Arithmetic of non-abelian Galois extensions

March 2014

Jiro Nomura

A Thesis for the Degree of Ph.D. in Science

Arithmetic of non-abelian Galois extensions

March 2014

Graduate School of Science and Technology
Keio University

Jiro Nomura

Contents

Chapter 1

Introduction

1.1 Overview

In this thesis, we study the non-abelian Galois action on arithmetic objects, such as ideal class groups and Selmer groups. More precisely, we study the relation between the non-abelian Galois module structure of arithmetic objects and the special values of L-functions. As a result, we obtain some results on Nickel's conjectures, which are generalizations of Stickelberger's classical result, and the annihilation of Selmer groups of abelian varieties. Not many results are obtained in the non-abelian cases; it is a very interesting problem to find phenomena which is unique to the non-abelian case. We hope our research would in the future also shed light to the abelian case. We start with the historical background.

In number theory, one of the most important subjects is to study the relation between analytic objects and arithmetic objects. There exists a classical and remarkable formula concerning this subject, which is called the *analytic class number formula*. We see the explicit statement in §4.1. By this formula, we can see that the special values of the *Dedekind zeta functions* know the orders of the *ideal class groups*, the *class numbers*. In other words, the Dedekind zeta functions know information on the ideal class groups as $\mathbb{Z}$-modules. The Dedekind zeta function is a generalization of the Riemann zeta function and a purely analytic object. On the other hand, the ideal class group is defined by a purely algebraic method and

the class number tells us how far the world of the ideals is from that of the usual numbers.

Let K/k be a finite Galois extension of number fields with Galois group G. Then the ideal class group of K has a natural G-action. We take a finite set S of places of k which contains all infinite places and all finite places of k which ramify in K. Stickelberger [34] proved that for an absolute abelian extension $K/\mathbb{Q}$, the *Stickelberger element* $\theta_{K/\mathbb{Q},S}$ in the group ring $\mathbb{Q}[G]$ "annihilates" the ideal class group of K. Since the Stickelberger elements are defined by using the special values of *Artin L-functions*, Stickelberger's theorem tells us that Artin L-functions know information on the structures of the ideal class groups as $\mathbb{Z}[G]$-modules. The Stickelberger elements and Artin L-functions are defined in §4.3 and §4.2, respectively.

For an arbitrary abelian extension K/k, Brumer formulated the following conjecture, which is a generalization of Stickelberger's theorem:

Conjecture 1.1.1 (Brumer's conjecture). *Stickelberger elements annihilates ideal class groups.*

For the precise statement of the above conjecture, see Remark 4.4.2 in §4.4. We remark that there exists a refinement of the above conjecture, the Brumer-Stark conjecture. We see the precise statement in Remark 4.4.6 in §4.4. There exists a large body of evidence of Brumer's conjecture and the Brumer-Stark conjecture. For example,

- if K/k is a quadratic extensions, the conjectures are true for K/k by Tate [36, §3, case(c)],

- if K/k is an extension with Galois group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the conjectures are true for K/k by Sands [32, Theorem 2.1],

- if K/k is an extension of degree 4 which is contained in some non-abelian extension of degree 8, the conjectures are true for K/k by Tate [36, §3, case(e)],

4

- if K/k is a cyclic extension of degree $2p$ for some odd prime p which satisfies some conditions concerning the ramifications of the primes of k, the conjectures are true for K/k by Greither, Roblot and Tangedal [11],

- if an odd prime p is "non-exceptional" and the Iwasawa μ-invariant of K vanishes, the "p-parts" of the conjectures are true for K/k and p by Nickel [21],

- if the set S contains all finite places of k above an odd prime p and the Iwasawa μ-invariant of $K(\zeta_p)$ vanishes, where ζ_p is a complex primitive p-th root of unity, the "p-parts" of the conjectures are true for K/k, S and p by Grither and Popescu [13].

Until quite recently, Brumer's conjecture and the Brumer-Stark conjecture were formulated only for abelian extensions. Andreas Nickel [23] recently formulated *non-abelian* generalizations of these conjectures, which are the most important objects in this thesis. The biggest difficulty of the conjectures are concerned with the "integrality" of the Stickelberger elements. In order to get over the difficulty, we have to introduce the "denominator ideal," which is defined in §2.2.4. With the denominator ideal, Nickel's generalization of Brumer's conjecture asserts

Conjecture 1.1.2. *Stickelberger elements multiplied by denominator ideals annihilate ideal class groups.*

We review the precise statements of his conjectures in §4.4. Nickel himself proved the following two results in [22] and [24]:

- if p is "non-exceptional" and the Iwasawa μ-invariant of K vanishes, the p-part of Conjecture 1.1.2 is true for K/k and p,

- if the set S contains all finite places of k above an odd prime p and the Iwasawa μ-invariant of $K(\zeta_p)$ vanishes, the p-part of Conjecture 1.1.2 is true for K/k and S.

The same claims are true for Nickel's non-abelian generalization of the Brumer-Stark conjecture. These results were proved via the *non-commutative Iwasawa*

main conjecture, which was proved independently by Kakde [16] and Ritter and Weiss [30] under Iwasawa's $\mu = 0$ conjecture. The conjecture claims that there exists a deep relation between the ideal class groups and the *p-adic L-functions* in certain p-adic Lie extensions, which are infinite Galois extensions.

Nickel [23] also formulated the "weak versions" of his conjectures. The term "weak" means the conjectures state a weaker integrality of the Stickelberger elements. Moreover, we replace the denominator ideal by the "central condcutor," which is also defined in §2.2.4. We note that the central conductor is always contained in the denominator ideal. With the central conductor, the weaker version of Conjecture 1.1.2 asserts

Conjecture 1.1.3. *Stickelberger elements multiplied by the central conductors annihilate ideal class groups.*

We review the precise statements of his weak conjectures in §4.4. Concerning this conjecture, he proved the following:

- if the places of K^+ above p do not split in K or $K^{cl} \not\subset (K^{cl})^+(\zeta_p)$, the p-part of Conjecture 1.1.3 is true for K/k and p, where superscripts ^{cl} and $^+$ mean the Galois closure over $\mathbb{Q}$ and the maximal real subfield, respectively.

The same claim is true for the weak version of Nickel's non-abelian generalization of the Brumer-Stark conjecture. This result is proved via the strong Stark conjecture.

In this thesis, we reduce Nickel's conjectures for certain non-abelian extensions to those for abelian extensions. We study the extensions whose Galois group is the dihedral group of order $4p$ with an odd prime p, the generalized quaternion group of 2-power order or the direct product of the alternating group on 4 letters and a cyclic group of order 2. We prove for primes l which do not split in certain cyclotomic fields, "l-parts" of Conjecture 1.1.3 for the extensions. We see the precise statement of these results in the next section. The proofs of these results are given in §4.4.2, §4.4.3. Thanks to the reduction step, we can avoid the assumption $\mu = 0$, and in fact we get slightly stronger results than Nickel's conjectures.

Moreover, we can prove the "2-parts" of the weak versions of Nickel's conjectures, which are excluded in known results.

The proof of the reduction step of Nickel's conjectures is purely algebraic and only needs the *Artin formalism* of Artin L-functions. We prove an abstract version of the reduction step in chapter 3. Using the result, we study the (classical) Selmer group of an abelian variety. As a result, we prove in §5.5.3 that a certain element which is defined by using the special values of the Hasse-Weil L-functions annihilates the Pontryagin dual of the p-primary part of the Selmer group for extensions whose Galois groups are the dihedral group of order $4p$. We remark that in order to prove the annihilation result we assume the Birch and Swinnerton-Dyer conjecture, the analytic continuation of the Hasse-Weil L-functions and Conjecture 5.3.5 in §5.3.2.

1.2 Main results on Nickel's conjectures

In this section we see the main results on Nickel's conjectures more precisely. For the explicit statements of Nickel's conjectures, see §4.4.

We first recall that a finite group G is called a *monomial group* if each of the irreducible characters of G is induced by a linear character of a subgroup of G. Then our first main results are

Theorem 1.2.1. *Let K/k be a finite Galois extension of number fields whose Galois group is monomial and S a finite set of places of k which contains all infinite places. Then if Conjecture 1.1.3 is true for the abelian subextensions of K/k, Conjecture 1.1.3 is true for K/k and S.*

Theorem 1.2.2. *Let p be a prime. Then the statement of Theorem 1.2.1 holds with "Conjecture 1.1.3" replaced by "the p-part of Conjecture 1.1.3".*

In the statements of the original versions of Nickel's conjectures, the finite set S has to contain not only all infinite places but also all finite places which ramify in K. Hence if we believe the weak versions of Nickel's conjectures for abelian extensions, we get stronger annihilation results than Nickel's conjectures. We could get this

fact because we directly compared conjectures for non-abelian extensions with conjectures for abelian extensions.

Although the above theorems are on Conjecture 1.1.3, in some cases Conjecture 1.1.3 is equivalent to Conjecture 1.1.2. For example,

- for a prime l which does not divide the order of G, the l-part of Conjecture 1.1.2 is equivalent to that of Conjecture 1.1.3,

- for a prime p, if $\mathcal{I}_p(G) = \zeta(\mathfrak{m}_p(G))$ and the degrees of all the irreducible characters of G are prime to p, the p-part of Conjecture 1.1.2 is equivalent to that of Conjecture 1.1.3, where $\mathcal{I}_p(G)$ is the module generated by the reduced norms of matrices over $\mathbb{Z}_p[G]$ and $\mathfrak{m}_p(G)$ is a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ which contains $\mathbb{Z}_p[G]$.

Remark 1.2.3. If G is isomorphic to the dihedral group of order $4p$ with an odd prime p, we see by [15, Example 6] (also see [26, Lemma 3.22]) that $\mathcal{I}_p(G) = \zeta(\mathfrak{m}_p(G))$. Moreover, all the irreducible characters of the dihedral groups are 1 or 2-dimensional. Hence the p-part of Conjecture 1.1.2 is equivalent to that of Conjecture 1.1.3 if G is isomorphic to the dihedral group of order $4p$.

As an application of Theorem 1.2.2, we get the following results:

Theorem 1.2.4. *Let K/k be a finite Galois extension of number fields whose Galois group is isomorphic to the dihedral group of order $4p$ with an odd prime p. We take a finite set S of places of k which contains all infinite places of k. Then*

(1) *for each odd prime l (l can be p) which does not split in $\mathbb{Q}(\zeta_p)$, the l-part of Conjecture 1.1.2 is true for K/k and S,*

(2) *if the prime 2 does not split in $\mathbb{Q}(\zeta_p)$, the 2-part of Conjecture 1.1.3 is true for K/k and S.*

Theorem 1.2.5. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to the generalized quaternion group of order 2^{n+2} with a natural number n and S be a finite set of places of k which contains all infinite places. Then the 2-part of Conjecture 1.1.3 is true for K/k and S.*

8

Theorem 1.2.6. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to the direct product of $\mathbb{Z}/2\mathbb{Z}$ and the alternating group on 4-letters, and S be a finite set of places of k which contains all infinite places. Then*

(1) *for each odd prime l such that $l \equiv 2 \mod 3$, the l-part of Conjecture 1.1.2 is true for K/k and S,*

(2) *the 2-part and the 3-part of Conjecture 1.1.3 are true for K/k and S.*

Remark 1.2.7. (1) The above three theorems say that the set S does not have to contain the places which ramify in K in contrast with Nickel's formulation. Hence we give stronger results than Nickel's works in [22] and [24] in the above special cases.

(2) Our results contain the 2-parts of conjectures which are excluded in known results.

(3) We use only the analytic class number formula to prove the above three results in contrast with known results which were proven via the non-commutative Iwasawa main conjecture or the strong Stark conjecture.

All the above theorems are true if we replace Conjectures 1.1.2 and 1.1.3 by Nickel's non-abelian generalization of the Brumer-Stark conjecture and its weak version, respectively.

We denote by D_{4p} the dihedral group of order $4p$ with an odd prime p. Then we know that the group ring $\mathbb{Z}_2[D_{4p}]$ is a "nice Fitting order". Using this fact, we can improve Theorem 1.2.4. More explicitly, we get the following:

Theorem 1.2.8. *Let K/k be a finite Galois CM-extension of number fields whose Galois group is isomorphic to D_{4p} and S a finite set of places of k which contains all infinite places and all finite places of k which ramify in K. Then if the prime 2 does not split in $\mathbb{Q}(\zeta_p)$, the 2-part of Conjecture 1.1.2 is true for K/k and S.*

1.3 Main results concerning Selmer groups of abelian varieties

Let K/k be a finite Galois extension of number fields with Galois group G, A an abelian variety over k and A^t the dual abelian variety of A. We set $A_K := A \times_k K$. Then the Galois group G acts naturally on the (classical) Selmer group $\mathrm{Sel}(A_K)$, which is a subgroup of $H^1(K, A_{tors})$. On the other hand, we can define the L-function $L(A, K/k, \chi, s)$ attached to A and a character χ in $\mathrm{Irr}\, G$, which is called the χ-twisted Hasse-Weil L-function (for the explicit definition, see §5.3.1). For a finite set S of places which contains all infinite places and all finite places of k which ramify in K, we define in §5.3.3 an element $\mathcal{L}_{A,K/k,S}$ in the center of $\mathbb{C}[G]$ by using the special values of the twisted Hasse-Weil L-functions. As an analogue of Nickel's conjectures, we formulate in §5.5.1 Problems 5.5.1, 5.5.2, 5.5.4 and 5.5.5 which state the element $\mathcal{L}_{A,K/k,S}$ annihilates $\mathrm{Sel}(A_K)^\vee = \mathrm{Hom}(\mathrm{Sel}(A_K), \mathbb{Q}/\mathbb{Z})$. Then we get the following exact analogue of Theorem 1.2.2:

Theorem 1.3.1. *We take an odd prime p. Let K/k be a finite Galois CM-extension of number fields whose Galois group G is monomial and A an abelian variety over k. Take a finite set S of places of k which contains all infinite places of k. We assume $A(K)[p] = A^t(K)[p] = 0$, namely the Mordell-Weil groups $A(K)$ and $A^t(K)$ have no non-trivial torsion points of order p. Then if Problems in §5.5.1 have the affirmative answers for all the abelian subextensions of K/k, the element $\mathcal{L}_{A,K/k,S}$ multiplied by the central conductor annihilates the p-primary part of $\mathrm{Sel}(A_K)^\vee$.*

Remark 1.3.2. (1) If $A(K)$ has non-trivial p-torsion points, we can prove in §5.5.2 a similar result to the above theorem, Theorem 5.5.11 but we need an extra "modification" of $\mathcal{L}_{A,K/k,S}$.

(2) In fact, we can prove the above theorem in a slightly general set up, that is, we only need the fact that k is totally real. In chapter 5, we prove the above theorem in the general set up.

For an odd prime p, we fix a Sylow p-subgroup P of G and set $N := K^P$. For a $\mathbb{Z}_p[G]$-module M, we set $M^\pm := \{m \in M \mid jm = \pm m\}$. Then as an

application of Theorem 1.3.1, we get the following theorem, which is an analogue of Theorem 1.2.4:

Theorem 1.3.3. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to the dihedral group of order $4p$ and $\alpha \in \{\pm\}$. We take a finite set of places of k which contains all infinite places of k. Let A be an abelian variety over k such that $Hyp_A(K/k, p)$ is satisfied ($Hyp_A(K/k, p)$ is defined in §5.2). We also assume that $A^t(N)[p] = 0$, $A(N)^\alpha$ is finite, $\text{III}_p(A_N)$ injects into $\text{III}_p(A_K)$ and the Birch and Swinnerton-Dyer conjecture holds for the intermediate fields of K/k. Then the element $\mathcal{L}_{A,K/k,S}$ multiplied by the central conductor annihilates the p-primary part of $(\text{Sel}(A_K)^\vee)^\alpha$.*

Remark 1.3.4. Burns, Macias Castillo and Wuthrich [4] proved that the p-part of the equivariant Tamagawa number conjecture for the pair $(h^1(A_K)(1), \mathbb{Z}[G])$ (with some technical assumptions) implies that the element $\mathcal{L}_{A,K/k,S}$ annihilates the Tate-Shafarevich group $\text{III}(A_K)^\vee$ if the set S contains all infinite places and all finite places which ramify in K. This implies Problem 5.5.5 with the same assumption as Theorem 1.3.3. In the statement of Theorem 1.3.3, the fixed set S need not contain finite places of k which ramify in K/k and in fact $\mathcal{L}_{A,F/k,S}$ is defined with $\tau(\mathbb{Q}, \text{Ind}_k^\mathbb{Q} \psi)$ rather than $\tau^*(\mathbb{Q}, \text{Ind}_k^\mathbb{Q} \psi)$, which are defined in §5.3.2. Therefore, this result is stronger than the result in [4] in this special set up (we also remove the assumption that p is unramified in $F/\mathbb{Q}$).

1.4 Abstract annihilation theorem

The key point of the proofs of Theorems 1.2.1, 1.2.2 and 1.3.1 is that the character components of $\theta_{K/k,S}$ and $\mathcal{L}_{A,K/k,S}$ satisfy the *Artin formalism*. We use a purely algebraic lemma for the proof of the above theorems. The key lemma, which is an abstract annihilation theorem, is stated in §3.2

Chapter 2

Algebraic preliminaries

Throughout this section, we let $\mathfrak{o}$ be a noetherian integrally closed domain with field F of quotients. We assume $\mathrm{char}(F) = 0$.

2.1 Character theory for finite groups

Let G be a finite group. In this section, we review the character theory for G over $\mathbb{C}$.

2.1.1 Restriction, inflation and induction of characters

For a $\mathbb{C}[G]$-module V of G with character χ, we write

$$\rho : G \to GL(V)$$

for the representation. If H is a subgroup of G, we get a representation

$$\rho_H : H \hookrightarrow G \to GL(V)$$

of H. We set

$$\mathrm{Res}_H^G \chi := \mathrm{Tr}(\rho_H).$$

This character is said to be the restriction of χ to H.

For a normal subgroup H of G and a $\mathbb{C}[G/H]$-module V' with character ψ, we write

$$\rho_{G/H} : G/H \to GL(V')$$

for the representation. Using this representation, we can define a representation G by

$$\hat{\rho}_{G/H} : G \to G/H \xrightarrow{\rho_{G/H}} GL(V).$$

Then we set

$$\mathrm{Inf}_H^G \psi := \mathrm{Tr}(\hat{\rho}_{G/H}).$$

This character is said to be the inflation of ψ to G.

For a subgroup of H and $\mathbb{C}[H]$-module W with character ϕ, we can define a representation G by

$$\hat{\rho} : G \to GL(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W).$$

We set

$$\mathrm{Ind}_H^G \phi := \mathrm{Tr}(\hat{\rho}).$$

This characters is said to be the character induced by ϕ. Induced characters have the following formula:

Proposition 2.1.1. *For each g in G, we have*

$$\mathrm{Ind}_H^G \phi(g) = \sum_{\substack{\tau \in G, \\ \tau g \tau^{-1} \in H}} \phi(\tau g \tau^{-1}).$$

2.1.2 Inner product of characters

For characters χ_1 and χ_2 of G, we define the inner product of χ_1 and χ_2 by

$$\langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1}) \in \mathbb{R}.$$

Then we have

- for an irreducible character χ, we have $\langle \chi, \chi \rangle = 1$,

- for two irreducible characters χ_1 and χ_2 of G with $\chi_1 \neq \chi_2$, we have $\langle \chi_1, \chi_2 \rangle = 0$,

- if a character χ' is of the form $\chi' = \sum_{\chi \in \mathrm{Irr}\, G} m_\chi \chi$ with $n_\chi \in \mathbb{Z}$, we have $\langle \chi', \chi \rangle = n_\chi$.

Concerning this inner product, we know the following theorem by Frobenius:

Theorem 2.1.2 (Frobenius reciprocity law). *Let H be a subgroup of G. We take characters ϕ of H and χ of G. Then we have*

$$\langle \mathrm{Ind}_H^G \phi, \chi \rangle_G = \langle \phi, \mathrm{Res}_H^G \chi \rangle_H.$$

2.1.3 Ring of virtual characters

We set
$$R_G := \bigoplus_{\chi \in \mathrm{Irr}\, G} \mathbb{Z}\chi = \{ \sum_{\chi \in \mathrm{Irr}\, G} a_\chi \chi \mid a_\chi \in \mathbb{Z} \}.$$

We call the elements of R_G *virtual characters* of G. We can define a ring structure of R_G by tensor products of characters. Thus we call R_G the *ring of virtual characters* of G. For each subgroup H of G we can define the map

$$R_H \to R_G, \ x \mapsto \mathrm{Ind}_H^G x.$$

If we write $R_{H,lin}$ for the subring of R_H generated by all linear characters of H, we have the following fundamental and strong result:

Theorem 2.1.3 (Brauer Induction Theorem). *The following map is surjective:*

$$\bigoplus_{H < G} R_{H,lin} \xrightarrow{\oplus \mathrm{Ind}_H^G} R_G. \tag{2.1}$$

Remark 2.1.4. In fact we can restrict subgroups H to "elementary subgroups". For details, see [8, §15B].

By this theorem, each of irreducible characters of G is a $\mathbb{Z}$-linear combination of induced characters. In some cases, it happens that each of irreducible characters is directly induced from a linear character of a subgroup of G.

Definition 2.1.5. *We say G is a monomial group (or monomial) if each of irreducible characters of G is induced by a linear character of a subgroup of G.*

In the case where G is monomial, we always fix the following notations:

- We write $\chi_1,\ \chi_2,\ \dots,\ \chi_{r_G}$ for the irreducible characters of G,

- For each $i \in \{1,\ 2,\ \dots,\ r_G\}$, we suppose that χ_i is induced from $\phi_{i,1},\ \phi_{i,2}$, $\dots,\ \phi_{i,s_i}$ of linear characters of H_i,

- We write $\phi'_{i,j}$ for the character of $H_i/\ker\phi_{i,j}$ whose inflation to H_i is $\phi_{i,j}$,

- For each i, we fix a representative $\phi_i \in \{\phi_{i,1},\ \phi_{i,2},\ \dots,\ \phi_{i,s_i}\}$,

- Finally, we set $\mathbb{G} := \{H_1,\ H_2,\ \dots,\ H_r\}$.

Additionally, if K/k is a finite Galois extension of number fields whose Galois group G is monomial, we fix the following notations:

- For each $i \in \{1,\ 2,\ \dots, r_G\}$ and $j \in \{1,\ 2,\ \dots,\ s_i\}$, we set $k_i = K^{H_i}$ and $K_{i,j} = K^{\ker\phi_{i,j}}$ (since $\phi_{i,j}$ is a linear character, $K_{i,j}/k_i$ is an abelian extension).

- We fix a representative $K_i \in \{K_{i,1},\ K_{i,2},\ \dots,\ K_{i,s_i}\}$ so that ϕ_i is a character of $\mathrm{Gal}(K_i/k_i)$.

- Finally, we set

$$
\begin{aligned}
\mathbb{K} :=\ & \{K_{1,1}/k_1,\ K_{1,2}/k_1,\ \dots,\ K_{1,s_1}/k_1, \\
& \quad K_{2,1}/k_2,\ K_{2,2}/k_2,\ \dots,\ K_{2,s_2}/k_2, \\
& \qquad\qquad\qquad \cdots \\
& \quad K_{r_G,1}/k_{r_G},\ K_{r_G,2}/k_{r_G},\ \dots,\ K_{r_G,s_{r_G}}/k_{r_G}\}.
\end{aligned}
\tag{2.2}
$$

15

2.2 Group rings

2.2.1 Wedderburn decompositions and idempotents

In this section, we study Wedderburn decompositions and idempotents of group rings.

For any finite group G, $\operatorname{Irr} G$ denotes the set of all the irreducible $\overline{F}$-valued characters of G. We put

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g, \ \operatorname{pr}_\chi := \frac{|G|}{\chi(1)} e_\chi = \sum_{g \in G} \chi(g^{-1})g, \ \chi \in \operatorname{Irr} G.$$

Then the elements e_χ are orthogonal central primitive idempotents of $\overline{F}[G]$ and pr_χ are associated projectors. For each $\chi \in \operatorname{Irr} G$, we fix a matrix representation

$$\rho_\chi : G \to M_{\chi(1)}(\overline{F}).$$

Then we have the Wedderburn decomposition

$$\overline{F}[G] = \bigoplus_{\chi \in \operatorname{Irr} G} \overline{F}[G]e_\chi \cong \bigoplus_{\chi \in \operatorname{Irr} G} M_{\chi(1)}(\overline{F}), \ \sum_{g \in G} \alpha_g g \mapsto (\sum_{g \in G} \alpha_g \rho_\chi(g))_{\chi \in \operatorname{Irr} G}$$

and this implies

$$\zeta(\overline{F}[G]) = \bigoplus_{\chi \in \operatorname{Irr} G} \overline{F}e_\chi \cong \bigoplus_{\chi \in \operatorname{Irr} G} \overline{F}.$$

From the above isomorphisms, we have

$$\zeta(F[G]) \cong \bigoplus_{\chi \in \operatorname{Irr} G/\sim} F(\chi), \tag{2.3}$$

where we put $F(\chi) := F(\chi(g) : g \in G)$ and the direct sum runs over all irreducible characters modulo $\operatorname{Gal}(\overline{F}/F)$-action. The orthogonal idempotents of $F[G]$ and associated projectors are given by

$$e_{[\chi]} := \sum_{\sigma \in \operatorname{Gal}(\overline{F}/F)} e_{\chi^\sigma}, \ \operatorname{pr}_{[\chi]} := \sum_{\sigma \in \operatorname{Gal}(\overline{F}/F)} \operatorname{pr}_{\chi^\sigma}, \ \chi \in \operatorname{Irr} G/\sim,$$

where we put $\chi^\sigma := \sigma \circ \chi$. We note that each element $(\alpha_\chi)_\chi$ in the right hand side of (2.3) corresponds to

$$\sum_{\chi \in \operatorname{Irr} G/\sim} \sum_{\sigma \in \operatorname{Gal}(F(\chi)/F)} \alpha_\chi^\sigma e_{\chi^\sigma}$$

in the left hand side. Let $\mathfrak{m}_{\mathfrak{o}}(G)$ be a maximal $\mathfrak{o}$-order in $F[G]$ which contains $\mathfrak{o}[G]$. Then $\zeta(\mathfrak{m}_{\mathfrak{o}}(G))$ is the unique maximal order in $F[G]$, and we have

$$\zeta(\mathfrak{m}_{\mathfrak{o}}(G)) \cong \bigoplus_{\chi \in \operatorname{Irr} G/\sim} \mathfrak{o}_\chi, \tag{2.4}$$

where $\mathfrak{o}_\chi$ denote the integral closure of $\mathfrak{o}$ in $F(\chi)$. From this isomorphism, we have the following:

Lemma 2.2.1. *Take an element $\alpha = \sum_{\chi \in \operatorname{Irr} G/\sim} \sum_{\sigma \in \operatorname{Gal}(F(\chi)/F)} \alpha_\chi^\sigma e_{\chi^\sigma}$ in $\zeta(F[G])$. Then α lies in $\mathfrak{m}_{\mathfrak{o}}[G]$ if and only if α_χ lies in $\mathfrak{o}_\chi$ for each χ in $\operatorname{Irr} G/\sim$.*

In the rest of this section, we study idempotents e_χ of $\overline{F}[G]$. For each 1-dimensional character $\chi \in \operatorname{Irr} G$, we take a subgroup Δ of $\ker \chi$ which is normal in G and let χ_Δ be the character of G/Δ whose inflation to G is χ. Then we have

$$e_\chi = e_{\chi_\Delta} \frac{1}{|\Delta|} \operatorname{Norm}_\Delta, \tag{2.5}$$

where $\operatorname{Norm}_{\ker \chi} := \sum_{h \in \ker \chi} h$. If χ is induced by a character of a subgroup of G, we can write down e_χ by the following lemma:

Lemma 2.2.2. *Let G be a finite group and let H be a subgroup of G. If an irreducible character χ of G is induced by an irreducible character of H, we have*

$$e_\chi = \sum_{\substack{\phi \in \operatorname{Irr} H/\sim_\chi, \\ \operatorname{Ind} \phi = \chi}} \sum_{h \in \operatorname{Gal}(F(\phi)/F(\chi))} e_{\phi^h},$$

where $\operatorname{Irr} H/\sim_\chi$ means $\operatorname{Irr} H$ modulo $\operatorname{Gal}(\overline{F}/F(\chi))$-action.

Proof. Since $\overline{F}[G]$ is a left and right $\overline{F}[H]$-algebra, $\overline{F}[G]$ decomposes into

$$\bigoplus_{\phi \in \operatorname{Irr} H} \overline{F}[G] e_\phi$$

17

and the components $F[G]e_\phi$ are left and right $\overline{F}[H]$-algebras. By the Frobenius reciprocity law, we have $\langle \chi, \operatorname{Ind}_H^G \phi \rangle_G = \langle \operatorname{Res}_H^G \chi, \phi \rangle_H$, where $\langle \ , \ \rangle$ is the usual inner product of characters. This implies that the simple component $\overline{F}[G]e_\chi$ of $\overline{F}[G]$ decomposes into

$$(\bigoplus_{\phi \in \operatorname{Irr} H} \overline{F}[G]e_\phi)e_\chi = \bigoplus_{\phi \in \operatorname{Irr} H, \ \operatorname{Ind}_H^G \phi = \chi} \overline{F}[G]e_\phi$$

as a left and right $\overline{F}[H]$-algebra. This implies

$$e_\chi = \sum_{\substack{\phi \in \operatorname{Irr} H, \\ \operatorname{Ind}_H^G \phi = \chi}} e_\psi. \qquad (2.6)$$

Take a character $\phi \in \operatorname{Irr} H$ such that $\operatorname{Ind}_H^G \phi = \chi$. Then for each $g \in G$, we have

$$\chi(g) = \sum_{\substack{\tau \in G, \\ \tau^{-1}g\tau \in H}} \phi(\tau^{-1}g\tau).$$

Hence, we have $\operatorname{Ind}_H^G \phi^\sigma = \operatorname{Ind}_H^G \phi$ for all $\sigma \in \operatorname{Gal}(F(\phi)/F(\chi))$. Combining this with (2.6), we have

$$e_\chi = \sum_{\substack{\phi \in \operatorname{Irr} H/\sim_\chi, \\ \operatorname{Ind}_H^G \phi = \chi}} \sum_{h \in \operatorname{Gal}(F(\phi)/F(\chi))} e_{\phi^h}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2.2 Nice Fitting orders

In this section, following [15], we introduce the notion "nice Fitting order" for the group ring $\mathfrak{o}[G]$. In [15], this notion is defined not only for group rings, however, we do not introduce the general definition. For details, see [15, §4].

