# Generalizations of Darmon's conjecture and the equivariant Tamagawa number conjecture

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# Chapter 1

# Introduction

In this thesis we discuss generalizations of Darmon's conjecture [Dar95]. More precisely, we formulate two different generalizations of Darmon's conjecture, and give some applications. Our first generalization concerns a relation between two different Rubin-Stark elements. We prove that, under some assumptions, most of this conjecture is a consequence of the "equivariant Tamagawa number conjecture (ETNC)" of Burns and Flach [BuFl01]. As an application, we give a full proof of Darmon's conjecture. Our second generalization concerns Euler systems defined for general $p$-adic representations. We prove this conjecture under the standard hypotheses (including that the core rank is equal to one) in the theory of Kolyvagin systems [MaRu04]. As an application, we give another proof for the important fact that an Euler system gives an upper bound of the size of the Selmer group [Rub00].

We begin with some historical background of our research.

## 1.1 Class number formulas

One of the main themes in number theory is the investigation of mysterious relations between zeta functions and arithmetic objects. A typical and classical example of arithmetic objects is the "class number". The notion of the class number was first introduced by Gauss in his famous magnum opus "Disquisitiones Arithmeticae", in his investigation of quadratic forms. Dirichlet found a formula, called the "class number formula", which relates the values of zeta functions with class numbers. It is said that Dirichlet highly adored Gauss, and investigated class numbers to find the class number formula. Later Dedekind generalized the notion of the class number for a general number field by using his theory of "ideals", and also generalized Dirichlet's class number formula for general number fields. The notion of ideals is a generalization of the notion of numbers. Dedekind's theory of ideals is still a foundation of modern algebraic number theory.

Thanks to Dedekind's theory of ideals, we have a definition of the "ideal class group" for a number field, whose cardinality is the class number. The ideal class group measures the discrepancy of the "uniqueness of the prime decomposition" for a number field in the following sense. We know that an integer is uniquely decomposed as a product of prime numbers. It is known that the uniqueness of the prime decomposition (more precisely, the uniqueness of the decomposition by "irreducible elements", which generalize prime numbers) fails in rings of integers of general number fields. But Dedekind proved that the decomposition by "prime ideals" is unique for general rings of integers. Roughly speaking, an ideal class group is defined as the quotient of the ideals by the numbers, so they measures the discrepancy between ideals and numbers. If the ideal class group is trivial (namely, the class number is one), then the uniqueness of the irreducible decomposition holds, and the converse is also true. This is the reason why the ideal class group measures the discrepancy of the uniqueness of the irreducible decomposition for a number field. Such an arithmetically interesting property of ideal class groups motivated many mathematicians to investigate them. Ideal class groups are regarded as typical arithmetic objects, and widely investigated even today.

The class number formula is stated as follows. Let $k$ be a number field. Let $\mathcal{O}_k$ denote the ring of integers of $k$. The zeta function of $k$, called the Dedekind zeta function, is defined by

$$\zeta_k(s) := \sum_{\mathfrak{a}} \frac{1}{\mathrm{N}\mathfrak{a}^s},$$

where $\mathfrak{a}$ runs over all non-zero ideals of $\mathcal{O}_k$, and $\mathrm{N}\mathfrak{a}$ denotes the cardinality of $\mathcal{O}_k/\mathfrak{a}$. It is known that the product in the right hand side converges when the real part of $s$ is greater than one, and $\zeta_k(s)$ is meromorphically continued on the whole complex plane. Let $\mu_k$ be the group of roots of unity in $k$. Let $r$ be the rank of the group $\mathcal{O}_k^\times/\mu_k$. Define the "regulator" of $k$ by

$$R_k := |\det(\log |u_i|_{v_j})_{1\leq i,j\leq r}|,$$

where $\{u_1, \ldots, u_r\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_k^\times/\mu_k$, $v_0, \ldots, v_r$ are all infinite places of $k$, and $|\cdot|_v$ is the normalized absolute value at $v$. The class number formula states that $r$ is equal to the order of zeros of $\zeta_k(s)$ at $s = 0$ and

$$\lim_{s\to 0} s^{-r}\zeta_k(s) = -\frac{h_k R_k}{|\mu_k|},$$

where $h_k$ denotes the class number of $k$. Thus, the class number formula relates the values of zeta functions with arithmetic objects such as class numbers and unit groups.

## 1.2 The equivariant Tamagawa number conjecture

At the moment, the most general conjecture in various generalizations of the classical class number formula is the "equivariant Tamagawa number conjecture (ETNC)", formulated by Burns and Flach in [BuFl01]. We shortly review the history of the ETNC.

In the 1960s, Birch and Swinnerton-Dyer proposed a conjectural analogue of the classical class number formula for elliptic curves in [BSD60s]. This conjecture, called the BSD conjecture for short, has not yet been solved completely, although it is supported by much evidence. The zeta functions of elliptic curves are generalized as the zeta functions of "motives". Motives are objects whose existence is dreamt by Grothendieck, from which many cohomology theories come. Definitions of the category of motives have been suggested by many mathematicians including Grothendieck, but many properties which should be satisfied are still conjectural. Thus, the "true definition" of general motives is still unclear, but Grothendieck's dream has been widely accepted in recent decades. Deligne attempted to generalize the BSD conjecture for general motives, and formulated a conjecture for critical motives in [Del79]. Deligne's conjecture is not just a generalization of the BSD conjecture, but its weak version. Later in [Bei85] Beilinson generalized Deligne's conjecture for general motives by constructing "higher regulators", usually referred as the "Beilinson regulators", which generalizes the classical regulators.

In [BlKa90], Bloch and Kato formulated a striking conjecture concerning the values of zeta functions of motives, which precisely generalizes the classical class number formula, the BSD conjecture and the Beilinson's conjecture simultaneously. Formulating the conjecture, Bloch and Kato introduced a notion of the "Tamagawa number for motives", which is an analogue of the Tamagawa number of algebraic groups. The conjecture of Bloch and Kato is called the "Tamagawa number conjecture (TNC)".

The ETNC is a generalization of the TNC for "equivariant coefficients". The terminology "equivariant" is used in the situation that a Galois group of number fields acts on a motive. Such a Galois action gives rise to the "equivariant zeta function" of the motive, and the ETNC concerns the values of the equivariant zeta function. The TNC is the special case of the ETNC that the Galois group is trivial.

In the case that the Galois group is abelian, the ETNC was first formulated by Kato [Kat93a], [Kat93b] and independently by Fontaine and Perrin-Riou [FoPe94]. In their formulation, the Tamagawa numbers introduced by Bloch and Kato do not appear, and ideas in Iwasawa theory are used. Iwasawa theory, which grew in the second half of the 20th century, is a powerful theory investigating ideal class groups with Galois actions. In the general case that the Galois group is not necessarily abelian, the ETNC was formulated by Burns and Flach in [BuFl01]. They combined the ideas of Kato and of Fontaine and Perrin-Riou with ideas in the Stark conjecture, which grew in Stark's seminal works [Sta71], [Sta75], [Sta76], [Sta80]. The Stark conjecture concerns the values of Artin $L$-functions, which is a direct generalization of the zeta function considered by Dirichlet. The Artin $L$-function is regarded as the equivariant zeta function of a particular motive, called the Tate motive. Also, Burns and Flach used ideas of Chinburg [Chi85] and Gruenberg, Ritter and Weiss [GRW99], in which the values of Artin $L$-functions are deeply investigated.

In some cases the ETNC can be solved by using Iwasawa theory. Burns, Greither, and Flach [BuGr03], [BuFl06], [Fla11] solved the ETNC for Tate motives for abelian extensions over $\mathbb{Q}$ by using the cyclotomic Iwasawa main conjecture proved by Mazur and Wiles in [MaWi84] and [Wil90]. This gives strong evidence for the validity of the ETNC, but at the moment in other cases only a few results on the ETNC are known.

## 1.3 Refined class number formulas

In [Gro88], Gross proposed an interesting conjectural analogue of the classical class number formula. Gross's conjecture is formulated as follows. Let $k$ be a number field. Let $L/k$ be a finite abelian extension, and $G$ be its Galois group. Let $S$ and $T$ be finite sets of places of $k$ satisfying certain conditions (see §3.1). Consider the Stickelberger element $\Theta_{L,S,T} \in \mathbb{Z}[G]$, which is defined as the value of the equivariant $(S,T)$-Artin $L$-function for $L/k$ at $s = 0$ (see §3.1). Let $I(G)$ denotes the augmentation ideal of $\mathbb{Z}[G]$. Gross's conjecture asserts that $\Theta_{L,S,T} \in I(G)^{|S|-1}$ and

$$\Theta_{L,S,T} \equiv -h_{k,S,T} R^{\mathrm{alg}}_{L,S,T} \pmod{I(G)^{|S|}},$$

where $h_{k,S,T}$ denotes the $(S,T)$-class number of $k$ and $R^{\mathrm{alg}}_{L,S,T} \in I(G)^{|S|-1}/I(G)^{|S|}$ is the "algebraic regulator", which is defined by using a basis of the $(S,T)$-unit group $\mathcal{O}^{\times}_{k,S,T}$ and the local reciprocity maps at places in $S$ (see [Gro88, Conjecture 4.1]). When $|S| = 1$, Gross's conjecture is equivalent to the classical class number formula. Thus, we can regard Gross's conjecture as a refinement of the class number formula, so it is called a "refined class number formula".

In [Dar95], Darmon formulated an analogue of Gross's conjecture for cyclotomic units.

Cyclotomic units are related to the values of Dirichlet's $L$-function, and Darmon's conjecture is also regarded as a refinement of the classical class number formula. For the precise formulation, see Theorem 4.1.1 (note that we slightly modified the formulation in [Dar95]).

Burns found that Gross's conjecture is a consequence of the ETNC for the untwisted Tate motive in [Bur07]. In particular, using the results of Burns and Greither [BuGr03] and of Flach [Fla11], Burns gave another proof of Gross's conjecture for abelian extensions over $\mathbb{Q}$, which was first proved by Aoki in [Aok91].

On the other hand, the "non-2-part" of Darmon's conjecture was recently solved by Mazur and Rubin in [MaRu11]. In the proof, they used their theory of "Kolyvagin systems" [MaRu04]. The theory of Kolyvagin systems is based on ideas in Kolyvagin's theory of "Euler systems" in [Kol90]. The system of cyclotomic units is a typical example of Euler systems. Mazur and Rubin proved that both sides of Darmon's conjectural equality form Kolyvagin systems. Then they proved that the equality holds by using the "uniqueness" of Kolyvagin systems, which is one of the main results in [MaRu04].

## 1.4 Main results

In this thesis, we formulate two different generalizations of Darmon's conjecture.

Our first conjecture is formulated as a refinement of the Rubin-Stark conjecture, proposed by Rubin in [Rub96] (see Conjecture 1 in §3.2). The Rubin-Stark conjecture predicts the existence of certain integral elements, called the Rubin-Stark elements, related to the values of Artin $L$-functions at $s = 0$. These elements are generalizations of the Stickelberger elements and the cyclotomic units.

We briefly sketch the formulation of the Rubin-Stark conjecture. Let $k$ be a fixed number field, and $L/k$ be a finite abelian extension with Galois group $G$. Take finite sets of places $S$ and $T$ of $k$ satisfying certain conditions (see §3.1). Also, choose a proper subset $V \subset S$ such that all $v \in V$ split completely in $L$. We denote the order of $V$ by $r$. It is known that the order of the equivariant $(S,T)$-Artin $L$-function $\Theta_{L,S,T}(s)$ for $L/k$ at $s = 0$ is greater than or equal to $r$, so we can consider the value

$$\Theta^{(r)}_{L,S,T}(0) := \lim_{s\to 0} \frac{1}{s^r}\Theta_{L,S,T}(s) \in \mathbb{C}[G].$$

The Rubin-Stark conjecture predicts that there exists a unique element $\varepsilon_{L,S,T,V}$ in a certain integral lattice of $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^r \mathcal{O}^\times_{L,S,T}$, which maps to $\Theta^{(r)}_{L,S,T}(0)$ under the regulator map $R_V : \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^r \mathcal{O}^\times_{L,S,T} \to \mathbb{C}[G]$. This element $\varepsilon_{L,S,T,V}$ is called the Rubin-Stark element for

$(L, S, T, V)$. The set $T$ is often considered to be fixed, and the Rubin-Stark element $\varepsilon_{L,S,T,V}$ is often denoted by $\varepsilon_{L,S,V}$. We remark that the Rubin-Stark conjecture is known to be true if $k = \mathbb{Q}$ (see [Bur07, Theorem A]).

We formulate a new conjecture on a relation between two different Rubin-Stark elements, which is a generalization of Darmon's conjecture. This is Conjecture 3 in §3.4. We give here a sketch of the formulation of Conjecture 3. Consider a tower of extensions $L'/L/k$ such that $L'/k$ is finite abelian. Consider two Rubin-Stark elements $\varepsilon_{L',S',V'}$ and $\varepsilon_{L,S,V}$. For simplicity, we explain the formulation in the case $S' = S$. It is known that, when $V' = V$, the norm map $\mathrm{N}_{L'/L}$ sends $\varepsilon_{L',S,V'}$ to $\varepsilon_{L,S,V}$ (the "norm relation", see Proposition 3.3.2). In the case $V' \subset V$, denoting the order of $V \setminus V'$ by $d$, we introduce the $d$th norm $\mathrm{N}^{(d)}_{L'/L}$, which generalizes the usual norm (see Definition 2.2.12 and Remark 2.2.13). Then Conjecture 3 predicts the following equality:

$$\mathrm{N}^{(d)}_{L'/L}(\varepsilon_{L',S,V'}) = \pm \mathrm{Rec}_{V \setminus V'}(\varepsilon_{L,S,V}), \tag{1.1}$$

where $\mathrm{Rec}_{V \setminus V'}$ is a map constructed by using the local reciprocity maps at places in $V \setminus V'$ (which is $i \circ (\bigwedge_{v \in V \setminus V'} \varphi_v)$ with the notation in §3.4). We remark that the sign in the right hand side of (1.1) can be determined explicitly. When $d = 0$ i.e. $V' = V$, the equality (1.1) is exactly the usual norm relation. Thus, Conjecture 3 gives a relation between two different Rubin-Stark elements $\varepsilon_{L',S,V'}$ and $\varepsilon_{L,S,V}$ even when $V' \neq V$.

Our first main result is as follows.

**Theorem 1.4.1** (Theorem 3.5.8)**.** *Let $p$ be a prime number not dividing $[L : k]$. Under the assumptions in Theorem 3.5.8, the $p$-part of our new conjecture (Conjecture 3) is deduced from the ETNC for the untwisted Tate motive.*

We remark that this result was later improved in a recent joint work of the author with Burns and Kurihara [BKS14]. It is proved in [BKS14] that Conjecture 3 is deduced from the ETNC completely (see Remark 3.5.9). In particular, using the result due to Burns, Greither and Flach [BuGr03], [Fla11], we know that Conjecture 3 for the fields $L'/L/k$ is true if $L'$ is abelian over $\mathbb{Q}$.

Using the above result, we prove the next theorem, which gives a complete solution to Darmon's conjecture. We explain the formulation of Darmon's conjecture (see §4.1 for the precise formulation). Let $F$ be a real quadratic field with conductor $f$. Let $n$ be a square-free positive integer which is prime to $f$. For simplicity, we assume that all prime divisors of $n$ split in $F$. Denote the number of prime divisors of $n$ by $\nu$. Let $F_n$ denote the maximal real subfield of $F(\zeta_n)$, where $\zeta_n$ is a primitive $n$th root of unity. Consider a cyclotomic unit $\beta_n \in F_n^\times$ (see §4.1), and define the "theta element" by

$$\theta_n := \sum_{\sigma \in \mathrm{Gal}(F_n/F)} \sigma\beta_n \otimes \sigma^{-1} \in F_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Gal}(F_n/F)].$$

Let $I_n$ be the augmentation ideal of $\mathbb{Z}[\mathrm{Gal}(F_n/F)]$. Darmon's conjecture predicts that the following equality holds in $(F^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} I_n^\nu/I_n^{\nu+1}$:

$$\theta_n = -h_n R_n,$$

where $h_n$ is the $n$-class number of $F$ (namely, the order of the Picard group $\mathrm{Pic}(\mathcal{O}_F[\frac{1}{n}])$), and $R_n \in F^\times \otimes_{\mathbb{Z}} I_n^\nu/I_n^{\nu+1}$ is the "algebraic regulator", which is defined by using the local reciprocity maps at prime divisors of $n$.

We prove in §4.2 that our new conjecture (Conjecture 3) is indeed a generalization of Darmon's conjecture. We have the following theorem.

**Theorem 1.4.2** (Theorem 4.1.1)**.** *Darmon's conjecture is deduced from Conjecture 3 for the tower of fields $F_n/F/\mathbb{Q}$. Consequently, Darmon's conjecture is true.*

Thus, we give a complete solution to Darmon's conjecture. This result is an improvement of the result of Mazur and Rubin in [MaRu11] (see Remark 4.1.2). We remark that Theorem 3.5.8 gives sufficient ingredients to prove the "non-2-part" of Darmon's conjecture (see [San14b]).

We remark that a conjecture essentially same to our new conjecture (Conjecture 3) is formulated independently by Mazur and Rubin in the recent preprint [MaRu13b].

Our second generalization of Darmon's conjecture is a generalization for Euler systems defined for general $p$-adic representations. This conjecture is not precisely a generalization of Darmon's conjecture, but a weak version of it. The formulation of this conjecture replaces the system of cyclotomic units $\{\beta_n\}_n$, which appears in Darmon's conjecture, by an Euler system for a general $p$-adic representation. For a given Euler system $c = \{c_n\}_n$, we define the theta element $\theta_n(c)$ as an analogue of Darmon's theta element (see Definition 5.1.4). We construct a module of algebraic regulators $\mathcal{R}_n$ (see Definition 5.1.2), and conjecture that

$$\theta_n(c) \in \mathfrak{h}_n \mathcal{R}_n, \tag{1.2}$$

where $\mathfrak{h}_n$ denotes the order of a certain $n$-modified Selmer group (this is denoted by $H^1_{(\mathcal{F}^*)_n}(\mathbb{Q}, A^*)$ in §5.1), which plays a role of $h_n$ in Darmon's conjecture. In the case that the $p$-adic representation is the Tate module of $\mathbb{G}_m$ twisted by the Dirichlet character associated with a real quadratic field $F$, we can take $c$ to be the Euler system of cyclotomic units, but in this case our conjecture (1.2) is weaker than the original conjecture of Darmon, since we do not give an explicit description of an algebraic regulator $\mathfrak{R}_n \in \mathcal{R}_n$ such that $\theta_n(c) = -\mathfrak{h}_n \mathfrak{R}_n$.

The theory of Kolyvagin systems, developed by Mazur and Rubin in [MaRu04], is a powerful theory investigating Selmer groups via Euler systems. We remark that many important properties of Kolyvagin systems, such as the "uniqueness" property which was used in the proof of the "non-2-part" of Darmon's conjecture in [MaRu11], are proved under the "standard hypotheses" including that the "core rank" is equal to one. We prove our generalized Darmon's conjecture for Euler systems under the standard hypotheses.

**Theorem 1.4.3** (Theorem 5.1.8)**.** *Assume that the standard hypotheses of the theory of Kolyvagin systems (including that the core rank is equal to one). Then our generalized Darmon's conjecture for Euler systems is true.*

As an application of this result, we give another proof for the important fact (see [Rub00]) that an Euler system gives an upper bound of the size of the Selmer group (see Corollary 5.1.9).

## 1.5 Expected overview

We mention an expected overview of further generalizations of Darmon's conjecture. It is believed that any important $p$-adic representation comes from a motive. So suppose that $T$ is a $p$-adic representation which comes from a motive $\mathcal{M}$. In some cases it is known (and believed in general) that, if the ETNC holds for $\mathcal{M}$, then we have a certain nice system in the $r$th exterior power (with some non-negative integer $r$) of the Galois cohomology groups of the dual of $T$. Such a system has properties like Euler systems, and called a "rank $r$ Euler system". In the case $r = 1$, this is exactly a usual Euler system. The integer $r$ is expected to be equal to the core rank of the dual of $T$. For example, when $\mathcal{M}$ is the untwisted Tate motive for an abelian field, in this case the ETNC is valid and $\mathcal{M}$ gives rise to the (rank 1) Euler system of cyclotomic units. A typical example of higher rank Euler systems is a system of conjectural Rubin-Stark elements, which come from the untwisted Tate motive for a general number field.

Our first generalization of Darmon's conjecture (Conjecture 3) is regarded as a conjecture for the higher rank Euler system which comes from the untwisted Tate motive, whereas the second (Theorem 5.1.8) is a conjecture for the rank one Euler systems which come from general motives. By this observation, it is natural to ask the following questions.

- Can we formulate a generalization of our new conjecture (Conjecture 3) for higher rank Euler systems which come from general motives?

- Is the conjecture deduced from the ETNC?

We also hope that our conjectures can be extended to the case of non-abelian Galois extensions. These expected generalizations should be done in future works.

## 1.6 Notation

For any abelian group $G$, $\mathbb{Z}[G]$-modules are simply called $G$-modules. The tensor product over $\mathbb{Z}[G]$ is denoted by

$$- \otimes_G -.$$

Similarly, the exterior power over $\mathbb{Z}[G]$, and Hom of $\mathbb{Z}[G]$-modules are denoted by

$$\bigwedge_G, \ \mathrm{Hom}_G(-,-)$$

respectively. We use the notations like this also for $\mathbb{Z}[G]$-algebras.

For any subgroup $H$ of $G$, we define the norm element $\mathrm{N}_H \in \mathbb{Z}[G]$ by

$$\mathrm{N}_H = \sum_{\sigma \in H} \sigma.$$

For any $G$-module $M$, we define

$$M^G = \{m \in M \mid \sigma m = m \text{ for all } \sigma \in G\}.$$

The maximal $\mathbb{Z}$-torsion subgroup of $M$ is denoted by $M_{\mathrm{tors}}$.

For any $G$-modules $M$ and $M'$, we endow $M \otimes_{\mathbb{Z}} M'$ with a structure of a $G$-bimodule by

$$\sigma(m \otimes m') = \sigma m \otimes m' \quad \text{and} \quad (m \otimes m')\sigma = m \otimes \sigma m',$$

where $\sigma \in G$, $m \in M$ and $m' \in M'$. If $\varphi \in \mathrm{Hom}_G(M, M'')$, where $M''$ is another $G$-module, we often denote $\varphi \otimes \mathrm{id} \in \mathrm{Hom}_G(M \otimes_{\mathbb{Z}} M', M'' \otimes_{\mathbb{Z}} M')$ by $\varphi$.

# Chapter 2

# Algebraic preliminaries

In this chapter, we summarize certain useful constructions concerning exterior powers and also prove algebraic results which are to be used in later chapters. The conventions in §2.1 are frequently used throughout this thesis. In §2.2, we study Rubin's lattices defined in [Rub96, §1.2]. The results in §2.2 are used in Chapter 3.

## 2.1 Exterior powers

Let $G$ be a finite abelian group. For a $G$-module $M$ and $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, there is a $G$-homomorphism

$$\bigwedge_G^r M \longrightarrow \bigwedge_G^{r-1} M$$

for all $r \in \mathbb{Z}_{\geq 1}$, defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^{r} (-1)^{i-1} \varphi(m_i) m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r.$$

This morphism is also denoted by $\varphi$.

This construction gives a morphism

$$\bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_G\left(\bigwedge_G^r M, \bigwedge_G^{r-s} M\right) \tag{2.1}$$

for all $r, s \in \mathbb{Z}_{\geq 0}$ such that $r \geq s$, defined by

$$\varphi_1 \wedge \cdots \wedge \varphi_s \mapsto (m \mapsto \varphi_s \circ \cdots \circ \varphi_1(m)).$$

By this construction, we often regard an element of $\bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G])$ as an element of $\mathrm{Hom}_G(\bigwedge_G^r M, \bigwedge_G^{r-s} M)$. Note that if $r = s$, $\varphi_1 \wedge \cdots \wedge \varphi_r \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$, and $m_1 \wedge \cdots \wedge m_r \in \bigwedge_G^r M$, then we have

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(m_1 \wedge \cdots \wedge m_r) = \det(\varphi_i(m_j))_{1 \le i,j \le r}.$$

For a $G$-algebra $Q$ and $\varphi \in \mathrm{Hom}_G(M, Q)$, there is a $G$-homomorphism

$$\bigwedge_G^r M \longrightarrow \left(\bigwedge_G^{r-1} M\right) \otimes_G Q$$

defined by

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^{r} (-1)^{i-1} m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r \otimes \varphi(m_i).$$

Similarly to the construction of (2.1), we have a morphism

$$\bigwedge_G^s \mathrm{Hom}_G(M, Q) \longrightarrow \mathrm{Hom}_G\left(\bigwedge_G^r M, \left(\bigwedge_G^{r-s} M\right) \otimes_G Q\right). \tag{2.2}$$

## 2.2 Rubin's lattices

In this section, we fix a finite abelian group $G$ and its subgroup $H$. Following Rubin [Rub96, §1.2], we give the following definition.

**Definition 2.2.1.** For a finitely generated $G$-module $M$ and $r \in \mathbb{Z}_{\ge 0}$, we define Rubin's lattice by

$$\bigcap_G^r M = \left\{ m \in \left(\bigwedge_G^r M\right) \otimes_{\mathbb{Z}} \mathbb{Q} \mid \Phi(m) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]) \right\}.$$

Note that $\bigcap_G^0 M = \mathbb{Z}[G]$.

**Remark 2.2.2.** We define $\iota : \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G]) \to \mathrm{Hom}_G(\bigwedge_G^r M, \mathbb{Z}[G])$ by $\varphi_1 \wedge \cdots \wedge \varphi_r \mapsto \varphi_r \circ \cdots \circ \varphi_1$ (see (2.1)). It is not difficult to see that

$$\bigcap_G^r M \xrightarrow{\sim} \mathrm{Hom}_G(\mathrm{im}\, \iota, \mathbb{Z}[G]); \quad m \mapsto (\Phi \mapsto \Phi(m))$$

is an isomorphism (see [Rub96, §1.2]).

Next, we study some more properties of Rubin's lattice.

Let $I_H$ (resp. $I(H)$) be the kernel of the natural map $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ (resp. $\mathbb{Z}[H] \to \mathbb{Z}$). Note that $I(H) \subset I_H$. For any $d \in \mathbb{Z}_{\geq 0}$, let $Q_H^d$ (resp. $Q(H)^d$) be the $d$th augmentation quotient $I_H^d/I_H^{d+1}$ (resp. $I(H)^d/I(H)^{d+1}$). Note that $Q_H^d$ has a natural $G/H$-module structure, since $\mathbb{Z}[G]/I_H \simeq \mathbb{Z}[G/H]$. It is known that there is a natural isomorphism of $G/H$-modules

$$\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^d \xrightarrow{\sim} Q_H^d \tag{2.3}$$

given by

$$\sigma \otimes \bar{a} \mapsto \overline{\widetilde{\sigma} a},$$

where $a \in I(H)^d$ and $\bar{a}$ denotes the image of $a$ in $Q(H)^d$, $\widetilde{\sigma} \in G$ is any lift of $\sigma \in G/H$, and $\overline{\widetilde{\sigma} a}$ denotes the image of $\widetilde{\sigma} a \in I_H^d$ in $Q_H^d$ ($\overline{\widetilde{\sigma} a}$ does not depend on the choice of $\widetilde{\sigma}$) (see [Pop11, Lemma 5.2.3(2)]). We often identify $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^d$ and $Q_H^d$.

The following lemma is well-known, and we omit the proof.

**Lemma 2.2.3.** *For a $G$-module $M$ and an abelian group $A$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(M, A) \xrightarrow{\sim} \mathrm{Hom}_G(M, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A); \quad \varphi \mapsto \left( m \mapsto \sum_{\sigma \in G} \sigma^{-1} \otimes \varphi(\sigma m) \right).$$

**Lemma 2.2.4.** *Let $M$ be a finitely generated $G/H$-module, and $\overline{M} = M/M_{\mathrm{tors}}$. For any $d \in \mathbb{Z}_{\geq 0}$, we have an isomorphism*

$$\mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} Q(H)^d \xrightarrow{\sim} \mathrm{Hom}_{G/H}(\overline{M}, Q_H^d); \quad \varphi \otimes a \mapsto (\bar{m} \mapsto \varphi(m)a).$$

*In particular,*

$$\mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \mathrm{Hom}_{G/H}(M, Q_H^d)$$

*is an injection.*

*Proof.* We have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} Q(H)^d \arrow[r] \arrow[d] & \mathrm{Hom}_{G/H}(\overline{M}, Q_H^d) \arrow[d] \\
\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes_{\mathbb{Z}} Q(H)^d \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\overline{M}, Q(H)^d),
\end{tikzcd}$$

where the bottom horizontal arrow is given by $\varphi \otimes a \mapsto (\bar{m} \mapsto \varphi(m)a)$, and the left and right vertical arrows are the isomorphisms given in Lemma 2.2.3 (note that we have a natural isomorphism $Q_H^d \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^d$, see (2.3)). The bottom horizontal arrow is an isomorphism, since $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\overline{M}, \mathbb{Z})$ and $\overline{M}$ is torsion-free by definition. Hence the upper horizontal arrow is also bijective. □

**Definition 2.2.5.** A finitely generated $G$-module $M$ is called a $G$-lattice if $M$ is torsion-free.

For example, for a finitely generated $G$-module $M$, $\mathrm{Hom}_G(M, \mathbb{Z}[G])$ is a $G$-lattice. Rubin's lattice $\bigcap_G^r M$ is also a $G$-lattice.

**Proposition 2.2.6.** *Let $M$ be a $G/H$-lattice, and $r, d \in \mathbb{Z}_{\geq 0}$ such that $r \geq d$. Then an element $\Phi \in \bigwedge_{G/H}^d \mathrm{Hom}_{G/H}(M, Q_H^1)$ induces a $G/H$-homomorphism*

$$\bigcap_{G/H}^{r} M \longrightarrow \left( \bigcap_{G/H}^{r-d} M \right) \otimes_{G/H} Q_H^d \left( \simeq \left( \bigcap_{G/H}^{r-d} M \right) \otimes_{\mathbb{Z}} Q(H)^d \right).$$

*Proof.* Note that $Q_H^1$ is the degree-1-part of the graded $G/H$-algebra $\bigoplus_{i \geq 0} Q_H^i$. We apply (2.2) to know that $\Phi$ induces the $G/H$-homomorphism

$$\bigwedge_{G/H}^{r} M \longrightarrow \left( \bigwedge_{G/H}^{r-d} M \right) \otimes_{G/H} Q_H^d. \tag{2.4}$$

We extend this map to Rubin's lattice $\bigcap_{G/H}^r M$. We may assume that there exist

$$\varphi_1, \ldots, \varphi_d \in \mathrm{Hom}_{G/H}(M, Q_H^1)$$

such that $\Phi = \varphi_1 \wedge \cdots \wedge \varphi_d$. Moreover, by Lemma 2.2.4, we may assume for each $1 \leq i \leq d$ that there exist $\psi_i \in \mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H])$ and $a_i \in Q(H)^1$ such that $\varphi_i = \psi_i(\cdot)a_i$. Put $\Psi = \psi_1 \wedge \cdots \wedge \psi_d \in \bigwedge_{G/H}^d \mathrm{Hom}_{G/H}(M, \mathbb{Z}[G/H])$. By the definition of Rubin's lattice, $\Phi$ induces a $G/H$-homomorphism

$$\bigcap_{G/H}^{r} M \longrightarrow \left( \bigcap_{G/H}^{r-d} M \right) \otimes_{\mathbb{Z}} Q(H)^d; \quad m \mapsto \Psi(m) \otimes a_1 \cdots a_d.$$

This extends the map (2.4). □

The following definition is due to [Bur07, §2.1].

