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$ and $(x'-x)/\pi^n \in \mathcal{O}_E$. Put $s = w = 0$ and $v = (x - x')/\pi^n \in \mathcal{O}_E$. Then we get

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x' w = x' - x.$$

By Lemma 4.2.9, M and M' are isomorphic as Λ_E-modules. In the case where $\ell < n-k$, we have $\overline{x} = \varepsilon\overline{x'}$ in $\mathcal{O}_E/\pi^n\mathcal{O}_E$ for some $\varepsilon \in 1+\pi^\ell\mathcal{O}_E$. Since $\mathrm{ord}_E(1-\varepsilon) \geq \ell$, we have $(1-\varepsilon)/(x'-\pi^m) \in \mathcal{O}_E$. Put $v = w = 0$ and $s = (1-\varepsilon)/(x'-\pi^m) \in \mathcal{O}_E$. Then we get

$$x'(x'-\pi^m)s - \pi^n v + \pi^n x' w = x' - \varepsilon x'.$$

By Lemma 4.2.9, we get $M(m,n,x) = M(m,n,\varepsilon x') \cong M(m,n,x')$. $\square$

The following propositions treat the case of $m = 0$ and that of $n = 0$.

Proposition 4.2.11. *Suppose that $m = 0, n \neq 0$, and $\mathrm{ord}_E(x) < n$. Then the following statements are equivalent:*

(i) *We have $M(0,n,x) \cong M(0,n,x')$ as Λ_E-modules.*

(ii) *We have $\overline{(0,n,x)} \sim \overline{(0,n,x')}$.*

Proof. Suppose that $M(0,n,x)$ is isomorphic to $M(0,n,x')$ as Λ_E-modules. By Proposition 4.2.5 and Lemma 4.2.6, there exist a_1 and $a_2 \in \mathcal{O}_E^\times$ satisfying (4.8) and (4.9). By the inequality (4.8), we have $\overline{x} = a_2\overline{x'}$. By the inequality (4.9), we have $\overline{1-a_2x'} = \overline{a_1(1-x')}$. Therefore we get

$$\mathrm{ord}_E(x) = \mathrm{ord}_E(x') \text{ and } \overline{1-x} = a_1\overline{(1-x')}.$$

Thus we get $\overline{(0,n,x)} \sim \overline{(0,n,x')}$. Conversely we suppose that $\overline{(0,n,x)} \sim \overline{(0,n,x')}$. There exists $a_1 \in \mathcal{O}_E^\times$ such that $\overline{1-x} = a_1\overline{(1-x')}$. Put $a_2 = x/x'$. Then we have (4.8) and (4.9). Indeed, we have $1-a_1-(a_2-a_1)\pi^{-m}\overline{x'} = 1-a_1-(a_2-a_1)\overline{x'} = \overline{0}$. By Proposition 4.2.5 and Lemma 4.2.6, $M(0,n,x)$ and $M(0,n,x')$ are isomorphic as Λ_E-modules. $\square$

Proposition 4.2.12. *Suppose that $n = 0$. The following statements are equivalent:*

(i) *We have $M(m,0,x) \cong M(m,0,x')$ as Λ_E-modules.*

(ii) *We have $\overline{(m,0,x)} \sim \overline{(m',0,x')}$.*

Proof. By Remark 4.1.4 (i), we have $M(m,0,x) = M(m,0,x') = M(m,0,0)$ and $\overline{(m,0,x)} = \overline{(m,0,x')} = \overline{(m,0,0)}$. $\square$

Now we can prove Theorem 1.

Proof of Theorem 1. For $[M(m,n,x)]_E \in \mathcal{M}^E_{f(T)}$, we may assume that $x = 0$ or $\mathrm{ord}_E(x) < n$ holds by Remark 4.1.4 (i). At first, Φ is well-defined by Corollary 4.2.2 and Propositions 4.2.8, 4.2.10, 4.2.11, and 4.2.12. The surjectivity follows from Proposition 4.1.3 and Remark 4.1.4. On the other hand, Φ is injective by Propositions 4.2.8, 4.2.10, 4.2.11, and 4.2.12. $\square$

Chapter 5

Proof of Theorem 2

In this chapter, we give a proof of Theorem 2. To state the theorem, we define the notion of "admissibility" and the describe statements (I) - (XII) in Section 4.1 and 4.2.

5.1 Some results

Let E be a finite extension over the field $\mathbb{Q}_p$ of p-adic numbers. Let $\mathcal{O}_E$, π, and ord_E be the ring of integers in E, a prime element, and the normalized additive valuation on E such that $\mathrm{ord}_E(\pi) = 1$, respectively. We put $\Lambda_E := \mathcal{O}_E[[T]]$, the ring of power series over $\mathcal{O}_E$.

In this chapter, we consider

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma)(T - \delta), \tag{5.1}$$

where α, β, γ, and δ are distinct elements of $\pi\mathcal{O}_E$. As in the previous chapter, by using the canonical isomorphism $\Lambda_E/(T - \alpha) \cong \mathcal{O}_E \quad (f(T) \longmapsto f(\alpha))$, we define an isomorphism

$$\iota : \mathcal{E} = \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta) \oplus \Lambda_E/(T - \gamma) \oplus \Lambda_E/(T - \delta) \longrightarrow \mathcal{O}_E^{\oplus 4}$$

by $(f_1(T), f_2(T), f_3(T), f_4(T)) \longmapsto (f_1(\alpha), f_2(\beta), f_3(\gamma), f_4(\delta))$. Let M be an $\mathcal{O}_E$-submodule of $\mathcal{E}$ with $\mathrm{rank}(M) = 4$.

$$M = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4), (d_1, d_2, d_3, d_4) \rangle_{\mathcal{O}_E} \ \subset \ \mathcal{E}.$$

In the same way as in the previous chapter, we have

$$M = \langle (\pi^s, a, b, c), (0, \pi^t, d, e), (0, 0, \pi^u, f), (0, 0, 0, \pi^v) \rangle_{\mathcal{O}_E}$$

for some non-negative integers s, t, u, v, and $a, b, c, d, e, f \in \mathcal{O}_E$. Further, by Lemma 4.1.2, we may assume that a Λ_E-module M is of the form

$$M = \langle (1, 1, 1, 1), (0, \pi^\ell, x, y), (0, 0, \pi^m, z), (0, 0, 0, \pi^n) \rangle_{\mathcal{O}_E} \ \subset \ \mathcal{E}$$

for some non-negative integers ℓ, m, n, and $x, y, z \in \mathcal{O}_E$. We define an $\mathcal{O}_E$-module M by

$$M(\ell, m, n; x, y, z) := \langle (1, 1, 1, 1), (0, \pi^\ell, x, y), (0, 0, \pi^m, z), (0, 0, 0, \pi^n) \rangle_{\mathcal{O}_E} \subset \mathcal{E},$$

where $\ell, m,$ and n are non-negative integers. We can prove the next lemma by the same method as Lemma 4.1.1

Lemma 5.1.1. *The following two statements are equivalent:*
(i) *The $\mathcal{O}_E$-module $M(\ell, m, n; x, y, z)$ is a Λ_E-module.*
(ii) *The integers $\ell, m, n,$ and $x, y, z \in \mathcal{O}_E$ satisfy*

$$\begin{cases} \text{(a)} & \ell & \leq \mathrm{ord}_E(\beta - \alpha), \\ \text{(b)} & m & \leq \mathrm{ord}_E\{(\gamma - \alpha) - (\beta - \alpha)\pi^{-\ell} x\}, \\ \text{(c)} & n & \leq \mathrm{ord}_E\left[(\delta - \alpha) - (\beta - \alpha)\pi^{-\ell} y - \{(\gamma - \alpha) - (\beta - \alpha)\pi^{-\ell} x\}\pi^{-m} z\right], \\ \text{(d)} & m & \leq \mathrm{ord}_E(\gamma - \beta) + \mathrm{ord}_E(x), \\ \text{(e)} & n & \leq \mathrm{ord}_E\{(\delta - \beta) y - (\gamma - \beta) x \pi^{-m} z\}, \text{ and} \\ \text{(f)} & n & \leq \mathrm{ord}_E(\delta - \gamma) + \mathrm{ord}_E(z). \end{cases}$$

Proposition 5.1.2. *Let $[M(\ell, m, n; x, y, z)]_E$ and $[M(\ell', m', n'; x', y', z')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. If $[M(\ell, m, n; x, y, z)]_E = [M(\ell', m', n'; x', y', z')]_E$, then we have $\ell = \ell'$, $m = m'$, and $n = n'$.*

Proof. We put $M = M(\ell, m, n; x, y, z)$ and $M' = M(\ell', m', n'; x', y', z')$. For every Λ-module M and $\xi \in \Lambda_E$, we define a map $\Pi_\xi = \Pi^M_\xi : M \longrightarrow M$ by $\Pi_\xi(y) = \xi y$. Then we have

$$\begin{aligned} \sharp\left(\mathrm{Ker}\left(\Pi^M_{(T-\alpha)}\right) / \mathrm{Im}\left(\Pi^M_{(T-\beta)}\right)\right) &= q^{\{\mathrm{ord}_E(\delta-\alpha) + \mathrm{ord}_E(\delta-\beta) + \mathrm{ord}_E(\delta-\gamma) - n\}}, \\ \sharp\left(\mathrm{Ker}\left(\Pi^M_{(T-\gamma)}\right) / \mathrm{Im}\left(\Pi^M_{(T-\alpha)(T-\beta)(T-\delta)}\right)\right) &= q^{\{\mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta) + \mathrm{ord}_E(\gamma-\delta) - m\}}. \end{aligned}$$

We put $N = \mathrm{Im}\left(\Pi^M_{(T-\gamma)(T-\delta)}\right)$. Then we have

$$\sharp\left(\mathrm{Ker}\left(\Pi^N_{(T-\beta)}\right)/\mathrm{Im}\left(\Pi^N_{(T-\alpha)}\right)\right) \;=\; q^{\{\mathrm{ord}_E(\beta-\alpha)-\ell\}}.$$

Since $M \cong M'$, we have $\mathrm{Ker}(\Pi^M_{(T-\gamma)}) \cong \mathrm{Ker}(\Pi^{M'}_{(T-\gamma)})$ and $\mathrm{Im}(\Pi^M_{(T-\alpha)(T-\beta)(T-\delta)}) \cong \mathrm{Im}(\Pi^{M'}_{(T-\alpha)(T-\beta)(T-\delta)})$. This implies $m = m'$. We get $\ell = \ell'$ and $n = n'$ by the same method. $\square$

For $M = M(\ell, m, n; x, y, z)$, we put $e_1 = (1,1,1,1)$, $e_2 = (0, \pi^\ell, x, y)$, $e_3 = (0,0,\pi^m, z)$, and $e_4 = (0,0,0,\pi^n)$. For $M' = M(\ell, m, n; x', y', z')$, we also put ${e_1}' = (1,1,1,1)$, ${e_2}' = (0,\pi^\ell, x', y')$, ${e_3}' = (0,0,\pi^m, z')$, ${e_4}' = (0,0,0,\pi^n)$ and

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pi^\ell & 0 & 0 \\ 1 & x & \pi^m & 0 \\ 1 & y & z & \pi^n \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \pi^\ell & 0 & 0 \\ 1 & x' & \pi^m & 0 \\ 1 & y' & z' & \pi^n \end{pmatrix}.$$

The matrix G is the transition matrix from the bases e_1, e_2, e_3, and e_4 to the bases $(1,0,0,0), (0,1,0,0), (0,0,1,0)$, and $(0,0,0,1)$. The matrix G' is the transition matrix from the basis ${e_1}', {e_2}', {e_3}'$, and ${e_4}'$ to the basis $(1,0,0,0), (0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$. Let $g : M \longrightarrow M'$ be a Λ_E-isomorphism. Since we have $g(Tx) = Tg(x)$ for $x \in M$ and $T(1,0,0,0) = (\alpha,0,0,0), T(0,1,0,0) = (0,\beta,0,0), T(0,0,1,0) = (0,0,\gamma,0)$, $T(0,0,0,1) = (0,0,0,\delta)$, we can prove the next proposition by the same method as Proposition 4.2.3.

Proposition 5.1.3. *Let $M = M(\ell, m, n;\ x, y, z)$ and $M' = M(\ell, m, n; x', y', z')$ be Λ_E-modules satisfying $[M]_E, [M']_E \in \mathcal{M}^E_{f(T)}$. Assume that $g : M \longrightarrow M'$ is a Λ_E-isomorphism. Let $\{e_1, e_2, e_3, e_4\}$ and $\{{e_1}', {e_2}', {e_3}', {e_4}'\}$ be the bases of M and M', respectively. Let A be the matrix corresponding to g with respect to the bases $\{e_1, e_2, e_3, e_4\}$ and $\{{e_1}', {e_2}', {e_3}', {e_4}'\}$. Then we have*

$$G'AG^{-1} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

for some $a_1, a_2, a_3,$ and $a_4 \in \mathcal{O}_E^\times$.

Put $A = (a_{ij})$, $1 \leq i, j \leq 4$. Using this proposition, we have $a_{ii} = a_i$ for $i = 1, 2, 3, 4$ and $a_{ij} = 0$ for $i < j$. Since we have $a_{ij} \in \mathcal{O}_E$ for $i > j$, we get the following proposition (cf. [12, Proposition 4.5 and Lemma 4.6] and [4, Lemma 2.1.2]). We note that we write a_1, a_2, and a_3 for $\frac{a_1}{a_4}, \frac{a_2}{a_4}$, and $\frac{a_3}{a_4}$, respectively, in the following

Proposition 5.1.4. *Let $[M(\ell, m, n; x, y, z)]_E$ and $[M(\ell, m, n; x', y', z')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. Then the following statements are equivalent:*

(i) *We have $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ as Λ_E-modules.*

(ii) *There exist a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying*

$$a_2 \equiv a_1 \mod \pi^\ell, \tag{5.2}$$

$$a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x' \equiv 0 \mod \pi^m, \tag{5.3}$$

$$\begin{aligned} &1 - a_1 - (a_2 - a_1)\pi^{-\ell}y' \\ &- \left\{a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x'\right\}\pi^{-m}z' \equiv 0 \mod \pi^n, \end{aligned} \tag{5.4}$$

$$a_3 x \equiv a_2 x' \mod \pi^m, \tag{5.5}$$

$$y - a_2 y' - (a_3 x - a_2 x')\pi^{-m}z' \equiv 0 \mod \pi^n, \text{ and} \tag{5.6}$$

$$z \equiv a_3 z' \mod \pi^n. \tag{5.7}$$

Let R be a set of complete representatives in $\mathcal{O}_E$ of the elements of the residue field $\mathcal{O}_E/(\pi)$. Namely, R is a subset of $\mathcal{O}_E$ and each class of $\mathcal{O}_E/(\pi)$ contains a unique element in R. We assume that R contains 0, 1 and fix this set R of complete representatives. For non-negative integers k, we set

$$\begin{aligned} S_k &= \left\{ \sum_{i=0}^{k-1} a_i \pi^i \;\middle|\; a_i \in R \text{ for } i = 0, 1, \ldots, k-1 \right\} \quad \text{if } k > 0, \\ S_0 &= \{0\} \quad \text{if } k = 0. \end{aligned}$$

Definition 5.1.5. Let $(\ell, m, n; x, y, z)$ be a 6-tuple with $\ell, m, n \in \mathbb{Z}_{\geq 0}$ and $x, y, z \in \mathcal{O}_E$ satisfying the conditions (a), (b), ..., and (f) in Lemma 5.1.1. We call a 6-tuple $(\ell, m, n; x, y, z)$ admissible if $x \in S_m$ and $y, z \in S_n$.

Proposition 5.1.6. *(1) If a* 6*-tuple* $(\ell, m, n; x, y, z)$ *is admissible, then* $M(\ell, m, n; x, y, z)$ *becomes a* Λ_E*-module and* $[M(\ell, m, n; x, y, z)] \in \mathcal{M}^E_{f(T)}$.
(2) Suppose that $[M] \in \mathcal{M}^E_{f(T)}$*. Then there is an admissible* 6*-tuple* $(\ell, m, n; x, y, z)$ *such that* $[M] = [M(\ell, m, n; x, y, z)]$.

Proof. Part (1) follows from Lemma 5.1.1.

Next, we prove part (2). We suppose that $[M] \in \mathcal{M}^E_{f(T)}$. Since we explained before Lemma 5.1.1, we can take a module $M(\ell, m, n; x', y', z')$ such that $[M] = [M(\ell, m, n; x', y', z')]$, where $\ell, m, n \geq 0$ and $x', y', z' \in \mathcal{O}_E$. We choose $x \in S_m$ and $y, z \in S_n$ satisfying $x' \equiv x \mod \pi^m$, $y' + (x - x')\pi^{-m}z' \equiv y \mod \pi^n$ and $z' \equiv z \mod \pi^n$. Then $(\ell, m, n; x, y, z)$ is admissible. Put $a_1 = a_2 = a_3 = 1$. Then equations (5.2), (5.3), (5.4), (5.5), (5.6), and (5.7) hold. By Proposition 5.1.4, we have $[M] = [M(\ell, m, n; x', y', z')] = [M(\ell, m, n; x, y, z)]$. Thus we get (2). □

5.2 The statements (I) - (XII)

In this section, we describe the statements (I), (II), …, and (XII) in Theorem 2. For two 6-tuples $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$, we set the following quantities. If $x' \neq 0, z' \neq 0$, we put

$$\begin{aligned} A &= \frac{\pi^n}{z'}\frac{x}{x'}y', \quad B = \frac{\pi^m}{x'}y' - z', \\ C &= -y + \frac{z}{z'}\frac{x}{x'}y', \quad D = x' - y', \\ E &= \pi^m - z', \qquad F = \pi^\ell - x' + (x' - y')\Big(1 - \frac{x}{x'}\Big), \\ \text{and } G &= -\pi^m + (\pi^m - z')\Big(1 - \frac{x}{x'}\Big). \end{aligned}$$

(I) If $x' \neq 0, z' \neq 0$ and $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(B)$, then either the following (I-1), (I-2), or (I-3) hold.

(I-1) All of the following (I-1-a), (I-1-b), (I-1-c), and (I-1-d) are satisfied:

$$\text{(I-1-a)} \qquad \min\left\{\operatorname{ord}_E\left(\frac{\pi^m}{x'}\right), \operatorname{ord}_E(F), \operatorname{ord}_E(G)\right\} = \operatorname{ord}_E\left(\frac{\pi^m}{x'}\right),$$

$$\text{(I-1-b)} \qquad \operatorname{ord}_E(A) \leq \operatorname{ord}_E(C),$$

$$\text{(I-1-c)} \qquad x = x',$$

$$\text{(I-1-d)} \qquad \min\left\{\operatorname{ord}_E\left(D + \frac{x'}{z'}A^{-1}BF\pi^{n-m}\right), \operatorname{ord}_E\left(E + \frac{x'}{z'}A^{-1}BG\pi^{n-m}\right),\right.$$
$$\left.\operatorname{ord}_E\left(\frac{\pi^n}{y'}\right)\right\} \leq \operatorname{ord}_E\left(1 - \frac{y}{y'}\right).$$

(I-2) All of the following (I-2-a), (I-2-b), (I-2-c), and (I-2-d) are satisfied:

$$\text{(I-2-a)} \qquad \min\left\{\operatorname{ord}_E\left(\frac{\pi^m}{x'}\right), \operatorname{ord}_E(F), \operatorname{ord}_E(G)\right\} = \operatorname{ord}_E(F),$$

$$\text{(I-2-b)} \qquad \operatorname{ord}_E(A) \leq \operatorname{ord}_E(C),$$

$$\text{(I-2-c)} \qquad \operatorname{ord}_E(F) \leq \operatorname{ord}_E\left(1 - \frac{x}{x'}\right),$$

$$\text{(I-2-d)} \qquad \min\left\{\operatorname{ord}_E\left(A^{-1}B\frac{\pi^n}{z'} + \frac{\pi^m}{x'}DF^{-1}\right), \operatorname{ord}_E(E - DF^{-1}G),\right.$$
$$\left.\operatorname{ord}_E\left(\frac{\pi^n}{y'}\right)\right\} \leq \operatorname{ord}_E\left(\frac{z}{z'} - 1 - A^{-1}C\frac{\pi^n}{z'} - \left(\frac{x}{x'} - 1\right)DF^{-1}\right).$$

(I-3) All of the following (I-3-a), (I-3-b), (I-3-c), and (I-3-d) are satisfied:

$$\text{(I-3-a)} \qquad \min\left\{\operatorname{ord}_E\left(\frac{\pi^m}{x'}\right), \operatorname{ord}_E(F), \operatorname{ord}_E(G)\right\} = \operatorname{ord}_E(G),$$

$$\text{(I-3-b)} \qquad \operatorname{ord}_E(A) \leq \operatorname{ord}_E(C),$$

$$\text{(I-3-c)} \qquad \operatorname{ord}_E(G) \leq \operatorname{ord}_E\left(1 - \frac{x}{x'}\right),$$

$$\text{(I-3-d)} \qquad \min\left\{\operatorname{ord}_E\left(A^{-1}B\frac{\pi^n}{z'} + \frac{\pi^m}{x'}EG^{-1}\right), \operatorname{ord}_E(D - EFG^{-1}),\right.$$
$$\left.\operatorname{ord}_E\left(\frac{\pi^n}{y'}\right)\right\} \leq \operatorname{ord}_E\left(\frac{z}{z'} - 1 - A^{-1}C\frac{\pi^n}{z'} - \left(\frac{x}{x'} - 1\right)EG^{-1}\right).$$

(II) If $x' \neq 0, z' \neq 0$ and $\operatorname{ord}_E(A) > \operatorname{ord}_E(B)$, then either the following (II-1), (II-2) or (II-3) holds.

(II-1) All of the following (II-1-a), (II-1-b), (II-1-c), and (II-1-d) are satisfied:

$$\text{(II-1-a)} \qquad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(D), \mathrm{ord}_E(E)\right\} = \mathrm{ord}_E\left(\frac{\pi^n}{z'}\right),$$

$$\text{(II-1-b)} \qquad \mathrm{ord}_E(B) \leq \mathrm{ord}_E(C),$$

$$\text{(II-1-c)} \qquad z = z',$$

$$\text{(II-1-d)} \qquad \min\left\{\mathrm{ord}_E\left(F + \frac{z'}{x'}AB^{-1}D\pi^{m-n}\right), \mathrm{ord}_E\left(G + \frac{z'}{x'}AB^{-1}E\pi^{m-n}\right),\right.$$
$$\left.\mathrm{ord}_E\left(\pi^n\Big(1 - \frac{x}{x'}\Big) + z'AB^{-1}\frac{\pi^m}{x'}\right), n + m - \mathrm{ord}_E(Bx')\right\}$$
$$\leq \mathrm{ord}_E\left(\frac{x}{x'} - 1 - B^{-1}C\frac{\pi^m}{x'}\right).$$

(II-2) All of the following (II-2-a), (II-2-b), (II-2-c), and (II-2-d) are satisfied:

$$\text{(II-2-a)} \qquad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(D), \mathrm{ord}_E(E)\right\} = \mathrm{ord}_E(D),$$

$$\text{(II-2-b)} \qquad \mathrm{ord}_E(B) \leq \mathrm{ord}_E(C),$$

$$\text{(II-2-c)} \qquad \mathrm{ord}_E(D) \leq \mathrm{ord}_E\left(1 - \frac{z}{z'}\right),$$

$$\text{(II-2-d)} \qquad \min\left\{\mathrm{ord}_E\left(AB^{-1}\frac{\pi^m}{x'} + \frac{\pi^n}{z'}D^{-1}F\right), \mathrm{ord}_E(G - D^{-1}EF),\right.$$
$$\left.n + \mathrm{ord}_E\left(-\Big(1 - \frac{x}{x'}\Big) + D^{-1}F\right), n + m - \mathrm{ord}_E(Bx')\right\}$$
$$\leq \mathrm{ord}_E\left(\frac{x}{x'} - 1 - B^{-1}C\frac{\pi^m}{x'} - \Big(\frac{z}{z'} - 1\Big)D^{-1}F\right).$$

(II-3) All of the following (II-3-a), (II-3-b), (II-3-c), and (II-3-d) are satisfied:

$$\text{(II-3-a)} \qquad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(D), \mathrm{ord}_E(E)\right\} = \mathrm{ord}_E(E),$$

$$\text{(II-3-b)} \qquad \mathrm{ord}_E(B) \leq \mathrm{ord}_E(C),$$

$$\text{(II-3-c)} \qquad \mathrm{ord}_E(E) \leq \mathrm{ord}_E\left(1 - \frac{z}{z'}\right),$$

$$\text{(II-3-d)} \qquad \min\left\{\mathrm{ord}_E\left(AB^{-1}\frac{\pi^m}{x'} + \frac{\pi^n}{z'}E^{-1}G\right), \mathrm{ord}_E(F - DE^{-1}G),\right.$$
$$\left.n + \mathrm{ord}_E\left(-\Big(1 - \frac{x}{x'}\Big) + E^{-1}G\right), n + m - \mathrm{ord}_E(Bx')\right\}$$
$$\leq \mathrm{ord}_E\left(\frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'} - \Big(\frac{z}{z'} - 1\Big)E^{-1}G\right).$$

(III) If $\ell \neq 0, m \neq 0$, and $n = 0$, then the following (III-a) holds.

$$\text{(III-a)} \qquad \min\left\{\operatorname{ord}_E\left(\frac{\pi^m}{x'}\right), \operatorname{ord}_E(\pi^\ell - x')\right\} \leq \operatorname{ord}_E\left(\frac{x}{x'} - 1\right).$$

(IV) If $\ell \neq 0$ and $m = 0$, then either the following (IV-1), (IV-2), or (IV-3) holds.

(IV-1) All of the following (IV-1-a), (IV-1-b), and (IV-1-c) are satisfied:

$$\begin{aligned}
&\text{(IV-1-a)} && y' \neq 0 \quad \text{and} \quad z' \neq 0,\\
&\text{(IV-1-b)} && \operatorname{ord}_E(y) = \operatorname{ord}_E(y'),\\
&\text{(IV-1-c)} && \min\left\{n, \operatorname{ord}_E\left((1 - z')\frac{\pi^n}{y'}\right), \operatorname{ord}_E(\pi^\ell(1 - z') - y')\right\}\\
& && \leq \operatorname{ord}_E\left(z - 1 - (z' - 1)\frac{y}{y'}\right).
\end{aligned}$$

(IV-2) All of the following (IV-2-a), (IV-2-b), and (IV-2-c) are satisfied:

$$\begin{aligned}
&\text{(IV-2-a)} && y' \neq 0 \quad \text{and} \quad z' = 0,\\
&\text{(IV-2-b)} && \operatorname{ord}_E(y) = \operatorname{ord}_E(y'),\\
&\text{(IV-2-c)} && \min\left\{\operatorname{ord}_E\left(\frac{\pi^n}{y'}\right), \operatorname{ord}_E(\pi^\ell - y')\right\} \leq \operatorname{ord}_E\left(\frac{y}{y'} - 1\right).
\end{aligned}$$

(IV-3) All of the following (IV-3-a) and (IV-3-b) are satisfied:

$$\begin{aligned}
&\text{(IV-3-a)} && y' = y = 0,\\
&\text{(IV-3-b)} && \operatorname{ord}_E(1 - z) = \operatorname{ord}_E(1 - z').
\end{aligned}$$

(V) If $\ell \neq 0, m \neq 0, n \neq 0$, and $z' = 0$, then either the following (V-1), (V-2), (V-3), (V-4), or (V-5) holds.