If $\mathfrak{o}$ is an integrally closed complete commutative noetherian local domain, we say $\mathfrak{o}$ is a *Fitting domain*. For such a domain, we say $\mathfrak{o}[G]$ is a *Fitting order* over $\mathfrak{o}$.

Definition 2.2.3 ([15], Definition 2). *Suppose that $\mathfrak{o}[G] = \bigoplus_{i=1}^{k} \Lambda_i$, where each Λ_i is either a maximal $\mathfrak{o}$-order or a ring of matrices over a commutative ring. Then we say that $\mathfrak{o}[G]$ is a nice Fitting order over $\mathfrak{o}$.*

We assume the residue degree of $\mathfrak{o}$ is p. Then if p does not divide the order of G, $\mathfrak{o}[G]$ is a maximal $\mathfrak{o}$-order in $F[G]$. Hence, by definition, $\mathfrak{o}[G]$ is a nice Fitting order. The following proposition enables us to find non-maximal nice Fitting orders.

Proposition 2.2.4 ([15], Proposition 4.4). *The group ring $\mathfrak{o}[G]$ is a nice Fitting order if and only if p does not divide the order of G'.*

For example, we see that $\mathbb{Z}_2[D_{4p}]$ is a (non-maximal) nice Fitting order since the commutator subgroup of D_{4p} is a cyclic group of order p.

2.2.3   Reduced norms

In this section we study the reduced norm map of the group ring $F[G]$ (for more general theory of the reduced norm map, see [31, §9] and [8, §7D]). First we assume $F[G]$ is a split semisimple F-algebra, that is,

$$F[G] = \bigoplus_{\chi \in \mathrm{Irr}\,G} M_{\chi(1)}(F).$$

Note that in this case all irreducible representations are realized over F. We fix a natural number n and take a matrix H in $M_n(F[G])$. For each χ in $\mathrm{Irr}\,G$, we define the reduced characteristic polynomial $f_{H,\chi}(X)$ of H as

$$f_{H,\chi}(X) := \det(X \cdot 1_{n\chi(1) \times n\chi(1)} - \rho_\chi(H)) = \sum_{i=1}^{n\chi(1)} a_{\chi,i} X^i \in F[X].$$

We write

$$a_{\chi,0} = (-1)^{n\chi(1)} \mathrm{nr}_{F[G],\chi}(H)$$

and set

$$\mathrm{nr}_{F[G]}(H) := \sum_{\chi \in \mathrm{Irr}\,G} \mathrm{nr}_{F[G],\chi}(H) e_\chi \in \zeta(F[G]).$$

19

We call this element the reduce norm of H. We can regard the reduced norm as the following composition map;

$$\text{nr}_{F[G]} : M_n(F[G]) \xrightarrow[\sim]{\oplus \rho_\chi} \bigoplus_{\chi \in \text{Irr}\, G} M_{n\chi(1)}(F) \xrightarrow{\oplus \det_F} \zeta(F[G]).$$

In the case where G is abelian, the reduced norm map is the usual determinant map. Hence if G is abelian, we have

$$\text{nr}_{F[G]}(M_n(\mathfrak{o}[G])) = \det_{F[G]}(M_n(o[G])) = \mathfrak{o}[G].$$

However in the case where G is non-abelian, the equality

$$\text{nr}_{F[G]}(M_n(\mathfrak{o}[G])) = \zeta(\mathfrak{o}[G])$$

does not hold in general. Besides, $\text{nr}_{F[G]}(M_n(\mathfrak{o}[G]))$ is not contained in $\zeta(\mathfrak{o}[G])$ in general. Then where does $\text{nr}_{F[G]}(M_n(\mathfrak{o}[G]))$ live ? The general answer is as follows:

Proposition 2.2.5. *We chose a maximal $\mathfrak{o}$-order $\mathfrak{m}_\mathfrak{o}(G)$ in $F[G]$ which contains $\mathfrak{o}[G]$. Then we have*
$$\text{nr}_{F[G]}(M_n(\mathfrak{o}[G])) \subset \zeta(\mathfrak{m}_\mathfrak{o}(G)).$$

Proof. Let V_χ be a $F[G]$-module with character $\chi \in \text{Irr}\, G$. Then we can choose a basis of V_χ so that $\rho_\chi(G) \subset GL_{\chi(1)}(\mathfrak{o})$. Then by the definition of the reduced characteristic polynomial for χ, we have $\text{nr}_{F[G],\chi}(M_n(\mathfrak{o}[G])) \in \mathfrak{o}_\chi$. Hence we have by (2.4)
$$\text{nr}_{F[G]}(M_n(\mathfrak{o}[G])) \subset \bigoplus_{\chi \in \text{Irr}\, G} \mathfrak{o}_\chi = \zeta(\mathfrak{m}_\mathfrak{o}(G)).$$

$\square$

Next we study the case where $F[G]$ is not a split algebra. In this case, we have

$$M_n(F[G]) \cong \bigoplus_{\chi \in \text{Irr}\, G/\sim} M_{nn_\chi}(D_\chi),$$

where D_χ is a skew filed with $\zeta(D_\chi) = F(\chi)$. We take a splitting field E of $F[G]$

20

which is finite Galois over F. Since we can regard H as an element in $M_n(E[G])$, for each χ in $\operatorname{Irr} G$, there exists the reduced characteristic polynomial $f_{H,\chi} \in E[X]$. We set

$$f_{H,[\chi]}(X) := \sum_{\sigma \in \operatorname{Gal}(E/F(\chi))} f_{H,\chi}^\sigma = \sum_{i=0}^{n\chi(1)} a_{[\chi],i} X^i \in F(\chi)[H].$$

This polynomial is independent of the choice of E. We call this polynomial the reduced characteristic polynomial of H for $\chi \in \operatorname{Irr} G/\sim$. We write

$$a_{[\chi],i} = (-1)^{n\chi(1)} \operatorname{nr}_{F[G],[\chi]}(H)$$

and define the reduced norm of H as

$$\operatorname{nr}_{F[G]}(H) = \sum_{\chi \in \operatorname{Irr} G/\sim} \sum_{\sigma \in \operatorname{Gal}(F(\chi)/F)} (\operatorname{nr}_{F[G],[\chi]}(H))^\sigma e_{\chi^\sigma} \in \zeta(F[G]).$$

We can regard the reduced norm as the following composition map;

$$\operatorname{nr}_{F[G]} : M_n(F[G]) \hookrightarrow M_n(E[G]) \xrightarrow[\sim]{\oplus \rho_\chi} \bigoplus_{\chi \in \operatorname{Irr} G} M_{n\chi(1)}(E) \xrightarrow{\oplus \det_{\overline{E}}} \zeta(F[G]).$$

Concerning the image of $M_n(\mathfrak{o}[G])$, the same result as Proposition 2.2.5 holds, that is, we get the following proposition:

Proposition 2.2.6. *We chose a maximal $\mathfrak{o}$-order $\mathfrak{m}_\mathfrak{o}(G)$ in $F[G]$ which contains $\mathfrak{o}[G]$. Then we have*

$$\operatorname{nr}_{F[G]}(M_n(\mathfrak{o}[G])) \subset \zeta(\mathfrak{m}_\mathfrak{o}(G)).$$

Proof. By the same manner as the proof of Proposition 2.2.5, we see that for each $\chi \in \operatorname{Irr} G/\sim$ the reduced characteristic polynomials of matrices in $M_n(\mathfrak{o}[G])$ belong to $\mathfrak{o}_\chi[X]$. Hence we have

$$\operatorname{nr}_{F[G]}(M_n(\mathfrak{o}[G])) \subset \bigoplus_{\chi \in \operatorname{Irr} G/\sim} \mathfrak{o}_\chi = \zeta(\mathfrak{m}_\mathfrak{o}(G)).$$

$\square$

We set

$$\mathcal{I}_{\mathfrak{o}}(G) := \langle \mathrm{nr}_{F[G]}(H) \mid \forall H \in M_n(\mathfrak{o}[G]),\ \forall n \in \mathbb{N} \rangle_{\zeta(\mathfrak{o}[G])} \subset \zeta(\mathfrak{m}_{\mathfrak{o}}(G)).$$

In the case $\mathfrak{o}$ is $\mathbb{Z}$ (resp. $\mathbb{Z}_p$ for some prime p), we abbreviate $\mathcal{I}_{\mathbb{Z}}(G)$ (resp. $\mathcal{I}_{\mathbb{Z}_p}(G)$) by $\mathcal{I}(G)$ (resp. $\mathcal{I}_p(G)$). If G is an abelian group, we have

$$\mathcal{I}_{\mathfrak{o}}(G) = \mathfrak{o}[G].$$

In contrast with this fact, if G is non-abelian, it is very hard to determine $\mathcal{I}_{\mathfrak{o}}(G)$. However, in some cases we can see what $\mathcal{I}_{\mathfrak{o}}(G)$ is. If the order of $|G|$ is invertible in $\mathfrak{o}$, the group ring $\mathfrak{o}[G]$ is a maximal order in $F[G]$ (cf. [8, Proposition 27.1]). Therefore we have

$$\mathcal{I}_{\mathfrak{o}}(G) = \zeta(\mathfrak{m}_{\mathfrak{o}}[G]) = \zeta(\mathfrak{o}[G]).$$

If $\mathfrak{o}[G]$ is a nice Fitting order, we have the following stronger result:

Proposition 2.2.7 ([15], Proposition 4.1)**.** *If $\mathfrak{o}[G]$ is a nice Fitting order, we have*

$$\mathcal{I}_{\mathfrak{o}}(G) = \zeta(\mathfrak{o}[G]).$$

2.2.4 Denominator ideals and central conductors

In commutative algebra, adjoint matrices are very useful tools, however, there is no such matrices in non-commutative algebra in general (we can not even take determinant maps). Johnston and Nickel [15] defined a non-commutative generalization of adjoint matrices for finite dimensional semisimple algebras. In this section, we first introduce the generalized adjoint matrix over $F[G]$.

We fix a natural number n and take a matrix H in $M_n(F[G])$. We write

$$H = \oplus_{\chi \in \mathrm{Irr}\, G/\sim} H_{[\chi]} \in \bigoplus_{\chi \in \mathrm{Irr}\, G/\sim} M_{nn_\chi}(D_\chi).$$

For each $\chi \in \mathrm{Irr}\, G/\sim$, we set

$$H^*_{[\chi]} := (-1)^{n\chi(1)+1} \sum_{i=1}^{n\chi(1)} a_{[\chi],i} H^{i-1}_{[\chi]}$$

and define the generalized adjoint matrix H^* of H by

$$H^* := \oplus_{\chi \in \mathrm{Irr}\, G/\sim} H^*_{[\chi]}.$$

If G is abelian, H^* coincides with the usual adjoint matrix of H.

Proposition 2.2.8. *For each matrix H in $M_n(F[G])$, we have*

$$HH^* = H^*H = \mathrm{nr}_{F[G]}(H) \cdot 1_{n \times n}.$$

Proof. It is enough to show $H_{[\chi]} H^*_{[\chi]} = \mathrm{nr}_{F[G],[\chi]}(H) \cdot 1_{n_\chi \times n_\chi}$. Since $f_{H,[\chi]}(H_{[\chi]}) = 0$, we have

$$\begin{aligned}
H_{[\chi]} H^*_{[\chi]} &= = (-1)^{n\chi(1)+1} \sum_{i=1}^{n\chi(1)} a_{[\chi],i} H^i_{[\chi]} \\
&= (-1)^{n\chi(1)+1}(-a_{[\chi],0}) \\
&= \mathrm{nr}_{F[G],[\chi]}\,.
\end{aligned}$$

$\square$

If G is abelian and H belongs to $M_n(\mathfrak{o}[G])$, we always have $H^* \in M_n(\mathfrak{o}[G])$. However, if G is not abelian, H^* does not belong to $M_n(\mathfrak{o}[G])$ in general. As well as the reduced norm map, we get the following proposition:

Proposition 2.2.9. *If H belongs to $M_n(\mathfrak{o}[G])$, we have*

$$H^* \in M_n(\mathfrak{m}_\mathfrak{o}[G]).$$

Proof. Since H belongs to $M_n(\mathfrak{o}[G])$, we see that $H_{[\chi]}$ belongs to $M_n(\mathfrak{m}_\mathfrak{o}(G)e_{[\chi]})$ and the coefficients $a_{[\chi],i}$ belong to $\mathfrak{o}_\chi$. Therefore we have $H^*_{[\chi]} \in M_n(\mathfrak{m}_\mathfrak{o}(G)e_{[\chi]})$.

Hence

$$H^* = \oplus_{\chi \in \operatorname{Irr} G/\sim} H^*_{[\chi]} \in \bigoplus_{\chi \in \operatorname{Irr} G/\sim} M_n(\mathfrak{m}_\mathfrak{o}(G)e_{[\chi]}) = M_n(\mathfrak{m}_\mathfrak{o}(G)).$$

□

Using the generalized adjoint matrices H^*, we define the *denominator ideal* $\mathcal{H}_\mathfrak{o}(G)$ by

$$\mathcal{H}_\mathfrak{o}(G) := \{x \in \zeta(\mathfrak{o}[G]) \mid xH^* \in M_n(\mathfrak{o}[G]), \ \forall H \in M_n(\mathfrak{o}[G]) \text{ and } \forall n \in \mathbb{N}\}.$$

In the case where $\mathfrak{o}$ is $\mathbb{Z}$ (resp. $\mathbb{Z}_p$ for some prime p), we abbreviate $\mathcal{H}_{\mathbb{Z}}(G)$ (resp. $\mathcal{H}_{\mathbb{Z}_p}(G)$) by $\mathcal{H}(G)$ (resp. $\mathcal{H}_p(G)$). By the definition of $\mathcal{H}_p(G)$ and Proposition 2.2.8, we have

$$\mathcal{H}_\mathfrak{o}(G)\mathcal{I}_\mathfrak{o}(G) \subset \zeta(\mathfrak{o}[G]).$$

The denominator ideal always has non-trivial elements, more precisely, the *central conductor* $\mathfrak{F}_\mathfrak{o}(G)$ of $\mathfrak{m}_\mathfrak{o}(G)$ over $\mathfrak{o}[G]$ defined by

$$\mathfrak{F}_\mathfrak{o}(G) := \{x \in \zeta(\mathfrak{o}[G]) \mid x\mathfrak{m}_\mathfrak{o}(G) \subset \mathfrak{o}[G]\}.$$

In the case where $\mathfrak{o}$ is $\mathbb{Z}$ (resp. $\mathbb{Z}_p$ for some prime p), we abbreviate $\mathfrak{F}_{\mathbb{Z}}(G)$ (resp. $\mathfrak{F}_{\mathbb{Z}_p}(G)$) by $\mathfrak{F}(G)$ (resp. $\mathfrak{F}_p(G)$). By Proposition 2.2.8 we have

$$\mathfrak{F}_\mathfrak{o}(G) \subset \mathcal{H}_\mathfrak{o}(G).$$

By Jacobinski's central conductor formula ([14, Theorem 3] also see [8, §27]), we see the explicit structure of $\mathfrak{F}_\mathfrak{o}(G)$ as

$$\mathfrak{F}_\mathfrak{o}(G) \cong \bigoplus_{\chi \in \operatorname{Irr} G/\sim} \frac{|G|}{\chi(1)} \mathfrak{D}^{-1}(F(\chi)/F), \tag{2.7}$$

where $\mathfrak{D}^{-1}(F(\chi)/F)$ is the inverse different of $F(\chi) := F(\chi(g); g \in G)$ over F and χ runs over all irreducible characters of G modulo $\operatorname{Gal}(\overline{F}/F)$-action. By contrast

24

to this formula, the structure of $\mathcal{H}_\mathfrak{o}(G)$ is not known in general. However, in some cases we can determine the structure of $\mathcal{H}_\mathfrak{o}(G)$. If the order of G is invertible in $\mathfrak{o}$ (that is, $\mathfrak{o}[G]$ is a maximal order), we have $\mathfrak{F}_\mathfrak{o}(G) = \zeta(\mathfrak{o}[G])$. Hence we have $\mathcal{H}_\mathfrak{o}(G) = \mathfrak{F}_\mathfrak{o}(G)$. Even if the order of G is not invertible, we have the following:

Proposition 2.2.10 ([15], Remark 6.5 and Corollary 6.20). *If $\mathcal{I}_\mathfrak{o}(G) = \zeta(\mathfrak{m}_\mathfrak{o}(G))$ and the degrees of all the irreducible characters of G are invertible in $\mathfrak{o}$, we have*

$$\mathcal{H}_\mathfrak{o}(G) = \mathfrak{F}_\mathfrak{o}(G).$$

In the case $\mathbb{Z}_p[G]$ is a nice Fitting order, we get the following stronger result:

Proposition 2.2.11 ([15], Proposition 4.1). *If $\mathfrak{o}[G]$ is a nice Fitting order, we have $\mathcal{H}_\mathfrak{o}(G) = \mathcal{I}_\mathfrak{o}(G) = \zeta(\mathfrak{o}[G])$.*

Finally, we prove the following technical lemma, which will be needed later:

Lemma 2.2.12. *Let χ be an irreducible character of G which is induced by an irreducible character of a subgroup H of G. Take an arbitrary element x in $\mathfrak{F}_\mathfrak{o}(G)$ of the form*

$$x = \sum_{\sigma \in \mathrm{Gal}(F(\chi)/F)} x_\chi^\sigma \, \mathrm{pr}_{\chi^\sigma}.$$

Then we have

$$x = \sum_{\substack{\phi \in \mathrm{Irr}\, H/\sim, \\ \exists \sigma \in \mathrm{Gal}(F(\chi)/F),\ \mathrm{Ind}\, \phi = \chi^\sigma}} \sum_{f \in \mathrm{Gal}(F(\phi)/F)} x_\chi^f \, \mathrm{pr}_{\phi^f}.$$

In particular, x also lies in $\mathfrak{F}_\mathfrak{o}(H)$.

Proof. For each $\sigma \in \mathrm{Gal}(F(\chi)/F)$, we fix an extension $\tilde{\sigma}$ to $\mathrm{Gal}(F(\phi)/F)$. Then

we have

$$\sum_{\substack{\phi\in\operatorname{Irr}H/\sim,\\ \exists\sigma\in\operatorname{Gal}(F(\chi)/F),\ \operatorname{Ind}\phi=\chi^\sigma}}\ \sum_{f\in\operatorname{Gal}(F(\phi)/F)} x_\chi^f\,\operatorname{pr}_{\phi^f}$$

$$=\sum_{\substack{\phi\in\operatorname{Irr}H/\sim,\\ \exists\sigma\in\operatorname{Gal}(F(\chi)/F),\ \operatorname{Ind}\phi=\chi^\sigma}}\ \sum_{\sigma\in\operatorname{Gal}(F(\chi)/F)}\ \sum_{h\in\operatorname{Gal}(F(\phi)/F(\chi))} x_\chi^{\tilde\sigma h}\,\operatorname{pr}_{\phi^{\tilde\sigma h}}$$

$$=\sum_{\substack{\phi\in\operatorname{Irr}H/\sim,\\ \exists\sigma\in\operatorname{Gal}(F(\chi)/F),\ \operatorname{Ind}\phi=\chi^\sigma}}\ \sum_{\sigma\in\operatorname{Gal}(F(\chi)/F)} \left(\sum_{h\in\operatorname{Gal}(F(\phi)/F(\chi))} x_\chi^h\,\operatorname{pr}_{\phi^h}\right)^{\tilde\sigma}$$

$$=\sum_{\substack{\phi\in\operatorname{Irr}H/\sim,\\ \exists\sigma\in\operatorname{Gal}(F(\chi)/F),\ \operatorname{Ind}\phi=\chi^\sigma}}\ \sum_{\sigma\in\operatorname{Gal}(F(\chi)/F)} \left(\sum_{h\in\operatorname{Gal}(F(\phi)/F(\chi))} x_\chi^h\,\operatorname{pr}_{\phi^h}\right)^{\sigma}$$

$$=\sum_{\sigma\in\operatorname{Gal}(F(\chi)/F)} x_\chi^\sigma \sum_{\substack{\phi\in\operatorname{Irr}H/\sim_{\chi^\sigma},\\ \operatorname{Ind}\phi=\chi^\sigma}} \left(\sum_{h\in\operatorname{Gal}(F(\phi)/F(\chi))} \operatorname{pr}_{\phi^h}\right)^{\sigma}$$

$$=\sum_{\sigma\in\operatorname{Gal}(F(\chi)/F)} x_\chi^\sigma \left(\sum_{\substack{\phi\in\operatorname{Irr}H/\sim_{\chi},\\ \operatorname{Ind}\phi=\chi}}\ \sum_{h\in\operatorname{Gal}(F(\phi)/F(\chi))} \operatorname{pr}_{\phi^h}\right)^{\sigma}$$

$$=\sum_{\sigma\in\operatorname{Gal}(F(\chi)/F)} x_\chi^\sigma\,\operatorname{pr}_{\chi^\sigma}\,.$$

The last equality follows from Lemma 2.2.2. Since x_χ also lies in $\mathcal{D}^{-1}(F(\phi)/F)$, x lies in $\mathfrak{F}_{\mathfrak{o}}(H)$. $\qquad\square$

Chapter 3

Abstract annihilation theorem

In this chapter, $\mathfrak{o}$ is a Dedekind domain with field F of quotients.

3.1 Equivariant elements

3.1.1 Functions over virtual characters

Let G be a finite group. For a subgroup H of G and a normal subgroup N of H, we choose a function

$$L^{H/N} : R_{H/N} \to \overline{F}.$$

We set

$$\mathbb{L}^G := \{(H/N, L^{H/N})\}_{H < G,\ N \triangleleft H}.$$

We refer to the following hypothesis for $\mathbb{L}^G$ as $Art(\mathbb{L}^G)$:

(Art1) If χ_1 and χ_2 are elements in R_H, we have

$$L^H(\chi_1 + \chi_2) = L^H(\chi_1)L^H(\chi_2).$$

(Art2) For each normal subgroup N of H and ψ in $R_{H/N}$, we have

$$L^H(\text{Inf } \psi) = L^{H/N}(\psi).$$

(Art3) For each subgroup J of H and ϕ in R_J, we have

$$L^H(\mathrm{Ind}_J^H \phi) = L^J(\phi).$$

(Art4) For each σ in $\mathrm{Gal}(\overline{F}/F)$ and χ inR_H^+, we have

$$L^H(\chi^\sigma) = L^H(\chi)^\sigma.$$

Note that the condition (Art4) implies that $L^H(\chi)$ lies in $F(\chi)$ for any χ in R_H^+.

Proposition 3.1.1. *If $\mathbb{L}^G$ satisfies conditions (Art1) and (Art3), we have*

$$L^{\{\mathrm{id}_G\}}(1) = \prod_{\chi \in \mathrm{Irr}\, G} L^G(\chi)^{\chi(1)}.$$

Proof. We recall that

$$\mathrm{Ind}_{\{\mathrm{id}_G\}}^G 1 = \sum_{\chi \in \mathrm{Irr}\, G} \chi(1)\chi.$$

Therefore, we have

$$
\begin{aligned}
L^{\{\mathrm{id}_G\}}(1) &= L^G(\mathrm{Ind}_{\{\mathrm{id}_G\}}^G 1_G) && \text{(by (Art3))} \\[2mm]
&= L^G\Big(\sum_{\chi \in \mathrm{Irr}\, G} \chi(1)\chi \Big) \\[2mm]
&= \prod_{\chi \in \mathrm{Irr}\, G} L^G(\chi)^{\chi(1)} && \text{(by (Art1)).}
\end{aligned}
$$

$\square$

3.1.2 Equivariant elements

For each finite group G, we define the element L^G as

$$\mathcal{L}^G := \sum_{\chi \in \mathrm{Irr}\, G} L^G(\check{\chi})e_\chi \in \zeta(\overline{F}[G]).$$

Proposition 3.1.2. *If $L^G(\cdot)$ satisfies the condition (Art4), we have*

$$\mathcal{L}^G \in \zeta(F[G]).$$

Proof.

$$
\begin{aligned}
L^G &= \sum_{\chi \in \mathrm{Irr}\, G} L^G(\check{\chi}) e_\chi \\[2mm]
&= \sum_{g \in G} \left(\sum_{\chi \in \mathrm{Irr}\, G} L^G(\check{\chi}) \frac{\chi(1)}{|G|} \chi(g^{-1}) \right) g \\[2mm]
&= \sum_{g \in G} \left(\sum_{\chi \in \mathrm{Irr}\, G/\sim} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} L^G(\check{\chi}^\sigma) \frac{\chi(1)^\sigma}{|G|} \chi(g^{-1})^\sigma \right) g \\[2mm]
&= \sum_{g \in G} \left(\sum_{\chi \in \mathrm{Irr}\, G/\sim} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} L^G(\check{\chi})^\sigma \frac{\chi(1)^\sigma}{|G|} \chi(g^{-1})^\sigma \right) g \quad \text{(by (Art4))} \\[2mm]
&= \sum_{g \in G} \left(\sum_{\chi \in \mathrm{Irr}\, G/\sim} \mathrm{Tr}_{F(\chi)/F} \left(L^G(\check{\chi}) \frac{\chi(1)}{|G|} \chi(g^{-1}) \right) \right) g.
\end{aligned}
$$

Clearly, $\sum_{\chi \in \mathrm{Irr}\, G/\sim} \mathrm{Tr}_{F(\chi)/F}(L^G(\check{\chi}) \frac{\chi(1)}{|G|} \chi(g^{-1}))$ lies in F. Hence L^G lies in $F[G]$. $\quad\square$

3.1.3 Integrality of $\mathcal{L}^G$

Definition 3.1.3 $(Int(\mathcal{L}^G))$**.** *If the element $\mathcal{L}^G$ has the following condition (3.1), we say that $\mathcal{L}^G$ satisfies $Int(\mathcal{L}^G)$:*

$$\mathcal{L}^G \in \mathcal{I}_\mathfrak{o}(G). \tag{3.1}$$

We take a maximal $\mathfrak{o}$-order $\mathfrak{m}_\mathfrak{o}(G)$ in $F[G]$ which contains $\mathfrak{o}[G]$.

Definition 3.1.4 $(Int_w(\mathcal{L}^G))$**.** *If the element $\mathcal{L}^G$ has the following condition (3.2),*

we say that $\mathcal{L}^G$ satisfies $Int_w(\mathcal{L}^G)$:

$$\mathcal{L}^G \in \zeta(\mathfrak{m}_\mathfrak{o}(G)). \tag{3.2}$$

Since $\mathcal{I}_\mathfrak{o}(G)$ is contained in $\zeta(\mathfrak{m}_\mathfrak{o}(G))$, $Int(\mathcal{L}^G)$ always implies $Int_w(\mathcal{L}^G)$. In general it is very hard to see whether $\mathcal{L}^G$ satisfies $Int(\mathcal{L}^G)$ or not. However, by Lemma 2.2.1, we get the following criterion to determine whether $\mathcal{L}^G$ lies in $\zeta(\mathfrak{m}_\mathfrak{o}(G))$:

Lemma 3.1.5. *We assume L^G satisfies the condition (Art4). Then $\mathcal{L}^G$ lies in $\zeta(\mathfrak{m}_\mathfrak{o}(G))$ if and only if $L^G(\chi)$ lies in $\mathfrak{o}_\chi$ for each χ in $\operatorname{Irr} G/\sim$.*

Proposition 3.1.6. *We assume G is a monomial group and $Art(\mathbb{L}^G)$ is satisfied. Then we have*

$$\mathcal{L}^G = \sum_{i=1}^r L^{H_i/\ker\phi_i}(\check{\phi}_i')e_{\chi_i}.$$

Moreover, if the element $\mathcal{L}^{H_i/\ker\phi_i}$ satisfies $Int_w(\mathcal{L}^{H_i/\ker\phi_i})$ for each abelian subquotients $H_i/\ker\phi_i$ of G, $\mathcal{L}^G$ satisfies $Int_w(\mathcal{L}^G)$.

Proof. Since $Art(\mathbb{L}^G)$ is satisfied, we have

$$
\begin{aligned}
L^G(\chi_i) &= L^G(\operatorname{Ind}_{H_i}^G \phi_i) \\
&= L^{H_i}(\phi_i) \\
&= L^{H/\ker\phi_i}(\phi_i').
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\mathcal{L}^G &= \sum_{i=1}^{r_G} L^G(\check{\chi}_i)e_{\chi_i} \\
&= \sum_{i=1}^{r_G} L^{H_i/\ker\phi_i}(\check{\phi}_i')e_{\chi_i}.
\end{aligned}
$$

If $Int_w(\mathcal{L}^{H_i/\ker\phi_i})$ is satisfied, we see that $\mathcal{L}^{H/\ker\phi_i}(\phi_i')$ lies in $\mathfrak{o}_{\phi_i'} = \mathfrak{o}_\phi$. Since $Art(\mathbb{L}^G)$ is satisfied, $L^G(\chi_i) = L^{H/\ker\phi_i'}(\phi_i')$ lies in $F(\chi_i)$. Recalling that $\mathfrak{o}_{\chi_i}$ is

30

integrally closed, we see that $L^G(\chi_i)$ actually lies in $\mathfrak{o}_{\chi_i}$. By Lemma 3.1.5, we see that $\mathcal{L}^G$ satisfies $Int_w(\mathcal{L}^G)$. $\qquad\square$

3.2   Abstract annihilation theorem

Let G be a finite group and M a $\mathfrak{o}[G]$-module. We refer to the following conditions as $Ab(G, M)$:

(i) G is a monomial group,

(ii) $Art(\mathbb{L}^G)$ is satisfied.

For each subgroup H_i in $\mathbb{G}$

(iii) $\mathcal{L}^{H_i/\ker\phi_{i,j}}$ satisfies $Int_w(\mathcal{L}^{H_i/\ker\phi_{i,j}})$,

(iv) $M^{\ker\phi_{i,j}}$ is annihilated by $\mathfrak{F}_\mathfrak{o}(H_i/\ker\phi_{i,j})\mathcal{L}^{H_i/\ker\phi_{i,j}}$.

Theorem 3.2.1. *Let G be a monomial group and M an $\mathfrak{o}[G]$-module which satisfies $Ab(G, M)$. Then $\mathcal{L}^G$ satisfies $Int_w(\mathcal{L}^G)$ and $\mathfrak{F}_\mathfrak{o}(G)\mathcal{L}^G$ annihilates M.*

Remark 3.2.2. (1) In the following proof, we do not need the condition $(Art1)$. (2) If the module M is a p-group for some prime p, we only need the conditions of $Ab(G, m)$ *modulo* p.

