Canonical Decompositions
Describing Structures of Matchings in Graphs

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主　論　文　要　旨

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主 論 文 題 目：

Canonical Decompositions Describing Structures of Matchings in Graphs

（グラフのマッチング構造を記述する標準分解）

（内容の要旨）

グラフとは道路網などネットワーク構造の数学的抽象化に相当する基本的な離散構造である．マッチングとはグラフの部分構造の一種であり，互いに素な枝の集合として定義される．マッチングは離散数学の最も代表的な研究対象の一つであり，長年に渡り関心を集めてきた．マッチングに関する研究の蓄積はマッチング理論と呼ばれるグラフ理論の一大分野を成している．本論文はマッチング理論の基盤に対し，標準分解・双対性に基づく構造解明・完全マッチングの数え上げ問題の三つの方向から貢献するものである．

従来マッチングの構造を把握する強力なツールとして，総称して標準分解と呼ばれるいくつかの分解型構造定理が重要な役割を果たしてきた．しかしこれらはそれぞれ特殊なクラスのグラフのみを実質的な適用対象としており，互いの相互関係やこれらをまとめ上げ統一して理解する方法は不明であった．これに対し本論文では任意のグラフを適用対象とし，かつ既存の標準分解をまとめ上げる新しい標準分解を提案する．これは既存の標準分解の洗練された記述も含んでおり，完全マッチングを持つ一般のグラフに対しても非自明な構造を明らかにするものである．

バリアもまたグラフの部分構造の一種であるが，これは最大マッチング問題の双対最適解の組合せ的解釈に相当しており，すなわちマッチングと対をなす概念である．双対性は組合せ最適化の理論体系の軸となる概念であり，事実バリアもマッチングの研究において重要な役割を果たす．しかしバリアについて知られていることは少なく，特に重要である極大バリアについてすらも解明は進んでいなかった．これに対し本論文では，一般のグラフに対し極大バリアの構造を明らかにする定理を与える．これは 1972 年に Lovász によって与えられた canonical partition の一般化に相当する．

グラフがもつ完全マッチングの総数を調べることは数え上げ組合せ論の代表的な問題の一つであり，様々な側面からの研究がなされている．カテドラル定理は飽和グラフの特徴づけを与えており，次数などグラフの構造に関するパラメタと完全マッチングの総数との関係を調べる際に有用である．カテドラル定理は 1972 年に Lovász によって与えられたのち，2001 年に Szigeti によって別証明が与えられている．本論文では，飽和グラフの性質を新しい標準分解を用いて精査することによってカテドラル定理のさらなる別証明を与える．この新しい証明では，カテドラル定理の背後にある構造を明らかにすることでより洗練された事実を副産物として与えつつ，非常に自然な形での別証明を与えている．

また，本論文で提案された新しい標準分解を計算する多項式時間アルゴリズムをいくつか提案する．これはカテドラル定理によって明らかにされる構造を計算するものにも対応する．

SUMMARY OF Ph.D. DISSERTATION

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Title

Canonical Decompositions Describing Structures of Matchings in Graphs

Abstract

Graphs are fundamental discrete structures, which arise as a mathematical formulation of network structures such as road networks. A *matching* is a kind of substructures of a graph; a set of edges is called a matching if any two of them are disjoint. The notion of matchings is one of the central concerns in discrete mathematics and has attracted attention for years. Studies on matchings form one of the largest branches of graph theory called *matching theory*. This dissertation is devoted to refining the foundation of matching theory, from three directions: canonical decompositions, structures of barriers, and the enumeration problem of perfect matchings.

A kind of decomposition of graphs called *canonical decompositions* in general has been a fundamental and powerful tool to see matchings. Several canonical decompositions have been known, however each of them can be substantially applicable to a special class of graphs, respectively; also, we have not had any way to know relationships between them or to integrate and unify them. In this dissertation, we give theorems that introduce a new canonical decomposition; this new decomposition can be applicable to any graph, describing much more detailed structures even for the general graphs with perfect matchings, and enables us to understand the other known canonical decompositions in a unified way.

A *barrier* is also a kind of substructures of graphs, which corresponds to a combinatorial interpretation of the dual optimal solutions of the maximum matching problem; that is to say, a barrier is a notion acting as a counterpart of matchings. Duality is a concept that supports the theory of combinatorial optimization, and indeed barriers play important roles when we investigate matchings. However, not so much has been known about barriers, even for those maximal, which are considered to be especially important. In this dissertation we give a theorem which describes structures of maximal barriers of general graphs. This structure theorem corresponds to a generalization of the canonical partition formulated by Lovász in 1972.

Enumerating all the perfect matchings of a given graph is one of the most fundamental problems in enumerative combinatorics and has been studied by various approaches. *The cathedral theorem* is a characterization of the *saturated graphs* and has been useful in investigating relations between the number of perfect matchings and some graph parameters such as degrees. The cathedral theorem was first given by Lovász in 1972, and later Szigeti gave another proof in 2001. Here in this dissertation we give yet another proof by considering the saturated graphs with the new canonical decomposition. In this new proof, we reveal the intrinsic structure exists behind the cathedral theorem and show it in quite a natural way providing more refined properties as by-products.

Moreover, we propose several polynomial time algorithms to compute the new canonical decompositions, which also correspond to algorithmic results of the cathedral theorem.

A Thesis for the Degree of Ph.D. in Science

Canonical Decompositions Describing Structures of Matchings in Graphs

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Preface

Graphs are fundamental discrete structures, which arise as a mathematical formulation of network structures such as road networks. A *matching* is a kind of substructures of a graph; a set of edges is called a matching if any two of them are disjoint. The notion of matchings is one of the central concerns in discrete mathematics and has attracted attention for years. Studies on matchings form one of the largest branches of graph theory called *matching theory*. This thesis is devoted to refining the foundation of matching theory, from three directions: canonical decompositions, structures of barriers, and the enumeration problem of perfect matchings.

A kind of decomposition of graphs called collectively *canonical decompositions* has been a fundamental and powerful tool to see matchings. Several canonical decompositions have been known, however each of them can be substantially applicable to a special class of graphs, respectively; also, we have not had any way to know relationships between them or to integrate and unify them. In this thesis, we give theorems that introduce a new canonical decomposition; this new decomposition can be applicable to any graph, describing much more detailed structures even for the general graphs with perfect matchings, and enables us to understand the other known canonical decompositions in a unified way.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Background

Given a graph, a *matching* (1-*matching*) is a set of edges no two of which have common vertices. This basic notion has gathered attention in discrete mathematics for years, and numerous related notions or problems including generalizations such as k-matchings have been studied extensively; these studies form one of the largest branches of graph theory: *matching theory* [30]. The notion of matchings is so fundamental that sometimes problems which do not appear to be related directly to matching problems can be solved with the help of matching theory, e.g., the Chinese postman problem, the Hamilton cycle problem, and so on [30]. Applications are not limited to graph theory; some linear algebraic problems are also such examples [4–6, 13].

What is more, we must mention that matching theory has played an important role at the heart of the rapid growth of combinatorial optimization in the past decades. Efficiency in the sense of computational costs is what matters most in combinatorial optimization, and since advocated by Edmonds [7], the concept of polynomial time solvability has been obtained broad acceptance as a measure of "well-solvability". The maximum matching problem is considered as the most fundamental one among problems solved in polynomial time, and it is said that

> matching theory serves as an archetypal example of how a well-solved problem can be studied [30].

In fact, fundamental structural results and algorithms on matchings let us pioneer the new paradigm, i.e., *polyhedral combinatorics*; here important notions and techniques that capture and handle well-solved problems somewhat comprehensively, e.g., min-max theorems and integrality of polyhedra, have originated out of matching theory [30, 34].

This thesis is devoted to contributing to the foundation of matching theory, and we approach it from three aspects: canonical decompositions, structures of barriers, and the enumeration problem of perfect matchings.

In the succeeding sections of this chapter, first we give fundamental definitions and notations in Section 1.2, then in Section 1.3 introduce concepts and theorems in matching theory related to the theme of this thesis such as barriers and canonical decompositions, and finally state an overview of this thesis in Section 1.4.

1.2 Fundamentals

1.2.1 Graphs

Graphs

A *graph* G is a pair $(V(G), E(G))$ of a set $V(G)$ and a multiset $E(G)$ disjoint from $V(G)$, each of whose elements is an unordered pair of not necessarily distinct two elements of $V(G)$. Each element of $V(G)$ and $E(G)$ is called a *vertex* and an *edge* of G, respectively. For an edge $\{x, y\}$, we say it *joins* x and y, and denote it by xy; we also say x is *adjacent* to y. For an edge e, we call the elements of e the *end vertices* or the *ends* of e. An edge is a *loop* if it is consists of identical vertices. Edges are *parallel* if they possess the same sets of end vertices and are distinguished respectively. The graphs we treat in this thesis are usually *multigraphs*, i.e., they might have loop edges or parallel edges. In this thesis we assume graphs are *finite*, i.e., $V(G)$ and $E(G)$ are finite sets. A graph whose vertex set and edge set are empty, i.e., $(\emptyset, \emptyset)$ is called *empty.* We assume all graphs under discussion is not empty unless otherwise specified. In figures, a graph is represented with each vertex indicated by a point and each edge by a line. Two graphs G_1 and G_2 are *identical* if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$ hold; we write $G_1 = G_2$. Two graphs G_1 and G_2 are *isomorphic* if there is a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$.