(V-1) All of the following (V-1-a), (V-1-b), and (V-1-c) are satisfied:

$$
\begin{aligned}
&\text{(V-1-a)} && x' \neq 0, y' \neq 0, \quad \text{and} \\
& && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} = \mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \\
&\text{(V-1-b)} && y = y', \\
&\text{(V-1-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^m}{x}\right), \right. \\
& && \left.\mathrm{ord}_E\left(\pi^\ell - x' - \frac{x'}{x} - (\pi^\ell - y')\left(1 - \frac{x'}{x}\right)\right)\right\} \leq \mathrm{ord}_E\left(1 - \frac{x'}{x}\right).
\end{aligned}
$$

(V-2) All of the following (V-2-a), (V-2-b), (V-2-c), and (V-2-d) are satisfied:

$$
\begin{aligned}
&\text{(V-2-a)} && x' \neq 0, y' \neq 0 \text{ and} \\
& && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} = \mathrm{ord}_E(\pi^\ell - y'), \\
&\text{(V-2-b)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \\
&\text{(V-2-c)} && \mathrm{ord}_E(\pi^\ell - y') \leq \mathrm{ord}_E\left(\frac{y}{y'} - 1\right), \\
&\text{(V-2-d)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\left(1 - \frac{x'}{x}\right) - \frac{\pi^n}{y'}\frac{\pi^\ell - x' - x'x^{-1}}{\pi^\ell - y'}\right)\right. \\
& && \left.\mathrm{ord}_E\left(\frac{\pi^n(\pi^\ell - x' - x'x^{-1})}{\pi^\ell - y'}\right), \mathrm{ord}_E\left(\frac{\pi^m}{x}\right)\right\} \\
& \leq && \mathrm{ord}_E\left(\frac{y}{y'}\left(1 - \frac{x}{x'}\right) - \left(\frac{y}{y'} - 1\right)\frac{\pi^\ell - x' - x'x^{-1}}{\pi^\ell - y'}\right).
\end{aligned}
$$

(V-3) All of the following (V-3-a), (V-3-b), and (V-3-c) are satisfied:

$$
\begin{aligned}
&\text{(V-3-a)} && x' \neq 0 \quad \text{and} \quad y' = 0, \\
&\text{(V-3-b)} && y = 0, \\
&\text{(V-3-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^m}{x}\right), \mathrm{ord}_E(\pi^\ell - x)\right\} \leq \mathrm{ord}_E\left(1 - \frac{x'}{x}\right).
\end{aligned}
$$

(V-4) All of the following (V-4-a), (V-4-b), and (V-4-c) are satisfied:

$$\text{(V-4-a)} \qquad x' = 0 \ \text{ and } \ y' \neq 0,$$
$$\text{(V-4-b)} \qquad \mathrm{ord}_E(y) = \mathrm{ord}_E(y'),$$
$$\text{(V-4-c)} \qquad \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y')\right\} \leq \mathrm{ord}_E\left(1 - \frac{y}{y'}\right).$$

(V-5) The following is satisfied:

$$x' = x = 0 \ \text{ and } \ y = y' = 0.$$

(VI) If $\ell \neq 0, m \neq 0, x' = 0$, and $z' \neq 0$, then either the following (VI-1), (VI-2), (VI-3), or (VI-4) holds.

(VI-1) All of the following (VI-1-a), (VI-1-b) and (VI-1-c) are satisfied:

$$\begin{aligned}
&\text{(VI-1-a)} && y' \neq 0 \ \text{ and} \\
& && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y'), \mathrm{ord}_E(z')\right\} = \mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \\
&\text{(VI-1-b)} && y = y', \\
&\text{(VI-1-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(y'), \mathrm{ord}_E(\pi^m - z')\right\} \leq \mathrm{ord}_E\left(1 - \frac{z}{z'}\right).
\end{aligned}$$

(VI-2) All of the following (VI-2-a), (VI-2-b), (VI-2-c), and (VI-2-d) are satisfied:

$$\begin{aligned}
&\text{(VI-2-a)} && y' \neq 0 \text{ and} \\
& && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y'), \mathrm{ord}_E(z')\right\} = \mathrm{ord}_E(\pi^\ell - y'), \\
&\text{(VI-2-b)} && \mathrm{ord}_E(\pi^\ell - y') \leq \mathrm{ord}_E\left(\frac{y}{y'} - 1\right), \\
&\text{(VI-2-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E\left(\pi^m - \frac{z'\pi^\ell}{\pi^\ell - y'}\right)\right\} \\
&\qquad \leq && \mathrm{ord}_E\left(\frac{z}{z'} - 1 + \frac{y - y'}{\pi^\ell - y'}\right), \\
&\text{(VI-2-d)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y').
\end{aligned}$$

(VI-3) All of the following (VI-3-a), (VI-3-b), (VI-3-c), and (VI-3-d) are satisfied:

$$\begin{aligned}
&\text{(VI-3-a)} && y' \neq 0 \ \text{ and} \\
& && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell - y'), \mathrm{ord}_E(z')\right\} = \mathrm{ord}_E(z'), \\
&\text{(VI-3-b)} && \mathrm{ord}_E(z') \leq \mathrm{ord}_E\left(\frac{y}{y'} - 1\right), \\
&\text{(VI-3-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}\frac{1}{z'}(\pi^m - z')\right), \mathrm{ord}_E\left(\frac{\pi^n}{z'}\right),\right. \\
& && \left.\mathrm{ord}_E\left(-y' + (\pi^\ell - y')\frac{1}{z'}(\pi^m - z')\right)\right\} \\
& \leq && \mathrm{ord}_E\left(\frac{z}{z'} - 1 + \left(\frac{y}{y'} - 1\right)\frac{1}{z'}(\pi^m - z')\right), \\
&\text{(VI-3-d)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y').
\end{aligned}$$

(VI-4) All of the following (VI-4-a) and (VI-4-b) are satisfied:

$$\begin{aligned}
&\text{(VI-4-a)} && y = y' = 0, \\
&\text{(VI-4-b)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(\pi^m - z')\right\} \leq \mathrm{ord}_E\left(\frac{z}{z'} - 1\right).
\end{aligned}$$

(VII) If $\ell = 0, m \neq 0, n \neq 0,\ x' \neq 0, 1,\ y' \neq 0$, and $z' = 0$, then the following (VII-a) and (VII-b) hold.

$$\begin{aligned}
&\text{(VII-a)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \quad \mathrm{ord}_E(1 - y) = \mathrm{ord}_E(1 - y'), \\
&\text{(VII-b)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}(1 - y')\right), \mathrm{ord}_E\left(\frac{\pi^m}{x}(1 - y')\right),\right. \\
& && \left.\mathrm{ord}_E\left(\frac{\pi^m}{1 - x'}(1 - y')\right), n\right\} \\
& \leq && \mathrm{ord}_E\left(1 - y - \frac{y}{y'}\frac{x'}{x}\frac{1 - x}{1 - x'}(1 - y')\right).
\end{aligned}$$

(VIII) If $\ell = 0, m \neq 0, n \neq 0,\ x' \neq 0, 1,\ y' = 0$, and $z' = 0$, then the following holds.

$$\text{(VIII-a)} \quad y = 0.$$

(IX) If $\ell = 0, m \neq 0, n \neq 0$, and $x' = 0$, then either the following (IX-1), (IX-2), (IX-3), or (IX-4) holds.

(IX-1) All of the following (IX-1-a), (IX-1-b), and (IX-1-c) are satisfied:

$$\begin{aligned}
&\text{(IX-1-a)} && y' \neq 0 \text{ and } z' \neq 0,\\
&\text{(IX-1-b)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y'),\\
&\text{(IX-1-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}(1-y')\right), n, \mathrm{ord}_E(\pi^m(1-y') - z')\right\}\\
&&\leq\ & \mathrm{ord}_E\left(y - 1 - \frac{z}{z'}(y'-1)\right).
\end{aligned}$$

(IX-2) All of the following (IX-2-a), and (IX-2-b) are satisfied:

$$\begin{aligned}
&\text{(IX-2-a)} && y' \neq 0 \text{ and } z' = 0,\\
&\text{(IX-2-b)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \quad \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y').
\end{aligned}$$

(IX-3) All of the following (IX-3-a), (IX-3-b), and (IX-3-c) are satisfied:

$$\begin{aligned}
&\text{(IX-3-a)} && y' = 0 \text{ and } z' \neq 0,\\
&\text{(IX-3-b)} && y = 0,\\
&\text{(IX-3-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(\pi^m - z')\right\} \leq \mathrm{ord}_E\left(\frac{z}{z'} - 1\right).
\end{aligned}$$

(IX-4) The following is satisfied:

$$\text{(IX-4-a)} \quad y = y' = 0 \ \text{ and } \ z = z' = 0.$$

(X) If $\ell = 0, m \neq 0, n \neq 0$, and $x' = 1$, then either the following (X-1) or (X-2) holds.

(X-1) All of the following (X-1-a), (X-1-b), and (X-1-c) are satisfied:

$$\begin{aligned}
&\text{(X-1-a)} && z' \neq 0,\\
&\text{(X-1-b)} && \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y'),\\
&\text{(X-1-c)} && \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}y'\right), \mathrm{ord}_E(\pi^m y' - z'), n\right\} \leq \mathrm{ord}_E\left(\frac{z}{z'}y' - y\right).
\end{aligned}$$

(X-2) All of the following (X-2-a) and (X-2-b) are satisfied:

$$\begin{aligned}
&\text{(X-2-a)} && z' = 0,\\
&\text{(X-2-b)} && \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \quad \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y').
\end{aligned}$$

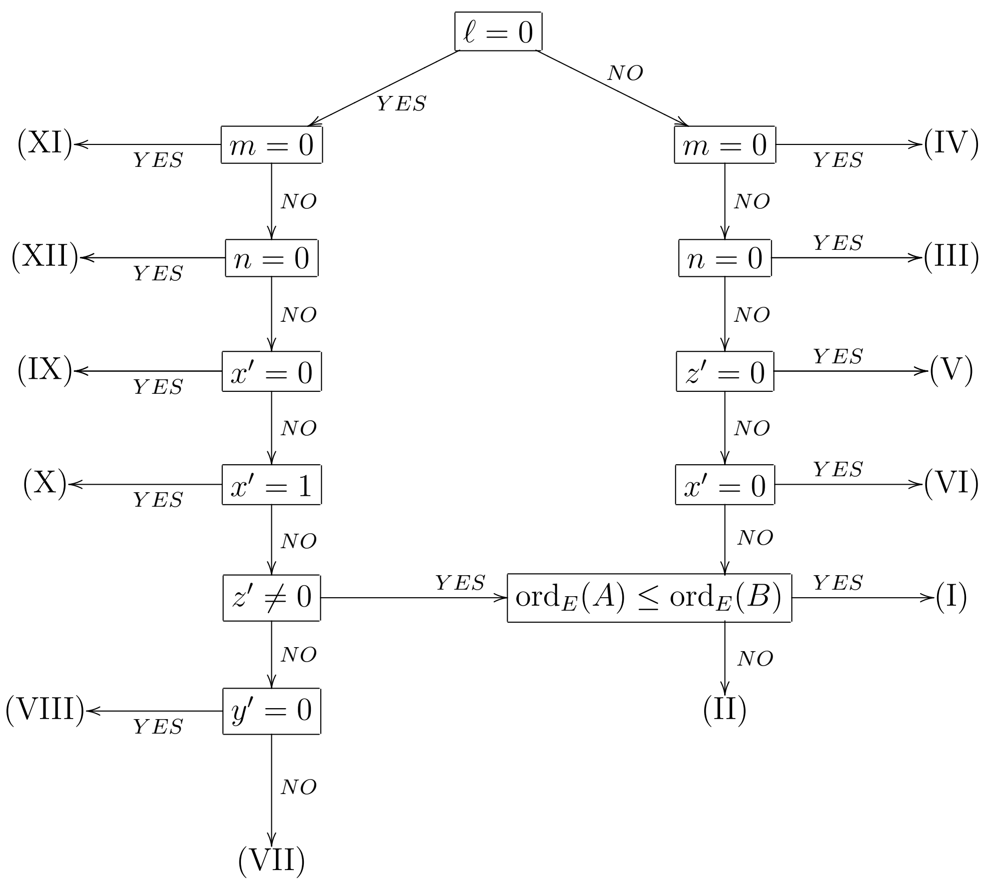
(XI) If $\ell = 0$ and $m = 0$, then the following (XI-a) and (XI-b) hold:

$$\begin{aligned}
&\text{(XI-a)} \qquad \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \\
&\text{(XI-b)} \qquad \mathrm{ord}_E(1 - y - z) = \mathrm{ord}_E(1 - y' - z').
\end{aligned}$$

(XII) $\ell = 0, m \neq 0$, and $n = 0$.

Remark 5.2.1. We can check the statements (I), (II), . . . , (XII) by calculating p-adic valuations of quantities described by using 6-tuples (ℓ, m, n, x, y, z) and (ℓ, m, n, x', y', z'). The following Table 5.1 is the algorithm of Theorem 5.3.1. This table can be used when we check whether two Λ_E-modules $M(\ell, m, n; x, y, z)$ and $M(\ell, m, n; x', y', z')$ are isomorphic.

Table 5.1:



$A = \dfrac{\pi^n}{z'}\dfrac{x}{x'}y'$ and $B = \dfrac{\pi^m}{x'}y' - z'$, which is defined before the statement of (I).

5.3 Proof of Theorem 2

In this section, we prove Theorem 2:

Theorem 5.3.1. *Let $(\ell, m, n;\ x, y, z)$ and $(\ell, m, n;\ x', y', z')$ be two admissible* 6-*tuples. Suppose that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$*, where* ord_E *is the normalized additive valuation on E such that* $\mathrm{ord}_E(\pi) = 1$*. Suppose also that* $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$ *if $\ell = 0$. Then the following statements are equivalent:*

(i) *We have $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ as Λ_E-modules.*

(ii) *One of* (I), (II), . . . , *and* (XII) *holds for $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y'$ $, z')$, where the statements* (I), (II), . . . , *and* (XII).

We fix notation. Let $M_{mn}(E)$ be the set of $m \times n$ matrices with entries in E and $GL_m(\mathcal{O}_E)$ the group of $m \times m$ matrices over $\mathcal{O}_E$ that are invertible. For A and $B \in M_{mn}(E)$, we write $A \sim B$ if there is a matrix $P \in GL_m(\mathcal{O}_E)$ such that $PA = B$. This is an equivalence relation on $M_{mn}(E)$.

First, we give necessary conditions for the two modules $M(\ell, m, n; x, y, z)$ and $M(\ell, m, n; x', y', z')$ to be isomorphic.

Proposition 5.3.2. *Let $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$ be admissible. Assume that $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ as Λ_E-modules. Then we have* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$.

Proof. We assume that $M(\ell, m, n; x, y, z) \cong M(\ell, m, n;\ x', y', z')$ as Λ_E-modules. Then we have (5.5) and (5.7) by Proposition 5.1.4. If $\mathrm{ord}_E(x) > \mathrm{ord}_E(x')$, then we get $\mathrm{ord}_E(a_3x - a_2x') = \mathrm{ord}_E(x') \geq m$ by (5.5). Since $(\ell, m, n;\ x', y', z')$ is admissible, this implies $x' = 0$. This contradicts $\mathrm{ord}_E(x) > \mathrm{ord}_E(x')$. By the same reason, $\mathrm{ord}_E(x) < \mathrm{ord}_E(x')$ does not hold. Therefore, we have $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. In the same way, we get $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$. $\square$

Further in the case $\ell = 0$, we have the following

Lemma 5.3.3. *Let $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$ be admissible. Then the following statements are equivalent:*

(i) *We have $M \cong M'$ as Λ_E-modules.*

(ii) *There exist* a_1, a_2, *and* $a_3 \in \mathcal{O}_E^\times$ *satisfying (5.4), (5.5), (5.6), and (5.7) in Proposition 5.1.4 and*

$$a_3(1-x) \equiv a_1(1-x') \mod \pi^m. \tag{5.8}$$

In particular, if (i) *holds, then we have*

$$\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x').$$

Proof. The conditions (5.8) and (5.3) are equivalent under the condition (5.5). Hence we get the conclusion. □

Proof of Theorem 5.3.1. By the Table 5.1 in Remark 5.2.1, for given two 6-tuples $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$, we have only to apply one statement among $(\mathrm{I}), (\mathrm{II}), \cdots$, and (XII). Using the following Propositions 5.3.4, 5.3.6, and 5.3.10, we can prove Theorem 5.3.1 in the case (I), (III), and (VII). By the same method as these Propositions, we can prove the remaining cases. This implies that our Theorem 5.3.1 holds. □

Let $[M(\ell, m, n;\, x, y, z)]$ be an element of $\mathcal{M}^E_{f(T)}$. We fix non-negative integers ℓ, m, and n.

Proposition 5.3.4. *Let* $(\ell, m, n; x, y, z)$ *and* $(\ell, m, n; x', y', z')$ *be admissible. Assume that* $x' \neq 0, z' \neq 0$ *if* $\ell \neq 0$ *and that* $x' \neq 0, 1, z' \neq 0$ *if* $\ell = 0$. *Suppose that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$ *and,* $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(B)$, *where* A, B *are defined before the statement* (I). *Suppose also that* $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$ *if* $\ell = 0$. *Then the following statements are equivalent:*

(i) *We have* $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ *as* Λ_E*-modules.*

(ii) *The statement* (I) *holds for* $(\ell, m, n; x, y, z)$ *and* $(\ell, m, n; x', y', z')$.

Proof. First, we prove (i) $\Rightarrow$ (ii). Let A, B, C, D, E, F, and $G \in \mathcal{O}_E$ be the elements defined before the statement (I). We note that these elements are all in

$\mathcal{O}_E$. By Proposition 5.1.4, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying

$$a_2 - a_1 = \pi^\ell v, \tag{5.9}$$

$$a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x' = \pi^m w, \tag{5.10}$$

$$\begin{aligned}&1 - a_1 - (a_2 - a_1)\pi^{-\ell}y' \\ &- \left\{a_3 - a_1 - (a_2 - a_1)\pi^{-\ell}x'\right\}\pi^{-m}z' = \pi^n\eta,\end{aligned} \tag{5.11}$$

$$a_3 x - a_2 x' = \pi^m \xi_x, \tag{5.12}$$

$$y - a_2 y' - \xi_x z' = \pi^n \xi_y, \text{ and} \tag{5.13}$$

$$z - a_3 z' = \pi^n \xi_z \tag{5.14}$$

for some v, w, η, ξ_x, ξ_y, and $\xi_z \in \mathcal{O}_E$. By the equations (5.9), (5.12), and (5.14), we have

$$\begin{aligned}
a_1 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x - \pi^\ell v,\\
a_2 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x, \text{ and}\\
a_3 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z.
\end{aligned}$$

By the equations (5.10), (5.11), (5.13), we have

$$\frac{\pi^n}{z'}(\frac{x}{x'} - 1)\xi_z + \frac{\pi^m}{x'}\xi_x + (\pi^\ell - x')v - \pi^m w = \frac{z}{z'}(\frac{x}{x'} - 1), \tag{5.15}$$

$$\frac{\pi^n}{z'}\frac{x}{x'}\xi_z + \frac{\pi^m}{x'}\xi_x + (\pi^\ell - y')v - z'w - \pi^n\eta = \frac{z}{z'}\frac{x}{x'} - 1, \tag{5.16}$$

$$\frac{\pi^n}{z'}\frac{x}{x'}y'\xi_z + (\frac{\pi^m}{x'}y' - z')\xi_x - \pi^n\xi_y = \frac{z}{z'}\frac{x}{x'}y' - y. \tag{5.17}$$

By the equations (5.15), (5.16), and (5.17), we obtain

$$\begin{pmatrix}
-\frac{\pi^n}{z'}(1 - \frac{x}{x'}) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0\\
\frac{\pi^n}{z'}\frac{x}{x'} & \frac{\pi^m}{x'} & \pi^\ell - y' & -z' & -\pi^n & 0\\
\frac{\pi^n}{z'}\frac{x}{x'}y' & \frac{\pi^m}{x'}y' - z' & 0 & 0 & 0 & -\pi^n
\end{pmatrix}
\begin{pmatrix}\xi_z\\ \xi_x\\ v\\ w\\ \eta\\ \xi_y\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{z}{z'}(1 - \frac{x}{x'})\\ \frac{z}{z'}\frac{x}{x'} - 1\\ \frac{z}{z'}\frac{x}{x'}y' - y\end{pmatrix}.$$

Therefore, the augmented matrix for the system of the equations (5.15), (5.16), and (5.17) is

$$\begin{pmatrix} -\frac{\pi^n}{z'}(1-\frac{x}{x'}) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0 & b_1 \\ \frac{\pi^n}{z'}\frac{x}{x'} & \frac{\pi^m}{x'} & \pi^\ell - y' & -z' & -\pi^n & 0 & b_2 \\ A & B & 0 & 0 & 0 & -\pi^n & C \end{pmatrix}, \tag{5.18}$$

where $b_1 = -\frac{z}{z'}(1-\frac{x}{x'})$ and $b_2 = \frac{z}{z'}\frac{x}{x'} - 1$. Performing row operations, the matrix in (5.18) is equivalent to

$$\begin{aligned} &\begin{pmatrix} -\frac{\pi^n}{z'}(1-\frac{x}{x'}) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0 & b_1 \\ \frac{\pi^n}{z'} & 0 & x' - y' & \pi^m - z' & -\pi^n & 0 & b_4 \\ A & B & 0 & 0 & 0 & -\pi^n & C \end{pmatrix} \\ \sim &\begin{pmatrix} A & B & 0 & 0 & 0 & -\pi^n & C \\ \frac{\pi^n}{z'} & 0 & D & E & -\pi^n & 0 & \frac{z}{z'} - 1 \\ -\frac{\pi^n}{z'}(1-\frac{x}{x'}) & \frac{\pi^m}{x'} & \pi^\ell - x' & -\pi^m & 0 & 0 & -\frac{z}{z'}(1-\frac{x}{x'}) \end{pmatrix} \\ \sim &\begin{pmatrix} A & B & 0 & 0 & 0 & -\pi^n & C \\ \frac{\pi^n}{z'} & 0 & D & E & -\pi^n & 0 & \frac{z}{z'} - 1 \\ 0 & \frac{\pi^m}{x'} & F & G & -\pi^n(1-\frac{x}{x'}) & 0 & \frac{x}{x'} - 1 \end{pmatrix}, \end{aligned} \tag{5.19}$$

where $b_4 = \frac{z}{z'} - 1$. By the matrix (5.19), we get $A\xi_z + B\xi_x - \pi^n\xi_y = C$. Since $\xi_x, \xi_y, \xi_z \in \mathcal{O}_E$ and $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(B)$, we have min $\{\mathrm{ord}_E(A),\, n\} \leq \mathrm{ord}_E(C)$. Further we have $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(C)$. Indeed, if $\mathrm{ord}_E(y') \geq \mathrm{ord}_E(z')$, we have $\mathrm{ord}_E(B) = \mathrm{ord}_E(z') < n$, since we assume that $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$ are admissible. If $\mathrm{ord}_E(y') < \mathrm{ord}_E(z')$, we have $\mathrm{ord}_E(A) < n$. Thus we get $\mathrm{ord}_E(A) \leq \mathrm{ord}_E(C)$. We prove that the statement (I) holds for $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$. First, we note that either (I-1-a), (I-2-a) or (I-3-a) holds. We suppose that (I-2-a) holds. By the matrix (5.19), we have $\frac{\pi^m}{x'}\xi_x + Fv + Gw - \pi^n(1-\frac{x}{x'})\eta = \frac{x}{x'} - 1$. Since we suppose (I-2-a), we get min

$$\left\{\mathrm{ord}_E\left(\frac{\pi^m}{x'}\right), \mathrm{ord}_E(F), \mathrm{ord}_E(G)\right\} = \mathrm{ord}_E(F).$$

This implies $\mathrm{ord}_E(F) \leq \mathrm{ord}_E(\frac{x}{x'} - 1)$. Thus we get the condition (I-2-c). Since $\mathrm{ord}_E(A) < n$ and $\mathrm{ord}_E(F) < m$, we have $A \neq 0$ and $F \neq 0$. Performing row operations for (5.19), we have

$$\begin{pmatrix} 1 & A^{-1}B & 0 & 0 & 0 & -A^{-1}\pi^n & c_1 \\ 0 & -A^{-1}B\frac{\pi^n}{z'} & D & E & -\pi^n & A^{-1}\frac{\pi^{2n}}{z'} & c_2 \\ 0 & \frac{\pi^m}{x'}F^{-1} & 1 & GF^{-1} & -\pi^n(1-\frac{x}{x'})F^{-1} & 0 & c_3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & A^{-1}B & 0 & 0 & 0 & -A^{-1}\pi^n & c_1 \\ 0 & \frac{\pi^m}{x'}F^{-1} & 1 & GF^{-1} & -\pi^n(1-\frac{x}{x'})F^{-1} & 0 & c_3 \\ 0 & U & 0 & E-GF^{-1}D & S & A^{-1}\frac{\pi^{2n}}{z'} & T \end{pmatrix},$$

where $T = -A^{-1}C\frac{\pi^n}{z'} + \frac{z}{z'} - 1 - (\frac{x}{x'} - 1)F^{-1}D$, $S = -\pi^n + \pi^n(1-\frac{x}{x'})F^{-1}D$, $U = -A^{-1}B\frac{\pi^n}{z'} - \frac{\pi^m}{x'}F^{-1}D$, $c_1 = A^{-1}C$, $c_2 = -A^{-1}C\frac{\pi^n}{z'} + \frac{z}{z'} - 1$, and $c_3 = (\frac{x}{x'} - 1)F^{-1}$. By the matrix above, we have

$$U\xi_x + (E - GF^{-1}D)w + S\eta + A^{-1}\frac{\pi^{2n}}{z'}\xi_y = T.$$

This implies that $\min\{\mathrm{ord}_E(U), \mathrm{ord}_E(E - DF^{-1}G), \mathrm{ord}_E(S), \mathrm{ord}_E(A^{-1}\ \frac{\pi^{2n}}{z'})\} \leq \mathrm{ord}_E(T)$. Since we have $\mathrm{ord}_E(A^{-1}\frac{\pi^{2n}}{z'}) = \mathrm{ord}_E(\frac{\pi^n}{y'})$, this is the condition (I-2-d). The condition (I-2-b) is already obtained after (5.19). Therefore (I-2) holds. We can prove the case of (I-1) and that of (I-3) by the same method. Thus we have obtained (ii).