Proof. The first claim follows form Proposition 3.1.6. Next we take an element x in $\mathfrak{F}_\mathfrak{o}(G)$. Then x is of the form

$$x = \sum_{\chi\in\mathrm{Irr}\,G/\sim}\ \sum_{\sigma\in\mathrm{Gal}(F(\chi)/F)} x_\chi^\sigma\,\mathrm{pr}_{\chi^\sigma},\ x_\chi\in\mathcal{D}^{-1}(F(\chi)/F).$$

By Lemma 2.2.12 , we have

$$\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma} = \sum_{\substack{\phi\in\mathrm{Irr}\,H_i/\sim \\ \exists\sigma\in\mathrm{Gal}(F(\chi_i)/F),\ \mathrm{Ind}_{H_i}^G(\phi)=\chi_i^\sigma}}\ \sum_{f\in\mathrm{Gal}(F(\phi)/F)} x_{\chi_i}^f\,\mathrm{pr}_{\phi f},$$

and there exists a subscript $j \in \{1, 2, \ldots, s_i\}$ such that

$$\sum_{f \in \mathrm{Gal}(F(\phi)/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi^f} = \sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi^f_{i,j}} \, .$$

This element also lies in $\mathfrak{F}_{\mathfrak{o}}(H_i)$ and its natural image

$$\sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi'^f_{i,j}}$$

in $\mathfrak{o}[H_i / \ker \phi_{i,j}]$ lies in $\mathfrak{F}(H_i / \ker \phi_{i,j})$. Also we have

$$
\begin{aligned}
\sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi^f_{i,j}} \, \mathcal{L}^{H_i} \; &= \sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} L^{H_i}(\phi^f_{i,j}) \, \mathrm{pr}_{\phi^f_{i,j}} \\
&= \sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} L^{H_i / \ker \phi_{i,j}}(\phi'^f_{i,j}) \, \mathrm{pr}_{\phi'^f_{i,j}} \, Norm_{\ker \phi_{i,j}} \\
&= \sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi'^f_{i,j}} \, \mathcal{L}^{H_i / \ker \phi_{i,j}} Norm_{\ker \phi_{i,j}} \, .
\end{aligned}
$$

Since $Norm_{\ker \phi_{i,j}}(M)$ is contained in $M^{\ker \phi_{i,j}}$, by the condition (iv) of $Ab(G, M)$, we have

$$\sum_{\sigma \in \mathrm{Gal}(F(\phi_{i,j})/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi^f_{i,j}} \, \mathcal{L}^{H_i} M = 0.$$

Therefore, we have

$$\sum_{\substack{\phi \in \mathrm{Irr}\, H_i / \sim \\ \exists \sigma \in \mathrm{Gal}(F(\chi_i)/F), \; \mathrm{Ind}^G_{H_i}(\phi) = \chi_i^{\sigma}}} \; \sum_{f \in \mathrm{Gal}(F(\phi)/F)} x^f_{\chi_i} \, \mathrm{pr}_{\phi^f} \, \mathcal{L}^{H_i} M = 0.$$

Since

$$\sum_{\substack{\phi\in\mathrm{Irr}\,H_i/\sim \\ \exists\sigma\in\mathrm{Gal}(F(\chi_i)/F),\ \mathrm{Ind}_{H_i}^G(\phi)=\chi_i^\sigma}}\ \sum_{f\in\mathrm{Gal}(F(\phi)/F)} x_{\chi_i}^f\,\mathrm{pr}_{\phi^f}\,\mathcal{L}^{H_i}$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\mathcal{L}^{H_i}$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\mathcal{L}^{H_i}\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} e_{\chi_i^\sigma}\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\mathcal{L}^{H_i}\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)}\left(\sum_{\substack{\phi\in\mathrm{Irr}\,H/\sim_\chi,\\ \mathrm{Ind}\,\phi=\chi}}\ \sum_{h\in\mathrm{Gal}(F(\phi)/F(\chi))} e_{\phi^h}\right)^\sigma\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)}\left(\sum_{\substack{\phi\in\mathrm{Irr}\,H/\sim_\chi,\ h\in\mathrm{Gal}(F(\phi)/F(\chi))\\ \mathrm{Ind}\,\phi=\chi}} L^{H_i}(\phi^h)e_{\phi^h}\right)^\sigma\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)}\left(\sum_{\substack{\phi\in\mathrm{Irr}\,H/\sim_\chi,\ h\in\mathrm{Gal}(F(\phi)/F(\chi))\\ \mathrm{Ind}\,\phi=\chi}} L^G(\chi)e_{\phi^h}\right)^\sigma\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)}\left(L^G(\chi)\sum_{\substack{\phi\in\mathrm{Irr}\,H/\sim_\chi,\ h\in\mathrm{Gal}(F(\phi)/F(\chi))\\ \mathrm{Ind}\,\phi=\chi}} e_{\phi^h}\right)^\sigma\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} L^G(\chi^\sigma)e_{\chi^\sigma}\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\mathcal{L}^G\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} e_{\chi^\sigma}\right)$$

$$=\left(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\right)\mathcal{L}^G,$$

we have

$$(\sum_{\sigma\in\mathrm{Gal}(F(\chi_i)/F)} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma})\mathcal{L}^G M = 0.$$

Thus we get

$$\sum_{\chi \in \operatorname{Irr} G/\sim} \sum_{\sigma \in \operatorname{Gal}(F(\chi)/F)} x_\chi^\sigma \operatorname{pr}_{\chi^\sigma} \mathcal{L}^G M = x\mathcal{L}^G M = 0.$$

This completes the proof. $\qquad\square$

Chapter 4

Ideal class groups

4.1 The analytic class number formula

Let K be a number field. In this section we review the analytic class number formula of the Dedekind zeta function $\zeta_K(s)$ of K. We first fix the following notations:

$\mathfrak{o}_K$	the ring of integers of K
$\mathfrak{o}_K^*$	the group of units in $\mathfrak{o}_K$
$Cl(K)$	the ideal class group of K
h_K	the class number of K i.e. $h_K = |Cl(K)|$
d_K	the discriminant of K
Reg_K	the regulator of K
$\mu(K)$	the group of the roots of unity in K
ω_K	the number of the roots of unity in K

The the following is the analytic class number formula:

Theorem 4.1.1.

$$\lim_{s \to 1}(s-1)\zeta_K(s) = \frac{(2)^{r_1}(2\pi)^{r_2}h_K \, \mathrm{Reg}_K}{\omega_K\sqrt{|d_K|}},$$

where r_1 and $2r_2$ are the numbers of real embeddings $K \hookrightarrow \mathbb{R}$ and complex embeddings $K \hookrightarrow \mathbb{C}$, respectively.

We set

$$A(s) := |d_K|^{s-\frac{1}{2}} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} (2(2\pi)^s \Gamma(s))^{r_1+2r_2},$$

where $\Gamma(s)$ is the gamma function. Then we get the following:

Theorem 4.1.2 (The functional equation of $\zeta_K(s)$)**.**

$$\zeta_K(1-s) = A(s)\zeta_K(s).$$

Combining this functional equation with the analytic class number formula, we get the following corollary:

Corollary 4.1.3. *The leading term of the Taylor expansion of $\zeta_K(s)$ at $s = 0$ is*

$$\zeta_K(s) = -\frac{h_k \operatorname{Reg}_K}{\omega_K} s^{r_1+r_2-1} + \cdots .$$

4.2 Artin L-functions

Let K/k be a finite Galois extension of number fields with Galois group G. For each finite place $\mathfrak{p}$ of k, we fix a finite place $\mathfrak{P}$ of K above $\mathfrak{p}$. We write $G_{\mathfrak{P}}$ (resp. $I_{\mathfrak{P}}$) for the decomposition subgroup (resp. inertia subgroup) of G at $\mathfrak{P}$. Finally, we fix a lift $\operatorname{Frob}_{\mathfrak{P}}$ of the Frobenius automorphism of $G_{\mathfrak{P}}/I_{\mathfrak{P}}$.

For each finite place $\mathfrak{p}$ of k and character χ, we define the local Artin L-function at $\mathfrak{p}$ attached to K/k and χ as

$$L_{\mathfrak{p}}(K/k, \chi, s) := \det(1 - \operatorname{Frob}_{\mathfrak{P}} N\mathfrak{p}^{-s} | V_{\chi}^{I_{\mathfrak{P}}}).$$

Take a finite set S of places of k. Using local L-functions, we define the S-truncated global Artin L-function attached to K/k and χ as

$$L_S(K/k, \chi, s) := \prod_{\mathfrak{p} \in S_f \setminus S} L_{\mathfrak{p}}(K/k, \chi, s).$$

This infinite product converges absolutely for all complex numbers s with real part

$\Re(s) > 1$. Moreover, it has an analytic continuation to a meromorphic function on the complex plane $\mathbb{C}$. For each finite set T of places of k such that $S \cap T = \emptyset$, we set

$$\delta_T(K/k, \chi, s) := \prod_{\mathfrak{p} \in T} \det(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} N\mathfrak{p}^{1-s} | V_\chi^{I_\mathfrak{P}})$$

and define the T-modified S-truncated Artin L-function $L_S^T(K/k, \chi, s)$ attached to K/k and χ as

$$L_S^T(K/k, \chi, s) = \delta_T(K/k, \check{\chi}, s) L_S(K/k, \chi, s).$$

This modified L-function has the following properties:

Proposition 4.2.1 (Artin formalism)**.**

(i) $L_S^T(K/k, 1_G, s) = \prod_{\mathfrak{p} \in T}(1 - N\mathfrak{p}^{1-s})\zeta_{k,S}(s)$,

(ii) *If χ_1 and χ_2 are characters of G, we have*

$$L_S^T(K/k, \chi_1 + \chi_2, s) = L_S^T(K/k, \chi_1, s) L_S^T(K/k, \chi_2, s),$$

(iii) *If F is an intermediate field of K/k such that F/k is Galois and ψ is a character of $\mathrm{Gal}(F/k)$, we have*

$$L_S^T(K/k, \mathrm{Inf}_{\mathrm{Gal}(F/k)}^G \psi, s) = L_S^T(F/k, \psi, s),$$

(iv) *If F is an intermediate field of K/k and ϕ is a character of $\mathrm{Gal}(K/F)$, we have*

$$L_S^T(K/k, \mathrm{Ind}_{\mathrm{Gal}(K/F)}^G \phi, s) = L_{S_F}^{T_F}(K/F, \phi, s).$$

Next we study the value of $L_S^T(K/k, \chi, s)$ at $s = 0$ for χ in R_G^+. First we easily see that

$$\delta_T(K/k, \chi^\sigma, 0) = \delta_T(K/k, \chi, 0)^\sigma, \forall \sigma \in \mathrm{Aut}(\mathbb{C}).$$

If we write $r_S(\chi)$ for the vanishing order of $L_S(K/k, \chi, s)$ at $s = 0$, we have by

[35, p24, Proposition 3.4]

$$r_S(\chi) = \sum_{\mathfrak{p} \in S} \dim V_\chi^{G_\mathfrak{P}} - \dim V_\chi^G \tag{4.1}$$

for each non-trivial character in R_G^+. This implies that if $L_S(K/k, \chi, 0)$ vanishes, $L_S(K/k, \chi^\sigma, 0)$ also vanishes for all σ. Therefore, we have

$$L_S(K/k, \chi^\sigma, 0) = L_S(K/k, \chi, 0)^\sigma, \ \forall \sigma \in \mathrm{Aut}(\mathbb{C}).$$

If $L_S(K/k, \chi, 0)$ does not vanish, we have

$$L_S(K/k, \chi^\sigma, 0) = L_S(K/k, \chi, 0)^\sigma, \ \forall \sigma \in \mathrm{Aut}(\mathbb{C})$$

by Stark's conjecture, which was proved by Siegel [33] if G is abelian with $r_S(\chi) = 0$, and the general result is given by Brauer induction [35, p70, Theorem 1.2]. Thus we have

$$L_S^T(K/k, \chi^\sigma, 0) = L_S^T(K/k, \chi, 0)^\sigma, \ \forall \sigma \in \mathrm{Aut}(\mathbb{C}).$$

This implies that $L_S^T(K/k, \chi^\sigma, 0)$ lies in $\mathbb{Q}(\chi)$ and

$$L_S^T(K/k, \chi^\sigma, 0) = L_S^T(K/k, \chi, 0)^\sigma, \ \forall \sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}). \tag{4.2}$$

For a Galois subextension K'/k' of K/k and a character ξ' of $\mathrm{Gal}(K'/k')$, we set

$$L^{\mathrm{Gal}(K'/k')}(\xi') := L_{S_{k'}}^{T_{k'}}(K'/k', \xi', 0)$$

and

$$\mathbb{L}^G := \{L^{\mathrm{Gal}(K'/k')}, \mathrm{Gal}(K'/k'))\}_{K'/k'}.$$

By Proposition 4.2.1 and the equation (4.2), we have the following:

Proposition 4.2.2. *For each finite Galois extension K/k of number fields with Galois group G, $Art(\mathbb{L}^G)$ is satisfied.*

Next we study the vanishing of Artin L-functions. By the formula (4.1), we

see that if k is not totally real or K is not totally imaginary, we always have $L_{S_\infty}(K/k, \chi, 0) = 0$ for all χ in $\operatorname{Irr} G$. Therefore, the only nontrivial case is the case that K/k is a CM-extension, which means that k is a totally real field, K is a CM-field and the complex conjugation induces a unique automorphism j belonging to the center of G. For a CM-extension, we can split the irreducible characters into odd and even characters. For each $\chi \in \operatorname{Irr} G$, we call χ is odd (resp. even) if $\chi(j) = -\chi(1)$ (resp. $\chi(j) = \chi(1)$). Hence in the case where K/k is a CM-extension the formula (4.1) implies that

$$
\begin{cases}
L_{S_\infty}(K/k, \chi, 0) = 0 \text{ if } \chi \text{ is even and } \chi \neq 1, \\
L_{S_\infty}(K/k, \chi, 0) \neq 0 \text{ if } \chi \text{ is odd.}
\end{cases}
$$

Since $L_{S_\infty}(K/k, 1, s) = \zeta_k(s)$, we see in the case where K/k is a CM-extension that

$$
L_{S_\infty}(K/k, 1, 0) \neq 0 \text{ if and only if } k = \mathbb{Q}.
$$

We set

$$
h_K^- = h_K / h_{K^+}.
$$

We conclude this section with the following lemma:

Lemma 4.2.3. *Let K/k be a finite Galois CM-extension of number fields with Galois group G. Then we have*

$$
h_K^- = 2^{-r-1} \mathcal{Q} \omega_K \prod_{\substack{\chi \in \operatorname{Irr} G, \\ \chi \text{ is odd}}} L(K/k, \chi, s)^{\chi(1)},
$$

where we set $\mathcal{Q} = [\mathfrak{o}_K^* : \mu(K)\mathfrak{o}_{K^+}^*]$ and $r = \frac{1}{2}[K : \mathbb{Q}] - 1$.

Remark 4.2.4. The *unit index* $\mathcal{Q}$ is equal to 1 or 2.

Proof. By Propositions 3.1.1 and 4.2.2, we have

$$
\begin{aligned}
\zeta_K(s) &= \prod_{\psi \in \operatorname{Irr} \operatorname{Gal}(K/K^+)} L(K/K^+, \psi, s) \\
&= \zeta_{K^+}(s) \cdot L(K/K^+, \psi_0, s),
\end{aligned}
$$

39

where ψ_0 is the non-trivial irreducible character of $\mathrm{Gal}(K/K^+)$. Since the vanishing orders of $\zeta_K(s)$ and $\zeta_K(K^+)$ are the same, we have

$$\lim_{s\to 0}\frac{\zeta_K(s)}{\zeta_{K^+}(s)} = \frac{h_K/h_{K^+}\,\mathrm{Reg}_K\,/\,\mathrm{Reg}_{K^+}}{\omega_K/\omega_{K^+}}$$
$$= 2\frac{\mathrm{Reg}_K\,/\,\mathrm{Reg}_{K^+}}{\omega_K}h_K^-.$$

By [37, Proposition 4.16], we have

$$\frac{\mathrm{Reg}_K}{\mathrm{Reg}_{K^+}} = \frac{1}{\mathcal{Q}}2^r.$$

Therefore, we have

$$\lim_{s\to 0}\frac{\zeta_K(s)}{\zeta_{K^+}(s)} = \frac{2^{r+1}}{\mathcal{Q}}h_K^-.$$

Now we only have to prove

$$L(K/K^+,\psi_0,s) = \prod_{\substack{\chi\in\mathrm{Irr}\,G,\\ \chi\text{ is odd}}} L(K/k,\chi,s)^{\chi(1)}.$$

This equality follows form the fact that we have

$$\mathrm{Ind}_{\mathrm{Gal}(K/K^+)}^{G}\psi_0 = \sum_{\substack{\chi\in\mathrm{Irr}\,G,\\ \chi\text{ is odd}}} \chi(1)\chi$$

and Proposition 4.2.1. $\qquad\square$

4.3   Stickelberger elements

Let K/k be a finite Galois extension of number fields with Galois group G. In this section we define the Stickelberger element and study its integrality.

4.3.1 Stickelberger elements

We take a finite set S of places of k which contains all infinite places. For each finite set T of places of k such that $S \cap T = \emptyset$, we define the (S, T)-modified Stickelberger element $\theta^T_{K.k,S}$ as

$$\theta^T_{K/k,S} := \sum_{\chi \in \mathrm{Irr}\, G} L^T_S(K/k, \check{\chi}, 0) e_\chi \in \mathbb{C}[G].$$

This element is characterized by the formula

$$\chi(\theta^T_{K/k,S}) = \chi(1) \delta_T(K/k, \chi, 0) L_S(K/k, \check{\chi}, 0). \tag{4.3}$$

When $S = S_{ram} \cup S_\infty$ and $T = \emptyset$, we put $\theta_{K/k} := \theta^T_{K/k,S}$. Moreover, in the case $k = \mathbb{Q}$ we will always omit the trivial character component of $\theta^T_{K/k,S}$.

By Proposition 3.1.2 and 4.2.2, we have the following proposition:

Proposition 4.3.1. $\theta^T_{K/k,S}$ *belongs to* $\zeta(\mathbb{Q}[G])$.

4.3.2 Integrality of Stickelberger elements

Let S and T be finite sets of places of k. We let $E_S(K)$ denote the group of $S(K)$-units of K and set $E^T_S(K) := \{x \in E_S(K) \mid x \equiv 1 \mod \prod_{\mathfrak{P} \in T_K} \mathfrak{P}\}$. We refer to the following hypothesis as $Hyp(S, T)$:

- S contains $S_{ram} \cup S_\infty$,

- $S \cap T = \emptyset$,

- $E^T_S(K)$ is torsion free.

Theorem 4.3.2 (Deligne and Ribet [9], Barsky [1], Cassou-Noguès [7])**.** *We assume K/k is an abelian extension. If finite sets S and T satisfy $Hyp(S, T)$, we have*

$$\theta^T_{K/k,S} \in \mathbb{Z}[G].$$

Remark 4.3.3. By [23, Lemma 2.2], we have

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K)) =_{\mathbb{Z}[G]} \langle \prod_{\mathfrak{p} \in T}(1 - \mathrm{Frob}_{\mathfrak{P}}^{-1} N\mathfrak{p}) \mid Hyp(S,T) \text{ is satisfied. } \rangle, \qquad (4.4)$$

where G is not necessarily abelian. Hence the above claim of Theorem 4.3.2 is equivalent to the claim that we have

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))\theta_{K/k,S} \subset \mathbb{Z}[G]$$

if K/k is abelian extension.

Now we introduce a conjecture by Nickel concerning the integrality of Stickelberger elements.

Conjecture 4.3.4 (Integrality of Stickelberger elements). *If finite sets S and T satisfy $Hyp(S,T)$, we have*

$$\theta_{K/k,S}^{T} \in \mathcal{I}(G).$$

Remark 4.3.5. If G is abelian, $\mathcal{I}(G)$ coincides with $\mathbb{Z}[G]$. Therefore, we can regard the above conjecture as a generalization of Theorem 4.3.2.

We choose a maximal $\mathbb{Z}$-order $\mathfrak{m}(G)$ in $\mathbb{Q}[G]$ which contains $\mathbb{Z}[G]$. He also conjectured the following weak version of the above conjecture:

Conjecture 4.3.6 (Weak integrality of Stickelberger elements). *If finite sets S and T satisfy $Hyp(S,T)$, we have*

$$\theta_{K/k,S}^{T} \in \zeta(\mathfrak{m}(G)).$$

In the case where G is monomial, we define abelian subextensions as follows: For each $i \in \{1, 2, \ldots, r_G\}$ and $j \in \{1, 2, \ldots, s_i\}$, we set $k_i = K^{H_i}$ and $K_{i,j} = K^{\ker \phi_{i,j}}$, noting that $K_{i,j}/k_i$ is an abelian extension since $\phi_{i,j}$ is a linear character. We fix a representative $K_i \in \{K_{i,1}, K_{i,2}, \ldots, K_{i,s_i}\}$ so that ϕ_i is a

character of $\mathrm{Gal}(K_i/k_i)$. Finally, we set

$$
\begin{aligned}
\mathbb{K} := \ \{ &K_{1,1}/k_1, \ K_{1,2}/k_1, \ \ldots, \ K_{1,s_1}/k_1, \\
&K_{2,1}/k_2, \ K_{2,2}/k_2, \ \ldots, \ K_{2,s_2}/k_2, \\
&\quad\quad\quad\quad\quad \cdots \\
&K_{r_G,1}/k_{r_G}, \ K_{r_G,2}/k_{r_G}, \ \ldots, \ K_{r_G,s_{r_G}}/k_{r_G} \}.
\end{aligned}
\tag{4.5}
$$

By Proposition 3.1.6, 4.2.2 and Theorem 4.3.2, we get the following:

Theorem 4.3.7. *Let S be a finite set of places of k which contains all infinite places. If G is a monomial group, we have*

$$
\theta_{K/k,S}^T = \sum_{i=1}^{r_G} \phi_i'(\theta_{K_i/k_i,S_{k_i}}^{T_{k_i}}) e_{\chi_i}.
$$

Moreover, if $Hyp(S \cup S_{ram}, T)$ is satisfied, we have

$$
\theta_{K/k,S}^T \in \zeta(\mathfrak{m}(G)).
$$

Remark 4.3.8. The above theorem says that the finite set S need not contain S_{ram} to lie in $\zeta(\mathfrak{m}(G))$ if G is a monomial group. Nickel [25] showed a stronger result for K/k whose Galois group is monomial but requires the condition S to contain S_{ram}.

4.4 Nickel's conjectures for non-abelian extensions

In this section we review the formulations of the non-abelian Brumer and Brumer-Stark conjectures by Andreas Nickel, for the details see [23].

First we introduce the non-abelian generalization of Brumer's conjecture by Nickel:

Conjecture 4.4.1 ($B(K/k, S)$)**.** *Let S be a finite set of places of k which contains all infinite places and all finite places which ramify in K. Then*

- $\mathfrak{A}_S \theta_S \subset \mathcal{I}(G)$,

- *For any $x \in \mathcal{H}(G)$, $x\mathfrak{A}_S \theta_S \subset \mathrm{Ann}_{\mathbb{Z}[G]}(Cl(K))$.*

Remark 4.4.2. We assume G is abelian. Then we have $\mathcal{I}(\mathbb{Z}[G]) = \mathcal{H}(\mathbb{Z}[G]) = \mathbb{Z}[G]$ and can take $x = 1$. Moreover we have $\mathfrak{A}_S = \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$. Hence the above claim is equivalent to

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))\theta_{K/k,S} \subset \mathrm{Ann}_{\mathbb{Z}[G]}(Cl(K)).$$

This is the exact claim of Brumer's conjecture. Therefore we can regard Conjecture4.4.1 as an generalization of Brumer's conjecture.

We take a maximal $\mathbb{Z}$-order $\mathfrak{m}(G)$ in $\mathbb{Q}[G]$ which contains $\mathbb{Z}[G]$. In [23], the author also formulated the following weak version of Conjecture 4.4.1:

Conjecture 4.4.3 ($B_w(K/k, S)$)**.** *Let S be a finite set of places of k which contains $S_{ram} \cup S_\infty$. Then*

- $\mathfrak{A}_S \theta_S \subset \zeta(\mathfrak{m}(G))$,

- *For any $x \in \mathfrak{F}(G)$, $x\mathfrak{A}_S \theta_S \subset \mathrm{Ann}_{\mathbb{Z}[G]}(Cl(K))$.*

Remark 4.4.4. Even if G is a nontrivial abelian group, we always have $\mathfrak{m}(G) \supsetneq \mathbb{Z}[G]$. Moreover, $\mathfrak{F}(G)$ does not contain the element 1. Hence we can not recover the usual Brumer's conjecture from the conjecture 4.4.3 even in the case where G is abelian. Roughly speaking, Conjecture 4.4.3 says $|G|\theta_{K/k,S}^T$ annihilates $Cl(K)$ if G is abelian.

Replacing $\mathbb{Z}$, $\mathbb{Q}$ and $Cl(K)$ with $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $Cl(K) \otimes \mathbb{Z}_p$ respectively, we can decompose $B(S, K/k)$ (resp. $B_w(S, K/k)$) into local conjectures $B(S, K/k, p)$ (resp. $B_w(S, K/k, p)$).

We call $\alpha \in K^*$ an anti-unit if $\alpha^{1+j} = 1$ and set $\omega_K = \mathrm{nr}(|\mu(K)|)$. We remark that w_k is no longer a rational integer but an element in $\zeta(\mathfrak{m}(G))$ of the form $\sum_{\chi \in \mathrm{Irr}\, G} |\mu(K)|^{\chi(1)} e_\chi$. We define

$$S_\alpha := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime of } k \text{ and } \mathfrak{p} \text{ divides } N_{K/k}\alpha\},$$

where $N_{K/k}$ is the usual norm of K over k. Then Nickel's non-abelian generalization of Brumer-Stark conjecture is as follows:

Conjecture 4.4.5 $(BS(K/k, S))$. *Let S be a finite set of places which contains $S_{ram} \cup S_\infty$. Then*

- *$\omega_K \theta_{K/k,S} \in \mathcal{I}(G)$,*

- *For any fractional ideal $\mathfrak{A}$ of K and for each $x \in \mathcal{H}(G)$, there exists an anti-unit $\alpha = \alpha(\mathfrak{A}, S, x)$ such that $\mathfrak{A}^{x\omega_K \theta_{K/k,S}} = (\alpha)$.*

Moreover, for any finite set T of places of k which satisfies $Hyp(S \cup S_\alpha, T)$, there exists $\alpha_T \in E_{S_\alpha}^T(K)$ such that

$$\alpha^{z\delta_T} = \alpha_T^{z\omega_K} \tag{4.6}$$

for each $z \in \mathcal{H}(G)$.

Remark 4.4.6. We assume G is abelian. Then we can take $x = z = 1$. Moreover, by [35, Proposition 1.2], the above statement is equivalent to the assertion that

$$\mathfrak{A}^{\omega_K \theta_{K/k,S}} = (\alpha) \text{ and } K(\alpha^{1/\omega_K})/k \text{ is an abelian extension.}$$

This is the exact claim of the Brumer-Stark conjecture. Hence we can regard Conjecture 4.4.5 as a non-abelian generalization of the Brumer-Stark conjecture.

As well as the non-abelian Brumer conjecture, there exits the following weak version of Conjecture 4.4.5.

Conjecture 4.4.7 $(BS_w(K/k, S))$. *Let S be a finite set of places which contains $S_{ram} \cup S_\infty$. Then*

- $\omega_K \theta_{K/k,S} \in \zeta(\mathfrak{m}(G))$,

- *For any fractional ideal $\mathfrak{A}$ of K and for each $x \in \mathfrak{F}(G)$, there exists an anti-unit $\alpha = \alpha(\mathfrak{A}, S, x)$ such that $\mathfrak{A}^{x\omega_K \theta_{K/k,S}} = (\alpha)$.*

Moreover, for any finite set T of places of k which satisfies $Hyp(S \cup S_\alpha, T)$, there exists $\alpha_T \in E_{S_\alpha}^T(K)$ such that

$$\alpha^{z\delta_T} = \alpha_T^{z\omega_K} \tag{4.7}$$

for each $z \in \mathfrak{F}(G)$.

Remark 4.4.8. For the same reason as Remark 4.4.4, we can not recover the usual Brumer-Stark conjecture from the Conjecture 4.4.7 in the case where G is abelian.

Let $\mathfrak{m}_p(G)$ be a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ which contains $\mathbb{Z}_p[G]$. Replacing $\mathfrak{m}(G)$ and $\mathfrak{A}$ with $\mathfrak{m}_p(G)$ and $\mathfrak{A}$ whose class in $Cl(K)$ is of p-power order respectively and in the equation (4.6), (4.7) replacing ω_K with $\omega_{K,p} := \mathrm{nr}(|\mu_K \otimes \mathbb{Z}_p|)$, we can decompose $BS(S, K/k)$ (resp. $BS_w(S, K/k)$) into local conjectures $BS(S, K/k, p)$ (resp. $BS_w(S, K/k, p)$).

For an intermediate field L of K/k and a set T of places of k, we write $Cl(L)^{T_L}$ for the ray class group of L to the ray $\prod_{\mathfrak{P}_F \in T_L} \mathfrak{P}_L$ and set $Cl(L)_p^{T_L} := Cl(L)^{T_L} \otimes \mathbb{Z}_p$. Then we can interpret Conjecture 4.4.5 as the annihilation of ray class groups as follows:

Proposition 4.4.9 ([28] Proposition 4.2 and [23] Proposition 3.8). *Let S be a finite set of places of k which contains S_∞ and S_{ram}. We assume $\theta_{K/k,S}^T$ belongs to $\mathcal{I}_p(G)$ for each finite set T of places which satisfies $Hyp(S,T)$. Then $BS(K/k, S, p)$ is true if and only if for each finite set T of places of k such that $Hyp(S,T)$ is satisfied, we have $\mathcal{H}_p(G)\theta_{K/k,S}^T \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(Cl(K)_p^{T_K})$.*

Remark 4.4.10. The following proof of the sufficiency is essentially the same as the proof of [23, Lemma 2.9].

