A Thesis for the Degree of Ph.D. in Science

Constructions of contact manifolds via reduction

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Contents

Chapter 1

Introduction

Sasaki-Einstein manifolds are real contact Einstein manifolds whose Riemannian cones are Ricci-flat Kähler manifolds. It has been acknowledged that they have an important role both in mathematics and physics. Moreover many manifolds are known to admit Sasaki-Einstein structures, nevertheless few of these manifolds have explicit representations. Complex contact manifolds, whose definition is analogous to that of real contact manifolds, are also expected to be important, but few examples of complex contact manifolds are known so far. In this thesis, we shall contribute to the study in those area by exhibiting new examples of real or complex contact manifolds via a reduction method.

Reduction techniques in symplectic geometry, developed in a text book such as Marsden and Ratiu [32], have natural analogues in the context of contact geometry. Depending on the geometric situation, various specializations have been considered by several authors, such as Geiges [14] in the Sasakian case, also Grantcharov and Ornea [16] and later Boyer and Galicki in [4] in the Sasaki-Einstein case.

The first result in this thesis is related to the Sasaki-Einstein metrics. Boyer and Galicki [4] have constructed a family of countably many Sasaki-Einstein metrics on the quotient space of the zero set of a moment map on a 7-dimensional sphere under a circle action. We shall present one of the metrics explicitly on a quotient space as an induced metric from the standard one on the 4-dimensional complex Euclidian space.

We describe the above construction in a explicit way and present the induced metric on the reduced space by using a projection from the zero set to the reduced space, which is diffeomorphic to $S^2 \times S^3$.

More precisely, we consider the following moment map on $\mathbf{C}^4$,

$$\mu(z_1, z_2, z_3, z_4) \ := \ |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2,$$

with the associated $U(1)$ action,

$$(z_1, z_2, z_3, z_4) \longmapsto (\, e^{i\theta} z_1, e^{i\theta} z_2, e^{-i\theta} z_3, e^{-i\theta} z_4 \,) \qquad (\theta \in \mathbf{R}),$$

and we show that $\mu^{-1}(0)|_{S^7}$ is diffeomorphic to $S^3 \times S^3$. Using this identification, we define a smooth projection π from $S^3 \times S^3$ to $(\mu^{-1}(0)|_{S^7})\,/S^1$ (see section 4.6):

$$\pi(z_1, z_2, z_3, z_4) := (2z_1\bar{z}_2,\ |z_1|^2 - |z_2|^2,\ z_1z_3 + \bar{z}_2\bar{z}_4,\ z_2z_3 - \bar{z}_1\bar{z}_4).$$

We show that this image is diffeomorphic to $S^2 \times S^3$. We notice that a $SU(2) \times SU(2)$ acts on $S^3 \times S^3$ naturally from the left, which makes π an equivariant map, that is, $S^2 \times S^3$ becomes a homogeneous space by this action. We then define an inner product $\langle \cdot, \cdot \rangle_o$ on $T_o(S^2 \times S^3)$ $\;(o = (0, 0, -1, 1, 0, 0, 0))$ and extend it at any point x in such a way that

$$\langle u, v \rangle_x := \left\langle dk^{-1}(u), dk^{-1}(v) \right\rangle_o \qquad (u, v \in T_x(S^2 \times S^3))$$

where k is a $(SU(2) \times SU(2))/U(1)$ free action such that $x = k \cdot o$. This is a representation of the metric, which is called the homogeneous Kobayashi-Tanno metric [39]. Our first result is an explicit discription of the metric [21].

Theorem A (**Theorem 4.5.1**) The Sasaki-Einstein metric $g_{1,0}$ on $S^2 \times S^3$ at any point x is given by the formula (4.5.1).

The second result is a construction of complex contact manifolds. To the best of our knowledge, the odd-dimensional complex projective space is only a known example of normal complex contact manifolds. In this thesis, we provide other examples of those manifolds, that is, we prove that there exists a normal complex almost contact metric structure on the quotient space of a hyperkähler manifold by holomorphic group actions. We also construct a complex almost contact metric structure on the product of the $(4p+3)$-dimensional and $(4q+3)$-dimensional spheres. This structure is obtained from the 3-Sasakian structures on each sphere. We also prove that this complex almost contact structure is not normal.

The theory of complex contact geometry started with the papers of Kobayashi [27] and Boothby [7], [8], as a variant of real contact geometry. More recent examples, including complex projective space and the complex Heisenberg group, are given in [3] and [6]. Ishihara and Konishi [24] defined the so-called I-K normality of complex contact manifolds as for Sasakian manifolds in real contact geometry. In this paper, we construct normal complex contact manifolds via reduction from hyperkähler manifolds. Referring to Definition 5.3.1 with the need terminologies on hyperkähler manifolds, we now state the second main result as follows [22]:

Theorem B (**Theorem 5.3.2**) Let $(\widetilde{M}, J_1, J_2, J_3, \tilde{g})$ be a hyperkähler manifold. Assume that $\mathbf{C}^*$ acts holomorphically with respect to the complex structure J_1 on $\widetilde{M}$. We also assume this action is proper and free. Then the quotient space $\widetilde{M}/\mathbf{C}^*$ is naturally equipped with a smooth manifold structure and the quotient map $\pi : \widetilde{M} \longrightarrow \widetilde{M}/\mathbf{C}^*$ canonically induces an I-K normal complex almost contact metric structure on $\widetilde{M}/\mathbf{C}^*$.

(For the structures of manifolds on the quotient spaces, see [30], [34].) Using this theorem, we construct a new example of a normal complex contact manifold (Example 5.3), a quotient space $M = (\mathbf{C}^4 \backslash \{z_1 z_2 z_3 z_4 = 0\})/\mathbf{C}^*$, where $\mathbf{C}^*$ acts on $\mathbf{C}^4 \backslash \{0\}$ by

$$\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \lambda^{-1} z_3, \lambda^{-1} z_4) \qquad (\lambda \in \mathbf{C}^*).$$

M is diffeomorphic to $\mathbf{C}^3 \backslash \{w_1 w_2 w_3 = 0\}$. This is a new example of an I-K normal complex almost contact metric manifold.

In studying complex contact manifolds, we found a remarkable property on the sectional curvature of an I-K normal complex contact metric manifold which will give a strong information [22]:

Proposition C (**Proposition 5.2.6.**) On I-K normal complex contact manifolds, we have

$$K(X, JX) + K(X, GX) + K(X, HX) = 6,$$

for any $X \in \mathrm{Ker}\ \omega$, where $K(X, Y)$ is the sectional curvatures of the plane spanned by $\{X, Y\}$, and G, H and J are associated to the complex contact metric structure (see Definition 5.1.2).

By this proposition, we can check whether any complex contact structure has I-K normality. For example, the odd dimentional complex projective space with the Fubini-Study metric is an I-K normal complex contact metric manifold, but there are no other known examples.

The third result is to construct complex almost contact metric structures. Calabi-Eckmann [10] and Morimoto [31] proved that $S^{2p+1} \times S^{2q+1}$ admits a complex structure J. In section 5.5, we show that 3-Sasakian structures on S^{4m+3} and S^{4n+3} induce a non-normal complex almost complex metric structure on $S^{4m+3} \times S^{4n+3}$ with respect to J [22].

Theorem D (**Theorem 5.5.1**.) The complex almost contact metric structure $(G_{m,n},\ H_{m,n}, J_{m,n}, u_{m,n}, v_{m,n}, U_{m,n}, V_{m,n}, g_{m,n})$ on $S^{4m+3} \times S^{4n+3}$ given by (5.5.1), (5.5.2), (5.5.4), (5.5.5) and (5.5.7) is not I-K normal.

This thesis is organized as follows. In section 2, we recall the definitions of real or complex manifolds. For more details, see [26], [37], [41] and [6], section2. In section 3, we discuss geometric structures, namely on the contact metric structures and Kähler structures. The contents of this section are based on [11], [14] and [41]. In section 4, as the special class of real contact metric manifolds, we define the Sasaki-Einstein manifolds and construct a Sasaki-Einstein metric explicitly on $S^2 \times S^3$. The contents of this section are based on [5]. In Section 5.1, we recall the definition of complex contact manifolds, which is a pair consisting of a manifold and a covering which admits a holomorphic 1-form ω. It is known that there exists a complex contact metric structure on any complex contact manifold. The complex (almost) contact metric structure looks like two (almost) contact metric structures which transform to each other via the fixed complex structure. In Section 5.2, we recall the notion of I-K normality defined by Ishihara and Konishi [24]. This normality implies any complex contact metric manifold is also a Kähler manifold. On the other hand, Korkmaz [29] has introduced a different notion of normality. As we show in section 5.2, the normality Korkmaz defines is weaker than I-K normality, and there are some manifolds which admit normal Korkmaz defines complex contact metric structures. In Section 5.3, we prove Theorem B and use it to construct new I-K normal almost complex contact metric manifolds by a projection from a hyperkähler manifold. Moreover, we give new examples of I-K normal complex contact metric manifolds. In section 5.4 and 5.5, we define the 3-Sasakian structure and construct a complex almost contact metric structure on $S^{4p+3} \times S^{4q+3}$ from the 3-Sasakian structures on each sphere. The contents of section 5 are based on [6] and [29]. In section 6, we discuss the further problems on real and complex contact manifolds.

Chapter 2

Preliminaries

2.1 Differentiable manifolds

Definition 2.1.1 Let M be a Hausdorff topological space. M is said an n-dimensional topological manifold if for any $p \in M$, there exists an open neighborhood U which is homeomorphic to an open subset V of $\mathbf{R}^n$.

Such a U is called a local coordinate neighborhood, and a homeomorphism $\psi : U \longrightarrow V$ is called a local coordinate function. We also say that (U, ψ) is local chart and regard $(x_1, \cdots, x_n) = \psi(p)$ as local coordinates for the manifold M.

Definition 2.1.2 Let M be a topological manifold. $\mathcal{A}$ is called an atlas for M if $\mathcal{A} = \{(U_\alpha, \psi_\alpha) \mid \alpha \in \Lambda\}$ where U_α is an open subset of M, ψ_α is an homeomorphism to an open subset V_α of M and $\cup_{\alpha\in\Lambda}\psi_\alpha = M$.

Definition 2.1.3 Let $\mathcal{A}$ be an atlas for M. If

$$\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap U_\beta) \longrightarrow \psi_\alpha(U_\alpha \cap U_\beta) \tag{2.1.1}$$

is a C^∞-map for all α, β satisfying $U_\alpha \cap U_\beta \neq \phi$, $\mathcal{A}$ is called C^∞-called atlas for M. We call $\psi_\alpha \circ \psi_\beta^{-1}$ a transition function.

Definition 2.1.4 A differentiable manifold $(M, \mathcal{A})$ is a pair of a topological manifold M and a atlas $\mathcal{A}$. We often write a differentiable manifold $(M, \mathcal{A})$ as M.

We recall that some topological spaces possess atlases. In section 2, the dimension of M is n unless otherwise specified.

Example 2.1 For the standard topological space $\mathbf{R}^n$, we have the trivial atlas $\mathcal{A} = \{(\mathbf{R}^n, id.)\}$. Then $(\mathbf{R}^n, \mathcal{A})$ is a differentiable manifold.

Example 2.2 Let S^n denote the unit sphere,

$$S^n = \left\{ (x_1, \cdots, x_{n+1}) \,\middle|\, \sum_{k=1}^{n+1} x_k^2 = 1 \right\}$$

equipped with the subset topology induced by $\mathbf{R}^{n+1}$. Put $U_N = S^n \backslash \{(1, 0, \cdots, 0)\}$, $U_S = S^n \backslash \{(-1, 0, \cdots, 0)\}$ and define $\psi_N : U_N \longrightarrow \mathbf{R}^n, \psi_S : U_S \longrightarrow \mathbf{R}^n$ by

$$\begin{aligned} \psi_N (x_1, \cdots, x_{n+1}) &= \tfrac{1}{1-x_1} (x_2, \cdots, x_{n+1}), \\ \psi_S (x_1, \cdots, x_{n+1}) &= \tfrac{1}{1+x_1} (x_2, \cdots, x_{n+1}). \end{aligned} \tag{2.1.2}$$

Then the transition maps $\phi_S \circ \phi_N^{-1},\ \phi_N \circ \phi_S^{-1} : \mathbf{R}^n \backslash \{(0, \cdots, 0)\} \longrightarrow \mathbf{R}^n \backslash \{(0, \cdots, 0)\}$ are given by

$$\begin{aligned} \psi_S \circ \psi_N^{-1} (y_1, \cdots, y_n) &= \frac{1}{1 + \sum_{k=1}^n y_k^2} (y_1, \cdots, y_n), \\ \psi_N \circ \psi_S^{-1} (y_1, \cdots, y_n) &= \frac{1}{1 + \sum_{k=1}^n y_k^2} (y_1, \cdots, y_n). \end{aligned}$$

So $\mathcal{A} = \{(U_N, \psi_N),\ (U_S, \psi_S)\}$ is a C^∞-atlas on S^n.

Example 2.3 On the set $\mathbf{R}^{n+1} \backslash \{0\}$, we define the equivalence relation $\sim$ by

$$x \sim y \text{ if and only if there exists a } \lambda \in \mathbf{R}^* \text{ such that } y = \lambda x.$$

Let $\mathbf{RP}^n$ be the quotient space $(\mathbf{R}^{n+1} \backslash \{0\}) / \sim$ and

$$\pi : \mathbf{R}^{n+1} \backslash \{0\} \longrightarrow \mathbf{RP}^n$$

be the natural projection mapping a point $(x_1, \cdots, x_{n+1}) \in \mathbf{R}^{n+1} \backslash \{0\}$ to the equivalence class $[x_1, \cdots, x_{n+1}] \in \mathbf{RP}^n$. Equip $\mathbf{RP}^n$ with the quotient topology induced by π and $\mathbf{R}^{n+1} \backslash \{0\}$. For $k \in \{1, 2, \cdots, n+1\}$, define the open subset U_k by

$$U_k = \{[x_1, \cdots, x_{n+1}] \in \mathbf{RP}^n \mid x_k \neq 0\}$$

and the coordinate functions $\psi_k : U_k \longrightarrow \mathbf{R}^n$ by

$$\psi_k(x_1, \cdots, x_{n+1}) \;=\; \left(\frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_{n+1}}{x_k}\right).$$

If $[x] = [y]$ then $y = \lambda x$ for some $\lambda \in \mathbf{R}^*$ so $\frac{x_l}{x_k} = \frac{y_l}{y_k}$ for all l. This means the map ψ_k is well-defined for all k. The corresponding transition maps

$$\psi_k \circ \psi_l^{-1} : \psi_l(U_k \cap U_l) \longrightarrow \psi_k(U_k \cap U_l)$$

are given by

$$(\psi_k \circ \psi_l^{-1})\left(\frac{x_1}{x_l}, \cdots, \frac{x_{l-1}}{x_l}, 1, \frac{x_{l+1}}{x_l}, \cdots, \frac{x_{n+1}}{x_l}\right) \;=\; \left(\frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_{n+1}}{x_k}\right).$$

So the collection $\mathcal{A} = \{(U_k, \psi_k) \mid k = 1, 2, \cdots, n+1\}$ is a C^∞-atlas on $\mathbf{RP}^n$. The differential manifold $(\mathbf{RP}^n, \mathcal{A})$ is called the n-dimensional real projective space.

Example 2.4 Let $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ be two differentiable manifolds. Let $M_1 \times M_2$ be the product space with the product topology. We define the set $\mathcal{A}$ on $M_1 \times M_2$by

$$\mathcal{A} = \{(U_\alpha \times U'_\beta,\, \psi_\alpha \times \psi'_\beta) \mid (U_\alpha, \psi_\alpha) \in \mathcal{A}_1,\; (U'_\beta, \psi'_\beta) \in \mathcal{A}_2\}$$

is a C^∞-atlas on $M_1 \times M_2$.

Definition 2.1.5 Let $(M, \{(U_\alpha, \psi_\alpha)\})$ be a smooth manifold and $F : M \longrightarrow \mathbf{R}$ be a function on M. F is smooth at $x \in M$ if $F \circ \psi_\alpha^{-1}$ is smooth at $\psi(x)$ for some local coordinate function $\psi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbf{R}^n$ with $x \in U_\alpha$ and F is a smooth map if F is smooth at any points of M.

In this situation, the function

$$F \circ \psi_\alpha^{-1} : (y_1, \cdots, y_n) \mapsto F(\psi_\alpha^{-1}(y_1, \cdots, y_n)) \tag{2.1.3}$$

is a C^∞-function on an open subset of $\mathbf{R}^n$. Therefore, the partial derivatives

$$\frac{\partial}{\partial y_1}(F \circ \psi_\alpha^{-1})\Big|_{\psi_\alpha(x)}, \cdots, \frac{\partial}{\partial y_n}(F \circ \psi_\alpha^{-1})\Big|_{\psi_\alpha(x)}$$

are defined.

The previous definition can be extended to the case of maps between manifolds.

Definition 2.1.6 Let $(M, \{(U_\alpha, \psi_\alpha)\}), (M', \{(U'_\beta, \psi'_\beta)\})$ be smooth manifolds with dim $M = n$, dim $M' = n'$ and $F : M \longrightarrow M'$ be a map. F is smooth at $x \in M$ if $\psi'_\beta \circ F \circ \psi_\alpha^{-1}$ is smooth at $\psi(x)$ for some local coordinate functions $\psi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbf{R}^n$, $\psi'_\beta : U'_\beta \longrightarrow V'_\beta \subset \mathbf{R}^{n'}$ with $x \in U_\alpha, \psi_\alpha(x) \in U'_\beta$ and F is a smooth map if F is smooth at any points of M.

With the notion of smooth maps, we define the diffeomorphism between two manifolds.

Definition 2.1.7 $F : M \longrightarrow M'$ is a diffeomorphism if F is smooth bijective map and F^{-1} is smooth map. We say that M and M' are diffeomorphic.

Example 2.5 Let us verify the function H on S^2,

$$H : S^2 \longrightarrow \mathbf{R}, \quad H(x, y, z) = x.$$

is a smooth function. it is necessary to compute the functions $H \circ \psi_N^{-1}$, $H \circ \psi_S^{-1}$ (where ψ_N and ψ_S are as in Exmaple 2.2). We have

$$\begin{aligned}
\left(H \circ \psi_N^{-1}\right)(p_1, p_2) &= \frac{1}{1 + p_1^2 + p_2^2}\left(p_1^2 + p_2^2 - 1, 2p_1, 2p_2\right) = \frac{p_1^2 + p_2^2 - 1}{p_1^2 + p_2^2 + 1}, \\
\left(H \circ \psi_S^{-1}\right)(q_1, q_2) &= \frac{1}{1 + q_1^2 + q_2^2}\left(1 - q_1^2 - q_2^2, 2q_1, 2q_2\right) = \frac{1 - q_1^2 - q_2^2}{1 + q_1^2 + q_2^2},
\end{aligned}$$

which are smooth on its domain $\psi_N(U_N), \psi_S(U_S)$ respectively. We conclude that H is smooth on S^2.

Example 2.6 We show that S^1 and $\mathbf{RP}^1$ are diffeomorphic. Define a map F from S^1 to $\mathbf{RP}^1$ by

$$F : S^2 \longrightarrow \mathbf{RP}^1, \quad F(x, y) = [x - 1, y].$$

We check that F satisfies the definition of the differomorphism. All the maps

$$(\psi_1 \circ F \circ \psi_N^{-1})(p) = \psi_1\left(F\left(\frac{-1 + p^2}{1 + p^2}, \frac{2p}{1 + p^2}\right)\right) = -p,$$

$$
\begin{aligned}
(\psi_2 \circ F \circ \psi_N^{-1})(p) &= \psi_2\left(F\left(\frac{-1+p^2}{1+p^2}, \frac{2p}{1+p^2}\right)\right) = -\frac{1}{p}, \\
(\psi_1 \circ F \circ \psi_S^{-1})(p) &= \psi_1\left(F\left(\frac{1-p^2}{1+p^2}, \frac{2p}{1+p^2}\right)\right) = -\frac{1}{p}, \\
(\psi_2 \circ F \circ \psi_S^{-1})(p) &= \psi_2\left(F\left(\frac{1-p^2}{1+p^2}, \frac{2p}{1+p^2}\right)\right) = -p
\end{aligned}
$$

are well-defined and smooth on their domains respectively. Conversely, the inverse map F^{-1} is given by

$$
F^{-1} : \mathbf{RP}^1 \longrightarrow S^2, \quad F^{-1}([X,Y]) = \left(\frac{-X^2+Y^2}{X^2+Y^2}, \frac{-2XY}{X^2+Y^2}\right).
$$

Similarly, we can show that all the maps

$$
\begin{aligned}
(\psi_N \circ F^{-1} \circ \psi_1^{-1})(p) &= \psi_N(F^{-1}([1,p])) = -p, \\
(\psi_N \circ F^{-1} \circ \psi_2^{-1})(p) &= \psi_N(F^{-1}([p,1])) = -\frac{1}{p}, \\
(\psi_S \circ F^{-1} \circ \psi_1^{-1})(p) &= \psi_S(F^{-1}([1,p])) = -\frac{1}{p}, \\
(\psi_S \circ F^{-1} \circ \psi_2^{-1})(p) &= \psi_S(F^{-1}([p,1])) = -p
\end{aligned}
$$

are well-defined and smooth on their domains respectively. So we conclude that $F : S^1 \longrightarrow \mathbf{RP}^1$ is a diffeomorphism.

Definition 2.1.8 Let $k < n$. A subspace N is a k-dimensional submanifold of M if each point $p \in N$ has a local chart $(U; x_1, \cdots, x_n)$ such that the intersection $M \cap N$ is determined by the equations $x_1 = \cdots = x_{n-k} = 0$. Moreover, considering $y_1 = x_{n-k+1}, \cdots, y_k = x_n$ as local coordinates on N, we define the structure of a smooth manifold on N. We introduce the following important notion.

Definition 2.1.9 A smooth map $F : M \longrightarrow M'$ is called regular at $p \in M$ (or p is called a regular point of the map F) if the rank of the Jacobian matrix of F at p written in some local coordinates $y_j = F_j(x_1, \cdots, x_n), j = 1, \cdots, k$, is equal to the dimension of N i.e.

$$
\operatorname{rank}\left(\frac{\partial F_j}{\partial x_j}\right) = \dim N.
$$

Examples of submanifolds are regular zero sets of smooth maps.