We next prove (ii) $\Rightarrow$ (i). Then either (I-1), (I-2), or (I-3) holds. We suppose that (I-2) holds. By the condition (I-2-d), there exist integers ξ_x, w, η, and $\xi_y \in \mathcal{O}_E$ satisfying

$$U\xi_x + (E - DF^{-1}G)w + S\eta + A^{-1}\frac{\pi^{2n}}{z'}\xi_y = T.$$

We put

$$\begin{aligned} v &= \left(\frac{x}{x'} - 1\right)F^{-1} - \frac{\pi^m}{x'}F^{-1}\xi_x - GF^{-1}w + \pi^n(1-\frac{x}{x'})F^{-1}\eta, \\ \xi_z &= A^{-1}C - A^{-1}B\xi_x + A^{-1}\pi^n\xi_y. \end{aligned}$$

By (I-2-a), (I-2-b), and (I-2-c), we have $v, \xi_z \in \mathcal{O}_E$. By the converse operation of the proof of (i) $\Rightarrow$ (ii), $\xi_x, \xi_y, \xi_z, w, \eta$, and v satisfy (5.15), (5.16), and (5.17). We also set

$$
\begin{aligned}
a_1 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x - \pi^\ell v, \\
a_2 &= \left(\frac{z}{z'} - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x, \\
\text{and } a_3 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z.
\end{aligned}
$$

Then a_1, a_2, and a_3 satisfy (5.9), (5.10), (5.11), (5.12), (5.13), and (5.14). In the case where $\ell \neq 0$, we can check a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ since we have $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, and $z' \neq 0$. In the case of $\ell = 0$, we have

$$
a_1 = \frac{z}{z'}\frac{1-x}{1-x'} - \frac{\pi^n}{z'}\frac{1-x}{1-x'}\xi_z + \frac{\pi^m}{1-x'}\xi_x - \frac{\pi^m}{1-x'}w.
$$

We note that we have $\mathrm{ord}_E\left(\frac{\pi^m}{1-x}\right) > 0$ since $x \in S_m$. Thus we have $a_1 \in \mathcal{O}_E^\times$. By the same method, we can show a_2 and $a_3 \in \mathcal{O}_E^\times$. Then a_1, a_2, and a_3 satisfy equalities (5.2), (5.3), (5.4), (5.5), (5.6), and (5.7). By Proposition 5.1.4, we obtain (i). If (I-1) or (I-3) holds, we can prove (i) by the same method. $\square$

Proposition 5.3.5. *Let $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$ be admissible. Assume that $x' \neq 0, z' \neq 0$ if $\ell \neq 0$ and that $x' \neq 0, 1, z' \neq 0$ if $\ell = 0$. Suppose that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, *and* $\mathrm{ord}_E(A) > \mathrm{ord}_E(B)$, *where A, B are defined before the statement (I). Suppose also that* $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$ *if $\ell = 0$. Then the following statements are equivalent:*

- (i) *We have $M(\ell, m, n; x, y, z) \cong M(\ell, m, n; x', y', z')$ as Λ_E-modules.*
- (ii) *The statement* (II) *holds for $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$.*

Proof. First, we assume (i). Let A, B, C, D, E, F, and $G \in \mathcal{O}_E$ be the same elements, which is defined before the condition (I). By Proposition 5.1.4, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying (5.9), (5.10), (5.11), (5.12), (5.13), and (5.14). In the same way as Proposition 5.3.4, we have the matrix (5.19), which is equivalent to

$$\begin{pmatrix} AB^{-1} & 1 & 0 & 0 & 0 & -\pi^n B^{-1} & CB^{-1} \\ \frac{\pi^n}{z'} & 0 & D & E & -\pi^n & 0 & \frac{z}{z'}-1 \\ 0 & \frac{\pi^m}{x'} & F & G & -\pi^n(1-\frac{x}{x'}) & 0 & \frac{x}{x'}-1 \end{pmatrix}$$
$$\sim \begin{pmatrix} AB^{-1} & 1 & 0 & 0 & 0 & -\pi^n B^{-1} & CB^{-1} \\ \frac{\pi^n}{z'} & 0 & D & E & -\pi^n & 0 & \frac{z}{z'}-1 \\ -AB^{-1}\frac{\pi^m}{x'} & 0 & F & G & c_4' & B^{-1}\frac{\pi^{m+n}}{x'} & c_4 \end{pmatrix}, \tag{5.20}$$

where $c_4 = \frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'}$, and $c_4' = -\pi^n(1-\frac{x}{x'})$. By the same methods as Proposition 5.3.4, we have $\mathrm{ord}_E(B) \leq \mathrm{ord}_E(C)$. We will prove that the statement (II) holds for $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x', y', z')$. We note that either (II-1-a), (II-2-a) or (II-3-a) holds. Then we have $z = z'$. Indeed, by the matrix (5.20) above, we have

$$\mathrm{ord}_E(\frac{\pi^n}{z'}) = \min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{z'}\right), \mathrm{ord}_E(D), \mathrm{ord}_E(E)\right\} \leq \mathrm{ord}_E(\frac{z}{z'} - 1).$$

Since we suppose z and $z' \in S_n$, this implies that $z = z'$. Further the matrix above (5.20) is equivalent to

$$\begin{pmatrix} AB^{-1} & 1 & 0 & 0 & 0 & -\pi^n B^{-1} & c_5 \\ 1 & 0 & \frac{z'}{\pi^n}D & \frac{z'}{\pi^n}E & -z' & 0 & 0 \\ -AB^{-1}\frac{\pi^m}{x'} & 0 & F & G & -\pi^n(1-\frac{x}{x'}) & \pi^n B^{-1}\frac{\pi^m}{x'} & c_6 \end{pmatrix}$$
$$\sim \begin{pmatrix} AB^{-1} & 1 & 0 & 0 & 0 & -\pi^n B^{-1} & c_5 \\ 1 & 0 & \frac{z'}{\pi^n}D & \frac{z'}{\pi^n}E & -z' & 0 & 0 \\ 0 & 0 & V & W & X & \pi^n B^{-1}\frac{\pi^m}{x'} & c_6 \end{pmatrix},$$

where $V = F + \frac{z'}{\pi^n}DAB^{-1}\frac{\pi^m}{x'}$, $W = G + \frac{z'}{\pi^n}EAB^{-1}\frac{\pi^m}{x'}$, $X = -\pi^n(1-\frac{x}{x'}) - z'AB^{-1}\frac{\pi^m}{x'}$, $c_5 = CB^{-1}$, and $c_6 = \frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'}$. Therefore there exist v, w, η, and $\xi_y \in \mathcal{O}_E$ satisfying

$$Vv + Ww + X\eta + \pi^n B^{-1}\frac{\pi^m}{x'}\xi_y = \frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'}.$$

This implies that

$$\min\left\{\mathrm{ord}_E(V), \mathrm{ord}_E(W), \mathrm{ord}_E(X), \mathrm{ord}_E\left(\pi^n B^{-1}\frac{\pi^m}{x'}\right)\right\}$$
$$\leq \mathrm{ord}_E\left(\frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'}\right).$$

Thus the condition (II-1-d) is satisfied. Therefore (II-1) holds. We can prove the case (II-2) and that of (II-3) by the same method. Thus we have obtained (ii).

Conversely, we prove (ii) $\Rightarrow$ (i). Then either (II-1), (II-2), or (II-3) holds. We suppose that (II-1) holds. By the condition (II-1-d), there exist v, w, η, and $\xi_y \in \mathcal{O}_E$ satisfying

$$Vv + Ww + X\eta + \pi^n B^{-1}\frac{\pi^m}{x'}\xi_y = \frac{x}{x'} - 1 - CB^{-1}\frac{\pi^m}{x'}.$$

Set $\xi_z = -\frac{z'}{\pi^n}Dv - \frac{z'}{\pi^n}Ew + z'\eta$ and $\xi_x = CB^{-1} - AB^{-1}\xi_z + \pi^n B^{-1}\xi_y$. We put

$$\begin{aligned} a_1 &= \left(1 - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x - \pi^\ell v, \\ a_2 &= \left(1 - \frac{\pi^n}{z'}\xi_z\right)\frac{x}{x'} - \frac{\pi^m}{x'}\xi_x, \text{ and} \\ a_3 &= 1 - \frac{\pi^n}{z'}\xi_z. \end{aligned}$$

Then a_1, a_2, and a_3 satisfy (5.9), (5.10), (5.11), (5.12), (5.13), and (5.14). In the case where $\ell \neq 0$, we can check a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ since $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $x' \neq 0$, and $z' \neq 0$. In the case of $\ell = 0$, we have a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ in the same way as Proposition 5.3.4. Then a_1, a_2, and a_3 satisfy equalities (5.2), (5.3), (5.4), (5.5), (5.6), and (5.7). By Proposition 5.1.4, we have (i). If (II-2) or (II-3) holds, we can prove (i) by the same method. $\square$

Next, we treat the case where $\ell \neq 0$ and $n = 0$. In this case, we have $y = z = 0$ for every admissible $(\ell, m, n; x, y, z)$.

Proposition 5.3.6. *Suppose that* $(\ell, m, 0; x, 0, 0)$ *and* $(\ell, m, 0; x', 0, 0)$ *are admissible. Assume that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$ *and* $\ell \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(\ell, m, 0; x, 0, 0) \cong M(\ell, m, 0; x', 0, 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (III) *holds for* $(\ell, m, 0; x, 0, 0)$ *and* $(\ell, m, 0; x', 0, 0)$.

Proof. We prove that (i) $\Rightarrow$ (ii). By Proposition 5.1.4, we have units a_1, a_2, and

$a_3 \in \mathcal{O}_E^\times$ satisfying

$$\begin{aligned} & a_2 \equiv a_1 \mod \pi^\ell, \\ & 1 - a_1 - (a_2 - a_1)\pi^{-\ell}x' \equiv 0 \mod \pi^m, \text{ and} \\ & x \equiv a_2 x' \mod \pi^m. \end{aligned}$$

By [12, Proposition 4.5 and Lemma 4.6], this is equivalent to saying that $M(\ell, n, x) \cong M(\ell, n, x')$, where $M(\ell, n, x) = \langle (1,1,1),\ (0, \pi^\ell, x),\ (0, 0, \pi^n)\ \rangle_{\mathcal{O}_E} \subset \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma)$ is defined in Section 4.1. By Corollary 1, this implies that (I') or (II') holds. This is the same as the statement (III). Hence we have (ii).

Next, we suppose (ii). Then we obtain $M(x, 0, 0) \cong M(x', 0, 0)$ by Theorem 1. Thus we have (i). □

Next, we consider the case where $\ell \neq 0$ and $m = 0$. In this case, we have $x = 0$ for every admissible $(\ell, m, n; x, y, z)$.

Proposition 5.3.7. *Let $(\ell, 0, n; 0, y, z)$ and $(\ell, 0, n; 0, y', z')$ be admissible. Suppose that $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$ and $\ell \neq 0$. Then the following statements are equivalent:*

(i) *We have $M(\ell, 0, n; 0, y, z) \cong M(\ell, 0, n; 0, y', z')$ as Λ_E-modules.*

(ii) *The statement* (IV) *holds for $(\ell, 0, n; 0, y, z)$ and $(\ell, 0, n; 0, y', z')$.*

Proof. First, we assume (i). We show (ii). We note that either (IV-1-a), (IV-2-a), or $y' = 0$ holds. We suppose that (IV-a-1) holds. By Proposition 5.1.4, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying (5.2), (5.4), (5.6), and (5.7). By the equation (5.2), (5.6), and (5.7), we have

$$a_1 = \frac{y}{y'} - \frac{\pi^n}{y'}\xi_y - \pi^\ell v, \tag{5.21}$$

$$a_2 = \frac{y}{y'} - \frac{\pi^n}{y'}\xi_y, \text{ and} \tag{5.22}$$

$$a_3 = \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z \tag{5.23}$$

for some v, ξ_y, and $\xi_z \in \mathcal{O}_E$. By (5.22), we get $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$. This is the condition (IV-1-b). Further by the equation (5.4) we obtain

$$\begin{aligned}
&\pi^n \xi_z + (1-z')\frac{\pi^n}{y'}\xi_y + \{\pi^\ell(1-z') - y'\}v - \pi^n \eta \\
&= z - 1 - (z'-1)\frac{y}{y'}. \qquad (5.24)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\min\left\{n, \mathrm{ord}_E\left((1-z')\frac{\pi^n}{y'}\right), \mathrm{ord}_E(\pi^\ell(1-z') - y')\right\} \\
\leq\ & \mathrm{ord}_E\left(z - 1 - (z'-1)\frac{y}{y'}\right).
\end{aligned}$$

This is the condition (IV-1-c). Therefore (IV-1) holds. We can prove the case of (IV-2) and that of (IV-3) by the same method.

Conversely, we prove that (ii) $\Rightarrow$ (i). Then either (IV-1), (IV-2), or (IV-3) holds. We suppose that (IV-1) holds. By the condition (IV-1-c), we have (5.24) for some ξ_y, ξ_z, v, and $\eta \in \mathcal{O}_E$. We put a_1, a_2, and a_3 the same as (5.21), (5.22), and (5.23), respectively. Then a_1, a_2, and a_3 are units and satisfy equalities (5.2), (5.4), (5.6), and (5.7) since we have $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$ and $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$. By Proposition 5.1.4, we obtain (i). We can show the conclusion by the same method when (IV-3) holds. Finally, we suppose that (IV-2) holds. In this case, we have $M(\ell, m, n; x, y, z) = M(\ell, 0, n; 0, y, 0) \cong \langle (1,1,1), (0, \pi^\ell, y), (0,0,\pi^n)\rangle_{\mathbb{Z}_p} \oplus \langle (0,0,1,0)\rangle_{\mathbb{Z}_p}$. Therefore (i) is equivalent to saying that

$$\langle (1,1,1), (0,\pi^\ell, y), (0,0,\pi^n)\rangle_{\mathbb{Z}_p} \cong \langle (1,1,1), (0,\pi^\ell, y'), (0,0,\pi^n)\rangle_{\mathbb{Z}_p}.$$

By Theorem 1, this is the same as the condition (IV-2). $\square$

Next, we treat the case where $\ell \neq 0, n \neq 0$, and $z' = 0$. Let $(\ell, m, n; x, y, z)$ and $(\ell, m, n; x, y', 0)$ be admissible. If we assume that $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, then we have $z = 0$.

Proposition 5.3.8. *Suppose that* $(\ell, m, n; x, y, 0)$ *and* $(\ell, m, n; x', y', 0)$ *are admissible. Assume that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\ell \neq 0$, $m \neq 0$, *and* $n \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(\ell, m, n; x, y, 0) \cong M(\ell, m, n; x', y', 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (V) *holds for* $(\ell, m, n; x, y, 0)$ *and* $(\ell, m, n; x', y', 0)$.

Proof. First, we prove that (ii) $\Rightarrow$ (ii). We suppose (i). By Proposition 5.1.4, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying (5.2), (5.3), (5.4), (5.5), and (5.6). By the equations (5.2), (5.5), and (5.6), we have

$$\begin{aligned} a_1 &= \frac{y}{y'} - \frac{\pi^n}{y'}\xi_y - \pi^\ell v, && (5.25)\\ a_2 &= \frac{y}{y'} - \frac{\pi^n}{y'}\xi_y, \text{ and} && (5.26)\\ a_3 &= \frac{y}{y'}\frac{x'}{x} - \frac{\pi^n}{y'}\frac{x'}{x}\xi_y + \frac{\pi^m}{x}\xi_x && (5.27) \end{aligned}$$

for some v, ξ_x, and $\xi_y \in \mathcal{O}_E$. By (5.26), we get $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$. By the equations (5.3) and (5.4) we obtain

$$\begin{aligned} &\frac{\pi^n}{y'}\left(1 - \frac{x'}{x}\right)\xi_y + \frac{\pi^m}{x}\xi_x + (\pi^\ell - x')v - \pi^m w = \frac{y}{y'}\left(1 - \frac{x'}{x}\right),\\ &\frac{\pi^n}{y'}\xi_y + (\pi^\ell - y')v - \pi^n\eta = \frac{y}{y'} - 1 \end{aligned}$$

for some η and $w \in \mathcal{O}_E$. In the same way as Proposition 5.3.4, we write the augmented matrix for the system of the equations above:

$$\begin{aligned} &\begin{pmatrix} \frac{\pi^n}{y'}(1 - \frac{x'}{x}) & \frac{\pi^m}{x} & \pi^\ell - x' & -\pi^m & 0 & \frac{y}{y'}(1 - \frac{x'}{x}) \\ \frac{\pi^n}{y'} & 0 & \pi^\ell - y' & 0 & -\pi^n & \frac{y}{y'} - 1 \end{pmatrix}\\ \sim &\begin{pmatrix} 0 & \frac{\pi^m}{x} & d_1 & -\pi^m & \pi^n(1 - \frac{x'}{x}) & 1 - \frac{x'}{x} \\ \frac{\pi^n}{y'} & 0 & \pi^\ell - y' & 0 & -\pi^n & \frac{y}{y'} - 1 \end{pmatrix}, \end{aligned} \qquad (5.28)$$

where $d_1 = \pi^\ell - x' - (\pi^\ell - y')(1 - \frac{x'}{x})$. We prove that the statement (V) holds for $(\ell, m, n; x, y, 0)$ and $(\ell, m, n; x', y', 0)$. Then we note that (V-1-a), (V-2-a), (V-3-a), (V-4-a), or (V-5-a) holds. We suppose that (V-1-a) holds. Then we get $\mathrm{ord}_E(\frac{\pi^n}{y'}) \leq \mathrm{ord}_E(\frac{y}{y'} - 1)$ by (5.28). This implies $y = y'$ since y and $y' \in S_n$. Further the matrix (5.28) is equivalent to

$$\begin{pmatrix} 0 & \frac{\pi^m}{x} & d_1 & -\pi^m & \pi^n(1 - \frac{x'}{x}) & 1 - \frac{x'}{x} \\ 1 & 0 & (\pi^\ell - y')\frac{y'}{\pi^n} & 0 & -y' & (\frac{y}{y'} - 1)\frac{y'}{\pi^n} \end{pmatrix}. \qquad (5.29)$$

Thus we have

$$\frac{\pi^m}{x}\xi_x + \left\{\pi^\ell - x' - (\pi^\ell - y')\left(1 - \frac{x'}{x}\right)\right\} v - \pi^m w + \pi^n \left(1 - \frac{x'}{x}\right)\eta$$
$$= 1 - \frac{x'}{x}. \tag{5.30}$$

This implies that

$$\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{x}\right), \mathrm{ord}_E\left(\pi^\ell - x' - (\pi^\ell - y')\left(1 - \frac{x}{x'}\right)\right)\right\} \le \mathrm{ord}_E\left(1 - \frac{x'}{x}\right).$$

Thus we get (V-1-c). Therefore we have obtained (ii). Next, we suppose that (V-2-a) holds. Then we have $\mathrm{ord}_E(\pi^\ell - y') \le \mathrm{ord}_E(\frac{y}{y'} - 1)$ by (5.28). Further the matrix (5.28) is equivalent to

$$\begin{pmatrix} \frac{\pi^n}{y'}(1 - \frac{x'}{x}) & \frac{\pi^m}{x} & \pi^\ell - x' & -\pi^m & 0 & \frac{y}{y'}(1 - \frac{x'}{x}) \\ \frac{\pi^n}{y'}\frac{1}{\pi^\ell - y'} & 0 & 1 & 0 & -\pi^n\frac{1}{\pi^\ell - y'} & (\frac{y}{y'} - 1)\frac{1}{\pi^\ell - y'} \end{pmatrix}$$
$$\sim \begin{pmatrix} d_2 & \frac{\pi^m}{x} & 0 & -\pi^m & \pi^n\frac{\pi^\ell - x'}{\pi^\ell - y'} & d_3 \\ \frac{\pi^n}{y'}\frac{1}{\pi^\ell - y'} & 0 & 1 & 0 & -\pi^n\frac{1}{\pi^\ell - y'} & (\frac{y}{y'} - 1)\frac{1}{\pi^\ell - y'} \end{pmatrix},$$

where $d_2 = \frac{\pi^n}{y'}(1 - \frac{x'}{x}) - \frac{\pi^n}{y'}\frac{\pi^\ell - x'}{\pi^\ell - y'}$ and $d_3 = \frac{y}{y'}(1 - \frac{x'}{x}) - (\frac{y}{y'} - 1)\frac{\pi^\ell - x'}{\pi^\ell - y'}$. This implies that

$$\min\left\{\mathrm{ord}_E\left(\frac{\pi^n}{y'}(1 - \frac{x}{x'}) - \frac{\pi^n}{y'}\frac{\pi^\ell - x'}{\pi^\ell - y'}\right), \mathrm{ord}_E\left(\pi^n\frac{\pi^\ell - x'}{\pi^\ell - y'}\right), \mathrm{ord}_E\left(\frac{\pi^m}{x}\right)\right\}$$
$$\le \mathrm{ord}_E\left(\frac{y}{y'}\left(1 - \frac{x}{x'}\right) - \left(\frac{y}{y'} - 1\right)\frac{\pi^\ell - x'}{\pi^\ell - y'}\right).$$

Thus we get (V-2-d). Therefore we have obtained (ii). The remaining cases are also showed by the same method as above.

Conversely, we prove that (ii) $\Rightarrow$ (i) holds. We suppose that (ii) holds. Then (V-1), (V-2), (V-3), (V-4), or (V-5) holds. We assume that (V-1) holds. Especially we assume that (V-1). By the condition (V-1-c), there exist ξ_x, v, w, and η satisfying (5.30). We put

$$\xi_y = \left(\frac{y}{y'} - 1\right)\frac{y'}{\pi^n} - (\pi^\ell - y')\frac{y'}{\pi^n}v + y'\eta$$

and set a_1, a_2, and a_3 the same as (5.25), (5.26), and (5.27), respectively. Then a_1, a_2, and a_3 are units and satisfy equalities (5.2), (5.3), (5.4), (5.5), (5.6), and

(5.7). By Proposition 5.1.4, we have (i). The remaining cases are proved by the same method as above. $\square$

Further we treat the case where $\ell \neq 0, m \neq 0, x' = 0$, and $z' \neq 0$. Let $(\ell, m, n; x, y, z)$ and $(\ell, m, n; 0, y', z')$ be admissible. If we assume that $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, then we have $x = 0$. In the same way as Proposition 5.3.8, we can show the following.

Proposition 5.3.9. *Suppose that* $(\ell, m, n; 0, y, z)$ *and* $(\ell, m, n; 0, y', z')$ *are admissible. Assume that* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, $\ell \neq 0$, $m \neq 0$, *and* $n \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(\ell, m, n; 0, y, z) \cong M(\ell, m, n; 0, y', z')$ *as* Λ_E*-modules.*

(ii) *The statement* (VI) *holds for* $M(\ell, m, n; 0, y, z)$ *and* $M(\ell, m, n; 0, y', z')$.

From now on, we treat the case of $\ell = 0$ and $z' = 0$. Let $(0, m, n; x, y, z)$ and $(0, m, n; x', y', 0)$ be admissible. If $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, then we have $z = 0$.

Proposition 5.3.10. *Suppose that* $(0, m, n; x, y, 0)$ *and* $(0, m, n; x', y', 0)$ *are admissible. Assume that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, $x' \neq 0, 1$, *and* $y' \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(0, m, n; x, y, 0) \cong M(0, m, n; x', y', 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (VII) *holds for* $(0, m, n; x, y, 0)$ *and* $(0, m, n; x', y', 0)$.

Proof. First we assume (i). By Lemma 5.3.3, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying (5.8), (5.4), (5.5), (5.6), and (5.7). By (5.6), we have $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$. Further using (5.4) and (5.6), we get

$$1 - y \equiv a_1(1 - y') \mod \pi^n. \tag{5.31}$$

Hence we have (VII-a). We show (VII-b). By (5.8), (5.5), and (5.6), we obtain

$$\begin{aligned} a_1 &= \left\{ \left(\frac{y}{y'} - \frac{\pi^n}{y'} \xi_y \right) \frac{x'}{x} + \frac{\pi^m}{x} \xi_x \right\} \frac{1-x}{1-x'} - \frac{\pi^m}{1-x'} w', &(5.32)\\ a_2 &= \frac{y}{y'} - \frac{\pi^n}{y'} \xi_y, \text{ and} &(5.33)\\ a_3 &= \left(\frac{y}{y'} - \frac{\pi^n}{y'} \xi_y \right) \frac{x'}{x} + \frac{\pi^m}{x} \xi_x &(5.34)\end{aligned}$$

for some ξ_x, ξ_y, and $w' \in \mathcal{O}_E$. By (5.31), we have $1 - y - a_1(1-y') = \pi^n \eta$ for some $\eta \in \mathcal{O}_E$. This implies that

$$\begin{aligned} & -\frac{\pi^n}{y'}\frac{1-x}{1-x'}\frac{x'}{x}(1-y')\xi_y + \frac{\pi^m}{x}\frac{1-x}{1-x'}(1-y')\xi_x + \frac{\pi^m}{1-x'}(1-y')w' + \pi^n\eta \\ = \ & 1 - y - \frac{y}{y'}\frac{x'}{x}\frac{1-x}{1-x'}(1-y'). \end{aligned} \tag{5.35}$$

This implies that (VII-b). Conversely, we suppose that (ii) holds. By (VII-b), there exist ξ_x, ξ_y, w', and $\eta \in \mathcal{O}_E$ satisfying (5.35). We put a_1, a_2, and a_3 as (5.32), (5.33), and (5.34), respectively. Since $(0, m, n; x, y, 0)$ and $(0, m, n; x', y', 0)$ are admissible and (VII-a) holds, $a_2, a_3 \in \mathcal{O}_E^\times$. Using $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, we have $a_1 \in \mathcal{O}_E^\times$. It is easy to check that a_1 and a_2, and a_3 satisfy (5.8), (5.3), (5.4), (5.5), (5.6), and (5.7). By Lemma 5.3.3, we get (i). □

Let $(0, m, n; x, y, z)$ and $(0, m, n; x', 0, 0)$ be admissible. If $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, then we have $z = 0$. In the same way as Proposition 5.3.10, we have the following.

Proposition 5.3.11. *Suppose that* $(0, m, n; x, y, 0)$ *and* $(0, m, n; x', 0, 0)$ *are admissible. Assume that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, $m \neq 0, n \neq 0$, *and* $x' \neq 0, 1$. *Then the following statements are equivalent:*

(i) *We have* $M(0, m, n; x, y, 0) \cong M(0, m, n; x', 0, 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (VIII) *holds for* $(0, m, n; x, y, 0)$ *and* $(0, m, n; x', 0, 0)$.

Next, we consider the case where $\ell = 0$, $m \neq 0$, $n \neq 0$, and $x' = 0$. Let $(0, m, n; x, y, z)$ and $(0, m, n; 0, y', z')$ be admissible. If $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, then we have $x = 0$.