Proof. Concerning the necessity, the same proof as [23, Proposition 3.8] works. Hence we only prove the sufficiency. We take a finite set T of places of k such that

$Hyp(S,T)$ is satisfied. Let $\mathfrak{A}$ be a fractional ideal of K coprime to the primes in T_K whose class in $Cl(K)^{T_K}$ has p-power order. Then for each $x \in \mathcal{H}_p(G)$, we have

$$\mathfrak{A}^{x\omega_K \theta_{K/k,S}} = (\alpha) \tag{4.8}$$

for some anti-unit $\alpha \in K^*$. Since $\mathfrak{A}$ is coprime to the primes in T_K, we see that $Hyp(S \cup S_\alpha, T)$ is satisfied. Hence there exists an element $\alpha_T \in E^T_{S_\alpha}(K)$ such that

$$\alpha^{z\delta_T} = \alpha_T^{z\omega_K} \tag{4.9}$$

for any z in $\mathcal{H}_p(G)$. Since $\mathrm{nr}(|\mu(K)|^{-1})$ belongs to $\zeta(\mathbb{Q}[G])$, there exists a natural number N such that $N\,\mathrm{nr}(|\mu(K)|^{-1}) \in \zeta(\mathbb{Z}[G])$. Then $N\,\mathrm{nr}(|\mu(K)|^{-1})\delta_T \in \zeta(\mathfrak{m}_p(G))$. Since $|G|$ is an element in $\mathfrak{F}_p(G) \subset \mathcal{H}_p(G)$, by (4.8) and (4.9) we have

$$
\begin{aligned}
(\mathfrak{A}^{x\omega_K \theta_{K/k,S}})^{|G|N\,\mathrm{nr}(|\mu(K)|^{-1})\delta_T} &= \mathfrak{A}^{x\theta^T_{K/k,S}|G|N} \\
&= (\alpha)^{|G|N\,\mathrm{nr}(|\mu(K)|^{-1})\delta_T} \\
&= (\alpha^{|G|\delta_T})^{N\,\mathrm{nr}(|\mu(K)|^{-1})} \\
&= (\alpha_T^{|G|\omega_K})^{N\,\mathrm{nr}(|\mu(K)|^{-1})} \\
&= (\alpha_T)^{|G||N|}.
\end{aligned}
$$

Since we assume $\theta^T_{K/k,S} \in \mathcal{I}_p(G)$ and the group of fractional ideals has no torsion, the above equation implies

$$\mathfrak{A}^{x\theta^T_{K/k,S}} = (\alpha_T).$$

This completes the proof. $\qquad\square$

In the abelian case, the Brumer-Stark conjecture implies Brumer's conjecture, and the same claim holds in the non-abelian case as follows:

Lemma 4.4.11 ([23], Lemma 2.9)**.**

- $BS(K/k, S)$ *(resp. $BS(K/k, S, p)$) implies $B(K/k, S)$ (resp. $B(K/k, S, p)$),*

- $BS_w(K/k, S)$ *(resp. $BS_w(K/k, S, p)$) implies $B_w(K/k, S,)$ (resp. $B_w(K/k, S, p)$).*

For the local conjectures, we can state the relation between usual conjectures and weaker conjectures as follows:

Lemma 4.4.12. *If $\mathcal{I}_p(G) = \zeta(\mathfrak{m}_p(G))$ and the degrees of all the irreducible characters of G are prime to p,*

- $B(K/k, S, p)$ *holds if and only if* $B_w(K/k, S, p)$ *holds,*

- $BS(K/k, S, p)$ *holds if and only if* $BS_w(K/k, S, p)$ *holds.*

Proof. If p does not divide the order of G (in this case, the degrees of irreducible characters are automatically prime to p, since they have to divide the order of G), by [23, Lemma 2.5 and Lemma 2.8], the equivalences hold. If p divides the order of G, by Proposition 2.2.10, we have $\mathcal{H}_p(G) = \mathfrak{F}_p(G)$, and hence we get the equivalences. $\qquad\square$

Let D_n denote the dihedral group of order n for any even natural number $n > 0$. Then as an application of Lemma 4.4.12, we get the following:

Lemma 4.4.13. *Let K/k be a finite Galois extension whose Galois group is isomorphic to D_{4p} for an odd prime p. Then we have for any odd prime l*

- $B(K/k, S, l)$ *holds if and only if* $B_w(K/k, S, l)$ *holds,*

- $BS(K/k, S, l)$ *holds if and only if* $BS_w(K/k, S, l)$ *holds.*

Proof. It is enough to treat the case $l = p$. First we recall that D_{4p} is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times D_{2p}$. We set $G = \mathbb{Z}/2\mathbb{Z} \times D_{2p}$ and j denotes the generator of $\mathbb{Z}/2\mathbb{Z}$. Since we have

$$\mathrm{nr}_{\mathbb{Q}_p[G]}\left(\frac{1+j}{2}\right) = \frac{1+j}{2} \quad \text{and} \quad \mathrm{nr}_{\mathbb{Q}_p[G]}\left(\frac{1-j}{2}\right) = \frac{1-j}{2},$$

we also have

$$\mathcal{I}(\mathbb{Z}_p[G]) = \mathcal{I}_p(D_{2p})\frac{1+j}{2} \oplus \mathcal{I}_p(D_{2p})\frac{1-j}{2}. \tag{4.10}$$

By [15, Example 6.22], $\mathcal{I}(D_{2p}) = \zeta(\Lambda'_{D_{2p}})$, where $\Lambda'_{D_{2p}}$ is a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[D_{2p}]$ which contains $\mathbb{Z}_p[D_{2p}]$. We set $\Lambda' := \Lambda'_{D_{2p}}\frac{1+j}{2} \oplus \Lambda'_{D_{2p}}\frac{1-j}{2}$, which is a

maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ which contains $\mathbb{Z}_p[G]$. Then we have

$$\mathcal{I}_p(G) = \zeta(\Lambda'_{D_{2p}})\frac{1+j}{2} \oplus \zeta(\Lambda'_{D_{2p}})\frac{1-j}{2} = \zeta(\Lambda').$$

By Lemma 4.4.12, this completes the proof. $\qquad\square$

4.4.1 Monomial CM-extensions

In this section, we prove the following theorem:

Theorem 4.4.14. *Let K/k be a finite Galois CM-extension whose Galois group is monomial. We take a finite set S of places of k which contains S_∞. Then if $B_w(K_{i,j}/k_i, S_{k_i})$ is true for all $K_{i,j}/k_i$ in $\mathbb{K}$, $B_w(K/k, S)$ is true.*

Remark 4.4.15. In the above statement the set S does not have to contain S_{ram}. Therefore, if we believe (the weak) Brumer's conjecture for abelian extensions, we get a stronger annihilation result than Nickel's conjecture.

Proof. We take another finite set T of places which satisfies $Hyp(S \cup S_{ram}, T)$. Then we have to show the following two things:

- $\theta^T_{K/k,S}$ lies in $\zeta(\mathfrak{m}(G))$,

- for all x in $\mathfrak{F}(G)$, we have $x\theta^T_{K/k,S} \cdot Cl(K) = 0$.

The first claim is true by Theorem 4.3.7. To show the second claim, we only have to show that the pair $(L^G, Cl(K))$ satisfies $Ab(L^G, Cl(K))$ by Theorem 3.2.1. The condition (i) of $Ab(L^G, Cl(K))$ is obviously satisfied. The condition (ii) is followed by Proposition 4.2.2. The condition (iii) is followed by the fact that the extensions $K_{i,j}/k_i$ are abelian extensions. The condition (iv) is followed by our assumption that $B_w(K_{i,j}/k_i, S_{k_i})$ is true for all $K_{i,j}/k_i$ in $\mathbb{K}$. $\qquad\square$

4.4.2 Extensions with group D_{4p}

In this section we study Nickel's conjectures for extensions with group D_{4p}. First we review the character theory of D_{4p}. After that we prove the main theorem of this section.

Character theory of D_{4p}

In this section, we review the character theory of D_{4p}. As is well known, all the irreducible characters of D_{4p} are four 1-dimensional characters and $p - 1$ 2-dimensional characters. The group D_{4p} is the direct product of $\mathbb{Z}/2\mathbb{Z}$ and D_{2p}. Hence if we use the presentation $D_{2p} = \langle \sigma, \ \tau \mid \sigma^p = \tau^2 = 1, \ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$, the commutator subgroup of D_{4p} is $\langle \sigma \rangle$ and we have $D_{4p}/\langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Moreover, the 1-dimensional characters are determined by the following table, where j is the generator of $\mathbb{Z}/2\mathbb{Z}$:

Table 4.1: 1-dimensional characters of D_{4p}

	σ	τ	j
χ_0	1	1	1
χ_1	1	1	-1
χ_2	1	-1	1
χ_3	1	-1	-1

Since the center of D_{4p} is $\{1, \ j\}$, the element j corresponds to the unique complex conjugation in the case where D_{4p} is the Galois group of some CM-extension of number fields. Hence we see that the only 1-dimensional odd (resp. even) characters are χ_1 and χ_3 (resp. χ_0 and χ_2). For $i = 1, \ 3$, we write χ_i^{ab} for the character of $\mathrm{Gal}(K^{ab}/k)$ whose inflation to G is χ_i. All the 2-dimensional odd characters are induced by the faithful odd characters of $\langle j\sigma \rangle$. For $m \in (\mathbb{Z}/p\mathbb{Z})^*$, let ϕ^m be the character of $\langle j\sigma \rangle$ which sends σ and j to ζ_p^m and -1, respectively. We set $\chi_{2m+3} = \mathrm{Ind}_{\langle j\sigma \rangle}^{D_{4p}} \phi^m$ (we use this numbering so that odd subscripts correspond to odd characters). Using the Frobenius reciprocity law and the fact that $\chi_{2m+3}(1) = 2$ and $\chi_{2m+3}(j) = -2$, we see that $\mathrm{Res}_{\langle j\sigma \rangle}^{D_{4p}} \chi_{2m+3} = \phi^m + \phi^{-m}$ and $\mathrm{Ind}_{\langle j\sigma \rangle}^{D_{4p}} \phi^m = \mathrm{Ind}_{\langle j\sigma \rangle}^{D_{4p}} \phi^{-m}$. Therefore, the number of 2-dimensional odd characters is $(p-1)/2$. All the even characters are induced by the characters ϕ^{2m} for $m = 1, \ 2, \ \ldots, \ p - 1$. We set $\chi_{2m+2} = \mathrm{Ind}_{\langle j\sigma \rangle}^{D_{4p}} \phi^{2m}$. Then by the same way as odd characters we see that $\mathrm{Res}_{\langle j\sigma \rangle}^{D_{4p}} \chi_{2m+2} = \phi^{2m} + \phi^{-2m}$ and $\mathrm{Ind}_{\langle j\sigma \rangle}^{D_{4p}} \phi^{2m} = \mathrm{Ind}_{\langle j\sigma \rangle}^{D_{4p}} \phi^{-2m}$. Finally we set $k_\phi := K^{\langle j\sigma \rangle}$.

Main Theorem

In this section we prove the following theorem:

Theorem 4.4.16 ([26], Theorem 5.1). *Let K/k be a finite Galois extension of number fields whose Galois group is isomorphic to D_{4p}. We take a finite set S of places of k which contains S_∞. Then*

(1) *for each odd prime l (l can be p) which does not split in $\mathbb{Q}(\zeta_p)$, the non-abelian Brumer and Brumer-Stark conjectures are true for K/k and S,*

(2) *if the prime 2 does not split in $\mathbb{Q}(\zeta_p)$, the weak non-abelian Brumer and Brumer-Stark conjectures are true for K/k and S.*

Remark 4.4.17. If S contains all finite places of k which ramify in K, we know the following:

(1) In the case of $k = \mathbb{Q}$, the above results except the 2-part is contained in Nickel's work [22], [24] if we assume $\mu = 0$; (2) If no prime above p splits in K/K^+ whenever $K^{cl} \subset (K^{cl})^+(\zeta_p)$, the above result holds for odd p by [25, Corollary 4.2].

The observation we made in the previous subsection tells us that we have only to verify the weak Brumer-Stark conjecture for two relative quadratic extensions K_3/k, K_4/k and the cyclic extension K/k_5. By [36, §3, case(c)], the Brumer-Stark conjecture is true for any relative quadratic extensions and hence true for K_3/k, K_4/k. In order to complete the proof of Theorem 4.4.16, we have to verify the l-part of the weak Brumer-Stark conjecture for K/k_5 for each prime l which does not split in $\mathbb{Q}(\zeta_p)$. Since K/k_5 is a cyclic extension of degree $2p$, it is enough to show the following:

Proposition 4.4.18. *Let l be a prime which does not split in $\mathbb{Q}(\zeta_p)$. Let K/F be any cyclic CM-extension of number fields of degree $2p$. Then the l-part of the weak Brumer-Stark conjecture for K/F is true.*

Remark 4.4.19. The method of the proof of this proposition is essentially the same as that of [11, Proposition 2.2 and Proposition 2.1] but we do not need the classifications in loc. cit because we only need a weaker annihilation results than the full Brumer-Stark conjecture.

Proof of Proposition 4.4.18. By abuse of notation, we also denote by σ and j elements of $\mathrm{Gal}(K/F)$ whose orders are p and 2, respectively. For a prime l which does not split in $\mathbb{Q}(\zeta_p)$, we take an element $x \in \mathfrak{F}(\mathrm{Gal}(K/F))$ of the form

$$
\begin{aligned}
x &= \sum_{\substack{\phi \in \mathrm{Irr}\, G/\sim \\ \phi \text{ is odd}}} \sum_{g \in \mathrm{Gal}(\mathbb{Q}(\phi)/\mathbb{Q})} x_\phi^g \, \mathrm{pr}_{\phi^g}, \ \ x_\phi \in \mathcal{D}^{-1}(\mathbb{Q}(\phi)/\mathbb{Q}) \\
&= x_\psi \, \mathrm{pr}_\psi + \sum_{\substack{\phi \in \mathrm{Irr}\, G/\sim \\ \phi \text{ is odd and } \phi(\sigma) \neq 1}} \sum_{g \in \mathrm{Gal}(\mathbb{Q}(\phi)/\mathbb{Q})} x_\phi^g \, \mathrm{pr}_{\phi^g},
\end{aligned}
$$

where ψ is the character of $\mathrm{Gal}(K/F)$ such that $\psi(j) = -1$ and $\psi(\sigma) = 1$ and x_ψ belongs to $\mathbb{Z}$. We set

$$
x_{[\phi]} := \sum_{\substack{\phi \in \mathrm{Irr}\, G/\sim \\ \phi \text{ is odd and } \phi(\sigma) \neq 1}} \sum_{g \in \mathrm{Gal}(\mathbb{Q}_l(\phi)/\mathbb{Q}_l)} x_\phi^g \, \mathrm{pr}_{\phi^g} \, .
$$

We take a finite set S of places of k which contains $S_{ram} \cup S_\infty$ and a fractional ideal $\mathfrak{A}$ of K whose class in $Cl(K)$ has l-th power order. We set $\omega_{K,l} := |\mu(K) \otimes \mathbb{Z}_l|$. With these notations, we prove the following two claims:

Claim 4.4.20. *There exists anti-unit α_1 such that*

$$
\mathfrak{A}^{x_\psi \, \mathrm{pr}_\psi \, \omega_K \theta_{K/k,S}} = (\alpha_1) \ and \ K(\alpha_1^{1/\omega_{K,l}})/F \ is \ abelian.
$$

Claim 4.4.21. *There exists anti-unit α_2 such that*

$$
\mathfrak{A}^{x_{[\phi]} \omega_K \theta_{K/k,S}} = (\alpha_2) \ and \ K(\alpha_2^{1/\omega_{K,l}})/F \ is \ abelian.
$$

If we assume the above two claims, we have

$$
\mathfrak{A}^{x \omega_K \theta_{K/k}} = (\alpha_1 \alpha_2) \ \text{and} \ K((\alpha_1 \alpha_2)^{1/\omega_{K,l}})/F \ \text{is abelian.}
$$

Hence in order to prove Proposition 4.4.18, it is enough to prove the above two claims. We first prove Claim 4.4.20.

Proof of Claim 4.4.20. We set $H := \langle \sigma \rangle$ and $E := K^H$. Then E/F is a quadratic extension. We denote by ψ' the nontrivial character of $\mathrm{Gal}(E/F)$. Then we have

$$
\begin{aligned}
x_\psi \, \mathrm{pr}_\psi \, \omega_K \theta_{K/k,S} &= x_\psi \, \mathrm{pr}_\psi \, \omega_K L_S(K/F,\psi,0) \, \mathrm{pr}_\psi \\
&= x_\psi \omega_K L_S(E/F,\psi',0) \, \mathrm{pr}_{\psi'} \, Norm_{\langle \sigma \rangle} \\
&= x_\psi \omega_K \theta_{E/F,S} Norm_{\langle \sigma \rangle}.
\end{aligned}
$$

By [36, §3, case(c)], the Brumer-Stark conjecture is true for any relative quadratic extensions. Hence there exists an anti-unit $\alpha_1' \in E^*$ such that

$$
Norm_{\langle \sigma \rangle}(\mathfrak{A})^{\omega_E \theta_{E/k,S}} = (\alpha') \text{ and } E(\alpha'^{1/\omega_{E,l}})/F \text{ is abelian}, \tag{4.11}
$$

where $\omega_{E,l} = |\mu(E) \otimes \mathbb{Z}_l|$. We set

$$
\alpha_1 := \alpha'^{x_\psi \frac{\omega_{K,l}}{\omega_{E,l}}}.
$$

Then α_1 is an anti-unit in K^* and we have by (4.11)

$$
\mathfrak{A}^{x_\psi \, \mathrm{pr}_\psi \, \omega_K \theta_{K/k,S}} = \mathfrak{A}^{x_\psi \omega_K \theta_{E/F,S} Norm_{\langle \sigma \rangle}} = (\alpha_1).
$$

Moreover, we have

$$
K(\alpha_1^{1/\omega_{K,l}}) = K(\alpha'^{x_\psi 1/\omega_{E,l}}).
$$

Since $x_\psi \in \mathbb{Z}$, we can conclude by (4.11) that

$$
K(\alpha_1^{1/\omega_{K,l}})/F \text{ is abelian.}
$$

$\square$

Proof of Claim 4.4.21. (i) First, we suppose $l = 2$. In this case, by [11, Theorem 3.2], Proposition 4.4.18 holds for $p = 3$ and exactly the same proof works for any odd prime p if 2 does not split in $\mathbb{Q}(\zeta_p)$. Hence Proposition 4.4.18 holds in this case.

(ii) In what follows we assume l is odd. Let ψ be the irreducible character of $\mathrm{Gal}(K/F)$ which sends σ and j to 1 and -1, respectively. Then this character is the inflation of the nontrivial character ψ' of $\mathrm{Gal}(E/F)$, where $E = K^H$ and $H = \langle \sigma \rangle$. We put

$$
\begin{aligned}
A_K &:= \frac{1-j}{2}(Cl(K) \otimes \mathbb{Z}_l), \\
A_E &:= \frac{1-j}{2}(Cl(E) \otimes \mathbb{Z}_l).
\end{aligned}
$$

Then by Lemma 4.2.3, we get

$$
\begin{aligned}
|A_K| &= \omega_{K,l} L(0,\psi, K/F) \prod_{j=1}^{p-1} L(0,\phi^j, K/F) \\
&= \omega_{K,l} L(0,\psi', E/F) N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(L(0,\phi, K/F)), \tag{4.12} \\
|A_E| &= \omega_{E,l} L(0,\psi', E/F), \tag{4.13}
\end{aligned}
$$

where the equalities are considered as equalities of the l-part and ϕ. If $l \neq p$, $|A_E|=|A_K^H|$ since A_E is canonically isomorphic to A_K^H. If $l = p$, by [11, Lemma 2.5] (also see the errata [12]), we know that

$$
\begin{aligned}
|A_K^H| &\geq \frac{1}{p}|A_E| \quad &&\text{if } \zeta_p \in K \text{ and } \omega_{K,p}/\omega_{E,p} = 1, \\
|A_K^H| &\geq |A_E| \quad &&\text{otherwise.}
\end{aligned}
$$

Since $x_{[\phi]} A_K^H = 0$, there is a natural surjection $A_K/A_K^H \twoheadrightarrow x_{[\phi]} A_K$. Hence we have

$$
\begin{aligned}
|x_{[\phi]} A_K| &\leq |A_K|/|A_K^H| \\
&\leq p^t |A_K|/|A_E| \\
&= p^t \frac{\omega_{K,l}}{\omega_{E,l}} N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(L(0,\phi, K/F)), \tag{4.14}
\end{aligned}
$$

54

where we set

$$t = 1 \qquad \text{if } l = p, \ \zeta_p \in K \text{ and } \omega_{K,p}/\omega_{E,p} = 1,$$
$$t = 0 \qquad \text{otherwise.}$$

Since the minus part of $\mathbb{Q}_l[G]$ is isomorphic to $\mathbb{Q}_l[H]$ by sending j to -1, in what follows, we identify the minus part of $\mathbb{Q}_l[G]$ with $\mathbb{Q}_l[H]$ just like [11, §2] (for example $\theta_{K/F}$ will be regarded as an element of $\mathbb{Q}_l[H]$ not of $\mathbb{Q}_l[G]$).

Case I. $l \neq p$.

In this case, we have $t = 0$ and the equality holds in (4.14). Moreover, we have $\omega_{K,l}/\omega_{E,l} = 1$ and hence the elements $L(0, \phi^m, K/F)$ are contained in $\mathbb{Z}_l[\zeta_p]$. Since $l \neq p$, we get an isomorphism

$$\mathbb{Z}_l[H] \cong \bigoplus_{\eta \in \mathrm{Irr}\, H/\sim} \mathbb{Z}_l[\eta],$$

where η runs over all the irreducible characters of H modulo $\mathrm{Gal}(\mathbb{Q}_l(\zeta_p)/\mathbb{Q}_l)$-action. Hence we have

$$A_K/A_K^H \cong (\sum_{m=1}^{p-1} e_{\phi^m}) A_K \cong \bigoplus_{\eta \in \mathrm{Irr}\, H \backslash \{1\}/\sim} \mathbb{Z}_l[\eta] \otimes_{\mathbb{Z}[H]} A_K.$$

By assumption that l does not split in $\mathbb{Q}(\zeta_p)$, we actually have

$$A_K/A_K^H \cong \mathbb{Z}_l[\eta] \otimes_{\mathbb{Z}[H]} A_K. \tag{4.15}$$

By (4.14), we have

$$|x_{[\phi]} A_K| \leq |A_K/A_K^H| = |\mathbb{Z}_l[\eta] \otimes_{\mathbb{Z}[H]} A_K| \ \ = \ \ [\mathbb{Z}_l[\zeta_p] : (L(0, \phi, K/F))]$$
$$= \ \ [\mathbb{Z}_l[\zeta_p] : (\overline{\theta}_{K/F})], \tag{4.16}$$

where $\overline{\theta}_{K/F}$ is the image of $\theta_{K/F}$ under the surjection $\mathbb{Z}_l[H] \twoheadrightarrow \mathbb{Z}_l[\zeta_p]$. Since we

have

$$(1 + \sigma + \sigma^2 + \cdots \sigma^{p-1})x_{[\phi]} = 0,$$

we can regard $x_{[\phi]}A_K$ as a $\mathbb{Z}_l[\zeta_p]$-module through the natural surjection $\mathbb{Z}_l[H] \twoheadrightarrow \mathbb{Z}_l[\eta] = \mathbb{Z}_l[\zeta_p]$. Moreover, since $x_{[\phi]}A_K$ is a torsion module, there exist natural numbers $n_1, n_2, \ldots, n_k$ such that

$$x_{[\phi]}A_K \cong \bigoplus_{i=1}^{k} \mathbb{Z}_l[\zeta_p]/(l)^{n_i}.$$

Combining the above isomorphism with (4.16), we have

$$|x_{[\phi]}A_K| = |\bigoplus_{i=1}^{k} \mathbb{Z}_l[\zeta_p]/(l)^{n_i}| \le |\mathbb{Z}_l[\zeta_p]/(\bar{\theta}_{K/F})|.$$

This inequality implies that $\theta_{K/F}$ annihilates $x_{[\phi]}A_K$. Therefore, for any fractional ideal $\mathfrak{A}$ of K whose class in $Cl(L)$ is of l-power order, $\mathfrak{A}^{x_{[\phi]}\omega_K\theta_{K/F}} = (\alpha^{\omega_K})$ for some $\alpha \in K^*$ and clearly $K((\alpha^{\omega_K})^{1/\omega_{K,l}})/F$ is abelian. This completes the proof of Proposition 4.4.18 in this case.

Case II. $l = p$ and $\zeta_p \notin K$.

In this case, we have $t = 0$. Hence by (4.14), we have

$$|x_{[\phi]}A_K| \le [\mathbb{Z}_p[\zeta_p] : (\bar{\theta}_{K/F})]. \tag{4.17}$$

Since $x_{[\phi]}A_K$ is a torsion module, there exists $n_1, n_2, \ldots, n_m \in \mathbb{N}$ such that

$$x_{[\phi]}A_K \cong \bigoplus_{i=1}^{m} \mathbb{Z}_p[\zeta_p]/(1 - \zeta_p)^{n_i}.$$

Combining this with (4.17), we have

$$|x_{[\phi]}A_K| = |\bigoplus_{i=1}^{m} \mathbb{Z}_p[\zeta_p]/(1 - \zeta_p)^{n_i}| \le |\mathbb{Z}_p[\zeta_p]/(\bar{\theta}_{K/F})|.$$

This implies $\theta_{K/F}$ annihilates $x_{[\phi]}A_K$. By the same argument as the final part of Case II, we obtain the conclusion in this case.

Case III. $l = p$ and $\zeta_p \in K$.

If $\omega_{K,p}/\omega_{E,p} = 1$, we have $t = 1$. Hence by (4.14),

$$
\begin{aligned}
|x_{[\phi]}A_K| &\leq p\frac{\omega_{K,p}}{\omega_{E,p}}N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(L(0,\phi,K/F)) \\
&= [\mathbb{Z}_p[\zeta_p] : (\zeta_p - 1)(\overline{\theta}_{K/F})].
\end{aligned}
$$

If $\omega_{K,p}/\omega_{E,p} \neq 1$, we have $t = 0$. We also see that $\omega_{K,p} = p^e$, $\omega_{E,p} = p^{e-1}$ for some $e \in \mathbb{N}$. Hence we have

$$
\begin{aligned}
|x_{[\phi]}A_K| &\leq \frac{\omega_{K,p}}{\omega_{E,p}}N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(L(0,\phi,K/F)) \\
&= [\mathbb{Z}_p[\zeta_p] : (\zeta_p - 1)(\overline{\theta}_{K/F})].
\end{aligned}
$$

In both cases we see that $(\sigma - 1)x_{[\phi]}\theta_{K/F}$ annihilates A_K. Hence for any fractional ideal $\mathfrak{A}$ of K whose class in $Cl(K)$ is of p-power order, there exists some $\beta \in K$ such that

$$
\mathfrak{A}^{\omega_{K,p}(\sigma-1)\theta_{K/F}x_{[\phi]}} = (\beta).
$$

In the last paragraph of [11, Proposition 2.2], the authors show that if $(\sum_{j=0}^{p-1}\sigma^j)\theta_{K/F} = 0$, there exists $\alpha \in \mathbb{Z}_p[H]$ such that

$$
p^e\theta_{K/F} = (\sigma - 1)\alpha\gamma\theta_{K/F},
$$

where $\gamma = \sigma^{p-1} + g\sigma^{p-2} + \cdots + g^{p-1}$ and g is the minimal positive integer which represents the action of σ on the p^eth-power root of unity in K. Since $(\sum_{j=0}^{p-1}\sigma^j)x_{[\phi]}\theta_{K/F} = 0$, replacing $\theta_{K/F}$ by $x_{[\phi]}\theta_{K/F}$, we get

$$
p^e x_{[\phi]}\theta_{K/F} = (\sigma - 1)\alpha\gamma x_{[\phi]}\theta_{K/F}
$$

for some $\alpha \in \mathbb{Z}_p[H]$. This implies

$$\mathfrak{A}^{p^e \theta_{K/F} x_{[\phi]}} = (\beta^{\alpha\gamma}).$$

To conclude the proof of Case IV, we use the following proposition:

Proposition 4.4.22 (Proposition 1.2, [35])**.** *Let L/k be an arbitrary abelian extension of number fields with Galois group G, $\{\sigma_i\}_{i \in I}$ be a system of generators of G, ζ be a primitive ω_Lth - root of unity. We suppose σ_i acts on ζ as $\zeta^{\sigma_i} = \zeta^{n_i}$. We take an element $\beta \in F$. Then for any natural number m, the following statement is equivalent to the condition that $F(\beta^{1/m})/K$ is abelian:*
There exists a system $\{\beta_i\}_{i \in I} \subset E_F$ such that

$$\begin{aligned}
\alpha_i^{\sigma_j - n_j} &= \alpha_j^{\sigma_i - n_i} \quad \text{for any } i, \, j \in I, \\
\beta^{\sigma_i - n_i} &= \alpha_i^m \qquad \quad \text{for any } i \in I.
\end{aligned}$$

Applying this proposition to our setting, we have

$$K((\beta^{\alpha\gamma})^{1/p^e})/F \text{ is abelian} \quad \text{if and only if} \quad \text{there exists } \alpha \in E_K \text{ such that}$$
$$(\beta^{\alpha\gamma})^{\sigma - g} = \alpha^{p^e}.$$

Since $(\beta^{\alpha\gamma})^{\sigma - g} = (\beta^\alpha)^{1 - g^p}$ and $1 - g^p$ is divisible by p^e, we can conclude that $K((\beta^{\alpha\gamma})^{1/p^e})/F$ is abelian. $\qquad\square$

$\qquad\square$

Improvement of the 2-part

As an application of Theorem 3.2.1, in the previous section we prove the 2-part of the *weak* Brumer-Stark conjecture for extensions with group D_{4p}. If we use the fact that $\mathbb{Z}_2[D_{4p}]$ is a nice Fitting order, we can improve the result. More precisely we can prove the following:

Theorem 4.4.23 ([28], Theorem 5.2)**.** *Let K/k be a finite Galois CM-extensions of number fields whose Galois group is isomorphic to the dihedral group of order*

$4p$. *Then if the prime* 2 *does not split in* $\mathbb{Q}(\zeta_p)$, *the* 2-*part of the (non-weak) non-abelian Brumer-Stark conjecture is true.*

Before proving this theorem, we prove the following:

Proposition 4.4.24 ([28], Proposition 5.1). *Let* K/k *be a finite Galois extension of number fields whose Galois group* G *is isomorphic to* D_{4p}. *We take two finite sets* S *and* T *of places of* k *such that* $Hyp(S,T)$ *is satisfied. Then we have*

$$\theta^T_{K/k,S} = \theta^T_{K^{ab}/k,S}\frac{1}{p}Norm_{G'} + \theta^{T_{k_\phi}}_{K/k_\phi,S_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}),$$

where G' *is the commutator subgroup of* G. *Moreover,* $\theta^T_{K/k,S}$ *belongs to* $\zeta(\mathbb{Z}_2[G])$.