Lemma 2.1.10 Let M, M' be n, k-dimensional manifolds, respectively. Let $c \in M'$. Suppose that $F : M \longrightarrow M'$ is a smooth map on M and its zero set $M_0 = F^{-1}(c)$ consists of regular points. Then M_0 is an $(n-k)$-dimensional submanifold of M.

Definition 2.1.11 Let $\mathcal{M}_p$ $(p \in M)$ be the set of all smooth real-valued functions, each of which is defined on some open neighborhood of p. A tangent vector to M at p is a map $v : \mathcal{M}_p \longrightarrow \mathbf{R}$ such that

$$\begin{aligned} &(1) \quad v(\lambda f + \mu g) = \lambda v(f) + \mu v(g), \\ &(2) \quad v(fg) = v(f)g(p) + f(p)v(g) \end{aligned}$$

for all $f, g \in \mathcal{M}_p$, $\lambda, \mu \in \mathbf{R}$. The set of all tangent vectors to M at p is denoted by T_pM. It is called the tangent space to M at p.

By the definition, T_pM is a vector space, whose dimension is equal to that of M.

Theorem 2.1.12 Let M be an n-dimensional differentiable manifold and (U, ψ) be a local chart on M. For $p \in M$, we define $\left(\frac{\partial}{\partial x_k}\right)_p$ in T_pM by

$$\left(\frac{\partial}{\partial x_k}\right)_p : f \mapsto \frac{\partial f}{\partial x_k}(p) = \frac{\partial}{\partial x_k}(f \circ \psi^{-1})\Big|_{\psi(p)}. \tag{2.1.4}$$

Then the set

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \right\}$$

is a basis for tangent space T_pM, i.e., dim $T_pM = n$ for any $p \in M$.

Example 2.7 Let $M = \{(x, y) \in \mathbf{R}^2 \mid 1 < x^2 + y^2 < 3\}$ and $p = (1, 1)$. We consider two charts at p, and two bases of T_pM.

(1) The inclusion map $\psi_1 : M \longrightarrow \mathbf{R}^2, \psi_1(x, y) = (x, y)$ is a local chart. If

we use the same standard coordinates x, y on M, and $\mathbf{R}^2$, then

$$\left(\frac{\partial}{\partial x}\right)_p, \quad \left(\frac{\partial}{\partial y}\right)_p$$

is a basis of T_pM.

(2) The map $\psi_2 : U \longrightarrow \mathbf{R}^2, \psi_2(x, y) = (\sqrt{x^2+y^2}, \tan^{-1}\frac{y}{x}) = (r, \theta)$ is a local chart, where $U = \{(x, y) \in M \mid x > 0, y > 0\}$. This time we use (r, θ) as standard coordinates for $\mathbf{R}^2$; thus

$$V = \psi_2(U) = \left\{(r, \theta) \in \mathbf{R}^2 \mid 1 < r < \sqrt{3},\ 0 < \theta < \frac{\pi}{2}\right\}$$

is an open rectangle. Then

$$\left(\frac{\partial}{\partial r}\right)_p, \quad \left(\frac{\partial}{\partial \theta}\right)_p$$

is a basis of T_pM. Now we consider the relation $\left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p$ and $\left(\frac{\partial}{\partial r}\right)_p, \left(\frac{\partial}{\partial \theta}\right)_p$. To find the linear transformation, we apply the tangent vectors to a function f:

$$\begin{aligned}
\left(\frac{\partial}{\partial r}\right)_p (f) &= \left(\frac{\partial}{\partial r}(f \circ \psi_2^{-1})\right)_{\psi_2(p)} \\
&= \left(\frac{\partial}{\partial r}(f \circ \psi_1^{-1}) \circ (\psi_1 \circ \psi_2^{-1})\right)_{\psi_2(p)} \\
&= \left(\frac{\partial}{\partial x}(f \circ \psi_1^{-1})\right)_{\psi_1(p)} \left(\frac{\partial x}{\partial r}\right)_{\psi_2(p)} + \left(\frac{\partial}{\partial y}(f \circ \psi_1^{-1})\right)_{\psi_1(p)} \left(\frac{\partial y}{\partial r}\right)_{\psi_2(p)}.
\end{aligned}$$

Here we use $(r, \theta) = (\psi_2 \circ \psi_1^{-1})(x, y) = (\sqrt{x^2+y^2}, \tan^{-1}\frac{y}{x})$. We have $\left(\frac{\partial x}{\partial r}\right)_{\psi_2(p)} = \frac{1}{\sqrt{2}}$ and $\left(\frac{\partial y}{\partial r}\right)_{\psi_2(p)} = \frac{1}{\sqrt{2}}$. Therefore,

$$\left(\frac{\partial}{\partial r}\right)_p = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}\right)_p + \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y}\right)_p$$

A similar calculation gives

$$\left(\frac{\partial}{\partial \theta}\right)_p = -\left(\frac{\partial}{\partial x}\right)_p + \left(\frac{\partial}{\partial y}\right)_p$$

Definition 2.1.13 Let M, M' be smooth manifolds with $\dim M = n$, $\dim M' = n'$, and $F : M \longrightarrow M'$ be a smooth map. We define the derivative of F at $x \in M$ to be the map

$$(dF)_x : T_xM \longrightarrow T_{F(x)}M'$$

given by

$$(dF)_x(v)(f) = v(f \circ F)$$

for any $v \in T_xM$ and any $f \in \mathcal{M}'_{F(x)}$.

If $F : M \longrightarrow M'$ and $F' : M' \longrightarrow M''$ are smooth maps, then the composition $F' \circ F : M \longrightarrow M''$ is also a smooth map. It follows that

$$(d(F' \circ F))_x = (dF')_{F(x)} \circ (dF)_x.$$

We introduce the tangent bundle TM of an n-dimensional manifold M. This is the object that we get by glueing at each point p of M the corresponding tangent space T_pM. The differentiable structure on M induces a natural differentiable structure on TM.

We saw that T_pM can be identified with $\mathbf{R}^n$. If we glue T_pM to $\mathbf{R}^n$, we obtain the tangent bundle $T\mathbf{R}^n$ of $\mathbf{R}^n$:

$$T\mathbf{R}^n = \{(p, v) \mid p \in \mathbf{R}^n,\ v \in T_p\mathbf{R}^n\}. \tag{2.1.5}$$

For this we have the natural projection $\pi : T\mathbf{R}^n \longrightarrow \mathbf{R}^n$ defined by $\pi(p, v) = p$ and for each point $p \in \mathbf{R}^n$, the fiber $\pi^{-1}(\{p\})$ over p is precisely $T_p\mathbf{R}^n$.

Classically, a vector field X on $\mathbf{R}^n$ is a smooth map $X : \mathbf{R}^n \longrightarrow T\mathbf{R}^n$ defined with abuse of notation by

$$X : p \mapsto (p, X(p)).$$

Let $(x_1, \cdots, x_n)$ be a basis of $\mathbf{R}^n$. Two vector fields X, Y can be written as

$$X = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k}, \quad Y = \sum_{k=1}^{n} b_k \frac{\partial}{\partial x_k}$$

where $a_k, b_k : \mathbf{R}^n \longrightarrow \mathbf{R}$ are smooth functions defined on $\mathbf{R}^n$. If $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is another such function the commutator $[X,Y]$ acts on f as follows:

$$\begin{aligned} [X,Y](f) &= X(Y(f)) - Y(X(f)) \\ &= \sum_{k,l=1}^{n} \left(a_k \frac{\partial}{\partial x_k} \left(b_l \frac{\partial}{\partial x_l} \right) - b_k \frac{\partial}{\partial x_k} \left(a_l \frac{\partial}{\partial x_l} \right) \right) (f) \qquad (2.1.6) \\ &= \sum_{k,l=1}^{n} \left(a_k \frac{\partial b_l}{\partial x_k} - b_k \frac{\partial a_l}{\partial x_k} \right) \left(\frac{\partial}{\partial x_l} \right) (f) \end{aligned}$$

This shows that $[X,Y]$ is a smooth vector field on $\mathbf{R}^n$.

We now generalize the tangent bundle $T\mathbf{R}^n$ to the manifold setting. This leads us first to the following notion of topological vector bundle.

Definition 2.1.14 Let E and M be topological manifolds and $\pi : E \longrightarrow M$ be a continuous surjective map. The triple (E, M, π) is called an n-dimensional topological vector bundle over M if
(1) for each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is an n-dimensional vector space,
(2) for each $p \in M$, there exists a bundle chart $(\pi^{-1}(U), \psi)$ consisting of the pre-image $\pi^{-1}(U)$ of an open neighborhood U of p and a homeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times \mathbf{R}^n$ such that for all $q \in U$ the map $\psi_q = \psi|_{E_q} : E_q \longrightarrow \{q\} \times \mathbf{R}^n$ is a vector space isomorphism. A continuous map $\sigma : M \longrightarrow E$ is called a section of the bundle (E, M, π) if $\pi \circ \sigma(p) = p$ for each $p \in M$.

Example 2.8 Let $(M, \mathcal{A})$ be an n-dimensional differentiable manifold. Define the set TM by

$$TM = \{(p,v) \mid p \in M,\ v \in T_pM\}$$

and let $\pi : TM \longrightarrow M$ be the projection map satisfying

$$\pi : (p,v) \mapsto p.$$

Then the fiber $\pi^{-1}(\{p\})$ over $p \in M$ is the n-dimensional tangent space T_pM. The triple (TM, M, π) is called the tangent bundle of M.

We define the vector fields on differentiable manifolds.

Definition 2.1.15 Let M be a differentiable manifold. A section $X : M \longrightarrow TM$ of the tangent bundle is called a vector field. The set of smooth vector fields is denoted by $C^\infty(TM)$.

In (2.1.6), we define the bracket product of vector fields on $\mathbf{R}^n$. Now we define the same one on a differentiable manifold.

Definition 2.1.16 Let M be a differentiable manifold. For two vector fields $X, Y \in C^\infty(TM)$, we define the Lie bracket $[X, Y]_p : C^\infty(M) \longrightarrow \mathbf{R}$ of X and Y at $p \in M$ by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)). \tag{2.1.7}$$

Proposition 2.1.17 Let M be a differentiable manifold and X, Y be vector fields on M. Then the section $[X, Y] : M \longrightarrow TM$ of tangent bundle given by $[X, Y] : p \longrightarrow [X, Y]_p$ is smooth.

For later use, we prove the following useful result.

Propostion 2.1.18 Let M be a differentiable manifold and $[,]$ be Lie bracket on TM. Then for all $X, Y \in C^\infty(TM)$ and $f \in C^\infty(M)$,

$$\begin{aligned} &(1) \quad [X, f\,Y] = X(f)\,Y + f\,[X, Y], \\ &(2) \quad [f\,X, Y] = f\,[X, Y] - Y(f)\,X. \end{aligned}$$

Let M be an n-dimensional manifold. The cotangent bundle T^*M is defined in a similar way to TM, using the dual vector spaces T_p^*M. There is a natural projection map $\pi^* : T^*M \longrightarrow M$. It can be proved that T^*M is a manifold of dimension $2n$. A differential 1-form on M is a map $\omega : M \longrightarrow \mathbf{R},\ p \mapsto \omega_p$ such that $\pi^* \circ \omega = id$. The vector space consisting of all 1-forms on M is denoted by $\mathcal{D}^1(M)$. More generally, for an integer $r > 1$, differential r-form ω is a map

$$\omega : C^\infty(TM) \times \cdots \times C^\infty(TM) \ \ (r \text{ times}) \ \longrightarrow \ C^\infty(M)$$

satisfying the following conditions:
(1) for each $j = 1, \cdots, n$, if for all of variables but X_j, $\omega(X_1, \cdots, X_j, \cdots, X_n)$ are

held constant, then $\omega(X_1, \cdots, X_j, \cdots, X_n)$ is a linear map of X_j,
(2) $\omega(X_1, \cdots, X_n)$ is skew-symmetric.

We denote the collection of differential r-forms on M by $\mathcal{D}^r(M)$. For $r = 0$ or $r > n$, we define $\mathcal{D}^0(M) = C^\infty(M), \mathcal{D}^r(M) = \{0\}$ respectively. Now we define the exterior product on $\mathcal{D}(M) = \oplus_{r=0}^n \mathcal{D}^r(M)$.

Definition 2.1.19 For $\omega_1 \in \mathcal{D}^r(M), \omega_2 \in \mathcal{D}^s(M)$, we define the exterior product $\omega_1 \wedge \omega_2$ by the alternatization of the map

$$(X_1, \cdots, X_r, X_{r+1}, \cdots, X_{r+s}) \ \mapsto \ \omega_1(X_1, \cdots, X_r)\, \omega_2(X_{r+1}, \cdots, X_{r+s}).$$

For example, in the case of $r = s = 1$, $\omega_1 \wedge \omega_2$ is given by

$$(\omega_1 \wedge \omega_2)(X, Y) = \frac{1}{2}\,(\omega_1(X)\,\omega_2(Y) - \omega_2(X)\,\omega_1(Y)).$$

For local coordinate $(U; x_1, \cdots, x_n)$, the differential r-form is given by

$$\omega = \sum_{j_1 < \cdots < j_r} \omega_{j_1 \cdots j_r} dx_{j_1} \wedge \cdots \wedge dx_{j_r}$$

where $\omega_{j_1 \cdots j_r} : C^\infty(U) \longrightarrow \mathbf{R}$ are functions for all $j_1, \cdots, j_r \in \{1, 2, \cdots, n\}$.

Proposition 2.1.20 We define the exterior differentiation d on $\mathcal{D}(M)$ as follows:
(1) for $f \in \mathcal{D}^0(M) = C^\infty(M)$, df is the derivative of f,
(2) for all r, $d\,|_{\mathcal{D}^r(M)} : \mathcal{D}^r(M) \longrightarrow \mathcal{D}^{r+1}(M)$ is the linear map,
(3) for $\omega_1 \in \mathcal{D}^r(M), \omega_2 \in \mathcal{D}^s(M)$, $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^r\, \omega_1 \wedge (d\omega_2)$
(4) $d^2 = 0$.

It is known that the differential operator d satisfying above is determined uniquely. If $\omega \in \mathcal{D}^r(M)$ is given by

$$\omega = \sum_{j_1 < \cdots < j_r} \omega_{j_1 \cdots j_r} dx_{j_1} \wedge \cdots \wedge dx_{j_r}$$

for the local coordinate $(U; x_1, \cdots, x_n)$, we denote $d\omega$ by

$$d\omega \ = \sum_{j_1 < \cdots < j_r} d\omega_{j_1 \cdots j_r} dx_{j_1} \wedge \cdots \wedge dx_{j_r}$$

$$= \sum_{j_1<\cdots<j_r} \left(\sum_{k=1}^{n} \frac{\partial \omega_{j_1\cdots j_r}}{\partial x_k} dx_k \right) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_r}.$$

Especially, if $\omega \in \mathcal{D}^1(M)$, for all $X, Y \in C^\infty(TM)$,

$$d\omega(X,Y) = \frac{1}{2}\,(X\,\omega(Y) - Y\,\omega(X) - \omega([X,Y])).$$

In the end of this subsection, we define the tensor field on the differential manifold M.

Definition 2.1.21 The tensor field K of type (r,s) on M is a map $K : (C^\infty(TM))^{\otimes r} \longrightarrow (C^\infty(TM))^{\otimes s}$ which is multi-linear over $C^\infty(M)$ i.e. satisfying

$$\begin{aligned} & K(X_1 \otimes \cdots \otimes X_{k-1} \otimes (f\,Y + g\,Z) \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ = \;& f\,K(X_1 \otimes \cdots \otimes X_{k-1} \otimes Y \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ & +g\,K(X_1 \otimes \cdots \otimes X_{k-1} \otimes Z \otimes X_{k+1} \otimes \cdots \otimes X_r) \end{aligned}$$

for all $X_1, \cdots, X_r, Y, Z \in C^\infty(TM)$, $f, g \in C^\infty(M)$ and $k = 1, 2, \cdots, r$.

For a tensor K, we shall by K_p denote the multi-linear restriction of K to the r-fold tensor product

$$T_pM \otimes \cdots \otimes T_pM$$

of the tangent vector space T_pM given by

$$K_p : ((X_1)_p, \cdots, (X_r)_p) \mapsto K(X_1, \cdots, X_r)(p).$$

Definition 2.1.22 Let M be a differential manifold. A Riemannian metric g on M is a tensor field

$$g : C^\infty(TM) \otimes C^\infty(TM) \longrightarrow C^\infty(M)$$

such that for each $p \in M$, the restriction

$$g_p = g|_{T_pM\otimes T_pM} : T_pM \otimes T_pM \longrightarrow \mathbf{R} \tag{2.1.8}$$

with

$$g_p : (X_p, Y_p) \mapsto g(X,Y)(p)$$

is an inner product on T_pM. The pair (M, g) is called a Riemannian manifold.

Example 2.9 The standard inner product on $\mathbf{R}^n$ given by

$$\langle X, Y\rangle_{\mathbf{R}^n} = \sum_{k=1}^{n} X_k Y_k$$

defines a Riemannian metric on $\mathbf{R}^n$. The Riemannian manifold $(\mathbf{R}^n, \langle,\rangle_{\mathbf{R}^n})$ is called the n-dimensional Euclidian space.

Example 2.10 Equip $\mathbf{R}^n$ with the Riemannian metric g given by

$$g_p(X, Y) = \frac{4}{(1 + |p|^2_{\mathbf{R}^n})^2}\langle X, Y\rangle_{\mathbf{R}^n}.$$

The Riemannian manifold $(\mathbf{R}^n, g)$ is called the n-dimensional punctured round sphere.

Example 2.11 Let $B_1^n(0)$ be the open unit ball in $\mathbf{R}^n$ given by

$$B_1^n(0) = \{p \in \mathbf{R}^n \mid |p|^2_{\mathbf{R}^n} < 1\}.$$

By the n-dimensional hyperbolic ball we mean $B_1^n(0)$ equipped with the Riemannian metric

$$g_p(X, Y) = \frac{4}{(1 - |p|^2_{\mathbf{R}^n})^2}\langle X, Y\rangle_{\mathbf{R}^n}.$$

The Lie derivative L_K of a (r, s)-tensor field K along a vector field X is defined by its values on r $X_1, \cdots, X_r$vector fields through the following formulas:
if K is a $(r, 0)$-tensor,

$$L_X K(X_1, \cdots, X_r) = XK(X_1, \cdots, X_r) - \sum_{j=1}^{r} K(X_1, \cdots, [X, X_j], \cdots, X_r);$$

if K is a $(r, 1)$-tensor,

$$L_X K(X_1, \cdots, X_r) = [X, K(X_1, \cdots, X_r)] - \sum_{j=1}^{r} K(X_1, \cdots, [X, X_j], \cdots, X_r);$$

and the rules L_X is linear and a derivation with respect to the tensor product $\otimes$. Finally, we recall that for a differential p-form ω, $L_X\omega$ is defined by

$$L_X\omega = \iota_X d\omega + d(\iota_X\omega),$$

where $\iota_X : \mathcal{D}^p(M) \longrightarrow \mathcal{D}^{p-1}(M)$ denotes the interior product, given by the formula

$$(\iota_X\alpha)(X_2, \cdots, X_p) = \alpha(X, X_2, \cdots, X_p).$$

2.2 Connections and curvatures

Definition 2.2.1 Let (E, M, π) be a smooth vector bundle over M. A connection on (E, M, π) is a map $\nabla : C^\infty(TM) \times C^\infty(E) \longrightarrow C^\infty(E)$ such that

$$\begin{aligned} &(1)\quad \nabla_X(\lambda v + \mu w) = \lambda\nabla_X v + \mu\nabla_X w, \\ &(2)\quad \nabla_X(fv) = X(f) + f\nabla_X v, \\ &(3)\quad \nabla_{(fX+gY)}v = f\nabla_X v + g\nabla_Y g \end{aligned}$$

for all $\lambda, \mu \in \mathbf{R}, X, Y \in C^\infty(TM), v, w \in C^\infty(E)$ and $f, g \in C^\infty$. A section $v \in C^\infty(E)$ of the vector bundle E is said to be parallel with respect to the connection ∇ if

$$\nabla_X v = 0$$

for all vector fields $X \in C^\infty(TM)$.

Definition 2.2.2 Let M be a smooth manifold and ∇ be a connection on the tangent bundle (TM, M, π). Then we define the torsion $T : C^\infty(TM) \times C^\infty(TM) \longrightarrow C^\infty(TM)$ of ∇ by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

where $[,]$ is the Lie bracket on $C^\infty(TM)$. The connection ∇ is said to be torsion-free if its torsion T vanishes i.e.

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

for all $X, Y \in C^\infty(TM)$.

Definition 2.2.3 Let (M, g) be a Riemannian manifold. Then a connection ∇ on the tangent bundle (TM, M, π) is said to be compatible with the metric if

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all $X, Y, Z \in C^\infty(TM)$.

Let (M, g) be a Riemannian manifold and ∇ be compatible with g and torsion-free connection on (TM, M, π). Then it is easily seen that the following equations hold

$$\begin{aligned} 2 \cdot g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) + Z(g(X, Y)) \\ &\quad + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]). \end{aligned} \tag{2.2.1}$$

Definition 2.2.4 Let (M, g) be a Riemannian manifold. Then the map $\nabla : C^\infty(TM) \times C^\infty(TM) \longrightarrow C^\infty(TM)$ given by is called the Levi-Civita connection on M.

Remark 2.2.5 The Levi-Civita connection on (M, g) is only depending the differentiable structure and its Riemannian metric, and the Levi-Civita connection is the unique compatible with and torsion-free connection on the tangent bundle (TM, M, π).