Proposition 5.3.12. *Suppose that* $(0, m, n; 0, y, z)$ *and* $(0, m, n; 0, y', z')$ *are admissible. Assume that* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, $m \neq 0$, *and* $n \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(0, m, n; 0, y, z) \cong M(0, m, n; 0, y', z')$ *as* Λ_E*-modules.*

(ii) *The statement* (IX) *holds for* $(0, m, n; 0, y, z)$ *and* $(0, m, n; 0, y', z')$.

Proof. First, we assume (i). We prove that (IX) holds for $(0, m, n; 0, y, z)$ and $(0, m, n; 0, y', z')$. By Lemma 5.3.3, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying

(5.8), (5.4), (5.6), and (5.7). By (5.8), we put $a_3 - a_1 = \pi^m w$. By the equation (5.6), (5.7), and (5.8), we have

$$\begin{aligned} a_1 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z - \pi^m w, & (5.36)\\ a_2 &= \frac{y}{y'} - \frac{\pi^n}{y'}\xi_y, \text{ and} & (5.37)\\ a_3 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z & (5.38)\end{aligned}$$

for some ξ_y, ξ_z, and $w \in \mathcal{O}_E$. Using (5.37), we have $\mathrm{ord}_E(y) = \mathrm{ord}_E(y')$. We note that (IX-1-a), (IX-2-a), (IX-3-a), or (IX-4-a) holds. We assume that (IX-1-a) holds. By (5.4), we have $1 - a_1 - (a_2 - a_1)y' - wz' = \pi^n \eta$ for some $\eta \in \mathcal{O}_E$. This implies that

$$\begin{aligned} &\frac{\pi^n}{z'}(1-y')\xi_z + \pi^n\xi_y + \{\pi^m(1-y') - z'\}w - \pi^n\eta \\ = \ & y - 1 - \frac{z}{z'}(y'-1). \qquad (5.39)\end{aligned}$$

Thus we have (IX-1-c) and get the conclusion. Therefore we have proved (ii). We can prove the remaining cases by the same method.

Conversely, we suppose that (ii) holds. Then either (IX-1), (IX-2), (IX-3), (IX-4), or (IX-5) holds. We assume that (IX-1) holds. By (IX-1-c), there exist ξ_y, ξ_z, w, and η satisfying (5.39). We define a_1, a_2, and a_3 by (5.36), (5.37), and (5.38), respectively. It is easy to check that a_1, a_2, and a_3 satisfy (5.8), (5.4), (5.5), (5.6), and (5.7). By Lemma 5.3.3, we get (i). We can prove the remaining cases in the same way. □

Next, we consider the case where $m \neq 0, m \neq 0, n \neq 0$, and $x' = 1$. Let $(0, m, n; x, y, z)$ and $(0, m, n; 1, y', z')$ be admissible. If $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, then we have $x = 1$.

Proposition 5.3.13. *Suppose that* $(0, m, n; 1, y, z)$ *and* $(0, m, n; 1, y', z')$ *are admissible. Assume that* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$, $m \neq 0$, *and* $n \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(0, m, n; 1, y, z) \cong M(0, m, n; 1, y', z')$ *as* Λ_E*-modules.*

(ii) *The statement* (X) *holds for* $(0, m, n; 1, y, z)$ *and* $(0, m, n; 1, y', z')$.

Proof. First, we assume (i). We prove that (ii) holds. We note that (X-1) or (X-2) holds. We assume that (X-1) holds. By Lemma 5.3.3, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying (5.8), (5.4), (5.5), (5.6), and (5.7). Further using (5.4) and (5.6), we obtain

$$1 - y \equiv a_1(1 - y') \mod \pi^n. \tag{5.40}$$

These imply (X-1-b). By (5.40), we put $1 - y - a_1(1 - y') = \pi^n \eta$ for some $\eta \in \mathcal{O}_E$. By the equation (5.4) and (5.5), we have

$$\begin{aligned} a_1 &= \frac{1-y}{1-y'} - \frac{\pi^n}{1-y'}\eta, && (5.41)\\ a_2 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z - \pi^m \xi_x, \text{ and} && (5.42)\\ a_3 &= \frac{z}{z'} - \frac{\pi^n}{z'}\xi_z && (5.43) \end{aligned}$$

for some ξ_x and $\xi_z \in \mathcal{O}_E$. By (5.6), we have

$$\frac{\pi^n}{z'} y' \xi_z + (\pi^m y' - z')\xi_x - \pi^n \xi_y = \frac{z}{z'} y' - y. \tag{5.44}$$

This implies (X-1-c). Thus we have conclusion. We can prove the case of (X-2) in the same way. It is easy to check that (ii) implies (i). □

Next, we consider the case of $\ell = 0$ and $m = 0$. If $(0, 0, n; x, y, z)$ and $(0, 0, n; x', y', z')$ are admissible, then we have $x = x' = 0$.

Proposition 5.3.14. *Suppose that* $(0, 0, n; 0, y, z)$ *and* $(0, 0, n; 0, y', z')$ *are admissible. Assume that* $\mathrm{ord}_E(z) = \mathrm{ord}_E(z')$. *Then the following statements are equivalent:*

(i) *We have* $M(0, 0, n; 0, y, z) \cong M(0, 0, n; 0, y', z')$ *as* Λ_E*-modules.*

(ii) *The statement* (XI) *holds for* $(0, 0, n; 0, y, z)$ *and* $(0, 0, n; 0, y', z')$.

Proof. First, we assume (i). We prove the statement (XI) holds for $(0, 0, n; 0, y, z)$ and $(0, 0, n; 0, y', z')$. By Lemma 5.3.3, we have units a_1, a_2, and $a_3 \in \mathcal{O}_E^\times$ satisfying (5.4), (5.6), and (5.7). By (5.4) and (5.6), we have (XI-a). By (5.7), we have $1 - y - z \equiv a_1(1 - y' - z') \mod \pi^n$. This implies (XI-b). Thus we get the conclusion. It is easy to see that (ii) implies (i). $\square$

Finally, we treat the case where $\ell = 0$, $m \neq 0$, and $n = 0$. If $(0, m, 0; x, y, z)$ and $(0, m, 0; x', y', z')$ are admissible, then we have $y = y' = z = z' = 0$.

Proposition 5.3.15. *Suppose that* $(0, m, 0; x, 0, 0)$ *and* $(0, m, 0; x', 0, 0)$ *are admissible. Assume that* $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$, $\mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x')$, *and* $m \neq 0$. *Then the following statements are equivalent:*

(i) *We have* $M(0, m, 0; x, 0, 0) \cong M(0, m, 0; x', 0, 0)$ *as* Λ_E*-modules.*

(ii) *The statement* (XII) *holds for* $(0, m, 0; x, 0, 0)$ *and* $(0, m, 0; x', 0, 0)$.

Proof. In the same way as Proposition 5.3.6, (i) is equivalent to saying that $M(0, m, x) \cong M(0, m, x')$, where

$$M(0, m, x) \subset \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma).$$

By Corollary 1, this is the condition (XII). $\square$

As an example, we classify all the elements of $\mathcal{M}_{f(T)}$ in the case of $E = \mathbb{Q}_p$ and $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\delta) = \mathrm{ord}_p(\delta-\alpha) = \mathrm{ord}_p(\beta-\delta) = \mathrm{ord}_p(\alpha-\gamma)=1$, where we write $\mathcal{M}_{f(T)}$ for $\mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ and ord_p for $\mathrm{ord}_{\mathbb{Q}_p}$. This example was also treated by C.Franks. We note that there is no distinguished polynomial which has this property in the case of $p = 2$ and 3. In the following, we take $R = \{0, 1, \ldots, p-1\}$, which is a set of complete representatives in $\mathbb{Z}_p$ of the elements of the residue field $\mathbb{Z}_p/p\mathbb{Z}_p$.

Corollary 5.3.16. *Suppose that* $p \geq 5$. *Let* $f(T)$ *be the same polynomial as (5.1) and put* $E = \mathbb{Q}_p$. *Assume that* $\mathrm{ord}_p(\alpha-\beta) = \mathrm{ord}_p(\beta-\gamma) = \mathrm{ord}_p(\gamma-\delta) = \mathrm{ord}_p(\delta-\alpha) = \mathrm{ord}_p(\beta-\delta) = \mathrm{ord}_p(\alpha-\gamma) = 1$. *Then we have* $\sharp\mathcal{M}_{f(T)} = 2p + 36$.

We note that this corollary holds for every totally ramified extensions of $\mathbb{Q}_p$.

Sketch of the proof of Corollary 5.3.16. For fixed non-negative integers ℓ, m, and n, we put

$$\mathcal{M}_{f(T)}^E(\ell, m, n) := \left\{[M(\ell', m', n'; x, y, z)] \in \mathcal{M}_{f(T)}^E \;\middle|\; x, y, z \in \mathbb{Z}_p\right\}$$

By Proposition 5.3.2, we have

$$\mathcal{M}^{E}_{f(T)} = \coprod_{\ell} \coprod_{n} \coprod_{m} \mathcal{M}^{E}_{f(T)}(\ell, m, n). \tag{5.45}$$

Using the conditions of Lemma 5.1.1, we have $0 \le \ell \le 1$, $0 \le m \le 2$, and $0 \le n \le 3$. Indeed, by (a), we have $0 \le \ell \le \mathrm{ord}_p(\beta - \alpha) = 1$. If $\mathrm{ord}_p(x) \ge 2$, we have $m \le 1$ by (b). If $\mathrm{ord}_p(x) \le 1$, we obtain $m \le 2$ by (d). These imply $0 \le m \le 2$. We can prove that $0 \le n \le 3$ by Lemma 5.1.1. In fact, by (f), we have $n \le 3$ in the case of $\mathrm{ord}_p(z) \le 2$. We suppose $\mathrm{ord}_p(z) \ge 3$. In the case of $\mathrm{ord}_p(y) \le 1$, we have $n \le 2$ by (e). If $\mathrm{ord}_p(y) \ge 2$, we have $n \le 1$ by (c). Thus we get $0 \le n \le 3$.

We denote $M(\ell, m, n; x, y, z)$ by $M(x, y, z)$ for the fixed triple $\ell, m,$ and n.

Then we get the following:

$$
\begin{aligned}
\mathcal{M}^E_{f(T)}(0,0,0) &= \{[M(0,0,0)]\}, \\
\mathcal{M}^E_{f(T)}(0,0,1) &= \left\{ \begin{array}{l} [M(0,2,p-1)], [M(0,1,1)], [M(0,0,0)], \\ [M(0,0,1)], [M(0,0,2)], [M(0,1,0)], \\ [M(0,2,0)] \end{array} \right\}, \\
\mathcal{M}^E_{f(T)}(0,1,0) &= \{[M(0,0,0)], [M(1,0,0)], [M(2,0,0)]\}, \\
\mathcal{M}^E_{f(T)}(0,1,1) &= \left\{ \begin{array}{l} [M(2,2,0)], \ldots, [M(p-1,2,0)], [M(p-2,4,0)], \\ [M(1,1,0)], [M(1,2,0)], [M(2,1,0)], [M(1,0,0)], \\ [M(0,0,0)], [M(0,1,0)], [M(0,2,0)], [M(2,0,0)] \end{array} \right\}, \\
\mathcal{M}^E_{f(T)}(0,1,2) &= \left\{ \begin{array}{l} \left[M(0,0,\tfrac{\delta-\alpha}{\gamma-\alpha}p)\right], [M(0,p,\tfrac{\delta-\alpha}{\gamma-\alpha}p)], \\ [M(1,1+p,\tfrac{\delta-\beta}{\gamma-\beta}p)], \left[M(1,1,\tfrac{\delta-\beta}{\gamma-\beta}p)\right], \\ \left[M(\tfrac{\beta-\delta}{\beta-\alpha}, \tfrac{\beta-\gamma}{\beta-\alpha}, p)\right] \end{array} \right\}, \\
\mathcal{M}^E_{f(T)}(1,0,0) &= \{[M(0,0,0)]\}, \\
\mathcal{M}^E_{f(T)}(1,0,1) &= \{[M(0,0,0)], [M(0,0,1)], [M(0,0,2)]\}, \\
\mathcal{M}^E_{f(T)}(1,0,2) &= \left\{ \left[M(0,\frac{\delta-\alpha}{\beta-\alpha}p,0)\right], \left[M(0,\frac{\delta-\alpha}{\beta-\alpha}p,p)\right] \right\}, \\
\mathcal{M}^E_{f(T)}(1,1,0) &= \{[M(0,0,0)]\}, \\
\mathcal{M}^E_{f(T)}(1,1,1) &= \left\{ [M(0,0,0)], \left[M(0,\frac{\gamma-\alpha}{\beta-\alpha},1)\right] \right\}, \\
\mathcal{M}^E_{f(T)}(1,1,2) &= \left\{ \left[M(0,0,\frac{\delta-\alpha}{\gamma-\alpha}p)\right] \right\} \\
&\quad \cup \left\{ \left[M(0,pu,\frac{\delta-\alpha}{\gamma-\alpha}p(1-\frac{\beta-\alpha}{\delta-\alpha}u))\right] \middle| u = 1, \ldots p-1 \right\}, \\
\mathcal{M}^E_{f(T)}(1,2,0) &= \left\{ \left[M(\frac{\gamma-\alpha}{\beta-\alpha}p,0,0)\right] \right\},
\end{aligned}
$$

$$\begin{aligned}
\mathcal{M}^E_{f(T)}(1,2,1) &= \left\{\left[M(\frac{\gamma-\alpha}{\beta-\alpha}p,0,0)\right]\right\},\\
\mathcal{M}^E_{f(T)}(1,2,2) &= \left\{\left[M(\frac{\gamma-\alpha}{\beta-\alpha}p,\frac{\delta-\alpha}{\beta-\alpha}p,0)\right]\right\},\\
\mathcal{M}^E_{f(T)}(1,2,3) &= \left\{\left[M(\frac{\gamma-\alpha}{\beta-\alpha}p,\frac{\delta-\alpha}{\beta-\alpha}p,\frac{(\delta-\alpha)(\delta-\beta)}{(\gamma-\alpha)(\gamma-\beta)}p^2)\right]\right\}.
\end{aligned}$$

The following table is the number of elements of $\mathcal{M}^E_{f(T)}(\ell,m,n)$ for each (ℓ,m,n). We pick up the case of $(\ell,m,n)=(1,0,0)$ and that of $(0,1,1)$ and determine

(ℓ,m,n)	$\sharp\mathcal{M}^E_{f(T)}(\ell,m,n)$
$(0,0,0)$	1
$(0,0,1)$	7
$(0,1,0)$	3
$(0,1,1)$	$p+7$
$(0,1,2)$	5
$(1,0,0)$	1
$(1,0,1)$	3
$(1,0,2)$	2
$(1,1,0)$	1
$(1,1,1)$	2
$(1,1,2)$	p
$(1,2,0)$	1
$(1,2,1)$	1
$(1,2,2)$	1
$(1,2,3)$	1

$\mathcal{M}^E_{f(T)}(1,0,0)$ and $\mathcal{M}^E_{f(T)}(0,1,1)$, using our Theorem 2. The remaining cases are proved by the same method as the case of $(1,0,0)$ and that of $(0,1,1)$. First, we consider the former. This is the simplest case. Since we have $m=0$ and $n=0$, we get $M(x,y,z)=M(x,0,0)=M(0,0,0)$. Thus we obtain the conclusion.

Next, we consider the case $(\ell,m,n)=(0,1,1)$. This is one of the most complicated cases. If $(0,1,1;x,y,z)$ is admissible, then we have $z=0$. Indeed we suppose that $(0,1,1;x,y,z)$ is admissible. Then x,y, and z satisfy (a), (b), (c), (d), (e), and

(f) in Lemma 5.1.1. We have $\mathrm{ord}_E(zx) \geq 1$ by (e). We have also $\mathrm{ord}_E(z) \geq 1$ by (c). Since $z \in S_1$, we have $z = 0$. We classify all the elements of $\mathcal{M}^E_{f(T)}(0,1,1)$. We note that $(0,1,1;x,y,0)$ is admissible for every x and $y \in S_1$. Let $(0,1,1;x',y',0)$ be admissible. We consider the following two cases:

$$\begin{cases} \text{(i)} \quad & x' \in \{0,1\} \ \text{ or } \ y' \in \{0,1\}, \\ \text{(ii)} \quad & x' \notin \{0,1\} \ \text{ and } \ y' \notin \{0,1\}. \end{cases}$$

(i) We suppose that $x' \in \{0,1\}$ or $y' \in \{0,1\}$. Then we have

$$\begin{aligned} M(x,y,0) \cong M(x',y',0) \quad \Leftrightarrow \quad & \mathrm{ord}_E(x) = \mathrm{ord}_E(x'), \mathrm{ord}_E(1-x) = \mathrm{ord}_E(1-x'), \\ & \mathrm{ord}_E(y) = \mathrm{ord}_E(y'), \ \text{and} \\ & \mathrm{ord}_E(1-y) = \mathrm{ord}_E(1-y'). \end{aligned}$$

Indeed, by the Table 5.1 in Remark 5.2.1, the 6-tuple $(0,1,1;x',y',0)$ corresponds to (VII), (VIII), (XI), or (X). Therefore the isomorphism classes of $M(x,y,0)$ satisfying (i) are

$$\left\{ \begin{array}{l} [M(0,0,0)], [M(0,1,0)], [M(0,2,0)], [M(1,0,0)], \\ [M(1,1,0)], [M(1,2,0)], [M(2,0,0)], [M(2,1,0)] \end{array} \right\}.$$

(ii) We suppose that $x' \notin \{0,1\}$ and $y' \notin \{0,1\}$. Then we have the following

Lemma 5.3.17. *Suppose* (ii). *Then we have*

$$\begin{aligned} M(x,y,0) \cong M(x',y',0) \quad \Leftrightarrow \quad & x \neq 0,1, y \neq 0,1 \quad \text{and} \\ & \frac{1-x}{x}\frac{y}{1-y} \equiv \frac{1-x'}{x'}\frac{y'}{1-y'} \mod p. \end{aligned}$$

Further we have

$$\frac{1-x'}{x'}\frac{y'}{1-y'} \mod p \equiv \begin{cases} 2 - \frac{2}{k} \mod p \\ \qquad \text{if } (x',y') = (k,2), \\ 2 \mod p \\ \qquad \text{if } (x',y') = (p-2,4). \end{cases}$$

Proof. Since we suppose (ii), the 6-tuple $(0,1,1;x',y',0)$ corresponds to (VII) by the Table 5.1. Since we assume that $x' \neq 0,1$ and $y' \neq 0,1$, the condition (VII-a) says that $x \neq 0,1$ and $y \neq 0,1$. By the same reason, the condition (VII-b) says that
$$\frac{1-x}{x}\frac{y}{1-y} \equiv \frac{1-x'}{x'}\frac{y'}{1-y'} \bmod p.$$
Thus we get the former. It is easy to show the latter. $\square$

By Lemma 5.3.17, the isomorphism classes of $M(x,y,0)$ satisfying (ii) are
$$\{[M(p-2,4,0)],[M(k,2,0)] \mid \ 2 \leq k \leq p-1\}.$$
Therefore we obtain $\sharp\mathcal{M}^E_{f(T)}(0,1,1) = p+7$,
$$\mathcal{M}^E_{f(T)}(0,1,1) \quad = \quad \left\{ \begin{array}{l} [M(2,2,0)],\ldots,[M(p-1,2,0)],[M(p-2,4,0)], \\ [M(1,1,0)],[M(1,2,0)],[M(2,1,0)],[M(1,0,0)], \\ [M(0,0,0)],[M(0,1,0)],[M(0,2,0)],[M(2,0,0)] \end{array} \right\}.$$
$\square$

Chapter 6

Higher Fitting ideals and Λ-modules

In this chapter, we state the relationships between Λ_E-modules and their higher Fitting ideals. By Lemma 3.2.1 in Chapter 3, the isomorphism class of a finitely generated torsion Λ_E-module M with $\mathrm{rank}_{\mathcal{O}_E}(M) = 2$ is determined by the Fitting ideals $\mathrm{Fitt}_{0,\Lambda_E}(M)$ and $\mathrm{Fitt}_{1,\Lambda_E}(M)$. However, in general, $\mathrm{Fitt}_{i,\Lambda_E}(M)$ $(i \geq 0)$ do not determine the isomorphism class of M (see Remark 6.1.1). In this chapter, we define Λ_E-invariants $m(M)$ and $n(M)$ for a Λ_E-module M. Our aim is to prove that $\mathrm{Fitt}_{1,\Lambda_E}(M)$, $m(M)$, and $n(M)$ determine the isomorphism class $[M]_E \in \mathcal{M}^E_{f(T)}$ (Theorem 6.1.2) for a fixed distinguished separable polynomial $f(T)$ with $\mathrm{deg} f(T) = 3$.

6.1 Higher Fitting ideals

In this chapter, we will use the higher Fitting ideals. For a commutative ring R and a finitely presented R-module M, we consider the following exact sequence

$$R^m \xrightarrow{f} R^n \to M \to 0,$$

where m and n are positive integers. For an integer $i \geq 0$ such that $0 \leq i < n$, the i-th Fitting ideal of M is defined to be the ideal of R generated by all $(n-i)\times(n-i)$ minors of the matrix corresponding to f. We denote the i-th Fitting ideal of M by

$\mathrm{Fitt}_{i,R}(M)$. This definition does not depend on the choice of the exact sequence above (see [16]).

We also define a notation. For A and $B \in M_3(\Lambda_E)$, we define

$$A \sim B \Longleftrightarrow PAQ = B \quad \text{for some } P, Q \in GL_3(\Lambda_E).$$

This is an equivalence relation on $M_3(\Lambda_E)$.

Remark 6.1.1. In general, $\mathrm{Fitt}_{i,\Lambda_E}(M)$ $(i \geq 0)$ do not determine the isomorphism class of M. Indeed, suppose that $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ with α, β, and $\gamma \in \mathbb{Z}_p$. We assume that $\mathrm{ord}_p(\alpha - \beta) = \mathrm{ord}_p(\beta - \gamma) = \mathrm{ord}_p(\gamma - \alpha) = 1$. For $[M(0,1,2)]$ and $[M(1,1,0)] \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$, we have

$$\mathrm{Fitt}_{1,\Lambda_{\mathbb{Q}_p}}(M(0,1,2)) = \mathrm{Fitt}_{1,\Lambda_{\mathbb{Q}_p}}(M(1,1,0)) = (p, T).$$

However, by Corollary 4.1.8, we have $[M(0,1,2)] \neq [M(1,1,0)]$.

In the following, we write $\mathrm{Fitt}_i(M)$ for $\mathrm{Fitt}_{i,\Lambda_E}(M)$ for simplicity. The main theorem in this chapter is the following, whose proof will be given in Section 6.2.

Theorem 6.1.2. *Let $[M(m,n,x)]_E$ and $[M(m',n',x')]_E$ be elements of $\mathcal{M}_{f(T)}^E$. Put $M = M(m,n,x)$ and $M' = M(m',n',x')$. The following statements are equivalent:*

(i) *We have $M \cong M'$ as Λ_E-modules.*

(ii) *We have $m(M) = m(M')$, $n(M) = n(M')$, and $\mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(M')$, where $m(M)$ and $n(M)$ are defined by*

$$m(M) = \mathrm{ord}_E(\beta - \alpha) - m, \quad n(M) = \mathrm{ord}_E(\gamma - \beta) + \mathrm{ord}_E(x) - n.$$

To prove Theorem 6.1.2, we prepare the following

Lemma 6.1.3. *There exists an exact sequence of Λ_E-modules*

$$0 \to \Lambda_E^3 \xrightarrow{\varphi} \Lambda_E^3 \to M \to 0$$

such that the matrix A_φ corresponding to the Λ_E-homomorphism φ is of the form

$$A_\varphi = \begin{pmatrix} T-\alpha & 0 & 0 \\ u_1 & T-\beta & 0 \\ w & u_2 & T-\gamma \end{pmatrix} \tag{6.1}$$

for some u_1, u_2, and $w \in \mathcal{O}_E$.

Proof. There exists an exact sequence

$$0 \to \Lambda_E \otimes_{\mathcal{O}_E} M \xrightarrow{\Phi} \Lambda_E \otimes_{\mathcal{O}_E} M \xrightarrow{\Psi} M \to 0,$$

where Φ and Ψ are defined as follows:

$$\begin{aligned} \Phi(a \otimes m) &= Ta \otimes m - a \otimes Tm, \\ \Psi(a \otimes m) &= am. \end{aligned}$$

We take $(1,1,1), (0, \pi^m, x)$, and $(0,0,\pi^n)$ as a basis of M. Then we have

$$\begin{aligned} T(1,1,1) &= \alpha(1,1,1) + (\beta-\alpha)\pi^{-m}(0,\pi^m,x) \\ &\quad + \{\gamma - \alpha - (\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0,0,\pi^n) \\ T(0,\pi^m,x) &= (0, \beta\pi^m, \gamma x) \\ &= \beta(0,\pi^m,x) + (\gamma-\beta)x\pi^{-n}(0,0,\pi^n), \text{ and} \\ T(0,0,\pi^n) &= \gamma(0,0,\pi^n). \end{aligned}$$

Therefore the matrix corresponding to Φ is

$$\begin{pmatrix} T-\alpha & 0 & 0 \\ -(\beta-\alpha)\pi^{-m} & T-\beta & 0 \\ -\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n} & -(\gamma-\beta)x\pi^{-n} & T-\gamma \end{pmatrix}. \tag{6.2}$$

Take $u_1 = -(\beta-\alpha)\pi^{-m}$, $u_2 = -(\gamma-\beta)x\pi^{-n}$, and $w = -\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}$. Since $\Lambda_E \otimes_{\mathcal{O}_E} M \cong \Lambda_E^{\oplus 3}$, we get the conclusion. □

Remark 6.1.4. (i) By elementary row and column operations, we can more simplify the matrix A_φ and get

$$A_\varphi \sim \begin{pmatrix} T-\alpha & 0 & 0 \\ \pi^m & T-\beta & 0 \\ x & \pi^n & T-\gamma \end{pmatrix}, \tag{6.3}$$

where m and n are non-negative integers and $x \in \mathcal{O}_E$. Indeed, let $u_1 = u_1'\pi^m$ and $u_2 = u_2'\pi^n$, where u_1 and $u_2 \in \mathcal{O}_E{}^\times$ and m, n are non-negative integers. Then we have

$$A_\varphi \sim \begin{pmatrix} T-\alpha & 0 & 0 \\ \pi^m & T-\beta & 0 \\ wu_1'^{-1}u_2'^{-1} & \pi^n & T-\gamma \end{pmatrix}.$$

(ii) If $m \geq \mathrm{ord}_E(\beta-\alpha)$ in the matrix (6.3), using elementary row and column operations, we find that A_φ is equivalent to

$$\begin{pmatrix} T-\alpha & 0 & 0 \\ \pi^{\mathrm{ord}_E(\beta-\alpha)} & T-\beta & 0 \\ * & \pi^n & T-\gamma \end{pmatrix},$$

where

$$* = \frac{\pi^{\mathrm{ord}_E(\beta-\alpha)}}{\beta-\alpha}\left\{x+\left(\frac{\pi^m}{\beta-\alpha}-1\right)\pi^n\right\}.$$

Thus we always may assume that $\pi^m \neq 0$ in other words, $m \neq \infty$. This implies that $0 \leq m \leq \mathrm{ord}_E(\alpha-\beta)$. By the same argument above, we may assume that $n \neq \infty$ and $0 \leq n \leq \mathrm{ord}_E(\beta-\gamma)$.