Subgraphs

Hereafter for a while let G be a graph and let $X \subseteq V(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq V(G)$ hold. In this thesis, if a graph H is a subgraph of G, we sometimes denote $H \subseteq G$. We say a subgraph H of G is *maximal* with respect to a certain property if for any subgraph $\hat{H}$ of G with $H \subseteq \hat{H}$, $\hat{H}$ satisfying the property yields $H = \hat{H}$. Given a set of edges $F \subseteq E(G)$ of G, the graph such that its vertices are the end vertices of F and the edge set is F is called the *subgraph of* G *determined by* F and is denoted by $G.F$. The graph such that its vertex set is X and its edges are the edges of G both of whose end vertices are in X is called the *subgraph of* G *induced by* X and is denoted by $G[X]$. We denote $G[V(G) \setminus X]$ by $G - X$.

Operations on Graphs

The graph obtained by regarding all the vertices of X as a single vertex is called the *contraction of* G *by* X and is denoted by G/X; more precisely, $V(G/X) = (V(G) \setminus X) \cup \{x\}$, where x is a new vertex disjoint from $V(G) \cup E(G)$, and $E(G/X)$ is obtained from $E(G)$ by removing all edges of $G[X]$ and replacing each edge $uv \in E(G)$ with $u \in X$ and $v \in V(G) \setminus X$ by xv.

Let G be a subgraph of a graph $\hat{G}$, and let $e = xy \in E(\hat{G})$. If G does not have an edge joining x and y with $x \neq y$, then we call xy a *complement edge* of G. The graph $G + e$ denotes the graph $(V(G) \cup \{x, y\}, E(G) \cup \{e\})$, and $G - e$ the graph $(V(G), E(G) \setminus \{e\})$. For $F = \{e_1, \ldots, e_k\} \subseteq E(\hat{G})$, we define $G + F := G + e_1 + \cdots + e_k$ and $G - F := G - e_1 - \cdots - e_k$. For simplicity, regarding the operations creating new graphs out of some given graphs, such as contraction and taking the union of graphs, we identify vertices, edges, and subgraphs of the newly created graph with those of old graphs that naturally correspond to them.

Graph Parameters

A *neighbor* of X is a vertex not in X which is joined to a vertex of X by an edge. We denote the set of neighbors of X by $N_G(X)$. Given $Y, Z \subseteq V(G)$, the notation $E_G[Y, Z]$ denotes the set of edges joining Y and Z, and $\delta_G(Y)$ denotes $E_G[Y, V(G) \setminus Y]$. For a vertex $v \in V(G)$, the *degree* of v is the number of edges adjacent to v, each loop counted as two edges.

We sometimes regard a graph as the set of its vertices. For example, given a subgraph H of G, we denote $N_G(V(H))$ by $N_G(H)$.

Special Classes of Graphs, Connectivity, Connected Components

A graph G without loops or multiple edges is a *path* if $|V(G)| = 1$ and $|E(G)| = 0$ holds or $|V(G)| \geq 2$ and exactly two vertices are of degree one and any other vertex is of degree two. We call the former type of paths *trivial*. Given a path P, vertices with degree no more than one are called *end vertices* or *ends* and vertices with degree two are called *internal* vertices of P. We say a path is *between x and y* if its end vertices are x and y; we also say a path *connects x and y*, or sometimes say it is *from x to y*.

A graph is called *connected* if for any two vertices x and y there is a path connects x and y. Given a graph G, a maximal connected subgraph is called a *connected component* or just a *component* of G. A graph G is 2-*connected* if it is connected and for any $v \in V(G)$ the graph $G - v$ is connected. Given a connected graph G, a *block* of G is a maximal 2-connected subgraph.

A graph is a *circuit* if it is connected and any of its vertices is of degree two. Note that in this thesis, we treat paths and circuits as graphs. A *tree* is a connected graph without circuits as its subgraphs.

A graph G is *bipartite* if there is a partition of $V(G)$ into two sets, say A and

B, *color classes*, which satisfies $E(G) = E_G[A, B]$; we write $G = (A, B; E(G))$.

1.2.2 Matchings

A set of edges without loops is called a *matching* if no two of them share end vertices. A matching of cardinality $|V(G)|/2$ (resp. $(|V(G)| - 1)/2$) is called a *perfect matching* (resp. a *near-perfect matching*). A graph is called *factorizable* if it has a perfect matching. Hereafter for a while let M be a matching of a graph G. We say M *exposes* (resp. *covers*) a vertex $v \in V(G)$ if $\delta_G(v) \cap M = \emptyset$ (resp. $\delta_G(v) \cap M \neq \emptyset$). For a matching M of G and $u \in V(G)$, u' denote the vertex such that $uu' \in M$, if u is not exposed. Such vertex u' is called *matched to u by M*. For $X \subseteq V(G)$, M_X denotes $M \cap E(G[X])$. A subgraph H of G is called *nice* if $G - V(H)$ is factorizable.

For a path or circuit Q of G, Q is *M-alternating* if $E(Q) \setminus M$ is a matching of Q, in other words, if edges of M and $E(Q) \setminus M$ appear alternately in Q. Let P be an M-alternating path of G with end vertices u and v. If P has an even number of edges and $M \cap E(P)$ is a near-perfect matching of P exposing only v, we call it an *M-balanced path* from u to v. We regard a trivial path, that is, a path composed of one vertex and no edges as an M-balanced path. If P has an odd number of edges and $M \cap E(P)$ (resp. $E(P) \setminus M$) is a perfect matching of P, we call it *M-saturated* (resp. *M-exposed*).

A path P of G is an *ear relative to X* if both end vertices of P are in X while internal vertices are not. Also, a circuit C is an ear relative to X if exactly one vertex of C is in X. For simplicity, we call the vertices of $V(P) \cap X$ *end vertices* of P, even if P is a circuit. Given $Y \subseteq V(G)$, an ear P relative to X is *through Y* if P has a vertex other than end vertices that is in Y. For an ear P of G relative to X, we call it an *M-ear* if $P - X$ is an M-saturated path.

A graph is called *factor-critical* if a deletion of an arbitrary vertex results in a factorizable graph. For convenience, we regard a graph with only one vertex as factor-critical.

An edge $e \in E(G)$ is called *allowed* if there is a perfect matching containing

e. Let $\hat{M}$ be the set of allowed edges of G. For each connected component C of the subgraph of G determined by $\hat{M}$, we call the subgraph of G induced by $V(C)$ as a *factor-connected component* or a *factor-component* for short. The set of all the factor-components of G is denoted by $\mathcal{G}(G)$. Therefore, a factorizable graph is composed of factor-components and some edges joining between different factor-components. A factorizable graph with exactly one factor-component is called *elementary*.

1.2.3 Digraphs

A *directed graph* or *digraph* D is a pair $(V(D), A(D))$ of a set $V(D)$ and a multiset $A(D)$ disjoint from $V(D)$, each of whose elements is an ordered pair of elements of $V(D)$. Here each element of $V(D)$ and $A(D)$ is a *vertex* and an *arc*. We denote an arc by (u, v) for some $u, v \in V(D)$, or sometimes by uv; here u (resp. v) is called the *tail* (resp. the *head*) of (u, v). Digraphs we treat in this thesis are generally *multidigraphs*, i.e., they might possess a *loop*, an arc (u, u) for some $u \in V(D)$ and there might be *parallel* arcs, the same as ordered pairs but are distinguished. In this thesis we assume that digraphs are *finite*, i.e., $V(D)$ and $A(D)$ are finite sets.

Analogous to graphs, a *subgraph* D' of a digraph D is a digraph with $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$. We say a subgraph D' of D is *maximal* with respect to a certain property if for any subgraph D'' of D such that D' is a subgraph of D'', D'' satisfying the property yields $V(D') = V(D'')$ and $E(D') = E(D'')$.

Given a digraph D, for $v \in V(D)$, *outdegree* (resp. *indegree*) of v is the number of edges whose tails (resp. heads) are v. A digraph P without loops or multiple arcs is a *dipath* if $|V(D)| = 1$ and $|A(D)| = 0$ hold or it has a single vertex of outdegree one and indegree zero and a single vertex of indegree one and outdegree zero and for any other vertex outdegree and indegree are respectively one. We say a dipath P is *from u to v* if $u = v$ and $V(P) = \{u\}$ hold or u (resp. v) is the vertex of P with outdegree one and indegree zero (resp. with indegree one and outdegree zero).