**Definition 2.2.7.** Let $M$ be a $G$-lattice. For $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, we define $\varphi^H \in \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ by

$$M^H \xrightarrow{\varphi} \mathbb{Z}[G]^H \xrightarrow{\sim} \mathbb{Z}[G/H],$$

where the last isomorphism is given by $\mathrm{N}_H \mapsto 1$. Similarly, for $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$ ($r \in \mathbb{Z}_{\geq 0}$), $\Phi^H \in \bigwedge_{G/H}^r \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ is defined. (If $r = 0$, we define $\Phi^H \in \mathbb{Z}[G/H]$ to be the image of $\Phi \in \mathbb{Z}[G]$ under the natural map.)

**Remark 2.2.8.** It is easy to see that

$$\varphi^H = \sum_{\sigma \in G/H} \varphi^1(\sigma(\,\cdot\,))\sigma^{-1},$$

where $\varphi^1 \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ corresponds to $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$ (see Lemma 2.2.3). If $r \geq 1$, then one also sees that

$$\Phi(m) = \Phi^H(\mathrm{N}_H^r m) \quad \text{in} \quad \mathbb{Z}[G/H] \tag{2.5}$$

for all $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$ and $m \in \bigcap_G^r M$.

**Lemma 2.2.9.** *If $M$ is a $G$-lattice, then the map*

$$\mathrm{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]); \quad \varphi \mapsto \varphi^H$$

*is surjective.*

*Proof.* By Remark 2.2.8, what we have to prove is that the restriction map

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M^H, \mathbb{Z})$$

is surjective. Therefore, it is sufficient to prove that $M/M^H$ is torsion-free. Take $m \in M$ such that $nm \in M^H$ for a nonzero $n \in \mathbb{Z}$. For any $\sigma \in H$, we have

$$n((\sigma - 1)m) = (\sigma - 1)nm = 0.$$

Since $M$ is a $G$-lattice, it is torsion-free. Therefore, we have $(\sigma - 1)m = 0$. This implies $m \in M^H$. $\square$

**Lemma 2.2.10.** *Let $M$ be a $G$-lattice, and $r, d \in \mathbb{Z}_{\geq 0}$. Then there is a canonical injection*

$$i : \bigcap_{G/H}^{r} M^H \longrightarrow \bigcap_{G}^{r} M.$$

*Furthermore, the maps*

$$\left(\bigcap_{G/H}^{r} M^H\right) \otimes_{\mathbb{Z}} Q(H)^d \xrightarrow{i} \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$$

*are both injective, where the first arrow is induced by $i$, and the second by the inclusion $Q(H)^d \hookrightarrow \mathbb{Z}[H]/I(H)^{d+1}$.*

*Proof.* Let

$$\iota : \bigwedge_{G}^{r} \operatorname{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \operatorname{Hom}_G\left(\bigwedge_{G}^{r} M, \mathbb{Z}[G]\right)$$

and

$$\iota_H : \bigwedge_{G/H}^{r} \operatorname{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \operatorname{Hom}_{G/H}\left(\bigwedge_{G/H}^{r} M^H, \mathbb{Z}[G/H]\right)$$

be the maps in Remark 2.2.2. It is easy to see that the map

$$\kappa : \operatorname{im} \iota \longrightarrow \operatorname{im} \iota_H; \quad \iota(\Phi) \mapsto \iota_H(\Phi^H)$$

is well-defined. By Lemma 2.2.9, the map

$$\bigwedge_{G}^{r} \operatorname{Hom}_G(M, \mathbb{Z}[G]) \longrightarrow \bigwedge_{G/H}^{r} \operatorname{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]); \quad \Phi \mapsto \Phi^H$$

is surjective. So the map $\kappa$ is also surjective. Hence, by Remark 2.2.2, we have an injection

$$i : \bigcap_{G/H}^{r} M^H \longrightarrow \bigcap_{G}^{r} M$$

(note that $\operatorname{Hom}_{G/H}(\operatorname{im} \iota_H, \mathbb{Z}[G/H]) \simeq \operatorname{Hom}_G(\operatorname{im} \iota_H, \mathbb{Z}[G])$ by Lemma 2.2.3). The cokernel of this map is isomorphic to a submodule of $\operatorname{Hom}_G(\ker \kappa, \mathbb{Z}[G])$, so it is torsion-free. Hence the map

$$i : \left(\bigcap_{G/H}^{r} M^H\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \left(\bigcap_{G}^{r} M\right) \otimes_{\mathbb{Z}} Q(H)^d$$

is injective. The injectivity of the map

$$\left(\bigcap_G^r M\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \left(\bigcap_G^r M\right) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$$

follows from the fact that $\bigcap_G^r M$ is torsion-free. $\square$

**Remark 2.2.11.** The canonical injection $i : \bigcap_{G/H}^r M^H \hookrightarrow \bigcap_G^r M$ constructed above does not coincide in general with the map induced by the inclusion $M^H \hookrightarrow M$. In fact, if $r \geq 1$, then we have

$$i(\mathrm{N}_H^r m) = \mathrm{N}_H m$$

for all $m \in \bigcap_G^r M$.

**Definition 2.2.12.** Let $M$ be a $G$-lattice, and $r, d \in \mathbb{Z}_{\geq 0}$. When $r \geq 1$, we define the $d$th norm

$$\mathrm{N}_H^{(r,d)} : \bigcap_G^r M \longrightarrow \left(\bigcap_G^r M\right) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$$

by

$$\mathrm{N}_H^{(r,d)}(m) = \sum_{\sigma \in H} \sigma m \otimes \sigma^{-1}.$$

When $r = 0$, we define

$$\mathrm{N}_H^{(0,d)} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]/I_H^{d+1}$$

to be the natural map.

**Remark 2.2.13.** The 0th norm is the usual norm :

$$\mathrm{N}_H^{(r,0)} = \begin{cases} \mathrm{N}_H & \text{if } r \geq 1, \\ \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H] & \text{if } r = 0. \end{cases}$$

**Proposition 2.2.14.** *Let $M$ be a $G$-lattice, $r, d \in \mathbb{Z}_{\geq 0}$, and $m \in \bigcap_G^r M$. Assume*

$$\mathrm{N}_H^{(r,d)}(m) \in \operatorname{im} i,$$

*where, in the case $r \geq 1$, $i : (\bigcap_{G/H}^r M^H) \otimes_{\mathbb{Z}} Q(H)^d \to (\bigcap_G^r M) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$ is defined to be the injection in Lemma 2.2.10, and in the case $r = 0$, $i : Q_H^d \hookrightarrow \mathbb{Z}[G]/I_H^{d+1}$ to be the inclusion. If $d = 0$ or $r = 0$ or $1$, then we have*

$$\Phi(m) = \Phi^H(i^{-1}(\mathrm{N}_H^{(r,d)}(m))) \quad \text{in} \quad Q_H^d$$

*for all* $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(M, \mathbb{Z}[G])$.

*Proof.* When $d = 0$, the proposition follows from Remarks 2.2.8, 2.2.11, and 2.2.13. When $r = 0$, the proposition is clear. So we suppose $r = 1$. Note that in this case the map $i$ is the inclusion

$$i : M^H \otimes_{\mathbb{Z}} Q(H)^d \hookrightarrow M \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}.$$

We regard $M^H \otimes_{\mathbb{Z}} Q(H)^d \subset M \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1}$.

Take any $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$. Then $\varphi^H$ is written by

$$\varphi^H = \sum_{\sigma \in G/H} \varphi^1(\sigma(\cdot))\sigma^{-1}$$

(see Remark 2.2.8). For each $\sigma \in G/H$, we fix a lifting $\widetilde{\sigma} \in G$, and put

$$\widetilde{\varphi} = \sum_{\sigma \in G/H} \varphi^1(\widetilde{\sigma}(\cdot))\widetilde{\sigma}^{-1} \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}[G]).$$

Then, by the assumption on $\mathrm{N}_H^{(1,d)}(m)$, we have

$$\varphi^H(\mathrm{N}_H^{(1,d)}(m)) = (\alpha \circ (\widetilde{\varphi} \otimes \mathrm{id}))(\mathrm{N}_H^{(1,d)}(m)) \in Q_H^d,$$

where

$$\alpha : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{d+1} \longrightarrow \mathbb{Z}[G]/I_H^{d+1}; \quad a \otimes \overline{b} \mapsto \overline{ab}.$$

It is easy to check that

$$\varphi(m) = (\alpha \circ (\widetilde{\varphi} \otimes \mathrm{id}))(\mathrm{N}_H^{(1,d)}(m)) \quad \text{in} \quad \mathbb{Z}[G]/I_H^{d+1}.$$

This can be checked by noting that

$$\varphi = \sum_{\sigma \in G/H} \sum_{\tau \in H} \varphi^1(\widetilde{\sigma}\tau(\cdot))\widetilde{\sigma}^{-1}\tau^{-1}.$$

Hence we have

$$\varphi(m) = \varphi^H(\mathrm{N}_H^{(1,d)}(m)) \quad \text{in} \quad Q_H^d.$$

$\square$

**Theorem 2.2.15.** *Let $M$ be a $G$-lattice, and $r, d \in \mathbb{Z}_{\geq 0}$. Then the map*

$$\left(\bigcap_{G/H}^{r} M^H\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \mathrm{Hom}_G\left(\bigwedge_{G}^{r} \mathrm{Hom}_G(M, \mathbb{Z}[G]), Q_H^d\right); \quad \alpha \mapsto (\Phi \mapsto \Phi^H(\alpha))$$

*is injective.*

*Proof.* Let

$$\iota_H : \bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \mathrm{Hom}_{G/H}\left(\bigwedge_{G/H}^{r} M^H, \mathbb{Z}[G/H]\right)$$

be the map defined in Remark 2.2.2 for $G/H$ and $M^H$. Taking $\mathrm{Hom}_{G/H}(-, \mathbb{Z}[G/H])$ to the exact sequence

$$0 \longrightarrow \ker \iota_H \longrightarrow \bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]) \longrightarrow \mathrm{im}\, \iota_H \longrightarrow 0,$$

we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \bigcap_{G/H}^{r} M^H \longrightarrow \mathrm{Hom}_{G/H}&\left(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), \mathbb{Z}[G/H]\right) \\ \longrightarrow \mathrm{Hom}_{G/H}&(\ker \iota_H, \mathbb{Z}[G/H]). \end{aligned}$$

Since $\mathrm{Hom}_{G/H}(\ker \iota_H, \mathbb{Z}[G/H])$ is torsion-free, the map

$$\left(\bigcap_{G/H}^{r} M^H\right) \otimes_{\mathbb{Z}} Q(H)^d \longrightarrow \mathrm{Hom}_{G/H}\left(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), \mathbb{Z}[G/H]\right) \otimes_{\mathbb{Z}} Q(H)^d$$

is injective. From Lemma 2.2.4, we have an injection

$$\begin{aligned} &\mathrm{Hom}_{G/H}\left(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), \mathbb{Z}[G/H]\right) \otimes_{\mathbb{Z}} Q(H)^d \\ \longrightarrow\ &\mathrm{Hom}_G\left(\bigwedge_{G/H}^{r} \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H]), Q_H^d\right). \end{aligned}$$

From Lemma 2.2.9, we also have an injection

$$\mathrm{Hom}_G\left(\bigwedge_{G/H}^{r}\mathrm{Hom}_{G/H}(M^H,\mathbb{Z}[G/H]),Q_H^d\right)\longrightarrow\mathrm{Hom}_G\left(\bigwedge_{G}^{r}\mathrm{Hom}_{G}(M,\mathbb{Z}[G]),Q_H^d\right).$$

The composition of the above three injections coincides with the map given in the theorem, hence we complete the proof. $\square$

# Chapter 3

# Refined abelian Stark conjectures

In this chapter, we formulate a new conjecture on a relation between two different Rubin-Stark elements (see Conjecture 3). In the next chapter, we show that our new conjecture (Conjecture 3) is indeed a generalization of Darmon's conjecture [Dar95].

## 3.1 Notation

Throughout this chapter, we fix a global field $k$. We also fix $T$, a finite set of places of $k$, containing no infinite place. For a finite separable extension $L/k$ and a finite set $S$ of places of $k$, $S_L$ denotes the set of places of $L$ lying above the places in $S$. For $S$ containing all the infinite places and disjoint to $T$, $\mathcal{O}_{L,S,T}^{\times}$ denotes the $(S,T)$-unit group of $L$, i.e.

$$\mathcal{O}_{L,S,T}^{\times} = \{a \in L^{\times} \mid \mathrm{ord}_w(a) = 0 \text{ for all } w \notin S_L \text{ and } a \equiv 1 \pmod{w'} \text{ for all } w' \in T_L\},$$

where $\mathrm{ord}_w$ is the (normalized) additive valuation at $w$. Let $Y_{L,S} = \bigoplus_{w \in S_L} \mathbb{Z}w$, the free abelian group on $S_L$, and $X_{L,S} = \{\sum a_w w \in Y_{L,S} \mid \sum a_w = 0\}$. Let

$$\lambda_{L,S} : \mathcal{O}_{L,S,T}^{\times} \longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} X_{L,S}$$

be the map defined by $\lambda_{L,S}(a) = -\sum_{w \in S_L} \log |a|_w w$, where $|\cdot|_w$ is the normalized absolute value at $w$.

Let $\Omega(= \Omega(k,T))$ be the set of triples $(L, S, V)$ satisfying the following:

- $L$ is a finite abelian extension of $k$,
- $S$ is a nonempty finite set of places of $k$ satisfying

- $S \cap T = \emptyset$,
- $S$ contains all the infinite places and all places ramifying in $L$,
- $\mathcal{O}_{L,S,T}^{\times}$ is torsion-free,

- $V$ is a subset of $S$ satisfying
  - any $v \in V$ splits completely in $L$,
  - $|S| \geq |V| + 1$.

We assume that $\Omega \neq \emptyset$. If $k$ is a number field, then the condition that $\mathcal{O}_{L,S,T}^{\times}$ is torsion-free is satisfied when, for example, $T$ contains two finite places of unequal residue characteristics.

Take $(L, S, V) \in \Omega$, and put $\mathcal{G}_L = \mathrm{Gal}(L/k)$, $r = r_V = |V|$. The equivariant $(S,T)$-Artin $L$-function for $L/k$ is defined by

$$\Theta_{L,S,T}(s) = \sum_{\chi \in \widehat{\mathcal{G}_L}} e_\chi L_{S,T}(s, \chi^{-1}),$$

where $\widehat{\mathcal{G}_L} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{G}_L, \mathbb{C}^\times)$, $e_\chi = \frac{1}{|\mathcal{G}_L|} \sum_{\sigma \in \mathcal{G}_L} \chi(\sigma)\sigma^{-1}$, and

$$L_{S,T}(s, \chi) = \prod_{v \in T} (1 - \chi(\mathrm{Fr}_v)\mathrm{N}v^{1-s}) \prod_{v \notin S} (1 - \chi(\mathrm{Fr}_v)\mathrm{N}v^{-s})^{-1},$$

where $\mathrm{Fr}_v \in \mathcal{G}_L$ is the arithmetic Frobenius at $v$, and $\mathrm{N}v$ is the cardinality of the residue field at $v$.

We define

$$\Lambda_{L,S,T}^r = \left\{ a \in \bigcap_{\mathcal{G}_L}^{r} \mathcal{O}_{L,S,T}^\times \mid e_\chi a = 0 \text{ for every } \chi \in \widehat{\mathcal{G}_L} \text{ such that } r(\chi) > r \right\},$$

where $r(\chi) = r(\chi, S) = \mathrm{ord}_{s=0} L_{S,T}(s, \chi)$ (for the definition of $\bigcap_{\mathcal{G}_L}^r$, see Definition 2.2.1). It is well-known that

$$r(\chi) = \begin{cases} |\{v \in S \mid v \text{ splits completely in } L^{\ker \chi}\}| & \text{if } \chi \text{ is nontrivial,} \\ |S| - 1 & \text{if } \chi \text{ is trivial,} \end{cases}$$

(see [Tat76, Proposition 3.4, Chpt. I]) so by our assumptions on $V$, we have $r(\chi) \geq r$ for every $\chi$. This implies that $s^{-r}\Theta_{L,S,T}(s)$ is holomorphic at $s = 0$. We define

$$\Theta_{L,S,T}^{(r)}(0) = \lim_{s \to 0} s^{-r} \Theta_{L,S,T}(s) \in \mathbb{C}[\mathcal{G}_L].$$

We fix the following:

- a bijection $\{\text{all the places of } k\} \simeq \mathbb{Z}_{\geq 0}$,

- for each place $v$ of $k$, a place of $\bar{k}$ (a fixed separable closure of $k$) lying above $v$.

From this fixed choice, we can regard $V$ as a totally ordered finite set with order $\prec$, and arrange $V = \{v_1, \ldots, v_r\}$ so that $v_1 \prec \cdots \prec v_r$. For each $v \in V$, there is a fixed place $w$ of $L$ lying above $v$, and define $v^* \in \mathrm{Hom}_{\mathcal{G}_L}(Y_{L,S}, \mathbb{Z}[\mathcal{G}_L])$ to be the dual of $w$, i.e.

$$v^*(w') = \sum_{\sigma w = w'} \sigma.$$

Thus, we often use slightly ambiguous notations such as follows: the fixed places of $L$ lying above $v, v', v_i$, etc. are denoted by $w, w', w_i$, etc. respectively. We define the analytic regulator map $R_V : \bigwedge_{\mathcal{G}_L}^r \mathcal{O}_{L,S,T}^\times \to \mathbb{R}[\mathcal{G}_L]$ by

$$R_V = \bigwedge_{v \in V} (v^* \circ \lambda_{L,S}),$$

where the exterior power in the right hand side means $(v_1^* \circ \lambda_{L,S} \wedge \cdots \wedge v_r^* \circ \lambda_{L,S})$ (defined similarly to (2.1)). Thus, when we take an exterior power on a totally ordered finite set, we always mean that the order is arranged to be ascending order. One can easily see that

$$v^* \circ \lambda_{L,S} = -\sum_{\sigma \in \mathcal{G}_L} \log |\sigma(\cdot)|_w \sigma^{-1},$$

so a more explicit definition of $R_V$ is as follows:

$$R_V(u_1 \wedge \cdots \wedge u_r) = \det \left( -\sum_{\sigma \in \mathcal{G}_L} \log |\sigma(u_i)|_{w_j} \sigma^{-1} \right).$$

## 3.2 The Rubin-Stark conjecture

We use the notations and conventions as in §3.1. Recall that the integral refinement of abelian Stark conjecture, which we call the Rubin-Stark conjecture, formulated by Rubin, is stated as follows:

**Conjecture 1** (Rubin [Rub96, Conjecture B′]). *For $(L, S, V) \in \Omega$, there is a unique*

$\varepsilon_{L,S,V} = \varepsilon_{L,S,T,V} \in \Lambda^r_{L,S,T}$ *such that*

$$R_V(\varepsilon_{L,S,V}) = \Theta^{(r)}_{L,S,T}(0).$$

The element $\varepsilon_{L,S,V}$ predicted by the conjecture is called the Rubin-Stark element.

**Remark 3.2.1.** When $r = 0$, Conjecture 1 is known to be true (see [Rub96, Theorem 3.3]). In this case we have $\varepsilon_{L,S,V} = \Theta_{L,S,T}(0) \in \mathbb{Z}[\mathcal{G}_L] = \bigcap^0_{\mathcal{G}_L} \mathcal{O}^\times_{L,S,T}$.

**Remark 3.2.2.** When $r < \min\{|S|-1, |\{v \in S \mid v \text{ splits completely in } L\}|\}$, we have $\Theta^{(r)}_{L,S,T}(0) = 0$, so Conjecture 1 is trivially true (namely, we have $\varepsilon_{L,S,V} = 0$).

**Remark 3.2.3.** When $k = \mathbb{Q}$, Conjecture 1 is true (see [Bur07, Theorem A]).

**Remark 3.2.4.** When $k$ is a function field, Conjecture 1 is true (see [Bur11, Corollary 1.2(iii)]).

## 3.3 Some properties of Rubin-Stark elements

In this section, we assume that Conjecture 1 holds for all $(L, S, V) \in \Omega$, and review some properties of Rubin-Stark elements.

**Lemma 3.3.1** ([Rub96, Lemma 2.7(ii)]). *Let $(L, S, V) \in \Omega$. Then $R_V$ is injective on $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda^r_{L,S,T}$.*

*Proof.* Since $\lambda_{L,S}$ induces an injection $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathcal{G}_L} \mathcal{O}^\times_{L,S,T} \to \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathcal{G}_L} X_{L,S}$, it is sufficient to prove that

$$\bigwedge_{v \in V} v^* : e_\chi \left( \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathcal{G}_L} X_{L,S} \right) \longrightarrow \mathbb{C}[\mathcal{G}_L]$$

is injective for every $\chi \in \widehat{\mathcal{G}_L}$ such that $r(\chi) = r$. It is well-known that $r(\chi) = \dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} X_{L,S}))$, so we have $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^r_{\mathcal{G}_L} X_{L,S})) = 1$. Take any $v' \in S \setminus V$, then we have

$$\left( \bigwedge_{v \in V} v^* \right) \left( e_\chi \bigwedge_{v \in V} (w - w') \right) = e_\chi \neq 0$$

(recall that $w$ (resp. $w'$) denotes the fixed place of $L$ lying above $v$ (resp. $v'$)), which proves the lemma. $\square$

**Proposition 3.3.2** ([Rub96, Proposition 6.1]). *Let*

$$(L, S, V), (L', S', V) \in \Omega,$$

*and suppose that $L \subset L'$ and $S \subset S'$. Then we have*

$$\mathrm{N}^r_{L'/L}(\varepsilon_{L',S',V}) = \left(\prod_{v \in S' \setminus S} (1 - \mathrm{Fr}_v^{-1})\right) \varepsilon_{L,S,V},$$

*where $\mathrm{N}_{L'/L} = \mathrm{N}_{\mathrm{Gal}(L'/L)}$, and if $r = 0$, then we regard $\mathrm{N}^r_{L'/L}$ as the natural map $\mathbb{Z}[\mathcal{G}_{L'}] \to \mathbb{Z}[\mathcal{G}_L]$.*

*Proof.* It is easy to see that $\mathrm{N}^r_{L'/L}(\varepsilon_{L',S',V}) \in \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda^r_{L,S',T}$. Hence, by Lemma 3.3.1, it is enough to check that

$$R_V(\mathrm{N}^r_{L'/L}(\varepsilon_{L',S',V})) = R_V\left(\left(\prod_{v \in S' \setminus S} (1 - \mathrm{Fr}_v^{-1})\right) \varepsilon_{L,S,V}\right).$$

The left hand side is equal to the image of $\Theta^{(r)}_{L',S',T}(0)$ in $\mathbb{R}[\mathcal{G}_L]$, and hence to $\prod_{v \in S' \setminus S}(1 - \mathrm{Fr}_v^{-1})\Theta^{(r)}_{L,S,T}(0)$ (see [Tat76, Proposition 1.8, Chpt. IV]). The right hand side is equal to $\prod_{v \in S' \setminus S}(1 - \mathrm{Fr}_v^{-1})\Theta^{(r)}_{L,S,T}(0)$, so we complete the proof. □

**Proposition 3.3.3** ([Rub96, Lemma 5.1(iv) and Proposition 5.2]). *Let*

$$(L, S, V), (L, S', V') \in \Omega,$$

*and suppose that $S \subset S'$, $V \subset V'$ and $S' \setminus S = V' \setminus V$. Put*

$$\Phi_{V',V} = \mathrm{sgn}(V', V) \bigwedge_{v \in V' \setminus V} \left(\sum_{\sigma \in \mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot))\sigma^{-1}\right) \in \bigwedge_{\mathcal{G}_L}^{r'-r} \mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}^\times_{L,S',T}, \mathbb{Z}[\mathcal{G}_L]),$$

*where $r = |V|$, $r' = |V'|$, and $\mathrm{sgn}(V', V) = \pm 1$ is defined by*

$$\left(\bigwedge_{v \in V} v^*\right) \circ \left(\bigwedge_{v \in V' \setminus V} v^*\right) = \mathrm{sgn}(V', V) \bigwedge_{v \in V'} v^* \quad \textit{in} \quad \mathrm{Hom}_{\mathcal{G}_L}\left(\bigwedge_{\mathcal{G}_L}^{r'} Y_{L,S'}, \mathbb{Z}[\mathcal{G}_L]\right).$$

*Then we have*

$$\Phi_{V',V}(\Lambda^{r'}_{L,S',T}) \subset \Lambda^r_{L,S,T}$$

*and*

$$\Phi_{V',V}(\varepsilon_{L,S',V'}) = \varepsilon_{L,S,V}.$$

*Proof.* Put $\Phi = \Phi_{V',V}$, for simplicity. First, we prove that

$$\Phi(\Lambda^{r'}_{L,S',T}) \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda^{r}_{L,S,T} \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.1}$$

There is a split exact sequence of $\mathbb{Q}[\mathcal{G}_L]$-modules:

$$0 \longrightarrow \mathcal{O}^{\times}_{L,S,T} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathcal{O}^{\times}_{L,S',T} \otimes_{\mathbb{Z}} \mathbb{Q} \stackrel{\bigoplus_{v\in S'\setminus S} \widetilde{w}}{\longrightarrow} \bigoplus_{v\in S'\setminus S} \mathbb{Q}[\mathcal{G}_L] \longrightarrow 0,$$

where $\widetilde{w} = \sum_{\sigma\in\mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot))\sigma^{-1}$. So we can choose a submodule $M \subset \mathcal{O}^{\times}_{L,S',T} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\mathcal{O}^{\times}_{L,S',T} \otimes_{\mathbb{Z}} \mathbb{Q} = (\mathcal{O}^{\times}_{L,S,T} \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus M$$

and

$$\bigoplus_{v\in S'\setminus S} \widetilde{w} : M \longrightarrow \bigoplus_{v\in S'\setminus S} \mathbb{Q}[\mathcal{G}_L]$$

is an isomorphism. Therefore, we have

$$\left(\bigwedge^{r'}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S',T}\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{i=0}^{r'} \left(\left(\bigwedge^{i}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S,T}\right) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \otimes_{\mathbb{Q}[\mathcal{G}_L]} \bigwedge^{r'-i}_{\mathbb{Q}[\mathcal{G}_L]} M.$$

If $i > r$ then $\Phi(((\bigwedge^{i}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S,T}) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}[\mathcal{G}_L]} \bigwedge^{r'-i}_{\mathbb{Q}[\mathcal{G}_L]} M) = 0$, and if $i < r$ then $\bigwedge^{r'-i}_{\mathbb{Q}[\mathcal{G}_L]} M = 0$. Hence we have

$$\Phi\left(\bigwedge^{r'}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S',T}\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\bigwedge^{r}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S,T}\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Now (3.1) follows by noting that $r(\chi, S') = r(\chi, S) + r' - r$ for every $\chi \in \widehat{\mathcal{G}_L}$.

For the first assertion, by (3.1), it is enough to prove that

$$\Phi\left(\bigcap^{r'}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S',T}\right) \subset \bigcap^{r}_{\mathcal{G}_L} \mathcal{O}^{\times}_{L,S,T}.$$

Since $\mathcal{O}^{\times}_{L,S',T}/\mathcal{O}^{\times}_{L,S,T}$ is torsion-free, we have a surjection

$$\mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}^{\times}_{L,S',T}, \mathbb{Z}[\mathcal{G}_L]) \to \mathrm{Hom}_{\mathcal{G}_L}(\mathcal{O}^{\times}_{L,S,T}, \mathbb{Z}[\mathcal{G}_L]).$$

Now the assertion follows from the definition of Rubin's lattice.

For the second assertion, it is enough to show that

$$R_V(\Phi(\varepsilon_{L,S',V'})) = \Theta^{(r)}_{L,S,T}(0).$$

It is easy to see that for $v \in V' \setminus V$

$$\log \mathrm{N}v \sum_{\sigma \in \mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot))\sigma^{-1} = v^* \circ \lambda_{L,S'},$$

and also that

$$\Theta^{(r')}_{L,S',T}(0) = \left(\prod_{v \in V' \setminus V} \log \mathrm{N}v\right) \Theta^{(r)}_{L,S,T}(0).$$

Therefore, we have

$$\begin{aligned} R_V(\Phi(\varepsilon_{L,S',V'})) &= \left(\prod_{v \in V' \setminus V} \log \mathrm{N}v\right)^{-1} R_{V'}(\varepsilon_{L,S',V'}) \\ &= \left(\prod_{v \in V' \setminus V} \log \mathrm{N}v\right)^{-1} \Theta^{(r')}_{L,S',T}(0) \\ &= \Theta^{(r)}_{L,S,T}(0). \end{aligned}$$

□

## 3.4 Refined conjectures

In this section, we propose the main conjectures. We keep the notations in §3.1. We also keep on assuming Conjecture 1 is true for all $(L, S, V) \in \Omega$. Fix $(L, S, V), (L', S', V') \in \Omega$ such that $L \subset L'$, $S \subset S'$, and $V \supset V'$. We also use the notations defined in Chapter 2, taking $G = \mathcal{G}_{L'}$ and $H = \mathrm{Gal}(L'/L)$. For convenience, we record the list of the notations here (some new notations are added).

- $\mathcal{G}_L = \mathrm{Gal}(L/k)$,
- $\mathcal{G}_{L'} = \mathrm{Gal}(L'/k)$,
- $G(L'/L) = \mathrm{Gal}(L'/L)$,

- $r = |V|$,
- $r' = |V'|$,
- $\varepsilon_{L,S,V} \in \bigcap_{\mathcal{G}_L}^{r} \mathcal{O}_{L,S,T}^{\times}$ (resp. $\varepsilon_{L',S',V'} \in \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times}$): the Rubin-Stark element for $(L, S, V)$ (resp. $(L', S', V')$) (see §3.2).
- $d = r - r'(\geq 0)$,
- $I_{L'/L} = I_{G(L'/L)} = \ker(\mathbb{Z}[\mathcal{G}_{L'}] \longrightarrow \mathbb{Z}[\mathcal{G}_L])$,
- $I(L'/L) = I(G(L'/L)) = \ker(\mathbb{Z}[G(L'/L)] \longrightarrow \mathbb{Z})$.

For $n \in \mathbb{Z}_{\geq 0}$,

- $Q_{L'/L}^n = Q_{G(L'/L)}^n = I_{L'/L}^n / I_{L'/L}^{n+1}$,
- $Q(L'/L)^n = Q(G(L'/L))^n = I(L'/L)^n / I(L'/L)^{n+1}$.