(iii) By elementary row and column operations for the matrix (6.2), we get the matrix

$$A = \begin{pmatrix} T-\alpha & 0 & 0 \\ (\beta-\alpha)\pi^{-m} & T-\beta & 0 \\ -\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n} & (\gamma-\beta)x\pi^{-n} & T-\gamma \end{pmatrix}. \tag{6.4}$$

In the following, we suppose $\mathrm{ord}_E(x) \leq n$ and $x \neq 0$ for a module $M(m,n,x)$.

Proposition 6.1.5. *Let $[M(m,n,x)]_E$ be an element of $\mathcal{M}^E_{f(T)}$. If we have a matrix corresponding to $M(m,n,x)$ of the form*

$$\begin{pmatrix} T-\alpha & 0 & 0 \\ \pi^{m'} & T-\beta & 0 \\ x' & \pi^{n'} & T-\gamma \end{pmatrix},$$

then we get

$$m' = \mathrm{ord}_E(\beta-\alpha) - m, \quad n' = \mathrm{ord}_E(\gamma-\beta) + \mathrm{ord}_E(x) - n.$$

Proof. We put $M = M(m,n,x)$. By assumptions, there is a basis of M e_1, e_2, and e_3 satisfying

$$\begin{aligned} (T-\alpha)e_1 &= -\pi^{m'}e_2 - x'e_3, \\ (T-\beta)e_2 &= -\pi^{n'}e_3, \text{ and} \\ (T-\gamma)e_3 &= 0. \end{aligned}$$

It is easy to see that

$$\begin{aligned}\mathrm{Fitt}_1(M) &= ((T-\beta)(T-\gamma),(T-\alpha)(T-\beta),(T-\alpha)(T-\gamma),\\ &\quad \pi^{m'+n'}-x'(T-\beta),\pi^{m'}(T-\gamma),\pi^{n'}(T-\alpha)).\end{aligned} \tag{6.5}$$

Since $(T-\beta)e_3=(\gamma-\beta)e_3$ and $n'\le \mathrm{ord}_E(\gamma-\beta)$, we have $(T-\beta)M=\langle (T-\beta)e_1,(T-\beta)e_2\rangle$. Further we have

$$\begin{aligned}T(T-\beta)e_1 &= (T-\beta)(\alpha e_1-\pi^{m'}e_2-x'e_3)\\ &= \alpha(T-\beta)e_1-\pi^{m'}(T-\beta)e_2+x'\frac{\gamma-\beta}{\pi^{n'}}(T-\beta)e_2\\ &= \alpha(T-\beta)e_1-\left(\pi^{m'}-\frac{\gamma-\beta}{\pi^{n'}}x'\right)(T-\beta)e_2,\\ T(T-\beta)e_2 &= (T-\beta)(\beta e_2-\pi^{n'}e_3)\\ &= \beta(T-\beta)e_2-\pi^{n'}(\gamma-\beta)e_3\\ &= \gamma(T-\beta)e_2.\end{aligned}$$

Thus we obtain

$$\mathrm{Fitt}_1((T-\beta)M)=\left(T-\gamma,\gamma-\alpha,\pi^{m'}-\frac{\gamma-\beta}{\pi^{n'}}x'\right). \tag{6.6}$$

Next, we take $(1,1,1),(0,\pi^m,x)$, and $(0,0,\pi^n)$ as a basis of M. Then we have the matrix (6.4) corresponding to a finite presentation of M and

$$\begin{aligned}\mathrm{Fitt}_1(M) &= ((T-\alpha)(T-\beta),(T-\alpha)(T-\gamma),(T-\beta)(T-\gamma),\\ &\quad (\beta-\alpha)\pi^{-m}(T-\gamma),(\gamma-\beta)x\pi^{-n}(T-\alpha),\Delta(T)),\end{aligned} \tag{6.7}$$

where

$$\Delta(T)=(\beta-\alpha)\pi^{-m}(\gamma-\beta)x\pi^{-n}+(T-\beta)\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}.$$

Since we have $\mathrm{ord}_E(x)\le n$, $(T-\beta)M$ is generated by

$$\begin{cases}(\alpha-\beta,0,\gamma-\beta) \text{ and } (0,0,(\gamma-\beta)x) & \text{if } x\ne 0,\\ (\alpha-\beta,0,\gamma-\beta) \text{ and } (0,0,(\gamma-\beta)\pi^n) & \text{if } x=0\end{cases}$$

and we obtain

$$\begin{aligned}T(\alpha-\beta,0,\gamma-\beta) &= \alpha(\alpha-\beta,0,\gamma-\beta)+(\gamma-\alpha)x^{-1}((0,0,(\gamma-\beta)x)),\\ T(0,0,(\gamma-\beta)x) &= \gamma(0,0,(\gamma-\beta)x).\end{aligned}$$

Thus we get

$$\mathrm{Fitt}_1((T-\beta)M) = (T-\gamma, (\gamma-\alpha)x^{-1}). \tag{6.8}$$

To get the conclusion, we consider the following case (I) and case (II).
(I) We suppose that $\mathrm{ord}_E(\gamma-\alpha) < \mathrm{ord}_E(\pi^{m'} - \frac{\gamma-\beta}{\pi^{n'}}x')$. First, we show $n' = \mathrm{ord}_E(\gamma-\beta) - n + \mathrm{ord}_E(x)$. By (6.6), we have

$$\mathrm{Fitt}_1((T-\beta)M) = (T-\gamma, \gamma-\alpha).$$

On the other hand, by (6.8), we get

$$\mathrm{Fitt}_1((T-\beta)M) = (T-\gamma, (\gamma-\alpha)x^{-1}).$$

Thus we obtain $\mathrm{ord}_E(x) = 0$. Further by (6.7) and $\Delta(\gamma) = (\gamma-\beta)(\gamma-\alpha)\pi^{-n}$ we have

$$\mathrm{Fitt}_1(M) \bmod (T-\gamma) = ((\gamma-\alpha)(\gamma-\beta)\pi^{-n}) \tag{6.9}$$

and we get

$$\mathrm{Fitt}_1(M) \bmod (T-\gamma) = ((\gamma-\alpha)\pi^{n'})$$

by (6.5) and the assumption $\mathrm{ord}_E(\gamma-\alpha)+n' < \mathrm{ord}_E(m'+n'-(\gamma-\beta)x')$. Therefore we obtain $n' = \mathrm{ord}_E(\gamma-\beta) - n = \mathrm{ord}_E(\gamma-\beta) - n + \mathrm{ord}_E(x)$. Next, we show $m' = \mathrm{ord}_E(\beta-\alpha) - m$. By (6.7) and $\mathrm{ord}_E(x) \leq n$, we have

$$\mathrm{Fitt}_1(M) \bmod (T-\beta) = ((\beta-\alpha)\pi^{-m}(\beta-\gamma)x\pi^{-n})$$

and we get

$$\mathrm{Fitt}_1(M) \bmod (T-\beta) = (\pi^{m'+n'})$$

by (6.5), $m \leq \mathrm{ord}_E(\alpha-\beta)$ and $n' \leq \mathrm{ord}_E(\gamma-\beta)$. Therefore we obtain $m'+n' = \mathrm{ord}_E(\beta-\alpha) - m + \mathrm{ord}_E(\gamma-\beta) + \mathrm{ord}_E(x) - n$ and $m' = \mathrm{ord}_E(\beta-\alpha) - m$.
(II) We suppose that $\mathrm{ord}_E(\gamma-\alpha) \geq \mathrm{ord}_E(\pi^{m'} - \frac{\gamma-\beta}{\pi^{n'}}x')$. First, we show $n' = \mathrm{ord}_E(\gamma-\beta) - n + \mathrm{ord}_E(x)$. By (6.6), we have

$$\mathrm{Fitt}_1((T-\beta)M) = \left(T-\gamma, \pi^{m'} - \frac{\gamma-\beta}{\pi^{n'}}x'\right).$$

On the other hand, by (6.8), we get

$$\mathrm{Fitt}_1((T-\beta)M) = (T-\gamma, (\gamma-\alpha)x^{-1}).$$

Thus we obtain $\mathrm{ord}_E(\gamma-\alpha) - \mathrm{ord}_E(x) = \mathrm{ord}_E(\pi^{m'} - \frac{\gamma-\beta}{\pi^{n'}}x')$. Further, by (6.7), we have

$$\mathrm{Fitt}_1(M) \bmod (T-\gamma) = ((\gamma-\alpha)(\gamma-\beta)\pi^{-n})$$

and by (6.5), we get

$$\mathrm{Fitt}_1(M) \bmod (T-\gamma) = \left(\pi^{n'}\left(\pi^{m'} - \frac{\gamma-\beta}{\pi^{n'}}x'\right)\right).$$

Therefore we obtain $n' = \mathrm{ord}_E(\gamma-\beta) - n + \mathrm{ord}_E(x)$. Finally, we show $m' = \mathrm{ord}_E(\beta-\alpha) - m$. By (6.7), we have

$$\mathrm{Fitt}_1(M) \bmod (T-\beta) = ((\beta-\alpha)\pi^{-m}(\beta-\gamma)x\pi^{-n})$$

and by (6.5) we get

$$\mathrm{Fitt}_1(M) \bmod (T-\beta) = (\pi^{m'+n'}).$$

Therefore we obtain $m'+n' = \mathrm{ord}_E(\beta-\alpha) - m + \mathrm{ord}_E(\gamma-\beta) + \mathrm{ord}_E(x) - n$ and $m' = \mathrm{ord}_E(\beta-\alpha) - m$. $\square$

By Proposition 6.1.5, we have the following

Corollary 6.1.6. *Let $[M(m,n,x)]_E$ be an element of $\mathcal{M}^E_{f(T)}$. If the matrices*

$$\begin{pmatrix} T-\alpha & 0 & 0 \\ \pi^{m_1} & T-\beta & 0 \\ x_1 & \pi^{n_1} & T-\gamma \end{pmatrix}$$

and

$$\begin{pmatrix} T-\alpha & 0 & 0 \\ \pi^{m_2} & T-\beta & 0 \\ x_2 & \pi^{n_2} & T-\gamma \end{pmatrix}$$

present the module M, then we get

$$m_1 = m_2 \quad \textit{and} \quad n_1 = n_2.$$

Put $M = M(m,n,x)$. By Corollary 6.1.6, we denote m_1, n_1, and x_1 by $m(M)$, $n(M)$, and $x(M)$, respectively. By Proposition 6.1.5, we have

$$m(M) = \mathrm{ord}_E(\beta-\alpha) - m, \quad n(M) = \mathrm{ord}_E(\gamma-\beta) + \mathrm{ord}_E(x) - n.$$

6.2 Proof of Theorem 6.1.2

In this section, we prove Theorem 6.1.2. First, by [12, Lemma 4.1], we get the following

Proposition 6.2.1. *Let $[M]_E$ and $[M']_E$ be elements of $\mathcal{M}^E_{f(T)}$. If M is isomorphic to M' as a Λ_E-module, then we have $m(M) = m(M')$ and $n(M) = n(M')$.*

Lemma 6.2.2. *Let $[M(m,n,x)]_E$ and $[M(m',n',x')]_E$ be elements of $\mathcal{M}^E_{f(T)}$. Suppose that $m(M) = m(M')$, $n(M) = n(M')$, and $\mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(M')$. Then we have $m = m'$, $n = n'$, and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$.*

Proof. We put $M = M(m,n,x)$ and $M' = M'(m',n',x')$. First, we show $m = m'$. By Proposition 6.1.5, we have $\mathrm{ord}_E(\beta-\alpha) - m = \mathrm{ord}_E(\beta-\alpha) = m'$. Thus we obtain $m = m'$. Next, we show $n = n'$ and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. By Proposition 6.1.5, we have

$$\mathrm{ord}_E(x) - n = \mathrm{ord}_E(x') - n'.$$

By the equation (6.9), we have

$$\mathrm{Fitt}_1(M) \bmod (T-\gamma) = ((\gamma-\alpha)(\gamma-\beta)\pi^{-n}).$$

Since $\mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(M')$, we get $n = n'$. Therefore we have $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. □

Lemma 6.2.3. *Suppose that $[M(m,n,x)]_E, [M(m',n',x')]_E \in \mathcal{M}^E_{f(T)}$. Put $M = M(m,n,x)$ and $M' = M'(m',n',x')$. If $m(M) = m(M')$, $n(M) = n(M')$, and $\mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(M')$, then there exist s, v, and $w \in \mathcal{O}_E$ satisfying*

$$x'(x'-\pi^m)s - \pi^n v + \pi^n x' w = x' - x. \tag{6.10}$$

Proof. By assumptions and Lemma 6.2.2, we have $m = m'$, $n = n'$, and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. We consider the following case (I) and case (II).

(I) We suppose $\mathrm{ord}_E(\beta-\alpha)\pi^{-m} \leq \mathrm{ord}_E(\gamma-\beta)x\pi^{-n}$, in other words $m(M) \leq$

$n(M)$. In this case, by (6.7), we obtain

$$\begin{aligned}
\mathrm{Fitt}_1(M) &= ((\beta-\alpha)\pi^{-m}(T-\gamma), \\
&\qquad (\gamma-\beta)x\pi^{-n}(\gamma-\alpha), \Delta(T), (T-\alpha)(T-\gamma)), \\
\mathrm{Fitt}_1(M') &= ((\beta-\alpha)\pi^{-m}(T-\gamma), \\
&\qquad (\gamma-\beta)x'\pi^{-n}(\gamma-\alpha), \Delta'(T), (T-\alpha)(T-\gamma)),
\end{aligned}$$

where

$$\begin{aligned}
\Delta(T) &= (\beta-\alpha)\pi^{-m}(\gamma-\beta)x\pi^{-n} + (T-\beta)\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}, \\
\Delta'(T) &= (\beta-\alpha)\pi^{-m}(\gamma-\beta)x'\pi^{-n} + (T-\beta)\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x'\}\pi^{-n}.
\end{aligned}$$

We note that

$$\begin{aligned}
\Delta(T) &= (T-\gamma)\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n} \\
&\quad +(\gamma-\alpha)(\gamma-\beta)\pi^{-n}, \qquad (6.11) \\
\Delta'(T) &= (T-\gamma)\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x'\}\pi^{-n} \\
&\quad +(\gamma-\alpha)(\gamma-\beta)\pi^{-n}. \qquad (6.12)
\end{aligned}$$

Since $\Delta(T) \in \mathrm{Fitt}_1(M')$, there exist $h_i(T) \in \Lambda$ $(i = 1, 2, 3, 4)$ satisfying

$$\begin{aligned}
\Delta(T) &= h_1(T)(\beta-\alpha)\pi^{-m}(T-\gamma) + h_2(T)(\gamma-\beta)x'\pi^{-n}(\gamma-\alpha) \\
&\quad +h_3(T)\Delta'(T) + h_4(T)(T-\alpha)(T-\gamma). \qquad (6.13)
\end{aligned}$$

By the equation (6.11), we have $\Delta(\gamma) = (\gamma-\alpha)(\gamma-\beta)\pi^{-n}$. By (6.13), we get

$$(\gamma-\alpha)(\gamma-\beta)\pi^{-n} = h_2(\gamma)(\gamma-\beta)x'\pi^{-n}(\gamma-\alpha) + h_3(\gamma)(\gamma-\alpha)(\gamma-\beta)\pi^{-n}.$$

Thus we obtain

$$1 = h_2(\gamma)x + h_3(\gamma).$$

Therefore, there exists a polynomial $g(T) \in \Lambda$ such that

$$h_3(T) = 1 - h_2(T)x' - (T-\gamma)g(T). \qquad (6.14)$$

Since $\Delta(\beta) = (\beta-\alpha)\pi^{-m}(\gamma-\beta)x\pi^{-n}$, by (6.13), we get

$$\begin{aligned}
(\beta-\alpha)\pi^{-m}(\gamma-\beta)x\pi^{-n} &= h_1(\beta)(\beta-\alpha)\pi^{-m}(\beta-\gamma) \\
&\quad +h_2(\beta)(\gamma-\beta)x'\pi^{-n}(\gamma-\alpha) \\
&\quad +h_3(\beta)(\beta-\alpha)\pi^{-m}(\gamma-\beta)x'\pi^{-n} \\
&\quad +h_4(\beta)(\beta-\alpha)(\beta-\gamma).
\end{aligned}$$

Therefore we have

$$\begin{aligned}(\beta-\alpha)\pi^{-m}x &= -h_1(\beta)(\beta-\alpha)\pi^{-m}\pi^n + h_2(\beta)x'(\gamma-\alpha)\\ &\quad +h_3(\beta)(\beta-\alpha)\pi^{-m}x' - h_4(\beta)(\beta-\alpha)\pi^n.\end{aligned}$$

Since $h_3(\beta) = 1 - h_2(\beta)x' - (\beta-\gamma)g(\beta)$ by (6.14), we get

$$\begin{aligned}(\beta-\alpha)\pi^{-m}x &= -h_1(\beta)(\beta-\alpha)\pi^{-m}\pi^n + h_2(\beta)x'(\gamma-\alpha)\\ &\quad +\{1 - h_2(\beta)x' - (\beta-\gamma)g(\beta)\}(\beta-\alpha)\pi^{-m}x'\\ &\quad -h_4(\beta)(\beta-\alpha)\pi^n.\end{aligned}$$

Thus we have

$$\begin{aligned}(x'-x) &= h_1(\beta)\pi^n - h_2(\beta)x'(\gamma-\alpha)(\beta-\alpha)^{-1}\pi^m\\ &\quad +\{h_2(\beta)x' + (\beta-\gamma)g(\beta)\}x' + h_4(\beta)\pi^n\pi^m\\ &= h_1(\beta)\pi^n - h_2(\beta)x'\left(1+\frac{\gamma-\beta}{\beta-\alpha}\right)\pi^m\\ &\quad +\{h_2(\beta)x' + (\beta-\gamma)g(\beta)\}x' + h_4(\beta)\pi^n\pi^m\\ &= x'(x'-\pi^m)h_2(\beta)\\ &\quad -\pi^n\left(-h_1(\beta) - h_4(\beta)\pi^m + \frac{\gamma-\beta}{\beta-\alpha}h_2(\beta)x'\pi^m\pi^{-n}\right).\end{aligned}$$

Put $s = h_2(\beta)$, $v = -h_1(\beta) - h_4(\beta)\pi^m + \frac{\gamma-\beta}{\beta-\alpha}h_2(\beta)x'\pi^m\pi^{-n}$, and $w = 0$. We note that we have $s \in \mathcal{O}_E$ by the assumption (I). Thus we get the conclusion.
(II) We suppose the case $\mathrm{ord}_E(\beta-\alpha)\pi^{-m} > \mathrm{ord}_E(\gamma-\beta)x\pi^{-n}$. In this case, the 1-st Fitting ideals of M and M' are

$$\begin{aligned}\mathrm{Fitt}_1(M) &= ((\beta-\alpha)\pi^{-m}(\alpha-\gamma), (\gamma-\beta)x\pi^{-n}(T-\alpha),\\ &\quad \Delta(T), (T-\alpha)(T-\gamma)),\\ \mathrm{Fitt}_1(M') &= ((\beta-\alpha)\pi^{-m}(\alpha-\gamma), (\gamma-\beta)x'\pi^{-n}(T-\alpha),\\ &\quad \Delta'(T), (T-\alpha)(T-\gamma)).\end{aligned}$$

Since $\Delta(T) \in \mathrm{Fitt}_1(M')$, there exist $h_i'(T) \in \Lambda$ for $i = 1,2,3,4$ satisfying

$$\begin{aligned}\Delta(T) &= h_1'(T)(\beta-\alpha)\pi^{-m}(\alpha-\gamma) + h_2'(T)(\gamma-\beta)x'\pi^{-n}(T-\alpha)\\ &\quad +h_3'(T)\Delta'(T) + h_4'(T)(T-\alpha)(T-\gamma).\end{aligned} \tag{6.15}$$

By (6.15), we get

$$(\gamma-\alpha)(\gamma-\beta)\pi^{-n} = h_1'(\gamma)(\beta-\alpha)\pi^{-m}(\alpha-\gamma) + h_2'(\gamma)(\gamma-\beta)x'\pi^{-n}(\gamma-\alpha) + h_3'(\gamma).$$

Thus we obtain

$$1 = -h_1'(\gamma)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)\pi^{-n}} + h_2'(\gamma)x' + h_3'(\gamma)(\gamma-\alpha)(\gamma-\beta)\pi^{-n}.$$

We note that we have $\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)\pi^{-n}} \in \mathcal{O}_E$ by the assumption (II). Therefore, there exists a polynomial $g'(T) \in \Lambda$ such that

$$h_3'(T) = 1 + h_1'(T)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)\pi^{-n}} - h_2'(T)x' + g'(T)(T-\gamma). \tag{6.16}$$

Since we have $\Delta(\alpha) = (\alpha-\gamma)(\beta-\alpha)(1-\pi^{-m}x)\pi^{-n}$ and (6.15), we obtain

$$\begin{aligned}(\alpha-\gamma)(\beta-\alpha)(1-\pi^{-m}x)\pi^{-n} &= h_1'(\alpha)(\beta-\alpha)\pi^{-m}(\alpha-\gamma)\\ &\quad +h_3'(\alpha)(\alpha-\gamma)(\beta-\alpha)(1-\pi^{-m}x')\pi^{-n}.\end{aligned}$$

Therefore we have

$$(1-\pi^{-m}x)\pi^{-n} = h_1'(\alpha)\pi^{-m} + h_3'(\alpha)(1-\pi^{-m}x')\pi^{-n}.$$

Since $h_3'(\alpha) = 1 + h_1'(\alpha)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)\pi^{-n}} - h_2'(\alpha)x' + g'(\alpha)(\alpha-\gamma)$ by (6.16), we get

$$\begin{aligned}(\pi^m - x) &= h_1'(\alpha)\pi^n\\ &\quad + \left\{1 + h_1'(\alpha)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)\pi^{-n}} - h_2'(\alpha)x' + g'(\alpha)(\alpha-\gamma)\right\}(\pi^m - x').\end{aligned}$$

Thus we have

$$\begin{aligned}(x'-x) &= \left\{h_1'(\alpha)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)\pi^{-n}} - h_2'(\alpha)x' + g'(\alpha)(\alpha-\gamma)\right\}(\pi^m - x')\\ &\quad +h_1'(\alpha)\pi^n\\ &= x'(x'-\pi^m)\left\{-h_1'(\alpha)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)x'\pi^{-n}} + h_2'(\alpha) - g'(\alpha)(\alpha-\gamma)x'^{-1}\right\}\\ &\quad +\pi^n h_1'(\alpha).\end{aligned}$$

Put $s = -h_1'(\alpha)\frac{(\beta-\alpha)\pi^{-m}}{(\gamma-\beta)x'\pi^{-n}} + h_2'(\alpha) - g'(\alpha)(\alpha-\gamma)x'^{-1} \in \mathcal{O}_E$ and $v = h_1'(\alpha)$. Thus we get the conclusion. $\square$

Lemma 6.2.4. *Let $[M]_E$ and $[M']_E$ be elements of $\mathcal{M}^E_{f(T)}$. Suppose that $M = M(0,n,x)$ and $M' = M'(0,n',x')$. Suppose also that $n(M) = n(M')$ and $\mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(M')$. Then we have $1-x \equiv \varepsilon(1-x') \mod \pi^n$ for some $\varepsilon \in \mathcal{O}_E^\times$.*

Proof. By Lemma 6.2.2, we have $n = n'$ and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. By (6.7), we have

$$\mathrm{Fitt}_1(M) \bmod (T-\alpha) = ((\alpha-\beta)(\alpha-\gamma), (\alpha-\beta)(\alpha-\gamma)(1-x)\pi^{-n})$$

and

$$\mathrm{Fitt}_1(M') \bmod (T-\alpha) = ((\alpha-\beta)(\alpha-\gamma), (\alpha-\beta)(\alpha-\gamma)(1-x')\pi^{-n}).$$

Since $\mathrm{Fitt}_1(M) = \mathrm{Fitt}_1(M')$, we get

$$1-x \equiv 0 \mod \pi^n \Longleftrightarrow 1-x' \equiv 0 \mod \pi^n.$$

Hence if $1-x \equiv 0 \mod \pi^n$, then we obtain $1-x \equiv \varepsilon(1-x') \mod \pi^n$ for some $\varepsilon \in \mathcal{O}_E^\times$. If $1-x \not\equiv 0 \mod \pi^n$, then we have $(\alpha-\beta)(\alpha-\gamma)(1-x)\pi^{-n} = \varepsilon(\alpha-\beta)(\alpha-\gamma)(1-x')\pi^{-n}$ for some $\varepsilon \in \mathcal{O}_E^\times$. Therefore we get $1-x \equiv \varepsilon(1-x') \mod \pi^n$. □

Proof of Theorem 6.1.2. We show that (ii) implies (i). Put $M = M(m,n,x)$ and $M' = M'(m',n',x')$. By Lemma 6.2.2, we have $m = m'$, $n = n'$, and $\mathrm{ord}_E(x) = \mathrm{ord}_E(x')$. Suppose that $m, n \neq 0$, and $\mathrm{ord}_E(x) \neq n$. Then we get $M \cong M'$, using Lemma 6.2.3 and [12, Lemma 4.9]. Suppose $m = 0$ and $n \neq 0$. Then we get $M \cong M'$ by Lemma 6.2.4 and [12, Proposition 4.11]. Suppose $n = 0$. Since $M(m,0,x) = M(m,0,0)$, we have $M(m,0,x) = M(m,0,x') = M(m,0,0)$. Therefore we get the conclusion. □

6.3 Complementary Properties

In this section, we show some propositions in order to determine the Iwasawa module associated to an imaginary quadratic field in Chapter 7.

For a non-negative integer n, we put $\omega_n = \omega_n(T) = (1+T)^{p^n} - 1$.

Proposition 6.3.1. *For a distinguished polynomial $f(T) \in \mathbb{Z}_p[T]$, let E be the splitting field of $f(T)$ over $\mathbb{Q}_p$. Then the natural map*

$$\Psi : \mathcal{M}_{f(T)}^{\mathbb{Q}_p} \longrightarrow \mathcal{M}_{f(T)}^{E} \qquad ([M] \longmapsto [M \otimes_\Lambda \Lambda_E]_E)$$

is injective.