A digraph D is *strongly-connected* if for any two vertices of u and v there is a dipath such that the vertex with indegree zero is u and the vertex with outdegree zero is v. Give a digraph, a *strongly-connected component* is a maximal strongly-connected subgraph of it.

A *dicircuit* is a digraph which is strongly-connected and any of its vertices satisfies outdegree one and indegree one.

1.3 Structure Theorems in Matching Theory

1.3.1 Barriers

In matching theory, the notion of barriers plays significant roles. Given a graph, we call a connected component of it with an odd (resp. even) number of vertices *odd component* (resp. *even component*). Given $X \subseteq V(G)$ of a graph G, we denote as $q_G(X)$ the number of odd components that the graph resulting from deleting X from G has; we denote the cardinality of a maximum matching of G as $\nu(G)$. There is a min-max theorem called the *Berge formula* [30] that for any graph G,

$$|V(G)| - 2\nu(G) = \max\{q_G(X) - |X| : X \subseteq V(G)\}.$$

A set of vertices that attains the maximum in the right side of the equation is called a *barrier*. Roughly speaking, barriers essentially coincide with dual optimal solutions of the maximum matching problem, and decompose graphs so that one can see the structures of maximum matchings.

Definition 1.3.1. Given a graph G and $X \subseteq V(G)$, we denote the vertices contained in the odd components of $G - X$ as D_X, and $V(G) \setminus X \setminus D_X$ as C_X.

The next proposition can be easily observed by the Berge formula.

Proposition 1.3.2 (folklore)**.** *Let G be a graph, and $X \subseteq V(G)$ be a barrier of G. Then for any maximum matching M of G,*

(i) *each vertex of X is matched to a vertex of D_X,*

(ii) *for each component* K *of* $G[D_X]$, M_K *is a near-perfect matching of* K, *accordingly* $|\delta(K) \cap M| \leq 1$,

(iii) *there exist exactly* $|V(G)| - 2\nu(G)$ *components with* $|\delta(K) \cap M| = 0$,

(iv) M *contains a perfect matching of* $G[C_X]$, *and*

(v) *no edge in* $E[X, C_X]$ *nor* $E(G[X])$ *is allowed.*

By Proposition 1.3.2, we see how a given barrier enables us to understand the size and structures of all maximum matchings at once. By this proposition, barriers are very useful in numerous contexts in matching theory. However, compared to numerous results on maximum matchings, "much less is known about barriers [30]".

1.3.2 Canonical Decompositions

Canonical Decompositions in Matching Theory

There is a fundamental and essential desire that we want to grasp in what way all the maximum matchings exist in graphs. In matching theory, there are theorems meeting this desire; these theorems decompose a given graph into subgraphs in some ways which are uniquely determined by the given graph and describe the structure of all the maximum matchings at once. These decompositions given by these theorems are called *canonical decompositions* in general, since the word *canonical* means "unique to given graphs" in matching theory. It seems barriers take on a role similar to canonical decompositions. Indeed, as we saw in the previous section, a barrier gives a decomposition describing all maximum matchings. However, a graph generally has a number of barriers and so decompositions they give are not canonical. To the best of our knowledge, three canonical decompositions have been known:

1. the Gallai-Edmonds decomposition (Gallai [12], Edmonds [7]),

2. the canonical partition (Kotzig [24–26], Lovász [27]), and

3. the Dulmage-Mendelsohn decomposition (Dulmage & Mendelsohn [4–6]).

In the following, we are going to explain them in detail.

The Gallai-Edmonds Structure Theorem

Definition 1.3.3. Given a graph G, we define $D(G)$ as the set of vertices that are exposed by some maximum matching, $A(G)$ as $N(D(G))$ and $C(G)$ as $V(G) \setminus (D(G) \cup A(G))$. This partition of $V(G)$ into three parts $D(G)$, $A(G)$, $C(G)$ is called the *Gallai-Edmonds partition*.

There is a famous theorem about the Gallai-Edmonds partition, called the *Gallai-Edmonds structure theorem*:

Theorem 1.3.4 (The Gallai-Edmonds structure theorem [30])**.** *Let G be a graph. Then, $D(G), A(G), C(G)$ satisfy the following:*

(i) *The set $A(G)$ is a barrier with $D_{A(G)} = D(G)$ and $C_{A(G)} = C(G)$.*

(ii) *Each odd component of $G - A(G)$ is factor-critical.*

(iii) *For each $e \in E[A(G), D(G)]$, there exists a maximum matching M of G with $e \in M$.*

When we refer to the decomposition of a graph determined naturally by the Gallai-Edmonds partition considering its properties given by Theorem 1.3.4, we call it the *Gallai-Edmonds decomposition.* Theorem 1.3.4 states that there is a special barrier, $A(G)$, existing canonically in a graph. The Gallai-Edmonds structure theorem tells us the size and structures of all maximum matchings in a canonical way, so it is useful in numerous contexts and is, without a doubt, the most powerful theorem in matching theory.

The first polynomial time algorithm for the maximum matching problem is Edmonds' one [7], which has given rise to more efficient algorithms proposed since then. Actually, to compute maximum matchings by Edmonds' algorithm is equivalent to compute the Gallai-Edmonds decomposition. Sometimes the Gallai-Edmonds decomposition offers a clue to develop a new algorithm [30]. Hence, the

Gallai-Edmonds decomposition lies in the essential part of the maximum matching algorithms. It is also related to some algebraic problems [3, 13].

The Canonical Partition

The *canonical partition* is a decomposition for elementary graphs and plays a crucial role in matching theory. First Kotzig introduced the canonical partition as a quotient set of a certain equivalence relation [24–26], and later Lovász redefined it from the point of view of barriers [30]. In fact, these are equivalent. For an elementary graph G and $u, v \in V(G)$, we say $u \sim v$ if $u = v$ or $G - u - v$ is not factorizable.

Theorem 1.3.5 (Kotzig [24–26])**.** *Let G be an elementary graph. Then $\sim$ is an equivalence relation on $V(G)$.*

The family of equivalence classes of $\sim$ is called the *canonical partition* of G, and denoted by $\mathcal{P}(G)$.

Theorem 1.3.6 (Lovász [30])**.** *Let G be an elementary graph. Then, the family of maximal barriers forms a partition of $V(G)$. Additionally, this partition coincides with the equivalence classes by $\sim$.*

In the polyhedral study of matchings, the notion of elementary graphs appears essential, and Lovász reformulated the canonical partition of elementary graphs so as to obtain the structural results of the perfect matching polytope. Thanks to this reformulation by Lovász, many graph theoretic results such as the *two ear theorem* of ear-decompositions of elementary graphs, *brick decompositions*, and *tight cut decompositions* have been obtained [8, 29, 30] (see also the survey article [2]), and together with the canonical partition itself, they have underlain the studies of the perfect matching polytope.

The Dulmage-Mendelsohn Decomposition

Factor-components of a bipartite factorizable graph are known to have the following partially ordered structure[1]:

Theorem 1.3.7 (The Dulmage-Mendelsohn Decomposition [4–6,30,32])**.** *Let* $G = (A, B; E)$ *be a bipartite factorizable graph, and let* $\mathcal{G}(G) =: \{G_i\}_{i\in I}$*. Let* $A_i := A \cap V(G_i)$ *and* $B_i := B \cap V(G_i)$ *for each* $i \in I$*. Then, there exists a partial order* $\preceq_A$ *on* $\mathcal{G}(G)$ *such that for any* $i, j \in I$*,*

(i) $E[B_j, A_i] \neq \emptyset$ *yields* $G_j \preceq_A G_i$*, and*

(ii) *if* $G_j \preceq_A H \preceq_A G_i$ *yields* $G_i = H$ *or* $G_j = H$ *for any* $H \in \mathcal{G}(G)$*, then* $E[B_j, A_i] \neq \emptyset$*.*

We call this decomposition of G into a poset *the Dulmage-Mendelsohn decomposition* (in short, *the DM-decomposition*), and each element of $\mathcal{G}(G)$, in this context, a *DM-component*. The DM-decomposition is uniquely determined by a graph, up to the choice of roles of color classes. In this thesis, we call the DM-decomposition of $G = (A, B; E)$ as in Theorem 1.3.7 the DM-decomposition *with respect to* A.

Dulmage and Mendelsohn introduced the DM-decomposition with an application to an efficient solution of the linear equations determined by large sparse matrices [4–6]. Another notable history about the DM-decomposition is its contribution to the theory of submodular function, namely, a branch of it about the *principal partition* [10, 33].