A vector field X on (M, g) induces the covariant derivative

$$\nabla_X : C^\infty(TM) \longrightarrow C^\infty(TM)$$

in the direction of X by

$$\nabla_X : Y \mapsto \nabla_X Y.$$

Example 2.12 Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Further let (U, x) be local coordinates on M and put $X_k = \partial/\partial x_k \in C^\infty(TU)$. Then $\{X_1, \cdots, X_n\}$ is a local orthonormal frame of TM on U. For (U, x), we define the Christoffel symbols $\Gamma^l_{jk} : U \longrightarrow \mathbf{R}$ of the connection ∇ with respect to (U, x) by

$$\sum_{k=1}^{n} \Gamma^l_{jk} X_l = \nabla_{X_j} X_k.$$

From the definition of the Levi-Civita connection, we now get

$$\begin{aligned}
\sum_{l=1}^{n} \Gamma_{jk}^{l} g_{lm} &= g\left(\sum_{l=1}^{n} \Gamma_{jk}^{l} X_k, X_l\right) \\
&= g(\nabla_{X_j} X_l, X_m) \\
&= \frac{1}{2}\left(X_j g(X_k, X_m) + X_k g(X_m, X_j) - X_m g(X_j, X_k)\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{km}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_m}\right).
\end{aligned}$$

If $g^{lm} = (g^{-1})_{lm}$, then

$$\Gamma_{jk}^{l} = \frac{1}{2}\sum_{m=1}^{n} g^{lm}\left(\frac{\partial g_{km}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_m}\right). \tag{2.2.2}$$

Now we define the Riemannian curvature tensor and the notion of sectional curvature of a Riemannian manifold. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then the curvature $R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \longrightarrow C^\infty(TM)$ is a tensor field on of type (3,1) defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.2.3}$$

The following result shows that the curvature tensor has many nice properties of symmetry.

Proposition 2.2.6 Let (M, g) be a Riemannian manifold. For vector fields $X, Y, Z, W \in C^\infty(TM)$ on M, we then have

(1) $R(X,Y)Z = -R(Y,X)Z$,
(2) $g(R(X,Y)Z, W) = -g(R(X,Y)W, Z)$,
(3) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$,
(4) $g(R(X,Y)Z, W) = -g(R(Z,W)X, Y)$.

Let (M, g) be a Riemannian manifold, $p \in M$ and $X, Y, Z, W \in T_pM$ be tangent vectors at p such that the two sections $\text{span}_{\mathbf{R}}\{X, Y\}$ and $\text{span}_{\mathbf{R}}\{Z, W\}$ are identical.

Definition 2.2.7 Let (M, g) be a Riemannian manifold and $p \in M$. Then the sectional curvature at p is given by

$$K_p(X, Y) = \frac{g(R(X, Y)Y, X)}{|X|^2\,|Y|^2 - g(X, Y)^2} \tag{2.2.4}$$

The next result shows how the curvature tensor can be expressed in terms of local coordinates.

Proposition 2.2.8 Let (M, g) be a Riemannian manifold and let (U, x) be local coordinates on M. For $j, k, l, m = 1, \cdots, n$, put

$$X_j = \frac{\partial}{\partial x_j},\ g_{jk} = g(X_j, X_k)\ \textit{and}\ R_{jklm} = g(R(X_j, X_k)X_l, X_m).$$

Then

$$R_{jklm} = \sum_{s=1}^{n} g_{sm} \left(\frac{\partial \Gamma^s_{kl}}{\partial x_j} - \frac{\partial \Gamma^s_{jl}}{\partial x_k} + \sum_{r=1}^{n} (\Gamma^r_{kl}\,\Gamma^s_{jr} - \Gamma^r_{jl}\,\Gamma^s_{kr}) \right),$$

where Γ^l_{jk} are the Christoffel symbols of the Levi-Civita connection ∇ of (M, g) with respect to (U, x).

Example 2.13 For n-dimensional vector space $\mathbf{R}^n$ equipped with the Euclidian metric g_0, the set $\{\partial/\partial x_1, \cdots, \partial/\partial x_n\}$ is a global frame for the tangent bundle $T\mathbf{R}^n$. In this situation, we have $g_{jk} = \delta_{jk}$, so Γ^l_{jk} by Example 2.10. This implies that $R \equiv 0$ so $\mathbf{R}^n$ is flat.

We define the Ricci and scalar curvatures of Riemannian manifolds. These are obtained by taking traces over the curvature tensor and play an important role in Riemnnian geometry.

Definition 2.2.9 Let (M, g) be a Riemannian manifold, then

(1) the Ricci curvature $\mathrm{Ric} : C^\infty(TM) \times C^\infty(TM) \longrightarrow C^\infty_0(TM)$ by

$$\mathrm{Ric}(X, Y) = \sum_{j=1}^{n} g(R(X, e_j)e_j, Y) \tag{2.2.5}$$

(2) the scalar curvature $s \in C^\infty(M)$ by

$$s = \sum_{j=1}^{n} \mathrm{Ric}(e_j, e_j) = \sum_{j=1}^{n}\sum_{k=1}^{n} g(R(e_j, e_k)e_k, e_j). \tag{2.2.6}$$

Here $\{e_1, \cdots, e_n\}$ is any local orthonormal frame for the tangent bundle.

2.3 Principal circle bundles

Definition 2.3.1 Let P and M be a C^∞ manifolds, $\pi : P \longrightarrow M$ a C^∞ map of P onto M, and G a Lie group acting on P to the right. Then is called a principal G-bundle if
(1) G acts freely on P,
(2) $\pi(p_1) = \pi(p_2)$ if and only if there exists $g \in G$ such that $p_1 g = p_2$,
(3) P is locally trivial over M, i.e. for every $m \in M$ there exists a neighborhood $\mathcal{U}$ of m and a map $F_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow G$ such that for every $p \in \pi^{-1}(\mathcal{U})$ and $g \in G, F_{\mathcal{U}}(pg) = (F_{\mathcal{U}}(p))g$, and such that the map $\psi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times G$ taking p to $(\pi(p), F_{\mathcal{U}}(p))$ is a diffeomorphism.

We turn to the case where $G = S^1$, in which case we say that P is a principal circle bundle over M and we study the group structure of the set $\mathcal{P}(M, S^1)$ of all principal circle bundles over M. Our treatment is based on .

Given $P, P' \in \mathcal{P}(M, S^1)$ with projections π, π', let

$$\Delta(P \times P') = \{(u, u') \in P \times P' \mid \pi(u) = \pi'(u')\}.$$

We say $(u_1, u_1') \sim (u_2, u_2')$ if there exists $s \in S^1$ such that $u_1 s = u_2$ and $u_1' s^{-1} = u_2'$. Note that since S^1 is abelian,

$$u_3 = u_2 t = u_1 st, \quad u_3' = u_2' t^{-1} = u_1' s^{-1} t^{-1} = u_1' (st)^{-1}.$$

Let $P + P' = \Delta(P \times P')/\sim$ and $\pi'' : P + P' \longrightarrow M$ the induced projection. S^1 acts on $\Delta(P \times P')$ by

$$(u, u')s = (us, u').$$

Now if $(u_1, u_1') \sim (u_2, u_2')$, $u_1 t = u_2$ and $u_1' t^{-1} = u_2'$, we have $u_2 s = u_1 ts = (u_1 s)t$. Therefore $(u_1 s, u_1') \sim (u_2 s, u_2')$ and hence S^1 acts on $P + P'$.

(1) S^1 acts freely : Suppose $u"s = u", u" \in P + P'$ and suppose (u, u') represents $u"$. Then $(u, u') \sim (us, u')$, so that $u' s^{-1} = u'$ and hence $s = 1 \in S^1$.

(2) S^1 acts transitively on fibers : Suppose $u"_1, u"_2 \in \pi"^{-1}(m)$ and $(u_1, u_1'), (u_2, u_2')$ are representatives. Then $u_2 = u_1 s, u_2' = u_1' s', s, s' \in S^1$. Now

$$(u_2, u_2') \sim (u_2 s, u_1') = (u_1 s s', u_1') = (u_1, u_1') s s'$$

and hence $u"_2 = u"_1 s s'$.

(3) $P + P'$ is locally trivial : If $F_{\mathcal{U}}(u) = g, F_{\mathcal{U}}'(u') = g'$, set $F"_{\mathcal{U}}(u, u') = g g'$. Then $F"_{\mathcal{U}}(us, u') = g s g' = g g' s$.

Theorem 2.3.2 Under the operation $+$, $\mathcal{P}(M, S^1)$ is an abelian group.

Proof. Let P_0 be the trivial bundle and $\alpha : P \longrightarrow P + P_0$ defined by

$$\alpha(u) = [(u, (\pi(u), 1))].$$

Then is a bundle isomorphism:

$$\begin{aligned} \alpha(us) &= [(us, (\pi(us), 1))] \\ &= [(u, (\pi(u), 1))s] \\ &= [(u, (\pi(u), 1))]s \\ &= \alpha(u)s, \\ \alpha^{-1}[(u, (\pi(u), g))] &= \alpha^{-1}[(ug^{-1}, (\pi(u), 1))] \; = \; ug^{-1}. \end{aligned}$$

Let $-P$ be a manifold diffeomorphic to P and $-u$ the point corresponding to u. Define $-\pi : -P \longrightarrow -M$ by $-\pi(-u) = \pi(u)$. S^1 acts on $-P$ by $(-u)s = -(us^{-1})$. Then $-P \in \mathcal{P}(M, S^1)$. Now let $(u_1, -u_2) \in \Delta(P \times (-P))$; then there exists a unique $s \in S^1$ such that $u_1 = u_2 s$. Let $\alpha : P + (-P) \longrightarrow P_0$ be defined by

$$\alpha([(u_1, -u_2)]) = (\pi(u_1), s).$$

Then α is a bundle isomorphism.

Let $\Delta(P \times P' \times P'') = \{(u, u', u'') \mid \pi(u) = \pi(u') = \pi(u'')\ \}$ and define the equivalence $\sim$ by $(u, u', u'') \sim (us, u's^{-1}s', u''s'^{-1})$. Then $\Delta(P \times P' \times P'')/\sim$ is naturally isomorphic to $(P + P') + P'', ((u's^{-1}, us)s', u''s'^{-1})$ and to $P + (P' + P''), (u's, (u's', u''s'^{-1})s^{-1})$. Now if $(u_1, u_1', u_1'') \sim (u_2, u_2', u_2'')$, then

$$u_2 = u_1 t, \quad u_2' = u_1' t^{-1} t', \quad u_2'' = u_1'' t'^{-1}.$$

Then $u_2s = u_1ts = (u_1s)t$ so that the right action preserves $\sim$. Finally, $P + P'$ is isomorphic to $P' + P$ by

$$[(u, u')] \longleftrightarrow [(u', u)], \quad (us, u') \sim (u, u's). \qquad \square$$

Let G_m be the cyclic subgroup of S^1 order m and $P \in \mathcal{P}(M, S^1)$. Since S^1 acts on P on the right, so does G_m. Then P/G_m is a principal bundle over M with group S^1/G_m. But $S^1/G_m \cong S^1$ and hence we can consider $P/G_m \in \mathcal{P}(M, S^1)$. More precisely, let $[u]$ be an element of P/G_m that is represented by $u \in P$. Define the action of S^1 on P/G_m by setting $[u]s = [us']$, where $s = (s')^m$. This definition is independent of the choice u of and s'. For if $g \in G_m$, then

$$[ug]s = [ugs'] = [us'g] = [us'] = [u]s,$$

and if $(s'')^m = s$, then

$$((s')^{-1}s'')^m = 1$$

so that $(s')^{-1}s'' \in G_m$ and hence

$$[us''] = [us'(s')^{-1}s''] = [us'].$$

Theorem 2.3.3 Let P, G_m and P/G_m be as above. Then $P/G_m \cong m \cdot P$.

Proof. From the definition above, it follows by induction that $m \cdot P$ can be defined by

$$\Delta(P \times \cdots \times P) = \{(u_1, \cdots, u_m) \in P \times \cdots \times P \mid \pi(u_1) = \cdots = \pi(u_m)\},$$

two elements of which, say $(u_1, \cdots, u_m)$ and $(u_1s_1, \cdots, u_ms_m)$, are equivalent if and only if $s_1 \cdots s_m = 1$. The quotient space of $\Delta(P \times \cdots \times P)$ by this relation is $m \cdot P$. The action of S^1 on $m \cdot P$ is given by

$$[(u_1, \cdots, u_m)] = [(u_1s, u_2, \cdots, u_m)] \;\; (s \in S^1).$$

Define $\phi : P/G_m \longrightarrow m \cdot P$ by

$$\phi([u]) = [(u, \cdots, u)],$$

which is independent of the choice of u, for if $g \in G_m$, then

$$\phi([ug]) = [(ug, \cdots, ug)] = [(u, \cdots, u)].$$

Now take $s \in S^1$ and s' such that $(s')^m = s$. Then

$$\begin{aligned} \phi([u]s) &= \phi[(us')] \\ &= [(us', \cdots, us')] \\ &= [(u, \cdots, u)s] \\ &= (\phi([u]))s. \end{aligned}$$

Therefore ϕ is a bundle isomorphism of P/G_m onto $m \cdot P$. $\square$

2.4 Complex manifolds

We recall the definition of complex manifolds. Roughly speaking, a complex manifold is a topological space that lotally looks like a neighborhood in $\mathbf{C}^n$.

Definition 2.4.1 A Hausdorff space M is called a complex manifold of complex dimension n, if M satisfies the following properties:

1) There exists an open covering $\{\mathcal{O}_\lambda\}$ of M and, for each λ, there exists a homeomorphism $\psi_\lambda : \mathcal{O}_\lambda \longrightarrow \psi_\lambda(\mathcal{O}_\lambda) \subset \mathbf{C}^n$;

2) For any two open sets $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ with nonempty intersection, maps

$$\begin{aligned} f_{\mu\lambda} = \psi_\mu \circ \psi_\lambda^{-1} : \psi_\lambda(\mathcal{O}_\lambda \cap \mathcal{O}_\mu) \longrightarrow \psi_\mu(\mathcal{O}_\lambda \cap \mathcal{O}_\mu) \\ f_{\lambda\mu} = \psi_\lambda \circ \psi_\mu^{-1} : \psi_\mu(\mathcal{O}_\lambda \cap \mathcal{O}_\mu) \longrightarrow \psi_\lambda(\mathcal{O}_\lambda \cap \mathcal{O}_\mu) \end{aligned} \tag{2.4.1}$$

are holomorphic.

The set $\{(\mathcal{O}_\lambda, \psi_\lambda)\}$ is called a system of holomorphic coordinate neighborhoods.

Example 2.14 We define the equivalent relation $\sim$ on $\mathbf{C}^{n+1}\backslash\{0\}$;

for $z, w \in \mathbf{C}^{n+1}\backslash\{0\}, z \sim w$ if there exists a non-zero complex number α such that $w = \alpha z$.

The complex projective space $\mathbf{CP}^n$ is the set of equivalence classes $\mathbf{C}^{n+1}\backslash\{0\}/\sim$ with the quotient topology from $\mathbf{C}^{n+1}\backslash\{0\}$.

Denote $\mathcal{O}_k = \{[z_1 : \cdots : z_k : \cdots : z_{n+1}] \in \mathbf{CP}^n \mid z_k \neq 0\}$ and let $\psi_k : \mathcal{O}_k \longrightarrow \mathbf{C}^n$ be the map defined by

$$\psi_k([z_1 : \cdots : z_k : \cdots : z_{n+1}]) = \left(\frac{z_1}{z_k}, \cdots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \cdots, \frac{z_{n+1}}{z_k}\right).$$

Then, $\psi_k^{-1}(w_1, \cdots, w_n) = [w_1 : \cdots : w_{k-1} : 1 : w_{k+1} : \cdots : w_n])$ and therefore

$$\psi_l \circ \psi_k^{-1}(z_1, \cdots, z_n) = \left(\frac{z_1}{z_l}, \cdots, \frac{z_{k-1}}{z_l}, \frac{1}{z_l}, \frac{z_k}{z_l}, \cdots, \frac{z_{l-1}}{z_l}, \frac{z_{l+1}}{z_l}, \cdots, \frac{z_{n+1}}{z_l}\right).$$

Thus, $\psi_l \circ \psi_k^{-1}$ is holomorphic and the complex projective space is a complex manifold.

Definition 2.4.2 Let $(\mathcal{O}, \psi)$ be a holomorphic coordinate neighborhood of a complex manifold M. A function $f : \mathcal{O} \longrightarrow \mathbf{C}$ is holomorphic if the function $f \circ \psi^{-1} : \psi(\mathcal{O}) \longrightarrow \mathbf{C}$ is holomorphic.

Definition 2.4.3 Let M, N be complex manifolds and $(\mathcal{O}, \psi)$ be a holomorphic coordinate neighborhood of $x \in M$. A continuous map $\phi : M \longrightarrow N$ is holomorphic if for any $x \in M$ and for any holomorphic coordinate neighborhood $(\mathcal{O}', \psi')$ of N such that $\phi(x) \in \mathcal{O}'$ and $\phi(\mathcal{O}) \subset \mathcal{O}'$, $\psi' \circ \phi \circ \psi^{-1} : \psi(\mathcal{O}) \longrightarrow \psi'(\mathcal{O}')$ is holomorphic.

Since the coordinate changes are biholomorphic the above definition of holomorphicity for maps is independent of the choice of local holomorphic neighborhood systems.

Definition 2.4.4 M is called a complex submanifold of a complex manifold $\overline{M}$, if M satisfies the following conditions:
(1) M is a submanifold of $\overline{M}$ as a differential manifold;
(2) the injection $\iota : M \longrightarrow \overline{M}$ is holomorphic.

Let M be an n-dimensional complex manifold. Identifying the local complex coordinates $(z_1, \cdots, z_n)$ with $(x_1, y_1, \cdots, x_n, y_n)$, where $z_k = x + iy \quad (k =$

$1, 2, \cdots, n$), we regard M as $2n$-dimensional differentiable manifold. The tangent space T_xM of M at a point $x \in M$ has a natural basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}$. For $k = 1, 2, \cdots, n$, we put

$$J_x\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad J_x\left(\frac{\partial}{\partial y_n}\right) = -\frac{\partial}{\partial x_n}. \tag{2.4.2}$$

Then J_x defines an isomorphism $J_x : T_xM \longrightarrow T_xM$. In fact, if we take other local complex coordinates $(w_1, \cdots, w_n)$, where $w_k = u_k + iv_k$, then they satisfy the Cauchy-Riemann equations,

$$\frac{\partial x_k}{\partial u_l} = \frac{\partial y_k}{\partial v_l}, \quad \frac{\partial x_k}{\partial v_l} = -\frac{\partial y_k}{\partial u_l}$$

for $k, l = 1, 2, \cdots, n$. Hence

$$J_x\left(\frac{\partial}{\partial u_k}\right) = \frac{\partial}{\partial v_k}$$

and

$$J_x\left(\frac{\partial}{\partial v_k}\right) = -\frac{\partial}{\partial u_k}$$

Thus J_x is independent of the choice of holomorphic coordinates and is well-defined. Regarding J as the map of the tangent bundle $TM = \cup_{x\in M} T_xM$, we call J the almost complex structure of M.

Definition 2.4.5 A differential manifold M is said to be an almost complex manifold if there exists a linear map $J : TM \longrightarrow TM$ satisfying (2.4.2) and J is said to be an almost complex structure of M.

As we have shown, a complex manifold M admits a naturally induced almost complex structure from complex structure, given by (2.4.2), and consequently M is an almost complex manifold.

Proposition 2.4.6 An almost complex manifold M is even-dimensional.

The Nijenhuis tensor N_J of an almost complex structure J is defined by

$$N_J(X, Y) = J[X, Y] - [JX, Y] - [X, JY] - J[JX, JY] \tag{2.4.3}$$

for any $X, Y \in TM$. An almost complex structure J on M is called integrable if there exists a complex structure on M and J is induced from the complex structure on M. The following theorem is due to Newlander and Nilenberg, see for example Appendix 8 in [26].

Definition 2.4.7 An almost complex structure J is integrable if and only if $N_J = 0$.

We recall some algebraic results on complex vector spaces, applied to tangent and cotangent spaces of complex manifolds.

Let M be an almost complex manifold with almost complex structure J. Then J can be extended an isomorphism of $T_x^C M = \{X + iY | X, Y \in T_x M\}$. We define $T_x^{(1,0)}M$ and $T_x^{(0,1)}M$ respectively by

$$\begin{aligned} T_x^{(1,0)}M &= \{X - iJX \mid X \in T_x M\}, \\ T_x^{(0,1)}M &= \{X + iJX \mid X \in T_x M\}. \end{aligned}$$

Then we have

$$T_x^C M = T_x^{(1,0)}M \oplus T_x^{(0,1)}M. \tag{2.4.4}$$

We note that $Z \in T^{(1,0)}M$ if and only if $JZ = iZ$, and that $Z \in T^{(1,0)}M$ if and only if $Z = -iZ$.

Definition 2.4.8 A vector field Z is said to be a vector field of type $(1,0)$ if $Z \in T^{(1,0)}M$ and of type $(0,1)$ if $Z \in T^{(0,1)}M$.

Let

$$T^C M = \bigcup_{x \in M} T_x^C M, \quad T^{(1,0)} = \bigcup_{x \in M} T_x^{(1,0)}M, \quad T^{(0,1)}M = \bigcup_{x \in M} T_x^{(0,1)}M.$$

Let M be an n-dimensional complex manifold and let $(z_1, \cdots, z_n)$ be complex coordinates in a neighborhood of x. We regard that M is a $2n$-dimensional differentiable manifold with local coordinates $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$, where $z_k = x_k + iy_k$. Then $\left\{ \dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial y_1}, \cdots, \dfrac{\partial}{\partial x_n}, \dfrac{\partial}{\partial y_n} \right\}$ is a basis of $T_x M$ and also a basis of $T_x^C M$. By definition of J which is induced from the complex structure of M, it follows

$$\begin{aligned}
\frac{\partial}{\partial x_k} &= \frac{1}{2}\left(\frac{\partial}{\partial x_k} - iJ\frac{\partial}{\partial x_k}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_k} + iJ\frac{\partial}{\partial x_k}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right).
\end{aligned}$$

We put

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$$

which yields

$$\frac{\partial}{\partial x_k} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k}, \quad \frac{\partial}{\partial y_k} = i\left(\frac{\partial}{\partial z_k} - \frac{\partial}{\partial \bar{z}_k}\right), \quad \overline{\frac{\partial}{\partial z_k}} = \frac{\partial}{\partial \bar{z}_k}. \tag{2.4.5}$$

From (2.4.5), we know that any $X \in T_x^C M$ can be expressed as a linear combination of $\frac{\partial}{\partial z_k}$ and $\frac{\partial}{\partial \bar{z}_k}$, $k = 1, 2, \cdots, n$. On the other hand, $\sum_{k=1}^n \left(a_k \frac{\partial}{\partial z_k} + b_k \frac{\partial}{\partial \bar{z}_k}\right) = 0$ implies $a_k = b_k = 0$ for $k = 1, 2, \cdots, n$. Therefore we conclude that

$$\left\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_n}\right\}$$

forms a basis of $T_x^C M$.