Recall that there is a natural isomorphism

$$\mathbb{Z}[\mathcal{G}_L] \otimes_{\mathbb{Z}} Q(L'/L)^n \simeq Q_{L'/L}^n$$

(see (2.3)).

Recall the definition of "higher norm" (Definition 2.2.12). In the case $r' \geq 1$, the $d$th norm

$$\mathrm{N}_{L'/L}^{(r',d)} = \mathrm{N}_{G(L'/L)}^{(r',d)} : \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times} \longrightarrow \left( \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[G(L'/L)]/I(L'/L)^{d+1}$$

is defined by

$$\mathrm{N}_{L'/L}^{(r',d)}(a) = \sum_{\sigma \in G(L'/L)} \sigma a \otimes \sigma^{-1},$$

and in the case $r' = 0$, $\mathrm{N}_{L'/L}^{(0,d)}$ is defined to be the natural map

$$\mathbb{Z}[\mathcal{G}_{L'}] \longrightarrow \mathbb{Z}[\mathcal{G}_{L'}]/I_{L'/L}^{d+1}.$$

In the case $r' \geq 1$, define

$$i : \left( \bigcap_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S',T}^{\times} \right) \otimes_{\mathbb{Z}} Q(L'/L)^d \hookrightarrow \left( \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S',T}^{\times} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[G(L'/L)]/I(L'/L)^{d+1}$$

to be the canonical injection in Lemma 2.2.10. In the case $r' = 0$, define

$$i : \left( \bigcap_{\mathcal{G}_L}^{0} \mathcal{O}_{L,S',T}^{\times} \right) \otimes_{\mathbb{Z}} Q(L'/L)^d \simeq Q_{L'/L}^d \hookrightarrow \mathbb{Z}[\mathcal{G}_{L'}]/I_{L'/L}^{d+1}$$

to be the inclusion.

**Conjecture 2.**

$$\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S',V'}) \in \operatorname{im} i.$$

**Remark 3.4.1.** When $d = 0$, Conjecture 2 is true by Remarks 2.2.11 and 2.2.13.

**Remark 3.4.2.** Conjecture 2 is related to the Kolyvagin's derivative construction, which is important in the theory of Euler systems ([Kol90], [Rub00]) and Mazur-Rubin's Kolyvagin systems ([MaRu04]). See [San14b, Remark 4.8] for the detail.

For $v \in V$, define

$$\varphi_v = \varphi_{v,L'/L} : L^{\times} \longrightarrow Q_{L'/L}^1$$

by $\varphi_v(a) = \sum_{\sigma \in \mathcal{G}_L} (\mathrm{rec}_w(\sigma a) - 1)\sigma^{-1}$, where $\mathrm{rec}_w$ is the local reciprocity map at $w$ (recall that $w$ is the fixed place of $L$ lying above $v$, see §3.1). Note that, by Proposition 2.2.6, $\bigwedge_{v \in V \setminus V'} \varphi_v \in \bigwedge_{\mathcal{G}_L}^d \operatorname{Hom}_{\mathcal{G}_L}(\mathcal{O}_{L,S,T}^{\times}, Q_{L'/L}^1)$ induces a morphism

$$\bigcap_{\mathcal{G}_L}^{r} \mathcal{O}_{L,S,T}^{\times} \longrightarrow \left( \bigcap_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S,T}^{\times} \right) \otimes_{\mathbb{Z}} Q(L'/L)^d.$$

We define $\operatorname{sgn}(V, V') = \pm 1$ by

$$\left( \bigwedge_{v \in V'} v^* \right) \circ \left( \bigwedge_{v \in V \setminus V'} v^* \right) = \operatorname{sgn}(V, V') \bigwedge_{v \in V} v^* \quad \text{in} \quad \operatorname{Hom}_{\mathcal{G}_L}\left( \bigwedge_{\mathcal{G}_L}^{r} Y_{L,S}, \mathbb{Z}[\mathcal{G}_L] \right).$$

The following conjecture predicts that $\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S',V'})$ is described in terms of $\varepsilon_{L,S,V}$.

**Conjecture 3.** *Conjecture 2 holds, and we have*

$$i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S',V'})) = \operatorname{sgn}(V, V') \left( \prod_{v \in S' \setminus S} (1 - \mathrm{Fr}_v^{-1}) \right) \left( \bigwedge_{v \in V \setminus V'} \varphi_v \right) (\varepsilon_{L,S,V}).$$

**Remark 3.4.3.** When $d = 0$, Conjecture 3 is true by the "norm relation" (Proposition 3.3.2). (See Remarks 2.2.11 and 2.2.13.)

**Remark 3.4.4.** When $r' = 0$, by Remark 3.2.1, one sees that Conjecture 3 is equivalent to the "Gross-type refinement of the Rubin-Stark conjecture" ([Pop11, Conjecture 5.3.3]), which generalizes Gross's conjecture ([Gro88, Conjecture 4.1]), see [Pop11, Proposition 5.3.6].

**Remark 3.4.5.** When $r' = 1$, Conjecture 3 is closely related to Darmon's conjecture ([Dar95, Conjecture 4.3]). The detailed explanation is given in Chapter 4.

**Proposition 3.4.6.** *It is sufficient to prove Conjecture 3 in the following case:*
$S = S'$,
$r = \min\{|S| - 1, |\{v \in S \mid v \text{ splits completely in } L\}|\} =: r_{L,S}$,
$r' = \min\{|S| - 1, |\{v \in S \mid v \text{ splits completely in } L'\}|\} =: r_{L',S}$.

*Proof.* From Proposition 3.3.2, we may assume $S = S'$. When $r < r_{L,S}$ and $r' < r_{L',S}$, Conjecture 3 is trivially true (see Remark 3.2.2). When $r < r_{L,S}$ and $r' = r_{L',S}$, we have

$$\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}) = 0$$

if Conjecture 3 is true when $r = r_{L,S}$ and $r' = r_{L',S}$. When $r = r_{L,S}$ and $r' < r_{L',S}$, we prove

$$\left(\bigwedge_{v \in V \setminus V'} \varphi_v\right)(\varepsilon_{L,S,V}) = 0.$$

If there exists $v \in V \setminus V'$ which splits completely in $L'$, this is clear. If all $v \in V \setminus V'$ do not split completely in $L'$, then there exists $v' \in S \setminus V$ which splits completely in $L'$, and we must have $V = S \setminus \{v'\}$. By the product formula, we see that

$$\sum_{v \in S \setminus V'} \varphi_{v,L'/L} = 0 \quad \text{on} \quad \mathcal{O}_{k,S,T}^{\times}.$$

Note that $\varepsilon_{L,S,V} \in e_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^r \mathcal{O}_{L,S,T}^{\times})$ in this case. Hence, choosing any $v'' \in V \setminus V'$, we have

$$\left(\bigwedge_{v \in V \setminus V'} \varphi_v\right)(\varepsilon_{L,S,V}) = \pm \left(\bigwedge_{v \in (S \setminus \{v''\}) \setminus V'} \varphi_v\right)(\varepsilon_{L,S,V}),$$

and the right hand side is 0 since $v'$ splits completely in $L'$. $\square$

From now on we assume $S = S'$, $r = r_{L,S}$, and $r' = r_{L',S}$.

**Proposition 3.4.7.** *If every place in $V \setminus V'$ is finite and unramified in $L'$, then Conjecture 3 is true.*

*Proof.* We treat the case $r' \geq 1$. The proof for $r' = 0$ is similar.

Put $W := V \setminus V'$ for simplicity. Note that $(L', S \setminus W, V') \in \Omega$. By Proposition 3.3.2, we have

$$\varepsilon_{L',S,V'} = \prod_{v \in W} (1 - \mathrm{Fr}_v^{-1}) \varepsilon_{L',S \setminus W,V'}.$$

Hence, we have

$$\begin{aligned} \mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}) &= \sum_{\sigma \in G(L'/L)} \sigma \prod_{v \in W} (1 - \mathrm{Fr}_v^{-1}) \varepsilon_{L',S \setminus W,V'} \otimes \sigma^{-1} \\ &= \sum_{\sigma \in G(L'/L)} \sigma \varepsilon_{L',S \setminus W,V'} \otimes \sigma^{-1} \prod_{v \in W} (1 - \mathrm{Fr}_v^{-1}) \\ &= \mathrm{N}_{L'/L} \varepsilon_{L',S \setminus W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1) \\ &\in \left( \mathrm{N}_{L'/L} \bigcap_{\mathcal{G}_{L'}}^{r'} \mathcal{O}_{L',S,T}^{\times} \right) \otimes_{\mathbb{Z}} Q(L'/L)^d. \end{aligned}$$

For every $v \in W$, we have

$$\varphi_v = \sum_{\sigma \in \mathcal{G}_L} \mathrm{ord}_w(\sigma(\cdot)) \sigma^{-1} (\mathrm{Fr}_v - 1)$$

(see [Ser79, Proposition 13, Chpt. XIII]), so by Proposition 3.3.3 we have

$$\mathrm{sgn}(V, V') \left( \bigwedge_{v \in W} \varphi_v \right) (\varepsilon_{L,S,V}) = \varepsilon_{L,S \setminus W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1).$$

By Proposition 3.3.2 and Remark 2.2.11, we have

$$\mathrm{N}_{L'/L} \varepsilon_{L',S \setminus W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1) = i \left( \varepsilon_{L,S \setminus W,V'} \prod_{v \in W} (\mathrm{Fr}_v - 1) \right),$$

hence the proposition follows. □

**Remark 3.4.8.** In [San15], it is proved that Conjecture 3 is true if the following three assumptions are satisfied:

- $V'$ contains all the infinite places of $k$,
- all $v \in S$ split completely in $L$,
- $G(L'/L) = \prod_{v \in S \setminus V'} J_v$, where $J_v \subset \mathcal{G}_{L'}$ is the inertia group at $v$.

The formulation of the following conjecture is a slight modification of [Bur07, Theorem 3.1] (see also Theorem 3.5.4 and Remark 3.5.6).

**Conjecture 4.** *For every* $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}])$*, we have*

$$\Phi(\varepsilon_{L',S,V'}) \in I_{L'/L}^{d}$$

*and*

$$\Phi(\varepsilon_{L',S,V'}) = \mathrm{sgn}(V,V')\Phi^{G(L'/L)}\left(\left(\bigwedge_{v \in V \setminus V'} \varphi_v\right)(\varepsilon_{L,S,V})\right) \quad \text{in} \quad Q_{L'/L}^{d}.$$

The following conjecture is motivated by the property of the higher norm described in Proposition 2.2.14.

**Conjecture 5.** *If Conjecture 2 holds, then we have*

$$\Phi(\varepsilon_{L',S,V'}) = \Phi^{G(L'/L)}(i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}))) \quad \text{in} \quad Q_{L'/L}^{d}$$

*for every* $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}])$.

**Remark 3.4.9.** When $d = 0$ or $r' = 0$ or 1, Conjecture 5 is true by Proposition 2.2.14.

## 3.5 Relation among the conjectures

We keep on assuming $S = S'$, $r = r_{L,S}$, and $r' = r_{L',S}$.

**Theorem 3.5.1.** *Assume Conjecture 5 holds. Then, Conjecture 3 holds if and only if Conjectures 2 and 4 hold.*

*Proof.* The "only if" part is clear. We prove the "if" part. Suppose that Conjectures 2 and 4 hold. Then, for every $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}])$, we have

$$\Phi^{G(L'/L)}(i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'}))) = \mathrm{sgn}(V,V')\Phi^{G(L'/L)}\left(\left(\bigwedge_{v \in V \setminus V'} \varphi_v\right)(\varepsilon_{L,S,V})\right) \quad \text{in} \quad Q_{L'/L}^{d}$$

by Conjectures 4 and 5. By Theorem 2.2.15, the map

$$\left(\bigcap_{\mathcal{G}_L}^{r'} \mathcal{O}_{L,S,T}^{\times}\right) \otimes_{\mathbb{Z}} Q(L'/L)^{d} \longrightarrow \mathrm{Hom}_{\mathcal{G}_{L'}}\left(\bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^{\times}, \mathbb{Z}[\mathcal{G}_{L'}]), Q_{L'/L}^{d}\right)$$

defined by $\alpha \mapsto (\Phi \mapsto \Phi^{G(L'/L)}(\alpha))$ is injective. Hence we have

$$i^{-1}(\mathrm{N}_{L'/L}^{(r',d)}(\varepsilon_{L',S,V'})) = \mathrm{sgn}(V,V') \left( \bigwedge_{v \in V \setminus V'} \varphi_v \right) (\varepsilon_{L,S,V}).$$

□

**Remark 3.5.2.** Since Conjecture 3 is closely related to Darmon's conjecture, as we mentioned in Remark 3.4.5, Theorem 3.5.1 gives a relation between Darmon's conjecture and Burns's conjecture (Conjecture 4). In [Hay04, Theorem 6.14], Hayward established a connection between these conjectures: he proved that Darmon's conjecture gives a "base change statement" for Burns's conjecture. More precisely, consider a real quadratic field $L$ and a real abelian field $\widetilde{L}$ which is disjoint to $L$. Put $L' := L\widetilde{L}$. Then Hayward proved that, assuming Darmon's conjecture for $L$, Burns's conjecture for $\widetilde{L}/\mathbb{Q}$ implies Burns's conjecture for $L'/L$ up to a power of 2. On the other hand, Theorem 3.5.1 gives an equivalence of Burns's conjecture and Darmon's conjecture, assuming Conjectures 2 and 5.

**Remark 3.5.3.** One can formulate for any prime number $p$ the "$p$-part" of Conjectures 2, 3, 4, and 5 in the obvious way. One sees that the "$p$-part" of Theorem 3.5.1 is also valid, namely, assuming the "$p$-part" of Conjecture 5, the "$p$-part" of Conjecture 3 holds if and only if the "$p$-part" of Conjectures 2 and 4 hold.

The following theorem gives evidence for the validity of Conjecture 4.

**Theorem 3.5.4** (Burns [Bur07, Theorem 3.1]). *If the conjecture in [Bur07, §6.3] holds for $L'/k$, then we have*

$$\Phi(\varepsilon_{L',S,V'}) \in I_{L'/L}^d$$

*for every $\Phi \in \bigwedge_{\mathcal{G}_{L'}}^{r'} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}_{L',S,T}^\times, \mathbb{Z}[\mathcal{G}_{L'}])$ and an equality*

$$\Phi(\varepsilon_{L',S,V'}) = \mathrm{sgn}(V,V')\Phi^{G(L'/L)} \left( \left( \bigwedge_{v \in V \setminus V'} \varphi_v \right) (\varepsilon_{L,S,V}) \right)$$

*in $\mathrm{coker}(\bigwedge_{v \in V \setminus V'} \varphi_v : (\bigwedge_{\mathcal{G}_L}^d L_T^\times)_{\mathrm{tors}} \to Q_{L'/L}^d)$, where $L_T^\times$ is the subgroup of $L^\times$ defined by*

$$L_T^\times = \{a \in L^\times \mid \mathrm{ord}_w(a-1) > 0 \text{ for all } w \in T_L\}.$$

**Remark 3.5.5.** In the number field case, as Burns mentioned in [Bur07, Remark 6.2], the conjecture in [Bur07, §6.3] for $L'/k$ is equivalent to the "equivariant Tamagawa number conjecture (ETNC)" ([BuFl01, Conjecture 4(iv)]) for the pair $(h^0(\mathrm{Spec}(L')), \mathbb{Z}[\mathcal{G}_{L'}])$, and known to be true if $L'$ is an abelian extension over $\mathbb{Q}$ by the works of Burns, Greither, and Flach ([BuGr03], [Fla11]).

**Remark 3.5.6.** In [Bur07, Theorem 3.1], Burns actually proved more: let

$$I^S_{L'/L} = \begin{cases} \prod_{v \in V \setminus V'} I_v & \text{if } d > 0, \\ \mathbb{Z}[\mathcal{G}_{L'}] & \text{if } d = 0, \end{cases}$$

where $I_v = \ker(\mathbb{Z}[\mathcal{G}_{L'}] \to \mathbb{Z}[\mathcal{G}_{L'}/\mathcal{G}_v])$ and $\mathcal{G}_v$ is the decomposition group of $w$ in $G(L'/L)$. Then Burns proved that, under the assumption that the conjecture in [Bur07, §6.3] holds for $L'/k$, $\Phi(\varepsilon_{L',S,V'}) \in I^S_{L'/L}$ for every $\Phi \in \bigwedge^{r'}_{\mathcal{G}_{L'}} \mathrm{Hom}_{\mathcal{G}_{L'}}(\mathcal{O}^\times_{L',S,T}, \mathbb{Z}[\mathcal{G}_{L'}])$ and an equality

$$\Phi(\varepsilon_{L',S,V'}) = \mathrm{sgn}(V, V')\Phi^{G(L'/L)}\left(\left(\bigwedge_{v \in V \setminus V'} \varphi_v\right)(\varepsilon_{L,S,V})\right)$$

holds in $\mathrm{coker}(\bigwedge_{v \in V \setminus V'} \varphi_v : (\bigwedge^d_{\mathcal{G}_L} L^\times_T)_{\mathrm{tors}} \to I^S_{L'/L}/I_{L'/L}I^S_{L'/L})$.

**Proposition 3.5.7.**

$$\left(\bigwedge^d_{\mathcal{G}_L} L^\times_T\right)_{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|\mathcal{G}_L|}\right] = 0.$$

*Proof.* Note that

$$\bigwedge^d_{\mathcal{G}_L} L^\times_T = \varinjlim \bigwedge^d_{\mathcal{G}_L} \mathcal{O}^\times_{L,\Sigma,T},$$

where $\Sigma$ runs over all finite sets of places of $k$, which contains all the infinite places and places ramifying in $L$, and is disjoint from $T$, and the direct limit is taken with respect to the map induced by the inclusion $\mathcal{O}_{L,\Sigma,T} \hookrightarrow \mathcal{O}_{L,\Sigma',T}$ $(\Sigma \subset \Sigma')$. So it is sufficient to prove that for such $\Sigma$, $\bigwedge^d_{\mathcal{G}_L} \mathcal{O}^\times_{L,\Sigma,T} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|\mathcal{G}_L|}]$ is torsion-free. Since $\mathcal{O}^\times_{L,S,T}$ is torsion-free, we see that $\mathcal{O}^\times_{L,\Sigma,T}$ is also torsion-free. It is well-known that a finitely generated $\mathbb{Z}[\frac{1}{|\mathcal{G}_L|}][\mathcal{G}_L]$-module is locally free if and only if it is torsion-free. So we see that $\mathcal{O}^\times_{L,\Sigma,T} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|\mathcal{G}_L|}]$ is locally free $\mathbb{Z}[\frac{1}{|\mathcal{G}_L|}][\mathcal{G}_L]$-module. Hence $\bigwedge^d_{\mathcal{G}_L} \mathcal{O}^\times_{L,\Sigma,T} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|\mathcal{G}_L|}]$ is also locally free, so it is torsion-free. □

Combining Theorem 3.5.1, Theorem 3.5.4, and Proposition 3.5.7, we have the following theorem (see also Remark 3.5.3).

**Theorem 3.5.8** ([San14b, Theorem 3.22])**.** *Let $p$ be a prime number not dividing $|\mathcal{G}_L|$. Assume the "p-part" of Conjecture 5 holds. If the conjecture in [Bur07, §6.3] for $L'/k$ and the "p-part" of Conjecture 2 hold, then the "p-part" of Conjecture 3 holds.*

**Remark 3.5.9.** In the joint work with Burns and Kurihara [BKS14], the authors proved that Conjectures 3 and 4 are equivalent under no assumptions. Furthermore, we proved that the conjecture in [Bur07, §6.3] for $L'/k$ implies Conjecture 3 directly. This result improves Theorems 3.5.8 and 3.5.4. Since the ETNC for the pair $(h^0(\mathrm{Spec}(L')), \mathbb{Z}[\mathcal{G}_{L'}])$ is known to be true if $L'$ is abelian over $\mathbb{Q}$, as we noted before, we have proved that Conjecture 3 is true if $L'$ is abelian over $\mathbb{Q}$.

# Chapter 4

# Darmon's conjecture

In this chapter, we show that Conjecture 3 is regarded as a generalization of Darmon's conjecture [Dar95]. By Remark 3.5.9, we know that Conjecture 3 is true if $L'$ is abelian over $\mathbb{Q}$. As an application of this fact, we give a full proof of Darmon's conjecture. Thus, we improve the main result of Mazur and Rubin in [MaRu11], where the "non-2-part" of Darmon's conjecture is proved.

## 4.1 The formulation

We formulate a slightly modified version of the conjecture of Darmon. First, we fix the following:

- a bijection $\{\text{all the places of } \mathbb{Q}\} \simeq \mathbb{Z}_{\geq 0}$ such that $\infty$ (the infinite place of $\mathbb{Q}$) corresponds to 0 (from this, we endow a total order on $\{\text{all the places of } \mathbb{Q}\}$),

- for each place $v$ of $\mathbb{Q}$, a place of $\overline{\mathbb{Q}}$ lying above $v$.

Let $F/\mathbb{Q}$ be a real quadratic field, and $\chi$ be the corresponding Dirichlet character with conductor $f$. Let $n$ be a square-free product of primes not dividing $f$. Put

$$n_{\pm} = \prod_{\ell | n, \chi(\ell) = \pm 1} \ell$$

(throughout this chapter, $\ell$ always denotes a prime number), and let $\nu_{\pm}$ be the number of prime divisors of $n_{\pm}$. For any positive integer $m$, $\mu_m$ denotes the group of $m$th roots of unity in $\overline{\mathbb{Q}}$, and $\zeta_m = e^{\frac{2\pi i}{m}}$ (the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is fixed above). Put $F_n := F(\mu_n)^+$, the maximal real subfield of $F(\mu_n)$. Define a cyclotomic unit by

$$\beta_n := \mathrm{N}_{F(\mu_n)/F_n}\left(\prod_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))}\sigma(1-\zeta_{nf})^{\chi(\sigma)}\right).$$

Put

$$\theta_n := \sum_{\sigma\in\mathrm{Gal}(F_n/F)}\sigma\beta_n\otimes\sigma^{-1}\in F_n^\times\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{Gal}(F_n/F)].$$

Let $I_n$ be the augmentation ideal of $\mathbb{Z}[\mathrm{Gal}(F_n/F)]$. Note that, since $F_n^\times/F^\times$ is torsion-free, the natural map

$$(F^\times/\{\pm1\})\otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_++1}\longrightarrow (F_n^\times/\{\pm1\})\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{Gal}(F_n/F)]/I_n^{\nu_++1}$$

is injective.

Next, write $n_+=\prod_{i=1}^{\nu_+}\ell_i$ so that $\ell_1\prec\cdots\prec\ell_{\nu_+}$ ("$\prec$" is the total order fixed above), and let $\lambda_i$ be the fixed place of $F$ lying above $\ell_i$. Let $\lambda_0$ be the fixed place of $F$ lying above $\infty$. Let $\tau$ be the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Take $u_0,\dots,u_{\nu_+}\in\mathcal{O}_F[\frac{1}{n}]^\times$ such that $\{(1-\tau)u_i\}_{0\le i\le\nu_+}$ forms a $\mathbb{Z}$-basis of $(1-\tau)\mathcal{O}_F[\frac{1}{n}]^\times$ (which is in fact a free abelian group of rank $\nu_++1$, see [MaRu11, Lemma 3.2(ii)]), and $\det(\log|(1-\tau)u_i|_{\lambda_j})_{0\le i,j\le\nu_+}>0$. Put

$$R_n := (\varphi^1_{\ell_1}\wedge\cdots\wedge\varphi^1_{\ell_{\nu_+}})((1-\tau)u_0\wedge\cdots\wedge(1-\tau)u_{\nu_+})\in(1-\tau)\mathcal{O}_F\left[\frac{1}{n}\right]^\times\otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_++1},$$

where

$$\varphi^1_{\ell_i}: F^\times\longrightarrow I_n/I_n^2$$

is defined by $\varphi^1_{\ell_i}=\mathrm{rec}_{\lambda_i}(\cdot)-1$, where $\mathrm{rec}_{\lambda_i}:F^\times\to\mathrm{Gal}(F_n/F)$ is the local reciprocity map at $\lambda_i$. Note that we have

$$R_n=\det\begin{pmatrix}(1-\tau)u_0 & \cdots & (1-\tau)u_{\nu_+}\\ \varphi^1_{\ell_1}((1-\tau)u_0) & \cdots & \varphi^1_{\ell_1}((1-\tau)u_{\nu_+})\\ \vdots & \ddots & \vdots\\ \varphi^1_{\ell_{\nu_+}}((1-\tau)u_0) & \cdots & \varphi^1_{\ell_{\nu_+}}((1-\tau)u_{\nu_+})\end{pmatrix}.$$

Finally, let $h_n$ denote the $n$-class number of $F$, i.e. the order of the Picard group of $\mathrm{Spec}\,\mathcal{O}_F[\frac{1}{n}]$.

Now Darmon's conjecture is stated as follows.

**Theorem 4.1.1** (Darmon's conjecture)**.**

$$\theta_n = -2^{\nu_-} h_n R_n \quad in \quad (F^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1}.$$

**Remark 4.1.2.** Let $J_n$ be the augmentation ideal of $\mathbb{Z}[\mathrm{Gal}(F(\mu_n)/F)]$. Note that the natural map $\mathrm{Gal}(F(\mu_n)/F) \to \mathrm{Gal}(F_n/F)$ induces the isomorphism

$$J_n^{\nu_+}/J_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right] \xrightarrow{\sim} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right].$$

Using this, it is not difficult to see that the following statement is equivalent to [MaRu11, Theorem 3.9]:

$$\theta_n = -2^{\nu_-} h_n R_n \quad \text{in} \quad (F^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{2}\right].$$

Thus, the "non-2-part" of the original conjecture of Darmon ([MaRu11, Conjecture 3.8]) is equivalent to the "non-2-part" of our modified conjecture of Darmon. Therefore, Theorem 4.1.1 improves [MaRu11, Theorem 3.9]. Note that, in the original conjecture of Darmon, the cyclotomic unit is defined by

$$\alpha_n := \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))} \sigma(1-\zeta_{nf})^{\chi(\sigma)},$$

whereas our cyclotomic unit is $\beta_n = \mathrm{N}_{F(\mu_n)/F_n}(\alpha_n)$. Since cyclotomic units, as Stark elements, lie in real fields, so it is natural to consider $\beta_n$.

## 4.2 Proof of Darmon's conjecture

We keep notation in the previous section, and also use notation defined in Chapter 3. We specialize the general setting of Chapter 3 into the following:

- $k = \mathbb{Q}$,
- $L = F$ (a real quadratic field),
- $L' = F_n$,
- $S = S' = \{\infty\} \cup \{\text{primes dividing } nf\}$,
- $V = \{\infty\} \cup \{\text{primes dividing } n_+\}$,

- $V' = \{\infty\}$,
- $T$: a finite set of places of $\mathbb{Q}$ such that
  - $S \cap T = \emptyset$,
  - $\mathcal{O}_{L',S,T}^{\times}$ is torsion-free.

Then one sees that $(L, S, V), (L', S, V') \in \Omega = \Omega(\mathbb{Q}, T)$.

It is known that the Rubin-Stark conjecture (Conjecture 1) for all the triples in $\Omega$ holds ([Bur07, Theorem A]). Let

$$\varepsilon_T = \varepsilon_{L,S,T,V} \in \bigcap_{\mathcal{G}_L}^{\nu_+ + 1} \mathcal{O}_{L,S,T}^{\times} \quad \left(\text{resp. } \varepsilon'_T = \varepsilon_{L',S,T,V'} \in \bigcap_{\mathcal{G}_{L'}}^{1} \mathcal{O}_{L',S,T}^{\times} = \mathcal{O}_{L',S,T}^{\times}\right)$$

denote the Rubin-Stark element for the triple $(L, S, V)$ (resp. $(L', S, V')$) (later we will vary $T$, so we keep in the notation the dependence on $T$).

By Remark 3.5.9, we know that Conjecture 3 is true if $L'$ is abelian over $\mathbb{Q}$. Hence we have the following theorem.

**Theorem 4.2.1.**

$$\mathrm{N}_{L'/L}^{(1,\nu_+)}(\varepsilon'_T) = (-1)^{\nu_+} \left( \bigwedge_{\ell | n_+} \varphi_\ell \right) (\varepsilon_T) \quad \textit{in} \quad L^{\times} \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+}.$$

We will deduce Darmon's conjecture (Theorem 4.1.1) from Theorem 4.2.1 by varying the set $T$.

The following proposition is well-known.

**Proposition 4.2.2.** *There exists a finite family $\mathcal{T}$ of $T$ such that $S \cap T = \emptyset$ and $\mathcal{O}_{L',S,T}^{\times}$ is torsion-free, and for every $T \in \mathcal{T}$, there is an $a_T \in \mathbb{Z}[\mathcal{G}_{L'}]$ such that*

$$2 = \sum_{T \in \mathcal{T}} a_T \delta_T \quad \textit{in} \quad \mathbb{Z}[\mathcal{G}_{L'}],$$

*where $\delta_T = \prod_{\ell \in T}(1 - \ell \mathrm{Fr}_\ell^{-1}) \in \mathbb{Z}[\mathcal{G}_{L'}]$.*

For the proof, see [Tat76, Lemme 1.1, Chpt. IV]. Take such a family $\mathcal{T}$ and $a_T$ for each $T \in \mathcal{T}$.

**Lemma 4.2.3.** (i)
$$(1-\tau)\sum_{T\in\mathcal{T}} a_T\varepsilon'_T = \beta_n \quad \textit{in} \quad L'^{\times}/\{\pm 1\},$$
*where $\tau$ is regarded as the generator of* $\mathrm{Gal}(L'/\mathbb{Q}(\mu_n)^+)$.