1.4 Overview

This thesis consists of six chapters including this chapter of introduction and Chapter 6, devoted to conclusion. Our main results are described in Chapter 2 to Chapter 5. Although matching theory has been studied extensively for years, there are

[1] Though this is sometimes presented as a theorem for general bipartite graphs, we introduce it as one for bipartite factorizable graphs.

still some lacks in its foundation. This thesis contributes to the foundation of matching theory, from three aspects: canonical decompositions, barriers, the enumeration problem of perfect matchings. In the following we give an overview of this thesis.

Canonical decompositions are the most fundamental tools to investigate matchings, and as we saw in Section 1.3.2, to the best of our knowledge, exactly three canonical decompositions have been known, i.e., the Gallai-Edmonds decomposition, the canonical partition, the DM-decomposition. However, each of them are substantially applicable to a special class of graphs, respectively; the Gallai-Edmonds decomposition can be applicable non-trivially only for non-factorizable graphs, the canonical partition is for a special class of factorizable graphs, i.e., elementary graphs, and the DM-decomposition is for bipartite graphs. Hence, we have not had any ways to see non-trivial structure of general factorizable graphs. Additionally, there have not been known any viewpoints to understand the known canonical decompositions in a unified way.

The factorizable graphs form such a wide class that we cannot give up obtaining non-trivial canonical decomposition. In this thesis, we present a new canonical decomposition, which is applicable and tells non-trivial structures for general factorizable graphs. By combining this new canonical decomposition with the Gallai-Edmonds decomposition, we can easily formulate a structure theorem applicable to any graph and give a refinement of the Gallai-Edmonds decomposition. The new canonical decomposition also give a generalization of the canonical partition for general graphs; our generalization here is based on Kotzig's, among two formulation of the canonical partition in Section 1.3.2. It also expresses a relationship between the factor-components in a partial order. This partially ordered structure is actually not a generalization of the DM-decomposition. A generalization of the DM-decomposition is not directly contained in the new decomposition, nor do we give it in this thesis, however in our coming work [17] we show that a generalization of the DM-decomposition for general graphs can be obtained with the new decomposition. Hence, in this sense, our new decomposition enables us to unify the three known canonical decompositions. In Chapter 2, we are going to introduce this new canonical decomposition.

The notion of barriers corresponds to the dual optimal solutions of the maximum matching problem and can be regarded as a counterpart of maximum matchings. Duality is a notion that supports the heart of polyhedral combinatorics and accordingly is one of the essential notions in combinatorial optimization. Maximal barriers seems especially important; one reason is that they are more useful in purely combinatorial or graph-theoretic arguments, and another reason is that canonical decompositions have some relationships with maximal barriers (recall Theorem 1.3.6, or see Theorem 3.3.4) and this fact implies that maximal barriers have something intrinsic. However, compared to numerous results about maximum matchings, "much less is known about barriers [30]". Even about maximal barriers, not so much has been known. In Chapter 3, we give a theorem that describes the structure of maximal barriers in general graphs, using the new canonical decomposition we give in Chapter 2. (Actually, we work on a wider notion called *odd-maximal barriers*.) This theorem turns out to be a generalization of Theorem 1.3.6, which indicates that our generalization of the canonical partition in Chapter 2 is reasonable considering both formulations of the canonical partition, by Kotzig and by Lovász.

Another important direction of the study of matchings is counting the number of perfect matchings [30]. Enumerating the perfect matchings of a given factorizable graph is one of the most fundamental problems in enumerative combinatorics and has been studied by various approaches. The *cathedral theorem* is a characterization of the *saturated graphs* and has been useful in investigating relations between the number of perfect matchings and some graph parameters such as degrees. The cathedral theorem was first given by Lovász in 1972 [27], and later Szigeti gave another proof in 2001 [36]. In Chapter 4, we give yet another proof by considering the saturated graphs with the new canonical decomposition. In this new proof, we reveal the intrinsic structure exists behind the cathedral theorem and show it in quite a natural way providing more refined properties as by-products.

We also give several polynomial-time algorithms to compute the new canonical decomposition. One algorithm is given in Chapter 2, and a bit more sophisticated algorithm is presented in Chapter 5 with the results in Chapter 3.

Chapter 2

Canonical Structures of Factorizable Graphs

2.1 Introduction

In this chapter, we show new results which give a new canonical decomposition. When we want to know structures or to develop algorithms on graphs, it is quite natural to decompose graphs appropriately into substructures so that we can consider the problems using this decomposition. There is a kind of decompositions that serve as fundamental tools to consider matchings, called *canonical decompositions* in general. In matching theory, we say something is *canonical* if it is a concept determined uniquely to a given graph. Canonical decompositions decompose a given graph in some way uniquely determined to the graph and tell us structures of all the maximum matchings at once.

As we mentioned in Chapter 1, to the best of our knowledge, exactly three theorems are known as those give canonical decompositions: the Gallai-Edmonds decomposition, the canonical partition, and the DM-decomposition [30]. They are useful tools that support the basis of matching theory. However each of them is substantially applicable only to a special class of graphs, respectively; the first one, the Gallai-Edmonds decomposition are substantially for non-factorizable graphs, the second one, the canonical partition is for elementary graphs, and the third one, the DM-decomposition is for bipartite graphs. Hence there are some classes of graphs do not have canonical decompositions to analyze them; the general factorizable graphs are such even though they form such a wide and fundamental class that we cannot give up investigating them. Moreover, we do not know any ways to see relations between the three canonical decompositions. In this chapter, we are going to show theorems that give a new canonical decomposition, which is applicable substantially and tells non-trivial structures for general factorizable graphs. This new canonical decomposition is given mainly by theorems which show

- a partially ordered structure on the factor-components,
- a generalization of the canonical partition, and
- a relationship between the above two notions.

The Gallai-Edmonds decomposition is by its definition applicable to all graphs, however the decompositions it gives are rather sparse and there are some classes of graphs that it treats as irreducible and does not give non-trivial decompositions; the factorizable graphs are such. If a graph is elementary factorizable, then we can analyze its matching-theoretic properties by the canonical partition. Otherwise, that is, if a graph is non-elementary factorizable, of course we can apply the canonical partition for each of its factor-components; however, this approach does not give enough combinatorial information in general because it fails to considering the overall structure of the graph. Here we generalize the canonical partition for general factorizable graphs including non-elementary graphs, not in a non-trivial way but considering the overall structure of given graphs. As we saw in Section 1.3.2, there are two equivalent formulation of the canonical partition by Kotzig [24–26] and Lovász [27, 30], respectively. In this chapter, we give a generalization based on Kotzig's formulation.

We should also investigate how the factor-components are related in a factorizable graphs. A factorizable graph consists of factor-components and some edges joining between them, so elementary graphs are fundamental building blocks of a factorizable graph. Conversely, suppose we are given some elementary graphs, say, $G_1, \ldots, G_k$, where $k > 0$, and create a new graph, say, G by joining these elementary graphs with edges in some way. The graph G will always be factorizable, but will the factor-components of G be $G_1, \ldots, G_k$? The answer is no. Therefore, there must be a certain non-trivial structure about relationship between the factor-components in a factorizable graph. In fact, the DM-decomposition reveals a relationship between the factor-components of bipartite factorizable graphs, stating that they form a poset. However, as for non-bipartite factorizable graphs, no result has been known. In this chapter, we reveal a partially ordered structure between factor-components of general factorizable graphs. It has some similar natures to the DM-decomposition, however they are distinct.

Actually, there is a relationship between the partial order on the factor-components and the generalization of the canonical partition. This relationship unites the two notions and so give rise to a new canonical decomposition; we will name it the *generalized cathedral decomposition* or just the *cathedral decomposition*, after the *cathedral theorem* for *saturated graphs*, given by Lovász [28, 30].

By combining the results here with the Gallai-Edmonds structure theorem, we can formulate the new canonical decomposition as one applicable to general graphs including those non-factorizable, which gives a refinement of the Gallai-Edmonds decomposition.

The results in this chapter are also found in papers by the author [19, 20]. The rest of this section is to explain the succeeding sections in this chapter. In Section 2.2, we give some basic definition and properties. In Section 2.3, we reveal a canonical partially ordered structure on factor-components of factorizable graphs. In Section 2.4, we give a generalization of the canonical partition based on Kotzig's way. In Section 2.5, we show some additional properties regarding the results in Sections 2.3 and 2.4, including a relationship between the partial order and the generalization of the canonical partition. In Section 2.6, we show some example and figures about results in previous sections. In Section 2.7, we show that the new canonical decomposition can be computed in polynomial time.

2.2 Preliminary Facts

2.2.1 Factor-critical Graphs

Here we show some basic facts about factor-critical graphs. Some of them are easy to see and might be regarded as folklores.

Property 2.2.1. *Let M be a near-perfect matching of a graph G that exposes $v \in V(G)$. Then, G is factor-critical if and only if for any $u \in V(G)$ there exists an M-balanced path from u to v.*

Proof. Take $u \in V(G)$ arbitrarily. Since G is factor-critical, there is a near-perfect matching M' of G exposing only u. Then, $G.M\triangle M'$ contains an M-balanced path from u to v, and the sufficiency part follows.