For a natural basis of tangent space $T_x M$ at $x \in M$ we consider its dual basis $\{dx_1, dy_1, \cdots, dx_n, dy_n\}$ in the cotangent space $T_x^* M$. We put

$$dz_k = dx_k + idy_k, \quad d\bar{z}_k = dx_k - idy_k.$$

consequently, it follows

$$dz_k\left(\frac{\partial}{\partial z_l}\right) = \delta_{kl}, \quad dz_k\left(\frac{\partial}{\partial \bar{z}_l}\right) = 0.$$

In the same way we have

$$d\bar{z}_k\left(\frac{\partial}{\partial z_l}\right) = 0, \quad d\bar{z}_k\left(\frac{\partial}{\partial \bar{z}_l}\right) = \delta_{kl}.$$

This shows that $\{dz_1, \cdots, dz_n, d\bar{z}_1, \cdots, d\bar{z}_n\}$ is the dual basis of $\left\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_n}\right\}$.

For a C^∞ function f defined on a neighborhood of $x \in M$, we have

$$df\left(\frac{\partial}{\partial z_k}\right) = \frac{\partial f}{\partial z_k}, \quad df\left(\frac{\partial}{\partial \bar{z}_l}\right) = \frac{\partial f}{\partial \bar{z}_l}.$$

and therefore

$$df = \sum_{k=1}^{n}\left(\frac{\partial f}{\partial z_k}dz_k + \frac{\partial f}{\partial \bar{z}_l}d\bar{z}_k\right). \tag{2.4.6}$$

Definition 2.4.9 Let r be a positive integer such that $r = p + q$ where p, q are nonnegative integers. Let an r-form ω on M be spanned by the set $\{dz_{k_1}, \cdots, dz_{k_p}, d\bar{z}_{l_1}, \cdots, d\bar{z}_{l_q}\}$, where $\{k_1, \cdots, k_p\}$ and $\{l_1, \cdots, l_q\}$ run over the setoff all increasing multi-indices of length p and q. Then ω is called a complex differential form of type (p, q).

Since an r-form of type (p, q), we have just defined, can be expressed as

$$\omega = \sum_{k_1<\cdots<k_p}^{n} \omega_{k_1\cdots k_p l_1\cdots l_q} dz_{k_1} \wedge \cdots \wedge dz_{k_p} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_q}. \tag{2.4.7}$$

We can easily prove the following.

Lemma 2.4.10 Let ω and η be complex differential forms.
(1) If ω is of type (p, q), then $\overline{\omega}$ is of type (q, p).
(2) If ω is of type (p, q) and η is of type (p', q'), then $\omega\wedge\eta$ is of type $(p+p', q+q')$.

Further, using (2.4.7), we compute the exterior differential dω of any complex r-form ω of type (p, q).

$$\begin{aligned} d\omega &= \sum_{\substack{k_1<\cdots<k_p \\ l_1<\cdots<l_q}} \sum_{k=1}^{n} \left(\frac{\partial \omega_{k_1\cdots k_p l_1\cdots l_q}}{\partial z_k} dz_k + \frac{\partial \omega_{k_1\cdots k_p l_1\cdots l_q}}{\partial \bar{z}_k} d\bar{z}_k\right) \wedge dz_{k_1} \wedge \cdots \\ &\qquad\qquad\qquad\qquad \cdots \wedge dz_{k_p} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_q} \\ &= \sum_{\substack{k_1<\cdots<k_{p+1} \\ l_1<\cdots<l_q}} \sum_{s=1}^{p+1} (-1)^{s-1} \frac{\partial \omega_{k_1\cdots \hat{k_s}\cdots k_{p+1} l_1\cdots l_q}}{\partial z_{k_s}} dz_{k_1} \wedge \cdots \wedge dz_{k_{p+1}} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_q} \end{aligned}$$

$$+(-1)^p \sum_{\substack{k_1<\cdots<k_p \\ l_1<\cdots<l_{q+1}}} \sum_{t=1}^{q+1} \frac{\partial \omega_{k_1\cdots k_p l_1\cdots \hat{l_t}\cdots l_{q+1}}}{\partial \bar{z}_{k_s}} dz_{k_1} \wedge \cdots \wedge dz_{k_p} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_{q+1}}.$$

Therefore, $d\omega$ is expressed as a sum of $(r+1)$-forms of type $(p+1,q)$ and of type $(p,q+1)$, denoted respectively by $\partial\omega$ and $\overline{\partial}\omega$. Thus we obtain two differential operators ∂ and $\overline{\partial}$, and this information is written as

$$d\omega = \partial\omega + \overline{\partial}\omega, \qquad d = \partial + \overline{\partial}.$$

Proposition 2.4.11 For differential operators $\partial, \overline{\partial}$ and r-form ω, we have

$$\partial^2\omega = \overline{\partial}^2\omega = 0, \quad (\partial\overline{\partial} + \overline{\partial}\partial)\omega = 0, \quad \partial\overline{\omega} = \overline{\overline{\partial}\omega}, \quad \overline{\partial}\overline{\omega} = \overline{\partial\omega}.$$

Proof. Since $d^2 = 0$, we compute

$$\begin{aligned} 0 &= d^2\omega = d(\partial\omega + \overline{\partial}\omega) \\ &= \partial(\partial\omega + \overline{\partial}\omega) + \overline{\partial}(\partial\omega + \overline{\partial}\omega) \\ &= \partial^2\omega + (\partial\overline{\partial} + \overline{\partial}\partial)\omega + \overline{\partial}^2\omega. \end{aligned}$$

As $\partial^2\omega$ is type of $(p+2,q)$, $(\partial\overline{\partial}+\overline{\partial}\partial)\omega$ is of type $(p+1,q+1)$ and $\overline{\partial}^2\omega$ is of type $(p,q+2)$, we conclude that each of them vanishes.

To prove the other relations, we recall the definition of $d\overline{\omega}$, that is, $d\overline{\omega} = \overline{d\omega}$. Therefore, $d\overline{\omega} = \overline{\partial\omega + \overline{\partial}\omega}$. On the other hand, using , it follows $d\overline{\omega} = \partial\overline{\omega} + \overline{\partial}\overline{\omega}$. Comparing the type of the right hand members of the last two equations, we get other two relations of the proposition. $\square$

Chapter 3

Geometrical Structures

3.1 Contact structures

Let $\mathcal{D} \subset TM$ be a field of hyperplane on M. Locally such a hyperplane field can always be written as the kernel of non-vanishing 1-form η. One way to see this is to choose an auxiliary Riemannian metric g on M and then to define $\eta = g(X, \cdot)$, where X is local non-zero section of the line bundle $\mathcal{D}^{\perp}$ (the orthogonal complement of $\mathcal{D}$ in TM). We see that the existence of a globally defined 1-form η with $\mathcal{D} = \ker \eta$ is equivalent to the orientability of $\mathcal{D}^{\perp}$. If satisfies the Frobenius integrability condition

$$\eta \wedge d\eta = 0,$$

then $\mathcal{D}$ is an integrable hyperplane field. Equivalently, this integrability condition can be written as

$$X, Y \in \mathcal{D} \Longrightarrow [X, Y] \in \mathcal{D}.$$

Contact structures are in a certain sense the exact opposite of integrable hyperplane fields.

Definition 3.1.1 Let M^{2n+1} be a $(2n+1)$-dimensional differential manifold. A contact structure on M^{2n+1} is a maximally non-integrable hyperplane field $\mathcal{D} = \ker \eta \subset TM^{2n+1}$, that is, the defining 1-form η is required to satisfy

$$\eta \wedge (d\eta)^n \neq 0, \tag{3.1.1}$$

Such a 1-form η is called a contact form. The pair $(M^{2n+1}, \mathcal{D})$ is called a contact manifold.

Observe that in this case $\eta \wedge (d\eta)^n \neq 0$ is a volume form on M^{2n+1}; in particular, M^{2n+1} needs to be orientable. The condition $\eta \wedge (d\eta)^n \neq 0$ is independent of the specific choice of and thus is indeed a property of $\mathcal{D} = \ker \eta$. Any other 1-form defining the same hyperplane field must be of the form $f\eta$ for some smooth function $f : M^{2n+1} \longrightarrow \mathbf{R}\backslash\{0\}$, and we have

$$f\eta \wedge (d(f\eta))^n = f\eta \wedge (fd\eta + df \wedge \eta)^n = f^{n+1}\, \eta \wedge (d\eta)^n \neq 0.$$

We see that if n is odd, the sign of this volume form depends only on $\mathcal{D}$, not the choice of η.

Definition 3.1.2 Associated with a contact form η one has the Reeb vector field ξ [35], defined by the equations

$$(1)\ \ d\eta(\xi, \cdot) \equiv 0, \quad (2)\ \ \eta(\xi) \equiv 1. \tag{3.1.2}$$

As a skew-symmetric form of maximal rank $2n$, the form $d\eta|_{T_pM}$ has a 1-dimensional kernel for each $p \in M^{2n+1}$. Hence equation (1) defines a unique line field $\langle \xi \rangle$ on M^{2n+1}. the contact condition $\eta \wedge (d\eta)^n \neq 0$ implies that η is non-trivial on that line field, so a global vector field is defined by additional normalization condition (2).

We now prove the classical theorem of Darboux (see also [37]).

Theorem 3.1.3 About each point of a contact manifold $(M^{2n+1}, \mathcal{D})$, there exist local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ with respect to which

$$\eta = dz - \sum_{k=1}^{n} y_k dx_k.$$

Proof. In some coordinate neighborhood, choose a $2n$-ball transverse to ξ; $d\eta$ is symplectic form on this ball, and hence there exist local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, u)$ such that $d\eta = \sum_{k=1}^{n} dx_k \wedge dy_k$. Now $d(\eta + \sum_{k=1}^{n} y_k dx_k) = 0$ so that $df = \eta + \sum_{k=1}^{n} y_k dx_k$ for some function f. Now

$$\eta \wedge (d\eta)^n = df \wedge dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \neq 0.$$

Therefore, df is independent of $dx_1, \cdots, dx_n, dy_1, \cdots, dy_n$, and hence we can regard x_k, y_k and $z = f$ as a coordinate system. $\square$

Example 3.1 In effect, we have already seen that $\mathbf{R}^{2n+1}$ with the Darboux form $\eta = dz - \sum_{k=1}^{n} y_k dx_k$ is a contact manifold. The Reeb vector field ξ is $\partial/\partial z$ and contact subbundle $\mathcal{D}$ is spanned by

$$\frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y_k}$$

for $k = 1, 2, \cdots, n$.

Turning to more standard examples, we prove the following theorem [17].

Theorem 3.1.4 Let $\iota : M \longrightarrow \mathbf{R}^{2n+2}$ be a smooth hupersurface immersed in $\mathbf{R}^{2n+2}$ and suppose that no tangent space of M contains the origin of $\mathbf{R}^{2n+2}$. Then M has a contact form.

Proof. Consider the 1-form η_0 on $\mathbf{R}^{2n+2}$ defined by

$$\eta_0 = x_1 dx_2 - x_2 dx_1 + \cdots + x_{2n-1} dx_{2n+2} - x_{2n+2} dx_{2n-1}$$

and let $V_1, \cdots, V_{2n+1}$ be $(2n+1)$ linearly independent vectors at a point $p = (p_1, \cdots, p_{2n+2})$ and define a vector $W = (W_1, \cdots, W_{2n+2})$ at p with components

$$W_k = *dx_k(V_1, \cdots, V_{2n+1})$$

where $*$ is the Hodge star operator of the euclidian metric on $\mathbf{R}^{2n+2}$. Then W is normal to the hypersurface spanned by $V_1, \cdots, V_{2n+1}$. Now regard p as a vector with components p_k. Then

$$\left(\eta \wedge (d\eta)^n\right)(V_1, \cdots, V_{2n+1}) = \sum_{k=1}^{2n+2} p_k W_k.$$

Thus if no tangent space of M regarded as a hyperplane in $\mathbf{R}^{2n+2}$ contains the origin, then $\eta = \iota^* \eta_0$ is a contact form on M. $\square$

As a special case, we see that an odd-dimensional sphere S^{2n+1} carries a contact form. Moreover, η_0 on S^{2n+1} is invariant under the reflection through the origin, $(x_1, \cdots, x_{2n+2}) \mapsto (-x_1, \cdots, -x_{2n+2})$ and hence the real projective space $\mathbf{RP}^{2n+1}$ is also a contact manifold. J. A. Wolf [42] then considered more general quotients of S^{2n+1} and proved that a complete connected odd-dimensional Riemannian manifold of positive constant curvature inherits a contact structure from the form η_0.

3.2 Contact metric structures and almost contact metric structures

Definition 3.2.1 Let M be a differentiable manifold of $\dim_{\mathbf{R}} = 2n+1$. M is said to have an almost contact structure (Φ, ξ, η) or (M, Φ, ξ, η) is said to be an almost contact manifold if it admits a $(1,1)$-tensor field Φ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \quad \text{and} \quad \Phi^2 = -id + \eta \otimes \xi. \tag{3.2.1}$$

First we recall some properties of almost contact manifolds.

Proposition 3.2.2 An almost contact manifold $(M; \Phi, \xi, \eta)$ admits a Riemannian metric g such that

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{3.2.2}$$

Definition 3.2.3 An almost contact metric structure (Φ, ξ, η, g) on M is an almost contact structure (Φ, ξ, η) with the associated metric g satisfying (3.2.2).

On a manifold M with a almost contact metric structure (Φ, ξ, η, g), we can find a particularly useful local orthonormal basis. Let U_α be a coordinate neighborhood and take X_1 a unit vector field on orthogonal to ξ. Then by (3.2.1) and (3.2.2), ΦX_1 is also a unit vector field orthogonal to both ξ and X_1. Now take X_2 to be a unit vector field on U_α orthogonal to ξ, X_1 and ΦX_1, then ΦX_2 is a unit vector field orthogonal to $\xi, X_1, \Phi X_1$ and X_2. Proceeding in this way, we obtain a local orthonormal basis

$$\{\xi, X_j, \Phi X_j \mid j = 1, 2, \cdots, n\} \tag{3.2.3}$$

called a Φ-basis.

Suppose M has an almost contact metric structure (Φ, ξ, η, g). We define a 2-form Ω on M by

$$\Omega(X, Y) = g(X, \Phi Y).$$

The skew-symmetry of Ω is immediate from equation (3.2.1) and (3.2.2) for

$$g(X, \Phi Y) = g(\Phi X, \Phi^2 Y) = -g(\Phi X, Y). \tag{3.2.4}$$

We call Ω the fundamental 2-form of the almost contact metric structure (Φ, ξ, η, g).

Proposition 3.2.4 Let M be a $(2n+1)$-dimensional differentiable manifold admitting a global 1-form η and a global 2-form Ω such that $\eta \wedge \Omega^n \neq 0$ everywhere. Then M admits an almost contact structure. If M is a contact manifold with a contact form η, then there exists an almost contact contact metric structure (Φ, ξ, η, g) such that the fundamental 2-form $\Omega = d\eta$.

By this proposition, a contact form η on M induces a almost contact metric structure (Φ, ξ, η, g) with $\Omega = d\eta$.

Definition 3.2.5 Let (Φ, ξ, η, g) be an almost contact metric structure on M. (Φ, ξ, η, g) is called a contact metric structure on M, or (M, Φ, ξ, η, g) is called a contact metric manifold if $d\eta$ becomes a fundamental 2-form on M.

Here we recall some examples of contact metric manifolds.

Example 3.2 We construct a contact metric structure on $\mathbf{R}^{2n+1} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \mid x_j, y_j, z \in \mathbf{R},\ j = 1, 2, \cdots, n\}$. For normalization convenience, we take as the standard contact form η on $\mathbf{R}^{2n+1}$,

$$\eta = \frac{1}{2}\left(dz - \sum_{k=1}^{n} y_k dx_k\right).$$

The Reeb vector field ξ is $2\frac{\partial}{\partial z}$, the Riemannian metric

$$g = \frac{1}{4}\left(\eta \otimes \eta + \sum_{k=1}^{n}(dx_k^2 + dy_k^2)\right)$$

and the tensor field Φ given by the matrix

$$\begin{pmatrix} \mathbf{0} & I_n & 0 \\ I_n & \mathbf{0} & 0 \\ 0 & y_1 \cdots y_n & 0 \end{pmatrix},$$

where I_n means a unit matrix, give a contact metric structure (Φ, ξ, η, g) on $\mathbf{R}^{2n+1}$.

Example 3.3 Tashiro [40] showed that a real hypersurface M in a almost complex manifold $(\overline{M}, J)$ has an almost contact structure. Let $\iota : M \longrightarrow \overline{M}$ be an immersion. We can choose a vector field C along M transverse to M in $\overline{M}$ such that JC is tangent to M. Indeed, if $J\iota_* X$ were tangent to M for every vector field X on M, then $J\iota_* X = \iota_* FX$ for a tensor field F on M. Applying J to this equation gives $F^2 = -id$, that is, F is an almost complex structure on M, which is a contradiction. Thus, there exists a vector field ξ on M such that $C = -J\iota_*\xi$ is transverse to M. Now define a tensor field Φ of type (1,1) and a 1-form η on M by

$$J\iota_* X = \iota_* \Phi X - \eta(X) C. \tag{3.2.5}$$

Then applying J we have

$$-\iota_* X = \iota_* \Phi^2 X - \eta(\Phi X) C - \eta(X) \iota_* \xi \tag{3.2.6}$$

and hence $\Phi^2 = -id + \eta \otimes \xi$ and $\eta \circ \Phi = 0$. Taking $X = \xi$ in equation (3.2.5) we have $C = \iota_* \Phi \xi + \eta(\xi) C$, so that $\Phi \xi = 0$ and $\eta(\xi) = 1$.Thus, (Φ, ξ, η) defines an almost contact structure on M.

Moreover, let G be a Hermitian metric on $\overline{M}$ and take C to be a unit normal to M. Then JC is tangent to M so that JC defines ξ and the rest of the above procedure is repeated. In this case the induced metric $g = \iota^* G$ is compatible with the almost contact structure (Φ, ξ, η) since

$$\begin{aligned} g(X, Y) &= G(\iota_* X, \iota_* Y) \\ &= G(J\iota_* X, J\iota_* Y) \\ &= g(\Phi X, \Phi Y) + \eta(X)\eta(Y). \end{aligned}$$

3.3 Kähler structures

Let (M, J) be a complex manifold. We say that a Riemannian metric g on M is Hermitian metric if

$$g(JX, JY) = g(X, Y).$$

We can then define a 2-form ω_g by

$$\omega_g(X, Y) = g(JX, Y)$$

Usually, we call such a 2-form the Kähler form of g.

Definition 3.3.1 A Kähler manifold (M, g, J) is a complex manifold (M, J) with a Hermitian metric g satisfying $d\omega_g = 0$.

Let (M, g, J) be a Kähler manifold. We can extend the metric g $\mathbf{C}$-linearly to $T^{\mathbf{C}}M$. Recalling that $T^{(1,0)}M$ and $T^{(0,1)}M$ are the $\pm i$-eigenspaces of J, we see that $g(X, Y) = 0$ for $X, Y \in T^{(1,0)}M$ or $X, Y \in T^{(0,1)}M$ (use compatibility of J with the metric). Define $h(X, Y) = g(X, \overline{Y})$ for $X, Y \in T^{(1,0)}M$, then this defines a Hermitian inner product on $T^{1,0}M$.

If (M, J) is a complex manifold and J is compatible with the metric, the from the above, we get that

$$g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l}\right) = g\left(\frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{z}_l}\right) = 0$$

and in local coordinates, we can write

$$g_{k\bar{l}} = g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right)$$

so we have

$$\omega_g = \frac{i}{2} \sum_{k,l} g_{k\bar{l}}\, dz_k \wedge d\bar{z}_l.$$

On a Kähler manifold M, a Kähler metric is uniquely determined by its Kähler form. So we often denote a Kähler metric g by its Kähler form ω_g. Note that

$N(J) = 0$ on M, so M is a complex manifold and $d\omega_g = 0$, that is, ω_g is a closed form.

Let (M, g, J) be a Kähler manifold and ∇ be its Levi-Civita connection. We extend ∇ in a $\mathbf{C}$-linear way to $\Gamma(T^C M)$. Since M is also a complex manifold, we have local coordinates $(z_1, \cdots, z_n)$ and a local basis $\left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_n}\right)$ for $T^C M$. We define the Christoffel symbols Γ^m_{kl} by

$$\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial z_l} = \Gamma^m_{kl} \frac{\partial}{\partial \bar{z}_m} + \Gamma^{\bar{m}}_{kl} \frac{\partial}{\partial \bar{z}_m} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial \bar{z}_l} = \Gamma^m_{k\bar{l}} \frac{\partial}{\partial \bar{z}_m} + \Gamma^{\bar{m}}_{k\bar{l}} \frac{\partial}{\partial \bar{z}_m}.$$

Because $\nabla J = 0$ and, $J \frac{\partial}{\partial z_k} = i \frac{\partial}{\partial z_k}$ and $J \frac{\partial}{\partial \bar{z}_k} = -i \frac{\partial}{\partial \bar{z}_k}$, we see that

$$\nabla_{\frac{\partial}{\partial z_k}} \left(J \frac{\partial}{\partial z_l} \right) = J \nabla_{\frac{\partial}{\partial z_k}} \left(\frac{\partial}{\partial z_l} \right)$$

implies

$$i \left(\Gamma^m_{kl} \frac{\partial}{\partial \bar{z}_m} + \Gamma^{\bar{m}}_{kl} \frac{\partial}{\partial \bar{z}_m} \right) = J \left(\Gamma^m_{kl} \frac{\partial}{\partial \bar{z}_m} + \Gamma^{\bar{m}}_{kl} \frac{\partial}{\partial \bar{z}_m} \right) = i \left(\Gamma^m_{kl} \frac{\partial}{\partial \bar{z}_m} - \Gamma^{\bar{m}}_{kl} \frac{\partial}{\partial \bar{z}_m} \right)$$

and therefore $\Gamma^{\bar{m}}_{kl} = 0$. Similarly, $\Gamma^{\bar{m}}_{k\bar{l}} = \Gamma^m_{k\bar{l}} = 0$, so the only possible non-zero terms are Γ^m_{kl} and $\Gamma^{\bar{m}}_{\bar{k}\bar{l}} = \overline{\Gamma^m_{kl}}$.