Proof. Recalling that Artin L-functions do not change by the inflation of characters, we have by (2.5) and (4.3)

$$
\begin{aligned}
\chi_1(\theta^T_{K/k,S}) &= \prod_{\mathfrak{p}\in T}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-1}N\mathfrak{p}|V_{\chi_1})L_S(K/k,\check{\chi}_1,0)e_{\chi_1}\\
&= \prod_{\mathfrak{p}\in T}\det(1-\mathrm{Frob}_{\mathfrak{P}}^{-1}N\mathfrak{p}|V_{\chi_1^{ab}})L_S(K^{ab}/k,\check{\chi}_1^{ab},0)e_{\chi_1^{ab}}\frac{1}{p}Norm_{G'}\\
&= \chi_1^{ab}(\theta^T_{K^{ab}/k,S})\frac{1}{p}Norm_{G'}. \qquad\qquad (4.18)
\end{aligned}
$$

The same is true for χ_3, that is, we have

$$\chi_3(\theta^T_{K/k,S}) = \chi_3^{ab}(\theta^T_{K^{ab}/k,S})\frac{1}{p}Norm_{G'}. \qquad\qquad (4.19)$$

Since χ_1^{ab} and χ_3^{ab} are the only odd characters of $\mathrm{Gal}(K^{ab}/k)$, we have by (4.18) and (4.19)

$$
\begin{aligned}
\chi_1(\theta^T_{K/k,S})e_{\chi_1} + \chi_3(\theta^T_{K/k,S})e_{\chi_3} &= (\chi_1^{ab}(\theta^T_{K^{ab}/k,S})e_{\chi_1^{ab}}\\
&\quad + \chi_3^{ab}(\theta^T_{K^{ab}/k,S})e_{\chi_3^{ab}})\frac{1}{p}Norm_{G'}\\
&= \theta^T_{K^{ab}/k,S}\frac{1}{p}Norm_{G'}.
\end{aligned}
$$

Next we compute $\chi_{2m+3}(\theta^T_{K/k,S})$ for $m = 1, 2, \ldots, (p-1)/2$. By the induction formula of Artin L-functions, we have

$$
\begin{aligned}
\chi_{2m+3}(\theta^T_{K/k,S}) &= \prod_{\mathfrak{p} \in T} \det(1 - \mathrm{Frob}_{\mathfrak{P}}^{-1} N\mathfrak{p} | V_{\chi_{2m+3}}) L_S(K/k, \chi_{2m+3}, 0) e_{\chi_{2m+3}} \\
&= \prod_{\mathfrak{p}_\phi \in T_{k_\phi}} \det(1 - \mathrm{Frob}_{\mathfrak{P}}^{-f_{\mathfrak{p}_\phi}} N\mathfrak{p}_\phi | V_{\phi^m}) L_{S_{k_\phi}}(K/k_\phi, \phi^m, 0) e_{\chi_{2m+3}},
\end{aligned}
\tag{4.20}
$$

where $f_{\mathfrak{p}_\phi}$ is the residue degree of $\mathfrak{p}_\phi$. We recall that $\chi_{2m+3} = \mathrm{Ind}_{\langle j\sigma \rangle}^G \phi^m = \mathrm{Ind}_{\langle j\sigma \rangle}^G \phi^{-m}$. Then we have

$$
L_{S_{k_\phi}}(K/k_\phi, \phi^m, 0) = L_{S_{k_\phi}}(K/k_\phi, \phi^{-m}, 0),
$$

$$
\prod_{\mathfrak{p}_\phi \in T_{k_\phi}} \det(1 - \mathrm{Frob}_{\mathfrak{P}}^{-f_{\mathfrak{p}_\phi}} N\mathfrak{p}_\phi | V_{\phi^m}) = \prod_{\mathfrak{p}_\phi \in T_{k_\phi}} \det(1 - \mathrm{Frob}_{\mathfrak{P}}^{-f_{\mathfrak{p}_\phi}} N\mathfrak{p}_\phi | V_{\phi^{-m}})
$$

and by Lemma 2.2.2 $e_{\chi_{2m+3}} = e_{\phi^m} + e_{\phi^{-m}}$. These equations imply that

$$
\chi_{2m+3}(\theta^T_{K/k,S}) e_{\chi_{2m+3}} = \theta^{T_{k_\phi}}_{K/k_\phi, S_{k_\phi}} e_{2m+3}
$$

and hence

$$
\sum_{m=1}^{\frac{p-1}{2}} \chi_{2m+3}(\theta^T_{K/k,S}) e_{\chi_{2m+3}} = \sum_{m=1}^{\frac{p-1}{2}} \theta^{T_{k_\phi}}_{K/k_\phi, S_{k_\phi}} e_{2m+3} = \theta^{T_{k_\phi}}_{K/k_\phi, S_{k_\phi}} \left(\sum_{m=1}^{\frac{p-1}{2}} e_{2m+3} \right).
$$

Combining this with (4.20), we get the first claim of Proposition 4.4.24.

Since K^{ab}/k is an abelian extension, $\theta^T_{K^{ab}/k, S}$ belongs to $\mathbb{Z}[G/G']$. Therefore, we see that $\theta^T_{K^{ab}/k, S} \frac{1}{p} Norm_{G'}$ belongs to $\mathbb{Z}_2[G]$. Next we show that $\theta^{T_{k_\phi}}_{K/k_\phi, S_{k_\phi}} (\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}})$ belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. First we write ψ for the character of $\mathrm{Gal}(K/k_\phi)$ which sends σ and j to 1 and -1, respectively. Since K/k_ϕ is

also an abelian extension, $\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}$ belongs to $\mathbb{Z}[\mathrm{Gal}(K/k_\phi)]$. Moreover, we have

$$
\begin{aligned}
\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} &= \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} e_\psi + \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \Big(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}\Big) \\
&= \theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p} Norm_{G'} + \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \Big(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}\Big).
\end{aligned}
$$

Since K^{ab}/k_ϕ is an abelian extension, $\theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}}$ belongs to $\mathbb{Z}[\mathrm{Gal}(K^{ab}/k_\phi)]$. Hence $\theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p} Norm_{G'}$ belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. Therefore, we see that

$$
\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \Big(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi_{2m+3}}\Big) = \theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}} - \theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p} Norm_{G'}
$$

belongs to $\mathbb{Z}_2[\mathrm{Gal}(K/k_\phi)]$. The above arguments imply that $\theta_{K/k,S}^{T}$ belongs to $\mathbb{Z}_2[G]$, in particular, to $\zeta(\mathbb{Z}_2[G])$. $\qquad\square$

Proof of Theorem 4.4.23. First we take two finite sets S and T of places of k such that $Hyp(S,T)$ is satisfied. Then it is enough to show the following two statements by Proposition 4.4.9:

$$
\theta_{K/k,S}^{T} \in \mathcal{I}_p(G) \text{ and } \mathcal{H}_2(G)\theta_{K/k,S}^{T} \subset \mathrm{Ann}_{\mathbb{Z}_2[G]}(Cl(K)_2^{T_K}).
$$

Since $\mathbb{Z}_2[G]$ is a nice Fitting order, this is equivalent to

$$
\theta_{K/k,S}^{T} \in \zeta(\mathbb{Z}_p[G]) \text{ and } \theta_{K/k,S}^{T} \subset \mathrm{Ann}_{\mathbb{Z}_2[G]}(Cl(K)_2^{T_K})
$$

by Proposition 2.2.11. The claim $\theta_{K/k,S}^{T} \in \zeta(\mathbb{Z}_p[G])$ is true by Proposition 4.4.24. Hence we only have to show $\theta_{K/k,S}^{T}$ annihilates $Cl(K)_2^{T_K}$. We have by Proposition

4.4.24

$$\theta_{K/k,S}^T(Cl(K)_2^{T_K}) \; = \; \theta_{K^{ab}/k,S}^T \frac{1}{p}Norm_{G'}(Cl(K)_2^{T_K})$$

$$+\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi 2m+3})(Cl(K)_2^{T_K}).$$

By [32, Theorem 2.1], the Brumer-Stark conjecture is true for biquadratic extensions and hence true for K^{ab}/k. Observing that

$$\frac{1}{p}Norm_{G'}(Cl(K)_2^{T_K}) \subset Cl(K^{ab})_2^{T_{K^{ab}}},$$

we have

$$\theta_{K^{ab}/k,S}^T \frac{1}{p}Norm_{G'}(Cl(K)_2^{T_K}) = 0. \qquad (4.21)$$

By [11, Theorem 3.2], the 2-part of the Brumer-Stark conjecture is true for cyclic extensions of degree 6. If 2 does not split in $\mathbb{Q}(\zeta_p)$, exactly the same proof works for cyclic extensions of degree $2p$. Hence we have

$$\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(Cl(K)_2^{T_K}) \; = \; \theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p}Norm_{G'}(Cl(K)_2^{T_K})$$

$$+\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi 2m+3})(Cl(K)_2^{T_K}) = 0.$$

By [36, §3, case(c)], the Brumer-Stark conjecture is true for quadratic extensions and hence true for K^{ab}/k_ϕ. Therefore, we have

$$\theta_{K^{ab}/k_\phi,S_{k_\phi}}^{T_{k_\phi}} \frac{1}{p}Norm_{G'}(Cl(K)_2^{T_K}) = 0$$

and hence

$$\theta_{K/k_\phi,S_{k_\phi}}^{T_{k_\phi}}(\sum_{m=1}^{\frac{p-1}{2}} e_{\chi 2m+3})(Cl(K)_2^{T_K}) = 0.$$

Combining this with (4.21), we have

$$\theta_{K/k,S}^T(Cl(K)_2^{T_K}) = 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

4.4.3 Extensions with group $Q_{2^{n+2}}$

Let K/k be a finite Galois extension whose Galois group is isomorphic to the quaternion group $Q_{2^{n+2}}$ of order 2^{n+2}. We use the presentation $Q_{2^{n+2}} = \langle x, y \mid x^{2^n} = y^2, \ x^{2^{n+1}} = 1, \ yxy^{-1} = x^{-1}\rangle$. Since the center of $Q_{2^{n+2}}$ is $\{1, x^{2^n}\}$, x^{2^n} corresponds to the unique complex conjugation j.

Characters of $Q_{2^{n+2}}$

$Q_{2^{n+2}}$ has two types of irreducible characters. One type is given through the natural surjection $Q_{2^{n+2}} \twoheadrightarrow Q_{2^{n+2}}/\langle x^n \rangle \simeq D_{2^{n+1}}$. Clearly, characters which are given in this way are even characters. The other type is two dimensional characters which are induced by the faithful odd characters of $\langle x \rangle$ (in fact, a character of $\langle x \rangle$ is faithful if and only if it is odd). Let ϕ be the character of $\langle x \rangle$ which sends x and x^n to $\zeta_{2^{n+1}}$ and -1 respectively. Then all faithful odd characters are of the form ϕ^m for $m \in (\mathbb{Z}/2^{n+1}\mathbb{Z})^*$. We set $\chi_m := \mathrm{Ind}_{\langle x \rangle}^{Q_{2^{n+2}}} \phi^m$. Then we have $\chi_m = \chi_{-m}$ and $k_m = k^{\langle x \rangle}$ for all m. Since ϕ^m is faithful, we conclude $K_{m,1} = K_{m,2} = K$.

Main Theorem

In this subsection, we prove the following:

Theorem 4.4.25. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to $Q_{2^{n+2}}$ and S be a finite set of places of k which contains all infinite places. Then the 2-part of the weak non-abelian Brumer conjecture and the weak non-abelian Brumer-Stark conjecture are true for K/k and S.*

Remark 4.4.26. (1) If no prime above p splits in K/K^+ whenever $K^{cl} \subset (K^{cl})^+(\zeta_p)$, the odd p-part of the above result holds by [25, Corollary 4.2].

(2) Since all the subgroups of $Q_{2^{n+2}}$ are normal and all the odd representations are faithful, $\theta_{K/k,S_\infty \cup S_{ram}}$ always coincides with $\theta_{K/k,S_\infty}$.

The observation in the previous section tells us that we have to verify the 2-part of the weak Brumer-Stark conjecture for $K/K^{\langle x \rangle}$. Since $K/K^{\langle x \rangle}$ is a cyclic extension of degree 2^{n+1}, it is enough to prove the following:

Proposition 4.4.27. *Let K/F be a cyclic CM-extension of degree 2^{n+1}. We assume F contains k so that (F/k is quadratic and) K/k is CM with Galois group $Q_{2^{n+2}}$. Then 2-part of the weak non-abelian Brumer-Stark conjecture is true for K/F.*

Before proving the above theorem, we prove the following lemma:

Lemma 4.4.28. *Let K/F be a cyclic CM-extension of degree 2^{n+1} which is contained in some $Q_{2^{n+2}}$-extension. Then all the roots of unity in K are ± 1.*

Proof of Lemma 4.4.28. Let ζ be a primitive ω_Kth roots of unity in K and assume $x(\zeta) = \zeta^{c_x}$ and $y(\zeta) = \zeta^{c_y}$ for some c_x, $c_y \in (\mathbb{Z}/\omega_K\mathbb{Z})^*$. Then we have $yxy^{-1}(\zeta) = \zeta^{c_y^{-1}c_x c_y} = \zeta^{c_x}$. On the other hand $yxy^{-1} = x^{-1}$, so we have $yxy^{-1}(\zeta) = \zeta^{c_x^{-1}}$. Hence we see that

$$c_x \equiv c_x^{-1} \mod \omega_K \Leftrightarrow c_x^2 \equiv 1 \mod \omega_K.$$

Therefore, we have $x^2(\zeta) = \zeta$ and hence $x^{2n}(\zeta) = \zeta$. This implies ζ lies in K^+. $\square$

Proof of Proposition 4.4.27. We define the group I_K^+ of the ambiguous ideals by

$$I_K^+ := \{\mathfrak{A} \mid \mathfrak{A} \text{ is an ideal of } K \text{ such that } \mathfrak{A}^j = \mathfrak{A}\},$$

where j is the unique complex conjugation in $\mathrm{Gal}(K/F)$. Also we define $A_K := \mathrm{Coker}(I_K^+ \to Cl(K)) \otimes \mathbb{Z}_2$. Then by Sands's formula [32, Proposition 3.2] (also see [11, §3]), we have

$$\omega_K \theta_{K^+/K} = 2^{[K^+:\mathbb{Q}]+d-2}|A_K|(1-j) \mod \mathbb{Z}_2^*, \qquad (4.22)$$

where d is the number of primes of K^+ which ramify in K. Let ξ be the non-trivial character of $\mathrm{Gal}(K/K^+)$. We set $M := \{a \mid 1 \leq a \leq 2^{n+1}, \ a \text{ is odd}\}$. Then we have $\mathrm{Ind}_{\mathrm{Gal}(K/K^+)}^{H}(\xi) = \sum_{m \in M} \phi^m$ and

$$\xi(\theta_{K/K^+}) = L(0, \xi, K/K^+) = \prod_{m \in M} L(0, \phi^m, K/F).$$

By (4.22), we have

$$
\begin{aligned}
|A_K| &= \omega_K \xi(\theta_{K/K^+}) 2^{-[K^+:\mathbb{Q}]-d+1} \\
&= \omega_K \prod_{m \in M} L(0, \phi^m, K/F) 2^{-[K^+:\mathbb{Q}]-d+1},
\end{aligned}
$$

where the equality is used in the sense that the 2-parts of the both sides coincide. Since $\omega_{K,2} = 2$ by Lemma 4.4.28, we also have

$$|A_K| = \prod_{m \in M} L(0, \phi^m, K/F) 2^{-[K^+:\mathbb{Q}]-d+2}.$$

Since $[K^+ : \mathbb{Q}] \geq 2^{n+1}$ (recalling that K/F is contained in some $Q_{2^{n+2}}$-extension), we get $-[K^+ : \mathbb{Q}] - d + 2 \leq -2^{n+1} + 2$. Hence we also get

$$
\begin{aligned}
|A_K| &\leq \prod_{m \in M} L(0, \phi^m, K/F) 2^{-2^{n+1}+2} \\
&= \frac{4}{2^{2^n}} N_{\mathbb{Q}(\zeta_{2^{n+1}})/\mathbb{Q}}\left(\frac{L(0, \phi, K/F)}{2}\right) \\
&\leq N_{\mathbb{Q}(\zeta_{2^{n+1}})/\mathbb{Q}}\left(\frac{L(0, \phi, K/F)}{2}\right) \quad\quad (4.23)
\end{aligned}
$$

and the right hand side of the last inequality lies in $\mathbb{Z}_2$ and hence $(1/2)L(0, \phi, K/F)$ lies in $\mathbb{Z}_2[\zeta_{2^{n+1}}]$. We take an element $x \in \mathfrak{F}_2(\mathrm{Gal}(K/F))$ of the form

$$x = \sum_{m \in M} x_{\phi^m} \, \mathrm{pr}_{\phi^m}, \ \ x_{\phi^m} \in \mathfrak{D}^{-1}(\mathbb{Q}_2(\phi^m)/\mathbb{Q}_2).$$

Since we have

$$Norm_{\mathrm{Gal}(K/F)} x = 0,$$

65

we can regard the module xA_K as a $\mathbb{Z}_2[\zeta_{2^{n+1}}]$-module. Then we have by (4.23)

$$|xA_K| \le [\mathbb{Z}_2[\zeta_{2^{n+1}}] : ((1/2)L(0, \phi, K/F))] = [\mathbb{Z}_2[\zeta_{2^{n+1}}] : ((1/2)\overline{\theta}_{K/F})],$$

where $\overline{\theta}_{K/F}$ is the image of $\theta_{K/F}$ under the surjection $\mathbb{Z}_2[H] \twoheadrightarrow \mathbb{Z}_2[\zeta_{2^{n+1}}]$. This implies $(1/2)x\theta_{K/F}$ annihilates A_K. Then for any fractional ideal $\mathfrak{A}$ of K whose class in $Cl(K)$ is of 2-power order, we have that $\mathfrak{A}^{(1/2)x\theta_{K/F}}$ lies in $P_K \cdot I_K^+$, where P_K is the group of principal ideals of K and hence we have $\mathfrak{A}^{(1/2)x\theta_{K/F}(1-j)} = \mathfrak{A}^{x\theta_{K/F}}$ lies in P_K^{1-j}. This completes the proof. $\qquad\square$

4.4.4 Extensions with group $\mathbb{Z}/2\mathbb{Z} \times A_4$

Let K/k be a finite Galois extension whose Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times A_4$, where A_4 is the alternating group on 4 letters. we regard A_4 as the group of even permutation of the set $\{1, 2, 3, 4\}$. Since the center of A_4 is trivial, the generator of $\mathbb{Z}/2\mathbb{Z}$ corresponds to the unique complex conjugation j.

Characters of $\mathbb{Z}/2\mathbb{Z} \times A_4$

We set $x = (12)(34)$ and $y = (123)$. The irreducible characters of $\mathbb{Z}/2\mathbb{Z} \times A_4$ are determined by the following character table, where $\{\cdot\}$ indicates conjugacy classes:

Table 4.2: The character table of $\mathbb{Z}/2\mathbb{Z} \times A_4$

	$\{1\}$	$\{x\}$	$\{yx\}$	$\{y^2x\}$	$\{j\}$	$\{jx\}$	$\{jyx\}$	$\{jy^2x\}$
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	ζ_3	ζ_3^2	1	-1	ζ_3	ζ_3^2
χ_4	1	1	ζ_3	ζ_3^2	-1	-1	$-\zeta_3$	$-\zeta_3^2$
χ_5	1	1	ζ_3^2	ζ_3	1	1	ζ_3^2	ζ_3
χ_6	1	1	ζ_3^2	ζ_3	-1	-1	$-\zeta_3^2$	$-\zeta_3$
χ_7	3	-1	0	0	3	-1	0	0
χ_8	3	-1	0	0	-3	1	0	0

From the above table, we see that the only odd characters are χ_2, χ_4, χ_6 and

χ_8. Since $\ker \chi_2$ has index 2, the corresponding subextension K_2/k is a quadratic extension, and since we have $\ker \chi_4 = \ker \chi_6$ and this subgroup has index 6, we have $K_4 = K_6$ and K_4/k is a cyclic extension of degree 6. Let V be Klein subgroup of A_4 and $\phi_{8,1}$, $\phi_{8,2}$ and $\phi_{8,3}$ be characters of $\mathbb{Z}/2\mathbb{Z} \times V$ whose restriction to V are non-trivial. Then we have $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z} \times V}^{\mathbb{Z}/2\mathbb{Z} \times A_k}(\phi_{8,i}) = \chi_8$ for $i = 1$, 2, 3 and the indices of their kernel in $\mathbb{Z}/2\mathbb{Z} \times V$ are 2. Hence we see that $k_8 = K^{\mathbb{Z}/2\mathbb{Z} \times V}$ and $K_{8,i}/k_8$ is a quadratic extension for all i.

Proof of conjectures for extensions with group $\mathbb{Z}/2\mathbb{Z} \times A_4$

In this subsection, we prove the following theorem:

Theorem 4.4.29. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times A_4$ and S be a finite set of places of k which contains all infinite places. Then*

(1) *for each odd prime l apart from 3 which does not split in $\mathbb{Q}(\zeta_3)$, the l-part of the non-abelian Brumer and Brumer-Stark conjecture are true for K/k and S,*

(2) *the 2-part and the 3-part of the weak non-abelian Brumer and Brumer-Stark conjectures are true for K/k and S.*

Remark 4.4.30. If S contains all finite places of k which ramify in K, we know the following as well as Theorem 4.4.16:
(1) In the case of $k = \mathbb{Q}$, the above result except the 2-part is contained in Nickel's work [22], [24] if we assume $\mu = 0$.
(2) If no prime above p splits in K/K^+ whenever $K^{cl} \subset (K^{cl})^+(\zeta_p)$, the above result holds for odd p by [25, Corollary 4.2].

The observation in the previous subsection tells us that we have only to verify the Brumer-Stark conjecture for K_2/k, K_4/k and $K_{8,i}/k_8$ for $i = 1$, 2, 3. By [36, §3, case(c)], the Brumer-Stark conjecture is true for any relative quadratic extensions and hence true for extensions K_2/k, $K_{8,i}/k_8$. In order to complete the proof of Theorem 4.4.29, it is enough to prove the following proposition:

Proposition 4.4.31. *Let l be a prime which does not split in $\mathbb{Q}(\zeta_3)$. Let F/k be any cyclic CM-extension of number fields of degree 6. Then the l-part of the weak non-abelian Brumer-Stark conjecture is true for F/k.*

Proof. Exactly the same proof as Proposition 4.4.18 works. $\qquad\qquad\square$

4.5   Numerical examples

In this section, we give some numerical examples for the non-abelian Brumer conjecture. Throughout this section, we use the same notation as in §4.4.2

4.5.1   Reduced norms of $\mathbb{Q}_p[D_{12}]$

We fix a prime p. In this section, we review the way how to compute the reduced norm of $\mathbb{Q}_p[D_{12}]$.

From Table 4.1, we see all the 1-dimensional representations of D_{12}. We set

$$\rho_{\chi_4}(\sigma) := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \; \rho_{\chi_4}(\tau) := \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \; \rho_{\chi_4}(j) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho_{\chi_5}(\sigma) := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \; \rho_{\chi_5}(\tau) := \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \; \rho_{\chi_5}(j) := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can easily see that these determine all the 2-dimensional representations (there is no deep reason we choose these forms). We also set $\rho_{\chi_i} := \chi_i$ for $i = 0, \, 1, \, 2, \, 3$. Then we have

$$\mathbb{Q}_p[D_{12}] \xrightarrow{\sim} \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus M_2(\mathbb{Q}_p) \oplus M_2(\mathbb{Q}_p), \; \alpha \mapsto \oplus\rho_{\chi_i}(\alpha).$$

The reduced norm map is defined by the following composition map:

$$\mathbb{Q}_p[D_{12}] \xrightarrow[\sim]{\oplus\rho_{\chi_i}} \bigoplus_{i=0}^{5} M_{\chi_i(1)}(\mathbb{Q}_p) \xrightarrow{\oplus \det} \bigoplus_{i=0}^{5} \mathbb{Q}_p \xrightarrow[\sim]{(\oplus\rho_{\chi_i})^{-1}} \zeta(\mathbb{Q}_p[D_{12}]).$$

Take an element

$$\alpha = A + B\sigma + C\sigma^2 + D\tau + E\sigma\tau + F\sigma^2\tau + Gj + H\sigma j + I\sigma^2 j + J\tau j + K\sigma\tau j + L\sigma^2\tau j$$

in $\mathbb{Q}_p[D_{12}]$. Then the coefficient of the identity of D_{12} is

$$\frac{1}{3}(A + 2A^2 + B - 2AB + 2B^2 + C - 2AC - 2BC + 2C^2 - 2D^2 + 2DE - 2E^2$$
$$+2DF + 2EF - 2F^2 + 2G^2 - 2GH + 2H^2 - 2GI - 2HI + 2I^2 - 2J^2 + 2JK$$
$$-2K^2 + 2JL + 2KL - 2L^2).$$

4.5.2 Stickelberger elements for D_{12}-extensions

We assume K/k is a finite Galois CM-extension whose Galois group G is isomorphic to D_{12}. As we observed in §4.4.2, D_{12} is the direct sum of $\mathbb{Z}/2\mathbb{Z}$ and $D_6 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ (D_6 coincides with the symmetric group $\mathfrak{S}_3$ of degree 3). As we have seen in §4.4.2, the only odd characters of D_{12} are χ_1, χ_3 and χ_5. By the definition of the Stickelberger elements, we have

$$
\begin{aligned}
\theta_{K/k,S} &= L_S(K/k, \chi_1, 0)e_{\chi_1} + L_S(K/k, \chi_3, 0)e_{\chi_3} + L_S(K/k, \chi_5, 0)e_{\chi_5} \\
&= \epsilon_{\chi_1,S} L_{S_\infty}(K/k, \chi_1, 0)e_{\chi_1} + \epsilon_{\chi_3,S} L_{S_\infty}(K/k, \chi_3, 0)e_{\chi_3} \\
&\quad + \epsilon_{\chi_5,S} L_{S_\infty}(K/k, \chi_5, 0)e_{\chi_5},
\end{aligned}
$$

where we set

$$\epsilon_{\chi_i,S} = \lim_{s \to 0} \prod_{\mathfrak{p} \in S \setminus S_\infty} \det(1 - \mathrm{Frob}_{\mathfrak{P}} N\mathfrak{p}^{-s} \mid V_{\chi_i}^{I_{\mathfrak{P}}}).$$

For $i = 1$, 3, we set $K_i := K^{\ker \chi_i}$ and write χ_i' for the character of $\mathrm{Gal}(K_i/k)$ whose inflation to G is χ_i. Then

$$
\begin{aligned}
\theta_{K/k,S} &= \epsilon_{\chi_1,S} L_{S_\infty}(K_1/k, \chi_1', 0)e_{\chi_1} + \epsilon_{\chi_3,S} L_{S_\infty}(K_3/k, \chi_3', 0)e_{\chi_3} \\
&\quad + \epsilon_{\chi_5,S} L_{S_\infty}(K/k_\phi, \phi, 0)e_{\chi_5} \\
&= \epsilon_{\chi_1,S} \chi_1'(\theta_{K_1/k})e_{\chi_1} + \epsilon_{\chi_3,S} \chi_3'(\theta_{K_3/k})e_{\chi_3} + \epsilon_{\chi_5,S} \phi(\theta_{K/k_\phi})e_{\chi_5}.
\end{aligned}
$$

$$(4.24)$$

This is a special case of [26, Lemma 3.1].

4.5.3 Integrality of Stickelberger elements

In the case that K/k is an abelian CM-extension, the first claim of Conjecture 4.4.1 is equivalent to

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))\theta_{K/k,S} \subset \mathbb{Z}_p[G]. \tag{4.25}$$

Hence one may expect the same strong integrality $\mathfrak{A}_S\theta_{K/k,S} \subset \zeta(\mathbb{Z}_p[G])$ holds even if G is non-abelian. However, the following example tells us that it is reasonable to conjecture that $\mathfrak{A}_S\theta_{K/k,S}$ is contained in $\mathcal{I}_p(G)$ but not in $\zeta(\mathbb{Z}_p[G])$.