Now suppose there is an M-balanced path P from u to v. Then, $M\triangle E(P)$ is a near-perfect matching of G exposing u. Hence, the necessity part follows. □

Property 2.2.2. *Let G be a graph. Then G is factor-critical if and only if each block of G is factor-critical.*

Proposition 2.2.3 (implicitly stated in [27])**.** *Let G be a factor-critical graph, $v \in V(G)$, and M be a near-perfect matching that exposes v. Then for any non-loop edge $e = vu \in E(G)$, there is a nice circuit C of G which is an M-ear relative to v and contains e.*

Theorem 2.2.4 (implicitly stated in [27])**.** *Let G be a factor-critical graph. For any nice factor-critical subgraph G' of G, G/G' is factor-critical.*

An *ear-decomposition* of a graph G is a sequence of subgraphs $G_0 \subseteq \cdots \subseteq G_k = G$ such that $G_0 = (\{r\}, \emptyset)$ for some $r \in V(G)$ and for each $i \geq 1$, G_i is obtained from G_{i-1} by adding an ear P_i relative to G_{i-1}. We sometimes regard an ear-decomposition as a family of ears $\mathcal{P} = \{P_1, \ldots, P_k\}$. An ear-decomposition is called *odd* if any of its ears has an odd number of edges.

Theorem 2.2.5 (Lovász [27])**.** *A graph is factor-critical if and only if it has an odd ear-decomposition.*

For a factor-critical graph G and a near-perfect matching M of G, an ear-decomposition of G is *alternating with respect to* M or just M*-alternating* if each ear is an M-ear.

Proposition 2.2.6 (Lovász [27]). *Let G be a factor-critical graph. Then for any near-perfect matching M of G, there is an M-alternating ear-decomposition of G.*

2.2.2 Other Facts

The following are fundamental and basic facts about matchings. These are easy to see. We use the these facts frequently all over in this thesis, sometimes without explicitly mentioning it. Readers familiar with matching theory might skip this section.

Property 2.2.7. *Let G be a factorizable graph, M be a perfect matching of G, and $e = xy \in E(G)$ be such that $e \notin M$. The following three properties are equivalent:*

(i) *The edge e is allowed in G.*

(ii) *There is an M-alternating circuit C such that $e \in E(C)$.*

(iii) *There is an M-saturated path between x and y.*

Proof. We first show that (i) and (ii) are equivalent. Let M' be a perfect matching of G such that $e \in M'$. Then, $G.M \triangle M'$ has a connected component which is an M-alternating circuit containing e. Hence, (i) yields (ii).

Now let $L := M \triangle E(C)$. Then, L is a perfect matching of G such that $e \in L$. Hence, (ii) yields (i); consequently, they are equivalent.

Since (ii) and (iii) are obviously equivalent, now we are done. □

Property 2.2.8. *Let G be a factorizable graph and M be a perfect matching of G, and let $u, v \in V(G)$. Then, $G - u - v$ is factorizable if and only if there is an M-saturated path of G between u and v.*

Proof. For the sufficiency part, let M' be a perfect matching of $G - u - v$. Then, $G.M \triangle M'$ has a connected component which is an M-saturated path between u

and v. For the necessity part, let P be an M-saturated path between u and v. Then, $M \triangle E(P)$ is a perfect matching of $G - u - v$, and we are done. □

Property 2.2.9. *Let G be a factorizable graph and M be a perfect matching of G, and let $u, v \in V(G)$. If there is an M-alternating circuit C with $u, v \in V(C)$, then all edges of $E(C)$ are allowed and therefore u and v are contained in the same factor-component of G.*

Proof. The set of edges $M \triangle E(C)$ is also a perfect matching of G. Therefore, each edge of $E(C)$ is allowed, and so u and v are connected by a path whose edges are all allowed. Hence, u and v belong to the same factor-component. □

Property 2.2.10. *Let G be a graph and M be a matching of G. Let $X \subseteq V(G)$ be such that M_X is a perfect matching of $G[X]$. Let P be an M-balanced path or an M-saturated path, one of its end vertices, say u, is covered by M. Trace P from u and let v be the first vertex we encounter that is in X. Then, the subpath uPv is an M-balanced path from u to v.*

Property 2.2.11. *Let G be a graph, M be a matching of G, and $X \subseteq V(G)$ be such that M_X is a perfect matching of $G[X]$. Let P be an M-exposed path with both end vertices in X or an M-alternating circuit with some vertices in X. Each connected component of $P - E(G[X])$ is an M-ear relative to X.*

2.3 A Partially Ordered Structure in Factorizable Graphs

In this section we show a relationship between factor-components of a given factorizable graph. We shall define a canonical binary relation on the factor-components, which captures matching-theoretic properties well, and show that in fact it is a partial order (Theorem 2.3.29).

Let G be a factorizable graph. For $X \subseteq V(G)$ we call X a *separating set* if for any $H \in \mathcal{G}(G)$, $V(H) \subseteq X$ or $V(H) \cap X = \emptyset$. The next property is easy to see by the definition.

Property 2.3.1. *Let G be a factorizable graph, and $X \subseteq V(G)$ be such that $X \neq \emptyset$. The following properties are equivalent:*

(i) *The set X is separating.*

(ii) *There exist $H_1, \ldots, H_k \in \mathcal{G}(G)$, where $k \geq 1$, such that X is a disjoint union of $V(H_1), \ldots, V(H_k)$.*

(iii) *For any perfect matching M of G, $\delta(X) \cap M = \emptyset$.*

(iv) *For any perfect matching M of G, M_X is a perfect matching of $G[X]$.*

Let $G_1, G_2 \in \mathcal{G}(G)$. We say a separating set X is a *critical-inducing set for* G_1 if $V(G_1) \subseteq X$ and $G[X]/G_1$ is a factor-critical graph. Moreover, we say X is a *critical-inducing set for* G_1 *to* G_2 if $V(G_1) \cup V(G_2) \subseteq X$ and $G[X]/G_1$ is a factor-critical graph.

Definition 2.3.2. Let G be a factorizable graph, and $G_1, G_2 \in \mathcal{G}(G)$. We say $G_1 \lhd G_2$ if there is a critical-inducing set for G_1 to G_2.

Lemma 2.3.3. *Let G be a factorizable graph and M be a perfect matching of G, and let $X \subseteq V(G)$ and $G_1 \in \mathcal{G}(G)$. Then, X is a critical-inducing set for G_1 if and only if for any $x \in X \setminus V(G_1)$ there exists $y \in V(G_1)$ such that there is an M-balanced path from x to y whose vertices except y are in $X \setminus V(G_1)$.*

Proof. The claim is rather easy from Property 2.2.1. The set X is a critical-inducing set for G_1 if and only if $G[X]/G_1$ is factor-critical. Note that $M_{X \setminus V(G_1)}$ forms a near-perfect matching of $G[X]/G_1$. Therefore, $G[X]/G_1$ is factor-critical if and only if for any $x \in X$ there is an M-balanced path from x to the contracted vertex g_1 corresponding to G_1. Therefore, the claim follows. $\square$

Proposition 2.3.4. *Let G be an elementary graph and M be a perfect matching of G. Then for any two vertices $u, v \in V(G)$ there is an M-saturated path between u and v, or an M-balanced path from u to v.*

Proof. Let $u \in V(G)$ be an arbitrary vertex. Let $U \subseteq V(G)$ be the set of vertices that can be reached from u by M-saturated or M-balanced paths. We are going to obtain the claim by showing $U = V(G)$. Suppose that it fails, namely that $U \subsetneq V(G)$.

Claim 2.3.5. *Let $v \in U$, and let P be an M-saturated path between u and v or an M-balanced path from u to v. Then $V(P) \subseteq U$.*

Proof. Let $w \in V(P)$. Then, uPw is an M-saturated path from u and w or an M-balanced path from u to w. Therefore, $w \in U$. Hence we have $V(P) \subseteq U$. □

Since G is connected, G has some edges joining U and $V(G) \setminus U$.