Proof. We suppose that $M \otimes_\Lambda \Lambda_E \cong M' \otimes_\Lambda \Lambda_E$ for $[M]$ and $[M'] \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$. Since $M \otimes_\Lambda \Lambda_E \cong M^n$ as Λ-modules, we get $M^n \cong M'^n$ as Λ-modules, where n is the degree of the extension $E/\mathbb{Q}_p$.

We assume that $M \ncong M'$ as Λ-modules. Since M is a finitely generated Λ-module, M is a profinite module and we have $M = \varprojlim M/\mathfrak{m}^n M$, where $\mathfrak{m} = (\pi, T)$. Since $M \ncong M'$, there exists a positive integer ℓ such that $M/\mathfrak{m}^\ell M \ncong M'/\mathfrak{m}^\ell M'$ [19, Proposition 5]. Since both $M/\mathfrak{m}^\ell M$ and $M'/\mathfrak{m}^\ell M'$ are of finite length, we can decompose these modules into indecomposable modules

$$M/\mathfrak{m}^\ell M = \bigoplus_i N_i^{\oplus e_i}, \quad M'/\mathfrak{m}^\ell M' = \bigoplus_i N_i^{\oplus e_i'},$$

where N_i's are indecomposable modules, $N_i \neq N_j$ $(i \neq j)$ and e_i, e_i' are non-negative integers. By Krull-Remak-Schmidt's theorem, there exists i such that $e_i \neq e_i'$. Furthermore we have

$$(M/\mathfrak{m}^\ell M)^n = \bigoplus_i N_i^{\oplus n e_i}, \quad (M'/\mathfrak{m}^\ell M')^n = \bigoplus_i N_i^{\oplus n e_i'}.$$

Thus we get $ne_i \neq ne_i'$ for some i. By Krull-Remak-Schmidt's theorem, we have $(M/\mathfrak{m}^\ell M)^n \ncong (M'/\mathfrak{m}^\ell M')^n$. This implies $M^n \ncong M'^n$. This contradicts our assumption. $\square$

Let $f(T) \in \mathbb{Z}_p[T]$ be a distinguished polynomial and E the splitting field of $f(T)$. We put

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where α, β, and $\gamma \in \pi\mathcal{O}_E$.

Proposition 6.3.2. *Let E and $f(T)$ be the same as above. Suppose that $[M]_E \in \mathcal{M}_{f(T)}^E$. If M is a cyclic Λ_E-module, then we have*

$$M \cong M(\mathrm{ord}_E(\beta - \alpha), \mathrm{ord}_E(\gamma - \alpha) + \mathrm{ord}_E(\gamma - \beta), u\pi^{\mathrm{ord}_E(\beta - \alpha)})$$

as Λ_E-modules, where $u = \dfrac{\gamma - \alpha}{\beta - \alpha}$.

Proof. Suppose that $M \cong M(m,n,x) \subset \mathcal{E}$. Suppose also that M is cyclic and put

$$M = \langle (a,b,c) \rangle_{\Lambda_E} \subset \mathcal{E}$$

for some a, b, and $c \in \mathcal{O}_E$. Since $(1,1,1) \in \langle (a,b,c) \rangle_{\Lambda_E}$, we have $(1,1,1) = h(T)(a,b,c) = (h(\alpha)a, h(\beta)b, h(\gamma)c)$ for some $h(T) \in \Lambda_E$. Therefore we get a, b, and $c \in \mathcal{O}_E^\times$. Since $(0,\pi^m,x)$ and $(0,0,\pi^n) \in \langle (a,b,c) \rangle_{\Lambda_E}$, we have

$$\begin{aligned} (0,\pi^m,x) &= q(T)(a,b,c) = (q(\alpha)a, q(\beta)b, q(\gamma)c), \\ (0,0,\pi^n) &= r(T)(a,b,c) = (r(\alpha)a, r(\beta)b, r(\gamma)c) \end{aligned}$$

for some $q(T)$ and $r(T) \in \Lambda_E$. Since $(T-\alpha)|q(T)$ and $(T-\alpha)(T-\beta)|r(T)$, we get $m = \mathrm{ord}_E(q(\beta)) \geq \mathrm{ord}_E(\beta-\alpha)$ and $n = \mathrm{ord}_E(r(\gamma)) \geq \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta)$. On the other hand, by Proposition 4.1.3 and Remark 6.1.4, we have $m \leq \mathrm{ord}_E(\beta-\alpha)$ and $n \leq \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta)$. Therefore we obtain $m = \mathrm{ord}_E(\beta-\alpha)$ and $n = \mathrm{ord}_E(\gamma-\alpha) + \mathrm{ord}_E(\gamma-\beta)$. Furthermore,

$$\begin{aligned} (T-\alpha)(1,1,1) &= (0, \beta-\alpha, \gamma-\alpha) \\ &= (\beta-\alpha)\pi^{-m}(0,\pi^m,x) \\ &\quad + \{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0,0,\pi^n). \end{aligned}$$

Since $\mathrm{ord}_E\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\} \geq n$, we have $x = \dfrac{\gamma-\alpha}{\beta-\alpha}\pi^m\left(1-\dfrac{\pi^n v}{\gamma-\alpha}\right)$ for some $v \in \mathcal{O}_E$. By Remark 6.1.4 (i) , we get

$$M(m,n,x) = M(\mathrm{ord}_E(\beta-\alpha), \mathrm{ord}_E(\gamma-\alpha)+\mathrm{ord}_E(\gamma-\beta), u\pi^{\mathrm{ord}_E(\beta-\alpha)}). \qquad \square$$

Proposition 6.3.3. *Let $f(T)$ be the same as above. Assume that* $\mathrm{ord}_E(\alpha-\beta) = \mathrm{ord}_E(\beta-\gamma) = \mathrm{ord}_E(\gamma-\alpha) = 1$ *and* $\mathrm{ord}_E(\alpha) \geq \mathrm{ord}_E(\beta) \geq \mathrm{ord}_E(\gamma)$*. Then, we have*

$$\mathcal{M}_{f(T)}^E = \{(0,0,0), (0,1,0), (1,0,0), (0,1,1), (1,2,u\pi), (1,1,0), (0,1,2)\},$$

where $u = \dfrac{\gamma-\alpha}{\beta-\alpha}$ *and (m,n,x) means $[M(m,n,x)]_E$. The following is the table of the structure of $\mathcal{O}_E$-modules $M/\omega_0 M$ for Λ_E-modules M.*

M	$M/\omega_0 M$
$M(0,0,0)$	$\mathcal{O}_E/(\alpha) \oplus \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\gamma)$
$M(0,1,0)$	$\mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$
$M(0,1,1)$	$\mathcal{O}_E/(\alpha) \oplus \mathcal{O}_E/(\beta\gamma)$
$M(0,1,2)$	$\mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$
$M(1,0,0)$	$\mathcal{O}_E/(\gamma) \oplus \mathcal{O}_E/(\alpha\beta)$
$M(1,1,0)$	$\mathcal{O}_E/(\gamma) \oplus \mathcal{O}_E/(\alpha\beta)$
$M(1,2,u\pi)$	$\mathcal{O}_E/(\alpha\beta\gamma)$

Proof. The former is Corollary 4.1.8. We show the latter. Let $[M]_E$ be an element of $\mathcal{M}^E_{f(T)}$. There exist m, n, and x such that

$$M = \langle (1,1,1), (0, \pi^m, x), (0,0,\pi^n) \rangle_{\mathcal{O}_E}.$$

Hence we have

$$\omega_0 M = \langle (\alpha, \beta, \gamma), (0, \beta\pi^m, \gamma x), (0,0,\gamma\pi^n) \rangle_{\mathcal{O}_E}.$$

Since $\mathcal{O}_E$ is a principal ideal domain, we can use the structure theorem over the principal ideal domain. We consider the map $\Pi_{\omega_0} : M \longrightarrow M$ and take $(1,1,1), (0,\pi^m,x)$, and $(0,0,\pi^n)$ as a basis of M. Then we have

$$\begin{aligned} T(1,1,1) &= \alpha(1,1,1) + (\beta-\alpha)\pi^{-m}(0,\pi^m,x) \\ &\quad + \{\gamma - \alpha - (\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0,0,\pi^n), \qquad (6.17) \\ T(0,\pi^m,x) &= (0,\beta\pi^m,\gamma x) \\ &= \beta(0,\pi^m,x) + (\gamma-\beta)x\pi^{-n}(0,0,\pi^n). \qquad (6.18) \end{aligned}$$

By the equalities (6.17) and (6.18), the matrix corresponding to Π_{ω_0} is

$$\begin{pmatrix} \alpha & 0 & 0 \\ (\beta-\alpha)\pi^{-m} & \beta & 0 \\ \{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n} & (\gamma-\beta)x\pi^{-n} & \gamma \end{pmatrix}.$$

In order to verify the table, we have only to transform this matrix by elementary row and column operations. For example, in the case of $M = M(0,1,0)$, we get the matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta-\alpha & \beta & 0 \\ (\gamma-\alpha)\pi^{-1} & 0 & \gamma \end{pmatrix}.$$

By the elementary row and column operations, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\gamma \end{pmatrix}.$$

Hence we get $M/\omega_0 M \cong \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$. The remaining cases of the table can be checked by the same method. $\square$

Proposition 6.3.4. *Put $f(T) = (T-\alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$. Let $g(T) \in \mathbb{Z}_p[T]$ be a distinguished irreducible polynomial of degree 2 and E the splitting field of $g(T)$ over $\mathbb{Q}_p$. If $[M(m,n,x)]_E \in$ Image $(\Psi : \mathcal{M}^{\mathbb{Q}_p}_{f(T)} \longrightarrow \mathcal{M}^{E}_{f(T)} \quad ([M] \longmapsto [M \otimes_\Lambda \Lambda_E]_E))$, we have*

$$\mathrm{ord}_E(x) = m.$$

Proof. Let $[M]$ be an element of $\mathcal{M}^{\mathbb{Q}_p}_{f(T)}$. We suppose that $M \otimes \Lambda_E \cong M(m,n,x) \subset \mathcal{E}$. There is a natural injective map

$$M \longrightarrow \Lambda/(f(T)) \longrightarrow \Lambda/(T-\alpha) \oplus \Lambda/(g(T))$$

[21, Lemma 13.8]. By this injective map, we have

$$M = \langle (a_1, b_1T + c_1), (a_2, b_2T + c_2), (a_3, b_3T + c_3) \rangle_{\mathbb{Z}_p} \subset \Lambda/(T-\alpha) \oplus \Lambda/(g(T))$$

for some a_i, b_i, and $c_i \in \mathbb{Z}_p$. Since we have

$$M \otimes_\Lambda \Lambda_E = \langle (a_1, b_1T + c_1), (a_2, b_2T + c_2), (a_3, b_3T + c_3) \rangle_{\mathcal{O}_E}$$

by the same argument before Lemma 5.1.1, we can write

$$M \otimes_\Lambda \Lambda_E = \langle (a'_1, b'_1T + c'_1), (0, b'_2T + c'_2), (0, c'_3) \rangle_{\mathcal{O}_E}$$

for some a'_i, b'_i, and $c'_i \in \mathbb{Z}_p$. Furthermore there is an injective map [21, Lemma 13.8]

$$\Lambda_E/(T-\alpha) \oplus \Lambda_E/(g(T)) \longrightarrow \mathcal{E}, \quad (s(t), u(t)) \longmapsto (s(\alpha), u(\beta), u(\gamma)),$$

where β and γ are the roots of $g(T)$ in E. By this map, $M \otimes_\Lambda \Lambda_E$ is isomorphic to the module

$$M' = \langle (a'_1, b'_1\beta + c'_1, b'_1\gamma + c'_1), (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

Since β and γ are conjugate, we have $\mathrm{ord}_E(b'_1\beta + c'_1) = \mathrm{ord}_E(b'_1\gamma + c'_1)$ and $\mathrm{ord}_E(b'_2\beta + c'_2) = \mathrm{ord}_E(b'_2\gamma + c'_2)$. By the same arguments after Lemma 4.1.2, we get

$$M' \cong \langle (1,1,1), (0, \pi^m, x), (0,0,\pi^n) \rangle_{\mathcal{O}_E}$$

for some m, n, and x which satisfy $m = \mathrm{ord}_E(x)$. Indeed, we may assume that $\mathrm{ord}_E(b'_2\beta + c'_2) \le \mathrm{ord}_E(c'_3)$. By Lemma 4.1.2, we have

$$M' \cong \langle (1, b'_1\beta + c'_1, b'_1\gamma + c'_1), (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E}.$$

In the case of $\mathrm{ord}_E(b'_1\beta + c'_1) \le \mathrm{ord}_E(b'_2\beta + c'_2)$, we have

$$M' \cong \left\langle \left(1, 1, b'_1\gamma + c'_1\right), \left(0, \frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}, b'_2\gamma + c'_2\right), \left(0, \frac{c'_3}{b'_1\beta + c'_1}, c'_3\right) \right\rangle_{\mathcal{O}_E}.$$

Since $\mathrm{ord}_E(b'_1\gamma + c'_1) \le \mathrm{ord}_E(b'_2\gamma + c'_2) \le \mathrm{ord}_E(c'_3)$, we get

$$\begin{aligned} M' &\cong \left\langle (1,1,1), \left(0, \frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}, \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right), \left(0, \frac{c'_3}{b'_1\beta + c'_1}, \frac{c'_3}{b'_1\gamma + c'_1}\right) \right\rangle_{\mathcal{O}_E} \\ &= \langle (1,1,1), s, t \rangle_{\mathcal{O}_E}, \end{aligned}$$

where

$$\begin{aligned} s &= \left(0, \frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}, \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right), \\ t &= \left(0, 0, \frac{c'_3}{b'_1\gamma + c'_1} - \frac{c'_3}{b'_2\beta + c'_2} \cdot \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right). \end{aligned}$$

Thus we get

$$\begin{aligned} m &= \mathrm{ord}_E\left(\frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}\right), \quad x = \pi^{-m}\frac{b'_1\beta + c'_1}{b'_2\beta + c'_2} \cdot \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}, \text{ and} \\ n &= \mathrm{ord}_E\left(\frac{c'_3}{b'_1\gamma + c'_1} - \frac{c'_3}{b'_2\beta + c'_2} \cdot \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right). \end{aligned}$$

Therefore we obtain $m = \mathrm{ord}_E(x)$. On the other hand, in the case of $\mathrm{ord}_E(b'_1\beta + c'_1) > \mathrm{ord}_E(b'_2\beta + c'_2)$, we have

$$\begin{aligned} M' &= \langle (a'_1, (b'_1 - b'_2)\beta + (c'_1 - c'_2), (b'_1 - b'_2)\gamma + (c'_1 - c'_2)), \\ &\quad (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3)\rangle_{\mathcal{O}_E}. \end{aligned}$$

Since $\mathrm{ord}_E(b'_1\beta + c'_1 - (b'_2\beta + c'_2)) = \mathrm{ord}_E(b'_2\beta + c'_2)$, we get the same conclusion as in the case of $\mathrm{ord}_E(b'_1\beta + c'_1) \leq \mathrm{ord}_E(b'_2\beta + c'_2)$. □

Proposition 6.3.5. *Suppose that $f(T) = (T - \alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$. Let $g(T) \in \mathbb{Z}_p[T]$ be an Eisenstein irreducible polynomial of degree 2 and E the splitting field of $g(T)$ over $\mathbb{Q}_p$. Assume that $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\beta - \gamma) = \mathrm{ord}_E(\gamma - \alpha) = 1$,*

$$M/\omega_0 M \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z} \qquad (i, j \in \mathbb{Z}_{\geq 1}).$$

Then we have

$$\Psi(M) = M \otimes_\Lambda \Lambda_E \cong M(0,1,1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma).$$

Proof. Since $M/\omega_0 M \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$, we have $M/\omega_0 M \otimes_\Lambda \Lambda_E \cong \mathcal{O}_E/(\pi^{2i}) \oplus \mathcal{O}_E/(\pi^{2j})$. Since $E/\mathbb{Q}_p$ is a totally ramified extension, $\mathrm{ord}_E(\alpha) = 2\mathrm{ord}_p(\alpha) \geq 2$. Thus we get $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$. Since $\mathrm{ord}_E(\pi^{2i}) = 2i$ and $\mathrm{ord}_E(\pi^{2j}) = 2j$ are even, we get

$$M \otimes_\Lambda \Lambda_E \cong M(0,1,1)$$

by the table of the Proposition 6.3.3. The isomorphism $M(0,1,1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma)$ is proved in [20, Lemma 3]. □

Corollary 6.3.6. *Let $f(T), g(T)$, and E be the same as in Propositions 6.3.5. Suppose that $[M]_{\mathbb{Q}_p} \in \mathcal{M}^{\mathbb{Q}_p}_{f(T)}$. Suppose also the same conditions of Proposition 6.3.5. Put $g(T) = T^2 + c_1 T + c_0$. Then the following* (a) *and* (b) *hold.*

(a) *Suppose $p \geq 5$. For $n \geq 0$, we have*

$$\sharp(M/\omega_n M \otimes \Lambda_E) = p^{\mathrm{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+2+\mathrm{ord}_E(\alpha)}.$$

Further we have

$$M/\omega_n M \otimes \Lambda_E \cong \mathcal{O}_E/(\pi^{\mathrm{ord}_E(\alpha)+2n}) \oplus \mathcal{O}_E/(\pi^{2n+2}) \oplus \mathcal{O}_E/(\pi^{2n}).$$

(b) *Suppose that $p=3$ and $(c_0,c_1)\neq(3,3)$. For $n\geq 1$, we have*

$$\sharp(M/\omega_n M\otimes\Lambda_E)=\begin{cases} p^{\mathrm{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))}=p^{6n+\mathrm{ord}_E(\alpha)+4\mathrm{ord}_3(c_0-3)-2} \\ \qquad\qquad \text{if} \quad \mathrm{ord}_3(c_0-3)\leq\mathrm{ord}_3(c_1-3), \\ p^{\mathrm{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))}=p^{6n+\mathrm{ord}_E(\alpha)+4\mathrm{ord}_3(c_1-3)} \\ \qquad\qquad \text{if} \quad \mathrm{ord}_3(c_0-3)>\mathrm{ord}_3(c_1-3). \end{cases}$$

Further we have

$$M/\omega_n M\otimes\Lambda_E\cong\begin{cases} \mathcal{O}_E/(\pi^{\mathrm{ord}_E(\alpha)+2n})\oplus\mathcal{O}_E/(\pi^{2\mathrm{ord}_3(c_0-3)+2n})\oplus \\ \mathcal{O}_E/(\pi^{2\mathrm{ord}_3(c_0-3)+2n-2}) \qquad \text{if} \quad \mathrm{ord}_3(c_0-3)\leq\mathrm{ord}_3(c_1-3), \\ \mathcal{O}_E/(\pi^{\mathrm{ord}_E(\alpha)+2n})\oplus\mathcal{O}_E/(\pi^{2\mathrm{ord}_3(c_1-3)+2n})\oplus \\ \mathcal{O}_E/(\pi^{2\mathrm{ord}_3(c_0-3)+2n}) \qquad \text{if} \quad \mathrm{ord}_3(c_0-3)>\mathrm{ord}_3(c_1-3). \end{cases}$$

Proof. Put $N=\langle(1,1,1),(0,1,1),(0,0,\pi)\rangle_{\mathcal{O}_E}\subset\mathcal{E}$. We have $M\otimes_\Lambda\Lambda_E\cong N$ as Λ_E-modules by Proposition 6.3.5. Thus we have

$$M/\omega_n M\otimes\Lambda_E\cong(M\otimes_\Lambda\Lambda_E)/\omega_n(M\otimes_\Lambda\Lambda_E)\cong N/\omega_n N$$

as $\Lambda_E/\omega_n\Lambda_E$-modules. By the same method as Proposition 6.3.3, we consider the map $\Pi_{\omega_n}:N\longrightarrow N$ and take $(1,0,0),(0,1,1)$ and $(0,0,\pi)$ as a basis of N. The matrix corresponding to Π_{ω_n} is

$$\begin{pmatrix} \omega_n(\alpha) & 0 & 0 \\ 0 & \omega_n(\beta) & 0 \\ 0 & (\omega_n(\beta)-\omega_n(\gamma))\pi^{-1} & \omega_n(\gamma) \end{pmatrix}.$$

We first consider the case (a). We have $\mathrm{ord}_E(\omega_n(\beta)-\omega_n(\gamma))=\mathrm{ord}_E(\beta-\gamma)+n\mathrm{ord}_E(3)=2n+1$ (cf. [9, Lemma 2.5]). Furthermore, we have $\mathrm{ord}_E(\omega_n(\alpha))=2n+\mathrm{ord}_E(\alpha)$ and we get $\mathrm{ord}_E\{(\omega_n(\beta)-\omega_n(\gamma))\pi^{-1}\}=2n<\mathrm{ord}_E(\omega_n(\beta))$ since $\mathrm{ord}_E(\omega_n(\beta))=\mathrm{ord}_E(\omega_n(\gamma))=2n+1$. Thus we can transform the matrix above into

$$\begin{pmatrix} \pi^{2n+\mathrm{ord}_E(\alpha)} & 0 & 0 \\ 0 & \pi^{2n} & 0 \\ 0 & 0 & \pi^{2n+2} \end{pmatrix}.$$

This implies $N/\omega_n N \cong \mathcal{O}_E/(\pi^{2n+\mathrm{ord}_E(\alpha)}) \oplus \mathcal{O}_E/(\pi^{2n}) \oplus \mathcal{O}_E/(\pi^{2n+2})$.

Next, we prove the case (b). For $n \geq 1$, we have

$$\mathrm{ord}_E(\omega_n(\beta)) = \begin{cases} 2\mathrm{ord}_3(c_0 - 3) + 2n - 1 & \text{if} \quad \mathrm{ord}_3(c_0 - 3) \leq \mathrm{ord}_3(c_1 - 3), \\ 2\mathrm{ord}_3(c_1 - 3) + 2n & \text{if} \quad \mathrm{ord}_3(c_0 - 3) > \mathrm{ord}_3(c_1 - 3). \end{cases}$$

On the other hand, for $n \geq 1$, we have

$$\mathrm{ord}_E(\omega_n(\beta) - \omega_n(\gamma)) \begin{cases} = 2\mathrm{ord}_3(c_0 - 3) + 2n - 1 \\ \qquad \text{if} \quad \mathrm{ord}_3(c_0 - 3) \leq \mathrm{ord}_3(c_1 - 3), \\ > 2\mathrm{ord}_3(c_1 - 3) + 2n \\ \qquad \text{if} \quad \mathrm{ord}_3(c_0 - 3) > \mathrm{ord}_3(c_1 - 3) \end{cases}$$

(cf. [9, Lemma 2.5]). The case (b) can be proved by the same method as the case (a). $\square$

Proposition 6.3.7. *Suppose that $f(T) = (T - \alpha)g(T)$, where $\alpha \in p\mathbb{Z}_p$. Let $g(T) \in \mathbb{Z}_p[T]$ be an irreducible polynomial of degree 2 and E the splitting field of $g(T)$ over $\mathbb{Q}_p$. Let $[M]_E$ be an element of $\mathcal{M}^E_{f(T)}$. Put $M = M(m, n, x)$.*

(1) Assume that $m = 0$ and $(\gamma - \beta)x\pi^{-n} \in \mathcal{O}_E^\times$. Then we have

$$\mathrm{Fitt}_{1,\Lambda}(M) = \begin{cases} (T - \alpha, (\alpha - \beta)(\alpha - \gamma)) & \text{if} \quad x = 1, \\ (T - \alpha, (\alpha - \beta)(\alpha - \gamma)(1 - x)\pi^{-n}) & \text{if} \quad x \neq 1. \end{cases}$$

(2) Assume that $n = 0$ and $(\beta - \alpha)\pi^{-m} \in \mathcal{O}_E^\times$. Then we have

$$\mathrm{Fitt}_{1,\Lambda}(M) = (T - \gamma, (\alpha - \gamma)(\beta - \gamma)).$$

(3) We have

$$\mathrm{Fitt}_{1,\Lambda}((T - \alpha)M) = \begin{cases} (T - \beta, (\beta - \gamma)\pi^{-n}) & \text{if} \quad n \leq \mathrm{ord}_E(\pi^m - x), \\ \left(T - \beta, \dfrac{\gamma - \beta}{\pi^m - x}\right) & \text{if} \quad n > \mathrm{ord}_E(\pi^m - x). \end{cases}$$

Proof. By the action of T, we have

$$\begin{aligned}
T(1,1,1) &= (\alpha,\beta,\gamma) \\
&= \alpha(1,1,1) + (\beta-\alpha)\pi^{-m}(0,\pi^m,x) \\
&\quad +\{\gamma-\alpha-(\beta-\alpha)\pi^{-m}x\}\pi^{-n}(0,0,\pi^n), \\
T(0,\pi^m,x) &= (0,\beta\pi^m,\gamma x) \\
&= \beta(0,\pi^m,x) + (\gamma-\beta)x\pi^{-n}(0,0,\pi^n), \text{ and} \\
T(0,0,\pi^n) &= \gamma(0,0,\pi^n).
\end{aligned}$$

Then we get the following matrix

$$\begin{pmatrix} T-\alpha & -(\beta-\alpha)\pi^{-m} & -\{(\gamma-\alpha)-(\beta-\alpha)\pi^{-m}x\}\pi^{-n} \\ 0 & T-\beta & -(\gamma-\beta)x\pi^{-n} \\ 0 & 0 & T-\gamma \end{pmatrix}.$$

We first show (1). Under the assumption of (1), the matrix is

$$\begin{pmatrix} T-\alpha & -\beta+\alpha & -\{(\gamma-\alpha)-(\beta-\alpha)x\}\pi^{-n} \\ 0 & T-\beta & -(\gamma-\beta)x\pi^{-n} \\ 0 & 0 & T-\gamma \end{pmatrix}.$$

By elementary row and column operations, we can transform the matrix above into

$$\begin{pmatrix} T-\alpha & (\alpha-\gamma)(1-x)\pi^{-n}(T-\beta) & 0 \\ 0 & (T-\beta)(T-\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore we get

$$\begin{aligned}
\mathrm{Fitt}_{1,\Lambda}(M) &= (T-\alpha,(\alpha-\beta)(\alpha-\gamma),(\alpha-\beta)(\alpha-\beta)(1-x)\pi^{-n}) \\
&= \begin{cases} (T-\alpha,(\alpha-\beta)(\alpha-\gamma)) & \text{if} \quad x=1, \\ (T-\alpha,(\alpha-\beta)(\alpha-\gamma)(1-x)\pi^{-n}) & \text{if} \quad x\neq 1. \end{cases}
\end{aligned}$$

Next, we show (2). Under the assumption of (2), the matrix is

$$\begin{pmatrix} T-\alpha & -(\beta-\alpha)\pi^{-m} & -(\gamma-\alpha)+(\beta-\alpha)\pi^{-m}x \\ 0 & T-\beta & -(\gamma-\beta)x \\ 0 & 0 & T-\gamma \end{pmatrix}.$$