Let α be a root of the cubic equation $x^3 - 11x + 7 = 0$ and set $K = \mathbb{Q}(\sqrt{-3}, \sqrt{4001}, \alpha)$. Then $K/\mathbb{Q}$ is a finite Galois CM-extension, K contains the 3rd roots of unity and its Galois group is isomorphic to

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{4001}, \alpha)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3 \cong D_{12}.$$

Using the same notations as §4.5.2, we see that

$$K_1 = \mathbb{Q}(\sqrt{-3}), \ K_3 = \mathbb{Q}(\sqrt{-12003}) \text{ and } k_\phi = \mathbb{Q}(\sqrt{4001}).$$

The only primes which ramify in $K/\mathbb{Q}$ are 3 and 4001. If we suitably choose the primes $\mathfrak{P}_3$ and $\mathfrak{P}_{4001}$ of K above 3 and 4001, we see that

$$G_{\mathfrak{P}_3} = \mathrm{Gal}(K/\mathbb{Q}(\alpha)) \cong \langle j \rangle \times \langle \tau \rangle, \quad I_{\mathfrak{P}_3} = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{4001}, \alpha)) \cong \langle j \rangle,$$
$$G_{\mathfrak{P}_{4001}} = \mathrm{Gal}(K/\mathbb{Q}(\alpha)) \cong \langle j \rangle \times \langle \tau \rangle, \quad I_{\mathfrak{P}_{4001}} = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-3}, \alpha)) \cong \langle \tau \rangle.$$

From this, we have

$$
\begin{aligned}
\epsilon_{\chi_1, S_{ram}} &= \lim_{s \to 0} \prod_{p \in S_{ram}} \det(1 - \mathrm{Frob}_{\mathfrak{P}} \, p^{-s} \mid V_{\chi_1}^{I_{\mathfrak{P}}}) \\
&= \lim_{s \to 0} \det(1 - \mathrm{Frob}_{\mathfrak{P}_{4001}} \, 4001^{-s} \mid V_{\chi_1}^{\langle \tau \rangle}) \\
&= \lim_{s \to 0} \det(1 - j 4001^{-s} \mid V_{\chi_1}) = 2.
\end{aligned}
$$

By the same way, we also have $\epsilon_{\chi_3, S_{ram}} = 1$ and $\epsilon_{\chi_5, S_{ram}} = 2$. By PARI/GP, we can compute L-values attached to χ_1, χ_3 and χ_5 as

$$
L_{S_\infty}(K_1/\mathbb{Q}, \chi_1', 0) = \frac{1}{3}, \ \ L_{S_\infty}(K_3/\mathbb{Q}, \chi_3', 0) = 30, \ \text{ and } \ L_{S_\infty}(K/k_\phi, \phi, 0) = 48.
$$

Hence we see from (4.24) that

$$
\theta_{K/\mathbb{Q}} = \frac{2}{3} e_{\chi_1} + 30 e_{\chi_3} + 96 e_{\chi_5} = \frac{1}{9}(1-j)(311 - 121(\sigma + \sigma^2) - 22(\tau + \sigma\tau + \sigma^2\tau)). \tag{4.26}
$$

Take the prime 7. This prime is completely decomposed in K and $Hyp(S_{ram} \cup S_\infty, \{7\})$ is satisfied. Also we have

$$
\delta_{\{7\}} = \mathrm{nr}(1 - \mathrm{Frob}_{\mathfrak{P}_7}^{-1} \, 7) = \mathrm{nr}(1 - 7) = \mathrm{nr}(-6).
$$

Then

$$
\delta_{\{7\}} \theta_{K/\mathbb{Q}} = \frac{1}{3}(1-j)(3410 - 1774(\sigma + \sigma^2) + 44(\tau + \sigma\tau + \sigma^2\tau)). \tag{4.27}
$$

Obviously this element does not belong to $\zeta(\mathbb{Z}_3[G])$ and hence we can not expect the strong inclusion $\mathfrak{A}_S \theta_{K/k,S} \subset \zeta(\mathbb{Z}_p[G])$ in general. However, we actually have

$$
\delta_{\{7\}} \theta_{K/\mathbb{Q}} = \mathrm{nr}((1-j)(-\frac{71}{2} + \frac{1}{2}\sigma - 11\sigma^2 + 19\tau + \frac{13}{2}\sigma\tau + \frac{37}{2}\sigma^2\tau)) \in \mathcal{I}_3(G). \tag{4.28}
$$

As long as we see this example, it seems reasonable to conjecture $\mathfrak{A}_S \theta_{K/k,S} \subset \mathcal{I}_p(G)$. In fact, by [26, Lemma 4.1] (and [26, Lemma 3.11]), if G is isomorphic

to D_{4p}, we always have $\mathfrak{A}_S \theta_{K/k,S} \subset \mathcal{I}_p(G)$. Note that the preimage of $\delta_{\{7\}} \theta_{K/\mathbb{Q}}$ is found in an ad hoc way, and as far as the author knows, there are no theoretical approaches to find concrete preimages of Stickelberger elements.

We have seen where $\mathfrak{A}_S \theta_{K/k,S}$ should live. Then where does the Stickelberger element $\theta_{K/k,S}$ itself live? First we return to the case where G is abelian. Since $|\mu(K)|$ belongs to $\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K))$, we have by (4.25)

$$|\mu(K)| \theta_{K/k,S} \in \mathbb{Z}_p[G]$$

or equivalently,

$$\theta_{K/k,S} \in \frac{1}{|\mu(K)|} \mathbb{Z}_p[G].$$

This implies the denominator of $\theta_{K/k,S}$ is at most $|\mu(K)|$. In the case where G is non-abelian, we see by (4.26) that the denominator of $\theta_{K/k,S}$ can not be bounded by $|\mu(K)|$. However, if we believe the first claim of Conjecture 4.4.5, we have

$$\omega_K \theta_{K/k,S} \in \mathcal{I}_p(G)$$

and hence

$$\theta_{K/k,S} \in \langle \mathrm{nr}(\frac{1}{|\mu(K)|} H) \mid H \in M_n(\mathbb{Z}_p[G]), n \in \mathbb{N} \rangle_{\zeta(\mathbb{Z}_p[G])}. \qquad (4.29)$$

Namely, the first claim of Conjecture 4.4.5 predicts that the denominators of preimages are at most $|\mu(K)|$ (not the denominators of $\theta_{K/k,S}$ itself). In fact, by (4.28), we see that

$$\theta_{K/\mathbb{Q}} = \mathrm{nr}(\frac{1}{6}(1-j)(\frac{71}{2} - \frac{1}{2}\sigma + 11\sigma^2 - 19\tau - \frac{13}{2}\sigma\tau - \frac{37}{2}\sigma^2\tau)). \qquad (4.30)$$

The reduced norm map is not injective, but the explicit computation of the reduced norm in §4.5.1 tells us the preimages of $\theta_{K/\mathbb{Q}}$ does not belong to $\mathbb{Z}_3[G]$. More explicitly we see that the preimages of $\theta_{K/\mathbb{Q}}$ must belong to $(1/3)\mathbb{Z}_3[G] = (1/|\mu(K)|)\mathbb{Z}_3[G]$. If we set $L = \mathbb{Q}(\sqrt{-2}, \sqrt{33}, \beta)$ (β satisfies $\beta^3 - 9\beta + 3 = 0$), we

have $\mu(L) = \{\pm 1\}$, $\mathrm{Gal}(L/\mathbb{Q}) \cong D_{12}$ and as computed in [26, §5.1.3]

$$\theta_{L/\mathbb{Q}} = \frac{2}{3}(1-j)(1 + \sigma + \sigma^2 - \tau - \sigma\tau - \sigma^2\tau).$$

Since L does not contain non-trivial roots of unity, we expect $\theta_{L/\mathbb{Q}}$ itself belongs to $\mathcal{I}_3(D_{12})$. In fact, we have

$$\theta_{L/\mathbb{Q}} = \mathrm{nr}(2(1-j)(-1 + \sigma + \sigma^2 - \tau + \sigma\tau - \sigma^2\tau)).$$

As long as we see these numerical examples, in the non-abelian cases it seems that the direct influence of the existence of the group $\mu(K)$ does not appear in the denominators of the Stickelberger elements themselves but in those of the preimages of Stickelberger elements.

Finally, we introduce an example which tells us that Stickelberger elements can belong to $\zeta(\mathbb{Z}_p[G])$ even if $\mathbb{Z}_p[G]$ is not a nice Fitting order. We take a root γ of the cubic equation $x^3 - 12x + 13 = 0$ and set $M = \mathbb{Q}(\sqrt{-6}, \sqrt{29}, \gamma)$. By the same manner as the calculation of $\theta_{K/\mathbb{Q}}$, we see that

$$\epsilon_{\chi_1, S_{ram}} = \epsilon_{\chi_5, S_{ram}} = 0 \text{ and } \epsilon_{\chi_3, S_{ram}} = 1,$$

and by PARI/GP

$$L_{S_\infty}(M/\mathbb{Q}, \chi_3, 0) = 12.$$

Therefore, we have

$$\theta_{M/\mathbb{Q}} = 12e_{\chi_3} = \mathrm{pr}_{\chi_3}.$$

Obviously, this element belongs to $\zeta(\mathbb{Z}_3[G])$. Moreover, $\theta_{M/\mathbb{Q}}$ comes from the reduced norm. In fact, we have

$$\theta_{M/\mathbb{Q}} = \mathrm{nr}(\mathrm{pr}_{\chi_3}).$$

Since M does not contain non-trivial roots of unity, this is also an example of the inclusion (4.29).

4.5.4 Annihilation of ideal class groups

As we have seen in the previous section, the elements $\delta_T \theta_{K/k,S}$ have denominators in general. Therefore, they can not act on the ideal class groups just as they are. This is one of the main reason why we adopt $\mathfrak{F}_p(G)$ and $\mathcal{H}_p(G)$ (in the latter half of this section, we will see that this is not the only reason). In this section, we see how Stickelberger elements annihilate ideal class groups with concrete Galois extensions appearing in the previous section.

First we study $K/\mathbb{Q}$, where we recall $K = \mathbb{Q}(\sqrt{-3}, \sqrt{4001}, \alpha)$ with $\alpha^3 - 11\alpha + 7 = 0$. By PARI/GP, we can see the structure as an abelian group of the ideal class group of K as follows:

$$Cl(K) \cong \mathbb{Z}/180\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}, \ Cl(K)_3 \cong \mathbb{Z}/9\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}.$$

We denote by c_1 and c_2 the basis of $Cl(K)_3$ which is chosen in the computation of PARI/GP. Then also using PARI/GP, we see the Galois action on $Cl(K)_3$ as follows:

$$\begin{cases} \sigma(c_1) = 4c_1 + c_2, & \tau(c_1) = -c_1, & j(c_1) = -c_1, \\ \sigma(c_2) = 6c_1 + c_2, & \tau(c_2) = c_2, & j(c_2) = -c_2. \end{cases} \quad (4.31)$$

The above relations imply that $Cl(K)_3$ is generated by c_1 as a $\mathbb{Z}_3[G]$-module.

By Proposition 2.2.10, $\mathcal{H}_3(G)$ coincides with $\mathfrak{F}_3(G)$, and hence, by (2.7) each element x in $\mathcal{H}_3(G)$ is of the form

$$x = \sum_{\chi \in \mathrm{Irr}\, G} x_\chi \, \mathrm{pr}_\chi, \ x_\chi \in \mathbb{Z}_3.$$

Then we have

$$x\delta_{\{7\}}\theta_{K/\mathbb{Q}} = -4x_{\chi_1}\, \mathrm{pr}_{\chi_1} - 180x_{\chi_3}\, \mathrm{pr}_{\chi_3} + 3456x_{\chi_5}\, \mathrm{pr}_{\chi_5}.$$

Obviously this element belongs to $\zeta(\mathbb{Z}_3[D_{12}])$. Since 180 and 3456 are multiples of 9, we have

$$180x_{\chi_3}\, \mathrm{pr}_{\chi_3}\, c_1 = 3456x_{\chi_5}\, \mathrm{pr}_{\chi_5}\, c_1 = 0.$$

Moreover, we see by (4.31) that

$$\mathrm{pr}_{\chi_1} c_1 = (1-j)(1+\sigma+\sigma^2)(1+\tau)c_1 = (1-j)(1+\sigma+\sigma^2)(1-1)c_1 = 0.$$

Hence

$$x\delta_{\{7\}}\theta_{K/\mathbb{Q}}c_1 = 0.$$

Thus thanks to the denominator ideal $\mathcal{H}_p(G)$ (and the central conductor $\mathfrak{F}_p(G)$), $\delta_{\{7\}}\theta_{K/k,S}$ becomes an element in $\zeta(\mathbb{Z}_3[G])$ and annihilates $Cl(K)_3$. Then what will happen in the case where the Stickelberger elements have no denominators? If $\mathbb{Z}_p[G]$ is a nice Fitting order, we do not need $\mathcal{H}_p(G)$. However, the following calculation tells us that we need $\mathcal{H}_p(G)$ in general.

We study $M/\mathbb{Q}$, where we recall $M = \mathbb{Q}(\sqrt{-6}, \sqrt{29}, \gamma)$ with $\gamma^3 - 12\gamma + 13 = 0$. By PARI/GP, we can see the explicit structure of the ideal class group of M and the Galois action on it as follows:

$$Cl(M) \cong \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}, \ \ Cl(M)_3 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

We denote by c_1, c_2 and c_3 the basis of $Cl(M)_3$ which is chosen in the computation of PARI/GP. Then we have

$$\begin{cases} \sigma(c_1) = -c_1 - c_2, & \tau(c_1) = -c_1, & j(c_1) = -c_1, \\ \sigma(c_2) = c_1 + c_3, & \tau(c_2) = c_1 + c_2 - c_3, & j(c_2) = -c_2, \\ \sigma(c_3) = c_3, & \tau(c_1) = -c, & j(c_3) = -c_3. \end{cases} \qquad (4.32)$$

By the above relations, we can see that $Cl(M)_3$ is generated by c_1 as a $\mathbb{Z}_3[G]$-module.

Take the prime 173. This prime is completely decomposed in M and satisfies $Hyp(S_{ram} \cup S_\infty, \{173\})$. Also we have

$$\delta_{\{173\}}\theta_{M/\mathbb{Q}} = \mathrm{nr}(-172)\,\mathrm{pr}_{\chi_3} = -172e_{\chi_3}\,\mathrm{pr}_{\chi_3} = -172\,\mathrm{pr}_{\chi_3}.$$

This element also belongs to $\zeta(\mathbb{Z}_3[G])$. However, from (4.32) we have

$$
\begin{aligned}
\delta_{\{173\}}\theta_{M/\mathbb{Q}}c_1 &= -172\,\mathrm{pr}_{\chi_3}\,c_1 = -172(1-j)(1+\sigma+\sigma^2)(1-\tau)c_1 \\
&= -172 \cdot 2 \cdot (-1) \cdot 2c_3 \neq 0.
\end{aligned}
$$

We take an element $x = \sum_{\chi\in\mathrm{Irr}\,G} x_\chi \,\mathrm{pr}_\chi \in \mathcal{H}_3(G)$. Then we have

$$
x\delta_{\{173\}}\theta_{M/\mathbb{Q}}c_1 = -172 \cdot 2 \cdot (-1) \cdot 2 \cdot 12 x_{\chi_3}c_3 = 0.
$$

Therefore, even in the case that Stickelberger elements do not have denominators, we need denominator ideal $\mathcal{H}_p(G)$.

Finally, we study why we need $\mathcal{H}_3(G)$. We recall that

$$
e_{\chi_3} = \frac{1}{12}\,\mathrm{pr}_{\chi_3} \text{ and } \theta_{M/\mathbb{Q}} = L_{S_\infty}(M/\mathbb{Q},\chi_3,0)e_{\chi_3} = 12e_{\chi_3} = \mathrm{pr}_{\chi_3}.
$$

The important thing here is that the L-value attached to χ_3 is canceled by the denominator of e_{χ_3} and hence $\theta_{M/\mathbb{Q}}$ has no information on the L-value. However, if we multiply $\theta_{M/\mathbb{Q}}$ by x, we have

$$
x\theta_{M/\mathbb{Q}} = x_{\chi_3}\,\mathrm{pr}_{\chi_3}\,\mathrm{pr}_{\chi_3} = x_{\chi_3}12\,\mathrm{pr}_{\chi_3} = x_{\chi_3}L_{S_\infty}(M/\mathbb{Q},\chi_3,0)\,\mathrm{pr}_{\chi_3}\,.
$$

In this way, thanks to the element x, we obtain information on the L-value from $\theta_{M/\mathbb{Q}}$. This is the reason why we need the denominator ideal $\mathcal{H}_3(G)$.

Chapter 5

Selmer groups of abelian varieties

Let K/k be a finite Galois extension of number fields with Galois group G and A be an abelian variety over k. In this chapter we study the Galois module structure of the (classical) Selmer group $\mathrm{Sel}(A_K)$ of A_K.

5.1   Selmer and Tate-Shafarevich groups

Let p be a prime. For each intermediate field L of K/k, we write $\mathrm{Sel}_p(A_L)$ and $\mathrm{III}_p(A_L)$ for the p-primary Selmer and the p-primary Tate-Shafarevich groups of A_L, respectively. Then there exists an exact sequence

$$0 \to A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_p(A_L) \to \mathrm{III}_p(A_L) \to 0$$

of $\mathbb{Z}_p[G]$-modules.

For any fields L and L' with $k \subset L \subset L' \subset F$, we set

$$\mathrm{III}_p^{L'}(A_L) := \ker\left(\mathrm{III}_p(A_L) \xrightarrow{\pi_L^{L'}} \mathrm{III}_p(A_{L'}) \right),$$

where $\pi_L^{L'}$ is the natural restriction map.

5.2 Some algebraic lemmas

In this subsection, we prove some technical lemmas. Let K/k be a finite Galois extension of number fields with Galois group G and p a rational prime. We fix a p-Sylow subgroup P of G and set $N := K^P$. We refer to the following conditions as $Hyp_A(K/k, p)$:

(a) $A(N)[p] = 0$,

(b) The Tamagawa number of A_N at each finite place of k is not divisible by p,

(c) A_N has good reduction for all p-adic places,

(d) For all p-adic places v that ramify in K/k, A has an ordinary reduction at $\mathfrak{p}$ and $A(\kappa_\mathfrak{p})[p] = 0$, where $\kappa_\mathfrak{p}$ is the residue field at $\mathfrak{p}$,

(e) No bad reduction place for A_k is ramified in K/k,

(f) p is odd,

(g) If a prime $\mathfrak{p}_k$ of k is ramified in K, we have $A(\kappa_{\mathfrak{p}_N})[p] = 0$ for any prime $\mathfrak{p}_N$ of N above $\mathfrak{p}_k$,

(h) $\mathrm{III}(A_K)$ is finite.

For each intermediate field L of K/k, we write $\mathrm{rk}(A(L))$ for the Mordell-Weil rank of $A(L)$. We first prove the following lemma:

Lemma 5.2.1. *Let K/k be a CM-extension of number fields and A an abelian variety such that $Hyp_A(K/k, p)$ is satisfied. We assume $\mathrm{III}_p^K(A_{K^H}) = 0$ for each subgroup H of P. Then for each $\alpha \in \{\pm\}$, if $A(N)^\alpha$ is finite, so is $A(K)^\alpha$.*

Proof. By [4, Proposition 2.7], $\mathrm{rk}(A(K)) \leq |P|\,\mathrm{rk}(A(N))$ holds. Therefore, we have

$$\mathrm{rk}(A(K)^+) + \mathrm{rk}(A(K)^-) \ \leq \ |P|(\mathrm{rk}(A(N)^+) + \mathrm{rk}(A(N)^-)),$$

and again by [4, Proposition 2.7] (and since $Hyp_A(K^+/k, p)$ is also satisfied), we have

$$\mathrm{rk}(A(K^+)) \ \leq \ |P|\,\mathrm{rk}(A(N^+)).$$

Since $\mathrm{rk}(A(K)^+) = \mathrm{rk}(A(K^+))$ and $\mathrm{rk}(A(N)^+) = \mathrm{rk}(A(N^+))$, we conclude the claim. □

Lemma 5.2.2. *Let K/k be a finite Galois extension of number fields and A an abelian variety such that the triple $(A, K/k, p)$ satisfies (b) and (e). Then, the Tamagawa number of $A_{K'}$ at each place in $S_{\mathrm{b}}^{K'}$ is not divisible by p for any intermediate field K' of K/k.*

Proof. First we prove the claim for K. Take a place $\mathfrak{p}$ at which A_N has bad reduction and let $\mathfrak{P}$ be a place of K above $\mathfrak{p}$. By the assumption (e), A_K still has bad reduction at $\mathfrak{P}$. Let $\mathcal{A}_{N_{\mathfrak{p}}}$ be the Néron model of $A_{N_{\mathfrak{p}}}$ over $\mathcal{O}_{N_{\mathfrak{p}}}$ and $\mathcal{A}_{N_{\mathfrak{p}}}^0$ the connected component of the identity of $\mathcal{A}_{N_{\mathfrak{p}}}$. We set $\tilde{\mathcal{A}}_{N_{\mathfrak{p}}} := \mathcal{A}_{N_{\mathfrak{p}}} \times_{\mathcal{O}_{N_{\mathfrak{p}}}} \kappa_{\mathfrak{p}}$ and $\tilde{\mathcal{A}}_{N_{\mathfrak{p}}}^0 := \mathcal{A}_{N_{\mathfrak{p}}}^0 \times_{\mathcal{O}_{N_{\mathfrak{p}}}} \kappa_{\mathfrak{p}}$, where $\kappa_{\mathfrak{p}}$ is the residue field at $\mathfrak{p}$. Then the Tamagawa number $c_{\mathfrak{p}}$ at $\mathfrak{p}$ is defined to be $|\Phi_{\mathfrak{p}}(\kappa_{\mathfrak{p}})|$, where $\Phi_{\mathfrak{p}}$ is a finite étale group scheme over $\kappa_{\mathfrak{p}}$ such that

$$1 \to \tilde{\mathcal{A}}_{N_{\mathfrak{p}}}^0 \to \tilde{\mathcal{A}}_{N_{\mathfrak{p}}} \to \Phi_{\mathfrak{p}} \to 1$$

is exact. Since $\mathfrak{P}$ is unramified in $K_{\mathfrak{P}}/N_{\mathfrak{p}}$, $\mathcal{A}_{N_{\mathfrak{p}}} \times \mathcal{O}_{K_{\mathfrak{P}}}$ is the Néron model of $A_{K_{\mathfrak{P}}}$ over $\mathcal{O}_{K_{\mathfrak{P}}}$ and hence we have $\Phi_{\mathfrak{P}} = \Phi_{\mathfrak{p}} \times \kappa_{\mathfrak{P}}$. Since $\kappa_{\mathfrak{P}}/\kappa_{\mathfrak{p}}$ is a p-extension, we see that $\Phi_{\mathfrak{P}}(\kappa_{\mathfrak{P}})[p] = \Phi_{\mathfrak{p}}(\kappa_{\mathfrak{p}})[p] = 0$. Therefore, $|\Phi_{\mathfrak{P}}(\kappa_{\mathfrak{P}})|$ is not divisible by p. Next, we take a subfield K' of K and let $\mathfrak{P}'$ be a place of K' at which $A_{K'}$ has bad reduction. Take a place $\mathfrak{P}$ of K above $\mathfrak{P}'$(again by the assumption (e), A_K has bad reduction at $\mathfrak{P}$). Since $K_{\mathfrak{P}}/K'_{\mathfrak{P}'}$ is unramified, we have $\Phi_{\mathfrak{P}} = \Phi_{\mathfrak{P}'} \times \kappa_{\mathfrak{P}}$. This implies there is natural inclusion $\Phi_{\mathfrak{P}'}(\kappa_{\mathfrak{P}'}) \hookrightarrow \Phi_{\mathfrak{P}}(\kappa_{\mathfrak{P}})$. Therefore, $|\Phi_{\mathfrak{P}'}(\kappa_{\mathfrak{P}'})|$ is not divisible by p. □

5.3 Equivariant Hasse-Weil L-functions

5.3.1 Twisted Hasse-Weil L-functions

For each finite place $\mathfrak{p}$ of k, we denote the Inertia subgroup of $G_{k_\mathfrak{p}}$ by $I_{k_\mathfrak{p}}$. We choose a rational prime l which is coprime to the character of $\kappa_\mathfrak{p}$. We denote by $T_l(A)$ the Tate-module of A at l and set $V_l(A) := \mathbb{Q}_l \otimes_{\mathbb{Z}_l} T_l(A)$. We define

$$P_\mathfrak{p}(X) := \det(1 - \mathrm{Frob}_\mathfrak{P}^{-1} X \mid \mathrm{Hom}(V_l A, \mathbb{Q}_l)^{I_{k_\mathfrak{p}}})^{-1}.$$

By Weil conjecture, the coefficients of this polynomial actually lie in $\mathbb{Z}$. Using this polynomial, we define the local L-function of A at $\mathfrak{p}$ to be

$$L_\mathfrak{p}(A, s) := P_\mathfrak{p}(|\kappa_\mathfrak{p}|^{-s}),$$

where s is the complex variable. For each character χ of G, we denote by V_χ a representation of G over $\mathbb{C}$ which has the character χ. We fix an isomorphism $i_l : \mathbb{C} \cong \mathbb{C}_l$ and define the χ-twisted local L-function of A at $\mathfrak{p}$ to be

$$L_\mathfrak{p}(A, K/k, \chi, s) := i_l^{-1}(\det(1 - \mathrm{Frob}_\mathfrak{P}^{-1} |\kappa_\mathfrak{p}|^{-s} \mid (i_l(V_\chi) \otimes_{\mathbb{Q}_l} \mathrm{Hom}(V_l A, \mathbb{Q}_l))^{I_{k_\mathfrak{p}}})^{-1}).$$

Finally, we define the global L-function and the χ-twisted global L-function to be

$$L(A, s) := \prod_{\mathfrak{p}:\text{finite}} L_\mathfrak{p}(A, s) \text{ and } L(A, K/k, \chi, s) := \prod_{\mathfrak{p}:\text{finite}} L_\mathfrak{p}(A, K/k, \chi, s).$$

This twisted L-function satisfies the following properties:

Proposition 5.3.1 (Artin formalism).

(LA1) $L(A, K/k, 1_G, s) = L(A, s)$,

(LA2) *If χ_1 and χ_2 are characters of G,*

$$L(A, K/k, \chi_1 + \chi_2, s) = L(A, K/k, \chi_1, s)L(A, K/k, \chi_2, s),$$

(LA3) *If L/k is a Galois extension with $L \subset K$, for each character ψ of $\operatorname{Gal}(L/k)$, we have*

$$L(A, K/k, \operatorname{Inf}^G_{\operatorname{Gal}(L/k)} \psi, s) = L(A, L/k, \psi, s),$$

(LA4) *For any intermediate field F of K/k and any character ϕ of $\operatorname{Gal}(K/F)$, we have*

$$L(A, K/k, \operatorname{Ind}^G_{\operatorname{Gal}(K/F)} \phi, s) = L(A, K/F, \phi, s).$$

Proof. The proof of these properties is just the same as that of the Artin L-function (cf. [19, Proposition 10.4]). $\qquad\square$

From the above proposition, we have

$$L(A_K, s) = L(A, s) \prod_{\substack{\chi \in \operatorname{Irr} G \\ \chi \neq 1_G}} L(A, K/k, \chi, s)^{\chi(1)}. \tag{5.1}$$

We set

$$L^\star(A, s) := (s-1)^{-r_A} L(A, s) \text{ and } L^\star(A, K/k, \chi, s) := (s-1)^{-r_A(\chi)} L(A, K/k, \chi, s),$$

where r_A and $r_A(\chi)$ are vanishing orders at $s = 1$ of $L(A, s)$ and $L(A, K/k, \chi, s)$, respectively. Then the same formulas as Proposition 5.3.1 are true for $L^\star(A, s)$ and $L^\star(A, K/k, \chi, s)$. Hence we have

$$L^\star(A_K, s) = L^\star(A, s) \prod_{\substack{\chi \in \operatorname{Irr} G \\ \chi \neq 1_G}} L^\star(A, K/k, \chi, s)^{\chi(1)}. \tag{5.2}$$

Finally, for any finite set S of places of k, we denote by $L_S(A, K/k, \chi, s)$ the χ-twisted S-truncated global L-function with complex variable s. In the case $S = S_\infty$, $L_S(A, K/k, \chi, s)$ coincides with $L(A, K/k, \chi, s)$.