Claim 2.3.6. *Let $v \in U \cap N(V(G) \setminus U)$. Then, there is no M-saturated path between u and v.*

Proof. Suppose the claim fails, and let P be an M-saturated path between u and $v \in U \cap N(V(G) \setminus U)$. By Claim 2.3.5, $V(P) \subseteq U$. Therefore, the vertex v' is in U, and by letting $w \in V(G) \setminus U$ be a vertex with $vw \in E(G)$ we have $vw \notin M$. Hence, $P + vw$ is an M-balanced path from u to w; this contradicts $w \notin U$. Hence we have this claim. □

Claim 2.3.7. *No edge joining U and $V(G) \setminus U$ is in M.*

Proof. Let $vw \in E[U, V(G) \setminus U]$ be an edge with $v \in U$ and $w \in V(G) \setminus U$. By Claim 2.3.5 and Claim 2.3.6, there is an M-balanced path P from u to v with $V(P) \subseteq U$. Hence, if $vw \in M$ then $P + vw$ is an M-saturated path between u and w, and this contradicts $w \notin U$. Therefore, $vw \notin M$, and we have this claim. □

Since G is elementary, of course some edges in $E[U, V(G) \setminus U]$ are allowed; let $e = vw$ be one of them. By Claim 2.3.7, $e \notin M$ holds, and so by Property 2.2.7, there is an M-saturated path Q between v and w. Trace P from u and let x be the first vertex we encounter that is in Q; such x surely exists under the current hypotheses since $v \in V(P) \cap V(Q)$. Note that by this definition of x, $uPx + xQ\alpha$ forms a path for each $\alpha \in \{v, w\}$.

Claim 2.3.8. *The path uPx is M-balanced from u to x.*

Proof. Suppose the claim fails, that is, uPx is an M-saturated path. Then, we have $x' \in V(uPx)$; however, at the same time, we have $x' \in V(Q)$, since $x \in V(Q)$ holds and Q is an M-saturated path. This contradicts the definition of x, and we have this claim. □

Note also that for α which equals either v or w, the subpath of Q between x and α is an M-saturated path. Hence, with Claim 2.3.8, for this α, it follows that $uPx + xQ\alpha$ is an M-saturated path between u and α. Thus $w \in U$ holds, a contradiction. Now we are done for this proposition. □

Let G be a factorizable graph and M be a perfect matching of G. We call a sequence of factor-components $S := (H_0, \ldots, H_k)$, where $k \geq 0$, an *M-ear sequence, from H_0 to H_k* if S satisfies the following three properties:

(i) for each $i = 0, \ldots, k$, $H_i \in \mathcal{G}(G)$,

(ii) for any $i, j \in \{0, \ldots, k\}$, $i \neq j$ yields $H_i \neq H_j$, and

(iii) if $k \geq 1$, then for each $i = 1, \ldots, k$ there is an M-ear P_i relative to H_{i-1} and through H_i.

We call k the *length* of S. The *distance* from H_0 to H_k is the length of the shortest M-ear sequence from H_0 to H_k. If $k \geq 1$, we call the sequence of M-ears $P := (P_1, \ldots, P_k)$ *associated with* S. If $k = 0$, an empty sequence, $P := ()$, is defined to be the M-ears associated with S, for convenience.

For S and P, we define the *sequence union* of S and P as $V(S;P) := \bigcup_{i=1}^{k} V(H_i) \cup \bigcup_{i=1}^{k} V(P_i) \setminus V(H_0)$, if $k \geq 1$. If $k = 0$, $V(S;P) := \emptyset$.

Proposition 2.3.9. *Let G be a factorizable graph and M be a perfect matching of G. Let $X \subseteq V(G)$, and let $H \in \mathcal{G}(G)$ be such that $V(H)$ is disjoint from X and there is an M-ear P relative to X and through H, whose end vertices are $u, v \in X$. Let $Y := V(H) \cup V(P) \setminus \{u, v\}$. Then, for any $x \in Y$,*

(i) *there exists an internal vertex y of P such that there is an M-balanced path Q from x to y with $V(Q) \subseteq Y$ and $V(Q) \cap V(P) = \{y\}$, and*

(ii) *for a vertex w which is identical to either u or v, $Q + yPw$ is an M-balanced path from x to w, whose vertices except w are contained in Y.*

Proof. If $x \in V(P) \setminus \{u, v\}$, the claims are obvious with the trivial path given by x. Let $x \in V(H) \setminus V(P)$. Then, by Proposition 2.3.4, for an arbitrarily chosen $z \in V(P) \cap V(H)$, there is a path R which is M-saturated between x and z or M-balanced from x to z, with $V(R) \subseteq V(H)$. Trace R from x and let y be the first vertex we encounter that is in $V(P)$. Then, xRy, which is an M-balanced path by Property 2.2.10, gives a desired path for (i), and the path $Q := xRy + yPw$, where w is either u or v, gives one for (ii). Therefore, we are done. □

Lemma 2.3.10. *Let G be a factorizable graph and M be a perfect matching of G. Let $G_1 \in \mathcal{G}(G)$ and $X \subseteq V(G)$ be a critical-inducing set for G_1. Suppose there exists an M-ear P relative to X, whose end vertices are $u, v \in V(G)$, and let $I_1, \ldots, I_s \in \mathcal{G}(G)$, where $s \geq 1$, be the factor-components that have common vertices with the internal vertices of P. Then, $X \cup \bigcup_{i=1}^{s} V(I_i)$ is also a critical-inducing set for G_1.*

Proof. We prove the claim by Lemma 2.3.3; let $Y := \bigcup_{i=1}^{s} V(I_i)$. By Lemma 2.3.3,

Claim 2.3.11. *for any $x \in X$ there exists $z \in V(G_1)$ such that there is an M-balanced path Q_x from x to z with $V(Q_x) \subseteq X$ and $V(Q_x) \cap V(G_1) = \{z\}$.*

Claim 2.3.12. *For any $y \in Y$ there exists $z \in V(G_1)$ such that there exists an M-balanced path Q_y from y to z with $V(Q_y) \subseteq X \cup Y$ and $V(Q_y) \cap V(G_1) = \{z\}$.*

Proof. Let $i \in \{1, \ldots, s\}$ be such that $y \in V(I_i)$. By applying Proposition 2.3.9 to X, I_i and P, for w which equals either u or v, there is an M-balanced path R from y to w such that $V(R) \setminus \{w\} \subseteq Y$. Therefore, $R + Q_w$ gives a desired path, namely, Q_y, where Q_w denotes an M-balanced path in Claim 2.3.11. □

Apparently by the definition $X \cup Y$ is a separating set, therefore with Claims 2.3.11 and 2.3.12 we can conclude that $X \cup Y$ is a critical-inducing set for G_1, by Lemma 2.3.3. □

Theorem 2.3.13. *Let G be a factorizable graph, M be a perfect matching of G, and let $G_1, G_2 \in \mathcal{G}(G)$. Then, $G_1 \triangleleft G_2$ if and only if there exists an M-ear sequence from G_1 to G_2.*

Proof. We first prove the sufficiency part of the theorem. Let $G_1 \triangleleft G_2$, and let $X \subseteq V(G)$ be a critical-inducing set for G_1 to G_2. Let us define the following three properties for $Y \subseteq X$:

C1(Y): The set Y is a critical-inducing set for G_1.

C2(Y): For each $H \in \mathcal{G}(G)$ with $V(H) \subseteq Y$, there is an M-ear sequence from G_1 to H.

Let X' be a maximal subset of X satisfying C1 and C2. Note that $X' \neq \emptyset$ holds because $V(G_1)$ satisfies C1 and C2. We are going to prove the sufficiency part of the theorem by showing $X' = X$. Suppose it fails, that is, $X' \subsetneq X$ holds. Then,

Claim 2.3.14. *there is an M-ear P relative to X' such that $V(P) \subseteq X$.*

Proof. The graph $G[X]/G_1$ is factor-critical, and $G[X']/G_1$ is a nice factor-critical subgraph of $G[X]/G_1$ by Property 2.3.1. Therefore, $G[X]/X'$ is factor-critical by Theorem 2.2.4, and $M_{X \setminus X'}$ forms a near-perfect matching of $G[X]/X'$ exposing only the contracted vertex x' corresponding to X'. By Proposition 2.2.3, in $G[X]/X'$ there is an M-ear P relative to x', and in G it corresponds to an M-ear relative to X' with $V(P) \subseteq X$. Thus, the claim follows. □

Let $u, v \in X'$ be the end vertices of P. Let $I_1, \dots, I_s \in \mathcal{G}(G)$ be the factor-components that have common vertices with internal vertices of P. We are going to prove that $X'' := X' \cup \bigcup_{i=1}^{s} V(I_i)$ satisfies C1 and C2.

Claim 2.3.15. *The set X'' satisfies C2.*

Proof. By Lemma 2.3.3, there exists an M-balanced path Q_u (resp. Q_v) from u (resp. v) to a vertex of $V(G_1)$, which is contained in X and whose vertices except the end vertex in $V(G_1)$ are disjoint from $V(G_1)$. Trace Q_u from u, and let r_u be the first vertex we encounter that is contained in a factor-component I_0 which has common vertices also with Q_v; such I_0 surely exists since both Q_u and Q_v have some vertices in G_1. Trace Q_v from v and let r_v be the first vertex we encounter that is in $V(I_0)$. For each $w \in \{u, v\}$, Property 2.2.10 yields that wQ_wr_w is an M-balanced path from w to r_w with $V(wQ_wr_w) \subseteq X'$ and $V(wQ_wr_w) \cap V(I_0) = \{r_w\}$, and it holds that $V(uQ_ur_u) \cap V(vQ_vr_v) \setminus \{r_u, r_v\} = \emptyset$. Therefore, $uQ_ur_u + P + vQ_vr_v$ is an M-ear relative to I_0 and through every $I_1, \ldots, I_s$. By the definition of X', there is an M-ear sequence from G_1 to I_0. Therefore, by adding subsequence (I_0, I_i) to it, we obtain an M-ear sequence from G_1 to I_i, for each $i = 1, \ldots, s$. Thus, we obtain the claim. □

Claim 2.3.16. *The set X'' satisfies C1.*

Proof. This is immediate by Lemma 2.3.10. □

With Claims 2.3.15 and 2.3.16, the set X'' contradicts the maximality of X'. Therefore, we obtain $X' = X$, accordingly the sufficiency part of the theorem follows.