By elementary row and column operations, we can transform the above matrix into

$$\begin{pmatrix} T-\alpha & 1 & 0 \\ 0 & T-\beta & 0 \\ 0 & 0 & T-\gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \mathrm{Fitt}_{1,\Lambda}(M) &= ((T-\alpha)(T-\beta), (T-\beta)(T-\gamma), (T-\alpha)(T-\gamma), (T-\gamma)) \\ &= (T-\gamma, (\alpha-\gamma)(\beta-\gamma)). \end{aligned}$$

Finally, we show (3). We note that

$$\begin{aligned} (T-\alpha)M &= \langle (0, \beta-\alpha, \gamma-\alpha), (0, (\beta-\alpha)\pi^m, (\gamma-\alpha)x), (0, 0, (\gamma-\alpha)\pi^n) \rangle_{\mathcal{O}_E} \\ &= \begin{cases} \langle (0, \beta-\alpha, \gamma-\alpha), (0, 0, (\gamma-\alpha)\pi^n) \rangle_{\mathcal{O}_E} \\ \quad \text{if} \quad n \leq \mathrm{ord}_E(\pi^m - x), \\ \langle (0, \beta-\alpha, \gamma-\alpha), (0, 0, (\gamma-\alpha)(\pi^m - x)) \rangle_{\mathcal{O}_E} \\ \qquad \text{if} \quad n > \mathrm{ord}_E(\pi^m - x). \end{cases} \end{aligned}$$

In the case of $n \leq \mathrm{ord}_E(\pi^m - x)$, by the action of T, we have

$$\begin{aligned} T(0, \beta-\alpha, \gamma-\alpha) &= (0, \beta(\beta-\alpha), \gamma(\gamma-\alpha)) \\ &= \beta(0, \beta-\alpha, \gamma-\alpha) + (\gamma-\beta)\pi^{-n}(0, 0, (\gamma-\alpha)\pi^n), \\ T(0, 0, (\gamma-\alpha)\pi^n) &= \gamma(0, 0, (\gamma-\alpha)\pi^n). \end{aligned}$$

Thus we get the following matrix

$$\begin{pmatrix} T-\beta & -(\gamma-\beta)\pi^{-n} \\ 0 & T-\gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \mathrm{Fitt}_{1,\Lambda}((T-\alpha)M) &= (T-\beta, T-\gamma, (\gamma-\beta)\pi^{-n}) \\ &= (T-\beta, (\gamma-\beta)\pi^{-n}). \end{aligned}$$

In the case of $n > \mathrm{ord}_E(\pi^m - x)$, by the same method as above, we get the following matrix

$$\begin{pmatrix} T-\beta & -\dfrac{\gamma-\beta}{\pi^m - x} \\ 0 & T-\gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \mathrm{Fitt}_{1,\Lambda}((T-\alpha)M) &= \left(T-\beta, T-\gamma, \frac{\gamma-\beta}{\pi^m - x}\right) \\ &= \left(T-\beta, \frac{\gamma-\beta}{\pi^m - x}\right). \quad \square \end{aligned}$$

Next, we consider the case of $\deg f(T) = 4$. Let $f(T) \in \mathbb{Z}_p[T]$ be a distinguished polynomial with $\deg f(T) = 4$. Then we have the following

Proposition 6.3.8. *Let E be the splitting field of $f(T)$ over $\mathbb{Q}_p$. Let $[M]_E$ be an element of $\mathcal{M}^E_{f(T)}$. Put $M = M(\ell, m, n; x, y, z)$. Then we have*

$$\begin{aligned} \mathrm{Fitt}_{1,\Lambda_E}(M) \mod (T-\delta) &= ((\delta-\alpha)(\delta-\beta)(\delta-\gamma)\pi^{-n}), \\ \mathrm{Fitt}_{1,\Lambda_E}(M) \mod (T-\gamma) &= \begin{cases} ((\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)z\pi^{-m-n}) \\ \qquad \text{if} \quad z \neq 0, \\ ((\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)\pi^{-m}) \\ \qquad \text{if} \quad z = 0. \end{cases} \end{aligned}$$

Proof. We put

$$\begin{aligned} e_1 &= (1,1,1,1), \\ e_2 &= (0, \pi^\ell, x, y), \\ e_3 &= (0, 0, \pi^m, z), \text{ and} \\ e_4 &= (0,0,0,\pi^n). \end{aligned}$$

By the action of T, we have

$$\begin{aligned}
Te_1 &= (\alpha, \beta, \gamma, \delta) \\
&= \alpha e_1 + (\beta - \alpha)\pi^{-\ell} e_2 + \{\gamma - \alpha - (\beta - \alpha)\pi^{-\ell} x\}\pi^{-m} e_3 \\
&\quad + \left[(\delta - \alpha) - (\beta - \alpha)\pi^{-\ell} y - \{(\gamma - \alpha) - (\beta - \alpha)\pi^{-\ell} x\}\pi^{-m} z\right] \frac{e_4}{\pi^n}, \\
Te_2 &= (0, \beta\pi^\ell, \gamma x, \delta y) \\
&= \beta e_2 + (\gamma - \beta) x \pi^{-m} e_3 + \{(\delta - \beta) y - (\gamma - \beta) x \pi^{-m} z\}\pi^{-n} e_4, \\
Te_3 &= (0, 0, \gamma\pi^m, \delta z) \\
&= \gamma e_3 + (\delta - \gamma) z \pi^{-n} e_4, \text{ and} \\
Te_4 &= \delta e_4.
\end{aligned}$$

Then we get the following matrix

$$\begin{pmatrix}
T - \alpha & -(\beta - \alpha)\pi^{-\ell} & -\{\gamma - \alpha - (\beta - \alpha)\pi^{-\ell} x\}\pi^{-m} & a_{14} \\
0 & T - \beta & -(\gamma - \beta) x \pi^{-m} & a_{24} \\
0 & 0 & T - \gamma & -(\delta - \gamma) z \pi^{-n} \\
0 & 0 & 0 & T - \delta
\end{pmatrix},$$

where

$$\begin{cases}
a_{24} &= -\{(\delta - \beta) y - (\gamma - \beta) x \pi^{-m} z\}\pi^{-n}, \\
a_{14} &= -\left[(\delta - \alpha) - (\beta - \alpha)\pi^{-\ell} y - \{(\gamma - \alpha) - (\beta - \alpha)\pi^{-\ell} x\}\pi^{-m} z\right] \pi^{-n}.
\end{cases}$$

We prove the former part. By the definition of Fitting ideals, we obtain

$$\begin{aligned}
&\mathrm{Fitt}_{1,\Lambda_E}(M) \mod (T - \delta) \\
&= (\widetilde{a_{41}}, (\delta - \alpha)(\delta - \beta)(\delta - \gamma) z \pi^{-n}, (\delta - \alpha)(\delta - \beta)(\delta - \gamma)\pi^{-n} y),
\end{aligned}$$

where

$$\widetilde{a_{41}} = \det \begin{pmatrix}
-(\beta - \alpha)\pi^{-\ell} & -\{\gamma - \alpha - (\beta - \alpha)\pi^{-\ell} x\}\pi^{-m} & a_{14} \\
T - \beta & -(\gamma - \beta) x \pi^{-m} & a_{24} \\
0 & T - \gamma & -(\delta - \gamma) z \pi^{-n}
\end{pmatrix}.$$

Since we have

$$\widetilde{a_{41}} \mod (T - \delta) = (\delta - \alpha)(\delta - \beta)(\delta - \gamma)\pi^{-n} \mod (T - \delta),$$

we obtain the conclusion. We can also prove the latter equation by the same method above. $\square$

Proposition 6.3.9. *Suppose that $f(T) = g(T)(T-\delta)$, where $\delta \in p\mathbb{Z}_p$. Let $g(T) \in \mathbb{Z}_p[T]$ be an Eisenstein polynomial of degree 3 and E the splitting field of $g(T)$ over $\mathbb{Q}_p$. Suppose that $[M]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ and $[M \otimes \Lambda_E] = [M(\ell, m, n; x, y, z)] \in \mathcal{M}_{f(T)}^{E}$. Assume that $\mathrm{ord}_E(\delta - \alpha) = \mathrm{ord}_E(\delta - \beta) = \mathrm{ord}_E(\delta - \gamma) = 1$ and*

$$M/TM \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z} \qquad (i, j \in \mathbb{Z}_{\geq 1}).$$

Then we have $n = 0$.

Proof. We have $\mathrm{Fitt}_{1,\Lambda_{\mathbb{Q}_p}}(M) \neq \Lambda_{\mathbb{Q}_p}$, since $\mathrm{Fitt}_{1,\mathbb{Z}_p}(M/TM) = (p^{\min\{i,j\}})$. By our assumption, $g(T)$ is an Eisenstein polynomial. Hence we have $\mathrm{Fitt}_{1,\Lambda_E}(M \otimes \Lambda_E) \mod (T-\delta) = (\pi^{3i})$ for some $i \geq 1$. Using Proposition 6.3.8, we obtain $\mathrm{Fitt}_{1,\Lambda_E}(M \otimes \Lambda_E) \mod (T-\delta) = (\pi^{3-n})$. This implies that $3i = 3 - n$. Thus we have $n = 0$. □

Chapter 7

Examples

In this chapter, we apply our Theorem 1 and Theorem 2 to Iwasawa Theory. We determine the isomorphism classes of Iwasawa modules associated to the cyclotomic $\mathbb{Z}_3$-extension of imaginary quadratic fields.

7.1 Numerical examples for $\lambda = 3$

In this section, we introduce some numerical examples which were computed using PARI/GP. We put $\Lambda = \mathbb{Z}_p[[T]]$.

We consider the case of $p = 3$ and $k = \mathbb{Q}(\sqrt{-d})$, where d is a positive square-free integer. For simplicity, let $d \not\equiv 2 \mod 3$. Our assumption $d \not\equiv 2 \mod 3$ implies that $p = 3$ is inert or ramifies in k. This assumption is also needed to get the isomorphism (7.1) below. In this section, we determine the Λ-isomorphism class of the Iwasawa module associated to $k = \mathbb{Q}(\sqrt{-d})$ in the range $1 < d < 10^5$ with $\lambda_p(k) = 3$, where $\lambda_p(k)$ is the Iwasawa λ-invariant with respect to the cyclotomic $\mathbb{Z}_p$-extension. There are 1109 imaginary quadratic fields satisfying these properties.

Let k_∞/k be the cyclotomic $\mathbb{Z}_p$-extension. For each $n \geq 0$, we denote by k_n the intermediate field of k_∞/k such that k_n is the unique cyclic extension over k of degree p^n. Let A_n be the p-Sylow subgroup of the ideal class group of k_n. We put $X_{k_\infty} = \varprojlim A_n$, where the inverse limit is taken with respect to the relative norms. Then X_{k_∞} becomes a $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$-module. Since there is a ring isomorphism between $\Lambda = \mathbb{Z}_p[[T]]$ and $\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$ which depends on the choice of a topological generator of $\mathrm{Gal}(k_\infty/k)$, X_{k_∞} becomes a finitely generated torsion Λ -module. Let $f(T)$ be the distinguished polynomial which generates $\mathrm{char}(X_{k_\infty})$. It is known that X_{k_∞} is a free $\mathbb{Z}_p$-module thus $[X_{k_\infty}]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ and we can apply Theorem 1 to the Iwasawa module X_{k_∞}.

We can calculate the polynomial $f(T)$ mod p^n for small n numerically. Let χ be the Dirichlet character associated to k, ω be the Teichimüler character, and f_0 be the least common multiple of p and conductor of χ. By the Iwasawa main conjecture, there exists a power series $g_{\chi^{-1}\omega}(T) \in \Lambda$ such that

$$\mathrm{char}(X_{k_\infty}) = (g_{\chi^{-1}\omega}(T)).$$

Here, $g_{\chi^{-1}\omega}(T)$ is the p-adic L-function constructed by Iwasawa. We can approximate $g_{\chi^{-1}\omega}(T)$ such as

$$g_{\chi^{-1}\omega}(T) \equiv -\frac{1}{2f_0p^n} \sum_{0<a<f_0p^n,(a,f_0p^n)=1} a\chi\omega^{-1}(a)(1+T)^{i_n(a)} \mod \omega_n,$$

where $i_n(a)$ is the unique integer such that $a\omega^{-1}(a) \equiv (1+p)^{i_n(a)} \mod p^{n+1}$ and $0 \leq i_n(a) < p^n$. By Weierstrass preparation theorem ([21, Theorem 7.3], there exists $u_{\chi^{-1}\omega} \in \Lambda^\times$ such that $g_{\chi^{-1}\omega}(T) = f(T)u_{\chi^{-1}\omega}(T)$. Thus we can get $f(T)$ approximately ([21, Proposition 7.2]. For the detail about computation of $g_{\chi^{-1}\omega}(T)$, see [2] and [6]. We computed $f(T)$ by Mizusawa's program Iwapoly.ub ([14, Research, Programing, Approximate Computation of Iwasawa Polynomials by UBASIC]), and referred Fukuda's table for the λ-invariants of imaginary quadratic fields [5].

Now we classify the Iwasawa module X_{k_∞}. There are two cases

$$\begin{cases} \text{(I)} & A_0 \text{ is a cyclic group ,} \\ \text{(II)} & A_0 \text{ is not a cyclic group .} \end{cases}$$

In order to determine the structure of X_{k_∞}, we use the following fact. In our case, exactly one prime ramifies in k_∞/k and it is totally ramified. Hence there are Λ-isomorphism

$$X_{k_\infty}/\omega_n X_{k_\infty} \cong A_n \tag{7.1}$$

for any non-negative integers [21, Proposition 13.22].

We determine the Λ-isomorphism class of X_{k_∞} by the information on the structures of A_n for some $n \geq 0$.

There are 1015 fields whose A_0 are cyclic groups among 1109 fields. First of all, we determine the isomorphism classes in the case (I). In this case, X_{k_∞} becomes a Λ_E-cyclic module by Nakayama's Lemma. Thus we can use Proposition 6.3.2 to get

$$M \cong M(\mathrm{ord}_E(\beta - \alpha), \mathrm{ord}_E(\gamma - \alpha) + \mathrm{ord}_E(\gamma - \beta), u\pi^{\mathrm{ord}_E(\beta-\alpha)}).$$

In the range of d above, no $f(T)$ splits completely in $\mathbb{Q}_p[T]$, thus we have to consider the minimal splitting field E of $f(T)$, which is quadratic over $\mathbb{Q}_p$.

Example 1. Put $k = \mathbb{Q}(\sqrt{-886})$. Then we have $A_0 \cong \mathbb{Z}/9\mathbb{Z}$ (cf. [17]). By using Mizusawa's program [14], we have

$$f(T) \equiv (T - 195)(T^2 + 291T + 429) \mod 3^6.$$

By Hensel's Lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T - \alpha)g(T),$$

where $\alpha \equiv 195 \bmod 3^5$ and $g(T) \equiv T^2 + 48T + 186 \bmod 3^5$. Since $g(T)$ is an Eisenstein polynomial, $E/\mathbb{Q}_p$ is a totally ramified extension. Let E be the minimal splitting field of $g(T)$. We put $g(T) = (T - \beta)(T - \gamma)$, where β and $\gamma \in E$. Since $\beta\gamma \equiv 186 \bmod 3^5$, we get $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$, and $\mathrm{ord}_E(\alpha-\gamma) = \mathrm{ord}_E(\alpha-\gamma) = 1$. Since $(\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma \equiv 1560 \bmod 3^5$, we have $\mathrm{ord}_E(\beta - \gamma) = 1$. By Proposition 6.3.1 and 6.3.2, we get $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(1, 2, u\pi)$, where $u = \dfrac{\gamma - \alpha}{\beta - \alpha}$.

Next, we determine the isomorphism classes in the case (II). There are 94 fields whose A_0 are not cyclic groups. There are 66 fields whose A_0 are not cyclic groups and whose $f(T)$ is reducible. We will determine $[X_{k_\infty}]_{\mathbb{Q}_p}$ for these 66 fields. We can determine the Λ-isomorphism class of X_{k_∞} for 60 fields by Proposition 6.3.5. The following example is the case where we can determine the Λ-isomorphism class of X_{k_∞} by Proposition 6.3.5.

Example 2. Put $k = \mathbb{Q}(\sqrt{-6583})$. In this case, we have $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (cf. [17]). We have

$$f(T) \equiv (T-96)(T^2+96T+696) \mod 3^6.$$

By Hensel's Lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T-\alpha)g(T),$$

where $\alpha \equiv 96 \bmod 3^5$ and $g(T) \equiv T^2+96T+210 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$. We put $g(T) = (T-\beta)(T-\gamma)$, where β and $\gamma \in E$. Then, $E/\mathbb{Q}_p$ is a totally ramified extension and we get $\mathrm{ord}_E(\alpha-\beta) = \mathrm{ord}_E(\beta-\gamma) = \mathrm{ord}_E(\gamma-\alpha) = 1$, $\mathrm{ord}_E(\alpha) = 2$, and $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$. Therefore we get $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,1,1)$ by Proposition 6.3.5.

There are remaining 6 fields which we cannot determine the structure of X_{k_∞} by Proposition 6.3.5. For these fields, we have to investigate the action of the group $\mathrm{Gal}(k_1/k)$. Explicitly, the remaining 6 fields are $\mathbb{Q}(\sqrt{-9574})$, $\mathbb{Q}(\sqrt{-30994})$, $\mathbb{Q}(\sqrt{-41631})$, $\mathbb{Q}(\sqrt{-64671})$, $\mathbb{Q}(\sqrt{-82774})$, and $\mathbb{Q}(\sqrt{-92515})$.

Example 3. Put $k = \mathbb{Q}(\sqrt{-9574})$. In this case, we have $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ (cf. [17]) and $A_1 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$. We have

$$f(T) \equiv (T-192)(T^2+1173T+1422) \mod 3^7.$$

By Hensel's Lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T-\alpha)g(T),$$

where $\alpha \equiv 192 \bmod 3^5$ and $g(T) \equiv T^2+201T+207 \bmod 3^5$. Let E be the splitting field of $g(T)$. We put $g(T) = (T-\beta)(T-\gamma)$, where β and $\gamma \in E$. Since the discriminant of $g(T)$ is $3^2 \cdot 4397 \bmod 3^7$ and 4397 is a quadratic nonresidue, $E/\mathbb{Q}_p$ is an unramified extension. Since the discriminant of $f(T)$ is $2^8 \cdot 3^6 \cdot 43 \cdot 89 \cdot 1039 \bmod 3^7$, we get $\mathrm{ord}_E(\alpha-\beta) = \mathrm{ord}_E(\beta-\gamma) = \mathrm{ord}_E(\gamma-\alpha) = 1$ and $\mathrm{ord}_E(\alpha) = \mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$. By checking the structures of A_0 and A_1 as $\mathcal{O}_E$-modules, we get

$$X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,1,1),\ M(0,1,2),\ M(1,0,0),\ \text{or } M(1,1,0).$$

Now we investigate the structure of A_1 as a $\mathrm{Gal}(k_1/k)$-module. We have an isomorphism $A_1 \cong \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Furthermore, PARI/GP gives explicit generators which give this isomorphism. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{a}_3$ be the generators which was computed. (We do not write down $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{a}_3$ because they are complicated.) Let σ be the generator of $\mathrm{Gal}(k_1/k)$, which was computed by PARI/GP. We compute,

$$\begin{aligned} (\sigma-1)\mathfrak{a}_1 &= 3\mathfrak{a}_2 - \mathfrak{a}_3, \\ (\sigma-1)\mathfrak{a}_2 &= 6\mathfrak{a}_2, \text{ and} \\ (\sigma-1)\mathfrak{a}_3 &= 18\mathfrak{a}_1 + 6\mathfrak{a}_2. \end{aligned}$$

There is a topological generator $\tilde{\sigma} \in \mathrm{Gal}(k_\infty/k)$ such that $\tilde{\sigma}$ is an extension of σ. By this topological generator, we have the isomorphism

$$\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]] \cong \Lambda = \mathbb{Z}_p[[T]] \text{ such that } \tilde{\sigma} \leftrightarrow 1+T.$$

We regard X_{k_∞} as a Λ-module by this isomorphism. We note that $f(T)$ depends on the choice of $\tilde{\sigma}$, but we can easily check that $\mathcal{M}^E_{f(T)}$ does not depend on the choice of $\tilde{\sigma}$. Because $\mathbb{Z}_p[[\mathrm{Gal}(k_1/k)]] \cong \Lambda/\omega_1\Lambda$, we get

$$\begin{aligned} \overline{T}\mathfrak{a}_1 &= 3\mathfrak{a}_2 - \mathfrak{a}_3, \\ \overline{T}\mathfrak{a}_2 &= 6\mathfrak{a}_2, \text{ and} \\ \overline{T}\mathfrak{a}_3 &= 18\mathfrak{a}_1 + 6\mathfrak{a}_2, \end{aligned}$$

where $\overline{T} = T \bmod \omega_1$. Now we have

$$\begin{aligned} \overline{(T^2+18)}\mathfrak{a}_1 + \overline{6}\mathfrak{a}_2 &= 0, \\ \overline{(T-6)}\mathfrak{a}_2 &= 0, \\ \overline{3T}\mathfrak{a}_1 &= 0, \\ \overline{27}\mathfrak{a}_1 &= 0, \text{ and} \\ \overline{9}\mathfrak{a}_2 &= 0. \end{aligned}$$

Therefore we can calculate the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T,3) \mod \omega_1,$$

where $\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E)$ is the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$ as a $\Lambda_E/\omega_1\Lambda_E$-module. On the other hand, by Proposition 6.3.7 (1) and (2) for $M(0,1,2)$, $M(1,0,0)$, and $M(0,1,1)$, we have

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(M/\omega_1 M) = \begin{cases} (T,3) & \mod \omega_1 \quad \text{if } M = M(0,1,2), \\ (T-\gamma, 9) & \mod \omega_1 \quad \text{if } M = M(1,0,0), \\ (T-\alpha, 9) & \mod \omega_1 \quad \text{if } M = M(0,1,1). \end{cases}$$

Therefore we have

$$X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,1,2) \text{ or } M(1,1,0).$$

We investigate the module $(T-\alpha)(M/\omega_1 M)$. By Proposition 6.3.7 (3), for $M(0,1,2)$ and $M(1,1,0)$ we get

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}((T-\alpha)(M/\omega_1 M)) = \begin{cases} (T,3) \mod \omega_1 & \text{if } M = M(0,1,2), \\ \Lambda_E/\omega_1\Lambda_E & \text{if } M = M(1,1,0). \end{cases}$$

We can compute the following from the data above

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\alpha)}A_1 \otimes \mathcal{O}_E) = (T,3) \mod \omega_1.$$

Therefore, we get $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,1,2)$.

By the same method as above, we can determine the isomorphism classes of X_{k_∞} of $\mathbb{Q}(\sqrt{-30994})$, $\mathbb{Q}(\sqrt{-82774})$, and $\mathbb{Q}(\sqrt{-92515})$. For the 3 fields, we can show that $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,1,2)$.

Finally, we determine the structure of X_{k_∞} for remaining 2 fields $\mathbb{Q}(\sqrt{-41631})$ and $\mathbb{Q}(\sqrt{-64671})$.

Example 4. Put $k = \mathbb{Q}(\sqrt{-41631})$. In this case, we have $A_0 \cong \mathbb{Z}/3^3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (cf. [17]) and $A_1 \cong \mathbb{Z}/3^4\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ by PARI/GP. We have

$$f(T) \equiv (T-42)(T^2 - 279T + 594) \mod 3^7.$$

By Hensel's Lemma, there exist $\alpha \in \mathbb{Z}_p$ and $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = (T-\alpha)g(T),$$

where $\alpha \equiv 42 \bmod 3^5$ and $g(T) \equiv T^2 + 36T + 108 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$. We put $g(T) = (T-\beta)(T-\gamma)$, where β and $\gamma \in E$. Then $E/\mathbb{Q}_p$ is a totally ramified extension with $\mathrm{ord}_E(\alpha-\beta) = \mathrm{ord}_E(\gamma-\alpha) = 2$, $\mathrm{ord}_E(\beta-\gamma) = 3$, $\mathrm{ord}_E(\alpha) = 2$, and $\mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 3$. Let π be a prime element of E. In this case, the elements $M(m,n,x) \in \mathcal{M}^E_{f(T)}$ which satisfy the conclusion of Proposition 6.3.4 are

$$\left\{ \begin{array}{l} (0,0,0),(0,1,1),(0,1,2),(0,2,1),(0,2,2),(0,2,1+\pi),(0,3,1), \\ (0,3,1+\pi),(0,3,1+\pi^2),(1,0,0),(1,1,0),(1,1,1),(1,2,\pi), \\ (1,2,2\pi),(1,3,\pi),(1,3,\pi+\pi^2),(1,3,\pi+2\pi^2),(1,4,u\pi), \\ (2,0,0),(2,1,0),(2,2,0),(2,3,u\pi^2),(2,4,u\pi^2),(2,5,u\pi^2) \end{array} \right\},$$

where $u = \dfrac{\gamma-\alpha}{\beta-\alpha}$. By checking the structures of A_0 and A_1 as $\mathcal{O}_E$-modules, we get

$$\begin{aligned} X_{k_\infty} \otimes_\Lambda \Lambda_E \;\cong\; & M(0,3,1),\ M(0,3,1+\pi),\ M(0,3,1+\pi^2), \\ & M(1,3,\pi+\pi^2),\ M(1,3,\pi+2\pi^2) \text{ or } M(2,3,u\pi^2). \end{aligned}$$

We have an isomorphism $A_1 \cong \mathbb{Z}/81\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{a}_3$ be the generators which were computed by PARI/GP. Further we have:

$$\begin{aligned} (\sigma-1)\mathfrak{a}_1 &= 54\mathfrak{a}_1 + 6\mathfrak{a}_2 + \mathfrak{a}_3, \\ (\sigma-1)\mathfrak{a}_2 &= 54\mathfrak{a}_1, \text{ and} \\ (\sigma-1)\mathfrak{a}_3 &= 54\mathfrak{a}_1 + 3\mathfrak{a}_2, \end{aligned}$$

for a certain generator σ of $\mathrm{Gal}(k_1/k)$ by PARI/GP. By the same method as $k = \mathbb{Q}(\sqrt{-9574})$, we fix a topological generator $\tilde{\sigma} \in \mathrm{Gal}(k_\infty/k)$ such that $\tilde{\sigma}$ is an extension of σ. Because $\mathbb{Z}_p[[\mathrm{Gal}(k_1/k)]] \cong \Lambda/\omega_1\Lambda$, we have

$$\begin{array}{rcl}\overline{(T^2-54T-54)}\mathfrak{a}_1 - \overline{3}\mathfrak{a}_2 & = & 0,\\ \overline{54}\ \mathfrak{a}_1 - \overline{T}\mathfrak{a}_2 & = & 0,\\ \overline{3T}\mathfrak{a}_1 \qquad\quad & = & 0,\\ \overline{81}\mathfrak{a}_1 \qquad\quad & = & 0, \text{ and}\\ \overline{9}\mathfrak{a}_2 & = & 0,\end{array}$$

where $\overline{T} = T \bmod \omega_1$. Therefore we get the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T,3) \mod \omega_1.$$

On the other hand, by Proposition 6.3.7 (1) and (2), we have

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(M/\omega_1 M) = \begin{cases}(T-\alpha,9) & \mod \omega_1 \quad \text{if} \quad M = M(0,3,1),\\ (T,3) & \mod \omega_1 \quad \text{if} \quad M = M(0,3,1+\pi),\\ (T-\alpha,\pi^3) & \mod \omega_1 \quad \text{if} \quad M = M(0,3,1+\pi^2)\end{cases}$$

for $M(0,3,1), M(0,3,1+\pi)$, and $M(0,3,1+\pi^2)$. Therefore we have

$X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,3,1+\pi),\ M(1,3,\pi+\pi^2),\ M(1,3,\pi+2\pi^2),$ or $M(2,3,u\pi^2)$.