(ii)
$$(1-\tau)\sum_{T\in\mathcal{T}} a_T\varepsilon_T = (-1)^{\nu_++1}2^{\nu_-}h_n(1-\tau)u_0\wedge\cdots\wedge u_{\nu_+} \quad \textit{in} \quad \mathbb{Q}\otimes_{\mathbb{Z}} \bigwedge_{\mathcal{G}_L}^{\nu_++1}\mathcal{O}_{L,S}^{\times}.$$

*Proof.* (i) From
$$2\varepsilon'_T = \delta_T \mathrm{N}_{\mathbb{Q}(\mu_{nf})/L'}(1-\zeta_{nf}),$$
we obtain
$$2\sum_{T\in\mathcal{T}} a_T\varepsilon'_T = 2\mathrm{N}_{\mathbb{Q}(\mu_{nf})/L'}(1-\zeta_{nf})$$
(see Proposition 4.2.2). We compute
$$\begin{aligned}(1-\tau)\mathrm{N}_{\mathbb{Q}(\mu_{nf})/L'}(1-\zeta_{nf}) &= \mathrm{N}_{L(\mu_n)/L'}((1-\tau)\mathrm{N}_{\mathbb{Q}(\mu_{nf})/L(\mu_n)}(1-\zeta_{nf}))\\ &= \beta_n,\end{aligned}$$
hence we have
$$(1-\tau)\sum_{T\in\mathcal{T}} a_T\varepsilon'_T = \beta_n \quad \text{in} \quad L'^{\times}/\{\pm 1\}.$$
(ii) By Lemma 3.3.1, $R_V$ is injective on $e_\chi(\mathbb{Q}\otimes_{\mathbb{Z}}\bigwedge_{\mathcal{G}_L}^{\nu_++1}\mathcal{O}_{L,S}^{\times})$, so it is sufficient to prove that
$$R_V\left((1-\tau)\sum_{T\in\mathcal{T}} a_T\varepsilon_T\right) = (-1)^{\nu_++1}2^{\nu_-}h_nR_V((1-\tau)u_0\wedge\cdots\wedge u_{\nu_+}).$$
By the characterization of $\varepsilon_T$, the left hand side is equal to $2(1-\tau)\Theta_{L,S}^{(\nu_++1)}(0)$. Hence, it is sufficient to prove that
$$2(1-\tau)\Theta_{L,S}^{(\nu_++1)}(0) = (-1)^{\nu_++1}2^{\nu_-}h_nR_V((1-\tau)u_0\wedge\cdots\wedge u_{\nu_+}). \tag{4.1}$$
If $\nu_- > 0$ and $\ell \mid n_-$, then we have
$$(1-\tau)\Theta_{L,S}^{(\nu_++1)}(0) = 2(1-\tau)\Theta_{L,S\setminus\{\ell\}}^{(\nu_++1)}(0),$$

since $\mathrm{Fr}_\ell = \tau$. Also, note that $h_n = h_{n/\ell}$ and $(1-\tau)\mathcal{O}_L[\frac{1}{n}]^\times = (1-\tau)\mathcal{O}_L[\frac{1}{n/\ell}]^\times$. Therefore, proving (4.1), we may assume $\nu_- > 0$. Using the well-known class number formulas for $n$-truncated Dedekind zeta functions of $L$ and $\mathbb{Q}$ (see [Gro88, §1]), we have

$$2(1-\tau)\Theta_{L,S}^{(\nu_+ +1)}(0) = 4h_n e_\chi \frac{R_{L,n}}{R_{\mathbb{Q},n}},$$

where $R_{L,n}$ and $R_{\mathbb{Q},n}$ are the usual $n$-regulators for $L$ and $\mathbb{Q}$ respectively. In Lemma 4.2.4, we will prove an equality

$$e_\chi R_{L,n} = (-1)^{\nu_+ +1} 2^{\nu_- -1} R_{\mathbb{Q},n} e_\chi R_V(u_0 \wedge \cdots \wedge u_{\nu_+}).$$

Hence we have

$$2(1-\tau)\Theta_{L,S}^{(\nu_+ +1)}(0) = (-1)^{\nu_+ +1} 2^{\nu_-} h_n R_V((1-\tau)u_0 \wedge \cdots \wedge u_{\nu_+}),$$

which completes the proof. □

**Lemma 4.2.4.**

$$e_\chi R_{L,n} = (-1)^{\nu_+ +1} 2^{\nu_- -1} R_{\mathbb{Q},n} e_\chi R_V(u_0 \wedge \cdots \wedge u_{\nu_+}).$$

*Proof.* (Compare the proof of [Rub96, Theorem 3.5].) There is an exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z}\left[\frac{1}{n}\right]^\times/\{\pm 1\} \longrightarrow \mathcal{O}_L\left[\frac{1}{n}\right]^\times/\{\pm 1\} \xrightarrow{1-\tau} (1-\tau)\mathcal{O}_L\left[\frac{1}{n}\right]^\times \longrightarrow 0.$$

Since $(1-\tau)\mathcal{O}_L[\frac{1}{n}]^\times$ is torsion-free (see [MaRu11, Lemma 3.2(ii)]), this exact sequence splits. So we can choose $\eta_1, \ldots, \eta_\nu \in \mathbb{Z}[\frac{1}{n}]^\times$ so that $\{\eta_1, \ldots, \eta_\nu, u_0, \ldots, u_{\nu_+}\}$ is a basis of $\mathcal{O}_L[\frac{1}{n}]^\times/\{\pm 1\}$ ($\nu$ is the number of prime divisors of $n$). Write $n_- = \prod_{i=1}^{\nu_-} \ell'_i$, where $\ell'_i$ is a prime number. Let $\lambda'_i$ be the (unique) place of $L$ lying above $\ell'_i$. We compute the regulator $R_{L,n}$ with respect to the basis $\{\eta_1, \ldots, \eta_\nu, u_0, \ldots, u_{\nu_+}\}$ of $\mathcal{O}_L[\frac{1}{n}]^\times/\{\pm 1\}$ and the places $\{\lambda'_2, \ldots, \lambda'_{\nu_-}, \lambda_0^\tau, \ldots, \lambda_{\nu_+}^\tau, \lambda_0, \ldots, \lambda_{\nu_+}\}$:

$$R_{L,n} = \pm \det \begin{pmatrix} \log|\eta|_{\lambda'} & \log|\eta|_{\lambda^\tau} & \log|\eta|_\lambda \\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_\lambda \end{pmatrix},$$

where we omit the subscript, for simplicity (for example, $\log|\eta|_{\lambda'}$ means the $\nu \times (\nu_- - 1)$-

matrix $(\log|\eta_i|_{\lambda'_j})_{1\le i\le \nu, 2\le j\le \nu_-})$. We may assume that the sign of the right hand side is positive (replace $\eta_1$ by $\eta_1^{-1}$ if necessary). We compute

$$\begin{aligned}
\det\begin{pmatrix} \log|\eta|_{\lambda'} & \log|\eta|_{\lambda^\tau} & \log|\eta|_{\lambda} \\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_{\lambda}\end{pmatrix} &= \det\begin{pmatrix} \log|\eta|_{\lambda'} & \log|\eta|_{\lambda} & \log|\eta|_{\lambda} \\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_{\lambda}\end{pmatrix} \\
&= \det\begin{pmatrix} \log|\eta|_{\lambda'} & \log|\eta|_{\lambda} & 0 \\ \log|u|_{\lambda'} & \log|u|_{\lambda^\tau} & \log|u|_{\lambda}-\log|u|_{\lambda^\tau}\end{pmatrix} \\
&= \det(\ \log|\eta|_{\lambda'} \quad \log|\eta|_{\lambda}\ )\det(\log|u|_{\lambda}-\log|u|_{\lambda^\tau}) \\
&= \det(\ 2\log|\eta|_{\ell'} \quad \log|\eta|_{\ell}\ )\det(\log|(1-\tau)u|_{\lambda}) \\
&= 2^{\nu_- -1}R_{\mathbb{Q},n}\det(\log|(1-\tau)u|_{\lambda}).
\end{aligned}$$

Hence we have

$$e_\chi R_{L,n} = 2^{\nu_- -1}R_{\mathbb{Q},n}e_\chi\det(\log|(1-\tau)u|_{\lambda}). \tag{4.2}$$

On the other hand, we compute

$$\begin{aligned}
e_\chi R_V(u_0\wedge\cdots\wedge u_{\nu_+}) &= (-1)^{\nu_+ +1}e_\chi\det(\log|u|_{\lambda}+\log|\tau(u)|_{\lambda}\tau) \\
&= (-1)^{\nu_+ +1}e_\chi\det(\log|(1-\tau)u|_{\lambda}+(1+\tau)\log|\tau(u)|_{\lambda}) \\
&= (-1)^{\nu_+ +1}e_\chi\det(\log|(1-\tau)u|_{\lambda}),
\end{aligned}$$

where the first equality follows by noting that $R_V = \bigwedge_{0\le i\le \nu_+}(-\log|\cdot|_{\lambda_i}-\log|\tau(\cdot)|_{\lambda_i}\tau)$ by definition (see §3.1), and the last equality follows from $e_\chi(1+\tau)=0$. Hence, by (4.2), we have the desired equality

$$e_\chi R_{L,n} = (-1)^{\nu_+ +1}2^{\nu_- -1}R_{\mathbb{Q},n}e_\chi R_V(u_0\wedge\cdots\wedge u_{\nu_+}).$$

□

Now we give the proof of Theorem 4.1.1.

*Proof of Theorem 4.1.1.* By Theorem 4.2.1, we have an equality

$$\mathrm{N}_{L'/L}^{(1,\nu_+)}(\varepsilon'_T) = (-1)^{\nu_+}\left(\bigwedge_{\ell\mid n_+}\varphi_\ell\right)(\varepsilon_T)$$

in $L^\times \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+}$. From this and Lemma 4.2.3, we deduce that an equality

$$\theta_n = -2^{\nu_-} h_n \left( \bigwedge_{\ell | n_+} \varphi_\ell \right) ((1-\tau) u_0 \wedge \cdots \wedge u_{\nu_+})$$

holds in $(L^\times / \{\pm 1\}) \otimes_{\mathbb{Z}} Q(L'/L)^{\nu_+}$. It is easy to see that

$$(-1)^{\nu_+} \left( \bigwedge_{\ell | n_+} \varphi_\ell \right) ((1-\tau) u_0 \wedge \cdots \wedge u_{\nu_+}) = R_n.$$

Hence we have the desired equality

$$\theta_n = -2^{\nu_-} h_n R_n.$$

□

**Remark 4.2.5.** By a similar argument to the proof of Theorem 4.1.1, we can show that Gross's "conjecture for tori" [Gro88, Conjecture 8.8] is also deduced from Conjecture 3. For the detail, see [San] or [BKS14].

# Chapter 5

# Euler systems and Kolyvagin systems

In this chapter, we give a generalization of Darmon's conjecture for Euler systems for general $p$-adic representations. The formulation of our conjecture is explained in §1.4 (see (1.2)). We prove this conjecture under the standard hypotheses in the theory of Kolyvagin systems (see Theorem 5.1.8).

A key observation lies in defining a notion of "algebraic Kolyvagin systems", which generalizes the notion of original Kolyvagin systems (see §5.2). We define four different modules of algebraic Kolyvagin systems, called $\theta$-Kolyvagin systems, derived-Kolyvagin systems, pre-Kolyvagin systems, and (simply) Kolyvagin systems. The $\theta$-Kolyvagin system is the system whose axioms are satisfied by the collection $\{\theta_n(c)\}_n$ of the theta elements (see Definition 5.1.4). The derived-Kolyvagin system is the system whose axioms are satisfied by the collection $\{\kappa'_n\}_n$ of the Kolyvagin's derivative classes of $c$. The pre-Kolyvagin system is an analogue of the $\theta$-Kolyvagin system. The system which we call simply Kolyvagin system is a direct generalization of the original Kolyvagin system. At a glance, these four modules of algebraic Kolyvagin systems may have different structures, but we prove that they are all isomorphic (see Theorem 5.2.17). This observation is useful in some aspects; firstly, we can prove that $\{\theta_n(c)\}_n$ is a $\theta$-Kolyvagin system by reducing to show that the Kolyvagin's derivative classes $\{\kappa'_n\}_n$ of $c$ satisfy the axioms of the derived-Kolyvagin systems (see Proposition 5.4.6); secondly, we can apply Mazur-Rubin's theory of Kolyvagin systems to other Kolyvagin systems.

In [MaRu04, Appendix B], Howard constructed "regulator-type" Kolyvagin systems. We extend this construction to other Kolyvagin systems. We introduce a new system, which we call "unit system", to treat Howard's construction more systematically (see Definition 5.3.3). We interpret Howard's construction as a "regulator map" from the module of unit systems to that of Kolyvagin systems (see Definition 5.3.5). We give analogues of this regulator map for other Kolyvagin systems, and prove the natural compatibility with the isomorphisms between different Kolyvagin systems (see Theorem 5.3.7). We apply Mazur-Rubin's theory to show that the regulator map is surjective (see Theorem 5.4.2). From this, we show that the system of the theta elements, which forms a $\theta$-Kolyvagin system, is in the image of the regulator map. This says in fact that $\theta_n(c) \in \mathfrak{h}_n \mathcal{R}_n$ holds (see (1.2)). Thus, we prove the main theorem in this chapter.

In this chapter, we use the following notation. For each place $v$ of $\mathbb{Q}$, we choose a place $w$ of $\overline{\mathbb{Q}}$ above $v$, and fix it. By the decomposition (resp. inertia) group of $v$ in $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we mean the decomposition (resp. inertia) group of $w$. The absolute Galois group of $\mathbb{Q}_v$ is identified with the decomposition group of $v$ in $G_{\mathbb{Q}}$.

For a field $F$, and a continuous $\mathrm{Gal}(\overline{F}/F)$-module $M$ (where $\overline{F}$ is a fixed separable closure of $F$), we denote

$$H^i(F, M) = H^i_{\mathrm{cont}}(\mathrm{Gal}(\overline{F}/F), M),$$

where $H^i_{\mathrm{cont}}$ is the continuous cochain cohomology ([Tat76]).

If $G$ is a profinite group, and $M$ is a continuous $G$-module, we denote for $\tau \in G$

$$M^{\tau=1} = \{a \in M \mid \tau a = a\}.$$

## 5.1 The statement

The aim of this section is to state the main theorem in this chapter (Theorem 5.1.8). First, we set some notation. Let $p$ be an odd prime, and fix a power of $p$, which is denoted by $M$. Let $T$ be a $p$-adic representation of the absolute Galois group of $\mathbb{Q}$ with coefficients in $\mathbb{Z}_p$, that is, $T$ is a free $\mathbb{Z}_p$-module of finite rank with a continuous $\mathbb{Z}_p$-linear action of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As usual, we assume that $T$ is unramified at almost all places of $\mathbb{Q}$, that is, for all but finitely many places $v$ of $\mathbb{Q}$, the inertia group of $v$ in $G_{\mathbb{Q}}$ acts trivially on $T$. We write $A = T/MT$. Fix $\Sigma$, a set of places of $\mathbb{Q}$, such that

$$\Sigma \subset \{\ell \mid \ell \text{ is a prime satisfying } (*)\},$$

where $(*)$ is as follows:

$$(*)\begin{cases} T \text{ is unramified at } \ell, \\ M \text{ divides } \ell-1, \\ A/(\mathrm{Fr}_\ell - 1)A \simeq \mathbb{Z}/M\mathbb{Z}, \end{cases}$$

where $\mathrm{Fr}_\ell$ is the arithmetic Frobenius at $\ell$.

Next, put $\mathcal{N} = \mathcal{N}(\Sigma) = \{$square-free products of primes in $\Sigma\}$. We suppose $1 \in \mathcal{N}$, for convention. Note that $\mathcal{N}$ is naturally identified with the family of all the finite subsets of $\Sigma$ (with this identification, $1 \in \mathcal{N}$ corresponds to the empty set $\emptyset \subset \Sigma$). This observation will be used later, in §5.2.

For every $\ell \in \Sigma$, put

$$P_\ell(x) = \det(1 - \mathrm{Fr}_\ell x | T) \in \mathbb{Z}_p[x],$$

where the right hand side means the characteristic polynomial with respect to the action of $\mathrm{Fr}_\ell$ on $T$. Note that $P_\ell(1) \equiv 0 \pmod{M}$, since $A/(\mathrm{Fr}_\ell - 1)A \simeq \mathbb{Z}/M\mathbb{Z}$ (see [MaRu04, Lemma 1.2.3]). Put

$$Q_\ell(x) = \frac{P_\ell(x) - P_\ell(1)}{x-1} \mod M \in \mathbb{Z}/M\mathbb{Z}[x].$$

This is the unique polynomial such that

$$(x-1)Q_\ell(x) \equiv P_\ell(x) \mod M$$

(see [Rub00, Lemma 4.5.2] or [MaRu04, Definition 1.2.2]).

Next, for every $n \in \mathcal{N}$, put

$$G_n = \mathrm{Gal}(\mathbb{Q}(n)/\mathbb{Q}),$$

where $\mathbb{Q}(n)$ is the maximal $p$-subextension of $\mathbb{Q}$ inside $\mathbb{Q}(\mu_n)$. Note that we have a natural isomorphism $G_n \simeq \bigoplus_{\ell | n} G_\ell$. (Note also that in this section $G_\ell$ does not mean the decomposition group at $\ell$.) For every $\ell \in \Sigma$, we define a generator $\sigma_\ell$ of $G_\ell$ as follows. Fix a generator $\xi$ of $\mathbb{Z}_p$-module $\varprojlim \mu_{p^m}$. Since we fixed the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, $\varprojlim \mu_{p^m}$ is also regarded as a subgroup of $\varprojlim \overline{\mathbb{Q}}_\ell^\times$. By Kummer theory, we have a canonical isomorphism

$$\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ur}}(\ell^{1/p^\infty})/\mathbb{Q}_\ell^{\mathrm{ur}}) \xrightarrow{\sim} \varprojlim \mu_{p^m}; \quad \sigma \mapsto \left( \frac{\sigma(\ell^{1/p^m})}{\ell^{1/p^m}} \right)_m,$$

where $\mathbb{Q}_\ell^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Q}_\ell$. We also have a natural surjection

$\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ur}}(\ell^{1/p^\infty})/\mathbb{Q}_\ell^{\mathrm{ur}}) \to G_\ell$, so we have a surjection $\varprojlim \mu_{p^m} \to G_\ell$. We define $\sigma_\ell \in G_\ell$ to be the image of $\xi \in \varprojlim \mu_{p^m}$ by this surjection.

For $n \in \mathcal{N}$, we denote $I_n$ the augmentation ideal of $\mathbb{Z}[G_n]$. Note that if $\ell \not| n$, then we have

$$P_\ell(\mathrm{Fr}_\ell) \otimes 1 \in I_n \otimes \mathbb{Z}/M\mathbb{Z},$$

since $P_\ell(1) \equiv 0 \pmod{M}$ as we mentioned above, where $\mathrm{Fr}_\ell$ is naturally regarded as an element of $G_n$ (note that since $\ell$ is prime to $n$, $\ell$ is unramified in $\mathbb{Q}(n)$). Therefore, we consider the image of $P_\ell(\mathrm{Fr}_\ell) \otimes 1$ in $I_n/I_n^2 \otimes \mathbb{Z}/M\mathbb{Z}$, and denote it also by $P_\ell(\mathrm{Fr}_\ell) \otimes 1$.

We next define important maps $v_\ell$, $u_\ell$, and $\varphi_\ell$ for $\ell \in \Sigma$. As a preliminary, we review some facts on Galois cohomology.

For $\ell \in \Sigma$, the unramified cohomology group at $\ell$ is defined by

$$H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, A) = H^1(\mathbb{Q}_\ell^{\mathrm{ur}}/\mathbb{Q}_\ell, A).$$

There is a canonical isomorphism:

$$H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, A) \simeq A/(\mathrm{Fr}_\ell - 1)A,$$

which is obtained by evaluating $\mathrm{Fr}_\ell \in \mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{ur}}/\mathbb{Q}_\ell)$ to 1-cocycles representing elements of $H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, A)$ (see [Rub00, Lemma B.2.8] or [MaRu04, Lemma 1.2.1(i)]).

There is a canonical decomposition:

$$H^1(\mathbb{Q}_\ell, A) \simeq H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, A) \oplus H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, A),$$

where $H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, A) := H^1(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell, A^{G_{\mathbb{Q}_\ell(\mu_\ell)}})$ is called the transverse cohomology group at $\ell$, and naturally identified with $\mathrm{Hom}(G_\ell, A^{\mathrm{Fr}_\ell=1})$ (see [MaRu04, Lemma 1.2.1(ii) and Lemma 1.2.4]). We remark that to get this decomposition, the assumption $M|\ell-1$ is needed.

Now we start to define $v_\ell$, $u_\ell$, and $\varphi_\ell$.

First, the definition of $v_\ell$ is as follows:

$$\begin{aligned} v_\ell : H^1(\mathbb{Q}, A) &\longrightarrow H^1(\mathbb{Q}_\ell, A) \\ &\longrightarrow H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, A) = \mathrm{Hom}(G_\ell, A^{\mathrm{Fr}_\ell=1}) \\ &\xrightarrow{\sim} A^{\mathrm{Fr}_\ell=1} \simeq \mathbb{Z}/M\mathbb{Z}, \end{aligned}$$

where the first arrow is the localization map at $\ell$, the second is the natural projection, the third isomorphism is obtained by evaluating $\sigma_\ell \in G_\ell$ (recall that $\sigma_\ell$ is the fixed generator of $G_\ell$), and the last (non-canonical) isomorphism follows by noting that $A/(\mathrm{Fr}_\ell - 1)A \simeq \mathbb{Z}/M\mathbb{Z}$ (see [MaRu04, Lemma 1.2.3]). We fix the last isomorphism.

Next, we define the map $u_\ell$ as follows:

$$\begin{aligned} u_\ell : H^1(\mathbb{Q}, A) &\longrightarrow H^1(\mathbb{Q}_\ell, A) \\ &\longrightarrow H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, A) = A/(\mathrm{Fr}_\ell - 1)A \\ &\xrightarrow{-Q_\ell(\mathrm{Fr}_\ell^{-1})} A^{\mathrm{Fr}_\ell = 1} = \mathbb{Z}/M\mathbb{Z}, \end{aligned}$$

where the first arrow is the localization at $\ell$, and the second is the natural projection. The third arrow is defined by

$$A/(\mathrm{Fr}_\ell - 1)A \longrightarrow A^{\mathrm{Fr}_\ell = 1}; \quad \bar{a} \mapsto -Q_\ell(\mathrm{Fr}_\ell^{-1})a$$

(the well-definedness is easily verified by using the Cayley-Hamilton theorem). This is in fact an isomorphism, see [Rub00, Corollary A.2.7] for the proof. Note that we use $-Q_\ell(\mathrm{Fr}_\ell^{-1})$ instead of $Q_\ell(\mathrm{Fr}_\ell^{-1})$ (this turns out to be meaningful when we see Example 5.1.1 below). The last identification $A^{\mathrm{Fr}_\ell=1} = \mathbb{Z}/M\mathbb{Z}$ in the definition of $u_\ell$ above is obtained by the fixed isomorphism when we defined $v_\ell$.

Finally, we define $\varphi_\ell$ as follows:

$$\varphi_\ell : H^1(\mathbb{Q}, A) \longrightarrow \varprojlim_{n \in \mathcal{N}} (I_n/I_n^2 \otimes \mathbb{Z}/M\mathbb{Z}); \quad a \mapsto -(\sigma_\ell - 1) \otimes u_\ell(a) - P_\ell(\mathrm{Fr}_\ell) \otimes v_\ell(a),$$

where the inverse limit in the right hand side is taken with respect to the natural restriction map of Galois groups, namely, if $n, m \in \mathcal{N}$ and $n|m$, the morphism from $I_m/I_m^2 \otimes \mathbb{Z}/M\mathbb{Z}$ to $I_n/I_n^2 \otimes \mathbb{Z}/M\mathbb{Z}$ is induced by the natural surjection $G_m \to G_n$. Note that $P_\ell(\mathrm{Fr}_\ell) \otimes 1$ is naturally regarded as an element of $\varprojlim_{n \in \mathcal{N}, \ell | n} (I_{n/\ell}/I_{n/\ell}^2 \otimes \mathbb{Z}/M\mathbb{Z})$. Since we have the canonical isomorphism

$$(I_\ell/I_\ell^2 \otimes \mathbb{Z}/M\mathbb{Z}) \oplus \varprojlim_{n \in \mathcal{N}, \ell | n} (I_{n/\ell}/I_{n/\ell}^2 \otimes \mathbb{Z}/M\mathbb{Z}) \simeq \varprojlim_{n \in \mathcal{N}} (I_n/I_n^2 \otimes \mathbb{Z}/M\mathbb{Z}),$$

we see that $-(\sigma_\ell - 1) \otimes u_\ell(a) - P_\ell(\mathrm{Fr}_\ell) \otimes v_\ell(a)$ lies in $\varprojlim_{n \in \mathcal{N}} (I_n/I_n^2 \otimes \mathbb{Z}/M\mathbb{Z})$, hence $\varphi_\ell$ is defined.

**Example 5.1.1.** Take $T = \mathbb{Z}_p(1) = \varprojlim \mu_{p^m}$, and $A = T/MT = \mu_M$. Take $\ell \in \Sigma$. Suppose

$a \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^M \simeq H^1(\mathbb{Q}, A)$, and

$$a = \ell^i e \quad \text{in } \mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^M,$$

where $i \in \mathbb{Z}/M\mathbb{Z}$ and $e \in \mu_M$ (note that $i$ and $e$ are uniquely determined for the image of $a$ in $\mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^M$). If we identify $\mathbb{Z}/M\mathbb{Z} = \mu_M$ by fixing a primitive $M$th root of unity, then we see that

$$v_\ell(a) = i$$

and

$$u_\ell(a) = e$$

(note that since $P_\ell(x) = 1 - \ell x \equiv 1 - x \pmod{M}$, we have $Q_\ell(x) = -1$). We see that $\varphi_\ell$ agrees with the following map:

$$\begin{array}{rcl} H^1(\mathbb{Q}, A) \simeq \mathbb{Q}^\times/(\mathbb{Q}^\times)^M & \longrightarrow & \mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^M \\ & \xrightarrow{\mathrm{rec}_\ell} & \varprojlim (G_n \otimes \mathbb{Z}/M\mathbb{Z}) \\ & \xrightarrow{\sim} & \varprojlim (I_n/I_n^2 \otimes \mathbb{Z}/M\mathbb{Z}), \end{array}$$

where $\mathrm{rec}_\ell$ is the map induced by the local reciprocity map at $\ell$, and the last isomorphism is given by $\sigma \mapsto \sigma - 1$.

We put

$$G(n) = \bigoplus_{i=0}^{\infty} I_n^i/I_n^{i+1} \otimes \mathbb{Z}/M\mathbb{Z}$$

for $n \in \mathcal{N}$, where $I_n^0$ is understood to be $\mathbb{Z}[G_n]$ (so we have $I_n^0/I_n^1 = \mathbb{Z}$). $G(n)$ has a structure of graded $\mathbb{Z}/M\mathbb{Z}$-algebra, and we can regard $\varphi_\ell$ as a homomorphism from $H^1(\mathbb{Q}, A)$ to a $\mathbb{Z}/M\mathbb{Z}$-module $\varprojlim_{n \in \mathcal{N}} G(n)$, that is, $\varphi_\ell \in \mathrm{Hom}_{\mathbb{Z}/M\mathbb{Z}}(H^1(\mathbb{Q}, A), \varprojlim_{n \in \mathcal{N}} G(n))$.

We define $\varphi_\ell^n$ to be the composition of the projection to $G(n)$ followed by $\varphi_\ell$, that is,

$$\varphi_\ell^n : H^1(\mathbb{Q}, A) \xrightarrow{\varphi_\ell} \varprojlim_{n \in \mathcal{N}} G(n) \longrightarrow G(n).$$

We denote throughout this paper $\mathcal{F}$ the canonical Selmer structure on $T$ in the sense of [MaRu04, Definition 3.2.1]. For $n \in \mathcal{N}$, we recall that the $n$-modified Selmer group $H^1_{\mathcal{F}^n}(\mathbb{Q}, A)$ is defined by

$$H^1_{\mathcal{F}^n}(\mathbb{Q}, A) = \{a \in H^1(\mathbb{Q}, A) \mid a_\ell \in H^1_{\mathcal{F}}(\mathbb{Q}_\ell, A) \text{ for any } \ell \nmid n\},$$

where $a_\ell$ is the image of $a$ by the localization at $\ell$. We also recall that the $n$-strict dual Selmer group $H^1_{(\mathcal{F}^*)_n}(\mathbb{Q}, A^*)$ is defined by

$$H^1_{(\mathcal{F}^*)_n}(\mathbb{Q}, A^*) = \{a \in H^1_{\mathcal{F}^*}(\mathbb{Q}, A^*) \mid a_\ell = 0 \text{ for any } \ell | n\},$$

where $A^* = \mathrm{Hom}(A, \mu_M)$ is the Kummer dual of $A$, and $\mathcal{F}^*$ is the dual Selmer structure of $\mathcal{F}$. See [MaRu04, Example 2.1.8 and Definition 2.3.1].

**Definition 5.1.2.** For $n \in \mathcal{N}$, we define a (module of) regulator $\mathcal{R}_n$ by

$$\mathcal{R}_n = \mathrm{im}\left(\varphi^n_{\ell_1} \wedge \cdots \wedge \varphi^n_{\ell_{\nu(n)}} : \bigwedge_{\mathbb{Z}_p}^{\nu(n)+1} H^1_{\mathcal{F}^n}(\mathbb{Q}, A) \longrightarrow H^1_{\mathcal{F}^n}(\mathbb{Q}, A) \otimes G(n)\right),$$

where $n = \ell_1 \cdots \ell_{\nu(n)}$ and $\nu(n)$ is the number of prime divisors of $n$. Note that $\mathcal{R}_n$ does not depend on the choice of the order of $\ell_1, \ldots, \ell_{\nu(n)}$.

We recall the definition of Euler systems ([Rub00, Definition 2.1.1], [MaRu04, Definition 3.2.2]). Note that the definition of Euler systems in [Rub00] and that of [MaRu04] are slightly different (see [MaRu04, Remark 3.2.3]). Our definition is due to the latter one.

**Definition 5.1.3.** A collection

$$\{c_F \in H^1(F, T) \mid \mathbb{Q} \subset F \subset \mathcal{K},\ F/\mathbb{Q}\text{: finite extension}\}$$

is an Euler system for $(T, \Sigma, \mathcal{K})$, where $\mathcal{K}$ is an abelian extension of $\mathbb{Q}$, if, whenever $F \subset F' \subset \mathcal{K}$ and $F'/\mathbb{Q}$ is finite,

$$\mathrm{Cor}_{F'/F}(c_{F'}) = \left(\prod P_\ell(\mathrm{Fr}_\ell^{-1})\right) c_F,$$

where the product runs over primes $\ell \in \Sigma$ which ramify in $F'$ but not in $F$.

We define an analogue of Darmon's "theta-element" ([Dar95, §4]) for a general Euler system.

**Definition 5.1.4.** Suppose $c = \{c_F \in H^1(F, T) \mid \mathbb{Q} \subset F \subset \mathcal{K},\ F/\mathbb{Q} : \text{finite extension}\}$ is an Euler system for $(T, \Sigma, \mathcal{K})$ such that $\mathbb{Q}(n) \subset \mathcal{K}$ for any $n \in \mathcal{N}$. We define the theta element $\theta_n(c)$ for $n \in \mathcal{N}$ by

$$\theta_n(c) = \sum_{\gamma \in G_n} \gamma c_n \otimes \gamma \in H^1(\mathbb{Q}(n), A) \otimes \mathbb{Z}[G_n],$$

where $c_n = c_{\mathbb{Q}(n)}$, which we regard as an element of $H^1(\mathbb{Q}(n), A)$ via the natural map $H^1(\mathbb{Q}(n), T) \to H^1(\mathbb{Q}(n), A)$, induced by the natural surjection $T \to A$.