From now on we prove the necessity. Let $(G_1 = H_0, \ldots, H_k = G_2)$, where $k \geq 0$, be the M-ear sequence from G_1 to G_2. We are going to prove that there is a critical-inducing set for G_1 to G_2. We proceed by induction on k. For the case $k = 0$, that is, $G_1 = G_2$, the statement apparently holds by taking $V(G_1)$.

Let $k > 0$, and suppose the statement holds for $k - 1$. Consider the M-ear subsequence $(H_0, \ldots, H_{k-1})$; by the induction hypothesis, there is a critical-inducing set X' for H_0 to H_{k-1}. If $V(H_k) \subseteq X'$ holds, then X' is a critical-inducing set of G_1 to H_k and the statement holds. Hence hereafter we consider the case of $V(H_k) \not\subseteq X'$; since X' is separating, this means $V(H_k) \cap X' = \emptyset$. Thus, by letting P_k be the associated M-ear relative to H_{k-1} and through H_k, we have $V(P_k) \setminus X' \neq \emptyset$ and the graph $P_k - E(G[X'])$ is not empty. By Property 2.2.11, each connected component $P_k - E(G[X'])$ is an M-ear relative to X', and one of them, which we call $\tilde{P}_k$, is through H_k. Therefore,

there is an M-ear $\tilde{P}_k$ relative to X' and through H_k.

Let $I_1, \ldots, I_s \in \mathcal{G}(G)$, where $s \geq 1$, be the factor-components that have common vertices with the internal vertices of $\tilde{P}_k$, and let $Y := \bigcup_{i=1}^{s} V(I_i)$. Then, by applying Lemma 2.3.10 to the critical-inducing set X' for G_1 and the M-ear $\tilde{P}_k$, we obtain that $X' \cup Y$ is a critical-inducing set for G_1 to H_k. This completes the proof. □

Let G be a factorizable graph and M be a perfect matching of G. Let $G_1, G_2 \in \mathcal{G}(G)$, and let $S := (G_1 = H_0, \ldots, H_k = G_2)$, where $k \geq 0$, be an M-ear sequence from G_1 to G_2, associated with M-ears P. For any i, j with $0 \leq i \leq j \leq k$, the subsequence $(H_i, \ldots, H_j)$ is an M-ear sequence, from H_i to H_j, and we denote it as $S[i, j]$. Additionally, if $i < j$, $(P_{i+1}, \ldots, P_j)$ is a sequence of M-ears associated with $S[i, j]$, and we denote it $P[i, j]$. If $i = j$, then the empty sequence is the one associated with $S[i, j]$, and it is also denoted as $P[i, j]$. We denote $S[0, j] =: S^j$, and $P[0, j] =: P^j$. Let us define in the following three properties for S and P:

D1(S, P)**:** If $k \geq 2$, then by letting $P = (P_1, \ldots, P_k)$, for each $i = 2, \ldots, k$, $V(P_i)$ is disjoint from $V(H_0)$.

D2(S, P)**:** If $k \geq 1$, by letting $P = (P_1, \ldots, P_k)$, for each $i = 1, \ldots, k$, for any $x \in V(S^i; P^i)$ there exists an internal vertex y of P_1 such that there is an M-balanced path Q from x to y with $V(Q) \subseteq V(S^i; P^i)$ and $V(Q) \cap V(P_1) = \{y\}$.

D3(S, P)**:** If $k \geq 1$, by letting $P = (P_1, \ldots, P_k)$, for each $i = 1, \ldots, k$, for any $x \in V(S^i; P^i)$, for w which equals either of the end vertices of P_1, there is an M-balanced path R from x to w such that $V(R) \setminus \{w\} \subseteq V(S^i; P^i)$.

Remark 2.3.17. By their definitions, if $k = 0$, then S and P trivially satisfy D1, D2 and D3.

Remark 2.3.18. If $k = 1$, then S and P satisfy D1 trivially and also $D2$ and $D3$ by Proposition 2.3.9.

Remark 2.3.19. D1, D2 and D3 are closed with respect to the substructures; if S and P satisfy D1, D2 and D3, then for any $i = 0, \dots, k$, so does S^i and P^i.

Proposition 2.3.20. *Let G be a factorizable graph and M be a perfect matching of G. Let S be an M-ear sequence, and P be a sequence of M-ears associated with S. Then, $M_{V(S;P)}$ is a perfect matching of $G[V(S;P)]$.*

Proof. If the length k of S equals zero, the claim is trivially true. Let $k \geq 1$, and let $S =: (H_0, \dots, H_k)$ and $P =: (P_1, \dots, P_k)$. Of course, $X := V(H_0) \dot\cup \cdots \dot\cup V(H_k)$ has a perfect matching M_X. For each P_i, the end vertices of P_i are in X and any other vertex is covered by M_{P_i}. Therefore, M contains a perfect matching of $Y := X \cup V(P_1) \cup \cdots \cup V(P_k)$. Accordingly, $V(S;P) = Y \setminus V(H_0)$ is covered by $M_{V(S;P)}$. □

Lemma 2.3.21. *Let G be a factorizable graph, and M be a perfect matching. Let $S := (H_0, \dots, H_k)$, where $k \geq 1$, be an M-ear sequence, associated with M-ears $P := (P_1, \dots, P_k)$. Suppose S^i and P^i satisfy D1, D2, and D3 for each $i = 0, \dots, k-1$, $V(H_k)$ is disjoint from $V(S^{k-1}; P^{k-1})$, and S and P satisfy D1. Then, S and P also satisfy D2 and D3.*

Proof. If $k = 1$, then by applying Proposition 2.3.9 to $V(H_0)$, P_1, and H_1, it holds that S and P satisfy D1, D2 and D3.

Hence hereafter let $k \geq 2$. First note that each connected component of $P_k - E(G[V(S^{k-1}; P^{k-1})])$ is an M-ear relative to $V(S^{k-1}; P^{k-1})$ by Property 2.2.11, and is disjoint from $V(H_0)$ since P_k is.

Take $x \in V(S;P) \setminus V(S^{k-1}; P^{k-1})$ arbitrarily, and let P_k^x be a connected component of $P_k - E(G[V(S^{k-1}; P^{k-1})])$ such that x is an internal vertex of P_k^x if $x \in V(P)$, or one through H_k if $x \in V(H_k) \setminus V(P)$.

Claim 2.3.22. *There exists $y \in V(S^{k-1}; P^{k-1})$ such that there exists an M-balanced path Q from x to y whose vertices except y are contained in $V(S;P) \setminus V(S^{k-1}; P^{k-1})$.*

Proof. By applying Proposition 2.3.9 to $V(S^{k-1}; P^{k-1})$, P_k^x, and H_k (if $x \in V(H_k)$), we obtain an internal vertex y of P_1 and an M-balanced path Q from

x to y with $V(Q) \setminus \{y\} \subseteq V(H_k) \cup V(P_k^x) \setminus V(S^{k-1}; P^{k-1})$. Since P_k is disjoint from $V(H_0)$, we can see $V(H_k) \cup V(P_k^x) \subseteq V(S; P)$. Therefore, $V(Q) \setminus \{y\} \subseteq V(S; P) \setminus V(S^{k-1}; P^{k-1})$, and the claim follows. □

Claim 2.3.23. *S and P satisfy D2.*

Proof. By the hypothesis on S^{k-1} and P^{k-1} there exists an internal vertex z of P_1 such that there is an M-balanced path R from y to z with $V(R) \subseteq V(S^{k-1}; P^{k-1})$ and $V(R) \cap V(P_1) = \{z\}$. Therefore, by Claim 2.3.22, $Q + R$ is an M-balanced path from x to z, whose vertices are contained in $V(S; P)$ and disjoint from P_1 except z.