As in the case where $k = \mathbb{Q}(\sqrt{-9574})$, we investigate the structure of $(T-\alpha)(M/\omega_1 M)$. By Proposition 6.3.7 (3), we get

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}((T-\alpha)(M/\omega_1 M)) = \begin{cases}(T,3) \mod \omega_1 & \text{if} \quad M = M(0,3,1+\pi),\\ \Lambda_E/\omega_1\Lambda_E & \text{if} \quad M = M(1,3,\pi+\pi^2),\\ (T,\pi) \mod \omega_1 & \text{if} \quad M = M(1,3,\pi+2\pi^2),\\ \Lambda_E/\omega_1\Lambda_E & \text{if} \quad M = M(2,3,u\pi^2).\end{cases}$$

We can compute from the data above

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\alpha)}A_1 \otimes \mathcal{O}_E) = (T,3) \mod \omega_1.$$

Therefore we get $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0,3,1+\pi)$.

We can determine the structure of $\mathbb{Q}(\sqrt{-64671})$ by the same method as above. For $\mathbb{Q}(\sqrt{-64671})$, we can show that $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(0, 3, 1 + \pi)$.

In the end of this chapter, we write down the table of the $X_{k_\infty} \otimes_\Lambda \Lambda_E$ for $p = 3$ and for the fields such that A_0 is not cyclic and $f(T)$ is reducible. On the table, m, n, x represent $X_{k_\infty} \otimes \Lambda_E \cong M(m, n, x)$, and ram./unram. means that $E/\mathbb{Q}_3$ is ramified /unramified extension, respectively. We marked $(*)$ on the remaining 6 fields for which we determined the structures in Example 3 and 4.

7.2 Numerical examples for $\lambda = 4$

Here, we consider the case of $p = 3$ and $k = \mathbb{Q}(\sqrt{-d})$, where $d = 5142$, 12453, 23683, 28477, and 78730. We also consider the case of $p = 5$ and $k = \mathbb{Q}(\sqrt{-15658})$. In this case, p does not split in k and we have $\lambda_p(k) = 4$, where $\lambda_p(k)$ is the Iwasawa λ-invariant with respect to the cyclotomic $\mathbb{Z}_p$-extension of k. As in the previous section, we determine the isomorphism class of X_{k_∞}.

For a non-negative integer n, we put $\omega_n = \omega_n(T) = (1+T)^{p^n} - 1$.

Example 5. Put $p = 3$ and $k = \mathbb{Q}(\sqrt{-12453})$. In this case, we have $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (cf. [17]). We have

$$f(T) \equiv (T^3 + 204T^2 + 567T + 426)(T + 525) \mod 3^6.$$

By Hensel's Lemma, there exist $\delta \in \mathbb{Z}_p$ and an irreducible polynomial $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = g(T)(T - \delta),$$

where $\delta \equiv 204 \bmod 3^5$ and $g(T) \equiv T^3 + 204T^2 + 81T + 183 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$. We put $g(T) = (T-\alpha)(T-\beta)(T-\gamma)$, where α, β, and $\gamma \in E$. Then $[E : \mathbb{Q}_p] = 3$ and the ramification index is 3 in $E/\mathbb{Q}_p$. Indeed, let $d(g)$ be the discriminant of $g(T)$. Then we have $d(g) \equiv (-1) \cdot 3^4 \cdot 13 \cdot 104 \equiv -162 \mod 3^5$. Thus we have $\sqrt{d(g)} \in \mathbb{Q}_p$. This implies that $[E : \mathbb{Q}_p] = 3$ and $E/\mathbb{Q}_p$ is a totally ramified extension. Further we have $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\beta - \gamma) = \mathrm{ord}_E(\gamma - \alpha) = 2$, $\mathrm{ord}_E(\alpha - \delta) = \mathrm{ord}_E(\beta - \delta) = \mathrm{ord}_E(\gamma - \delta) = 1$, $\mathrm{ord}_E(\alpha) = \mathrm{ord}_E(\beta) = \mathrm{ord}_E(\gamma) = 1$ and $\mathrm{ord}_E(\delta) = 3$. Suppose that $[X_{k_\infty} \otimes_\Lambda \Lambda_E] = [M(\ell, m, n; x, y, z)] \in \mathcal{M}^E_{f(T)}$. By Proposition 6.3.9, we have $n = 0$. Therefore we may assume that $[X_{k_\infty} \otimes_\Lambda \Lambda_E] = [M(\ell, m, 0; x, 0, 0)] = [M(\ell, m, x) \oplus \langle (0,0,0,1) \rangle_{\mathbb{Z}_p}]$, where $M(\ell, m, x)$ are defined before Theorem 1. Since we have $X_{k_\infty}/TX_{k_\infty} \otimes \mathcal{O}_E \cong A_0 \otimes \mathcal{O}_E \cong \mathcal{O}_E/(\pi^3) \oplus \mathcal{O}_E/(\pi^3)$, $M(\ell, m, x)/TM(\ell, m, x)$ is a cyclic module. Then M becomes a Λ_E-cyclic module by Nakayama's Lemma. Using Proposition 6.3.2, we have $M(\ell, m, x) = M(2, 4, u\pi^2)$, where $u = \dfrac{\gamma - \alpha}{\beta - \alpha}$. Hence we obtain $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(2, 4, 0; u\pi^2, 0, 0) \cong \Lambda_E/(T - \delta) \oplus \Lambda_E/(g(T))$.

By the same method as above, we can determine the isomorphism classes of X_{k_∞} of $\mathbb{Q}(\sqrt{-5142})$, $\mathbb{Q}(\sqrt{-23683})$, and $\mathbb{Q}(\sqrt{-28477})$. Next we consider the case of $p = 5$.

Example 6. Put $p = 5$ and $k = \mathbb{Q}(\sqrt{-15658})$. In this case, we have $A_0 \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ (cf. [17]). We have

$$f(T) \equiv (T^3 + 11740T^2 + 8565T + 14160)(T + 3295) \mod 5^6.$$

By Hensel's Lemma, there exist $\delta \in \mathbb{Z}_p$ and an irreducible polynomial $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = g(T)(T - \delta),$$

where $\delta \equiv 3295 \bmod 5^6$ and $g(T) \equiv T^3 + 11740T^2 + 8565T + 14160 \bmod 5^6$. In the same way as in the proof of Example 5, we obtain

$$X_{k_\infty} \otimes_\Lambda \Lambda_E \cong \Lambda_E/(T - \delta) \oplus \Lambda_E/(g(T)),$$

where E is the minimal splitting field of $g(T)$.

The following is an example that we have to investigate the action of the group $\mathrm{Gal}(k_1/k)$.

Example 7. Put $p = 3$ and $k = \mathbb{Q}(\sqrt{-78730})$. In this case, we have $A_0 \cong \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ (cf. [17]). We have

$$f(T) \equiv (T^2 + 4068T + 5817)(T + 3189)(T + 888) \mod 3^8.$$

By Hensel's Lemma, there exist γ, $\delta \in \mathbb{Z}_p$, and an irreducible polynomial $g(T) \in \mathbb{Z}_p[T]$ such that

$$f(T) = g(T)(T - \gamma)(T - \delta),$$

where $\gamma \equiv 84 \bmod 3^5$, $\delta \equiv 213 \bmod 3^5$, and $g(T) \equiv T^2 + 180T + 228 \bmod 3^5$. Let E be the minimal splitting field of $g(T)$. We put $g(T) = (T - \alpha)(T - \beta)$, where α and $\beta \in E$. Since $g(T)$ is an Eisenstein polynomial, the extension $E/\mathbb{Q}_p$ is a totally ramified extension. Therefore, we have $\mathrm{ord}_E(\alpha) = \mathrm{ord}_E(\beta) = 1$, $\mathrm{ord}_E(\gamma) = \mathrm{ord}_E(\delta) = 2$, $\mathrm{ord}_E(\gamma - \delta) = 2$, and $\mathrm{ord}_E(\alpha - \beta) = \mathrm{ord}_E(\beta - \gamma) = \mathrm{ord}_E(\beta - \delta) = \mathrm{ord}_E(\alpha - \delta) = \mathrm{ord}_E(\gamma - \alpha) = 1$. By Proposition 6.3.8, we obtain

$\mathrm{Fitt}_{1,\Lambda_E}(X_{k_\infty}\otimes\Lambda_E) \mod (T-\delta) = (\pi^{4-n})$. Since we have $A_0 \cong \mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$, we obtain $\mathrm{Fitt}_{1,\Lambda}(X_{k_\infty}) \mod (T-\delta) \neq \Lambda$. We put $\mathrm{Fitt}_{1,\Lambda}(X_{k_\infty}) \mod (T-\delta)$ $= (p^i)$ for some $i \geq 1$. Then we have $(\pi^{4-n}) = (\pi^{2i})$. This implies $4-n = 2i$. Clearly, we have $n = 0$ or $n = 2$. Using Proposition 6.3.8, we get

$$\mathrm{Fitt}_{1,\Lambda_E}(X_{k_\infty}\otimes\Lambda_E) \mod (T-\gamma) = \begin{cases} (\pi^{\mathrm{ord}_E(z)+4-m-n}) & \text{if} \quad z\neq 0, \\ (\pi^{4-m}) & \text{if} \quad z = 0. \end{cases}$$

Therefore we may consider the only three cases

$$(\natural)\begin{cases} n=2 \ \ \textit{and} \ \ m=\mathrm{ord}_E(z), \\ n=2 \ \ \textit{and} \ \ z=0, \ \text{and} \\ n=0. \end{cases}$$

The isomorphism classes of Λ_E-module $M(\ell,m,n;x,y,z)$ satisfying $(\natural)$are

$$\begin{aligned} &\left\{ \begin{array}{l} [M(0,1,2;0,0,\pi)], [M(0,1,2;0,\pi,\pi)], [M(0,1,2;1,1,\pi)], \\ [M(0,1,2;1,1+\pi,\pi)], [M(0,1,2;2,2,\pi)], [M(0,1,2;2,2+\pi,\pi)], \\ [M(0,1,2;2,2+2\pi,\pi)], [M(1,0,2;0,0,1)], [M(1,0,2;0,\pi,2)], \\ [M(1,0,2;0,0,1+\pi)], [M(1,1,2;0,\pi,2\pi)], [M(1,1,2;0,0,\pi)], \\ [M(1,0,2;0,2\pi,0)], [M(1,2,2;2\pi,2\pi,0)] \end{array} \right\} \\ &\cup \ \{ \ [N\oplus\Lambda_E/(T-\delta)\Lambda_E] \mid [N]\in\mathcal{M}^E_{(T-\alpha)(T-\beta)(T-\gamma)} \} \\ &\cup \ \{ \ [M(0,0,2;0,y,z)] \mid \mathrm{ord}_E(z)=0 \}. \end{aligned} \tag{7.2}$$

It is easy to see that $M = N\oplus\Lambda_E/(T-\delta)\Lambda_E$ does not satisfy $M/TM \cong \mathcal{O}_E/\pi^4\mathcal{O}_E\oplus\mathcal{O}_E/\pi^2\mathcal{O}_E$ if $N \ncong M(1,2,u\pi)$, where $u = \dfrac{\gamma-\alpha}{\beta-\alpha}$. We note that $M(1,2,u\pi)\cong\Lambda_E/(T-\alpha)(T-\beta)(T-\gamma)\Lambda_E$ by Proposition 6.3.2. We can also check $M/TM \ncong \mathcal{O}_E/\pi^4\mathcal{O}_E\oplus\mathcal{O}_E/\pi^2\mathcal{O}_E$ for $[M]\in\{[M(0,0,2;0,y,z)] \mid \ \mathrm{ord}_E(z)=0\}$ and $[M(0,1,2;0,0,\pi)]$, $[M(0,1,2;1,1,\pi)]$, and $[M(1,1,2;0,0,\pi)]$.

Now we investigate the structure of A_1 as a $\mathrm{Gal}(k_1/k)$-module. We have an isomorphism $A_1 \cong \mathbb{Z}/27\mathbb{Z}\oplus\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$. Furthermore, PARI/GP gives explicit generators which give this isomorphism. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, and $\mathfrak{a}_4$ be the generators PARI/GP computed. (We do not write down $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, and $\mathfrak{a}_4$ because they are complicated.) Let σ be a generator of $\mathrm{Gal}(k_1/k)$. By PARI/GP, we compute

$$
\begin{aligned}
(\sigma-1)\mathfrak{a}_1 &= 6\mathfrak{a}_1-\mathfrak{a}_2+\mathfrak{a}_3,\\
(\sigma-1)\mathfrak{a}_2 &= 3\mathfrak{a}_2+4\mathfrak{a}_3,\\
(\sigma-1)\mathfrak{a}_3 &= 9\mathfrak{a}_1+6\mathfrak{a}_2+6\mathfrak{a}_3, \text{ and}\\
(\sigma-1)\mathfrak{a}_4 &= 6\mathfrak{a}_2.
\end{aligned}
$$

There is a topological generator $\tilde{\sigma} \in \mathrm{Gal}(k_\infty/k)$ such that $\tilde{\sigma}$ is an extension of σ. By this topological generator, we have an isomorphism

$$\mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]] \cong \Lambda = \mathbb{Z}_p[[T]] \text{ such that } \tilde{\sigma} \leftrightarrow 1+T.$$

We regard X_{k_∞} as a Λ-module by this isomorphism. Since $\mathbb{Z}_p[\mathrm{Gal}(k_1/k)] \cong \Lambda/\omega_1\Lambda$, we get

$$
\begin{aligned}
\overline{T}\mathfrak{a}_1 &= 6\mathfrak{a}_1-\mathfrak{a}_2+\mathfrak{a}_3,\\
\overline{T}\mathfrak{a}_2 &= 3\mathfrak{a}_2+4\mathfrak{a}_3,\\
\overline{T}\mathfrak{a}_3 &= 9\mathfrak{a}_1+6\mathfrak{a}_2+6\mathfrak{a}_3, \text{ and}\\
\overline{T}\mathfrak{a}_4 &= 6\mathfrak{a}_2,
\end{aligned}
$$

where $\overline{T} = T \bmod \omega_1$. Now we have

$$
\begin{cases}
\overline{(T^2-12T)}\mathfrak{a}_1+\overline{(T-12)}\mathfrak{a}_2 &= 0,\\
\overline{(4T-24)}\mathfrak{a}_1-\overline{(T-7)}\mathfrak{a}_2 &= 0,\\
\overline{6}\mathfrak{a}_2-\overline{T}\mathfrak{a}_4 &= 0,\\
\overline{27}\mathfrak{a}_1 &= 0,\\
\overline{9T}\mathfrak{a}_1 &= 0,\\
\overline{9}\mathfrak{a}_2 &= 0, \text{ and}\\
\overline{3}\mathfrak{a}_4 &= 0.
\end{cases}
\tag{7.3}
$$

Therefore, we can calculate the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$;

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1\otimes\mathcal{O}_E) \mod 9 = (T,3) \mod (\omega_1,9), \tag{7.4}$$

where $\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E)$ is the 1-st Fitting ideal of $A_1 \otimes \mathcal{O}_E$. Then $M(0,1,2;0,\pi,\pi)$, $M(1,0,2;0,0,1)$, $M(1,0,2;0,0,1+\pi)$, and $M(1,1,2;0,\pi,2\pi)$ do not satisfy (7.4). Therefore we get

$$\begin{aligned} X_{k_\infty} \otimes_\Lambda \Lambda_E \quad \cong \quad & M(0,1,2;2,2+\pi,\pi),\ M(0,1,2;1,1+\pi,\pi),\ M(1,0,2;0,\pi,2), \\ & M(0,1,2;2,2,\pi), M(0,1,2;2,2+2\pi,\pi),\ M(1,0,2;0,2\pi,0), \\ & M(1,2,2;2\pi,2\pi,0), \text{or } M(1,2,0;u\pi,0,0). \end{aligned}$$

Further, using the above relations (7.3), we get

$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\gamma)}A_1 \otimes \mathcal{O}_E) \mod 9 \quad = \quad (T,3) \mod (\omega_1, 9), \tag{7.5}$$
$$\mathrm{Fitt}_{1,\Lambda_E/\omega_1\Lambda_E}(\overline{(T-\delta)}A_1 \otimes \mathcal{O}_E) \mod 9 \quad = \quad (T,3) \mod (\omega_1, 9). \tag{7.6}$$

Then only $M(1,0,2;0,\pi,2)$ satisfies (7.5) and (7.6). Hence we obtain $X_{k_\infty} \otimes_\Lambda \Lambda_E \cong M(1,0,2;0,\pi,2)$.

Table 7.1:

	d	$\mathrm{ord}_E(\alpha-\beta)$	$\mathrm{ord}_E(\beta-\gamma)$	$\mathrm{ord}_E(\gamma-\alpha)$	$E/\mathbb{Q}_3$	m	n	x	A_0
	6583	1	1	1	ram.	0	1	1	$(3,3)$
	8751	1	1	1	ram.	0	1	1	$(3,3)$
	9069	1	1	1	ram.	0	1	1	$(3,3)$
$(*)$	9574	1	1	1	unram.	0	1	2	$(3^2,3)$
	12118	1	1	1	ram.	0	1	1	$(3,3)$
	16627	1	1	1	ram.	0	1	1	$(3,3)$
	21018	1	1	1	ram.	0	1	1	$(3,3)$
	23178	1	1	1	ram.	0	1	1	$(3,3)$
	24109	1	1	1	ram.	0	1	1	$(3,3)$
	25122	1	1	1	ram.	0	1	1	$(3,3)$
	29569	1	1	1	ram.	0	1	1	$(3,3)$
	29778	1	1	1	ram.	0	1	1	$(3,3)$
	29994	1	1	1	ram.	0	1	1	$(3,3)$
$(*)$	30994	1	1	1	unram.	0	1	2	$(3^2,3)$
	31999	1	1	1	ram.	0	1	1	$(3,3)$
	34507	1	1	1	ram.	0	1	1	$(3,3)$
	34867	1	1	1	ram.	0	1	1	$(3,3)$
	35539	1	1	1	ram.	0	1	1	$(3,3)$
	37213	1	1	1	ram.	0	1	1	$(3,3)$
	37237	1	1	1	ram.	0	1	1	$(3,3)$
	38226	1	1	1	ram.	0	1	1	$(3,3)$
	38553	1	1	1	ram.	0	1	1	$(3,3)$
	38926	1	1	1	ram.	0	1	1	$(3,3)$
	40299	1	1	1	ram.	0	1	1	$(3,3)$
	41583	1	1	1	ram.	0	1	1	$(3,3)$
$(*)$	41631	2	3	2	ram.	0	3	$1+\pi$	$(3^3,3)$
	41671	1	1	1	ram.	0	1	1	$(3,3)$
	45210	1	1	1	ram.	0	1	1	$(3,3)$
	45753	1	1	1	ram.	0	1	1	$(3,3)$
	45942	1	1	1	ram.	0	1	1	$(3,3)$
	46198	1	1	1	ram.	0	1	1	$(3,3)$
	47199	1	1	1	ram.	0	1	1	$(3^2,3)$
	48667	1	1	1	ram.	0	1	1	$(3,3)$

Table 7.2:

	d	$\mathrm{ord}_E(\alpha-\beta)$	$\mathrm{ord}_E(\beta-\gamma)$	$\mathrm{ord}_E(\gamma-\alpha)$	$E/\mathbb{Q}_3$	m	n	x	A_0
	49074	1	1	1	ram.	0	1	1	$(3,3)$
	51142	1	1	1	ram.	0	1	1	$(3,3)$
	52858	1	1	1	ram.	0	1	1	$(3,3)$
	53839	1	1	1	ram.	0	1	1	$(3,3)$
	53862	1	1	1	ram.	0	1	1	$(3,3)$
	54319	1	1	1	ram.	0	1	1	$(3,3)$
	54853	1	1	1	ram.	0	1	1	$(3,3)$
	56773	1	1	1	ram.	0	1	1	$(3,3)$
	59478	1	1	1	ram.	0	1	1	$(3,3)$
	59578	1	1	1	ram.	0	1	1	$(3,3)$
	60099	1	1	1	ram.	0	1	1	$(3,3)$
(∗)	64671	2	3	2	ram.	0	3	$1+\pi$	$(3^2,3)$
	68314	1	1	1	ram.	0	1	1	$(3,3)$
	72591	1	1	1	ram.	0	1	1	$(3,3)$
	75273	1	1	1	ram.	0	1	1	$(3,3)$
	75354	1	1	1	ram.	0	1	1	$(3^2,3)$
	75790	1	1	1	ram.	0	1	1	$(3,3)$
	75841	1	1	1	ram.	0	1	1	$(3,3)$
	78181	1	1	1	ram.	0	1	1	$(3^2,3)$
	80233	1	1	1	ram.	0	1	1	$(3,3)$
	80242	1	1	1	ram.	0	1	1	$(3^2,3)$
	80746	1	1	1	ram.	0	1	1	$(3,3)$
(∗)	82774	1	1	1	unram.	0	1	2	$(3^2,3)$
	87727	1	1	1	ram.	0	1	1	$(3,3)$
	87979	1	1	1	ram.	0	1	1	$(3^2,3)$
	88134	1	1	1	ram.	0	1	1	$(3^2,3)$
	88242	1	1	1	ram.	0	1	1	$(3,3)$
(∗)	92515	1	1	1	unram.	0	1	2	$(3^2,3)$
	94998	1	1	1	ram.	0	1	1	$(3,3)$
	95691	1	1	1	ram.	0	1	1	$(3,3)$
	97555	1	1	1	ram.	0	1	1	$(3,3)$
	98277	1	1	1	ram.	0	1	1	$(3,3)$
	98929	1	1	1	ram.	0	1	1	$(3,3)$

Acknowledgments

I am sincerely grateful to my supervisor, Professor Masato Kurihara. As his Ph.D student I could have opportunity to attend many conferences and could get to know many mathematicians. I am grateful to him for his valuable suggestions during my research, which led me to get the results in this paper.

I would like to express my gratitude to professor Yoshitaka Hachimori for inviting me to Iwasawa theory and for giving me valuable comments, especially Proposition 4.2.3 in this thesis.

I would like to express my deepest appreciation to professor Taka-aki Tanaka, who read the first version of this thesis carefully, pointed out many mistakes, and taught me patiently how to write papers in mathematics.

I would like to thank to professor Kenichi Bannai and professor Hiroyasu Izeki for their kind encouragement and helpful advice on the manuscript.

I owe a very important debt to Hiroki Aoki and Takeshi Goto, who invited me to number theory.

Finally, I am very grateful to Rei Otsuki, Satoshi Fujii, Keiji Okano, Takashi Miura, Takayuki Morisawa, Jiro Nomura, Takahiro Kitajima, Takamichi Sano, and Toshihiro Shirakawa for discussion with them.

Bibliography

[1] Bourbaki, N.: *Commutative algebra*, Hermann/Addison-Wesley, 1972.

[2] Ernvall, R. and Metsankyla, T.: Computation of the zero of p-adic L-functions, Math. Comp. **58** (1992), 815-830.

[3] Ferrero, B. and Washington, L.: The Iwasawa invariant μ_p vanishes for abelian number fields, Ann. of Math. **109** (1979), 377-395.

[4] Franks, C.: Classifying Λ-modules up to Isomorphism and Applications to Iwasawa Theory, PhD Dissertation, Arizona State University, 2011.

[5] Fukuda, T.: Iwasawa λ-invariants of imaginary quadratic fields, J. College Industrial Technology Nihon Univ. **27** (1994), 35-88.

[6] Ichimura, H. and Sumida, H.: On the Iwasawa invariants of certain real abelian fields II, Int. J. Math. **7** (1996), 721-744.

[7] Iwasawa, K.: On $\mathbb{Z}_\ell$-extensions of algebraic number fields, Ann. of Math. **98** (1973), 246-326.

[8] Iwasawa, K.: On Γ-extensions of algebraic number fields, Bull. Amer. Math. Soc. **65** (1959), 183-226.

[9] Koike, M.: On the isomorphism classes of Iwasawa modules associated to imaginary quadratic fields with $\lambda = 2$, J. Math. Sci. Univ. Tokyo **6** (1999), 371-396.

[10] Kurihara, M.: Iwasawa theory and Fitting ideals, J. reine angew. Math. **561** (2003), 39-86.

[11] Mazur, B. and Wiles, A.: Class fields of abelian extensions of $\mathbb{Q}$, Invent. Math. **76** (1984), 179-330.

[12] Murakami, K.: On the isomorphism classes of Iwasawa modules with $\lambda = 3$ and $\mu = 0$, Osaka J. Math. **51** (2014), 829-865.

[13] Murakami, K.: Isomorphism classes of modules over Iwasawa algebra with $\lambda = 4$, to appear, in Tokyo J. Math.

[14] Mizusawa, Y.: http://mizusawa.web.nitech.ac.jp/index.html

[15] Neukirch, J., Schmidt, A., and Wingberg, K.: *Cohomology of number fields*, Second edition, Grundlehren der Mathematischen Wissenschaften **323**, Springer, 2008.

[16] Northcott, D. G.: *Finite free resolutions*, Cambridge University Press, Cambridge-New York, 1976.

[17] Saito, M. and Wada, H.: A table of ideal class groups of imaginary quadratic fields, Sophia Kokyuroku in Math. 1988.

[18] Serre, J. P.: Classes des corps cyclotomiques, Seminaire Bourbaki, Exp. No. **174**, (1958-1959), 83-93.

[19] Sumida, H.: Greenberg's conjecture and the Iwasawa polynomial, J. Math. Soc. Japan **49** (1997), 689-711.

[20] Sumida, H.: Isomorphism Classes and Adjoints of Certain Iwasawa Modules, Abh. Math. Sem. Univ. Hamburg **70** (2000), 113-117.

[21] Washington, L.: *Introduction to cyclotomic fields*, Second edition, Graduate Texts in Mathematics **83**, Springer-Verlag, New York 1997.

[22] Wiles, A.: Modular elliptic curves and Fermat's last theorem, Ann. of Math. **141** (1995), 443-551.