Moreover, if $g_{k\bar{l}} = g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right)$ denote the metric tensor in local coordinates, then

$$\frac{\partial g_{l\bar{m}}}{\partial z_k} = \frac{\partial}{\partial z_k} g\left(\frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_m}\right) = g\left(\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_m}\right) = g\left(\Gamma^j_{kl} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_m}\right) = \Gamma^j_{kl} g_{j\bar{m}}$$

and hence

$$\Gamma^j_{kl} = g^{j\bar{m}} \frac{\partial g_{l\bar{m}}}{\partial z_k} = g^{j\bar{m}} \frac{\partial g_{k\bar{m}}}{\partial z_l}$$

so if the Kähler metric is given by $\{g_{k\bar{l}}\}$, its connection ∇ is given by $\{\Gamma^j_{kl}\}$.

Given a Kähler manifold (M, g, J) with its Kähler form

$$\omega_g = \frac{i}{2} \sum_{k,l} g_{k\bar{l}} \, dz_k \wedge d\bar{z}_l$$

and its compatible connection ∇, the Riemannian curvature tensor is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and can be extended in a $\mathbf{C}$-linear way to $T^{\mathbf{C}}M$.

Note that because J is parallel, that is $\nabla J = 0$, we have that

$$R(X,Y)JZ = JR(X,Y)Z.$$

Defining

$$R(X,Y,Z,W) = g(R(X,Y)W,Z),$$

we can easily see that

$$R(X,Y,JZ,JW) = R(X,Y,Z,W)$$

and because of the splitting $T^{\mathbf{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ into the $\pm i$-eigenspaces of J, we can deduce that $R(X,Y,Z,W) = 0$ unless Z and W are of different type.

In local coordinates $(z_1, \cdots, z_n)$, this means that the only possibly non-vanishing terms are essentially

$$R_{j\bar{k}l\bar{m}} = R\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_m}\right).$$

Using that

$$\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k} = g^{l\bar{m}} \frac{\partial g_{j\bar{m}}}{\partial z_k} \frac{\partial}{\partial z_l},$$

we can deduce that

$$R_{j\bar{k}l\bar{m}} = -\frac{\partial^2 g_{j\bar{k}}}{\partial z_l \partial \bar{z}_m} + g^{s\bar{t}} \frac{\partial g_{s\bar{k}}}{\partial z_l} \frac{\partial g_{j\bar{t}}}{\partial \bar{z}_m}.$$

We define the Ricci curvature $\mathrm{Ric}_{l\bar{m}}$ to be the trace of this, so we get

$$\mathrm{Ric}_{l\bar{m}} = g^{j\bar{k}} R_{j\bar{k}l\bar{m}} = -\frac{\partial^2}{\partial z_l \partial \bar{z}_m} (\log \det g_{j\bar{k}}).$$

In complex coordinates, we have found a nice expression for the Ricci curvature, but we need to check that it is still the same as that in the Riemannian case. So choose an orthonormal basis $(e_1, \cdots, e_{2n})$ such that $Je_j = e_{n+j}$ for $j = 1, \cdots, n$ and set $u_j = \frac{1}{\sqrt{2}}(e_j - iJe_j)$, then $\{u_j\}$ is a unitary basis. It follow that

$$R(u_j, \bar{u}_j) = \sum_k R(u_j, \bar{u}_j, u_k, \bar{u}_k) = \mathrm{Ric}\,(e_j, e_j).$$

Here we have used the first Bianchi identity for R:

$$R(e_j, Je_j, e_k, Je_k) + R(Je_j, e_k, e_j, Je_k) + R(e_k, e_j, Je_j, Je_k) = 0.$$

This shows that the Ricci curvature defined above is the same as the one in Riemannian geometry.

Recall that if $|X| = |Y| = 1$ and X is perpendicular to Y, then $R(X, Y, Y, X)$ is the sectional curvature of the plane spanned by X, Y. Set now

$$Z = \frac{1}{\sqrt{2}}(X - iJX) \text{ and } W = \frac{1}{\sqrt{2}}(Y - iJY),$$

then

Definition 3.3.2 The bisectional curvature is defined to be

$$R(Z, \overline{Z}, W, \overline{W}) = R(X, Y, Y, X) + R(X, JY, JY, X).$$

Proposition 3.3.3 A Kähler manifold (M, g, J) is said to be of constant bisectional curvature if there exists a constant λ such that in any local coordinates of M,

$$R_{j\bar{k}l\bar{m}} = \lambda(g_{j\bar{k}}g_{l\bar{m}} + g_{j\bar{m}}g_{l\bar{k}}).$$

Now we give some examples of Kähler manifolds.

Example 3.4 Since $\mathbf{C}^n$ can be identified with $\mathbf{R}^{2n}$, let $\langle , \rangle$ be the Euclidian metric of $\mathbf{R}^{2n}$. Then we have

$$\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle = \left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k} \right\rangle = \delta_{jk}, \quad \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k} \right\rangle = 0.$$

This, together with $J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$ and $J\left(\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial x_j}$, implies that is a Hermitian metric of $(\mathbf{C}^n, J)$. We put

$$\Omega = \sum_{j,k=1}^{n} (a_{jk} dx_j \wedge dx_k + b_{jk} dx_j \wedge dy_k + c_{jk} dy_j \wedge dy_k)$$

and note that

$$dx_j \wedge dy_k \left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_m}\right) = dx_j \wedge dy_k \left(\frac{\partial}{\partial y_l}, \frac{\partial}{\partial y_m}\right) = 0,$$
$$dx_j \wedge dy_k \left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial y_m}\right) = \delta_{jl}\, \delta_{km}.$$

Then

$$\Omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \sum_{l,m=1}^{n} a_{lm} \delta_{jl}\, \delta_{km} = a_{jk}.$$

On the other hand, it follows

$$\Omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = J\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k} = 0.$$

Hence we have $a_{jk} = 0$, In entirely the same way, we conclude $b_{jk} = \delta_{jk}$ and $c_{jk} = 0$. Thus, the Kähler form of $(\mathbf{C}^n, J)$ is represented by

$$\Omega = \sum_{k=1}^{n} dx_k \wedge dy_k.$$

From we conclude that $d\Omega = 0$ and that $(\mathbf{C}^n, J)$, with usual Euclidian metric, is a Kähler manifold.

Example 3.5 Let $M = B^n = \{z \in \mathbf{C}^n \mid |z| < 1\}$ and let

$$\omega_g = \frac{i}{2} \partial\overline{\partial} \log(1 - |z|^2).$$

Then $R_{j\bar{k}l\bar{m}} = -(g_{j\bar{k}} g_{l\bar{m}} + g_{j\bar{m}} g_{l\bar{k}})$ and (B^n, g) is Kähler manifold of constant bisectional curvature -1.

The end of this subsection, we recall the definition of hyperkähler manifolds.

Definition 3.3.4 (M, J_1, J_2, J_3, g) is a hyperkähler manifold if J_1, J_2, J_3 are complex structures on a complex manifold M satisfying

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -id,$$

and g is an Hermitian metric on M with respect to J_1, J_2 and J_3. We produce a normal complex contact metric manifold from hyperkähler manifolds in section 5.3.

Chapter 4

Sasakian Manifolds

4.1 Normal almost contact structures

Let (M, Φ, ξ, η) be an almost contact manifold and consider the manifold $M \times \mathbf{R}_+$. We denote a vector field on $M \times \mathbf{R}_+$ by $\left(X, f\frac{\partial}{\partial t}\right)$ where X is tangent to M, t the coordinate of $\mathbf{R}_+$ and f a C^∞-function on $M \times \mathbf{R}_+$. Define an almost complex structure J on $M \times \mathbf{R}_+$ by

$$J\left(X,\, f\,\frac{\partial}{\partial t}\right) = \left(\Phi X - f\xi,\ \eta(X)\,\frac{\partial}{\partial t}\right). \tag{4.1.1}$$

that $J^2 = -id$ is easy to check. If now J is integrable, we say that the almost contact structure (Φ, ξ, η) is normal .

As the vanishing of the Nijenhuis tensor N_J is a necessary and sufficient condition of normality in terms of the Nijenhuis tensor N_Φ. Since N_J is a $(1,2)$-tensor, it suffices to compute $N_J((X,0),(Y,0))$ and $N_J((X,0),(0,\frac{\partial}{\partial t}))$ for vector fields X and Y on M.

$$\begin{aligned} N_J((X,0),(Y,0)) &= -[(X,0),(Y,0)] + \left[\left(\Phi X,\ \eta(X)\,\frac{\partial}{\partial t}\right),\left(\Phi Y,\ \eta(Y)\,\frac{\partial}{\partial t}\right)\right] \\ &\quad -J\left[\left(\Phi X,\ \eta(X)\,\frac{\partial}{\partial t}\right),(Y,0)\right] - J\left[(X,0),\left(\Phi Y,\ \eta(Y)\,\frac{\partial}{\partial t}\right)\right] \\ &= -([X,Y],0) + \left([\Phi X,\Phi Y],\ (\Phi X\eta(Y) - \Phi Y\eta(X))\frac{\partial}{\partial t}\right) \\ &\quad -\left(\Phi[\Phi X,Y] + (Y\eta(X))\xi,\ \eta([\Phi X,Y])\frac{\partial}{\partial t}\right) \end{aligned}$$

$$
\begin{aligned}
& -\left(\Phi[X,\Phi Y]-(X\eta(Y))\xi,\ \eta([X,\Phi Y])\frac{\partial}{\partial t}\right)\\
= & \left(N_\Phi(X,Y)+2d\eta(X,Y)\xi,\ ((L_{\Phi X}\eta)(Y)-(L_{\Phi Y}\eta)(X))\frac{\partial}{\partial t}\right),
\end{aligned}
$$

$$
\begin{aligned}
N_J\left((X,0),\left(0,\frac{\partial}{\partial t}\right)\right) &= -\left[(X,0),\left(0,\frac{\partial}{\partial t}\right)\right]+\left[\left(\Phi X,\ \eta(X)\frac{\partial}{\partial t}\right),(\xi,0)\right]\\
&\quad -J\left[\left(\Phi X,\ \eta(X)\frac{\partial}{\partial t}\right),\left(0,\frac{\partial}{\partial t}\right)\right]-J[(X,0),(-\xi,0)]\\
&= \left(-[\Phi X,\xi],\ (\xi\eta(X))\frac{\partial}{\partial t}\right)+\left(\Phi[X,\xi],\ \eta([X,\xi])\frac{\partial}{\partial t}\right)\\
&= \left((L_\xi\Phi)X,\ (L_\xi\eta)(X)\frac{\partial}{\partial t}\right).
\end{aligned}
$$

We are thus lead to define four tensors N_1, N_2, N_3 and N_4 by

$$
\begin{aligned}
N_1(X,Y) &= N_\Phi(X,Y)+2d\eta(X,Y)\xi,\\
N_2(X,Y) &= (L_{\Phi X}\eta)(Y)-(L_{\Phi Y}\eta)(X),\\
N_3(X,Y) &= (L_\xi\Phi)X,\\
N_4(X,Y) &= (L_\xi\eta)(X).
\end{aligned}
$$

It is clear that the almost contact structure (Φ,ξ,η) is normal if and only if these four tensors vanish. However, we will show that the vanishing of N_1 implies the vanishing N_2, N_3 and N_4, so that the normality condition is

$$
N_\Phi(X,Y)+2d\eta(X,Y)\xi=0. \tag{4.1.2}
$$

The reminder of this subsection is devoted to poring this and other important properties of the tensors N_1, N_2, N_3 and N_4.

Proposition 4.1.1 For an almost contact structure (Φ,ξ,η), the vanishing of N_1 implies the vanishing of N_2, N_3 and N_4.

Proof. Since $N_1=0$, we have

$$
\begin{aligned}
0 &= N_\Phi(X,Y)+2d\eta(X,Y)\xi,\\
&= -[X,\xi]+\eta([X,\xi])\xi-\Phi[\Phi X,\xi]+(X\eta(\xi))\xi-(\xi\eta(X))\xi-\eta([X,\xi])\xi\\
&= [\xi,X]+\Phi[\xi,\Phi X]-(\xi\eta(X))\xi.
\end{aligned} \tag{4.1.3}
$$

Applying η to this equation, we obtain

$$\begin{aligned} 0 &= \eta([\xi, X]) + \eta(\Phi[\xi, \Phi X]) - (\xi\eta(X))\eta(\xi) \qquad (4.1.4) \\ &= \eta([\xi, X]) - \xi\eta(X) \\ &= \eta(L_\xi X) - (L_\xi \eta)(X) - \eta(L_\xi X) \\ &= -(L_\xi \eta)(X) \end{aligned}$$

which is just $N_4(X, Y) = L_\xi \eta = 0$. Note also at this point that if we replace X by ΦX we have

$$\begin{aligned} 0 &= \eta([\xi, \Phi X]) - \xi\eta(\Phi X) \\ &= \eta([\xi, \Phi X]). \end{aligned}$$

Now applying Φ to (4.1.3), we have

$$\begin{aligned} 0 &= \Phi[\xi, X] + \Phi^2[\xi, \Phi X] - (\xi\eta(X))\Phi\xi \\ &= \Phi[\xi, X] - [\xi, \Phi X] + \eta([\xi, \Phi X])\xi \\ &= \Phi L_\xi X - L_\xi(\Phi X) \\ &= -(L_\xi \Phi)X \end{aligned}$$

and hence $N_3 = 0$. Finally using $N_1 = 0$ again,

$$\begin{aligned} 0 &= N_\Phi(\Phi X, Y) + 2d\eta(\Phi X, Y)\xi, \\ &= -[\Phi X, Y] + \eta([\Phi X, Y])\xi + [-X + \eta(X)\xi, \Phi Y] - \Phi[-X + \eta(X)\xi, Y] \\ &\quad -\Phi[\Phi X, \Phi Y] + (\Phi X\eta(Y))\xi - (Y\eta(\Phi X))\xi - \eta([\Phi X, Y])\xi \\ &= -[\Phi X, Y] - [X, \Phi Y] - (\Phi Y\eta(X))\xi - \eta(X)[\Phi Y, \xi] - \Phi[-X + \eta(X)\xi, Y] \\ &\quad -\Phi[\Phi X, \Phi Y] + (\Phi X\eta(Y))\xi. \end{aligned}$$

Applying η to this and using (4.1.4), we have

$$\begin{aligned} 0 &= -\eta([\Phi X, Y]) - \eta([X, \Phi Y]) - \Phi Y\eta(X) - \eta(X)\eta([\Phi Y, \xi]) + \Phi X\eta(Y) \\ &= (L_{\Phi X}\eta)(Y) - (L_{\Phi Y}\eta)(X) \qquad (4.1.5) \end{aligned}$$

giving $N_2 = 0$. $\square$

We now consider the case of a contact manifold with contact form η and associated almost contact metric structure (Φ, ξ, η, g).

Proposition 4.1.2 Let (Φ, ξ, η, g) be a contact metric structure. Then the tensors N_2 and N_4 vanish. Moreover, N_3 vanishes if and only if the Reeb vector field ξ is Killing with respect to g, i.e. $L_\xi g = 0$.

Proof. In view of the above discussion, to show that vanishing of N_2, it suffices to show that equation (4.1.5) holds. We have

$$\begin{aligned} d\eta(\Phi X, \Phi Y) &= \Omega(\Phi X, \Phi Y) \\ &= g(\Phi X, \Phi^2 Y) \\ &= -g(X, \Phi^3 Y) \\ &= g(X, \Phi Y) \\ &= d\eta(X, Y). \end{aligned}$$

In section 3.1, we saw that $N_4 = L_\xi \eta = 0$ for any contact structure.

Now since $N_4 = 0$, we automatically have

$$(L_\xi g)(X, \xi) = \xi\eta(X) - \eta([\xi, X]) = (L_\xi \eta)(X) = 0.$$

We also saw in section 3.1 that $d\eta$ is invariant under the 1-parameter group of ξ and hence

$$\begin{aligned} 0 &= (L_\xi \Omega)(X, Y) \\ &= \xi g(X, \Phi Y) - g([\xi, X], \Phi Y) - g(X, \Phi[\xi, Y]) \\ &= \xi g(X, \Phi Y) - g([\xi, X], \Phi Y) - g(X, L_\xi(\Phi Y) - (L_\xi \Phi) Y) \\ &= (L_\xi g)(X, \Phi Y) + g(X, (L_\xi \Phi) Y). \end{aligned}$$

Thus $N_3 = L_\xi \Phi = 0$ if and only if ξ is Killing vector field. $\square$

Next we will establish a formula for the covariant derivative of Φ for general almost contact metric structure (ϕ, ξ, η, g).

Lemma 4.1.3 For an almost contact metric structure (ϕ, ξ, η, g), the covariant derivative of Φ is given by

$$\begin{aligned} 2\, g((\nabla_X \Phi) Y, Z) &= 3\, d\Omega(X, \Phi Y, \Phi Z) - 3\, d\Omega(X, Y, Z) + g(N_1(Y, Z), \Phi X) \\ &= N_2(Y, Z)\eta(X) + 2\, d\eta(\Phi Y, X)\eta(Z) - 2\, d\eta(\Phi Z, X)\eta(Y). \end{aligned}$$

Proof. Recall that the Riemannian connection ∇ of g is given in (2.2.1);

$$\begin{aligned} 2\,g(\nabla_X Y, Z) &= X\,g(Y,Z) + Y\,g(X,Z) - Z\,g(X,Y) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X). \end{aligned}$$

and that the coboundary formula for d on a 2-form Ω is

$$\begin{aligned} d\Omega(X,Y,Z) &= \frac{1}{3}\,(X\,\Omega(Y,Z) + Y\,\Omega(Z,X) + Z\,\Omega(X,Y) \\ &\quad -\Omega([X,Y],Z) - \Omega([Z,X],Y) - \Omega([Y,Z],X))\,. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\,g((\nabla_X\Phi)Y,Z) &= 2\,g(\nabla_X(\Phi Y),Z) + 2\,g(\nabla_X Y,\Phi Z) \\ &= X\,g(\Phi Y,Z) + (\Phi Y)\,g(X,Z) - Z\,g(X,\Phi Y) \\ &\quad + g([X,\Phi Y],Z) + g([Z,X],\Phi Y) - g([\Phi Y,Z],X) \\ &\quad + X\,g(Y,\Phi Z) + Y\,g(X,\Phi Z) - (\Phi Z)\,g(X,Y) \\ &\quad + g([X,Y],\Phi Z) + g([\Phi Z,X],Y) - g([Y,\Phi Z],X) \\ &= -X\Omega(Y,Z) + (\Phi Y)(\Omega(\Phi Z,X) + \eta(Z)\eta(X)) - Z\,\Omega(X,Y) \\ &\quad -\Omega([X,\Phi Y],\Phi Z) + \eta([X,\Phi Y])\eta(Z) \\ &\quad +\Omega([Z,X],Y) - g(\Phi[\Phi Y,Z],\Phi X) + \eta(X)\eta([Z,\Phi Y]) \\ &\quad +X\,\Omega(\Phi Y,\Phi Z) - Y\,\Omega(Z,X) - (\Phi Z)(\Omega(\Phi Y,X) + \eta(Y)\eta(X)) \\ &\quad +\Omega([X,Y],Z) - \Omega([\Phi Z,X],\Phi Y) + \eta([\Phi Z,X])\eta(Y) \\ &\quad -g(\Phi[Y,\Phi Z],\Phi X) + \eta(X)\eta([\Phi Z,Y]) + \Omega([Y,Z],X) \\ &\quad -g([Y,Z],\phi X) - \Omega([\Phi Y,\Phi Z],X) + g([\Phi Y,\Phi Z],\Phi X) \\ &\quad +g(2d\eta(Y,Z)\xi,\Phi X) \\ &= 3\,d\Omega(X,\Phi Y,\Phi Z) - 3\,d\Omega(X,Y,Z) + g(N_1(Y,Z),\Phi X) \\ &\quad +N_2(Y,Z)\eta(X) + 2\,d\eta(\Phi Y,X)\eta(Z) - 2\,d\eta(\Phi Z,X)\eta(Y). \qquad \square \end{aligned}$$

In the case of a contact metric structure $\Omega = d\eta$ and $N_2 = N_4 = 0$, so the above formula becomes

$$2\,g((\nabla_X\Phi)Y,Z) = g(N_1(Y,Z),\Phi X) + 2\,d\eta(\Phi Y,X)\eta(Z) - 2\,d\eta(\Phi Z,X)\eta(Y).$$

In particular, setting $X = \xi$,

$$\begin{aligned}
2\,g((\nabla_\xi \Phi)Y, Z) &= g(N_1(Y,Z), \Phi\xi) + 2\,d\eta(\Phi Y, \xi)\eta(Z) - 2\,d\eta(\Phi Z, \xi)\eta(Y) \\
&= \eta(Z)(\Phi Y \eta(\xi) - \xi\eta(\Phi Y) - \eta([\Phi Y, \xi])) \\
&\quad -\eta(Y)(\Phi Z\eta(\xi) - \xi\eta(\Phi Z) - \eta([\Phi Z, \xi])) \\
&= \eta(Z)\eta(L_\xi(\Phi Y)) - \eta(Y)\eta(L_\xi(\Phi Z)) \\
&= \eta(Z)(L_\xi(\eta(\Phi Y)) - (L_\xi\eta)(\Phi Y)) \\
&\quad -\eta(Y)(L_\xi(\eta(\Phi Z)) - (L_\xi\eta)(\Phi Z)) \\
&= 0
\end{aligned}$$

gives $\nabla_\xi \Phi = 0$. It is also easy to see that on a contact metric manifold the integral curves of ξ are geodesics. Clearly $g(\nabla_\xi \xi, \xi) = 0$ and for X orthogonal to ξ,

$$\begin{aligned}
g(\nabla_\xi \xi, X) &= -g(\xi, \nabla_\xi X) \\
&= -g(\xi, \nabla_X \xi + [\xi, X]) \\
&= -\eta([\xi, X]) \\
&= d\eta(\xi, X) \\
&= 0.
\end{aligned}$$

4.2 The Boothby-Wang fibration

Definition 4.2.1 Let N be an even-dimensional differentiable manifold. A symplectic structure on N is a closed nondegenerate differentiable 2-form ω on N:

$$d\omega = 0, \quad \omega(X, X) = 0 \text{ if and only if } X = 0.$$

The pair (N, ω) is called a symplectic manifold.