**Lemma 5.1.5.** *Suppose $d, n \in \mathcal{N}$ and $d|n$. Then we have*

$$\pi_d(\theta_n(c)) = \theta_d(c) \prod_{\ell | n/d} P_\ell(\mathrm{Fr}_\ell),$$

*where $\pi_d$ is the map induced by the natural projection $G_n \to G_d$.*

*Proof.* We may assume $d = n/\ell$, where $\ell$ is a prime divisor of $n$. We compute

$$\begin{aligned}
\pi_{n/\ell}(\theta_n(c)) &= \pi_{n/\ell}\left(\sum_{\gamma \in G_n} \gamma c_n \otimes \gamma\right) \\
&= \sum_{\alpha \in G_{n/\ell}} \sum_{\beta \in G_\ell} \alpha\beta c_n \otimes \alpha \\
&= \sum_{\alpha \in G_{n/\ell}} \alpha \cdot \mathrm{N}_{\mathbb{Q}(n)/\mathbb{Q}(n/\ell)}(c_n) \otimes \alpha \\
&= \sum_{\alpha \in G_{n/\ell}} \alpha \cdot P_\ell(\mathrm{Fr}_\ell^{-1}) c_{n/\ell} \otimes \alpha \\
&= \sum_{\alpha \in G_{n/\ell}} \alpha c_{n/\ell} \otimes \alpha \cdot P_\ell(\mathrm{Fr}_\ell) \\
&= \theta_{n/\ell}(c) P_\ell(\mathrm{Fr}_\ell),
\end{aligned}$$

where $\mathrm{N}_{\mathbb{Q}(n)/\mathbb{Q}(n/\ell)}$ is the norm from $\mathbb{Q}(n)$ to $\mathbb{Q}(n/\ell)$ (note that $\mathrm{N}_{\mathbb{Q}(n)/\mathbb{Q}(n/\ell)}$ is equal to $\mathrm{Res}_{\mathbb{Q}(n)/\mathbb{Q}(n/\ell)} \circ \mathrm{Cor}_{\mathbb{Q}(n)/\mathbb{Q}(n/\ell)}$). This proves the lemma. $\square$

The following proposition is an analogue of [Dar95, Theorem 4.5(2)].

**Proposition 5.1.6.** *Let the notations be as in Definition 5.1.4. We have*

$$\theta_n(c) \in H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)},$$

*and if we regard $\theta_n(c) \in H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1}$, then there is a canonical inverse image of $\theta_n(c)$ under the restriction map*

$$H^1(\mathbb{Q}, A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1} \longrightarrow H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1},$$

*namely, there is a canonical element* $x_n \in H^1(\mathbb{Q}, A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1}$ *such that*

$$\mathrm{Res}_{\mathbb{Q}(n)/\mathbb{Q}}(x_n) = \theta_n(c).$$

*Proof.* We prove this proposition by induction on $\nu(n)$. When $\nu(n) = 0$ (i.e. $n = 1$), we have $I_1^0 = \mathbb{Z}$ and $\theta_1(c) = c_1 \in H^1(\mathbb{Q}, A)$, so there is nothing to prove (since $x_1 = c_1$). Suppose $\nu(n) > 0$. We write every $\gamma \in G_n$ uniquely as

$$\gamma = \prod_{\ell|n} \gamma_\ell,$$

where $\gamma_\ell \in G_\ell$. We compute

$$\begin{aligned} \sum_{\gamma \in G_n} \gamma c_n \otimes \prod_{\ell|n}(\gamma_\ell - 1) &= \theta_n(c) + \sum_{d|n, d\neq n} (-1)^{\nu(n/d)} \left( \sum_{\gamma \in G_n} \gamma c_n \otimes \prod_{\ell|d} \gamma_\ell \right) \\ &= \theta_n(c) + \sum_{d|n, d\neq n} (-1)^{\nu(n/d)} \theta_d(c) \prod_{\ell|n/d} P_\ell(\mathrm{Fr}_\ell), \end{aligned}$$

where the first equality follows by direct computation, and the second by Lemma 5.1.5. This shows $\theta_n(c) \in H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)}$, since by the inductive hypothesis we have

$$\theta_d(c) \prod_{\ell|n/d} P_\ell(\mathrm{Fr}_\ell) \in H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)}$$

if $d|n$ and $d \neq n$.

We compute

$$\sum_{\gamma \in G_n} \gamma c_n \otimes \prod_{\ell|n}(\gamma_\ell - 1) = \left( \prod_{\ell|n} D_\ell \right) c_n \otimes \prod_{\ell|n}(\sigma_\ell - 1) \text{ in } H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1},$$

where

$$D_\ell = \sum_{i=1}^{|G_\ell|-1} i\sigma_\ell^i,$$

(recall that $\sigma_\ell$ is the fixed generator of $G_\ell$). It is well-known that $\left(\prod_{\ell|n} D_\ell\right) c_n$ has a canonical inverse image in $H^1(\mathbb{Q}, A)$, which is usually called Kolyvagin's derivative class (see [Rub00, Definition 4.4.10]). We denote it by $\kappa'_n$ (in [Rub00, §4.4], it is denoted by

$\kappa_{[\mathbb{Q},n,M]}$). Hence we have

$$\theta_n(c) = \kappa'_n \otimes \prod_{\ell|n}(\sigma_\ell - 1) - \sum_{d|n, d\neq n} (-1)^{\nu(n/d)}\theta_d(c) \prod_{\ell|n/d} P_\ell(\mathrm{Fr}_\ell). \tag{5.1}$$

By the inductive hypothesis, we see that $\theta_n(c) \in H^1(\mathbb{Q}(n), A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1}$ has a canonical inverse image in $H^1(\mathbb{Q}, A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1}$. □

**Remark 5.1.7.** By the proof of Proposition 5.1.6, we know that the element

$$x_n \in H^1(\mathbb{Q}, A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1}$$

such that $\mathrm{Res}_{\mathbb{Q}(n)/\mathbb{Q}}(x_n) = \theta_n(c)$ is inductively constructed by

$$x_n = \kappa'_n \otimes \prod_{\ell|n}(\sigma_\ell - 1) - \sum_{d|n, d\neq n} (-1)^{\nu(n/d)} x_d \prod_{\ell|n/d} P_\ell(\mathrm{Fr}_\ell).$$

Since $\kappa'_n$ is a canonical element, we can say that $x_n$ is also canonical. So we can naturally regard $\theta_n(c) \in H^1(\mathbb{Q}, A) \otimes I_n^{\nu(n)}/I_n^{\nu(n)+1}$.

We summarize here the standard hypotheses (H.0)-(H.6) of Kolyvagin systems for the triple $(A, \mathcal{F}, \Sigma)$ ([MaRu04, §3.5]):

(H.0) $A$ is a free $\mathbb{Z}/M\mathbb{Z}$-module of finite rank.

(H.1) $A/pA$ is an absolutely irreducible $\mathbb{F}_p[G_\mathbb{Q}]$-representation.

(H.2) There is a $\tau \in G_\mathbb{Q}$ such that $\tau = 1$ on $\mu_{p^\infty}$ and $A/(\tau - 1)A \simeq \mathbb{Z}/M\mathbb{Z}$.

(H.3) $H^1(\mathbb{Q}(A)\mathbb{Q}(\mu_{p^\infty}), A/pA) = H^1(\mathbb{Q}(A)\mathbb{Q}(\mu_{p^\infty}), A^*[p]) = 0$, where $\mathbb{Q}(A)$ is the fixed field in $\overline{\mathbb{Q}}$ of the kernel of the map $G_\mathbb{Q} \to \mathrm{Aut}(A)$, and $A^*[p] = \{a \in A^* \mid pa = 0\}$.

(H.4) Either

(H.4a) $\mathrm{Hom}_{\mathbb{F}_p[[G_\mathbb{Q}]]}(A/pA, A^*[p]) = 0$, or

(H.4b) $p > 4$.

(H.5) $\Sigma_t \subset \Sigma \subset \Sigma_1$ for some $t \in \mathbb{Z}_{>0}$, where for $k \in \mathbb{Z}_{>0}$ $\Sigma_k$ is the set of all the primes $\ell$ satisfying $(*)$ for $M$ replaced by $p^k$.

(H.6) For every $\ell \in \{\ell \mid T$ is ramified at $\ell\} \cup \{p, \infty\}$, the local condition $\mathcal{F}$ at $\ell$ is cartesian (see [MaRu04, Definition 1.1.4]) on the category $\mathrm{Quot}_{\mathbb{Z}/M\mathbb{Z}}(A)$ (see [MaRu04, Example 1.1.3]).

Note that, in our case, (H.0) is always satisfied.

Now, our main theorem is as follows:

**Theorem 5.1.8** ([San14a, Theorem 3.8]). *Suppose that there exists an Euler system $c$ for $(T, \Sigma, \mathcal{K})$. Assume the following:*

(i) *the standard hypotheses* (H.0)-(H.6) *of Kolyvagin systems are satisfied for the triple $(A, \mathcal{F}, \Sigma)$,*

(ii) *$\mathcal{K}$ contains the maximal abelian $p$-extension of $\mathbb{Q}$ which is unramified outside of $p$ and $\Sigma$,*

(iii) *$T/(\mathrm{Fr}_\ell - 1)T$ is a cyclic $\mathbb{Z}_p$-module for every $\ell \in \Sigma$,*

(iv) *$\mathrm{Fr}_\ell^{p^k} - 1$ is injective on $T$ for every $\ell \in \Sigma$ and $k \geq 0$,*

(v) *the core rank $\chi(A, \mathcal{F}) = 1$ ([MaRu04, Definition 4.1.11]),*

*(*(ii)-(iv) *are the assumptions of the first statement of [MaRu04, Theorem 3.2.4], and* (iii) *is satisfied since we assumed $A/(\mathrm{Fr}_\ell - 1)A \simeq \mathbb{Z}/M\mathbb{Z}$). Then we have*

$$\theta_n(c) \in h_n \mathcal{R}_n,$$

*where $h_n = |H^1_{(\mathcal{F}^*)_n}(\mathbb{Q}, A^*)|$.*

From this, we obtain the following corollary, which is a special case of [Rub00, Theorem 2.2.2] and [MaRu04, Corollary 4.4.5] (see also Remark 5.4.7).

**Corollary 5.1.9.** *Under the same assumptions in Theorem 5.1.8, we have $c_\mathbb{Q} = \theta_1(c) \in H^1_\mathcal{F}(\mathbb{Q}, A)$ and*

$$\mathrm{ord}_p(h_1) \leq \mathrm{ind}(c),$$

*where $\mathrm{ord}_p(h_1)$ is defined by $h_1 = p^{\mathrm{ord}_p(h_1)}$, and*

$$\mathrm{ind}(c) = \sup\{m \mid c_\mathbb{Q} \in p^m H^1_\mathcal{F}(\mathbb{Q}, A)\}.$$

*Proof.* Take $n = 1$ in Theorem 5.1.8, then we have

$$c_{\mathbb{Q}} = \theta_1(c) \in h_1 \mathcal{R}_1 = h_1 H^1_{\mathcal{F}}(\mathbb{Q}, A).$$

Hence we have the desired inequality $\mathrm{ord}_p(h_1) \leq \mathrm{ind}(c)$. □

## 5.2 Algebraic Kolyvagin systems

In this section, we introduce a notion of "algebraic Kolyvagin systems". The aim of this section is to prove Theorem 5.2.17. Our Kolyvagin systems are defined for a 7-tuple $(\mathcal{O}, \Sigma, H, t, v, u, P)$ satisfying the following:

- $\mathcal{O}$: a commutative ring (with unity),
- $\Sigma$: a countable set,
- $H$: an $\mathcal{O}$-module,
- $t = \{t_{\mathfrak{q}}\}_{\mathfrak{q}} \in \prod_{\mathfrak{q} \in \Sigma} \mathbb{Z}_{\geq 1}$,
- $v = \{v_{\mathfrak{q}}\}_{\mathfrak{q}} \in \prod_{\mathfrak{q} \in \Sigma} \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O})$,
- $u = \{u_{\mathfrak{q}}\}_{\mathfrak{q}} \in \prod_{\mathfrak{q} \in \Sigma} \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O}/(t_{\mathfrak{q}}))$ ($(t_{\mathfrak{q}})$ denotes the ideal $t_{\mathfrak{q}}\mathcal{O}$),
- $P = \{P_{\mathfrak{q}}\}_{\mathfrak{q}} \in \prod_{\mathfrak{q} \in \Sigma} G(\Sigma \setminus \mathfrak{q})_1$ (we often denote $\Sigma \setminus \{\mathfrak{q}\}$ by $\Sigma \setminus \mathfrak{q}$),

where for any subset $\Sigma' \subset \Sigma$,

$$G(\Sigma')_i = \varprojlim_{\mathfrak{n} \in \mathcal{N}(\Sigma')} \left( I_{\mathfrak{n}}^i / I_{\mathfrak{n}}^{i+1} \otimes_{\mathbb{Z}} \mathcal{O} \right),$$

and where $\mathcal{N}(\Sigma') = \{\mathfrak{n} \subset \Sigma' \mid \nu(\mathfrak{n}) := |\mathfrak{n}| < \infty\}$, and $I_{\mathfrak{n}}$ is the augmentation ideal of $\mathbb{Z}[\bigoplus_{\mathfrak{q} \in \mathfrak{n}} \mathbb{Z}/t_{\mathfrak{q}}\mathbb{Z}]$.

Note that $G(\Sigma')_1$ is canonically isomorphic to $\prod_{\mathfrak{q} \in \Sigma'} \mathcal{O}/(t_{\mathfrak{q}})$, since

$$I_{\mathfrak{n}}/I_{\mathfrak{n}}^2 \otimes_{\mathbb{Z}} \mathcal{O} \simeq \bigoplus_{\mathfrak{q} \in \mathfrak{n}} \mathbb{Z}/t_{\mathfrak{q}}\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O} \simeq \bigoplus_{\mathfrak{q} \in \mathfrak{n}} \mathcal{O}/(t_{\mathfrak{q}})$$

for any $\mathfrak{n} \in \mathcal{N}(\Sigma')$, where the first isomorphism is induced by the inverse of

$$\bigoplus_{\mathfrak{q} \in \mathfrak{n}} \mathbb{Z}/t_{\mathfrak{q}}\mathbb{Z} \xrightarrow{\sim} I_{\mathfrak{n}}/I_{\mathfrak{n}}^2; \quad \sigma \mapsto \sigma - 1.$$

So if $\Sigma'' \subset \Sigma'$, then $G(\Sigma'')_1$ is regarded as an $\mathcal{O}$-submodule of $G(\Sigma')_1$, and also its quotient. We put

$$G(\Sigma') = \varprojlim_{\mathfrak{n} \in \mathcal{N}(\Sigma')} \left( \bigoplus_{i=0}^{\infty} I_{\mathfrak{n}}^i / I_{\mathfrak{n}}^{i+1} \otimes_{\mathbb{Z}} \mathcal{O} \right).$$

Note that if $\Sigma'' \subset \Sigma'$, then there is a natural map from $G(\Sigma'')$ to $G(\Sigma')$ induced by the inclusion $I_{\mathfrak{n}} \hookrightarrow I_{\mathfrak{m}}$, where $\mathfrak{n} \in \mathcal{N}(\Sigma'')$ and $\mathfrak{m} \in \mathcal{N}(\Sigma')$ with $\mathfrak{n} \subset \mathfrak{m}$. So any element of $G(\Sigma'')$ is naturally regarded as an element of $G(\Sigma')$.

From now on we fix a 7-tuple $(\mathcal{O}, \Sigma, H, t, v, u, P)$ satisfying above, and give some more notations for it. We denote simply $\mathcal{N} = \mathcal{N}(\Sigma)$. If $\Sigma' \subset \Sigma$, there is a natural projection map from $G(\Sigma)$ to $G(\Sigma')$, which we denote by $(\cdot)|_{\Sigma'}$. In particular, for $\mathfrak{n} \in \mathcal{N}$, which is by definition a subset of $\Sigma$, we denote the projection map to $G(\mathfrak{n})$ by $\pi_{\mathfrak{n}}$ (namely, $\pi_{\mathfrak{n}} := (\cdot)|_{\mathfrak{n}} : G(\Sigma) \to G(\mathfrak{n})$).

If $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m} \subset \mathfrak{n}$, we denote $\mathfrak{n}/\mathfrak{m}$ instead of the set theoretic notation $\mathfrak{n} \setminus \mathfrak{m}$. If $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{q} \in \Sigma$ such that $\mathfrak{q} \notin \mathfrak{n}$, we denote $\mathfrak{n}\mathfrak{q}$ instead of $\mathfrak{n} \cup \mathfrak{q}$. We also denote 1 instead of $\emptyset \in \mathcal{N}$.

For each $\mathfrak{q} \in \Sigma$, fix a generator $x_{\mathfrak{q}}$ of $G(\mathfrak{q})_1 (\simeq \mathcal{O}/(t_{\mathfrak{q}}))$ (as an $\mathcal{O}$-module).

**Definition 5.2.1.** For any $\mathfrak{q} \in \Sigma$, we define an $\mathcal{O}$-homomorphism

$$\varphi_{\mathfrak{q}} : H \longrightarrow G(\Sigma)_1$$

by $\varphi_{\mathfrak{q}}(a) = -u_{\mathfrak{q}}(a) x_{\mathfrak{q}} - v_{\mathfrak{q}}(a) P_{\mathfrak{q}}$. For $\mathfrak{n} \in \mathcal{N}$, we denote the composition map $\pi_{\mathfrak{n}} \circ \varphi_{\mathfrak{q}}$ by $\varphi_{\mathfrak{q}}^{\mathfrak{n}}$.

Note that if $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{n} = \mathfrak{d} \sqcup \mathfrak{m}$, we have $\varphi_{\mathfrak{q}}^{\mathfrak{n}} = \varphi_{\mathfrak{q}}^{\mathfrak{d}} + \varphi_{\mathfrak{q}}^{\mathfrak{m}}$ for any $\mathfrak{q} \in \Sigma$, since $G(\mathfrak{n})_1 (\simeq \bigoplus_{\mathfrak{q}' \in \mathfrak{n}} \mathcal{O}/(t_{\mathfrak{q}'})) \simeq G(\mathfrak{d})_1 \oplus G(\mathfrak{m})_1$.

**Example 5.2.2.** The setting in §5.1 fits into this general setting. Use the notations as in §5.1, take $(\mathcal{O}, \Sigma, H, t, v, u, P)$ as follows:

- $\mathcal{O} = \mathbb{Z}/M\mathbb{Z}$,
- $\Sigma$: as in §5.1,
- $H = \bigcup_{n \in \mathcal{N}(\Sigma)} H^1_{\mathcal{F}^n}(\mathbb{Q}, A)$,
- $t_\ell$: the maximal $p$-power dividing $\ell - 1$,
- $v_\ell$: as in §5.1,
- $u_\ell$: as in §5.1,

- $P = (P_\ell(\mathrm{Fr}_\ell) \otimes 1)_\ell \in \prod_{\ell \in \Sigma} \varprojlim_{n \in \mathcal{N}(\Sigma), \ell | n} (I_{n/\ell}/I_{n/\ell}^2 \otimes \mathbb{Z}/M\mathbb{Z}) = \prod_{\ell \in \Sigma} G(\Sigma \setminus \ell)_1$.

If we set $x_\ell = (\sigma_\ell - 1) \otimes 1$, then $\varphi_\ell$ in the above definition is the same as in §5.1.

Now, for $r \in \mathbb{Z}_{\geq 1}$, we define algebraic Kolyvagin systems of "rank $r$". Recall that for $\mathfrak{n} \in \mathcal{N}$, we put $\nu(\mathfrak{n}) = |\mathfrak{n}|$ and $G(\mathfrak{n})_{\nu(\mathfrak{n})} = I_{\mathfrak{n}}^{\nu(\mathfrak{n})}/I_{\mathfrak{n}}^{\nu(\mathfrak{n})+1} \otimes_{\mathbb{Z}} \mathcal{O}$. In what follows, for any $\mathcal{O}$-module $G$, we denote $(\bigwedge_{\mathcal{O}}^r H) \otimes_{\mathcal{O}} G$ by $\bigwedge^r H \otimes_{\mathcal{O}} G$ for simplicity.

By the construction in §2.1, for every $\mathfrak{q} \in \Sigma$ and $\mathfrak{n} \in \mathcal{N}$, $v_{\mathfrak{q}} \in \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O})$ induces the map

$$v_{\mathfrak{q}} : \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \longrightarrow \bigwedge^{r-1} H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})}.$$

Similarly, $u_{\mathfrak{q}} \in \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O}/(t_{\mathfrak{q}}))$ induces the map

$$u_{\mathfrak{q}} : \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \longrightarrow \bigwedge^{r-1} H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \otimes_{\mathcal{O}} \mathcal{O}/(t_{\mathfrak{q}}),$$

and $\varphi_{\mathfrak{q}} \in \mathrm{Hom}_{\mathcal{O}}(H, G(\Sigma)_1)$ induces the map

$$\varphi_{\mathfrak{q}} : \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \longrightarrow \bigwedge^{r-1} H \otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})+1}.$$

**Definition 5.2.3.** A collection

$$\left\{ \kappa_{\mathfrak{n}} \in \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \;\middle|\; \mathfrak{n} \in \mathcal{N} \right\}$$

is a Kolyvagin system of rank $r$ if the following axioms (K1)-(K4) are satisfied:

(K1) if $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}$, then $v_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) = 0$,

(K2) if $\mathfrak{q} \in \mathfrak{n}$, then $u_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) = 0$,

(K3) if $\mathfrak{q} \in \mathfrak{n}$, then $v_{\mathfrak{q}}(\kappa_{\mathfrak{n}}) = \varphi_{\mathfrak{q}}(\kappa_{\mathfrak{n}/\mathfrak{q}})$,

(K4) if $\mathfrak{q} \in \mathfrak{n}$, then $\pi_{\mathfrak{n}/\mathfrak{q}}(\kappa_{\mathfrak{n}}) = 0$.

We denote the $\mathcal{O}$-module consisting of all Kolyvagin systems of rank $r$ by $\mathrm{KS}_r$. This is an $\mathcal{O}$-submodule of $\prod_{\mathfrak{n} \in \mathcal{N}} \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})}$.

We will see that our Kolyvagin systems generalize the notion of original Kolyvagin systems in [MaRu04] (see Proposition 5.4.1).

We define the other three algebraic Kolyvagin systems, in Definitions 5.2.5, 5.2.6, and 5.2.7, which we call $\theta$-Kolyvagin systems, pre-Kolyvagin systems, and derived-Kolyvagin systems respectively. The $\mathcal{O}$-module consisting of all $\theta$-Kolyvagin systems (resp. pre-Kolyvagin systems, resp. derived-Kolyvagin systems) of rank $r$ is denoted by $\mathrm{TKS}_r$ (resp. $\mathrm{PKS}_r$, resp. $\mathrm{DKS}_r$).

The following definition is due to [MaRu11, Definition 6.1].

**Definition 5.2.4.** Let $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{d} \subset \mathfrak{n}$. When $\mathfrak{d} \neq 1$, define

$$\mathcal{D}_{\mathfrak{n},\mathfrak{d}} = \det \begin{pmatrix} -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_1}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_1}) & \cdots & & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_1}) \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{q}_3}(P_{\mathfrak{q}_2}) & \cdots & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_2}) \\ \vdots & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_3}) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_\nu}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_\nu}) & \cdots & & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_\nu}) \end{pmatrix} \in G(\mathfrak{n})_{\nu(\mathfrak{d})},$$

where $\{\mathfrak{q}_1, \dots, \mathfrak{q}_\nu\} = \mathfrak{d}$ ($\nu = \nu(\mathfrak{d})$). When $\mathfrak{d} = 1$, define

$$\mathcal{D}_{\mathfrak{n},1} = 1 \in \mathcal{O} = G(\mathfrak{n})_0.$$

Note that $\mathcal{D}_{\mathfrak{n},\mathfrak{d}}$ does not depend on the choice of the order $\mathfrak{q}_1, \dots, \mathfrak{q}_\nu$ of the elements of $\mathfrak{d}$.

We put

$$\mathcal{D}_{\mathfrak{d}} = \pi_{\mathfrak{d}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) = \det \begin{pmatrix} 0 & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_1}) & \cdots & & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_1}) \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_2}) & 0 & -\pi_{\mathfrak{q}_3}(P_{\mathfrak{q}_2}) & \cdots & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_2}) \\ \vdots & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_3}) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_\nu}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_\nu}) & \cdots & & 0 \end{pmatrix} \in G(\mathfrak{d})_{\nu(\mathfrak{d})}.$$

Clearly, $\mathcal{D}_{\mathfrak{d}}$ does not depend on $\mathfrak{n}$.

**Definition 5.2.5.** A collection

$$\left\{ \theta_{\mathfrak{n}} \in \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \;\middle|\; \mathfrak{n} \in \mathcal{N} \right\}$$

is a $\theta$-Kolyvagin system of rank $r$ if the following axioms (TK1)-(TK4) are satisfied:

(TK1) if $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}$, then $v_{\mathfrak{q}}(\theta_{\mathfrak{n}}) = 0$, (TK2) if $\mathfrak{q} \in \mathfrak{n}$, then $u_{\mathfrak{q}}(\sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n}, \mathfrak{n}/\mathfrak{d}}) = 0$,

(TK3) if $\mathfrak{q} \in \mathfrak{n}$, then $v_{\mathfrak{q}}(\sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{d}}(\theta_{\mathfrak{n}})) = \varphi_{\mathfrak{q}}(\sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q})} \pi_{\mathfrak{d}}(\theta_{\mathfrak{n}/\mathfrak{q}}))$,

(TK4) if $\mathfrak{q} \in \mathfrak{n}$, then $\pi_{\mathfrak{n}/\mathfrak{q}}(\theta_{\mathfrak{n}}) = \theta_{\mathfrak{n}/\mathfrak{q}} \cdot \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})$.

**Definition 5.2.6.** A collection

$$\left\{ \widetilde{\kappa}_{\mathfrak{n}} \in \bigwedge^{r} H \otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})} \;\middle|\; \mathfrak{n} \in \mathcal{N} \right\}$$

is a pre-Kolyvagin system of rank $r$ if the following axioms (PK1)-(PK5) are satisfied:

(PK1) if $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}$, then $v_{\mathfrak{q}}(\widetilde{\kappa}_{\mathfrak{n}}) = 0$,

(PK2) if $\mathfrak{q} \in \mathfrak{n}$, then $u_{\mathfrak{q}}(\sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}' \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}'}(P_{\mathfrak{q}'})) = 0$,

(PK3) if $\mathfrak{q} \in \mathfrak{n}$, then $v_{\mathfrak{q}}(\widetilde{\kappa}_{\mathfrak{n}}) = \varphi_{\mathfrak{q}}(\widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}})$,

(PK4) if $\mathfrak{q} \in \mathfrak{n}$, then $\widetilde{\kappa}_{\mathfrak{n}}|_{\Sigma \setminus \mathfrak{q}} = \widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}}|_{\Sigma \setminus \mathfrak{q}} \cdot P_{\mathfrak{q}}$,

(PK5) $\widetilde{\kappa}_{\mathfrak{n}} = \sum_{\mathfrak{d} \subset \mathfrak{n}} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma \setminus \mathfrak{n}}$.

**Definition 5.2.7.** A collection

$$\left\{ \kappa'_{\mathfrak{n}} \in \bigwedge^{r} H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})} \;\middle|\; \mathfrak{n} \in \mathcal{N} \right\}$$

is a derived-Kolyvagin system of rank $r$ if the following axioms (DK1)-(DK4) are satisfied:

(DK1) if $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}$, then $v_{\mathfrak{q}}(\kappa'_{\mathfrak{n}}) = 0$,

(DK2) if $\mathfrak{q} \in \mathfrak{n}$, then $u_{\mathfrak{q}}(\sum_{\mathfrak{d} \subset \mathfrak{n}} \kappa'_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n}/\mathfrak{d}}) = 0$,

(DK3) if $\mathfrak{q} \in \mathfrak{n}$, then $v_{\mathfrak{q}}(\kappa'_{\mathfrak{n}}) = \varphi_{\mathfrak{q}}(\kappa'_{\mathfrak{n}/\mathfrak{q}})$,

(DK4) if $\mathfrak{q} \in \mathfrak{n}$, then $\pi_{\mathfrak{n}/\mathfrak{q}}(\kappa'_{\mathfrak{n}}) = 0$.

**Remark 5.2.8.** The notion of "pre-Kolyvagin systems" first appeared in [MaRu11, Definition 6.2]. Note that the notion which generalizes pre-Kolyvagin systems in [MaRu11] is what we call $\theta$-Kolyvagin systems in this paper. We use the terminology "pre-Kolyvagin system" for a different system.

Next we define morphisms between these Kolyvagin systems. In the following definition, the meaning of the subscript of $F_{PT}$ is "from pre-Kolyvagin systems to $\theta$-Kolyvagin systems", and that of $F_{PK}$, $F_{TK}$, etc. are similar (see Theorem 5.2.17).

**Definition 5.2.9.** We define homomorphisms $F_{PT}$ and $F_{PK}$ from $\prod_{\mathfrak{n}\in\mathcal{N}} \bigwedge^r H \otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})}$ to $\prod_{\mathfrak{n}\in\mathcal{N}} \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})}$ by

$$F_{PT}(\{a_{\mathfrak{n}}\}_{\mathfrak{n}}) = \{\pi_{\mathfrak{n}}(a_{\mathfrak{n}})\}_{\mathfrak{n}},$$

$$F_{PK}(\{a_{\mathfrak{n}}\}_{\mathfrak{n}}) = \left\{\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(a_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right\}_{\mathfrak{n}}.$$

We define endomorphisms $F_{TK}, F_{TD}$, and $F_{DK}$ of $\prod_{\mathfrak{n}\in\mathcal{N}} \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})}$ by

$$F_{TK}(\{a_{\mathfrak{n}}\}_{\mathfrak{n}}) = \left\{\sum_{\mathfrak{d}\subset\mathfrak{n}} a_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}\right\}_{\mathfrak{n}},$$

$$F_{TD}(\{a_{\mathfrak{n}}\}_{\mathfrak{n}}) = \left\{\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}a_{\mathfrak{d}}\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{d}}(P_{\mathfrak{q}})\right\}_{\mathfrak{n}},$$

and

$$F_{DK}(\{a_{\mathfrak{n}}\}_{\mathfrak{n}}) = \left\{\sum_{\mathfrak{d}\subset\mathfrak{n}} a_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n}/\mathfrak{d}}\right\}_{\mathfrak{n}}.$$

**Proposition 5.2.10.** *$F_{TK}, F_{TD}$, and $F_{DK}$ are injective.*

*Proof.* We only show for $F_{TK}$. One can show the injectivity for the others by the same method. Suppose $\{a_{\mathfrak{n}}\}_{\mathfrak{n}} \in \ker F_{TK}$, i.e.

$$\sum_{\mathfrak{d}\subset\mathfrak{n}} a_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = 0$$

for all $\mathfrak{n} \in \mathcal{N}$. We show by induction on $\nu(\mathfrak{n})$ that $a_{\mathfrak{n}} = 0$. When $\nu(\mathfrak{n}) = 0$, i.e. $\mathfrak{n} = 1$, we have $\sum_{\mathfrak{d}\subset\mathfrak{n}} a_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = a_1$ and this is 0 by the assumption. When $\nu(\mathfrak{n}) > 0$, by the inductive hypothesis we have

$$\sum_{\mathfrak{d}\subset\mathfrak{n}} a_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = a_{\mathfrak{n}}\mathcal{D}_{\mathfrak{n},1} = a_{\mathfrak{n}}.$$

Since the left hand side is 0 by the assumption that $F_{TK}(\{a_{\mathfrak{n}}\}_{\mathfrak{n}}) = 0$, we get $a_{\mathfrak{n}} = 0$. $\square$

We define the following useful operator $s_{\mathfrak{m},\mathfrak{n}}$.