5.3.2   Period and Galois Gauss sum

We fix Néron models $\mathcal{A}$ of A over $\mathcal{O}_k$ and $\mathcal{A}_{k_\mathfrak{p}}$ of $A_{k_\mathfrak{p}}$ over $\mathcal{O}_{k_\mathfrak{p}}$ for each p-adic place $\mathfrak{p}$ of k. We take a k-basis $\{\omega_1, \omega_2, \ldots, \omega_d\}$ of $H^0(A, \Omega^1_A)$ such that they give an

$\mathcal{O}_{k_\mathfrak{p}}$-basis of $H^0(A_{k_\mathfrak{p}}, \Omega^1_{A_{k_\mathfrak{p}}})$ for each p-adic place of k. For each real place v of k, we fix $\mathbb{Z}$-bases $\{\gamma^+_{v,1},\ \gamma^+_{v,2},\ \ldots,\ \gamma^+_{v,d}\}$ of $H_1(\sigma_v(A)(\mathbb{C}), \mathbb{Z})^{c=1}$ and $\{\gamma^-_{v,1},\ \gamma^-_{v,2},\ \ldots,\ \gamma^-_{v,d}\}$ of $H_1(\sigma_v(A)(\mathbb{C}), \mathbb{Z})^{c=-1}$, where c denotes the complex conjugation. We set

$$\Omega^+_v(A) := \left| \det\left(\int_{\gamma^+_{v,a}} \omega_b \right)_{1 \le a,b \le d} \right|, \quad \Omega^-_v(A) := \left| \det\left(\int_{\gamma^-_{v,a}} \omega_b \right)_{1 \le a,b \le d} \right|.$$

For each place v in $S^k_\mathbb{C}$, we take a $\mathbb{Z}$-basis $\{\gamma_{v,1},\ \gamma_{v,2},\ \ldots,\ \gamma_{v,2d}\}$ of $H_1(\sigma_v(A)(\mathbb{C}), \mathbb{Z})$ and set

$$\Omega_v(A) := \left| \det\left(\int_{\gamma^+_{v,a}} \omega_b,\ c\left(\left(\int_{\gamma^-_{v,a}} \omega_b\right)\right) \right)_{\substack{1 \le a \le d \\ 1 \le b \le 2d}} \right|.$$

For each $\chi \in \mathrm{Irr}\,G$, we set $\chi^+_v(1) := \dim_\mathbb{C} V_\chi^{G_{k_v}}$ and $\chi^-_v(1) := \chi(1) - \chi^+_v(1)$. We define the periods

$$\Omega_v(A, \chi) := \begin{cases} \Omega^+_v(A)^{\chi^+_v(1)} \Omega^-_v(A)^{\chi^-_v(1)} & \text{if } v \in S^k_\mathbb{R}, \\ \Omega_v(A)^{\chi(1)} & \text{if } v \in S^k_\mathbb{C} \end{cases}$$

and $\Omega(A, \chi) = \prod_{v \in S_\infty} \Omega_v(A, \chi)$. The periods $\Omega(A, \chi)$ satisfy the following properties:

Proposition 5.3.2. *We use the same notation as* $(LA1) \sim (LA4)$. *Then we have*

(P1) $\Omega(A, 1_G) = \prod_{v \in S^k_\mathbb{R}} \Omega^+_v(A) \prod_{v \in S^k_\mathbb{C}} \Omega_v(A)$,

(P2) $\Omega(A, \chi_1 + \chi_2) = \Omega(A, \chi_1)\Omega(A, \chi_2)$,

(P3) $\Omega(A, \mathrm{Inf}^G_{\mathrm{Gal}(L/k)} \psi) = \Omega(A, \psi)$,

(P4) *If k is totally real,* $\Omega(A, \mathrm{Ind}^G_{\mathrm{Gal}(K/F)} \phi) = \Omega(A, \phi)$ *up to the 2-primary part.*

Proof. (P1) $\sim$ (P3) are obvious. For (P4), it is enough to show that up to the 2-primary part, we have

$$\Omega_v(A, \mathrm{Ind}^G_{\mathrm{Gal}(K/F)} \phi) = \prod_{\substack{w' \in S^F_\infty \\ w' | v}} \Omega_{w'}(A, \phi).$$

82

The same method as the proof of [19, Proposition 12.1]) implies

$$
\begin{aligned}
\Omega_v(A, \mathrm{Ind}_{\mathrm{Gal}(K/F)}^G \phi) &= \prod_{\substack{w' \in S_{\mathbb{C}}^{K'} \\ w'|v}} (\Omega_v^+(A)\Omega_v^-(A))^{\phi(1)} \prod_{\substack{w' \in S_{\mathbb{R}}^{K'} \\ w'|v}} \Omega_v^+(A)^{\phi_{w'}^+(1)} \Omega_v^-(A)^{\phi_{w'}^-(1)} \\
&= \prod_{\substack{w' \in S_{\mathbb{C}}^{F} \\ w'|v}} (\Omega_v^+(A)\Omega_v^-(A))^{\phi(1)} \prod_{\substack{w' \in S_{\mathbb{R}}^{F} \\ w'|v}} \Omega_{w'}^+(A)^{\phi_{w'}^+(1)} \Omega_{w'}^-(A)^{\phi_{w'}^-(1)}.
\end{aligned}
$$

Since k is totally real, ω_1, $\omega_2, \ldots, \omega_d$ are defined over $\mathbb{R}$. Therefore, we have

$$
c\left(\int_{\gamma_{v,a}^+} \omega_b \right) = \int_{c(\gamma_{v,a}^+)} \omega_b = \int_{\gamma_{v,a}^+} \omega_b, \quad c\left(\int_{\gamma_{v,a}^-} \omega_b \right) = \int_{c(\gamma_{v,a}^-)} \omega_b = -\int_{\gamma_{v,a}^-} \omega_b
$$

and hence (suitable elementary column operations imply)

$$
\Omega_{w'}(A) = \left| \det \left(\int_{\gamma_{v,a}} \omega_b, \ c\left(\int_{\gamma_{v,a}} \omega_b \right) \right)_{\substack{1 \le a \le d \\ 1 \le b \le 2d}} \right| = 2^d \Omega_v^+(A)\Omega_v^-(A).
$$

From this, we finally get, up to the 2-primary part,

$$
\begin{aligned}
\Omega_v(A, \mathrm{Ind}_{\mathrm{Gal}(K/F)}^G \phi) &= \prod_{\substack{w' \in S_{\mathbb{C}}^{F} \\ w'|v}} \Omega_{w'}(A)^{\phi(1)} \prod_{\substack{w' \in S_{\mathbb{R}}^{F} \\ w'|v}} \Omega_{w'}^+(A)^{\phi_{w'}^+(1)} \Omega_{w'}^-(A)^{\phi_{w'}^-(1)} \\
&= \prod_{\substack{w' \in S_{\infty}^{F} \\ w'|v}} \Omega_{w'}(A, \phi).
\end{aligned}
$$

$\square$

From this proposition, if k is totally real, we have up to the 2-primary part

$$
\prod_{w \in S_{\mathbb{C}}^K} \Omega_w(A) \prod_{w \in S_{\mathbb{R}}^K} \Omega_w^+(A) = \prod_{\chi \in \mathrm{Irr}\, G} \Omega(A, \chi)^{\chi(1)}. \tag{5.3}
$$

We set

$$
\omega_v(\chi) := \begin{cases} i^{\chi_v^-(1)} & \text{if } v \in S_{\mathbb{R}}^k, \\ i^{\chi(1)} & \text{if } v \in S_{\mathbb{C}}^k \end{cases}
$$

83

and $\omega_\infty(\chi) := \prod_{v \in S_\infty} \omega_v(\chi)$. The elements $\omega_\infty(\chi)$ satisfy the following proposition:

Proposition 5.3.3. (O1) $\omega_\infty(1_G) = i^{|S_\mathbb{C}^k|}$,

(O2) $\omega_\infty(\chi_1 + \chi_2) = \omega_\infty(\chi_1)\omega_\infty(\chi_2)$,

(O3) $\omega_\infty(\mathrm{Inf}_{\mathrm{Gal}(L/k)}^G \psi) = \omega_\infty(\psi)$,

(O4) $\omega_\infty(\mathrm{Ind}_{\mathrm{Gal}(K/F)}^G \phi) = \omega_\infty(\phi)$.

Proof. The properties (O1) $\sim$ (O3) are obvious and (O4) follows from the same method as the proof of Proposition 5.3.2. $\qquad\qquad\square$

For each $\chi \in \mathrm{Irr}\, G$, we denote by $\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \chi)$ the Galois Gauss sum for $\mathrm{Ind}_k^\mathbb{Q} \chi$ which is defined in [17], where $\mathrm{Ind}_k^\mathbb{Q}$ means $\mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}/k)}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$. Then we have

Proposition 5.3.4. *We use the same notation as (LA1) $\sim$ (LA4). Then, we have*

(G1) $\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} 1_G) = i^{|S_\mathbb{C}^k|}\sqrt{|d_k|}$,

(G2) $\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q}(\chi_1 + \chi_2)) = \tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \chi_1)\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \chi_2)$,

(G3) $\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \mathrm{Inf}_{\mathrm{Gal}(L/K)}^G \psi) = \tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \psi)$,

(G4) $\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \mathrm{Ind}_{\mathrm{Gal}(K/F)}^G \phi) = \tau(\mathbb{Q}, \mathrm{Ind}_F^\mathbb{Q} \phi)$.

Proof. (G4) is obvious. (G2) and (G3) follow from the definition of the Galois Gauss sum. By [17, Theorem 8.1], we have

$$\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} \chi) = \tau(k, \chi)(i^{|S_\mathbb{C}^k|}\sqrt{|d_k|})^{\chi(1)}.$$

Since $\tau(k, 1_G) = 1$, we have

$$\tau(\mathbb{Q}, \mathrm{Ind}_k^\mathbb{Q} 1_G) = \tau(k, 1_G)i^{|S_\mathbb{C}^k|}\sqrt{|d_k|} = i^{|S_\mathbb{C}^k|}\sqrt{|d_k|}.$$

$\qquad\qquad\square$

Combining the properties (G1) $\sim$ (G4) with (O1) $\sim$ (O4), we have

$$\sqrt{|d_F|} = \prod_{\chi \in \mathrm{Irr}\, G} \frac{\tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^{\chi(1)}}{\omega_\infty(\chi)^{\chi(1)}}. \tag{5.4}$$

For each finite place $\mathfrak{p}$ of k, we fix a place $\mathfrak{P}$ of F above $\mathfrak{p}$. We denote by $G_{\mathfrak{P}}$ (resp. $I_{\mathfrak{P}}$) the decomposition subgroup (resp. the inertia subgroup) of G. For each place $\mathfrak{p} \in S_r^k$ and $\chi \in \mathrm{Irr}\, G$, we set

$$u_{\mathfrak{p}}(\chi) := \det(- \mathrm{Frob}_{\mathfrak{P}}^{-1} |V_\chi^{I_{\mathfrak{P}}})$$

and

$$u(\chi) := \prod_{\mathfrak{p} \in S_r^k} u_{\mathfrak{p}}(\chi). \tag{5.5}$$

Finally, we define

$$\tau^*(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}}, \chi) = u_{\mathfrak{p}}(\chi)\tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}}, \chi).$$

Note that if χ is faithful and $I_{\mathfrak{P}}$ is normal in G, we have $\tau^*(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}}, \chi) = \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}}, \chi)$.

In what follows, we always assume the following conjecture:

Conjecture 5.3.5. *For each $\chi \in \mathrm{Irr}\, G$,*

$$\frac{L(A, K/k, \chi, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d} \in \mathbb{Q}(\chi)$$

and for each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$,

$$\frac{L(A, K/k, \chi^\sigma, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi^\sigma)^d}{\Omega(A, \chi^\sigma) \cdot \omega_\infty(\chi^\sigma)^d} = \left(\frac{L(A, K/k, \chi, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d} \right)^\sigma.$$

Remark 5.3.6. This conjecture is a special case of the rationality conjecture [5, Conjecture 4(iii)], that is, a special case of the Deligne - Beilinson conjecture. For the Tate motives, this conjecture corresponds to Stark's conjecture.

For each Galois subextension K'/k' of K/k and χ in $R_{\mathrm{Gal}(K'/k')}$, we set

$$L^{\mathrm{Gal}(K'/k')}(\chi) := \frac{L_S(A, K/k, \check{\chi}, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d}$$

and

$$\mathbb{L}^G := \{L^{\mathrm{Gal}(K'/k')}, \mathrm{Gal}(K'/k'))\}_{K'/k'}.$$

By Propositions 5.3.1, 5.3.2, 5.3.3 and 5.3.4, we get the following:

Proposition 5.3.7. *We assume k is totally real. Then $\mathrm{Art}(\mathbb{L}^G)$ is satisfied up to the 2-primary part.*

5.3.3 Equivariant L-functions

Let S be a finite set of places of k which contains S_∞^k and S_r^k. We set

$$\mathcal{L}_{A,K/k,S} := \sum_{\chi \in \mathrm{Irr}\, G} e_\chi \frac{L_S(A, K/k, \check{\chi}, 1) \cdot \tau^*(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d}.$$

In [4], this element is used as an annihilator of Tate-Shafarevich groups, however, in this paper, we use

$$\mathcal{L}'_{A,K/k,S} := \sum_{\chi \in \mathrm{Irr}\, G} e_\chi \frac{L_S(A, K/k, \check{\chi}, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d}.$$

The relation of these two elements is

$$\mathcal{L}'_{A,K/k,S} = \Big(\sum_{\chi \in \mathrm{Irr}\, G} u(\chi) e_\chi \Big) \mathcal{L}_{A,K/k,S}, \tag{5.6}$$

where $u(\chi)$ is defined in (5.5). In the following, we do not assume S contains S_r^k (S has only to contain S_∞^k). If $L(A, K/k, \chi, 1)$ does not vanish for any $\chi \in \mathrm{Irr}\, G$, we have

$$\mathcal{L}'_{A,K/k,S_\infty} = \sum_{\chi \in \mathrm{Irr}\, G} e_\chi \frac{L^\star(A, K/k, \check{\chi}, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}_k^{\mathbb{Q}} \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d}.$$

Moreover, we have by (5.2), (5.3) and (5.4)

$$\frac{L^\star(A_K,1)(\sqrt{|d_K|})^d}{\Omega(A_K)} = \prod_{\chi\in\operatorname{Irr}G}\left(\frac{L^\star(A,K/k,\check{\chi},1)\cdot\tau(\mathbb{Q},\operatorname{Ind}_k^{\mathbb{Q}}\chi)^d}{\Omega(A,\chi)\cdot\omega_\infty(\chi)^d}\right)^{\chi(1)}, \qquad (5.7)$$

where $\Omega(A_K)=\prod_{\omega\in S_{\mathbb{R}}^K}\Omega_\omega^+(A_K)\prod_{\omega\in S_{\mathbb{C}}^k}\Omega_\omega(A_K)$. Now, we assume K/k is a CM-extension with the unique complex conjugation j. Set $\operatorname{Irr}^\pm G=\{\chi\in\operatorname{Irr}G\mid\chi(j)=\pm\chi(1)\}$. We define

$$\mathcal{L}_{A,K/k,S}^\pm=(\sum_{\chi\in\operatorname{Irr}^\pm G}e_\chi)\mathcal{L}'_{A,K/k,S}.$$

If $L(A,K/k,\chi,1)$ does not vanish for all $\chi\in\operatorname{Irr}G$, we have

$$\mathcal{L}_{A,K/k,S_\infty}^\pm=\sum_{\chi\in\operatorname{Irr}^\pm G}e_\chi\frac{L^\star(A,K/k,\check{\chi},1)\cdot\tau(\mathbb{Q},\operatorname{Ind}_k^{\mathbb{Q}}\chi)^d}{\Omega(A,\chi)\cdot\omega_\infty(\chi)^d}.$$

We set

$$L^\star(A_K,1)^+:=L^\star(A_{K^+},1),\ \sqrt{|d_K|}^+:=\sqrt{|d_{K^+}|},\ \Omega(A_K)^+:=\Omega(A_{K^+})$$

and

$$L^\star(A_K,1)^-:=\frac{L^\star(A_K,1)}{L^\star(A_K,1)^+},\ \sqrt{|d_K|}^-:=\frac{\sqrt{|d_K|}}{\sqrt{|d_K|}^+},\ \Omega(A_K)^-:=\frac{\Omega(A_K)}{\Omega(A_K)^+}.$$

Recalling the properties (LA3), (P3), (O3) and (G3), we have,

$$\frac{L^\star(A_K,1)^+(\sqrt{|d_K|}^+)^d}{\Omega(A_K)^+} = \prod_{\chi\in\operatorname{Irr}^+G}\left(\frac{L^\star(A,K/k,\check{\chi},1)\cdot\tau(\mathbb{Q},\operatorname{Ind}_k^{\mathbb{Q}}\chi)^d}{\Omega(A,\chi)\cdot\omega_\infty(\chi)^d}\right)^{\chi(1)} \qquad (5.8)$$

by the same method as (5.7). Dividing (5.7) by (5.8), we also have

$$\frac{L^\star(A_K,1)^-(\sqrt{|d_K|}^-)^d}{\Omega(A_K)^-} = \prod_{\chi\in\operatorname{Irr}^-G}\left(\frac{L^\star(A,K/k,\check{\chi},1)\cdot\tau(\mathbb{Q},\operatorname{Ind}_k^{\mathbb{Q}}\chi)^d}{\Omega(A,\chi)\cdot\omega_\infty(\chi)^d}\right)^{\chi(1)}. \qquad (5.9)$$

In the next section, we compare these formulas with the Birch and Swinnerton-Dyer conjecture of A.

Thanks to Conjecture 5.3.5, we have

$$\mathcal{L}_{A,K/k,S} \in \zeta(\mathbb{Q}[G]).$$

Therefore, Propositions 3.1.6 and 5.3.7 imply the following:

Proposition 5.3.8. *We assume k is totally real and G is monomial. We take a finite set S of places of k which contains S_∞. Then if $\mathcal{L}'_{A,K_{i,j}/k_i,S}$ belongs to $\zeta(\mathfrak{m}_p(\mathrm{Gal}(K_{i,j}/k_i)))$ for all $K_{i,j}/k_i$ in $\mathbb{K}$, $\mathcal{L}'_{A,K/k,S}$ belongs to $\zeta(\mathfrak{m}_p(G))$.*

We assume K/k is a CM-extension. Take $\alpha \in \{\pm 1\}$. Then by the same way as Proposition 5.3.8 we get the following:

Proposition 5.3.9. *We assume G is monomial. We take a finite set S of places of k which contains S_∞. Then if $\mathcal{L}^\alpha_{A,K_{i,j}/k_i,S}$ belongs to $\zeta(\mathfrak{m}_p(\mathrm{Gal}(K_{i,j}/k_i)))^\alpha$ for all $K_{i,j}/k_i$ in $\mathbb{K}$, $\mathcal{L}^\alpha_{A,K/k,S}$ belongs to $\zeta(\mathfrak{m}_p(G))^\alpha$.*

5.4 The Birch and Swinnerton-Dyer Conjecture

In this section, we review the formulation of the Birch and Swinnerton-Dyer conjecture for abelian varieties and prove some propositions needed in the next section. In what follows, we use the same notation as §5.3.

We set $\omega_A := \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_d$ and take a fractional ideal $\mathfrak{a}_K$ of K so that $\omega_A \mathfrak{a}_K = \bigwedge^d H^0(\mathcal{A}_K, \Omega^1_{\mathcal{A}_K})$. We let $\mathrm{Reg}(A_K)$ denote the regulator of A_K defined by the Néron-Tate pairing of A_K. For each finite place $\mathfrak{P}$ of K, we denote by $c_{\mathfrak{P}}$ the Tamagawa number of A_K at $\mathfrak{P}$. Finally, we set $\Omega(A_K) := \prod_{w \in S^K_{\mathbb{R}}} \Omega^+_w(A_K) \prod_{w \in S^K_{\mathbb{C}}} \Omega_w(A_K)$. Now, the Birch and Swinnerton-Dyer conjecture of A over K asserts that

Conjecture 5.4.1.

(1) *The order of vanishing at $s = 1$ of $L(A_K, s)$ is equal to $\mathrm{rk}(A(K))$,*

(2) $\text{III}(A_K)$ *is finite,*

(3) *We have*

$$|\text{III}(A_K)| = \frac{L^\star(A_K, 1)\sqrt{|d_K|}^d |A(K)_{\text{tors}}||A^t(K)_{\text{tors}}|}{\Omega(A_K)\prod_{\mathfrak{P}\in S_f^K}|\mathfrak{a}_K^{-1}|_{\mathfrak{P}}\prod_{\mathfrak{P}\in S_f^K}c_{\mathfrak{P}}\,\text{Reg}(A_K)}. \tag{5.10}$$

If $A^t(K)[p] = 0$ and the triple $(A, K/k, p)$ satisfies (a), (b), (e) and (f), by Lemma 5.2.2, we can derive from (5.10),

$$|\text{III}_p(A_K)| = \text{the } p\text{-part of } \frac{L^\star(A_K, 1)\sqrt{|d_K|}^d}{\Omega(A_K)\,\text{Reg}(A_K)},$$

where we can omit $\prod_{\mathfrak{P}\in S_f^K}|\mathfrak{a}_K^{-1}|_{\mathfrak{P}}$ since we took the elements ω_i so that they are also $\mathcal{O}_{k_{\mathfrak{p}}}$-bases of $H^0(\mathcal{A}_{k_{\mathfrak{p}}}, \Omega^1_{\mathcal{A}_{k_{\mathfrak{p}}}})$ for each p-adic place $\mathfrak{p}$ of k.

Now, we assume K/k is a CM-extension. We set

$$\text{III}_p(A_K)^+ := \text{III}_p(A_{K^+}), \ A(K)^+ = A(K^+), \ \text{Reg}(A_K)^+ = \text{Reg}(A_{K^+})$$

and

$$\text{III}_p(A_K)^- := \frac{\text{III}_p(A_K)}{\pi_{K^+}^K(\text{III}_p(A_K)^+)}, \ A(K)^- = \frac{A(K)}{A(K)^+}, \ \text{Reg}(A_K)^- := \frac{\text{Reg}(A_K)}{\text{Reg}(A_K)^+}.$$

Then for each $\alpha \in \{\pm\}$, if $A(K)^\alpha$ is finite, we can see $\text{Reg}(A_K)^\alpha = 1$. Therefore, by (5.8) and (5.9), we have the following proposition:

Proposition 5.4.2. *We assume that $A^t(K)[p] = 0$ and the triple $(A, K/k, p)$ satisfies (a), (b), (e) and (f). Then if $A(K)^\alpha$ is finite and the Birch and Swinnerton-Dyer conjecture is true for A_K, we have*

$$\begin{aligned}
|\text{III}_p(A_K)^\alpha| &= \frac{L^\star(A_K, 1)^\alpha \cdot (\sqrt{|d_K|}^\alpha)^d}{\Omega(A_K)^\alpha} \\
&= \prod_{\chi\in\text{Irr}^\alpha G}\left(\frac{L^\star(A, K/k, \check{\chi}, 1)\cdot\tau(\mathbb{Q}, \text{Ind}_k^{\mathbb{Q}}\chi)^d}{\Omega(A, \chi)\cdot\omega_\infty(\chi)^d}\right)^{\chi(1)}.
\end{aligned}$$

5.5 Annihilation problems

5.5.1 Formulations

Let K/k be a finite Galois extension of number fields with Galois group G. We fix a rational prime p. We write g_{A_F} for the minimal number of generators of $A(K)[p^\infty]^\vee$ as a $\mathbb{Z}_p[G]$-module, where $A(K)[p^\infty]$ is the p-power torsion points of $A(K)$. Now, we consider the following problem:

Problem 5.5.1.

(i) *Does $a_G(A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}_{A,K/k,S}$ lie in $\mathcal{I}_p(G)$?*

(ii) *Does $\mathcal{H}_p(G)a_G((A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}_{A,K/k,S}$ annihilate $\mathrm{Sel}_p(A_K)^\vee$?*

We chose a maximal $\mathbb{Z}_p$-order $\mathfrak{m}_p(G)$ in $\mathbb{Q}_p[G]$ which contains $\mathbb{Z}_p[G]$. Then by the same method as in [23] we can get the following weaker versions of Problem 5.5.1

Problem 5.5.2.

(i) *Does $a_G(A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}_{A,K/k,S}$ lie in $\zeta(\mathfrak{m}_p(G))$?*

(ii) *Does $\mathfrak{F}_p(G)a_G((A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}_{A,K/k,S}$ annihilate $\mathrm{Sel}_p(A_K)^\vee$?*

If the prime p does not divide the order of G, the above two Problems are equivalent. Even in the case p divides the order of G, we get the following relation by Proposition 2.2.10.

Proposition 5.5.3. *With the same assumption as Proposition 2.2.10, Problem 5.5.1 is equivalent to Problem 5.5.2.*

We assume K/k is a CM-extension with the unique complex conjugation j. We take $\alpha \in \{\pm 1\}$. Then we can consider the following problems:

Problem 5.5.4.

(i) *Does $a_G(A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}^\alpha_{A,K/k,S}$ lie in $\zeta(\mathfrak{m}^\alpha_p(G))$?*

(ii) *Does* $\mathfrak{F}_p(G)^\alpha a_G((A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}^\alpha_{A,K/k,S}$ *annihilate* $(\mathrm{Sel}_p(A_K)^\vee)^\alpha$?

Problem 5.5.5.

(i) *Does* $a_G(A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}^\alpha_{A,K/k,S}$ *lie in* $\zeta(\mathfrak{m}_p(G))^\alpha$?

(ii) *Does* $\mathfrak{F}_p(G)^\alpha a_G((A(K)[p^\infty]^\vee)^{g_{A_K}} \mathcal{L}^\alpha_{A,K/k,S}$ *annihilate* $(\mathrm{Sel}_p(A_K)^\vee)^\alpha$?

As well as Proposition 5.5.3, we get the following:

Proposition 5.5.6. *With the same assumption as Proposition 2.2.10, Problem 5.5.4 is equivalent to Problem 5.5.5.*

Next we consider the case $G = D_{4p}$ with odd prime p. Each of the irreducible characters of D_{4p} is 1-dimensional or 2-dimensional, and as in the proof of [26, Lemma 2.1] (also see [15, Example 6.22]) we have $\mathcal{I}_p(D_{4p}) = \zeta(\mathfrak{m}_p(D_{4p}))$. Therefore, we have by Proposition 2.2.10

Proposition 5.5.7. *If G is isomorphic to D_{4p}, Problem 5.5.4 is equivalent to Problem 5.5.5.*

5.5.2   Monomial extensions

In this section we prove the following theorem:

Theorem 5.5.8. *We take an odd prime p. Let k be a totally real number field, K/k be a Galois extension of number fields whose Galois group G is monomial and A an abelian variety over k. We assume $A(K)[p] = A^t(K)[p] = 0$. Then if Problem 5.5.2 has the affirmative answers for subextensions in $\mathbb{K}$, Problem 5.5.2 has the affirmative answer for K/k.*

Proof. We take a finite set S of places of k which contains S_∞. We have to prove the following two things:

- $\mathcal{L}_{A,K/k,S}$ belongs to $\zeta(\mathfrak{m}_p(G))$,

- $\mathfrak{F}_p(G)\mathcal{L}_{A,K/k,S}$ annihilates $\mathrm{Sel}_p(A_K)^\vee$.

The first claim is true by Proposition 5.3.8. To prove the second claim, we only have to show that the pair $(L^G, \mathrm{Sel}_p(A_K)^\vee)$ satisfies $Ab(L^G, \mathrm{Sel}_p(A_K)^\vee)$ by Theorem 3.2.1. The condition (i) of $Ab(L^G, \mathrm{Sel}_p(A_K)^\vee)$ is obviously satisfied. Concerning the condition (ii), we only need the $Art(L^G)$ modulo p by Remark 3.2.2. By Proposition 5.3.1, we see that $Art(L^G)$ modulo p is true (since p is odd). The conditions (iii) and (iv) are followed by our assumption that Problem 5.5.2 has the affirmative answers for all $K_{i,j}/k_i$ in $\mathbb{K}$. $\qquad\square$

By the same proof as Theorem 5.5.8, we get the following:

Theorem 5.5.9. *We take an odd prime p. Let K/k be a Galois CM-extension of number fields whose Galois group G is monomial and A an abelian variety over k. We assume $A(K)[p] = A^t(K)[p] = 0$. Then if Problem 5.5.5 has the affirmative answers for subextensions in $\mathbb{K}$, Problem 5.5.5 has the affirmative answer for K/k.*

Next we study the case where $A(K)[p]$ does not vanish. For each $K_{i,j}/k_i$ in $\mathbb{K}$, we write $g_{A_{K_{i,j}}}$ for the minimal number of generators of $A(K_{i,j})[p^\infty]^\vee$ as a $\mathbb{Z}_p[H_{i,j}/\ker\phi_{i,j}]$-module and set

$$g := \mathrm{l.c.m}_{1 \leq i \leq s}(g_{A_{K_{i,j}}}). \tag{5.11}$$

If $A(K)[p]$ does not vanish, we can not apply Theorem 3.2.1 as it is. However, we can prove the following:

Lemma 5.5.10. *We take an odd prime p. Let k be a totally real number field, K/k be a Galois extension of number fields whose Galois group G is monomial and A an abelian variety over k. We take a finite set S of places of k which contains S_∞^k. If $\phi_i'(a_{H_i/\ker\phi_i}(A(K_i)[p^\infty]^\vee)^{g_{A_{K_i}}} \mathcal{L}_{A,K_i/k_i,S_\infty^{k_i}})e_{\phi_i'}$ lies in $\mathfrak{m}_p(H_i/\ker\phi_i)e_{\phi_i'}$ for all K_i/k_i in $\mathbb{F}$, $a_G(A(K)[p^\infty]^\vee)^{g^2}\mathcal{L}_{A,K/k,S}'$ lies in $\zeta(\mathfrak{m}_p(G))$.*

Theorem 5.5.11. *We take an odd prime p. Let k be a totally real number field, K/k be a Galois extension of number fields whose Galois group G is monomial and A an abelian variety over k. We take a finite set S of places of k which contains S_∞^k. Then if Problem 5.5.2 has the affirmative answers for subextensions*

92

in $\mathbb{F}$,

$$\mathfrak{F}_p(G)a_G(A(K)[p^\infty]^\vee)^{g^2}\mathcal{L}'_{A,K/k,S}$$

lies in $\zeta(\mathbb{Z}_p[G])$ *and annihilates* $\mathrm{Sel}_p(A_K)^\vee$.