Here we recall some examples of symplectic manifolds.

Example 4.1 Consider the vector space

$$\mathbf{R}^{2n} = \{(p_1, \cdots, p_n, q_1, \cdots, q_n) \mid p_j, q_j \in \mathbf{R}\}$$

and define a closed nondegenerate 2-form $\omega_0 = \sum_j dp_j \wedge dq_j$. Then $(\mathbf{R}^{2n}, \omega_0)$ is a symplectic manifold.

Example 4.2 Let V be an n-dimensional manifold. The cotangent bundle T^*V has a natural structure of a differentiable manifold of dimension $2n$. If $(q_1, \cdots, q_n)$ is a choice of local coordinates for points in V, the such a form is given by its n components $(p_1, \cdots, p_n)$. Together, the $2n$ numbers $(p_1, \cdots, p_n, q_1, \cdots, q_n)$ form a collection of local coordinates for points in T^*V. There is a natural projection $\pi : T^*V \longrightarrow V$ sending every 1-form on TV_x to the point $x \in V$. The projection π is differentiable and surjective. The pre image of a point $x \in V$ under π is the cotangent space $(T^*V)_x$.

Theorem 4.2.2 The cotangent bundle T^*V has a natural symplectic structure, In the local coordinates described above, this symplectic structure ω is given by the formula

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j = dp_1 \wedge dq_n + \cdots + dp_n \wedge dq_n.$$

Proof. First, we define a distinguished form on T^*V. Let $\eta \in T(T^*V)$ be a vector tangent to the cotangent bundle at the point $p \in T^*V$. The derivative $f_* : T(T^*V) \longrightarrow TV$ takes η to a vector $f_*\eta$ tangent to V at x. We define a 1-form θ on T^*V by the relation $\theta(\eta) = p(f_*\eta)$. In the local coordinates described above, this form is $\theta = \sum_j p_j dq_j$. By the example 2.7.2, the closed form $\omega = d\theta$ is nondegenarate. $\square$

Definition 4.2.3 Let ω be a closed nondegenerate differentiable 2-form in a neighborhood of a point x in the space $\mathbf{R}^{2n}$. Then in some neighborhood of x, one can choose a coordinate system $(p_1, \cdots, p_n, q_1, \cdots, q_n)$ such that the form ω has the standard form:

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j.$$

This theorem allows us to extend to all symplectic manifolds any assertion of a local character which is invariant with respect to canonical transformation and is proven for the standard space $(\mathbf{R}^{2n}, \omega)$.

We now give an important example, namely certain principal circle bundles over symplectic manifolds whose symplectic 2-forms have integral period and conversely we will prove that a compact regular contact metric manifold is of this type. A contact metric structure is regular if its Reeb vector field is regular. Let N be a symplectic manifold with a symplectic form ω and $\pi : M \longrightarrow N$ the corresponding principal circle bundle. If η' is a connection form on M, then there exists a 2-form ω' on M such that $d\eta' = \pi^*\omega'$. However, the characteristic class $[\omega']$ is independent of the choice of connections, so that $[\omega] = [\omega']$. Thus, there exists a 1-form τ on N such that $\omega' = \omega + d\tau$. Now $\pi^*\tau$ is horizontal and equivariant, and S^1 is abelian so that

$$\pi^*\tau \circ (R_t)_* = \mathrm{ad}\,(t^{-1})\pi^*\tau = \pi^*\tau$$

where R_t denotes right transformation by $t \in S^1$ and $(R_t)_*$ its differential. Thus setting $\eta = \eta' + \pi^*\tau$, we have

$$\eta \circ (R_t)_* = \eta \quad \text{and} \quad \mathrm{d}\eta = \pi^*\omega.$$

Moreover, if ξ is a vertical vector field such that $\eta'(\xi) = 1$, $\eta(\xi) = 1$ since $(\pi^*\tau)(\xi) = \tau(\pi_*\xi) = 0$. Now if at any point of M, $X_1, \cdots, X_{2n}$ are linearly independent horizontal vectors, $(\eta \wedge (d\eta)^n)(\xi, X_1, \cdots, X_{2n})$ is non-zero. Thus, we see that η is a contact form on M. Conversely, we now give the theorem of Boothby and Wang [9].

Theorem 4.2.4 Let M be a compact regular contact manifold with the contact form η'. Then there exists a contact form $\eta = f\eta'$ for some non-vanishing function f whose characteristic vector field ξ generates a free effective S^1 action on M. Moreover, M is the bundle space of principal circle bundle $\pi : M \longrightarrow N$ over a symplectic manifold N whose fundamental 2-form ω determines an integral cocycle on N. η is a connection form on the bundle with curvature form $d\eta = \pi^*\omega$.

4.3 Sasakian manifolds

In section 4.2, we showed that a contact manifold M carries an almost contact metric structure (Φ, ξ, η, g) with $\Omega = d\eta$. If a contact metric structure (Φ, ξ, η, g) on M is normal, we call it a Sasakian structure.

Definition 4.3.1 $(M; \Phi, \xi, \eta, g)$ is a Sasakian manifold if $(M \times \mathbf{R}_+, J, t^2 g + dt^2)$ is a Kähler manifold where J is a complex structure on $M \times \mathbf{R}$ defined in ;

$$J\left(X,\, f\frac{\partial}{\partial t}\right) = \left(\Phi X - f\xi,\ \eta(X)\frac{\partial}{\partial t}\right).$$

A Sasakian structure is in some sense an analogue of a Kähler structure. This point of view is suggested in the following formulation of the Sasakian condition.

Definition 4.3.2 An almost contact metric structure (Φ, ξ, η, g) is Sasakian structure if and only if

$$(\nabla_X \Phi)Y = g(X,Y)\xi - \eta(Y)X \tag{4.3.1}$$

where ∇ denotes the Riemannian connection of g.

Lemma 4.3.3 On a Sasakian manifold, for a unit vector field X, Y, Z orthogonal to ξ, we have

$$\begin{aligned} &(1) \quad \eta(\nabla_X Y) = -d\eta(X,Y), \\ &(2) \quad \eta([X, \nabla_Y Z]) = 0, \\ &(3) \quad R(X,\xi)X = -\xi. \end{aligned}$$

Proof. By direct computations,

$$\begin{aligned} (1) \quad \eta(\nabla_X Y) &= g(\nabla_X Y, \xi) \\ &= X(g(Y,\xi)) - g(Y, \nabla_X \xi) \\ &= g(Y, \Phi X) \\ &= -g(X, \Phi Y) \\ &= -d\eta(X,Y). \end{aligned}$$

$$\begin{aligned} (2) \quad \eta([X, \nabla_Y Z]) &= \eta(\nabla_X \nabla_Y Z - \nabla_{\nabla_Y Z} X) \\ &= g(\nabla_X \nabla_Y Z, \xi) - g(\nabla_{\nabla_Y Z} X, \xi) \\ &= -d\eta(X, \nabla_Y Z) - (\nabla_Y Z)(g(X,\xi)) + g(X, \nabla_{\nabla_Y Z}\xi) \\ &= -d\eta(X, \nabla_Y Z) + g(X, \Phi\nabla_Y Z) \end{aligned}$$

$$
\begin{aligned}
&= -d\eta(X, \nabla_Y Z) + d\eta(X, \nabla_Y Z) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
(3) \quad & g(R(X,\xi)X, Y) = -g(R(X,Y)\xi, X) \\
&= -g(\nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi, X) \\
&= -g(-\nabla_X(\Phi Y) + \nabla_Y(\Phi X) + \Phi[X,Y], X) \\
&= g((\nabla_X \Phi)Y - (\nabla_Y \Phi)X, X) \\
&= -g(\eta(Y)X - \eta(X)Y, X) \\
&= -\eta(Y) \\
&= -g(\xi, Y). \quad \square
\end{aligned}
$$

Sasakian manifolds have many properties analogous to Kähler manifolds. Let (M, Φ, ξ, η, g) be a Sasakian manifold and define a $(0,4)$-tensor field P on M by

$$
\begin{aligned}
P(X, Y; Z, W) \;\; = \;\; & d\eta(X,Z)g(Y,W) - d\eta(X,W)g(Y,Z) \\
& -d\eta(Y,Z)g(X,W) - d\eta(Y,W)g(X,Z).
\end{aligned}
$$

Note that

$$P(X, Y; Z, W) = -P(Z, W; X, Y).$$

Lemma 4.3.4 On a Sasakian manifold, for X, Y, Z and W orthogonal to ξ, we have

$$
\begin{aligned}
&(1) \quad (\nabla_X \Omega)(Y, Z) = g(X,Z)\eta(Y) - g(Y,X)\eta(Z). \\
&(2) \quad g(R(\Phi X, \Phi Y)\Phi Z, \Phi W) = g(R(X,Y)Z, W).
\end{aligned}
$$

Thus choosing a Φ-basis $\{X_j,\ X_{n+j} = \Phi X_j,\ \xi\}$, we have

$$
\begin{aligned}
\mathrm{Ric}\,(\Phi X, \Phi Y) \;\; &= \;\; \sum_{j=1}^{2n} g(R(\Phi X, X_j)X_j, \Phi Y) + g(R(\Phi X, \xi)\xi, \Phi Y) \\
&= \;\; \sum_{j=1}^{2n} g(R(\Phi X, \Phi X_j)\Phi X_j, \Phi Y) + g(X, Y) \\
&= \;\; \mathrm{Ric}\,(\mathrm{X}, \mathrm{Y}).
\end{aligned}
$$

Example 4.3 In example 3.2, we gave explicitly an associated almost contact metric structure (Φ, ξ, η, g) to the canonical contact structure $\eta = \frac{1}{2}\left(dz - \sum_{j=1}^{n} y_j dx_j\right)$ on $\mathbf{R}^{2n+1}$. From the matrix expression for Φ, it is easy to check that $N_\Phi + 2\, d\eta \otimes \xi = 0$ and hence this contact metric structure is Sasakian.

Example 4.4 We can construct a Sasakian structure on an odd-dimensional sphere in this way. Let $S^{2n+1}(r)$ be a sphere of radius r in $\mathbf{R}^{2n+2}$ with its usual Kähler structure J, i.e. J is parallel with respect to the Riemannian connection D of the Euclidian metric on $\mathbf{R}^{2n+2}$. Then the structure induced as above with respect to the unit outer normal vector N is an almost contact metric structure (Φ, ξ, η, g) and clearly η is the standard contact form . Since S^{2n+1} is umbilical in $\mathbf{R}^{2n+2}$, the second fundamental form h satisfies $h = -\frac{1}{r}g$. Thus, using the fact that J is parallel and the Gauss-Weingarten equations, we have

$$\begin{aligned} 0 &= (D_{\iota_* X} J)\iota_*\xi \\ &= D_{\iota_* X} N - J(\iota_* \nabla_X \xi + h(X,\xi)N) \\ &= \frac{1}{r}\iota_* X - \iota\Phi\nabla_X\xi - \frac{1}{r}\eta(X)\iota_*\xi. \end{aligned}$$

Applying Φ, we have $\nabla_X \xi = -\frac{1}{r}\Phi X$. Since Φ has rank $2n$, we again see that $\eta \wedge (d\eta)^n \neq 0$. The almost contact metric structure (Φ, ξ, η, g) is not an associated one for $r \neq 1$ as $d\eta = \frac{1}{r}\Omega$, but the structure $\bar{\eta} = \frac{1}{r}$, $\bar{\xi} = r\xi, \overline{\Phi} = \Phi$ and the homothetic change of metric $\bar{g} = \frac{1}{r^2}g$ gives a contact metric structure $(\overline{\Phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ on M. Alternatively the metric

$$g' = \frac{1}{r} + \left(1 - \frac{1}{r}\right)\eta \otimes \eta$$

is an associated one for the induced contact form η on $S^{2n+1}(r)$. $(\overline{\Phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is Sasakian structure on $S^{2n+1}(r)$ because of the expression of the second fundamental form.

4.4 Sasakian reduction

In this section, we recall the definition of Sasaki-Einstein manifolds and the special Sasakian reduction constructed by Boyer and Galicki in [4]. In particular, they focus on $n = 4$ case.

Definition 4.4.1 A Sasaki-Einstein manifold is a $(2n+1)$-dimentional Riemannian manifold (M, g) whose metric cone $(C(M), r^2 g + dr^2, J)$ is a Ricci-flat (i.e. Ricci curvature is equal to 0) Kähler manifold.

Example 4.5 An odd-dimentional sphere S^{2n+1} with induced metric g_0 from $\mathbf{C}^{n+1}$ is Sasaki-Einstein, as its cone $(C(S^{2n+1}), r^2 g_0 + dr^2)$ is isometric to $(\mathbf{C}^{n+1}, g_{std})$, where g_{std} is the standard Ricci-flat Kähler metric on $\mathbf{C}^n$.

We recall the special Sasakian reduction constructed by Boyer and Galicki in [4]. In particular, they focus on $n = 4$ case.

Definition 4.4.2 Let $p, q \in \mathbf{Z}_{\geq 0}$ be coprime and $p > q$, or $p = 1, q = 0$. We define a moment map $\mu_{p,q} : \mathbf{C}^4 \longrightarrow \mathbf{R}$ as follows

$$\mu_{p,q}(z_1, z_2, z_3, z_4) := p|z_1|^2 + p|z_2|^2 - (p-q)|z_3|^2 - (p+q)|z_4|^2,$$

and $S^1_{p,q}$ is the associated S^1 action on $(\mathbf{C}^*)^4$,

$$(z_1, z_2, z_3, z_4) \longmapsto (z_1\, e^{ip\theta}, z_2\, e^{ip\theta}, z_3\, e^{-i(p-q)\theta}, z_4\, e^{-i(p+q)\theta}).$$

Theorem 4.4.3 We set an inclusion ι and a projection π as

$$\begin{aligned} \iota &: \mu^{-1}_{p,q}(0)|_{S^7} \hookrightarrow S^7 \\ \pi &: \mu^{-1}_{p,q}(0)|_{S^7} \longrightarrow \left(\mu^{-1}_{p,q}(0)|_{S^7}\right)/S^1_{p,q}. \end{aligned}$$

Then we have the following:
(1) $\mu^{-1}_{p,q}(0)|_{S^7}$ is diffeomorphic to $S^3 \times S^3$.
(2) $\left(\mu^{-1}_{p,q}(0)|_{S^7}\right)/S^1_{p,q}$ is diffeomorphic to $S^2 \times S^3$.
(3) There is a Sasaki-Einstein metric $g_{p,q}$ on $\left(\mu^{-1}_{p,q}(0)|_{S^7}\right)/S^1_{p,q}$ satisfying $\iota^* g_0 = \pi^* g_{p,q}$, where g_0 is the induced metric on S^7 from $\mathbf{C}^4$.

4.5 Calculation in the case of $p = 1, q = 0$

Let us restrict our attention for the case of $p = 1$ and $q = 0$, and consider the zero level set

$$\mu^{-1}_{1,0}(0)|_{S^7} = \{(z_1, z_2, z_3, z_4) \in S^7; |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 = \frac{1}{2}\}$$

$$= \quad S^3\left(\frac{1}{\sqrt{2}}\right) \times S^3\left(\frac{1}{\sqrt{2}}\right).$$

For any point in $\mu_{1,0}^{-1}(0) \in S^3 \times S^3$, we identify S^3 and $SU(2)$ as follows:

$$(z_1, z_2) \in S^3 \;\longleftrightarrow\; \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in SU(2).$$

The reduced space $S^3 \times S^3/S^1$ is diffeomorphic to $S^2 \times S^3$ with a projection π defined by,

$$\pi(h_1, h_2) := ([h_1], {h_1}^t h_2)$$

where $h_1, h_2 \in SU(2)$ and $[\cdot]$ is the equivalence class $\sim$ given by

$$h_1 \sim h_2 \iff h_2 = h_1 \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

This equivalence relation is the same as in the definition of the projective space $\mathbf{C}P^1$. In complex coordinates, π is given explicitly by

$$\pi(z_1, z_2, z_3, z_4) := (2z_1\bar{z}_2,\ |z_1|^2 - |z_2|^2,\ z_1 z_3 + \bar{z}_2\bar{z}_4,\ z_2 z_3 - \bar{z}_1\bar{z}_4).$$

Then we have a left $SU(2) \times SU(2)$ action $\phi = (\phi_1, \phi_2)$ on $S^3 \times S^3$,

$$\phi(h_1, h_2) := (\phi_1 h_1, \phi_2 h_2) \quad (\phi_1, \phi_2 \in SU(2)).$$

Let us define a $(SU(2) \times SU(2))/U(1)$ action $\tilde{\phi} = ([\tilde{\phi}_1], \tilde{\phi}_2)$ on $S^2 \times S^3$ as follows

$$\tilde{\phi}([h_1], {h_1}^t h_2) := ([\tilde{\phi}_1 h_1], \tilde{\phi}_1 {h_1}^t {h_2}^t \tilde{\phi}_2) \quad (\tilde{\phi}_1, \tilde{\phi}_2 \in SU(2)),$$

such that ϕ induces $\tilde{\phi}$, and π is $(\phi, \tilde{\phi})$-equivariant :

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{\phi} & S^3 \times S^3 \\ \pi \downarrow & & \downarrow \pi \\ S^2 \times S^3 & \xrightarrow{\tilde{\phi}} & S^2 \times S^3. \end{array}$$

Since $S^2 \times S^3$ is a homogeneous space for $(SU(2) \times SU(2))/U(1)$, we can define an inner product $\langle \cdot, \cdot \rangle_o$ on $T_o(S^2 \times S^3)$, where o is written with an unit matrix I_2,

$$o := (0, 0, -1, 1, 0, 0, 0) = ([I_2], I_2) = \pi(I_2, I_2) = \pi(1, 0, 0, 0, 1, 0, 0, 0),$$

for the Sasaki-Einstein metric $g_{1,0}$. By Theorem 4.4.3, the inner product $\langle\cdot,\cdot\rangle_o$ satisfies a condition:

$$d\pi\left(\{\text{an orthonormal basis of } T_{(I_2,I_2)}(S^3\times S^3)\}\right)$$
$$=\{\text{an orthonormal basis of } T_o(S^2\times S^3)\}.$$

By this, if we choose $\left\{\frac{\partial}{\partial s_2},\frac{\partial}{\partial s_3},\frac{\partial}{\partial s_4},\frac{\partial}{\partial s_6},\frac{\partial}{\partial s_7},\frac{\partial}{\partial s_8}\right\}$ an orthonormal basis of $T_{(I_2,I_2)}(S^2\times S^3)$, thus

$$\left\{d\pi\left(\frac{\partial}{\partial s_2}\right)=d\pi\left(\frac{\partial}{\partial s_6}\right)=\left(\frac{\partial}{\partial x_5}\right)_o,\right.$$
$$d\pi\left(\frac{\partial}{\partial s_3}\right)=2\left(\frac{\partial}{\partial x_1}\right)_o+\left(\frac{\partial}{\partial x_6}\right)_o,d\pi\left(\frac{\partial}{\partial s_4}\right)=2\left(\frac{\partial}{\partial x_2}\right)_o+\left(\frac{\partial}{\partial x_7}\right)_o,$$
$$\left.d\pi\left(\frac{\partial}{\partial s_7}\right)=-\left(\frac{\partial}{\partial x_6}\right)_o,d\pi\left(\frac{\partial}{\partial s_8}\right)=\left(\frac{\partial}{\partial x_7}\right)_o\right\}$$

is an orthonormal basis of $T_o(S^2\times S^3)$. Then the metric $g_o(\cdot,\cdot)=\langle\cdot,\cdot\rangle_o$ defined by

$$\left(\left\langle\left(\frac{\partial}{\partial x_i}\right)_o,\left(\frac{\partial}{\partial x_j}\right)_o\right\rangle_o\right)_{ij}=\begin{pmatrix}\frac{1}{2}&0&0&-\frac{1}{2}&0\\0&\frac{1}{2}&0&0&-\frac{1}{2}\\0&0&1&0&0\\-\frac{1}{2}&0&0&1&0\\0&-\frac{1}{2}&0&0&1\end{pmatrix},\quad(i,j=1,2,5,6,7).$$

Choosing the local chart (U_0,ψ_0) such that

$$U_0=\{(x_1,\cdots,x_7)\in S^2\times S^3;x_3<0,x_4>0\},$$
$$\psi_0:(x_1,\cdots,x_7)\mapsto(x_1,x_2,x_5,x_6,x_7),$$

we extend this metric to any point $x:=([k_1],k_2)$ by another $(SU(2)\times SU(2))/U(1)$ action on $S^2\times S^3$: for $k=(k_1,k_2)$,

$$k([h_1],h_2):=([k_1h_1],k_1h_2k_1^{-1}k_2),$$

noting that $x=k\cdot o$. We define the metric g at x by

$$g_x(u,v):=g_0(dk^{-1}(u),dk^{-1}(v))\qquad(u,v\in T_x(S^2\times S^3)).$$

For $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7) \in U_o$, we can write k^{-1} as

$$\begin{aligned}
k^{-1}(y) &= \Big(k_1^{-1}(y), k_2^{-1}(y), k_3^{-1}(y), k_4^{-1}(y), k_5^{-1}(y), k_6^{-1}(y), k_7^{-1}(y)\Big) \\
&= \Bigg(\frac{(1-x_3-x_1^2)y_1 - x_1x_2y_1 + x_1(1-x_3)y_3}{1-x_3}, \\
&\quad \frac{-x_1x_2y_1 + (1-x_3-x_2^2)y_2 + x_2(1-x_3)y_3}{1-x_3},\ -x_1y_1 - x_2y_2 - x_3y_3, \\
&\quad \frac{X_1Y_1 + X_2Y_2 + X_3Y_3 + X_4Y_4}{2(1-x_3)},\ \frac{-X_2Y_1 - X_1Y_2 + X_4Y_3 - X_3Y_4}{2(1-x_3)}, \\
&\quad \frac{-X_3Y_1 - X_4Y_2 - X_1Y_3 + X_2Y_4}{2(1-x_3)},\ \frac{-X_4Y_1 + X_3Y_2 - X_2Y_3 - X_1Y_4}{2(1-x_3)}\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
&X_1 = (1-x_3)x_4 + x_1x_6 + x_2x_7,\ X_2 = x_2x_6 - x_1x_7 - (1-x_3)x_5 \\
&X_3 = x_1x_4 - x_2x_5 - (1-x_3)x_6,\ X_4 = x_1x_5 + x_2x_4 - (1-x_3)x_7 \\
&Y_1 = (1-x_3)y_4 + x_1y_6 + x_2y_7,\ Y_2 = x_2y_6 - x_1y_7 - (1-x_3)y_5 \\
&Y_3 = x_1y_4 - x_2y_5 - (1-x_3)y_6,\ Y_4 = x_1y_5 + x_2y_4 - (1-x_3)y_7.
\end{aligned}$$