**Definition 5.2.11.** For $\mathfrak{n}, \mathfrak{m} \in \mathcal{N}$ such that $\mathfrak{n} \subset \mathfrak{m}$, we define an operator $s_{\mathfrak{m},\mathfrak{n}}$ on $G(\mathfrak{m})$ by

$$s_{\mathfrak{m},\mathfrak{n}}(g) = \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{d})} \pi_{\mathfrak{m}/\mathfrak{d}}(g).$$

This is an $\mathcal{O}$-endomorphism of $G(\mathfrak{m})$. When $\mathfrak{m} = \mathfrak{n}$, put $s_{\mathfrak{n}} = s_{\mathfrak{n},\mathfrak{n}}$.

**Lemma 5.2.12.** *Let $\mathcal{M}$ be an $\mathcal{O}$-module, and $\mathfrak{n}, \mathfrak{m} \in \mathcal{N}$ such that $\mathfrak{n} \subset \mathfrak{m}$. We regard $s_{\mathfrak{m},\mathfrak{n}}$ as an operator on $\mathcal{M} \otimes_{\mathcal{O}} G(\mathfrak{m})$. Then we have the following:*

(i)
$$s_{\mathfrak{m},\mathfrak{n}}(\mathcal{M} \otimes_{\mathcal{O}} G(\mathfrak{m})) \subset \mathcal{M} \otimes_{\mathcal{O}} \left( \prod_{\mathfrak{q} \in \mathfrak{n}} x_{\mathfrak{q}} \right),$$

*where $x_{\mathfrak{q}}$ is the fixed generator of $G(\mathfrak{q})_1$ and $(\prod_{\mathfrak{q} \in \mathfrak{n}} x_{\mathfrak{q}})$ is the (principal) ideal of $G(\mathfrak{m})$ generated by $\prod_{\mathfrak{q} \in \mathfrak{n}} x_{\mathfrak{q}}$.*

*In particular, we have $\pi_{\mathfrak{n}/\mathfrak{q}} \circ s_{\mathfrak{m},\mathfrak{n}} = 0$ for all $\mathfrak{q} \in \mathfrak{n}$.*

(ii) *If $\mathfrak{d}, \mathfrak{n} \in \mathcal{N}$, $\mathfrak{d} \subset \mathfrak{n}$, $g \in \mathcal{M} \otimes_{\mathcal{O}} G(\mathfrak{d})_{\nu(\mathfrak{d})}$, and $h \in G(\mathfrak{n})_{\nu(\mathfrak{n}/\mathfrak{d})}$, then we have*

$$s_{\mathfrak{n}}(gh) = s_{\mathfrak{d}}(g) s_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}(h).$$

*Proof.* (i) Suppose $\mathfrak{n} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\nu\}$ ($\nu = \nu(\mathfrak{n})$). Take any generator of $\mathcal{M} \otimes_{\mathcal{O}} G(\mathfrak{m})$, and write it as follows:

$$\sum_{\alpha} m_\alpha \otimes g_\alpha x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu},$$

where $\alpha$ runs over $\mathbb{Z}_{\geq 0}^{\nu}$, $m_\alpha \in \mathcal{M}$, and $g_\alpha \in G(\mathfrak{m}/\mathfrak{n})$. Put $\mathfrak{d}_\alpha = \{\mathfrak{q}_i \in \mathfrak{n} \mid \alpha_i = 0\}$. We have

$$\begin{aligned} s_{\mathfrak{m},\mathfrak{n}} \left( \sum_{\alpha} m_\alpha \otimes g_\alpha x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu} \right) &= \sum_{\alpha} m_\alpha \otimes \left( \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{d})} g_\alpha \pi_{\mathfrak{m}/\mathfrak{d}}(x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu}) \right) \\ &= \sum_{\alpha} m_\alpha \otimes \left( \sum_{\mathfrak{d} \subset \mathfrak{d}_\alpha} (-1)^{\nu(\mathfrak{d})} g_\alpha x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu} \right) \end{aligned}$$

(note that since $g_\alpha \in G(\mathfrak{m}/\mathfrak{n})$, we have $\pi_{\mathfrak{m}/\mathfrak{d}}(g_\alpha) = g_\alpha$ for any $\mathfrak{d} \subset \mathfrak{n}$). If $\nu(\mathfrak{d}_\alpha) > 0$, then we have

$$\sum_{\mathfrak{d} \subset \mathfrak{d}_\alpha} (-1)^{\nu(\mathfrak{d})} = (1-1)^{\nu(\mathfrak{d}_\alpha)} = 0.$$

Hence we have

$$s_{\mathfrak{m},\mathfrak{n}}\left(\sum_{\alpha} m_\alpha \otimes g_\alpha x_{\mathfrak{q}_1}^{\alpha_1}\cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu}\right) = \sum_{\alpha,\alpha_i\geq 1} m_\alpha \otimes g_\alpha x_{\mathfrak{q}_1}^{\alpha_1}\cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu} \in \mathcal{M}\otimes_{\mathcal{O}}\left(\prod_{\mathfrak{q}\in\mathfrak{n}} x_{\mathfrak{q}}\right).$$

(ii) Suppose $\mathfrak{d} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\mu\}$, and $\mathfrak{n}/\mathfrak{d} = \{\mathfrak{q}'_1, \ldots, \mathfrak{q}'_\nu\}$ ($\mu = \nu(\mathfrak{d})$, $\nu = \nu(\mathfrak{n}/\mathfrak{d})$). Write $g$ and $h$ as follows:

$$g = \sum_{|\alpha|=\mu} m_\alpha \otimes x_{\mathfrak{q}_1}^{\alpha_1}\cdots x_{\mathfrak{q}_\mu}^{\alpha_\mu}$$

and

$$h = \sum_{|\beta+\gamma|=\nu} a_{\beta,\gamma} x_{\mathfrak{q}_1}^{\beta_1}\cdots x_{\mathfrak{q}_\mu}^{\beta_\mu} x_{\mathfrak{q}'_1}^{\gamma_1}\cdots x_{\mathfrak{q}'_\nu}^{\gamma_\nu},$$

where $m_\alpha \in \mathcal{M}$, $a_{\beta,\gamma} \in \mathcal{O}$, $|\alpha|$ means $\alpha_1 + \cdots + \alpha_\mu$, and $|\beta+\gamma|$ is similar. As in the proof of (i), we have

$$s_{\mathfrak{d}}(g) = m_{(1,\ldots,1)} \otimes x_{\mathfrak{q}_1}\cdots x_{\mathfrak{q}_\mu},$$

$$s_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}(h) = a_{(0,\ldots,0),(1,\ldots,1)} x_{\mathfrak{q}'_1}\cdots x_{\mathfrak{q}'_\nu},$$

and

$$s_{\mathfrak{n}}(gh) = a_{(0,\ldots,0),(1,\ldots,1)} m_{(1,\ldots,1)} \otimes x_{\mathfrak{q}_1}\cdots x_{\mathfrak{q}_\mu} x_{\mathfrak{q}'_1}\cdots x_{\mathfrak{q}'_\nu}.$$

Hence we have

$$s_{\mathfrak{n}}(gh) = s_{\mathfrak{d}}(g) s_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}(h).$$

□

**Corollary 5.2.13.** *Let $\mathfrak{n}, \mathfrak{m} \in \mathcal{N}$ with $\mathfrak{n} \subset \mathfrak{m}$ and $g \in \mathcal{M}\otimes_{\mathcal{O}} G(\mathfrak{m})_{\nu(\mathfrak{n})}$. If $\pi_{\mathfrak{m}/\mathfrak{q}}(g) = 0$ for every $\mathfrak{q} \in \mathfrak{n}$, then we have*

$$g \in \mathcal{M}\otimes_{\mathcal{O}}\left\langle \prod_{\mathfrak{q}\in\mathfrak{n}} x_{\mathfrak{q}} \right\rangle_{\mathcal{O}},$$

*where $\langle\prod_{\mathfrak{q}\in\mathfrak{n}} x_{\mathfrak{q}}\rangle_{\mathcal{O}}$ is the $\mathcal{O}$-submodule of $G(\mathfrak{m})$ generated by $\prod_{\mathfrak{q}\in\mathfrak{n}} x_{\mathfrak{q}}$.*

*In particular, we have $g \in \mathcal{M}\otimes_{\mathcal{O}} G(\mathfrak{n})$.*

*Proof.* Suppose $\mathfrak{n} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\nu\}$ ($\nu = \nu(\mathfrak{n})$). Write $g$ as

$$g = \sum_j \sum_\alpha m_j \otimes g_{j,\alpha} x_{\mathfrak{q}_1}^{\alpha_1}\cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu},$$

where $m_j \in \mathcal{M}$ and $g_{j,\alpha} \in G(\mathfrak{m}/\mathfrak{n})$. As in the proof of Lemma 5.2.12, we have

$$s_{\mathfrak{m},\mathfrak{n}}\left(\sum_j \sum_\alpha m_j \otimes g_{j,\alpha} x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu}\right) = \sum_j \sum_{\alpha, \alpha_i \geq 1} m_j \otimes g_{j,\alpha} x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu}.$$

Since $\pi_{\mathfrak{m}/\mathfrak{q}}(g) = 0$ for every $\mathfrak{q} \in \mathfrak{n}$ by the assumption, we have $s_{\mathfrak{m},\mathfrak{n}}(g) = g$ (by the definition of $s_{\mathfrak{m},\mathfrak{n}}$). Hence we have

$$g = \sum_j \sum_{\alpha, \alpha_i \geq 1} m_j \otimes g_{j,\alpha} x_{\mathfrak{q}_1}^{\alpha_1} \cdots x_{\mathfrak{q}_\nu}^{\alpha_\nu}.$$

Since $g \in \mathcal{M} \otimes_{\mathcal{O}} G(\mathfrak{m})_\nu$ ($g$ is "homogeneous of degree $\nu$"), each $\alpha_i$ must be equal to 1, and hence the right hand side must be in $\mathcal{M} \otimes_{\mathcal{O}} \langle \prod_{\mathfrak{q} \in \mathfrak{n}} x_\mathfrak{q} \rangle_{\mathcal{O}}$. $\square$

**Lemma 5.2.14.** *If* $\{\widetilde{\kappa}_\mathfrak{n} \in \bigwedge^r H \otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})} \mid \mathfrak{n} \in \mathcal{N}\}$ *satisfies* (PK4)*, then we have the following: if* $\mathfrak{n} \subset \mathfrak{m}$*, then for every* $\mathfrak{q} \in \mathfrak{n}$*, we have*

$$\pi_{\mathfrak{m}/\mathfrak{q}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_\mathfrak{m}(\widetilde{\kappa}_\mathfrak{d}) \prod_{\mathfrak{q}' \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}'}(P_{\mathfrak{q}'})\right) = 0.$$

*Proof.*

$$\begin{aligned}
& \pi_{\mathfrak{m}/\mathfrak{q}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_\mathfrak{m}(\widetilde{\kappa}_\mathfrak{d}) \prod_{\mathfrak{q}' \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}'}(P_{\mathfrak{q}'})\right) \\
= \; & \pi_{\mathfrak{m}/\mathfrak{q}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_\mathfrak{m}(\widetilde{\kappa}_\mathfrak{d}) \prod_{\mathfrak{q}' \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}'}(P_{\mathfrak{q}'})\right. \\
& \left. + \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q})} \pi_\mathfrak{m}(\widetilde{\kappa}_{\mathfrak{d}\mathfrak{q}}) \prod_{\mathfrak{q}'' \in \mathfrak{n}/\mathfrak{d}\mathfrak{q}} \pi_{\mathfrak{m}/\mathfrak{q}''}(P_{\mathfrak{q}''})\right) \\
= \; & \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{m}/\mathfrak{q}}\left(\widetilde{\kappa}_\mathfrak{d} \prod_{\mathfrak{q}' \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}'}(P_{\mathfrak{q}'})\right) \\
& + \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q})} \pi_{\mathfrak{m}/\mathfrak{q}}\left(\widetilde{\kappa}_{\mathfrak{d}\mathfrak{q}} \prod_{\mathfrak{q}'' \in \mathfrak{n}/\mathfrak{d}\mathfrak{q}} \pi_{\mathfrak{m}/\mathfrak{q}''}(P_{\mathfrak{q}''})\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{m}/\mathfrak{q}}\left(\widetilde{\kappa}_{\mathfrak{d}} \prod_{\mathfrak{q}'\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}'}(P_{\mathfrak{q}'})\right) \\
&\quad + \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q})}\pi_{\mathfrak{m}/\mathfrak{q}}\left(\widetilde{\kappa}_{\mathfrak{d}} \prod_{\mathfrak{q}''\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}''}(P_{\mathfrak{q}''})\right) \\
&= 0,
\end{aligned}
$$

where the third equality follows from (PK4). □

**Proposition 5.2.15.** (i) (PK5) *is equivalent to the following:*

$$
(\mathrm{PK5})' \textit{ if } \mathfrak{n}\subset\mathfrak{m}, \textit{ then } \widetilde{\kappa}_{\mathfrak{n}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} \pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{m}}.
$$

(ii) *If* $\{\widetilde{\kappa}_{\mathfrak{n}} \in \bigwedge^r H\otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})} \mid \mathfrak{n}\in\mathcal{N}\}$ *satisfies* (PK4)*, then we have the following: if* $\mathfrak{n}\subset\mathfrak{m}$*, then we have an equality in* $\bigwedge^r H\otimes_{\mathcal{O}} G(\mathfrak{m})_{\nu(\mathfrak{n})}$*:*

$$
\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}}(P_{\mathfrak{q}}) = \sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}).
$$

*Proof.* (i) One sees immediately that (PK5)′ implies (PK5) (take $\mathfrak{m}=\mathfrak{n}$ in (PK5)′, this is (PK5)). Suppose (PK5) and we show (PK5)′ by induction on $\nu(\mathfrak{n})$. When $\nu(\mathfrak{n})=0$, i.e. $\mathfrak{n}=1$, we have

$$
\widetilde{\kappa}_1 = \pi_{\mathfrak{m}}(\widetilde{\kappa}_1)
$$

for any $\mathfrak{m}$ since $\widetilde{\kappa}_1 \in \bigwedge^r H\otimes_{\mathcal{O}} G(\Sigma)_0 = \bigwedge^r H$, and we have

$$
\sum_{\mathfrak{d}\subset 1} \pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in 1/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{m}} = \pi_{\mathfrak{m}}(\widetilde{\kappa}_1)
$$

so (PK5)′ is satisfied in this case. When $\nu(\mathfrak{n})>0$, we prove (PK5)′ by induction on $\nu(\mathfrak{m}/\mathfrak{n})$. When $\nu(\mathfrak{m}/\mathfrak{n})=0$, i.e. $\mathfrak{m}=\mathfrak{n}$, there is nothing to prove because it is (PK5). When $\nu(\mathfrak{m}/\mathfrak{n})>0$, take any $\mathfrak{q}\in\mathfrak{m}/\mathfrak{n}$. We have for any $\mathfrak{d}\subset\mathfrak{n}$

$$
\pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) = \sum_{\mathfrak{c}\subset\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}}(\widetilde{\kappa}_{\mathfrak{c}}) \prod_{\mathfrak{q}'\in\mathfrak{d}/\mathfrak{c}} \pi_{\mathfrak{q}}(P_{\mathfrak{q}'}). \tag{5.2}
$$

To see this, if $\mathfrak{d}\neq\mathfrak{n}$, we get this equality by the inductive hypothesis on $\nu(\mathfrak{n})$ (replace $\mathfrak{n}$, $\mathfrak{m}$ in (PK5)′ by $\mathfrak{d}$, $\mathfrak{m}/\mathfrak{q}$ respectively, then apply $\pi_{\mathfrak{m}}$). If $\mathfrak{d}=\mathfrak{n}$, we get the equality by the inductive hypothesis on $\nu(\mathfrak{m}/\mathfrak{n})$ (replace $\mathfrak{m}$ in (PK5)′ by $\mathfrak{m}/\mathfrak{q}$, then apply $\pi_{\mathfrak{m}}$).

Hence we have

$$\begin{aligned}
\sum_{\mathfrak{d}\subset\mathfrak{n}} \pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}'\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}'}|_{\Sigma\setminus\mathfrak{m}} &= \sum_{\mathfrak{d}\subset\mathfrak{n}} \sum_{\mathfrak{c}\subset\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}}(\widetilde{\kappa}_{\mathfrak{c}}) \prod_{\mathfrak{q}''\in\mathfrak{d}/\mathfrak{c}} \pi_{\mathfrak{q}}(P_{\mathfrak{q}''}) \prod_{\mathfrak{q}'\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}'}|_{\Sigma\setminus\mathfrak{m}} \\
&= \sum_{\mathfrak{d}\subset\mathfrak{n}} \pi_{\mathfrak{m}/\mathfrak{q}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}'\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}'}|_{\Sigma\setminus(\mathfrak{m}/\mathfrak{q})} \\
&= \widetilde{\kappa}_{\mathfrak{n}},
\end{aligned}$$

where the first equality is obtained by (5.2), and the second by the direct computation (note that $P_{\mathfrak{q}'}|_{\Sigma\setminus(\mathfrak{m}/\mathfrak{q})} = P_{\mathfrak{q}'}|_{\Sigma\setminus\mathfrak{m}} + \pi_{\mathfrak{q}}(P_{\mathfrak{q}'})$), and the last is by the inductive hypothesis on $\nu(\mathfrak{m}/\mathfrak{n})$ (replace $\mathfrak{m}$ in (PK5)$'$ by $\mathfrak{m}/\mathfrak{q}$). This completes the proof of (i).

(ii) From Lemma 5.2.14 and Corollary 5.2.13, we have

$$\sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}}(P_{\mathfrak{q}}) \in \bigwedge^{r} H \otimes_{\mathcal{O}} G(\mathfrak{n}),$$

so the left hand side does not change when we apply $\pi_{\mathfrak{n}}$. Hence we have

$$\begin{aligned}
\sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}}(P_{\mathfrak{q}}) &= \pi_{\mathfrak{n}} \left( \sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{m}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{m}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \\
&= \sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}).
\end{aligned}$$

□

**Proposition 5.2.16.** *Suppose $\mathfrak{d}, \mathfrak{n} \in \mathcal{N}$ and $\mathfrak{d} \subset \mathfrak{n}$.*

(i) *If $\mathfrak{q} \in \mathfrak{d}$, then $\pi_{\mathfrak{n}/\mathfrak{q}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) = -\mathcal{D}_{\mathfrak{n}/\mathfrak{q},\mathfrak{d}/\mathfrak{q}} \cdot \pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}})$.*

(ii) *If $\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}$, then $\pi_{\mathfrak{n}/\mathfrak{q}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) = \mathcal{D}_{\mathfrak{n}/\mathfrak{q},\mathfrak{d}}$.*

(iii) *$s_{\mathfrak{n},\mathfrak{d}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) = \mathcal{D}_{\mathfrak{d}}$.*

*Proof.* (i) Suppose $\mathfrak{d} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_\nu\}$ and $\mathfrak{q} = \mathfrak{q}_\nu$. By the definition of $\mathcal{D}_{\mathfrak{n},\mathfrak{d}}$ (see Definition

5.2.4), we have

$$\begin{aligned}
\pi_{\mathfrak{n}/\mathfrak{q}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) &= \pi_{\mathfrak{n}/\mathfrak{q}}\left(\det\begin{pmatrix} -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_1}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_1}) & \cdots & & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_1}) \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{q}_3}(P_{\mathfrak{q}_2}) & \cdots & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_2}) \\ \vdots & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_3}) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_\nu}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_\nu}) & \cdots & & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_\nu}) \end{pmatrix}\right) \\
&= \det\begin{pmatrix} -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_1}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_1}) & \cdots & & 0 \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{q}_3}(P_{\mathfrak{q}_2}) & \cdots & 0 \\ \vdots & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_3}) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_\nu}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_\nu}) & \cdots & & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_\nu}) \end{pmatrix} \\
&= -\mathcal{D}_{\mathfrak{n}/\mathfrak{q},\mathfrak{d}/\mathfrak{q}} \cdot \pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}}).
\end{aligned}$$

(ii) Suppose $\mathfrak{d} = \{\mathfrak{q}_1, \dots, \mathfrak{q}_\nu\}$. By the definition of $\mathcal{D}_{\mathfrak{n},\mathfrak{d}}$, we have

$$\begin{aligned}
\pi_{\mathfrak{n}/\mathfrak{q}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) &= \pi_{\mathfrak{n}/\mathfrak{q}}\left(\det\begin{pmatrix} -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_1}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_1}) & \cdots & & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_1}) \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{q}_3}(P_{\mathfrak{q}_2}) & \cdots & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_2}) \\ \vdots & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_3}) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_\nu}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_\nu}) & \cdots & & -\pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}_\nu}) \end{pmatrix}\right) \\
&= \det\begin{pmatrix} -\pi_{\mathfrak{n}/\mathfrak{d}\mathfrak{q}}(P_{\mathfrak{q}_1}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_1}) & \cdots & & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_1}) \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{n}/\mathfrak{d}\mathfrak{q}}(P_{\mathfrak{q}_2}) & -\pi_{\mathfrak{q}_3}(P_{\mathfrak{q}_2}) & \cdots & -\pi_{\mathfrak{q}_\nu}(P_{\mathfrak{q}_2}) \\ \vdots & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_3}) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -\pi_{\mathfrak{q}_1}(P_{\mathfrak{q}_\nu}) & -\pi_{\mathfrak{q}_2}(P_{\mathfrak{q}_\nu}) & \cdots & & -\pi_{\mathfrak{n}/\mathfrak{d}\mathfrak{q}}(P_{\mathfrak{q}_\nu}) \end{pmatrix} \\
&= \mathcal{D}_{\mathfrak{n}/\mathfrak{q},\mathfrak{d}}.
\end{aligned}$$

(iii) As in the proof of Lemma 5.2.12(i), $s_{\mathfrak{n},\mathfrak{d}}$ eliminates all the terms other than "$\prod_{\mathfrak{q}\in\mathfrak{d}} x_{\mathfrak{q}}$-terms". When we expand the determinant $\mathcal{D}_{\mathfrak{n},\mathfrak{d}}$, the sum of its "$\prod_{\mathfrak{q}\in\mathfrak{d}} x_{\mathfrak{q}}$-terms" is equal to $\mathcal{D}_{\mathfrak{d}}$. Hence we have $s_{\mathfrak{n},\mathfrak{d}}(\mathcal{D}_{\mathfrak{n},\mathfrak{d}}) = \mathcal{D}_{\mathfrak{d}}$. □

**Theorem 5.2.17** ([San14a, Theorem 4.17]). *The following diagram is commutative and* *all the morphisms are isomorphisms:*

$$\begin{tikzcd} \mathrm{PKS}_r \arrow[r, "F_{PT}"] \arrow[d, "F_{PK}"'] & \mathrm{TKS}_r \arrow[d, "F_{TD}"] \arrow[dl, "F_{TK}"'] \\ \mathrm{KS}_r & \mathrm{DKS}_r. \arrow[l, "F_{DK}"] \end{tikzcd}$$

**Remark 5.2.18.** It is shown in [MaRu11, Proposition 6.5] that $F_{TK}$ induces isomorphism $\mathrm{TKS}_r \simeq \mathrm{KS}_r$ in a special case. Theorem 5.2.17 is its generalization.

*Proof.* The strategy of the proof is as follows. The proof is divided into 5 steps.

In Steps 1, 2, and 3, we show that $F_{PK}$, $F_{TD}$, and $F_{TK}$ are isomorphisms respectively.

In Steps 4 and 5, we show that $F_{DK} \circ F_{TD} = F_{TK}$ and $F_{TK} \circ F_{PT} = F_{PK}$ respectively.

By Steps 1, 3, 5, and Proposition 5.2.10, we see that $F_{PT}$ is an isomorphism. By Steps 2, 3 and 4, we see that $F_{DK}$ is an isomorphism. Hence by all the steps, we complete the proof.

**Step 1.** We show that $F_{PK}$ is an isomorphism. Step 1 is divided into 3 steps.

In Step 1.1, we show $F_{PK}(\mathrm{PKS}_r) \subset \mathrm{KS}_r$.

In Step 1.2, we construct the inverse $G_{PK}$ of $F_{PK}$ and show $G_{PK}(\mathrm{KS}_r) \subset \mathrm{PKS}_r$.

In Step 1.3, we show $G_{PK} \circ F_{PK} = F_{PK} \circ G_{PK} = \mathrm{id}$, and this completes Step 1.

**Step 1.1.**

Suppose $\widetilde{\kappa} = \{\widetilde{\kappa}_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{PKS}_r$. Put

$$\kappa_{\mathfrak{n}} = F_{PK}(\widetilde{\kappa})_{\mathfrak{n}} = \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}).$$

We show that $\kappa = \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n}} = F_{PK}(\widetilde{\kappa}) \in \mathrm{KS}_r$. We see that $\kappa$ satisfies the axioms (K1)-(K4).

(K1) Suppose $\mathfrak{q}' \in \Sigma \setminus \mathfrak{n}$. We have

$$v_{\mathfrak{q}'}(\kappa_{\mathfrak{n}}) = \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}})) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) = 0,$$

since $v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) = 0$ for every $\mathfrak{d} \subset \mathfrak{n}$, by (PK1). This shows (K1).

From now on we suppose $\mathfrak{q}' \in \mathfrak{n}$.

(K2) By (PK2), we have

$$u_{\mathfrak{q}'}(\kappa_{\mathfrak{n}}) = u_{\mathfrak{q}'}\left( \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) = 0.$$

This shows (K2).

(K3) We have

$$\begin{aligned}
v_{\mathfrak{q}'}(\kappa_{\mathfrak{n}}) &= \sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}))\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \\
&= \sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{q}'\in\mathfrak{d}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}))\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \\
&= \varphi^{\mathfrak{n}}_{\mathfrak{q}'}\left(\sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'}(-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right) \\
&= \varphi^{\mathfrak{n}}_{\mathfrak{q}'}\left(\sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'}(-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')}\pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'}\pi_{\mathfrak{n}/\mathfrak{q}\mathfrak{q}'}(P_{\mathfrak{q}})\right) \\
&= \varphi^{\mathfrak{n}}_{\mathfrak{q}'}(\kappa_{\mathfrak{n}/\mathfrak{q}'}) \\
&= \varphi_{\mathfrak{q}'}(\kappa_{\mathfrak{n}/\mathfrak{q}'}),
\end{aligned}$$

where the second equality follows from (PK1), that is, $v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) = 0$ unless $\mathfrak{q}' \in \mathfrak{d}$, and the third from (PK3), that is, $v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) = \varphi_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}/\mathfrak{q}'})$, the fourth from proposition 5.2.15(ii), the fifth by definition, and the last from (K1).

(K4) By Lemma 5.2.14, we have

$$\pi_{\mathfrak{n}/\mathfrak{q}'}(\kappa_{\mathfrak{n}}) = \pi_{\mathfrak{n}/\mathfrak{q}'}\left(\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right) = 0.$$

Hence we have $\kappa \in \mathrm{KS}_r$.

**Step 1.2.**

We construct the inverse $G_{PK}$ of $F_{PK}$. Suppose $\kappa = \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{KS}_r$ is given. Put

$$\widetilde{\kappa}_1 = \kappa_1,$$

and define $\widetilde{\kappa}_{\mathfrak{n}} \in \bigwedge^r H \otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})}$ inductively by

$$\widetilde{\kappa}_{\mathfrak{n}} = \kappa_{\mathfrak{n}} + \sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n}}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\left\{\left(\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}}\right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\left(\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right)\right\}. \tag{5.3}$$

We define $G_{PK}(\kappa) = \{\widetilde{\kappa}_{\mathfrak{n}}\}_{\mathfrak{n}}$. We show first that $\widetilde{\kappa} = \{\widetilde{\kappa}_{\mathfrak{n}}\}_{\mathfrak{n}} = G_{PK}(\kappa) \in \mathrm{PKS}_r$ (in Step 1.3 we show that $G_{PK} \circ F_{PK} = F_{PK} \circ G_{PK} = \mathrm{id}$).

(PK1) We show by induction on $\nu(\mathfrak{n})$ that $v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}})=0$ for $\mathfrak{q}'\in\Sigma\setminus\mathfrak{n}$. When $\nu(\mathfrak{n})=0$ i.e. $\mathfrak{n}=1$, this is clear by (K1) since $\widetilde{\kappa}_1=\kappa_1$. When $\nu(\mathfrak{n})>0$, we have for $\mathfrak{q}'\in\Sigma\setminus\mathfrak{n}$

$$\begin{aligned}&v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}})\\ =\;& v_{\mathfrak{q}'}(\kappa_{\mathfrak{n}})+\sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n}}\pi_{\mathfrak{n}}(v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}))\left\{\left(\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}}\right)-(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\left(\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right)\right\}\\ =\;&0,\end{aligned}$$

by (K1) and the inductive hypothesis. This shows (PK1).
(PK2) Applying $\pi_{\mathfrak{n}}$ to the both sides of (5.3), we obtain

$$\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{n}})=\kappa_{\mathfrak{n}}-\sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}). \tag{5.4}$$

Hence by (K2) we have

$$u_{\mathfrak{q}'}\left(\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right)=u_{\mathfrak{q}'}(\kappa_{\mathfrak{n}})=0$$

for any $\mathfrak{q}'\in\mathfrak{n}$. This shows (PK2).

Next we show (PK5), (PK4), and finally (PK3).
(PK5) By (5.4), we have

$$\kappa_{\mathfrak{n}}=\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}). \tag{5.5}$$

Substituting this to (5.3), we obtain

$$\begin{aligned}\widetilde{\kappa}_{\mathfrak{n}}\;=\;&\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\\ &+\sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n}}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\left\{\left(\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}}\right)-(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\left(\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\right)\right\}\\ =\;&\sum_{\mathfrak{d}\subset\mathfrak{n}}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}}.\end{aligned}$$

This is (PK5).
(PK4) We show by induction on $\nu(\mathfrak{n})$ that $\widetilde{\kappa}_{\mathfrak{n}}|_{\Sigma\setminus\mathfrak{q}'}=\widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}'}|_{\Sigma\setminus\mathfrak{q}'}\cdot P'_{\mathfrak{q}}$ for any $\mathfrak{q}'\in\mathfrak{n}$. When

$\nu(\mathfrak{n}) = 1$, say $\mathfrak{n} = \mathfrak{q}'$, we have by (5.3)

$$\widetilde{\kappa}_{\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{q}'} = \kappa_{\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{q}'} + \pi_{\mathfrak{q}'}(\widetilde{\kappa}_1) P_{\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{q}'} = \widetilde{\kappa}_1 \cdot P_{\mathfrak{q}'},$$

so (PK4) holds in this case. When $\nu(\mathfrak{n}) > 1$, take $\mathfrak{q}' \in \mathfrak{n}$. By (5.3) and the fact that $\kappa_{\mathfrak{n}}|_{\Sigma\backslash\mathfrak{q}'} = 0$ (this follows from (K4)), we have

$$\begin{aligned}
\widetilde{\kappa}_{\mathfrak{n}}|_{\Sigma\backslash\mathfrak{q}'} &= \sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n}} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}/\mathfrak{q}'} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\} \\
&= \sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n},\mathfrak{q}'\in\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}\mathfrak{q}'}(P_{\mathfrak{q}}) \right) \right\} \\
&\quad + \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}/\mathfrak{q}'} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\} \\
&= \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}',\mathfrak{d}\neq\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \pi_{\mathfrak{n}/\mathfrak{q}'}(P_{\mathfrak{q}'}) \\
&\quad \times \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}\mathfrak{q}'}(P_{\mathfrak{q}}) \right) \right\} \\
&\quad + \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}/\mathfrak{q}'} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\} \\
&= \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \pi_{\mathfrak{n}/\mathfrak{q}'}(P_{\mathfrak{q}'}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} + \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\backslash\mathfrak{n}} \\
&= \widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{q}'} (\pi_{\mathfrak{n}/\mathfrak{q}'}(P_{\mathfrak{q}'}) + P_{\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{n}}) \\
&= \widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{q}'} P_{\mathfrak{q}'},
\end{aligned}$$

where the third equality follows by the inductive hypothesis, and the fifth by (PK5).