Proof of Lemma 5.5.10. If we set

$$t_S := \prod_{\mathfrak{p}\in S\setminus S_\infty^k} L_\mathfrak{p}(A,K/k,\check{\chi},1)^{-1}e_\chi,$$

then t_S lies in $\zeta(\mathfrak{m}_p(G))$. Hence it suffices to show the claim for $\mathcal{L}'_{A,K/k,S_\infty^k}$. Take an element α in $a_G(A(K)[p^\infty]^\vee)$ and write

$$\alpha = \sum_{\chi\in\mathrm{Irr}\,G} \alpha_\chi e_\chi$$

as an element in $\mathbb{C}_p[G]$. Note that α_χ is an algebraic integer in $\mathbb{Q}_p(\chi)$. Thanks to Conjecture 5.3.5 in §5.3.3, it is sufficient to show that for each $\chi_i \in \mathrm{Irr}\,G$

$$\alpha_{\chi_i}^{g^2} \frac{L_{S_\infty^k}(A,K/k,\check{\chi}_i,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_k^\mathbb{Q}(\chi_i))^d}{\Omega(A,\chi_i)\cdot\omega_\infty(\chi_i)^d}$$

is an algebraic integer. Since this factor does not change by inflation and induction of characters except α_χ as we saw in §5.3, we have

$$
\begin{aligned}
\frac{L_{S_\infty^k}(A,K/k,\check{\chi}_i,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_k^\mathbb{Q}(\chi_i))^d}{\Omega(A,\chi_i)\cdot\omega_\infty(\chi_i)^d} &= \frac{L_{S_\infty^{k_i}}(A,F/k_i,\check{\phi}_i,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_{k_i}^\mathbb{Q}(\phi_i))^d}{\Omega(A,\phi_i)\cdot\omega_\infty(\phi_i)^d} \\
&= \frac{L_{S_\infty^{k_i}}(A,K_i/k_i,\check{\phi}'_i,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_{k_i}^\mathbb{Q}(\phi'_i))^d}{\Omega(A,\phi'_i)\cdot\omega_\infty(\phi'_i)^d} \\
&= \phi'_i(\mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}}).
\end{aligned}
$$

Since α_{χ_i} lies in $\mathbb{Q}_p(\chi_i) \subset \mathbb{Q}_p(\phi_i)$, by Lemma 2.2.2, $\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma}$ lies in $\zeta(\mathbb{Q}_p[H_i])$ and hence lies in $\zeta(\mathfrak{m}_p(H_i))$ (recall that α_{χ_i} is an algebraic integer). This implies that $\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma}$ lies in $\mathrm{Ann}_{\zeta(\mathfrak{m}_p(H_i))}(\zeta(\mathfrak{m}_p(H_i))\otimes A(K)[p^\infty]^\vee)$. Let $(\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma})|_{K_i}$ be the natural image of $\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma}$ under the natural surjection $\mathbb{Z}_p[H_i] \twoheadrightarrow \mathbb{Z}_p[H_i/\ker\phi_i]$. In

what follows we set $\overline{H}_i := H_i/\ker\phi_i$. Then, we have

$$(\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma})|_{K_i} \in \mathrm{Ann}_{\mathfrak{m}_p(\overline{H}_i)}(\mathfrak{m}_p(\overline{H}_i)\otimes A(K_i)[p^\infty]^\vee). \tag{5.12}$$

Moreover, by the definition of g and [29, Theorem5], we have

$$\begin{aligned}\mathrm{Ann}_{\mathfrak{m}_p(\overline{H}_i)}(\mathfrak{m}_p(\overline{H}_i)\otimes A(K_i)[p^\infty]^\vee))^{g^2} &\subset (\mathrm{Fitt}_{\mathfrak{m}_p(\overline{H}_i)}(\mathfrak{m}_p(\overline{H}_i)\otimes A(K_i)[p^\infty]^\vee))^g \\ &= (\mathfrak{m}_p(\overline{H}_i)\cdot\mathrm{Fitt}_{\mathbb{Z}_p[\overline{H}_i]}(A(K_i)[p^\infty]^\vee))^g,\end{aligned} \tag{5.13}$$

where $\mathrm{Fitt}_{\mathbb{Z}_p[\overline{H}_i]}(\cdot)$ denotes the initial Fitting ideal over $\mathbb{Z}_p[\overline{H}_i]$. By our assumption, we have

$$\phi_i'(a_{\overline{H}_i}(A(K_i)[p^\infty]^\vee)^g \mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}})e_{\phi_i'} \in \mathfrak{m}_p(\overline{H}_i)e_{\phi'}.$$

Hence we also have

$$\phi_i'((\mathfrak{m}_p(\overline{H}_i)\cdot a_{\overline{H}_i}(A(K_i)[p^\infty]^\vee))^g \mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}})e_{\phi_i'} \in \mathfrak{m}_p(\overline{H}_i)e_{\phi'}.$$

Combining this with (5.13) (and recalling the fact that the initial Fitting ideal is contained in the annihilator ideal), we know that

$$\begin{aligned}\phi_i'((\sum_{\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma})^{g^2}|_{K_i}\mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}}) &= \phi_i'(\alpha_{\chi_i}^{g^2}\mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}}) \\ &= \alpha_{\chi_i}^{g^2}\cdot\phi_i'(\mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}})\end{aligned}$$

is an algebraic integer. This completes the proof. $\qquad\square$

Proof of Theorem 5.5.11. Since t_S lies in $\zeta(\mathfrak{m}_p(G))$ and $\mathfrak{F}_p(G)$ is an ideal of $\zeta(\mathfrak{m}_p(G))$, it is sufficient to show the claim for $\mathcal{L}'_{A,F/k,S_\infty^k}$. Take an element α in $a_G(A(K)[p^\infty]^\vee)$. As in the proof of Lemma 5.5.10, we write,

$$\alpha = \sum_{\chi\in\mathrm{Irr}\,G} \alpha_\chi e_\chi.$$

94

By Lemma 5.5.10, we see that $\alpha^{g^2} \mathcal{L}'_{A,F/k,S_\infty}$ lies in $\zeta(\mathfrak{m}_p(G))$. We take an element $x \in \mathfrak{F}_p(G)$. Then x is of the form

$$\sum_{\chi \in \mathrm{Irr}\, G/\sim} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p)} x_\chi^\sigma \, \mathrm{pr}_{\chi^\sigma}$$

with $x_\chi \in \mathfrak{D}^{-1}(\mathbb{Q}_p(\chi)/\mathbb{Q}_p)$. By Lemma 2.2.12, we have

$$\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} x_{\chi_i}^\sigma \, \mathrm{pr}_{\chi_i^\sigma} = \sum_{\substack{\phi \in \mathrm{Irr}\, H_i/\sim \\ \exists \sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p),\ \mathrm{Ind}_{H_i}^G(\phi)=\chi_i^\sigma}} \sum_{f \in \mathrm{Gal}(\mathbb{Q}_p(\phi)/\mathbb{Q}_p)} x_{\chi_i}^f \, \mathrm{pr}_{\phi^f},$$

and there exists a subscript $j \in \{1,\ 2,\ \ldots,\ s_i\}$ such that

$$\sum_{f \in \mathrm{Gal}(\mathbb{Q}_p(\phi)/\mathbb{Q}_p)} x_{\chi_i}^f \, \mathrm{pr}_{\phi^f} = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \, \mathrm{pr}_{\phi_{i,j}^f}.$$

This element also lies in $\mathfrak{F}_p(H_i)$ and its restriction to $K_{i,j}$ lies in $\mathfrak{F}_p(\overline{H}_{i,j})$, where $\overline{H}_{i,j} = H_i/\ker \phi_{i,j}$. By (5.12) and (5.13), we see that

$$(\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma})^{g^2}|_{K_{i,j}} \in (\mathfrak{m}_p(\overline{H}_{i,j}) \cdot a_{\overline{H}_{i,j}}(A(K_{i,j})[p^\infty]^\vee))^g.$$

Since we have $\mathfrak{F}_p(\overline{H}_{i,j})\mathfrak{m}_p(\overline{H}_{i,j}) \subset \mathfrak{F}_p(\overline{H}_{i,j})$, the product

$$(\sum_{f \in \mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \, \mathrm{pr}_{\phi_{i,j}^f})|_{K_{i,j}} (\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p)} \alpha_{\chi_i}^\sigma e_{\chi_i^\sigma})^{g^2}|_{K_{i,j}}$$
$$= \sum_{f \in \mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \alpha_{\chi_i}^{fg^2} \, \mathrm{pr}_{\phi_{i,j}'^f}$$

lies in $\mathfrak{F}_p(\overline{H}_{i,j})a_{\overline{H}_{i,j}}(A(K_{i,j})[p^\infty]^\vee)$. Hence, by our assumption,

$$(\sum_{f \in \mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \alpha_{\chi_i}^{fg^2} \, \mathrm{pr}_{\phi_{i,j}'^f})\mathcal{L}_{A,K_{i,j}/k_i,S_\infty^{k_i}}$$
$$= \sum_{f \in \mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \alpha_{\chi_i}^{fg^2} \phi_{i,j}'^f(\mathcal{L}_{A,K_{i,j}/k_i,S_\infty^{k_i}}) \, \mathrm{pr}_{\phi_{i,j}'^f} \qquad (5.14)$$

95

annihilates $\mathrm{Sel}_p(A_{K_{i,j}})^\vee$. We write $i^\sharp$ for the involution map on $\mathbb{Q}_p[G]$ which sends each element in G to its inverse. Then

$$i^\sharp\Big(\sum_{f\in\mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \alpha_{\chi_i}^{fg^2} \phi_{i,j}'^f(\mathcal{L}_{A,K_{i,j}/k_i,S_\infty^{k_i}})\,\mathrm{pr}_{\phi_{i,j}'^f}\Big)$$

annihilates $\mathrm{Sel}_p(A_{K_{i,j}})$. Since we have $\phi_{i,j}'^f(\mathcal{L}'_{A,K_i/k_i,S_\infty^{k_i}}) = \phi_{i,j}^f(\mathcal{L}'_{A,F/k_i,S_\infty^{k_i}})$ and $\mathrm{pr}_{\phi_{i,j}} = \mathrm{pr}_{\phi_{i,j}'}(\sum_{h\in\ker\phi_{i,j}} h)$,

$$i^\sharp\Big(\sum_{f\in\mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \alpha_{\chi_i}^{fg^2} \phi_{i,j}'^f(\mathcal{L}_{A,K_{i,j}/k_i,S_\infty^{k_i}})\,\mathrm{pr}_{\phi_{i,j}^f}\Big)$$

annihilates $\mathrm{Sel}_p(A_F)$, and hence

$$\sum_{f\in\mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f \alpha_{\chi_i}^{fg^2} \phi_{i,j}'^f(\mathcal{L}_{A,K_{i,j}/k_i,S_\infty^{k_i}})\,\mathrm{pr}_{\phi_{i,j}^f}$$

annihilates $\mathrm{Sel}_p(A_F)^\vee$. We recall that $\chi_i^f = \mathrm{Ind}_{H_i}^G(\phi_i^f)$. Then we have $\phi_i^f(\mathcal{L}'_{A,F/k_i,S_\infty^{k_i}}) = \chi_i^f(\mathcal{L}_{A,F/k,S_\infty^k})/\chi_i^f(1)$, and hence

$$\sum_{f\in\mathrm{Gal}(\mathbb{Q}_p(\phi_{i,j})/\mathbb{Q}_p)} x_{\chi_i}^f\,\mathrm{pr}_{\phi_{i,j}^f}\,\alpha_{\chi_i}^{fg^2}\,\frac{\chi_i^f(\mathcal{L}'_{A,F/k,S_\infty^k})}{\chi_i^f(1)}$$

annihilates $\mathrm{Sel}_p(A_F)^\vee$. Therefore,

$$\sum_{\substack{\phi\in\mathrm{Irr}\,H_i/\sim \\ \exists\sigma\in\mathrm{Gal}(\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p),\ \mathrm{Ind}_{H_i}^G(\phi)=\chi_i^\sigma}} \sum_{f\in\mathrm{Gal}(\mathbb{Q}_p(\phi)/\mathbb{Q}_p)} x_{\chi_i}^f\,\mathrm{pr}_{\phi^f}\,\alpha_{\chi_i}^{fg^2}\,\frac{\chi_i^f(\mathcal{L}'_{A,F/k,S_\infty^k})}{\chi_i^f(1)}$$

$$= \Big(\sum_{\sigma\in\mathbb{Q}_p(\chi_i)/\mathbb{Q}_p} x_{\chi_i}^\sigma\,\mathrm{pr}_{\chi_i^\sigma}\Big)\alpha^{g^2}\mathcal{L}'_{A,F/k,S_\infty^k}$$

annihilates $\mathrm{Sel}_p(A_F)^\vee$. Hence, we know that

$$\Big(\sum_{\chi\in\mathrm{Irr}\,G/\sim}\sum_{\sigma\in\mathbb{Q}_p(\chi)/\mathbb{Q}_p} x_\chi^\sigma\,\mathrm{pr}_{\chi^\sigma}\Big)\alpha^{g^2}\mathcal{L}'_{A,F/k,S_\infty^k} = x\alpha^{g^2}\mathcal{L}'_{A,F/k,S_\infty^k}$$

96

annihilates $\mathrm{Sel}_p(A_F)^\vee$. This completes the proof. $\qquad\square$

By the same method, we obtain the following:

Theorem 5.5.12. *Let F/k be a CM-extension whose Galois group is monomial and A an abelian variety over k. We take a finite set S of places of k which contains S_∞^k. Then if Problem 5.5.4 has the affirmative answers for subextensions in $\mathbb{K}$,*

$$\mathfrak{F}_p(G)^\alpha a_G(A(K)[p^\infty]^\vee)^{g^2}\mathcal{L}_{A,F/k,S}^\alpha$$

lies in $\zeta(\mathbb{Z}_p[G])^\alpha$ and annihilates $(\mathrm{Sel}_p(A_F)^\vee)^\alpha$.

5.5.3 Extensions with group D_{4p}

In this section, we use the same notations as in §4.4.2. We set

$$
\begin{aligned}
\mathbb{K}' &:= \{K_1,\ K_2,\ K_3,\ldots,\ K_{(p-1)/2}\} \\
&= \{k,\ K_2,\ K_3,\ K_4,\ K,\ K^+\}. \tag{5.15}
\end{aligned}
$$

Then our main result in this section is

Theorem 5.5.13. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to the dihedral group of order $4p$. Let A be an abelian variety over k such that $Hyp_A(K/k,p)$ is satisfied. We also assume that $A^t(N)[p] = 0$, $A(N)^\alpha$ is finite, $\mathrm{III}_p^K(A_N)=0$ and the Birch and Swinnerton-Dyer conjecture holds for intermediate fields in $\mathbb{K}'$. Then Problem 5.5.4 has the affirmative answer.*

By Theorem 5.5.9, in order to prove this theorem, we only have to show annihilation results corresponding to Problem 5.5.4 for subextensions K_i/k_i.

For $i = 1, 2, 3, 4$, the triples $(A, K_i/k_i, p)$ do not satisfy $Hyp_A(A, K_i/k_i, p)$ but still satisfy (a), (b), (e) and (f) by Lemma 5.2.2. Therefore, for these extensions, it is enough to show the following:

Proposition 5.5.14. *Let K/k be a quadratic CM-extension and A be an abelian variety over k such that the triple $(A, K/k, p)$ satisfies (a), (b), (e) and (f).*

We also assume that $A(K)^t[p] = 0$ and $A(K)^\alpha$ is finite. Then if the Birch and Swinnerton-Dyer conjecture is true for A_K, Problem 5.5.4 has the affirmative answer for A and K/k.

Proof. First we prove for $\mathcal{L}^\alpha_{A,K/k,S^k_\infty}$. We take the unique element $\chi \in \mathrm{Irr}^\alpha G$. Then we have by Proposition 5.4.2,

$$
\begin{aligned}
|(\mathrm{Sel}_p(A_K)^\vee)^\alpha| &= |(\mathrm{III}_p(A_F)^\vee)^\alpha| \\
&= |\mathrm{III}_p(A_F)^\alpha| \\
&= \frac{L^\star(A_K, 1)^\alpha \cdot (\sqrt{d_F}^{-\alpha})^d}{\Omega(A_K)^\alpha} \\
&= \frac{L^\star(A, K/k, \chi, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}^\mathbb{Q}_k \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d} \\
&= \frac{L(A, K/k, \chi, 1) \cdot \tau(\mathbb{Q}, \mathrm{Ind}^\mathbb{Q}_k \chi)^d}{\Omega(A, \chi) \cdot \omega_\infty(\chi)^d} \\
&= \chi(\mathcal{L}'_{A,K/k,S^k_\infty}).
\end{aligned}
$$

This implies that $\mathcal{L}^\alpha_{A,K/k,S^k_\infty}$ lies in $\mathbb{Z}_p[\mathrm{Gal}(K/k)]$ and

$$
\mathcal{L}^\alpha_{A,K/k,S^k_\infty} \cdot (\mathrm{Sel}_p(A_K)^\vee)^\alpha = 0. \tag{5.16}
$$

This completes the proof for S^k_∞. Take an arbitrary finite set S^k of places of k which contains S^k_∞. Then we have

$$
\mathcal{L}^\alpha_{A,K/k,S^k} = t_{S^k} \mathcal{L}^\alpha_{A,K/k,S^k_\infty},
$$

where t_{S^k} is an element in $\zeta(\mathfrak{m}_p(G))$ defined in the proof of Lemma 5.5.10. Take any element x in $\mathfrak{F}_p(G)^\alpha$. Then $x t_{S^k}$ lies in $\zeta(\mathbb{Z}_p[G])^\alpha$. Therefore, we have by (5.16)

$$
x \mathcal{L}^\alpha_{A,K/k,S^k} \cdot (\mathrm{Sel}_p(A_K)^\vee)^\alpha = x t_{S^k} \mathcal{L}^\alpha_{A,K/k,S^k_\infty} \cdot (\mathrm{Sel}_p(A_K)^\vee)^\alpha = 0.
$$

This completes the proof. $\qquad\qquad\square$

Next we consider the case of $i = 5, 6, \ldots, p+3$. Since extensions K_i/k_i satisfies

$Hyp_A(K_i/k_i, p)$ for $i = 5, 6, \ldots, p+3$, it is enough to show the following:

Proposition 5.5.15. *Let K/F be a cyclic CM-extension of degree $2p$ with Galois group G and A an abelian variety over k such that $Hyp_A(K/F, p)$ is satisfied. We assume that $A(N)^\alpha$ is finite and $\text{III}_p^K(A_N) = 0$. Then if the Birch and Swinnerton-Dyer conjecture is true for A_K and A_N, Problem 5.5.5 has the affirmative answer for K/k.*

Proof. In what follows, we use the same notation as in the proof of Proposition 4.4.18. By the same argument as in the proof of Theorem 5.5.11, it is enough to prove the claim for $\mathcal{L}^\alpha_{A,K/F,S^k_\infty}$. We observe that $A(K)^\alpha$ is finite by Lemma 5.2.1. Take an element x in $\mathfrak{F}_p(G)$. Then x is of the form

$$
\begin{aligned}
x &= \sum_{\substack{\phi \in \text{Irr}\, G/\sim \\ \phi \text{ is odd}}} \sum_{g \in \text{Gal}(\mathbb{Q}(\phi)/\mathbb{Q})} x_\phi^g \, \text{pr}_{\phi^g}, \ \ x_\phi \in \mathcal{D}^{-1}(\mathbb{Q}(\phi)/\mathbb{Q}) \\
&= x_\psi \, \text{pr}_\psi + \sum_{\substack{\phi \in \text{Irr}\, G/\sim \\ \phi \text{ is odd and } \phi(\sigma) \neq 1}} \sum_{g \in \text{Gal}(\mathbb{Q}(\phi)/\mathbb{Q})} x_\phi^g \, \text{pr}_{\phi^g},
\end{aligned}
$$

where ψ is the character of $\text{Gal}(K/k)$ such that $\psi(j) = -1$ and $\psi(\sigma) = 1$ and x_ψ belongs to $\mathbb{Z}$. We set

$$
x_{[\phi]} := \sum_{\substack{\phi \in \text{Irr}\, G/\sim \\ \phi \text{ is odd and } \phi(\sigma) \neq 1}} \sum_{g \in \text{Gal}(\mathbb{Q}_l(\phi)/\mathbb{Q}_l)} x_\phi^g \, \text{pr}_{\phi^g}.
$$

Since $P = \langle \sigma \rangle$, we have $N := K^{\langle \sigma \rangle}$. Moreover, N/F is a quadratic extension. We denote by ψ' the nontrivial character of $\text{Gal}(N/F)$. Then we have

$$
x_\psi \, \text{pr}_\psi \, \mathcal{L}_{A,K/k,S} = x_\psi \mathcal{L}_{A,N/F,S} Norm_{\langle \sigma \rangle}.
$$

Since $A(K)^\alpha$ is finite, we also have $A(N)^\alpha$ is finite. Moreover, the triple $(A, N/F, p)$ satisfies the conditions (a), (b), (e) and (f). Therefore, we have by Proposition 5.5.14

$$
x_\psi \, \text{pr}_\psi \, \mathcal{L}_{A,K/F,S}(\text{Sel}_p(A_K)^\vee)^\alpha = 0.
$$

Next we show
$$x_{[\phi]}\mathcal{L}_{A,K/F,S}(\mathrm{Sel}_p(A_K)^\vee)^\alpha = 0.$$

Since $A(K)^\alpha$ is finite, we have by Proposition 5.4.2

$$
\begin{aligned}
|(\mathrm{Sel}_p(A_K)^\vee)^\alpha| &= |(\mathrm{III}_p(A_K)^\vee)^\alpha| \\
&= |\mathrm{III}_p(A_K)^\alpha| \\
&= \frac{L^\star(A_K,1)^\alpha \cdot (\sqrt{d_K}^{-\alpha})^d}{\Omega(A_K)^\alpha} \\
&= \prod_{\phi\in\mathrm{Irr}^\alpha G} \frac{L^\star(A,K/F,\check{\phi},1) \cdot \tau(\mathbb{Q},\mathrm{Ind}_F^{\mathbb{Q}}\phi)^d}{\Omega(A,\phi) \cdot \omega_\infty(\phi)^d} \\
&= \prod_{\phi\in\mathrm{Irr}^\alpha G} \frac{L(A,K/F,\check{\phi},1) \cdot \tau(\mathbb{Q},\mathrm{Ind}_F^{\mathbb{Q}}\phi)^d}{\Omega(A,\phi) \cdot \omega_\infty(\phi)^d}. \quad (5.17)
\end{aligned}
$$

We take the unique element $\psi \in \mathrm{Irr}^\alpha \mathrm{Gal}(N/F)$. Then by the same way, we have

$$
\begin{aligned}
|(\mathrm{Sel}_p(A_N)^\vee)^\alpha| &= |(\mathrm{III}_p(A_N)^\vee)^\alpha| \\
&= |\mathrm{III}_p(A_N)^\alpha| \\
&= \frac{L^\star(A,N/F,\psi,1) \cdot \tau(\mathbb{Q},\mathrm{Ind}_F^{\mathbb{Q}}\psi)^d}{\Omega(A,\psi) \cdot \omega_\infty(\psi)^d} \\
&= \frac{L(A,N/F,\psi,1) \cdot \tau(\mathbb{Q},\mathrm{Ind}_F^{\mathbb{Q}}\psi)^d}{\Omega(A,\psi) \cdot \omega_\infty(\psi)^d}. \quad (5.18)
\end{aligned}
$$

Since $\mathrm{Inf}_{\mathrm{Gal}(K/k)}^G \psi$ is the element in $\mathrm{Irr}\, G$ whose kernel contains P, dividing (5.17) by (5.18), we have

$$\frac{|(\mathrm{Sel}_p(A_F)^\vee)^\alpha|}{|(\mathrm{Sel}_p(A_K)^\vee)^\alpha|} = \prod_{\substack{\phi\in\mathrm{Irr}^\alpha G \\ \phi(P)\neq 1}} \frac{L(A,K/F,\check{\phi},1) \cdot \tau(\mathbb{Q},\mathrm{Ind}_k^{\mathbb{Q}}\phi)^d}{\Omega(A,\phi) \cdot \omega_\infty(\phi)^d}.$$

Recalling that $x_{[\phi]}$ is of the form

$$x_{[\phi]} := \sum_{\substack{\phi\in\mathrm{Irr}\, G/\sim \\ \phi \text{ is odd and } \phi(\sigma)\neq 1}} \sum_{g\in\mathrm{Gal}(\mathbb{Q}_l(\phi)/\mathbb{Q}_l)} x_\phi^g\, \mathrm{pr}_{\phi^g},$$

we see that P does not act trivially on $x\,\mathrm{Sel}_p(A_K)^\alpha$. Since we assume $\Sha_p^K(A_N) = 0$, we have

$$
\begin{aligned}
|x_{[\phi]}(\mathrm{Sel}_p(A_K)^\vee)^\alpha| \;&\leq\; \frac{|(\mathrm{Sel}_p(A_K)^\vee)^\alpha|}{|(\mathrm{Sel}_p(A_K)^\vee)^{\alpha P}|} \\[4pt]
&=\; \frac{|(\Sha_p(A_K)^\vee)^\alpha|}{|(\Sha_p(A_K)^\vee)^{\alpha P}|} \\[4pt]
&\leq\; \frac{|(\Sha_p(A_K)^\vee)^\alpha|}{|(\Sha_p(A_N)^\vee)^\alpha|} \\[4pt]
&=\; \frac{|(\mathrm{Sel}_p(A_K)^\vee)^\alpha|}{|\,\mathrm{Sel}_p(A_N)^\vee)^\alpha|} \\[4pt]
&=\; \prod_{\substack{\phi\in\mathrm{Irr}^\alpha G \\ \phi(P)\neq 1}} \frac{L(A,K/F,\check\phi,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_k^{\mathbb{Q}}\phi)^d}{\Omega(A,\phi)\cdot\omega_\infty(\phi)^d} \\[4pt]
&=\; N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\Big(\frac{L(A,K/F,\check\phi_0,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_k^{\mathbb{Q}}\phi_0)^d}{\Omega(A,\phi_0)\cdot\omega_\infty(\phi_0)^d}\Big), \quad (5.19)
\end{aligned}
$$

where ϕ_0 is a generator of $\mathrm{Irr}^\alpha G$ and the last equality follows from Conjecture 5.3.5. The last three equalities of (5.19) imply that for each $\phi \in \mathrm{Irr}^\alpha G$ such that $\phi(P)\neq 1$, the element

$$
\frac{L(A,\check\phi,K/F,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_k^{\mathbb{Q}}\phi)}{\Omega(A,\phi)\cdot\omega_\infty(\phi)}
$$

actually lies in $\mathbb{Z}_p[\zeta_p]$ and hence $\phi(\mathcal{L}^\alpha_{A,K/F,S_\infty^k})$ lies in $\zeta(\mathfrak{m}_p(\mathrm{Gal}(K/F)))e_\phi$. In what follows, we regard $x(\mathrm{Sel}_p(A_F)^\vee)^\alpha$ as a $\mathbb{Z}_p[\zeta_p]$-module (this is possible because $(\sum_{\sigma\in P}\sigma)x_{[\phi]}(\mathrm{Sel}_p(A_K)^\vee)^\alpha = 0$). Then we have by (5.19),

$$
\begin{aligned}
|x_{[\phi]}(\mathrm{Sel}_p(A_K)^\vee)^\alpha| \;&\leq\; [\mathbb{Z}_p[\zeta_p] : (\frac{L(A,\check\phi_0,K/F,1)\cdot\tau(\mathbb{Q},\mathrm{Ind}_k^{\mathbb{Q}}\phi_0)}{\Omega(A,\phi_0)\cdot\omega_\infty(\phi_0)})] \\[4pt]
&\leq\; [\mathbb{Z}_p[\zeta_p] : (\overline{\mathcal{L}}^\alpha_{A,K/F,S_\infty^k})],
\end{aligned}
$$

where $\overline{\mathcal{L}}^\alpha_{A,K/F,S_\infty^k}$ is the image of $\mathcal{L}^\alpha_{A,K/F,S_\infty^k}$ under the natural surjection $\mathbb{Q}_p[G]^\alpha \cong \mathbb{Q}_p[P]^\alpha \twoheadrightarrow \mathbb{Q}_p[\zeta_p]$. This implies that $x_{[\phi]}\mathcal{L}^\alpha_{A,K/F,S_\infty^k}$ annihilates $(\mathrm{Sel}_p(A_K)^\vee)^\alpha$. This completes the proof. $\qquad\square$

Acknowledgement

Bibliography

[1] Barsky, D.: *Fonctions zêta p-adiques d'une classe de rayon des corps de nombres totalement réels* , Groupe d'Etude d'Analyse Ultrametrique (1977/78), Exp. No. 16.

[2] Burns, D.: *On derivatives of Artin L-series*, Invent. math. 186 (2011) 291-371.

[3] Burns, D.: *On derivatives of p-adic L-series at s=0*, preprint.

[4] Burns, D., Castillo, D. M., Wuthrich, C.: *On Mordell-Weil groups and congruences between derivatives of twisted Hasse-Weil L-functions*, preprint.

[5] Burns, D., Flach, M.:*Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. 6.

[6] Burns, D., Johnston, H.: *A non-abelian Stickelberger Theorem*, Compositio Math. 147 (2011), p. 35-55.

[7] Cassou-Noguès, P.: *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques*, Invent. Math. 51 (1979), 29-59.

[8] Curtis, C. W., Reiner, I.:*Methods of Representation Theory with applications to finite groups and orders*, Vol. 1, John Wiley & Sons, (1981).

[9] Deligne, P., Ribet, K.:*Values of abelian L-functions at negative integers over totally real fields*, Invent. Math. 59 (1980), 227-286.

[10] Greither, C.: *Determining Fitting ideals of minus class groups via the Equivariant Tamagawa Number Conjecture*, Compos. Math. 143, No. 6 (2007), 1399-1426.

[11] Greither, C., Roblot, X.-F., Tangedal, B.: *The Brumer-Stark Conjecture in some families of extensions of specified degree*, Math. Comp. 73 (2004), 297-315.

[12] Greither, C., Roblot, X.-F., Tangedal, B.: *Corrigendum to the paper "The Brumer-Stark Conjecture in some families of extensions of specified degree"*, available at http://math.univ-lyon1.fr/ roblot/resources/corrigendum-bsfam.pdf.

[13] Greither, C., Popescu, C.: *An equivariant main conjecture in Iwasawa theory and applications*, to appear in Journal of Algebraic Geometry.

[14] Jacobinski, H.: *On extensions of lattices*, Michigan Math. J. 13 (1966), 471-475. MR 0204538 (34 # 4377).

[15] Johnston, H.,Nickel,A.:*Noncommutative Fitting invariants and improved annihilation results*, J. Lond. Math. Soc. 88 (1), (2013), 137-160

[16] Kakde, M.: *The main conjecture of Iwasawa theory for totally real fields*, Invent. Math. 193 (2013) 539-626.

[17] Martinet, J.:*Character theory and Artin L-functions*, Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London.

[18] Mazur, B., Wiles, A.:*Class fields of abelian extensions of* $\mathbb{Q}$, Invent. math. 76 (1984), 179-330.

[19] Neukirch,J.:*Algebraic number theory*, Springer (1999).

[20] Nickel, A.: *Non-commutative Fitting invariants and annihilation of class groups*, J. Algebra 323 (10) (2010), 2756-2778.

[21] Nickel, A.: *On the Equivariant Tamagawa Number Conjecture in tame CM-extensions*, Math. Z. (2010) DOI 10.1007/s00209-009-0658-9.

[22] Nickel, A.: *On the equivariant Tamagawa number conjecture in tame CM-extensions II*, Compos. Math. 147 (4) (2011), 1179-1204.

[23] Nickel, A.*On non-abelian Stark-type conjectures*, Ann. Inst. Fourier 61 (6), (2011), 2577-2608. Fourier.

[24] Nickel, A.: *Equivariant Iwasawa theory and non-abelian Stark-type conjectures*, Proc. Lond. Math. Soc, **106 (6)**, (2013), 1223-1247.

[25] Nickel, A.:*Integrality of Stickelberger elements and the equivariant Tamagawa number conjecture*, preprint.

[26] Nomura, J.:*On non-abelian Brumer and Brumer-Stark conjectures for monomial CM-extensions*, to appear in International Journal of Number theory.

[27] Nomura, J.:*Annihilation of Selmer groups for monomial CM-extensions*, to appear in Journal of Number Theory.

[28] Nomura, J.:*The 2-part of the Brumer-Stark conjecture for extensions with group D_{4p} and numerical examples of the conjecture*, to appear in RIMS Kôkyûroku Bessatsu.

[29] Northcott, D.G.: *Finite free resolutions*, Cambridge Tracts in Math 71, Cambridge Univ. Press, Cambridge-New York (1976).

[30] Ritter, J., Weiss, A.:*On the 'main conjecture' of equivariant Iwasawa theory*, J. Amer. Math. Soc. 24 (2011), 1015-1050.

[31] Reiner, I.: *Maximal orders* L.M.S. Monographs, Academic Press, London (1975).

[32] Sands, J.:*Galois groups of exponent two and the Brumer-Stark conjecture*, J. Reine Angew. Math. 349 (1984), 129-135.

[33] Siegel, C.:*Über die Zetafunktionen indefiniter quadratischen Formen*, I, II, Math. Z., 43 (1938), 682-708; 44 (1939), 398-426. 15-56.

[34] Stickelberger, L.*Über eine Verallgemeinerung der Kreistheilung*, Math. Annalen 37 (1890), 321-367

[35] Tate, J.:*Les Conjectures de Stark sur les Fonctions L d'Artin en s = 0*, Birkhauser, 1984.

[36] Tate, J.:*Brumer-Stark-Stickelberger, Séminaire de Théorie des Nombres*, Université de Bordeaux I, U. E. R. de mathématiques et d'information exposé no. 24.

[37] Washington, L.:*Introduction to cyclotomic fields*, Graduate Texts in Math. 83, Springer-Verlag, 1982.

[38] Wiles, A.:*The Iwasawa conjecture for totally real fields*, Ann. of Math. 131 (1990), 493-540.