Next we calculate $g_x\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5}\right)$. Let us first consider the derivation dk^{-1},

$$\begin{aligned}
dk^{-1}\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial k_1^{-1}}{\partial y_1}(x)\left(\frac{\partial}{\partial x_1}\right)_o + \frac{\partial k_2^{-1}}{\partial y_1}(x)\left(\frac{\partial}{\partial x_2}\right)_o \\
&\quad + \frac{\partial k_5^{-1}}{\partial y_1}(x)\left(\frac{\partial}{\partial x_5}\right)_o + \frac{\partial k_6^{-1}}{\partial y_1}(x)\left(\frac{\partial}{\partial x_6}\right)_o + \frac{\partial k_7^{-1}}{\partial y_1}(x)\left(\frac{\partial}{\partial x_7}\right)_o \\
&= \frac{x_2^2 + x_3 - 1}{x_3(1-x_3)}\left(\frac{\partial}{\partial x_1}\right)_o + \frac{-x_1x_2}{x_3(1-x_3)}\left(\frac{\partial}{\partial x_2}\right)_o
\end{aligned}$$

and

$$\begin{aligned}
&dk^{-1}\left(\frac{\partial}{\partial x_5}\right) \\
= &\frac{\partial k_1^{-1}}{\partial y_5}(x)\left(\frac{\partial}{\partial x_1}\right)_o + \frac{\partial k_2^{-1}}{\partial y_5}(x)\left(\frac{\partial}{\partial x_2}\right)_o \\
&+ \frac{\partial k_5^{-1}}{\partial y_5}(x)\left(\frac{\partial}{\partial x_5}\right)_o + \frac{\partial k_6^{-1}}{\partial y_5}(x)\left(\frac{\partial}{\partial x_6}\right)_o + \frac{\partial k_7^{-1}}{\partial y_5}(x)\left(\frac{\partial}{\partial x_7}\right)_o
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{-x_3(x_4^2+x_5^2)+x_1(x_4x_6+x_5x_7)+x_2(x_4x_7-x_5x_6)}{x_4}\left(\frac{\partial}{\partial x_5}\right)_o \\
& + \frac{(1-x_3-x_2^2)(x_5x_6-x_4x_7)+(1-x_3)x_2(x_4^2+x_5^2)+x_1x_2(x_4x_6+x_5x_7)}{(1-x_3)x_4}\left(\frac{\partial}{\partial x_6}\right)_o \\
& + \frac{(1-x_3-x_1^2)(x_4x_6+x_5x_7)-(1-x_3)x_1(x_4^2+x_5^2)+x_1x_2(x_5x_6-x_4x_7)}{(1-x_3)x_4}\left(\frac{\partial}{\partial x_7}\right)_o .
\end{aligned}
$$

Then the coeffcient of dx_1dx_5 and dx_5dx_1 is given by

$$
\begin{aligned}
& g_x\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5}\right) \\
= \quad & g_0\left(dk^{-1}\left(\frac{\partial}{\partial x_1}\right), dk^{-1}\left(\frac{\partial}{\partial x_5}\right)\right) \\
= \quad & \frac{-(x_1^2+x_3^2)(x_4x_7-x_5x_6)-x_2x_3(x_4^2+x_5^2)+x_1x_2(x_4x_6+x_5x_7)}{2x_3x_4}.
\end{aligned}
$$

Also we can find the coefficient of dx_idx_j and dx_jdx_i by calculating $g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. As the result, with the local coordinates $x = (x_1, x_2, x_5, x_6, x_7)$ on U_0, we have the formula:

$$
\begin{aligned}
g_x \quad = \quad & \sum_{i=1}^{2}\frac{x_i^2+x_3^2}{2x_3^2}dx_i^2 + \frac{x_1x_2}{x_3^2}dx_1dx_2 + \sum_{i=5}^{7}\frac{x_4^2+x_i^2}{x_4^2}dx_i^2 \qquad (4.5.1) \\
& + \frac{2x_5x_6}{x_4^2}dx_5dx_6 + \frac{2x_5x_7}{x_4^2}dx_5dx_7 + \frac{2x_6x_7}{x_4^2}dx_6dx_7 \\
& + \frac{-(x_1^2+x_3^2)(x_4x_7-x_5x_6)-x_2x_3(x_4^2+x_5^2)+x_1x_2(x_4x_6+x_5x_7)}{x_3x_4}dx_1dx_5 \\
& + \frac{(x_1^2+x_3^2)(x_4^2+x_6^2)-x_1x_2(x_4x_5-x_6x_7)-x_2x_3(x_4x_7+x_5x_6)}{x_3x_4}dx_1dx_6 \\
& + \frac{x_1x_2(x_4^2+x_7^2)+(x_1^2+x_3^2)(x_4x_5+x_6x_7)+x_2x_3(x_4x_6-x_5x_7)}{x_3x_4}dx_1dx_7 \\
& + \frac{x_1x_3(x_4^2+x_5^2)-x_1x_2(x_4x_7-x_5x_6)+(x_2^2+x_3^2)(x_4x_6+x_5x_7)}{x_3x_4}dx_2dx_5 \\
& + \frac{x_1x_2(x_4^2+x_6^2)+x_1x_3(x_4x_7+x_5x_6)-(x_2^2+x_3^2)(x_4x_5-x_6x_7)}{x_3x_4}dx_2dx_6 \\
& + \frac{(x_2^2+x_3^2)(x_4^2+x_7^2)+x_1x_2(x_4x_5+x_6x_7)-x_1x_3(x_4x_6-x_5x_7)}{x_3x_4}dx_2dx_7.
\end{aligned}
$$

On other open sets $U_{i^\pm j^\pm}$ of $S^2 \times S^3$ defined by for $i \in \{1,2,3\}, j \in \{4,5,6,7\}$, i.e.

$$U_{i^+j^+} = \{x_i > 0, x_j > 0\},\ U_{i^-j^+} = \{x_i < 0, x_j > 0\},$$
$$U_{i^+j^-} = \{x_i > 0, x_j < 0\} \text{ and } U_{i^-j^-} = \{x_i < 0, x_j < 0\},$$

we can calculate the metric the same way as the previous case. This is an explicit representation at x of the Sasaki-Einstein metric $g_{1,0}$ called the homogeneous Kobayashi-Tanno metric by Boyer and Galicki in [4].

Theorem 4.5.1 The Sasaki-Einstein metric $g_{1,0}$ on $S^2 \times S^3$ at any point x is given by the formula (4.5.1).

Chapter 5

Complex Contact Manifolds

5.1 Definitions

We first recall the notion of complex contact metric manifolds.

Definition 5.1.1 Let M be a complex manifold with $\dim_{\mathbf{C}} M = 2n+1$ and J the complex structure on M. M is called a complex contact manifold if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\lambda\}$ of M such that:

(1) On each $\mathcal{O}_\lambda$ there is a holomorphic 1-form ω_λ with $\omega_\lambda \wedge (d\omega_\lambda)^n \neq 0$ everywhere;
(2) If $\mathcal{O}_\lambda \cap \mathcal{O}_\mu \neq \phi$, there is a nonvanishing holomorphic function $h_{\lambda\mu}$ on $\mathcal{O}_\lambda \cap \mathcal{O}_\mu$ such that

$$\omega_\lambda = h_{\lambda\mu}\omega_\mu \ \text{ in } \mathcal{O}_\lambda \cap \mathcal{O}_\mu. \tag{5.1.1}$$

For each $\mathcal{O}_\lambda$, we define a distribution $\mathcal{H}_\lambda = \{X \in T\mathcal{O}_\lambda \mid \omega_\lambda(X) = 0\}$. Note that the $h_{\lambda\mu}$ are nonvanishing, and $\mathcal{H}_\lambda = \mathcal{H}_\mu$ on $\mathcal{O}_\lambda \cap \mathcal{O}_\mu$. Thus $\mathcal{H} = \cup\mathcal{H}_\lambda$ is a holomorphic, nonintegrable subbundle on M, called the horizontal subbundle.

Definition 5.1.2 Let M be a complex manifold with $\dim_{\mathbf{C}} = 2n+1$ and J a complex structure. Let g be a Hermitian metric. M is called a complex almost contact metric manifold if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\lambda\}$ of M such that:

(1) On each $\mathcal{O}_\lambda$ there are 1-forms u_λ and $v_\lambda = u_\lambda J$, (1,1) tensors G_λ and

$H_\lambda = G_\lambda J$, unit vector fields U_λ and $V_\lambda = -JU_\lambda$ such that

$$
\begin{aligned}
&G_\lambda J_\lambda = -J_\lambda G_\lambda, \quad H_\lambda^2 = G_\lambda^2 = -id + u_\lambda \otimes U_\lambda + v_\lambda \otimes V_\lambda,\\
&g(G_\lambda X, Y) = -g(X, G_\lambda Y), \quad g(U_\lambda, X) = u_\lambda(X),\\
&G_\lambda U_\lambda = 0, \quad u_\lambda(U_\lambda) = 1;
\end{aligned}
\tag{5.1.2}
$$

(2) If $\mathcal{O}_\lambda \cap \mathcal{O}_\mu \neq \phi$, there are functions a, b on $\mathcal{O}_\lambda \cap \mathcal{O}_\mu$ such that

$$
\begin{aligned}
&u_\mu = au_\lambda - bv_\lambda, \quad v_\mu = bu_\lambda + av_\lambda,\\
&G_\mu = aG_\lambda - bH_\lambda, \quad H_\mu = bG_\lambda + aH_\lambda,\\
&a^2 + b^2 = 1.
\end{aligned}
\tag{5.1.3}
$$

Definition 5.1.3 Let $(M, \{\omega_\lambda\})$ be a complex contact manifold with complex contact structure J and Hermitian metric g. We call (M, u, v, U, V, J, g) a complex contact metric manifold if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\lambda\}$ of M such that (here and below $G = G_\lambda$, etc) :

(1) On each $\mathcal{O}_\lambda$ there is a local (1,1) tensor G such that $(u_\lambda, v_\lambda, U_\lambda, V_\lambda, G_\lambda, H_\lambda = G_\lambda J, g)$ is an almost contact metric structure on M;

(2) $g(X, G_\lambda Y) = du_\lambda(X, Y) + (\sigma_\lambda \wedge v_\lambda)(X, Y)$ and $g(X, H_\lambda Y) = dv_\lambda(X, Y) - (\sigma_\lambda \wedge u_\lambda)(X, Y)$, where $\sigma_\lambda(X) = g(\nabla_X U_\lambda, V_\lambda)$ with ∇ the Levi-Civita connection with respect to g.

Remark 5.1.4 Foreman [13] showed the existence of complex contact metric structures on complex contact manifolds.

Definition 5.1.5 We can locally choose orthonormal vectors $X_1, \cdots, X_n$ in $\mathcal{H}$ such that $\{X_i, JX_i, GX_i, HX_i, U, V \mid 1 \leq i \leq n\}$ is an orthonormal basis of the tangent spaces of U_α.

5.2 Normality of complex contact manifolds

We recall the definition of I-K normality introduced by Ishihara and Konishi [23] for (almost) complex contact metric structures. We set the two tensor fields S

and T by,

$$\begin{aligned} S(X,Y) &= [G,G](X,Y) + 2g(X,GY)U - 2g(X,HY)V \qquad (5.2.1)\\ &\quad +2v(Y)HX - 2v(X)HY + \sigma(GY)HX\\ &\quad -\sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,\\ T(X,Y) &= [H,H](X,Y) - 2g(X,GY)U + 2g(X,HY)V \qquad (5.2.2)\\ &\quad +2u(Y)GX - 2u(X)GY + \sigma(HX)GY\\ &\quad -\sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX. \end{aligned}$$

Definition 5.2.1 A complex contact manifold M is I-K normal if the tensors S and T both vanish.

Proposition 5.2.2 I-K normality implies that the underlying Hermitian manifold (M, J, g) is a Kähler manifold (cf. [23]).

We recall properties obtained by Korkmaz [29].

Proposition 5.2.3 On an I-K normal complex contact manifold, for $X, Y, Z \in \mathcal{H}$, we have

$$\begin{aligned} g((\nabla_X G)Y, Z) = &-\sigma(X)g(HX,Y) + v(X)\Omega(GZ,GY)\\ &-2v(X)g(HGY,Z) - u(Y)g(X,Z) - v(Y)g(JX,Z)\\ &+u(Z)g(X,Y) - v(Z)g(X,JY), \end{aligned}$$

and

$$\begin{aligned} g((\nabla_X H)Y, Z) = &-\sigma(X)g(GX,Y) + u(X)\Omega(HZ,GHY)\\ &-2u(X)g(HGY,Z) + u(Y)g(JX,Z) - v(Y)g(X,Z)\\ &+u(Z)g(X,JY) + v(Z)g(X,Y), \end{aligned}$$

where ∇ is the Levi-Civita connection with respect to g.

Lemma 5.2.4 Under the same assumptions as in Proposition 5.2.3, we have

$$\begin{aligned} g(R(X,GX)Y,GY) = \ &g(R(X,Y)X,Y) + g(R(X,GY)X,GY)\\ &+4g(JX,Y)\Omega(X,Y) - 4g(HX,Y)\Omega(GX,Y)\\ &-2g(GX,Y)^2 - 4g(HX,Y)^2 - 2g(X,Y)^2\\ &+2g(X,X)g(Y,Y) - 4g(JX,Y)^2, \end{aligned}$$

and

$$\begin{aligned}g(R(X,HX)Y,HY) = \; & g(R(X,Y)X,Y) + g(R(X,HY)X,HY) \\ & +4g(JX,Y)\Omega(X,Y) + 4g(GX,Y)\Omega(HX,Y) \\ & -2g(HX,Y)^2 - 4g(GX,Y)^2 - 2g(X,Y)^2 \\ & +2g(X,X)g(Y,Y) - 4g(JX,Y)^2.\end{aligned}$$

Lemma 5.2.5 On a I-K normal complex contact manifold, for X $\in \mathcal{H}$, we have

$$\Omega(JX,X) = -2g(X,X).$$

Proof. Since J is parallel for ∇,

$$\begin{aligned}g(R(X,HX)JX,GX) &= g(JR(X,HX)X,JHX) \\ &= -g(R(X,HX)HX,X).\end{aligned} \tag{5.2.3}$$

By Lemma 5.2.4, we get

$$\begin{aligned}g(R(X,HX)JX,GX) = \; & -g(R(X,HX)HX,X) \\ & -2g(X,X)(\Omega(JX,X) + 2g(X,X)).\end{aligned} \tag{5.2.4}$$

Comparing the right hand sides of (5.2.3) and (5.2.4), we get the lemma. □

Finally, we have the following property of sectional curvatures.

Proposition 5.2.6 On I-K normal complex contact manifolds, we have

$$K(X,JX) + K(X,GX) + K(X,HX) = 6,$$

for any horizontal vector field X.

Proof. Since J is parallel for ∇,

$$\begin{aligned}g(R(X,GX)JX,GJX)) &= -g(JR(X,GX)X,JGX) \\ &= g(R(X,GX)GX,X).\end{aligned} \tag{5.2.5}$$

On the other hand, by Lemmas 5.2.4 and 5.2.5,

$$\begin{aligned}g(R(X,GX)JX,GJX) &= -g(R(X,JX)JX,X) - g(R(X,HX)HX,X) \\ &\quad -4g(X,X)\Omega(JX,X) - 2g(X,X)^2 \\ &= -g(R(X,JX)JX,X) - g(R(X,HX)HX,X) \\ &\quad +6g(X,X)^2.\end{aligned} \tag{5.2.6}$$

This gives the conclusion. $\square$

The notion of I-K normality seems too strong, since the complex Heisenberg group admits no I-K normal contact metric structure while the real Heisenberg group admits a normal contact metric structure. Korkmaz introduced a weaker version of normality as follows.

Definition 5.2.7 A complex contact metric structure is normal in the sense of [29] if

$$\begin{cases} S(X,Y) = T(X,Y) = 0 & \text{for every } X, Y \in \mathcal{H}, \\ S(U,Y) = T(V,Y) = 0 & \text{for every } Y. \end{cases}$$

From now on, we use this definition of normality.

The following lemma is obtained by Korkmaz [29].

Lemma 5.2.8 If X is a horizontal vector field, then

$$\begin{aligned} g(R(X,JX)JX,X) + g(R(X,GX)GX,X) + g(R(X,HX)HX,X) \\ = -6g(X,X)(\Omega(JX,X) + g(X,X)). \end{aligned}$$

Example 5.1 We introduce the example of the complex Heisenberg group, the closed subgroup $\mathbf{H_C}$ of $GL(3, \mathbf{C})$ given by

$$\left\{ \left. \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \right| b_{12}, b_{13}, b_{23} \in \mathbf{C} \right\}.$$

Blair [3] defined the following complex contact metric structure on $\mathbf{H_C}$ (see also [6]). Let z_1, z_2, z_3 be the coordinates on $\mathbf{H_C} \cong \mathbf{C}^3$, defined by $z_1(A) = b_{23}$, $z_2(A) = b_{12}$, $z_3(A) = b_{13}$ for A in $\mathbf{H_C}$. Then the Hermitian metric

$$g = \frac{1}{8} \left(\begin{array}{ccc|ccc} & & & 1+|z_2|^2 & 0 & -z_2 \\ & 0 & & 0 & 1 & 0 \\ & & & -\bar{z}_2 & 0 & 1 \\ \hline 1+|z_2|^2 & 0 & -\bar{z}_2 & & & \\ 0 & 1 & 0 & & 0 & \\ -z_2 & 0 & 1 & & & \end{array} \right)$$

is a left invariant metric on $\mathbf{H_C}$. We define real 1-forms u, v and unit vector fields U, V by decomposing the holomorphic 1-form $\theta = (dz_3 - z_2 dz_1)/2$ and the complex vector field $X = 4(\partial/\partial z_3)$ into their real parts and the imaginary parts:

$$\theta = u - iv, \quad X = U + iV.$$

Also define two type-$(1,1)$ tensors

$$G = \left(\begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & z_2 & 0 \\ \hline 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & \bar{z}_2 & 1 & & & \end{array}\right),$$

$$H = \left(\begin{array}{ccc|ccc} & & & 0 & -i & 0 \\ & 0 & & i & 0 & 0 \\ & & & 0 & -iz_2 & 0 \\ \hline 0 & i & 0 & & & \\ -i & 0 & 0 & & 0 & \\ 0 & i\bar{z}_2 & 1 & & & \end{array}\right).$$

Then one can check that (u, v, U, V, G, H, J, g) is a normal complex contact metric structure on $\mathbf{H_C}$ [29].

5.3 Construction from Hyperkähler manifolds via reduction

In this section, we construct a normal complex contact structure on the quotient space of a hyperkähler manifold via a $\mathbf{C}^*$ action. We first recall the definition of hyperkähler manifolds.

Definition 5.3.1 (M, J_1, J_2, J_3, g) is a hyperkähler manifold if J_1, J_2, J_3 are complex structures on a complex manifold M satisfying

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -id,$$

and g is an Hermitian metric on M with respect to J_1, J_2 and J_3.

We can produce a normal complex almost contact metric manifold from hyperkähler manifolds.

Theorem 5.3.2 Let $(\widetilde{M}, J_1, J_2, J_3, \tilde{g})$ be a hyperkähler manifold. We assume that $\mathbf{C}^*$ acts holomorphically with respect to J_1 on $\widetilde{M}$. We also assume this action is proper and free. Then the quotient space $\widetilde{M}/\mathbf{C}^*$ is naturally equipped with a smooth manifold structure and the quotient map $\pi : \widetilde{M} \longrightarrow \widetilde{M}/\mathbf{C}^*$ canonically induces an I-K normal complex almost contact metric structure on $\widetilde{M}/\mathbf{C}^*$.