(PK3) We show by induction on $\nu(\mathfrak{n})$ that $v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}}) = \varphi_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}'})$ for any $\mathfrak{q}' \in \mathfrak{n}$. When $\nu(\mathfrak{n}) = 1$, say $\mathfrak{n} = \mathfrak{q}'$, we have

$$\begin{aligned}
v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{q}'}) &= v_{\mathfrak{q}'}(\kappa_{\mathfrak{q}'}) + \pi_{\mathfrak{q}'}(v_{\mathfrak{q}'}(\widetilde{\kappa}_1)) P_{\mathfrak{q}'}|_{\Sigma\backslash\mathfrak{q}'} \\
&= v_{\mathfrak{q}'}(\kappa_{\mathfrak{q}'}) \\
&= \varphi_{\mathfrak{q}'}(\kappa_1) \\
&= \varphi_{\mathfrak{q}'}(\widetilde{\kappa}_1),
\end{aligned}$$

where the first equality follows by (5.3), the second by (PK1), the third by (K3), and the last by the definition of $\widetilde{\kappa}_1$. When $\nu(\mathfrak{n}) > 1$, take $\mathfrak{q}' \in \mathfrak{n}$. Then we have

$$\begin{aligned}
v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}}) &= v_{\mathfrak{q}'}(\kappa_{\mathfrak{n}}) + \sum_{\mathfrak{d}\subset\mathfrak{n},\mathfrak{d}\neq\mathfrak{n}} \pi_{\mathfrak{n}}(v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}})) \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\} \\
&= \varphi^{\mathfrak{n}}_{\mathfrak{q}'}(\kappa_{\mathfrak{n}/\mathfrak{q}'}) \\
&\quad + \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}',\mathfrak{d}\neq\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \left\{ \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')} \left( \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\},
\end{aligned}$$

where the first equality follows by (5.3), the second by (K3) and by the inductive hypothesis (note that $v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{d}}) = 0$ unless $\mathfrak{q}' \in \mathfrak{d}$, by (PK1)). By (5.5) and Proposition 5.2.15(ii) (note that we have already proved (PK4)), we have

$$\kappa_{\mathfrak{n}/\mathfrak{q}'} = \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}).$$

Substituting this to the above, we have

$$\begin{aligned}
v_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}}) &= \varphi^{\mathfrak{n}}_{\mathfrak{q}'} \left( \sum_{\mathfrak{d}\subset\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}\mathfrak{q}'} P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}} \right) \\
&= \varphi^{\mathfrak{n}}_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}'}) \\
&= \varphi_{\mathfrak{q}'}(\widetilde{\kappa}_{\mathfrak{n}/\mathfrak{q}'}),
\end{aligned}$$

where the second equality follows by (PK5) and Proposition 5.2.15(i), and the last by (PK1).

Hence $\kappa$ satisfies the axioms (PK1)-(PK5), and we have completed Step 1.2.

**Step 1.3.**

In this step, we show $G_{PK} \circ F_{PK} = F_{PK} \circ G_{PK} = \mathrm{id}$.

We first show $G_{PK} \circ F_{PK} = \mathrm{id}$. Take any $\widetilde{\kappa} = \{\widetilde{\kappa}_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{PKS}_r$. We show by induction on $\nu(\mathfrak{n})$ that $(G_{PK} \circ F_{PK})(\widetilde{\kappa})_{\mathfrak{n}} = \widetilde{\kappa}_{\mathfrak{n}}$. When $\nu(\mathfrak{n}) = 0$, i.e. $\mathfrak{n} = 1$, by the definitions of $F_{PK}$ and $G_{PK}$, we have

$$(G_{PK} \circ F_{PK})(\widetilde{\kappa})_1 = F_{PK}(\widetilde{\kappa})_1 = \widetilde{\kappa}_1.$$

When $\nu(\mathfrak{n}) > 0$, we have

$$
\begin{aligned}
& (G_{PK} \circ F_{PK})(\widetilde{\kappa})_{\mathfrak{n}} \\
= \; & F_{PK}(\widetilde{\kappa})_{\mathfrak{n}} + \sum_{\mathfrak{d} \subset \mathfrak{n}, \mathfrak{d} \neq \mathfrak{n}} \pi_{\mathfrak{n}}((G_{PK} \circ F_{PK})(\widetilde{\kappa})_{\mathfrak{d}}) \\
& \times \left\{ \left( \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma \setminus \mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \left( \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\} \\
= \; & \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \\
& + \sum_{\mathfrak{d} \subset \mathfrak{n}, \mathfrak{d} \neq \mathfrak{n}} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \left\{ \left( \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma \setminus \mathfrak{n}} \right) - (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \left( \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \right) \right\} \\
= \; & \sum_{\mathfrak{d} \subset \mathfrak{n}} \pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma \setminus \mathfrak{n}} \\
= \; & \widetilde{\kappa}_{\mathfrak{n}},
\end{aligned}
$$

where the first equality follows by the definition of $G_{PK}$ (see (5.3)), the second by the definition of $F_{PK}$ (see Definition 5.2.9) and the inductive hypothesis, and the last by (PK5).

Next we show $F_{PK} \circ G_{PK} = \mathrm{id}$. Take any $\kappa = \{\kappa_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{KS}_r$. By (5.5), we have

$$\kappa_{\mathfrak{n}} = \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(G_{PK}(\kappa)_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}),$$

but the right hand side is by definition equal to $F_{PK}(G_{PK}(\kappa))_{\mathfrak{n}}$. We have completed Step 1.3.

**Step 2.**

We show that $F_{TD}$ induces an isomorphism $\mathrm{TKS}_r \simeq \mathrm{DKS}_r$. Step 2 is divided into 3 steps, as in Step 1.

In Step 2.1, we show $F_{TD}(\mathrm{TKS}_r) \subset \mathrm{DKS}_r$.

In Step 2.2, we construct the inverse $G_{TD}$ of $F_{TD}$, and show $G_{TD}(\mathrm{DKS}_r) \subset \mathrm{TKS}_r$.

In Step 2.3, we show $G_{TD} \circ F_{TD} = F_{TD} \circ G_{TD} = \mathrm{id}$.

**Step 2.1.**

Take $\theta = \{\theta_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{TKS}_r$. We show that $F_{TD}(\theta) \in \mathrm{DKS}_r$. Put

$$\kappa'_{\mathfrak{n}} = F_{TD}(\theta)_{\mathfrak{n}} = \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \theta_{\mathfrak{d}} \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}).$$

Note that by (TK4) we have
$$\theta_{\mathfrak{d}} \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) = \pi_{\mathfrak{d}}(\theta_{\mathfrak{n}}),$$
so we have
$$\kappa'_{\mathfrak{n}} = s_{\mathfrak{n}}(\theta_{\mathfrak{n}}) \tag{5.6}$$
(see Definition 5.2.11 for the definition of $s_{\mathfrak{n}}$). We see that $\{\kappa'_{\mathfrak{n}}\}_{\mathfrak{n}}$ satisfies the axioms (DK1)-(DK4).

(DK1) For any $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}$, we have by (TK1)
$$v_{\mathfrak{q}}(\kappa'_{\mathfrak{n}}) = \sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{d}}(v_{\mathfrak{q}}(\theta_{\mathfrak{n}})) = 0.$$
This is (DK1).

(DK2) It is sufficient to show that
$$\sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = \sum_{\mathfrak{d} \subset \mathfrak{n}} \kappa'_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n}/\mathfrak{d}}. \tag{5.7}$$
(From this, (DK2) follows from (TK2)). Take $\mathfrak{q} \in \mathfrak{n}$. We have
$$\begin{aligned} \pi_{\mathfrak{n}/\mathfrak{q}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}\right) &= \pi_{\mathfrak{n}/\mathfrak{q}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} \theta_{\mathfrak{d}\mathfrak{q}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}\mathfrak{q}} + \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}\right) \\ &= \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} \theta_{\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) \mathcal{D}_{\mathfrak{n}/\mathfrak{q},\mathfrak{n}/\mathfrak{d}\mathfrak{q}} - \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n}/\mathfrak{q},\mathfrak{n}/\mathfrak{d}\mathfrak{q}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) \\ &= 0, \end{aligned}$$
where the second equality follows by Proposition 5.2.16(i), (ii) and (TK4). So we have by the definition of $s_{\mathfrak{n}}$
$$s_{\mathfrak{n}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}\right) = \sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}.$$
On the other hand, by Lemma 5.2.12(ii), Proposition 5.2.16(iii), and (5.6), we have
$$s_{\mathfrak{n}}\left(\sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}\right) = \sum_{\mathfrak{d} \subset \mathfrak{n}} \kappa'_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n}/\mathfrak{d}}.$$
Hence we have $\sum_{\mathfrak{d} \subset \mathfrak{n}} \theta_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = \sum_{\mathfrak{d} \subset \mathfrak{n}} \kappa'_{\mathfrak{d}} \mathcal{D}_{\mathfrak{n}/\mathfrak{d}}$.

(DK3) Since $\kappa'_{\mathfrak{n}} = s_{\mathfrak{n}}(\theta_{\mathfrak{n}})$, (DK3) follows from (TK3).

(DK4) Again since $\kappa'_{\mathfrak{n}} = s_{\mathfrak{n}}(\theta_{\mathfrak{n}})$, (DK4) follows from Lemma 5.2.12(i).

Hence we have completed Step 2.1.

**Step 2.2.**

We construct the inverse $G_{TD}$ of $F_{TD}$. Suppose $\kappa' = \{\kappa'_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{DKS}_r$. Put

$$\theta_1 = \kappa'_1$$

and we define $\theta_{\mathfrak{n}}$ inductively by

$$\theta_{\mathfrak{n}} = \kappa'_{\mathfrak{n}} - \sum_{\mathfrak{d} \subset \mathfrak{n}, \mathfrak{d} \neq \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \theta_{\mathfrak{d}} \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}). \tag{5.8}$$

Define $G_{TD}(\kappa') = \{\theta_{\mathfrak{n}}\}_{\mathfrak{n}}$, and we show that $G_{TD}(\kappa') \in \mathrm{TKS}_r$.

(TK1) follows from (DK1) by induction on $\nu(\mathfrak{n})$.

(TK4) We show by induction on $\nu(\mathfrak{n})$. When $\nu(\mathfrak{n}) = 1$, say $\mathfrak{n} = \mathfrak{q}'$, we have

$$\pi_1(\theta_{\mathfrak{q}'}) = 0 = \theta_1 \pi_1(P_{\mathfrak{q}'})$$

(note that $\pi_1(G(\mathfrak{q}')_1) = 0$). When $\nu(\mathfrak{n}) > 1$, for any $\mathfrak{q}' \in \mathfrak{n}$ we have by (5.8)

$$\begin{aligned}
\pi_{\mathfrak{n}/\mathfrak{q}'}(\theta_{\mathfrak{n}}) &= \pi_{\mathfrak{n}/\mathfrak{q}'}(\kappa'_{\mathfrak{n}}) - \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}'} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \theta_{\mathfrak{d}} \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) \\
&\quad - \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}', \mathfrak{d} \neq \mathfrak{n}/\mathfrak{q}'} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')} \pi_{\mathfrak{d}}(\theta_{\mathfrak{d}\mathfrak{q}'}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}\mathfrak{q}'} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) \\
&= - \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}'} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \theta_{\mathfrak{d}} \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) - \sum_{\mathfrak{d} \subset \mathfrak{n}/\mathfrak{q}', \mathfrak{d} \neq \mathfrak{n}/\mathfrak{q}'} (-1)^{\nu(\mathfrak{n}/\mathfrak{d}\mathfrak{q}')} \theta_{\mathfrak{d}} \pi_{\mathfrak{d}}(P_{\mathfrak{q}'}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}\mathfrak{q}'} \pi_{\mathfrak{d}}(P_{\mathfrak{q}}) \\
&= \theta_{\mathfrak{n}/\mathfrak{q}'} \pi_{\mathfrak{n}/\mathfrak{q}'}(P_{\mathfrak{q}'}),
\end{aligned}$$

where the second equality follows from (DK4) and the inductive hypothesis. This shows (TK4).

(TK2) By (5.8) and (TK4), we have

$$\theta_{\mathfrak{n}} = \kappa'_{\mathfrak{n}} - \sum_{\mathfrak{d} \subset \mathfrak{n}, \mathfrak{d} \neq \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{d}}(\theta_{\mathfrak{n}}).$$

Hence,

$$\kappa'_{\mathfrak{n}} = \mathfrak{s}_n(\theta_{\mathfrak{n}}). \tag{5.9}$$

Using (5.9) and (TK4), we repeat the argument in the proof of (DK2) in Step 2.1 to show

$\sum_{\mathfrak{d}\subset\mathfrak{n}} \theta_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} \kappa'_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n}/\mathfrak{d}}$. Hence (TK2) follows from (DK2).
(TK3) By (5.9), (TK3) follows from (DK3).

We have completed Step 2.2.

**Step 2.3.**

To show that $F_{TD}$ induces isomorphism from $\mathrm{TKS}_r$ to $\mathrm{DKS}_r$, since we already know by Proposition 5.2.10 that $F_{TD}$ is injective, it suffices to show $F_{TD} \circ G_{TD} = \mathrm{id}$. Suppose $\kappa' = \{\kappa'_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{DKS}_r$. By (5.9) we have

$$\kappa'_{\mathfrak{n}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{d}}(G_{TD}(\kappa')_{\mathfrak{n}}) = s_{\mathfrak{n}}(G_{TD}(\kappa')_{\mathfrak{n}}).$$

By (5.6) we have

$$F_{TD}(G_{TD}(\kappa'))_{\mathfrak{n}} = s_{\mathfrak{n}}(G_{TD}(\kappa')_{\mathfrak{n}}),$$

which completes Step 2.3.

**Step 3.**

Since the bijectivity of $F_{TK}$ is shown similarly as in Step 2 (or in the proof of [MaRu11, Proposition 6.5]), we omit the proof.

**Step 4.**

We show $F_{DK} \circ F_{TD} = F_{TK}$. Take $\theta = \{\theta_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{TKS}_r$. We have to show

$$\sum_{\mathfrak{d}\subset\mathfrak{n}} F_{TD}(\theta)_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n}/\mathfrak{d}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} \theta_{\mathfrak{d}}\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}.$$

But this is (5.7), which has been already shown. Hence $F_{DK} \circ F_{TD} = F_{TK}$.

**Step 5.**

Our final task is to prove $F_{TK} \circ F_{PT} = F_{PK}$. Take $\widetilde{\kappa} = \{\widetilde{\kappa}_{\mathfrak{n}}\}_{\mathfrak{n}} \in \mathrm{PKS}_r$. We have to prove

$$\sum_{\mathfrak{d}\subset\mathfrak{n}} \pi_{\mathfrak{d}}(\widetilde{\kappa}_{\mathfrak{d}})\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}). \tag{5.10}$$

By (PK5), we have for $\mathfrak{d} \subset \mathfrak{n}$

$$\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}}) = \sum_{\mathfrak{c}\subset\mathfrak{d}} \pi_{\mathfrak{d}}(\widetilde{\kappa}_{\mathfrak{c}}) \prod_{\mathfrak{q}\in\mathfrak{d}/\mathfrak{c}} \pi_{\mathfrak{n}/\mathfrak{d}}(P_{\mathfrak{q}}).$$

Using this relation repeatedly, we arrange the right hand side of (5.10), and sum up the "coefficients" of each $\pi_{\mathfrak{d}}(\widetilde{\kappa}_{\mathfrak{d}})$ to obtain

$$\begin{aligned}
&\sum_{\mathfrak{d}\subset\mathfrak{n}}(-1)^{\nu(\mathfrak{n}/\mathfrak{d})}\pi_{\mathfrak{n}}(\widetilde{\kappa}_{\mathfrak{d}})\prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}) \\
= &\sum_{\mathfrak{d}\subset\mathfrak{n}}\left(\sum_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)\in\Delta(\mathfrak{n}/\mathfrak{d})}(-1)^{\nu(\mathfrak{c}_k)}\prod_{\mathfrak{q}\in\mathfrak{c}_k}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\prod_{\mathfrak{q}\in\mathfrak{c}_{k-1}}\pi_{\mathfrak{c}_k}(P_{\mathfrak{q}})\prod_{\mathfrak{q}\in\mathfrak{c}_{k-2}}\pi_{\mathfrak{c}_{k-1}}(P_{\mathfrak{q}})\cdots\prod_{\mathfrak{q}\in\mathfrak{c}_1}\pi_{\mathfrak{c}_2}(P_{\mathfrak{q}})\right) \\
&\times\pi_{\mathfrak{d}}(\widetilde{\kappa}_{\mathfrak{d}}),
\end{aligned}$$

where

$$\Delta(\mathfrak{n}/\mathfrak{d})=\{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)\mid\emptyset\neq\mathfrak{c}_i\subset\mathfrak{n}/\mathfrak{d},\ \mathfrak{n}/\mathfrak{d}=\coprod_{i=1}^{k}\mathfrak{c}_i,\ k\in\mathbb{Z}_{\geq1}\}.$$

Hence it is sufficient to show

$$\mathcal{D}_{\mathfrak{n},\mathfrak{n}/\mathfrak{d}}=\sum_{(\mathfrak{c}_1,\ldots,\mathfrak{c}_k)\in\Delta(\mathfrak{n}/\mathfrak{d})}(-1)^{\nu(\mathfrak{c}_k)}\prod_{\mathfrak{q}\in\mathfrak{c}_k}\pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})\prod_{\mathfrak{q}\in\mathfrak{c}_{k-1}}\pi_{\mathfrak{c}_k}(P_{\mathfrak{q}})\prod_{\mathfrak{q}\in\mathfrak{c}_{k-2}}\pi_{\mathfrak{c}_{k-1}}(P_{\mathfrak{q}})\cdots\prod_{\mathfrak{q}\in\mathfrak{c}_1}\pi_{\mathfrak{c}_2}(P_{\mathfrak{q}}).$$

This is reduced to the following

**Lemma 5.2.19.** *Suppose $A=(a_{ij})$ is a $\nu\times\nu$-matrix with entries in a commutative ring. Then we have*

$$\begin{aligned}
&(-1)^{\nu}\det A \\
= &\sum_{(C_1,\ldots,C_k)\in\Delta(\nu)}(-1)^{|C_k|}\prod_{i\in C_k}\left(\sum_{j=1}^{\nu}a_{ij}\right)\prod_{i\in C_{k-1}}\left(\sum_{j\in C_k}a_{ij}\right) \\
&\times\prod_{i\in C_{k-2}}\left(\sum_{j\in C_{k-1}}a_{ij}\right)\cdots\prod_{i\in C_1}\left(\sum_{j\in C_2}a_{ij}\right),
\end{aligned}$$

*where $\Delta(\nu)=\{(C_1,\ldots,C_k)\mid\emptyset\neq C_i\subset\{1,\ldots,\nu\},\ \{1,\ldots,\nu\}=\coprod_{i=1}^{k}C_i,\ k\in\mathbb{Z}_{\geq1}\}$.*

*Proof.* Fix a map $\tau:\{1,\ldots,\nu\}\to\{1,\ldots,\nu\}$. We see that the coefficient of $\prod_{i=1}^{\nu}a_{i,\tau(i)}$ of the left (resp. right) hand side of the equality in the lemma is

$$\sum_{\sigma\in\mathfrak{S}_\nu}\mathrm{sgn}(\sigma)\prod_{i=1}^{\nu}(-\delta_{\tau(i),\sigma(i)})$$

$$\left(\text{resp.}\quad\sum_{(C_1,\ldots,C_k)\in\Delta(\tau)}(-1)^{|C_k|}\right),$$

where $\delta_{\tau(i),\sigma(i)}$ denotes Kronecker's delta, and

$$\Delta(\tau) = \{(C_1, \ldots, C_k) \in \Delta(\nu) \mid \tau(i) \in C_{j+1} \text{ for all } 1 \leq j \leq k-1 \text{ and } i \in C_j\}.$$

So it is sufficient to show that

$$\sum_{\sigma \in \mathfrak{S}_\nu} \operatorname{sgn}(\sigma) \prod_{i=1}^{\nu} (-\delta_{\tau(i),\sigma(i)}) = \sum_{(C_1,\ldots,C_k) \in \Delta(\tau)} (-1)^{|C_k|}.$$

For every map $\mu : \{1, \ldots, \nu\} \to \{1, \ldots, \nu\}$ we set

$$\operatorname{Fix}(\mu) = \{i \in \{1, \ldots, \nu\} \mid \mu(i) = i\}.$$

We compute

$$\begin{aligned}
& \sum_{\sigma \in \mathfrak{S}_\nu} \operatorname{sgn}(\sigma) \prod_{i=1}^{\nu} (-\delta_{\tau(i),\sigma(i)}) \\
= & \sum_{\sigma \in \mathfrak{S}_\nu} \operatorname{sgn}(\sigma) \prod_{i \in \operatorname{Fix}(\tau)} (-\delta_{i,\sigma(i)}) \prod_{i \notin \operatorname{Fix}(\tau)} (-\delta_{\tau(i),\sigma(i)}) \\
= & \sum_{\sigma \in \mathfrak{S}_\nu, \operatorname{Fix}(\tau) \subset \operatorname{Fix}(\sigma)} \operatorname{sgn}(\sigma) (-1)^{|\operatorname{Fix}(\tau)|} \prod_{i \notin \operatorname{Fix}(\tau)} (-\delta_{\tau(i),\sigma(i)}) \\
= & \sum_{\sigma \in \mathfrak{S}_\nu, \operatorname{Fix}(\tau) \subset \operatorname{Fix}(\sigma)} \operatorname{sgn}(\sigma) (-1)^{|\operatorname{Fix}(\tau)|} (1-1)^{|\operatorname{Fix}(\sigma) \setminus \operatorname{Fix}(\tau)|} \prod_{i \notin \operatorname{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}) \\
= & \sum_{\sigma \in \mathfrak{S}_\nu, \operatorname{Fix}(\tau) \subset \operatorname{Fix}(\sigma)} \operatorname{sgn}(\sigma) \sum_{D \subset \operatorname{Fix}(\sigma) \setminus \operatorname{Fix}(\tau)} (-1)^{|D| + |\operatorname{Fix}(\tau)|} \prod_{i \notin \operatorname{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}) \\
= & \sum_{\operatorname{Fix}(\tau) \subset C \subset \{1,\ldots,\nu\}} (-1)^{|C|} \sum_{\sigma \in \mathfrak{S}_\nu, C \subset \operatorname{Fix}(\sigma)} \operatorname{sgn}(\sigma) \prod_{i \notin \operatorname{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}).
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{(C_1,\ldots,C_k) \in \Delta(\tau)} (-1)^{|C_k|} \\
= & \sum_{C \subset \{1,\ldots,\nu\}} (-1)^{|C|} |\{(C_1, \ldots, C_k) \in \Delta(\tau) \mid C_k = C\}| \\
= & \sum_{\operatorname{Fix}(\tau) \subset C \subset \{1,\ldots,\nu\}} (-1)^{|C|} |\{(C_1, \ldots, C_k) \in \Delta(\tau) \mid C_k = C\}|.
\end{aligned}$$

Hence, it is sufficient to show for each set $C$ with $\mathrm{Fix}(\tau) \subset C \subset \{1, \ldots, \nu\}$ that

$$\sum_{\sigma \in \mathfrak{S}_\nu, C \subset \mathrm{Fix}(\sigma)} \mathrm{sgn}(\sigma) \prod_{i \notin \mathrm{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}) = |\{(C_1, \ldots, C_k) \in \Delta(\tau) \mid C_k = C\}|.$$

Note that the right hand side is equal to 1 or 0. Suppose first that the right hand side is equal to 1. Then we see that $\prod_{i \notin \mathrm{Fix}(\sigma)}(-\delta_{\tau(i),\sigma(i)}) = 0$ unless $\sigma = \mathrm{id}$. Indeed, suppose $\sigma \neq \mathrm{id}$ and let $(C_1, \ldots, C_k)$ be the unique element of $\{(C_1, \ldots, C_k) \in \Delta(\tau) \mid C_k = C\}$. Note that in this case we must have $k \geq 2$, since $C \subset \mathrm{Fix}(\sigma)$. We see that there exists an integer $j$ with $1 \leq j \leq k-1$ such that $C_j \not\subset \mathrm{Fix}(\sigma)$ and $C_{j+1} \subset \mathrm{Fix}(\sigma)$. This shows that there exists $i \in C_j$ such that $i \notin \mathrm{Fix}(\sigma)$ and $\tau(i) \neq \sigma(i)$ (since $\sigma$ is injective). Hence we have shown that $\prod_{i \notin \mathrm{Fix}(\sigma)}(-\delta_{\tau(i),\sigma(i)}) = 0$ unless $\sigma = \mathrm{id}$. Therefore we have

$$\begin{aligned}
\sum_{\sigma \in \mathfrak{S}_\nu, C \subset \mathrm{Fix}(\sigma)} \mathrm{sgn}(\sigma) \prod_{i \notin \mathrm{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}) &= \mathrm{sgn}(\mathrm{id}) \\
&= 1 \\
&= |\{(C_1, \ldots, C_k) \in \Delta(\tau) \mid C_k = C\}|.
\end{aligned}$$

Next, suppose that $|\{(C_1, \ldots, C_k) \in \Delta(\tau) \mid C_k = C\}| = 0$. In this case we must have $C \neq \{1, \ldots, \nu\}$, and we see that there exist $j \in \{1, \ldots, \nu\} \setminus C$ and a positive integer $m$ such that $\tau^{m+1}(j) = j$, that $j, \tau(j), \ldots, \tau^m(j)$ are different each other and not contained in $C$. We set $\mu = (j \ \tau(j) \ \cdots \ \tau^m(j)) \in \mathfrak{S}_\nu$. If we put

$$\mathfrak{S}_\nu(\tau, C) = \{\sigma \in \mathfrak{S}_\nu \mid C \subset \mathrm{Fix}(\sigma),\ \tau(i) = \sigma(i) \text{ for all } i \notin \mathrm{Fix}(\sigma)\},$$

then we have

$$\sum_{\sigma \in \mathfrak{S}_\nu, C \subset \mathrm{Fix}(\sigma)} \mathrm{sgn}(\sigma) \prod_{i \notin \mathrm{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}) = \sum_{\sigma \in \mathfrak{S}_\nu(\tau, C)} \mathrm{sgn}(\sigma)(-1)^{\nu - |\mathrm{Fix}(\sigma)|}.$$

It is easy to see that

$$\{\sigma \in \mathfrak{S}_\nu(\tau, C) \mid \sigma(j) \neq j\} = \mu\{\sigma \in \mathfrak{S}_\nu(\tau, C) \mid \sigma(j) = j\},$$

and therefore we have

$$\mathfrak{S}_\nu(\tau, C) = \mu\{\sigma \in \mathfrak{S}_\nu(\tau, C) \mid \sigma(j) = j\} \sqcup \{\sigma \in \mathfrak{S}_\nu(\tau, C) \mid \sigma(j) = j\}.$$

So we have

$$\begin{aligned}
& \sum_{\sigma\in\mathfrak{S}_\nu(\tau,C)} \mathrm{sgn}(\sigma)(-1)^{\nu-|\mathrm{Fix}(\sigma)|} \\
= & \sum_{\sigma\in\mathfrak{S}_\nu(\tau,C),\sigma(j)=j} \mathrm{sgn}(\mu\sigma)(-1)^{\nu-|\mathrm{Fix}(\mu\sigma)|} + \sum_{\sigma\in\mathfrak{S}_\nu(\tau,C),\sigma(j)=j} \mathrm{sgn}(\sigma)(-1)^{\nu-|\mathrm{Fix}(\sigma)|} \\
= & (\mathrm{sgn}(\mu)(-1)^{m+1}+1) \sum_{\sigma\in\mathfrak{S}_\nu(\tau,C),\sigma(j)=j} \mathrm{sgn}(\sigma)(-1)^{\nu-|\mathrm{Fix}(\sigma)|} \\
= & ((-1)^m(-1)^{m+1}+1) \sum_{\sigma\in\mathfrak{S}_\nu(\tau,C),\sigma(j)=j} \mathrm{sgn}(\sigma)(-1)^{\nu-|\mathrm{Fix}(\sigma)|} \\
= & 0.
\end{aligned}$$

Hence we have

$$\sum_{\sigma\in\mathfrak{S}_\nu, C\subset\mathrm{Fix}(\sigma)} \mathrm{sgn}(\sigma) \prod_{i\notin\mathrm{Fix}(\sigma)} (-\delta_{\tau(i),\sigma(i)}) = 0 = |\{(C_1,\ldots,C_k)\in\Delta(\tau) \mid C_k = C\}|.$$

This completes the proof. □

Hence we have completed all the steps of the proof of Theorem 5.2.17.

□

## 5.3 Regulator Kolyvagin systems

In this section, we construct Kolyvagin systems by "regulators". We construct an $\mathcal{O}$-module $\mathrm{US}_r$, which we call "unit systems" (see Definition 5.3.3 below), and maps from unit systems to Kolyvagin systems (see Theorem 5.3.7). The idea of our method in this section is due to [MaRu04, Appendix B]. We keep the notations in §5.2.