Proof. Let $\{\widetilde{O}_\lambda\}$ be an open covering of $\widetilde{M}$. We choose local sections $s_\lambda : \pi(\widetilde{O}_\lambda) \longrightarrow \widetilde{O}_\lambda$. Then we define type $(1,1)$ tensors J, G and H on $T(\widetilde{M}/\mathbf{C}^*)$, and 1-forms u, v by

$$\begin{cases} J_1(s_\lambda)_*X = (s_\lambda)_*JX, \\ J_2(s_\lambda)_*X = (s_\lambda)_*GX + u(X)\nu + v(X)J_1\nu, \\ J_3(s_\lambda)_*X = (s_\lambda)_*HX - v(X)\nu + u(X)J_1\nu, \end{cases} \tag{5.3.1}$$

where ν and $J_1\nu$ are unit tangent vectors to the orbit by $\mathbf{C}^*$. For example, u and v are explicitly given by

$$u(X) = \tilde{g}(J_2 s_* X, \nu), \quad v(X) = \tilde{g}(J_2 s_* X, J_1\nu) = -\tilde{g}(J_3 s_* X, \nu). \tag{5.3.2}$$

Finally, we define the unit vector fields U, V on $\widetilde{M}/\mathbf{C}^*$ by

$$U = -\pi_*(J_2\nu), \quad V = \pi_*(J_3\nu). \tag{5.3.3}$$

It is seen that the structure (u, v, U, V, G, H, J, g) (with g the metric induced by $\tilde{g}$) satisfies Definition 5.1.2 and Definition 5.2.1. $\square$

Example 5.2 By Theorem 5.3.2, the complex projective space $\mathbf{C}P^{2n+1}$ with the Fubini-Study metric admits an I-K normal complex contact structure. Now we express this structure analytically in the case of $n = 1$.
Let $\mathbf{C}^4\backslash\{0\}$ have the hyperkähler structure $(J_1, J_2, J_3, \langle,\rangle)$, where J_1, J_2 and J_3 act on the position vector $p = (z_1, z_2, z_3, z_4)$ by

$$\begin{aligned} J_1 p &= (iz_1, iz_2, iz_3, iz_4), \\ J_2 p &= (\bar{z}_3, \bar{z}_4, -\bar{z}_1, -\bar{z}_2), \\ J_3 p &= (i\bar{z}_3, i\bar{z}_4, -i\bar{z}_1, -i\bar{z}_2), \end{aligned}$$

and $\langle , \rangle$ is the standard metric on $\mathbf{C}^4\backslash\{0\}$. We denote the norm $\sqrt{\sum_{k=1}^4 z_k\bar{z}_k}$ of $z = (z_1, z_2, z_3, z_4)$ by $\|z\|$.
$\mathbf{C}^*$ acts on $\mathbf{C}^4\backslash\{0\}$ by $\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4)$, which acts freely and commutes with J_1. We can easily check that at z, the orbit space of $\mathbf{C}^*$ has tangent space at z spanned by vectors

$$\begin{aligned} \nu &= \frac{1}{2\|z\|}\sum_{j=1}^{4}\left(z_j\frac{\partial}{\partial z_j} + \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right), \\ J_1\nu &= \frac{i}{2\|z\|}\sum_{j=1}^{4}\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right). \end{aligned}$$

By calculating with the inner product $\langle , \rangle$, u and v are given by

$$\begin{aligned} u &= \frac{1}{2\|z\|}\sum_{j=1}^{2}\left(-z_{2j-1}dz_{2j} - \bar{z}_{2j-1}d\bar{z}_{2j} + z_{2j}dz_{2j-1} + \bar{z}_{2j}d\bar{z}_{2j-1}\right), \\ v &= \frac{-i}{2\|z\|}\sum_{j=1}^{2}\left(z_{2j-1}dz_{2j} - \bar{z}_{2j-1}d\bar{z}_{2j} - z_{2j}dz_{2j-1} + \bar{z}_{2j}d\bar{z}_{2j-1}\right). \end{aligned}$$

Then in complex coordinates G and H are given by

$$G = \frac{1}{\|z\|^2}\begin{pmatrix} O & A \\ \bar{A} & O \end{pmatrix}, \quad H = \frac{1}{\|z\|^2}\begin{pmatrix} O & iA \\ -i\bar{A} & O \end{pmatrix},$$

where

$$A = i\begin{pmatrix} z_2\bar{z}_1 & -\|z\|^2 + z_2\bar{z}_2 & z_2\bar{z}_3 & z_2\bar{z}_4 \\ \|z\|^2 - z_1\bar{z}_1 & -z_1\bar{z}_2 & -z_1\bar{z}_3 & -z_1\bar{z}_4 \\ z_4\bar{z}_1 & z_4\bar{z}_2 & z_4\bar{z}_3 & -\|z\|^2 + z_4\bar{z}_4 \\ -z_3\bar{z}_1 & -z_3\bar{z}_2 & \|z\|^2 - z_3\bar{z}_3 & -z_3\bar{z}_4 \end{pmatrix}.$$

Finally, the two vector fields U and V are given by

$$\begin{aligned} U &= \frac{1}{2\|z\|}\sum_{j=1}^{2}\left(-\bar{z}_{2j-1}dz_{2j} - z_{2j-1}d\bar{z}_{2j} + \bar{z}_{2j}dz_{2j-1} + z_{2j}d\bar{z}_{2j-1}\right), \\ V &= \frac{i}{2\|z\|}\sum_{j=1}^{2}\left(\bar{z}_{2j-1}dz_{2j} - z_{2j-1}d\bar{z}_{2j} - \bar{z}_{2j}dz_{2j-1} + z_{2j}d\bar{z}_{2j-1}\right). \end{aligned}$$

With the Fubini-Study metric g, we find that this complex contact metric structure (u, v, U, V, G, H, J, g) is I-K normal and satisfies Proposition 5.2.6.

By modifying the $\mathbf{C}^*$ action on $\mathbf{C}^4\backslash\{0\}$, we can give another example of normal complex contact metric manifolds.

Example 5.3 We consider another $\mathbf{C}^*$ action on $\mathbf{C}^4\backslash\{z_1z_2z_3z_4=0\}$ by

$$\lambda\cdot(z_1,z_2,z_3,z_4)=(\lambda z_1,\lambda z_2,\lambda^{-1}z_3,\lambda^{-1}z_4),$$

which also acts freely and commutes with J_1. This orbit space has tangent spaces at z spanned by the vectors

$$\begin{aligned}\nu &= \frac{1}{2\|z\|}\Big(z_1\frac{\partial}{\partial z_1}+\bar z_1\frac{\partial}{\partial \bar z_1}+z_2\frac{\partial}{\partial z_2}+\bar z_2\frac{\partial}{\partial \bar z_2}\\ &\qquad -z_3\frac{\partial}{\partial z_3}-\bar z_3\frac{\partial}{\partial \bar z_3}-z_4\frac{\partial}{\partial z_4}-\bar z_4\frac{\partial}{\partial \bar z_4}\Big),\\ J_1\nu &= \frac{i}{2\|z\|}\Big(z_1\frac{\partial}{\partial z_1}-\bar z_1\frac{\partial}{\partial \bar z_1}+z_2\frac{\partial}{\partial z_2}-\bar z_2\frac{\partial}{\partial \bar z_2}\\ &\qquad -z_3\frac{\partial}{\partial z_3}+\bar z_3\frac{\partial}{\partial \bar z_3}-z_4\frac{\partial}{\partial z_4}+\bar z_4\frac{\partial}{\partial \bar z_4}\Big).\end{aligned}$$

We check that the quotient space $M=(\mathbf{C}^4\backslash\{z_1z_2z_3z_4=0\})/\mathbf{C}^*$ is a complex manifold. We define a biholomorphic map F on M by

$$F([z_1,z_2,z_3,z_4])=\left(\frac{z_2}{z_1},\ z_1z_3,\ z_1z_4\right).$$

This map shows that M is diffeomorphic to $\mathbf{C}^3\backslash\{w_1w_2w_3=0\}$. By a direct computation of the standard inner product $\langle,\rangle$, u and v are given by

$$\begin{aligned}u &= \frac{-i}{2\|z\|}(-z_1dz_4+\bar z_1d\bar z_4+z_2dz_3-\bar z_2d\bar z_3\\ &\quad -z_3dz_2+\bar z_3d\bar z_2-z_4dz_1+\bar z_4d\bar z_1),\\ v &= \frac{1}{2\|z\|}(-z_1dz_4-\bar z_1d\bar z_4+z_2dz_3+\bar z_2d\bar z_3\\ &\quad +z_3dz_2+\bar z_3d\bar z_2-z_4dz_1-\bar z_4d\bar z_1).\end{aligned}$$

Then in complex coordinates G and H are given as follows:

$$G=\frac{1}{\|z\|^2}\begin{pmatrix} O & A\\ \bar A & O\end{pmatrix},\quad H=\frac{1}{\|z\|^2}\begin{pmatrix} O & iA\\ -i\bar A & O\end{pmatrix},$$

where

$$A = i \begin{pmatrix} z_1\bar{z}_4 & -z_1\bar{z}_3 & -z_1\bar{z}_2 & -\|z\|^2 + z_1\bar{z}_1 \\ z_2\bar{z}_4 & -z_2\bar{z}_3 & \|z\|^2 - z_2\bar{z}_2 & z_2\bar{z}_1 \\ -z_3\bar{z}_4 & -\|z\|^2 + z_3\bar{z}_3 & z_3\bar{z}_2 & -z_3\bar{z}_1 \\ \|z\|^2 - z_4\bar{z}_4 & z_4\bar{z}_3 & z_4\bar{z}_2 & -z_4\bar{z}_1 \end{pmatrix}.$$

Finally, the two vector fields U and V are given by

$$\begin{aligned} U &= \frac{i}{2\|z\|}\Big(z_1\frac{\partial}{\partial \bar{z}_4} - \bar{z}_1\frac{\partial}{\partial z_4} - z_2\frac{\partial}{\partial \bar{z}_3} + \bar{z}_2\frac{\partial}{\partial z_3} \\ &\quad -z_3\frac{\partial}{\partial \bar{z}_2} + \bar{z}_3\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial \bar{z}_1} - \bar{z}_4\frac{\partial}{\partial z_1}\Big), \\ V &= \frac{-1}{2\|z\|}\Big(z_1\frac{\partial}{\partial \bar{z}_4} + \bar{z}_1\frac{\partial}{\partial z_4} - z_2\frac{\partial}{\partial \bar{z}_3} - \bar{z}_2\frac{\partial}{\partial z_3} \\ &\quad -z_3\frac{\partial}{\partial \bar{z}_2} - \bar{z}_3\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial \bar{z}_1} + \bar{z}_4\frac{\partial}{\partial z_1}\Big). \end{aligned}$$

With the induced metric g from the standard inner product $\langle , \rangle$ on $\mathbf{C}^4$, we can check that this complex almost contact metric structure (u, v, U, V, G, H, J, g) is I-K normal. Thus we get a new example of a normal complex almost contact metric manifold.

5.4 3-Sasakian manifolds

We recall the 3-Sasakian structures on M^{4p+3}.

Definition 5.4.1 Let M^{4n+3} be a real $(4n+3)$-dimensional manifold. The 3-Sasakian structure on M^{4n+3} is a triple of Sasakian structures $\{\Phi_i, \xi_i, \eta_i\}_{i=1,2,3}$ on M^{4n+3} satisfying

$$\begin{aligned} &\Phi_k = \Phi_i\Phi_j - \eta_j \otimes \xi_i = -\Phi_j\Phi_i + \eta_i \otimes \xi_j, \\ &\eta_i \circ \Phi_j = \eta_k, \quad \eta_i(\xi_j) = \delta_{ij}, \end{aligned}$$

where $\{i, j, k\}$ is one of the cyclic permutations of $\{1, 2, 3\}$. Define M^{4n+3} is called the 3-Sasakian manifold if there exists a 3-Sasakian structure on it.

The typical example of 3-sasakian manifold is S^{4n+3} obtained by taking as a hyper surface in the quaternion vector space $\mathbf{H}^{n+1}$. Each of three almost complex structure on $\mathbf{H}^{n+1}$ applied to the outer normal vector of the sphere gives a vector field $\xi_i, i = 1, 2, 3$, on S^{4n+3}. These three vector fields are orthogonal each other and give rise to the standard 3-Sasakian structure on S^{4n+3}.

5.5 Complex almost contact metric structure on $S^{4p+3} \times S^{4q+3}$

We show that 3-Sasakian structures on S^{4m+3} and S^{4n+3} induce a complex almost contact metric structure on $S^{4m+3} \times S^{4n+3}$. Let $\{\Phi_i^m, \xi_i^m, \eta_i^m\}_{i=1,2,3}$ and $\{\Phi_i^n, \xi_i^n, \eta_i^n\}_{i=1,2,3}$ be 3-Sasakian structures on S^{4m+3} and S^{4n+3}, respectively. We first define an almost complex structure on $S^{4m+3} \times S^{4n+3}$ by

$$J_{m,n}(X, Y) = (\Phi_1^m X - \eta_1^n(Y)\xi_1^m, \Phi_1^n Y + \eta_1^m(X)\xi_1^m), \tag{5.5.1}$$

where $(X, Y) \in T\,(S^{4m+3} \times S^{4n+3})$. Since $J_{m,n}$ is integrable [31] (see also [10]), then $(S^{4m+3} \times S^{4n+3}, J_{m,n})$ is a complex manifold. Moreover, it is also proved that the product space of two normal almost contact metric manifolds is a complex manifold with the above $J_{m,n}$. Next, we define a metric $g_{m,n}$ on $S^{4m+3} \times S^{4n+3}$ by

$$\begin{aligned} g_{m,n}\left((X, Y), (X', Y')\right) &= g_m(X, X') + \eta_1^m(X)\eta_1^m(X') \\ &\quad + g_n(Y, Y') + \eta_1^n(Y)\eta_1^n(Y'), \end{aligned} \tag{5.5.2}$$

where g_m and g_n are the associated metrics to 3-Sasakian structures on S^{4m+3} and S^{4n+3} respectively. It is easily checked that $g_{m,n}$ is a Hermitian metric with respect to $J_{m,n}$.

$X \in TM^{4m+3}$ and $Y \in TM^{4n+3}$ are decomposed to the subspace spanned by $\{\xi_1^m, \xi_2^m, \xi_3^m\}$, $\{\xi_1^n, \xi_2^n, \xi_3^n\}$ and their orthogonal complements uniquely as follows.

$$\left\{ \begin{array}{l} X = X_0 + \eta_1^m(X)\xi_1^m + \eta_2^m(X)\xi_2^m + \eta_3^m(X)\xi_3^m, \\ Y = Y_0 + \eta_1^n(X)\xi_1^n + \eta_2^n(X)\xi_2^n + \eta_3^n(X)\xi_3^n, \end{array} \right. \tag{5.5.3}$$

where $X_0 \in Span\{\xi_1^m, \xi_2^m, \xi_3^m\}^\perp$, $Y_0 \in Span\{\xi_1^n, \xi_2^n, \xi_3^n\}^\perp$. With this decomposition, we define the two type-$(1,1)$ tensors $G_{m,n}$ and $H_{m,n}$ by

$$\begin{aligned}
&G_{m,n}(X,Y) = \qquad\qquad (5.5.4)\\
&\quad\Bigg(\Phi_2^m X_0 - \frac{\eta_3^m(X) - \eta_3^n(Y)}{2}\xi_1^m + \eta_1^n(Y)\xi_2^m + \eta_1^m(X)\xi_3^m,\\
&\qquad \Phi_2^n Y_0 - \frac{\eta_2^m(X) - \eta_2^n(Y)}{2}\xi_1^n - \eta_1^m(Y)\xi_2^n - \eta_1^m(X)\xi_3^n\Bigg),\\
&H_{m,n}(X,Y) = J_{m,n}\, G_{m,n}(X,Y), \qquad\qquad (5.5.5)
\end{aligned}$$

where $(X,Y) \in T\,(S^{4m+3} \times S^{4n+3})$. We can check that G and H satisfy the condition to be a complex almost contact metric structure. Using the formula (5.16), we get

$$\begin{aligned}
G_{m,n}^2(X,Y) &= \Bigg((\Phi_2^m)^2 X_0 - \eta_1^m(X)\xi_1^m + \frac{\eta_2^m(X) - \eta_2^n(Y)}{2}\xi_2^m - \frac{\eta_3^m(X) - \eta_3^n(Y)}{2}\xi_3^m,\\
&\qquad (\Phi_2^n)^2 Y_0 - \eta_1^n(Y)\xi_1^n + \frac{\eta_2^m(X) - \eta_2^n(Y)}{2}\xi_2^n + \frac{\eta_3^m(X) - \eta_3^n(Y)}{2}\xi_3^n\Bigg)\\
&= \big(- X_0 - \eta_1^m(X)\xi_1^m - \eta_2^m(X)\xi_2^m - \eta_3^m(X)\xi_3^m,\\
&\qquad -Y_0 - \eta_1^n(Y)\xi_1^n - \eta_2^n(Y)\xi_2^n - \eta_3^n(Y)\xi_3^n\ \big)\\
&\quad + \frac{\eta_2^m(X) + \eta_2^n(Y)}{2}\ (\xi_2^m, \xi_2^n) + \frac{\eta_3^m(X) + \eta_3^n(Y)}{2}\ (\xi_3^m, \xi_3^n), \quad (5.5.6)
\end{aligned}$$

where $(X,Y) \in T\,(S^{4m+3} \times S^{4n+3})$. Here we define 1-forms $u_{m,n}, v_{m,n}$ and dual orthonormal vector fields $U_{m,n}, V_{m,n}$ which satisfy Definition 5.1.2. by

$$\begin{cases} u_{m,n} = \frac{1}{\sqrt{2}}(\eta_3^m + \eta_3^n), \quad v_{m,n} = \frac{1}{\sqrt{2}}(\eta_2^m + \eta_2^n),\\ U_{m,n} = \frac{1}{\sqrt{2}}(\xi_3^m, \xi_3^n), \quad V_{m,n} = \frac{1}{\sqrt{2}}(\xi_2^m, \xi_2^n). \end{cases} \qquad (5.5.7)$$

With these elements, we get

$$G_{m,n}^2(X,Y) = -(X,Y) + u_{m,n}(X,Y) \otimes U + v_{m,n}(X,Y) \otimes V. \qquad (5.5.8)$$

Moreover, by (5.5.1), (5.5.2) and (5.5.4), we have

$$\begin{aligned}
&J_{m,n} G_{m,n}(X,Y) = -G_{m,n} J_{m,n}(X,Y) \qquad\qquad (5.5.9)\\
&= \Bigg(\Phi_3^m X_0 + \frac{\eta_2^m(X) - \eta_2^n(Y)}{2}\xi_1^m - \eta_1^m(X)\xi_2^m + \eta_1^n(Y)\xi_3^m,
\end{aligned}$$

$$\Phi_3^n Y_0 - \frac{\eta_3^m(X) - \eta_3^n(Y)}{2}\xi_1^n + \eta_1^m(X)\xi_2^n - \eta_1^n(Y)\xi_3^n\Bigg) .$$

$$\begin{aligned} & g_{m,n}\left(G_{m,n}(X,Y),(X',Y')\right) = -g_{m,n}\left((X,Y),G_{m,n}(X',Y')\right) \qquad (5.5.10) \\ = \;& g_m(\Phi_2^m X_0, X_0') + g_n(\Phi_2^n Y_0, Y_0') + \eta_1^m(X)\left(\eta_3^m(X') - \eta_3^n(Y')\right) \\ & -\eta_1^m(X')\left(\eta_3^m(X') - \eta_3^n(Y')\right) + \eta_1^n(Y)\left(\eta_2^m(X') - \eta_2^n(Y')\right) \\ & -\eta_1^n(Y')\left(\eta_2^m(X) - \eta_3^n(Y)\right) . \end{aligned}$$

Note that this complex almost contact metric structure is not I-K normal. Finally we conclude as follows.

Theorem 5.5.1 The complex almost contact metric structure on $S^{4m+3} \times S^{4n+3}$, $(G_{m,n},\ H_{m,n}, J_{m,n}, u_{m,n}, v_{m,n}, U_{m,n}, V_{m,n}, g_{m,n})$ given by (5.5.1), (5.5.2), (5.5.4), (5.5.5) and (5.5.7) is not I-K normal.

Chapter 6

Conclusions and Further Problems

In this section, we summarize the results in thesis and discuss further problems about constructions of real or complex contact manifolds.

In section 4.5, we presented a Sasaki-Einstein metric $g_{1,0}$ on $S^2 \times S^3$, but it is not known how to describe them explicitly in other cases.

Theorem A (**Theorem 4.5.1**) The Sasaki-Einstein metric $g_{1,0}$ on $S^2 \times S^3$ at any point x is given by the formula (4.5.1).

Problem 6.1 For relatively prime $p > q$, calculate $g_{p,q}$.

In section 5.3, we constructed I-K normal complex almost contact manifolds from hyperkähler manifolds via reduction.

Theorem B (**Theorem 5.3.2**) Let $(\widetilde{M}, J_1, J_2, J_3, \tilde{g})$ be a hyperkähler manifold. Assume that $\mathbf{C}^*$ acts holomorphically with respect to the complex structure J_1 on $\widetilde{M}$. We also assume this action is proper and free. Then the quotient space $\widetilde{M}/\mathbf{C}^*$ is naturally equipped with a smooth manifold structure and the quotient map $\pi : \widetilde{M} \longrightarrow \widetilde{M}/\mathbf{C}^*$ canonically induces an I-K normal complex almost contact metric structure on $\widetilde{M}/\mathbf{C}^*$.

Using this result, we construct a new example of I-K normal complex almost contact metric manifold (Example 5.3).

In studying complex contact manifolds, we found a remarkable property on the sectional curvature of an I-K normal complex contact metric manifold which will give a strong information [22]:

Proposition C (**Proposition 5.2.6.**) On I-K normal complex contact manifolds, we have

$$K(X, JX) + K(X, GX) + K(X, HX) = 6, \tag{6.0.1}$$

for any $X \in \operatorname{Ker} \omega$, where $K(X, Y)$ is the sectional curvatures of the plane spanned by $\{X, Y\}$, and G, H and J are associated to the complex contact metric structure.

By this proposition, any I-K normal complex contact metric manifold satisfies (6.0.1). Conversely, does any complex (almost) contact metric manifold satisfying (6.0.1) admit I-K normality?

Problem 6.2 Find examples of complex contact metric manifolds satisfying (6.0.1) except for $\mathbf{CP}^{2n+1}$ (Example 5.2). Are these manifolds I-K normal?

In example 4.4, the standard Kähler structure on $\mathbf{R}^{2n+2}$ induces the Sasakian structure on S^{2n+1}. Similarly, we expect that the standard hyperkähler structure on $\mathbf{C}^{2n+2}$ induces a complex (almost) contact metric structure on some complex hypersurfaces of $\mathbf{C}^{2n+2}$.

Problem 6.3 Find the complex hypersurfaces of a hyperkähler manifold which admit complex (almost) contact metric structures. More generally, does any complex submanifolds of a hyperkähler manifold which admit such structures exist?

In section 5.5, we construct a non-normal almost contact metric structure on $S^{4m+3} \times S^{4n+3}$. More generally, it is expected that the product space of two 3-Sasakian manifolds admits complex almost contact metric structures.

Theorem D (**Theorem 5.5.1.**) The complex almost contact metric structure

$(G_{m,n},\, H_{m,n}, J_{m,n}, u_{m,n}, v_{m,n}, U_{m,n}, V_{m,n}, g_{m,n})$ on $S^{4m+3} \times S^{4n+3}$ given by (5.5.1), (5.5.2), (5.5.4), (5.5.5) and (5.5.7) is not I-K normal.

Problem 6.4 Does any complex (almost) contact metric structure on the product space of two 3-Sasakian manifolds exist? Especially, does the structure have normality?

Complex (almost) contact metric structures have good relevance to the triple of geometrical structures. We also expect that complex submanifolds of quaternionic Kähler manifolds admit complex (almost) contact metric structures.

Problem 6.5 Find the complex submanifolds of quaternionic Kähler manifolds which admit complex (almost) contact metric structures.

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