**Definition 5.3.1.** For $\mathfrak{n}\in\mathcal{N}$, we define "$\mathfrak{n}$-modified Selmer group" by

$$S^{\mathfrak{n}} = \{a\in H \mid v_{\mathfrak{q}}(a) = 0 \text{ for every } \mathfrak{q}\in\Sigma\setminus\mathfrak{n}\}.$$

**Remark 5.3.2.** In the setting of Example 5.2.2, we have

$$S^n = H^1_{\mathcal{F}^n}(\mathbb{Q}, A).$$

**Definition 5.3.3.** Define a partially ordered set

$$\begin{aligned} & \mathscr{I} \\ = \ & \{(s,\mathcal{U}) \mid s = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots) : \text{a sequence of all the elements in } \Sigma,\ \mathcal{U} \subset \mathcal{N} \text{ satisfying } (*)\}, \end{aligned}$$

where

$$(*)\ \mathcal{U} = \{\mathfrak{n}_1, \mathfrak{n}_2, \ldots\}, \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} \mathfrak{n}_i = \Sigma, \text{ and } \mathfrak{n}_i = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_{\nu(\mathfrak{n}_i)}\} \text{ for any } i \geq 1,$$

and we define the order on $\mathscr{I}$ by

$$(s,\mathcal{U}) \leq (s',\mathcal{U}') \text{ if and only if } s = s' \text{ and } \mathcal{U}' \subset \mathcal{U}.$$

We define the module $\mathrm{US}_r$ of unit systems of rank $r$ by

$$\mathrm{US}_r = \varinjlim_{(s,\mathcal{U}) \in \mathscr{I}} \varprojlim_{\mathfrak{n} \in \mathcal{U}} \bigwedge^{\nu(\mathfrak{n})+r} S^{\mathfrak{n}},$$

where the morphisms of the inverse limit are defined by

$$(-v_{\mathfrak{q}_{\nu(\mathfrak{n}_{i+1})}}) \wedge \cdots \wedge (-v_{\mathfrak{q}_{\nu(\mathfrak{n}_i)+1}}) : \bigwedge^{\nu(\mathfrak{n}_{i+1})+r} H \longrightarrow \bigwedge^{\nu(\mathfrak{n}_i)+r} H,$$

and that of the direct limit by the natural projection maps.

**Remark 5.3.4.** The assumption that $\Sigma$ is countable is used here.

**Definition 5.3.5.** Suppose $(s,\mathcal{U}) \in \mathscr{I}$, say $s = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots)$, $\mathcal{U}$ is as $(*)$ above, and $\varepsilon = \{\varepsilon_{\mathfrak{n}}\}_{\mathfrak{n}} \in \varprojlim_{\mathfrak{n} \in \mathcal{U}} \bigwedge^{\nu(\mathfrak{n})+r} S^{\mathfrak{n}}$. For $\mathfrak{n} \in \mathcal{N}$, take $\mathfrak{n}_i \in \mathcal{U}$ so that $\mathfrak{n} \subset \mathfrak{n}_i$ (this is possible since $\mathcal{U}$ consists of an increasing sequence of elements in $\mathcal{N}$ which covers $\Sigma$). Define regulators $R_P(\varepsilon)_{\mathfrak{n}}$, $R_T(\varepsilon)_{\mathfrak{n}}$, and $R_K(\varepsilon)_{\mathfrak{n}}$ by

$$R_P(\varepsilon)_{\mathfrak{n}} = (\psi^{(\mathfrak{n})}_{P,\nu(\mathfrak{n}_i)} \wedge \cdots \wedge \psi^{(\mathfrak{n})}_{P,1})(\varepsilon_{\mathfrak{n}_i}),$$

$$R_T(\varepsilon)_{\mathfrak{n}} = (\psi^{(\mathfrak{n})}_{T,\nu(\mathfrak{n}_i)} \wedge \cdots \wedge \psi^{(\mathfrak{n})}_{T,1})(\varepsilon_{\mathfrak{n}_i}),$$

and

$$R_K(\varepsilon)_{\mathfrak{n}} = (\psi^{(\mathfrak{n})}_{K,\nu(\mathfrak{n}_i)} \wedge \cdots \wedge \psi^{(\mathfrak{n})}_{K,1})(\varepsilon_{\mathfrak{n}_i}),$$

where

$$\psi^{(\mathfrak{n})}_{P,j} \text{ (resp. } \psi^{(\mathfrak{n})}_{T,j}, \text{ resp. } \psi^{(\mathfrak{n})}_{K,j}) = \begin{cases} \varphi_{\mathfrak{q}_j} \text{ (resp. } \varphi^{\mathfrak{n}}_{\mathfrak{q}_j}, \text{ resp. } \varphi^{\mathfrak{q}_j}_{\mathfrak{q}_j}) & \text{if } \mathfrak{q}_j \in \mathfrak{n}, \\ -v_{\mathfrak{q}_j} & \text{if } \mathfrak{q}_j \in \mathfrak{n}_i/\mathfrak{n} \end{cases}$$

(for the definition of $\varphi_{\mathfrak{q}}$, see Definition 5.2.1). One sees by definition that $R_P(\varepsilon)_{\mathfrak{n}}$, $R_T(\varepsilon)_{\mathfrak{n}}$, and $R_K(\varepsilon)_{\mathfrak{n}}$ do not depend on the choice of $\mathfrak{n}_i$. Indeed, if we take another $\mathfrak{n}_{i'} \in \mathcal{U}$, say $\mathfrak{n} \subset \mathfrak{n}_i \subset \mathfrak{n}_{i'}$, then we have

$$\begin{aligned} (\psi^{(\mathfrak{n})}_{\nu(\mathfrak{n}_{i'})} \wedge \cdots \wedge \psi^{(\mathfrak{n})}_1)(\varepsilon_{\mathfrak{n}_{i'}}) &= (\psi^{(\mathfrak{n})}_{\nu(\mathfrak{n}_i)} \wedge \cdots \wedge \psi^{(\mathfrak{n})}_1)((-v_{\mathfrak{q}_{\nu(\mathfrak{n}_{i'})}}) \wedge \cdots \wedge (-v_{\mathfrak{q}_{\nu(\mathfrak{n}_i)+1}})(\varepsilon_{\mathfrak{n}_{i'}})) \\ &= (\psi^{(\mathfrak{n})}_{\nu(\mathfrak{n}_i)} \wedge \cdots \wedge \psi^{(\mathfrak{n})}_1)(\varepsilon_{\mathfrak{n}_i}), \end{aligned}$$

where $\psi^{(\mathfrak{n})}_j$ denotes any of $\psi^{(\mathfrak{n})}_{P,j}$, $\psi^{(\mathfrak{n})}_{T,j}$, and $\psi^{(\mathfrak{n})}_{K,j}$. $R_P$ (resp. $R_T$ and $R_K$) define(s) a homomorphism from $\mathrm{US}_r$ to $\prod_{\mathfrak{n}\in\mathcal{N}} \bigwedge^r H \otimes_{\mathcal{O}} G(\Sigma)_{\nu(\mathfrak{n})}$ (resp. $\prod_{\mathfrak{n}\in\mathcal{N}} \bigwedge^r H \otimes_{\mathcal{O}} G(\mathfrak{n})_{\nu(\mathfrak{n})}$).

**Remark 5.3.6.** The idea of defining the unit systems and the regulators above is due to [MaRu04, Appendix B].

**Theorem 5.3.7** ([San14a, Theorem 5.7])**.** *We have the following commutative diagram:*

$$\begin{tikzcd} \mathrm{US}_r \arrow[dr, "R_P"'] \arrow[drr, bend left, "R_T"] \arrow[ddr, bend right, "R_K"'] & & \\ & \mathrm{PKS}_r \arrow[r, "F_{PT}"'] \arrow[d, "F_{PK}"] & \mathrm{TKS}_r \arrow[dl, "F_{TK}"] \\ & \mathrm{KS}_r & \end{tikzcd}$$

*Proof.* We first show the commutativity of the diagram, and then prove the image of the map $R_P$ is in $\mathrm{PKS}_r$. This completes the proof of the theorem, since by Theorem 5.2.17 we know that $F_{PT}(\mathrm{PKS}_r) = \mathrm{TKS}_r$ and $F_{PK}(\mathrm{PKS}_r) = \mathrm{KS}_r$.

Take $\varepsilon = \{\varepsilon_{\mathfrak{n}}\}_{\mathfrak{n}} \in \varprojlim_{\mathfrak{n}\in\mathcal{U}} \bigwedge^{\nu(\mathfrak{n})+r} S^{\mathfrak{n}}$. To prove the commutativity of the diagram, we have to show $R_T(\varepsilon)_{\mathfrak{n}} = F_{PT}(R_P(\varepsilon))_{\mathfrak{n}}$ and $R_K(\varepsilon)_{\mathfrak{n}} = F_{PK}(R_P(\varepsilon))_{\mathfrak{n}}$ for any $\mathfrak{n} \in \mathcal{N}$ (note that $F_{TK} \circ F_{PT} = F_{PK}$ was already proved in Theorem 5.2.17). Note that by definition $F_{PT}(R_P(\varepsilon))_{\mathfrak{n}} = \pi_{\mathfrak{n}}(R_P(\varepsilon)_{\mathfrak{n}})$ (see Definition 5.2.9), and that $\varphi^{\mathfrak{n}}_{\mathfrak{q}} = \pi_{\mathfrak{n}} \circ \varphi_{\mathfrak{q}}$, so we have $R_T(\varepsilon)_{\mathfrak{n}} = F_{PT}(R_P(\varepsilon))_{\mathfrak{n}}$ by the definitions of $R_T$ and $R_P$. Next, to see $R_K(\varepsilon)_{\mathfrak{n}} = F_{PK}(R_P(\varepsilon))_{\mathfrak{n}}$, note that by definition

$$F_{PK}(R_P(\varepsilon))_{\mathfrak{n}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(R_P(\varepsilon)_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}),$$

and that

$$\varphi_{\mathfrak{q}}^{\mathfrak{q}} = \varphi_{\mathfrak{q}}^{\mathfrak{n}} - \varphi_{\mathfrak{q}}^{\mathfrak{n}/\mathfrak{q}} = \pi_{\mathfrak{n}} \circ \varphi_{\mathfrak{q}} - (-v_{\mathfrak{q}} \cdot \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}))$$

holds for $\mathfrak{q} \in \mathfrak{n}$. Then we see again by definition $R_K(\varepsilon)_{\mathfrak{n}} = F_{PK}(R_P(\varepsilon))_{\mathfrak{n}}$ holds (substitute $\varphi_{\mathfrak{q}}^{\mathfrak{q}} = \pi_{\mathfrak{n}} \circ \varphi_{\mathfrak{q}} - (-v_{\mathfrak{q}} \cdot \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}}))$ to the definition of $R_K$, and expand it, then we obtain $\sum_{\mathfrak{d} \subset \mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \pi_{\mathfrak{n}}(R_P(\varepsilon)_{\mathfrak{d}}) \prod_{\mathfrak{q} \in \mathfrak{n}/\mathfrak{d}} \pi_{\mathfrak{n}/\mathfrak{q}}(P_{\mathfrak{q}})$).

We prove $R_P(\varepsilon) \in \mathrm{PKS}_r$. Take $\mathfrak{n}_i \in \mathcal{U}$ so that $\mathfrak{n} \subset \mathfrak{n}_i$. We show that $R_P(\varepsilon)_{\mathfrak{n}}$ satisfies axioms (PK1)-(PK5).

(PK1) If $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}_i$, we have

$$v_{\mathfrak{q}}(R_P(\varepsilon)_{\mathfrak{n}}) = (\psi_{P,\nu(\mathfrak{n}_i)}^{(\mathfrak{n})} \wedge \cdots \wedge \psi_{P,1}^{(\mathfrak{n})} \wedge v_{\mathfrak{q}})(\varepsilon_{\mathfrak{n}_i}) = 0,$$

since any element $a \in S^{\mathfrak{n}_i}$ satisfies $v_{\mathfrak{q}}(a) = 0$ by definition (see Definition 5.3.1). If $\mathfrak{q} \in \mathfrak{n}_i \setminus \mathfrak{n}$, say $\mathfrak{q} = \mathfrak{q}_j$, $1 \leq j \leq \nu(\mathfrak{n}_i)$ (recall $\mathfrak{n}_i = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_{\nu(\mathfrak{n}_i)}\}$, see $(*)$ in Definition 5.3.3), we have

$$\begin{aligned} v_{\mathfrak{q}_j}(R_P(\varepsilon)_{\mathfrak{n}}) &= (\psi_{P,\nu(\mathfrak{n}_i)}^{(\mathfrak{n})} \wedge \cdots \wedge \psi_{P,1}^{(\mathfrak{n})} \wedge v_{\mathfrak{q}_j})(\varepsilon_{\mathfrak{n}_i}) \\ &= (\cdots \wedge (-v_{\mathfrak{q}_j}) \wedge \cdots \wedge v_{\mathfrak{q}_j})(\varepsilon_{\mathfrak{n}_i}) \\ &= 0, \end{aligned}$$

since $(\cdots \wedge (-v_{\mathfrak{q}_j}) \wedge \cdots \wedge v_{\mathfrak{q}_j}) = 0$. Hence we have $v_{\mathfrak{q}}(R_P(\varepsilon)_{\mathfrak{n}}) = 0$ for any $\mathfrak{q} \in \Sigma \setminus \mathfrak{n}$.

(PK2) Take any $\mathfrak{q} \in \mathfrak{n}$. We prove $u_{\mathfrak{q}}(R_K(\varepsilon)_{\mathfrak{n}}) = 0$ (note that we have already proved $F_{PK}(R_P(\varepsilon))_{\mathfrak{n}} = R_K(\varepsilon)_{\mathfrak{n}}$, so (PK2) is equivalent to $u_{\mathfrak{q}}(R_K(\varepsilon)_{\mathfrak{n}}) = 0$). We have

$$\begin{aligned} u_{\mathfrak{q}}(R_K(\varepsilon)_{\mathfrak{n}}) &= (\cdots \wedge \varphi_{\mathfrak{q}}^{\mathfrak{q}} \wedge \cdots \wedge u_{\mathfrak{q}})(\varepsilon_{\mathfrak{n}_i}) \\ &= (\cdots \wedge (-u_{\mathfrak{q}} \cdot x_{\mathfrak{q}}) \wedge \cdots \wedge u_{\mathfrak{q}})(\varepsilon_{\mathfrak{n}_i}) \\ &= 0, \end{aligned}$$

where the second equality holds since $\varphi_{\mathfrak{q}}^{\mathfrak{q}} = -u_{\mathfrak{q}} \cdot x_{\mathfrak{q}}$ by definition (see Definition 5.2.1).

(PK3) For any $\mathfrak{q} \in \mathfrak{n}$, we have

$$\begin{aligned} v_{\mathfrak{q}}(R_P(\varepsilon)_{\mathfrak{n}}) &= (\cdots \wedge \varphi_{\mathfrak{q}} \wedge \cdots \wedge v_{\mathfrak{q}})(\varepsilon_{\mathfrak{n}_i}) \\ &= (\cdots \wedge (-v_{\mathfrak{q}}) \wedge \cdots \wedge \varphi_{\mathfrak{q}})(\varepsilon_{\mathfrak{n}_i}) \\ &= \varphi_{\mathfrak{q}}(R_P(\varepsilon)_{\mathfrak{n}/\mathfrak{q}}), \end{aligned}$$

where the second equality is obtained by reversing $v_{\mathfrak{q}}$ and $\varphi_{\mathfrak{q}}$ (note that then the sign is changed), and the last by the definition of $R_P(\varepsilon)_{\mathfrak{n}/\mathfrak{q}}$.

(PK4) For any $\mathfrak{q} \in \mathfrak{n}$, we have

$$\begin{aligned} R_P(\varepsilon)_{\mathfrak{n}}|_{\Sigma\setminus\mathfrak{q}} &= ((\cdots \wedge \varphi_{\mathfrak{q}} \wedge \cdots)(\varepsilon_{\mathfrak{n}_i}))|_{\Sigma\setminus\mathfrak{q}} \\ &= ((\cdots \wedge (-v_{\mathfrak{q}} \cdot P_{\mathfrak{q}}) \wedge \cdots)(\varepsilon_{\mathfrak{n}_i}))|_{\Sigma\setminus\mathfrak{q}} \\ &= R_P(\varepsilon)_{\mathfrak{n}/\mathfrak{q}}|_{\Sigma\setminus\mathfrak{q}} \cdot P_{\mathfrak{q}}, \end{aligned}$$

where the second equality follows by noting $(\cdot)|_{\Sigma\setminus\mathfrak{q}} \circ \varphi_{\mathfrak{q}} = -v_{\mathfrak{q}} \cdot P_{\mathfrak{q}}$.
(PK5) Note that we have

$$\varphi_{\mathfrak{q}} = \pi_{\mathfrak{n}} \circ \varphi_{\mathfrak{q}} + (-v_{\mathfrak{q}}) \cdot P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}}$$

for any $\mathfrak{q} \in \mathfrak{n}$. Substitute this into the definition of $R_P(\varepsilon)_{\mathfrak{n}}$, and expand it, then we have

$$R_P(\varepsilon)_{\mathfrak{n}} = \sum_{\mathfrak{d}\subset\mathfrak{n}} \pi_{\mathfrak{n}}(R_P(\varepsilon)_{\mathfrak{d}}) \prod_{\mathfrak{q}\in\mathfrak{n}/\mathfrak{d}} P_{\mathfrak{q}}|_{\Sigma\setminus\mathfrak{n}},$$

which is (PK5).

□

## 5.4 The proof of Theorem 5.1.8

In this section, we prove Theorem 5.1.8 by using the general theory developed in §§5.2 and 5.3. Recall that the setting of the main theorem is the one as in Example 5.2.2, so we assume in this section that 7-tuple $(\mathcal{O}, \Sigma, H, t, v, u, P)$ to be as in Example 5.2.2.

**Proposition 5.4.1.** $\mathrm{KS}_1$ *and* $\mathrm{KS}(A, \mathcal{F}, \Sigma)$ *in [MaRu04, Definition 3.1.3] are naturally isomorphic.*

*Proof.* We use the following fact: there is a natural isomorphism

$$\left\langle \prod_{\ell|n} x_\ell \right\rangle_{\mathbb{Z}/M\mathbb{Z}} \xrightarrow{\sim} \bigotimes_{\ell|n} G_\ell \otimes \mathbb{Z}/M\mathbb{Z}; \quad \prod_{\ell|n} x_\ell \mapsto \bigotimes_{\ell|n} \sigma_\ell \otimes 1.$$

For the proof, see [MaRu11, Proposition 4.2(iv)].

Suppose $\kappa = \{\kappa_n\}_n \in \mathrm{KS}_1$. By (K4) and Corollary 5.2.13, we have

$$\kappa_n \in H \otimes \left\langle \prod_{\ell|n} x_\ell \right\rangle_{\mathbb{Z}/M\mathbb{Z}},$$

so from the above fact we can naturally regard

$$\kappa_n \in H \otimes \left( \bigotimes_{\ell | n} G_\ell \right).$$

Since each $G_\ell \otimes \mathbb{Z}/M\mathbb{Z}$ is isomorphic to $\mathbb{Z}/M\mathbb{Z}$, we see that $H \otimes \left( \bigotimes_{\ell | n} G_\ell \right)$ is isomorphic to $H$. By this observation, we see that axioms (K1) and (K2) say

$$\kappa_n \in H^1_{\mathcal{F}(n)}(\mathbb{Q}, A) \otimes \left( \bigotimes_{\ell | n} G_\ell \right).$$

One sees by definition that (K3) is equivalent to the relation in [MaRu04, (5) in Definition 3.1.3]. Hence we naturally get a Kolyvagin system of [MaRu04] from our Kolyvagin system. Conversely, the Kolyvagin systems of [MaRu04] satisfies the axioms of our Kolyvagin systems (K1)-(K4), with the identification $\langle \prod_{\ell | n} x_\ell \rangle_{\mathbb{Z}/M\mathbb{Z}} = \bigotimes_{\ell | n} G_\ell \otimes \mathbb{Z}/M\mathbb{Z}$. $\square$

**Theorem 5.4.2** ([San14a, Theorem 6.2]). *Suppose the assumptions in Theorem 5.1.8 hold. Then the map*

$$R_K : \mathrm{US}_1 \longrightarrow \mathrm{KS}_1$$

*is surjective.*

*Proof.* First note that by Proposition 5.4.1 we can identify $\mathrm{KS}_1$ and $\mathrm{KS}(A, \mathcal{F}, \Sigma)$. By the proof of [MaRu04, Theorem B.7], we can take $(s, \mathcal{U}) \in \mathscr{I}$ for each $m \in \mathcal{N}$ so that the composed map

$$\varprojlim_{n \in \mathcal{U}} \bigwedge^{\nu(n)+1} S^n \xrightarrow{R_K} \operatorname{im} R_K \xrightarrow{\kappa \mapsto \kappa_m} \mathcal{H}'(m)$$

is surjective, where $\mathcal{H}' = \mathcal{H}'_{(A,\mathcal{F},\Sigma)}$ is the sheaf of stub Selmer modules (for the definition, see [MaRu04, Definition 4.3.1]). By the proof of [MaRu04, Corollary 4.3.5], if $m$ is core (see [MaRu04, Definition 4.1.8] for definition), then we have an isomorphism

$$\Gamma(\mathcal{H}') \xrightarrow{\sim} \mathcal{H}'(m); \quad \kappa \mapsto \kappa_m,$$

where $\Gamma(\mathcal{H}')$ is the global section of $\mathcal{H}'$ (see [MaRu04, Definition 3.1.1]). By [MaRu04, Theorem 4.4.1], the natural inclusion $\Gamma(\mathcal{H}') \hookrightarrow \mathrm{KS}_1$ induces an isomorophism

$$\Gamma(\mathcal{H}') \xrightarrow{\sim} \mathrm{KS}_1.$$

Hence we have $\operatorname{im} R_K = \mathrm{KS}_1$. $\square$

**Remark 5.4.3.** The proof of [MaRu04, Theorem B.7] actually shows that we can take $(s, \mathcal{U})$ satisfying above so that every $n \in \mathcal{U}$ is core. We will use this fact later.

**Proposition 5.4.4.** *Suppose $(s, \mathcal{U}) \in \mathscr{I}$, $\varepsilon \in \varprojlim_{n \in \mathcal{U}} \bigwedge^{\nu(n)+1} S^n$ (see Definition 5.3.3), and every $n \in \mathcal{U}$ is core. Then we have for any $n \in \mathcal{N}$*

$$R_T(\varepsilon)_n \in h_n \mathcal{R}_n.$$

This proposition is reduced to the following lemma (note that if $m$ is core, then $h_m = 1$):

**Lemma 5.4.5.** *Suppose $n = \ell_1 \cdots \ell_{\nu(n)}$, $m = n\ell_{\nu(n)+1} \cdots \ell_{\nu(m)} \in \mathcal{N}$.*
*If $\varepsilon \in \bigwedge^{\nu(m)+1} H^1_{\mathcal{F}^m}(\mathbb{Q}, A)$, then we have*

$$((-v_{\ell_{\nu(m)}}) \wedge \cdots \wedge (-v_{\ell_{\nu(n)+1}}) \wedge \varphi^n_{\ell_{\nu(n)}} \wedge \cdots \wedge \varphi^n_{\ell_1})(\varepsilon) \in \frac{h_n}{h_m} \mathcal{R}_n.$$

*Proof.* We prove by induction on $\nu(m/n)$. When $\nu(m/n) = 0$, i.e. $m = n$, it is clear by the definition of $\mathcal{R}_n$ (see Definition 5.1.2). When $\nu(m/n) > 0$, put $\ell = \ell_{\nu(m)}$ for simplicity.

We claim that there are $\varepsilon' \in \bigwedge^{\nu(m/\ell)+1} H^1_{\mathcal{F}^{m/\ell}}(\mathbb{Q}, A)$, $\varepsilon'' \in \bigwedge^{\nu(m)+1} H^1_{\mathcal{F}^{m/\ell}}(\mathbb{Q}, A)$, and $\delta \in H^1_{\mathcal{F}^m}(\mathbb{Q}, A)$ satisfying

$$\varepsilon = \varepsilon' \wedge \delta + \varepsilon'', \text{ and } (v_\ell(\delta)) = \left(\frac{h_{m/\ell}}{h_m}\right) \text{ (as ideal of } \mathbb{Z}/M\mathbb{Z}).$$

This claim is shown as follows. First note that by definition we have an exact sequence

$$0 \longrightarrow H^1_{\mathcal{F}^{m/\ell}}(\mathbb{Q}, A) \longrightarrow H^1_{\mathcal{F}^m}(\mathbb{Q}, A) \xrightarrow{v_\ell} \mathbb{Z}/M\mathbb{Z}.$$

So we see that there is a $\delta \in H^1_{\mathcal{F}^m}(\mathbb{Q}, A)$ such that $\bar{\delta}$ generates $H^1_{\mathcal{F}^m}(\mathbb{Q}, A)/H^1_{\mathcal{F}^{m/\ell}}(\mathbb{Q}, A)$. Since $v_\ell(\delta)$ generates $\operatorname{im}(H^1_{\mathcal{F}^m}(\mathbb{Q}, A) \xrightarrow{v_\ell} \mathbb{Z}/M\mathbb{Z})$, we have by the global duality

$$(v_\ell(\delta)) = \left(\frac{h_{m/\ell}}{h_m}\right)$$

(see [MaRu04, Theorem 2.3.4] or [Rub00, Theorem 1.7.3]). Since $\bar{\delta}$ generates the group $H^1_{\mathcal{F}^m}(\mathbb{Q}, A)/H^1_{\mathcal{F}^{m/\ell}}(\mathbb{Q}, A)$, any $\eta \in H^1_{\mathcal{F}^m}(\mathbb{Q}, A)$ can be written as the following form: $\eta = \eta' + a\delta$, where $\eta' \in H^1_{\mathcal{F}^{m/\ell}}(\mathbb{Q}, A)$ and $a \in \mathbb{Z}$. Hence $\varepsilon \in \bigwedge^{\nu(m)+1} H^1_{\mathcal{F}^m}(\mathbb{Q}, A)$ can be written as claimed above.

By the claim, we have

$$\begin{aligned}
&((-v_{\ell_{\nu(m)}}) \wedge \cdots \wedge (-v_{\ell_{\nu(n)+1}}) \wedge \varphi^n_{\ell_{\nu(n)}} \wedge \cdots \wedge \varphi^n_{\ell_1})(\varepsilon) \\
= \;& \pm v_\ell(\delta)((-v_{\ell_{\nu(m/\ell)}}) \wedge \cdots \wedge (-v_{\ell_{\nu(n)+1}}) \wedge \varphi^n_{\ell_{\nu(n)}} \wedge \cdots \wedge \varphi^n_{\ell_1})(\varepsilon') \\
\in \;& v_\ell(\delta) \cdot \frac{h_n}{h_{m/\ell}} \mathcal{R}_n = \frac{h_n}{h_m} \mathcal{R}_n,
\end{aligned}$$

where the first equality follows from that $v_\ell(\varepsilon) = \pm v_\ell(\delta)\varepsilon'$ (by definition), and the next from the inductive hypothesis. Hence we have completed the proof. □

**Proposition 5.4.6.**

$$\{\theta_n(c)\}_n \in \mathrm{TKS}_1.$$

*Proof.* By (5.1) in the proof of Proposition 5.1.6, we have

$$\sum_{d|n} (-1)^{\nu(n/d)} \theta_d(c) \prod_{\ell | n/d} P_\ell(\mathrm{Fr}_\ell) = \kappa'_n \otimes \prod_{\ell|n} (\sigma_\ell - 1).$$

Note that the left hand side is equal to $F_{TD}(\{\theta_n(c)\}_n)_n$ (see definition 5.2.9). By Theorem 5.2.17 and Proposition 5.2.10, it is reduced to show

$$\left\{ \kappa'_n \otimes \prod_{\ell|n} (\sigma_\ell - 1) \right\}_n \in \mathrm{DKS}_1.$$

(DK1) and (DK3) are well-known properties of Kolyvagin's derivatives (see [Rub00, Theorem 4.5.1 and Theorem 4.5.4]). (DK2) is shown in [MaRu04, Proof of Theorem 3.2.4 in Appendix A] (note that

$$\sum_{d|n} \left( \kappa'_d \otimes \prod_{\ell|d} (\sigma_\ell - 1) \right) \mathcal{D}_{n/d} \;=\; \sum_{\tau \in \mathfrak{S}(n)} \mathrm{sgn}(\tau) \left( \kappa'_{d_\tau} \otimes \prod_{\ell | d_\tau} (\sigma_\ell - 1) \right) \prod_{\ell | n/d_\tau} \pi_\ell(P_{\tau(\ell)}(\mathrm{Fr}^{-1}_{\tau(\ell)})),$$

where $\mathfrak{S}(n)$ is the set of permutations of the prime divisors of $n$, and $d_\tau = \prod_{\tau(\ell)=\ell} \ell$). (DK4) is clearly satisfied. □

**Remark 5.4.7.** From the above, we see that the Kolyvagin's derivative class $\kappa'_n$ satisfies

$$\kappa'_n \otimes \prod_{\ell|n} (\sigma_\ell - 1) = s_n(\theta_n(c))$$

(for the definition of $s_n$, see Definition 5.2.11). So if we admit Theorem 5.1.8, then we have

$$s_n(\theta_n(c)) \in h_n s_n(\mathcal{R}_n) \subset h_n H^1_{\mathcal{F}^n}(\mathbb{Q}, A) \otimes \left\langle \prod_{\ell | n} (\sigma_\ell - 1) \right\rangle.$$

Hence we have the following upper bound of $h_n$:

$$\mathrm{ord}_p(h_n) \leq \sup\{m \mid \kappa'_n \in p^m H^1_{\mathcal{F}^n}(\mathbb{Q}, A)\}.$$

This generalizes Corollary 5.1.9, since $\kappa'_1 = c_{\mathbb{Q}}$.

Now we prove the main theorem.

*Proof of Theorem 5.1.8.* By Proposition 5.4.6, Theorem 5.2.17, Theorem 5.3.7 and Theorem 5.4.2, there exists $\varepsilon \in \varprojlim_{n \in \mathcal{U}} \bigwedge^{\nu(n)+1} S^n$ such that

$$R_T(\varepsilon)_n = \theta_n(c).$$

Here note that by Remark 5.4.3 every $n \in \mathcal{U}$ is taken to be core. Hence by Proposition 5.4.4 we have

$$R_T(\varepsilon)_n \in h_n \mathcal{R}_n.$$

This completes the proof. □

**Remark 5.4.8.** We expect that Theorem 5.1.8 can be generalized for higher rank Euler systems. If the core rank of $T$ is greater than one, the theory of Kolyvagin systems in [MaRu04] does not work well. Recently, Mazur and Rubin initiated the theory of higher rank Kolyvagin systems, which works well in the higher core rank case (see [MaRu13a]). But we point out two difficulties for the generalization of Theorem 5.1.8. Firstly, if the core rank $r$ is greater than one, then the natural inclusion

$$\Gamma(\mathcal{H}') \hookrightarrow \mathrm{KS}_r$$

is not surjective, where $\mathcal{H}'$ is the sheaf of stub Selmer modules (see [MaRu13a, Remark 11.9]). By this fact, we cannot expect that the map

$$R_K : \mathrm{US}_r \longrightarrow \mathrm{KS}_r$$

constructed in Theorem 5.3.7 is surjective, namely, a natural generalization of Theorem

5.4.2 would be false. Secondly, a connection between higher rank Euler systems and higher rank Kolyvagin systems, which would be a generalization of Proposition 5.4.6, is still mysterious (see [MaRu13a, Introduction]). By these obstacles, it seems difficult to generalize Theorem 5.1.8 in the higher core rank case.

On the other hand, since a typical example of higher rank Euler systems is the system of Rubin-Stark elements, Conjecture 3 in Chapter 3 is regarded as a generalization of Darmon's conjecture for higher rank Euler systems. We note that in this case the $p$-adic representation comes from $\mathbb{G}_m$. So we expect that a generalization of Theorem 5.1.8 for higher rank Euler systems can be established by generalizing Conjecture 3 for general $p$-adic representations (or motives). As we mentioned in Introduction, it is expected that the generalization of Conjecture 3 is related with the ETNC.

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