Since x is chosen arbitrarily from $V(S; P) \setminus V(S^{k-1}; P^{k-1})$, we obtain that S and P satisfy D2. □

By similar arguments, we can say that S and P satisfy D3 too, and the claim follows. □

Proposition 2.3.24. *Let G be a factorizable graph and M be a perfect matching of G. Let $G_1, G_2 \in \mathcal{G}(G)$ be such that $G_1 \triangleleft G_2$. Then, there exists an M-ear sequence S from G_1 to G_2 of shortest length and M-ears P associated with S such that D1(S, P), D2(S, P), and D3(S,P) hold.*

Proof. We proceed by induction on the distance k from G_1 to G_2. If $k \in \{0, 1\}$, then the statement apparently follows; see Remark 2.3.17 and Remark 2.3.18.

Let $k \geq 2$, and suppose the statement is true for any two factor-components with distance at most $k - 1$. Take arbitrarily an M-ear sequence $S = (G_1 = H_0, \ldots, H_k = G_2)$ from G_1 to G_2 of shortest length, and M-ears $P = (P_1, \ldots, P_k)$ associated with it. Let u_1 and v_1 be the end vertices of P_1.

Claim 2.3.25. *Without loss of generality we can assume that S and P are chosen so that S^{k-1} and P^{k-1} satisfy D1, D2, and D3.*

Proof. Since S^{k-1} and P^{k-1} give an M-ear sequence from H_0 to H_{k-1} of length $k-1$, the distance from H_0 to H_{k-1} is at most $k-1$. Therefore, by the induction hypothesis, there is an M-ear sequence S' from H_0 to H_{k-1}, associated with M-ears P', whose length are at most $k-1$, with D1, D2, D3 satisfied. By adding H_k and P_k at the ends of S' and P', we obtain an M-ear sequence from G_1 to G_2 of length at most k (in fact exactly k) and associated M-ears satisfying the claim. □

Claim 2.3.26. *The vertices of H_k are disjoint from $V(S^{k-1}; P^{k-1})$.*

Proof. Suppose the claim fails, in other words, $V(H_k) \cap V(S^{k-1}; P^{k-1}) \neq \emptyset$ holds. Since H_k is distinct from any factor-component of S, there exists $i \in \{1, \ldots, k-1\}$ such that P_i has some internal vertices in $V(H_k)$; in other words, P_i is an M-ear relative to H_{i-1} and through H_k. Then, $(H_0, \ldots, H_{i-1}, H_k)$ gives an M-ear sequence from G_1 to G_2, associated with M-ears $(P_1, \ldots, P_i)$, whose length is i, which is less than k; this contradicts the minimality of k, and we are done for this claim. □

Since P_k is an M-ear relative to H_{k-1} and through H_k, Claim 2.3.26 yields $V(P_k) \not\subseteq V(S^{k-1}; P^{k-1})$ and accordingly together with Property 2.2.9 it follows that connected components of $P_k - E(G[V(S^{k-1}; P^{k-1})])$ are M-ears relative to $V(S^{k-1}; P^{k-1})$, one of which, say, Q is through H_k.

Claim 2.3.27. *The M-ear Q is disjoint from $V(H_0)$.*

Proof. Take $x \in V(Q) \cap V(H_k)$ arbitrarily, and let u and v be the end vertices of Q. Trace xQu from x, and let y be the first vertex we encounter that is in $V(H_0) \cup \{u\}$. On the other hand, trace xQv from x and let z be the first vertex we encounter that is in $V(H_0) \cap \{v\}$. Then, yQz is an M-exposed path, whose internal vertices contains $x \in V(H_k)$, and whose vertices except the end vertices y and z are disjoint from $V(H_0) \cup V(S^{k-1}; P^{k-1})$. We are going to prove this claim by showing $y = u$ and $z = v$. First suppose the case where $y, z \in V(H_0)$. Then, yQz is an M-ear relative to H_0 and through H_k, which means (H_0, H_k) forms an M-ear sequence of length one, contradicting the definition of k, since $k \geq 2$.

Second suppose the case where $y \in V(H_0)$ and $z = v$. Since S^{k-1} and P^{k-1} satisfy D3, it follows that for either $w \in \{u_1, v_1\}$ there is an M-balanced path R

from z to w with $V(R) \setminus \{w\} \subseteq V(S^{k-1}; P^{k-1})$. Therefore, $yQz + R$ is an M-ear relative to H_0 and through H_k, again letting (H_0, H_k) be an M-ear sequence, a contradiction.

In the remaining case, where $y = u$ and $z \in V(H_0)$, by symmetric arguments we are again lead to a contradiction. Therefore, we obtain $y = u$ and $z = v$, which is equivalent to Q being disjoint from $V(H_0)$. $\square$

Since S^{k-1} and P^{k-1} satisfy D2, for each $\alpha \in \{u, v\}$ there is an M-balanced path Q_α from α to r_α, where r_α is a internal vertex of P_1, satisfying $V(Q_\alpha) \subseteq V(S^{k-1}; P^{k-1})$. Trace Q_u from u, and let s be the first vertex we encounter that is contained in a factor-component $I \in \{H_1, \ldots, H_{k-1}\}$, which has common vertices also with $V(Q_v)$; such I surely exists since both Q_u and Q_v have vertices in H_1. Trace Q_u (resp. Q_v) from u (resp. v), and let s (resp. t) be the first vertex we encounter that is in $V(I)$.

Claim 2.3.28. *Let $\tilde{Q} := uQ_us + Q + vQ_zt$. Then $\tilde{Q}$ is an M-ear relative to I and through H_k, which is disjoint from H_0; accordingly, $I = H_{k-1}$.*

Proof. By Property 2.2.10, the path uQ_us and vQ_zt are M-balanced from u to s and from v to t, respectively, both of whose vertices are in $V(S^{k-1}; P^{k-1})$, satisfying $(V(uQ_us) \cap V(vQ_zt)) \setminus \{s, t\} = \emptyset$. Therefore, $\tilde{Q}$ is an M-ear relative to I and through H_k. Since Claim 2.3.27 says Q is disjoint from H_0, we also have that $\tilde{Q}$ is disjoint from H_0.

If $k = 2$, then trivially we have $I = H_{k-1}$. Consider the case of $k > 2$, and suppose I equals H_i for some $i \in \{1, \ldots, k-2\}$. Then, $(H_0, \ldots, H_i, H_k)$ forms an M-ear sequence from G_1 to G_2, associated with M-ears $(P_1, \ldots, P_{k-2}, \tilde{Q})$, whose length is at most $k - 1$, a contradiction. Hence we have $I = H_{k-1}$. $\square$

By Claim 2.3.28, we obtain the M-ear sequence $S = (H_0, \ldots, H_k)$ associated with M-ears $P' = (P_1, \ldots, P_{k-1}, \tilde{Q})$ such that S^{k-1} and P'^{k-1} satisfy D1, D2, D3 and S and P' satisfy $D1$. Therefore, by Lemma 2.3.21, S and P' also satisfy $D2$ and $D3$, and we are done for the theorem. $\square$

Theorem 2.3.29. *For any factorizable graph G, the binary relation $\lhd$ is a partial order on $\mathcal{G}(G)$.*

Proof. The reflexivity is obvious from the definition. The transitivity obviously follows from Theorem 2.3.13. Hence, we shall prove the antisymmetry. Let $G_1, G_2 \in \mathcal{G}(G)$ be factor-components with $G_1 \lhd G_2$ and $G_2 \lhd G_1$. Suppose the antisymmetry fails, that is, $G_1 \neq G_2$ holds. Let M be a perfect matching of G. By Proposition 2.3.24, there exist an M-ear sequence from G_1 to G_2, say $S := (G_1 = H_0, \ldots, H_k = G_2)$, where $k \geq 1$, and associated M-ears $P := (P_1, \ldots, P_k)$ satisfying D1, D2, and D3. Let u_1 and v_1 be the end vertices of P_1.

By Lemma 2.3.3, there exists $w \in V(G_2)$ such that there is an M-balanced path Q from u_1 to w. Trace Q from u_1, and let x be the first vertex we encounter that is in $(V(S;P) \cup \{v_1\}) \setminus \{u_1\}$; such a vertex surely exists since $V(G_2) \subseteq V(S;P)$.

Claim 2.3.30. *Without loss of generality we can assume that $x \neq v_1$, i.e., $x \in V(S;P)$ holds and u_1Qx is a path with $v_1 \notin V(u_1Qx) \setminus \{u_1\}$, which is M-balanced from u_1 to x.*

Proof. Suppose the claim fails, that is, $x = v_1$ holds. Then, $u_1 \neq v_1$ holds by the definition of x. If u_1Qv_1 is an M-saturated path, then $P_1 + u_1Qv_1$ forms an M-alternating circuit, containing non-allowed edges of $E(P_1) \cap \delta(G_1)$, a contradiction by Property 2.2.9. Otherwise, that is, if u_1Qv_1 is an M-balanced path from u_1 to v_1, then v_1Qw is an M-balanced path from v_1 to w, disjoint from u_1. Now redefine x as the first vertex we encounter that is in $V(S;P)$ if we trace v_1Qw from v_1. Then, v_1Qx is a path disjoint from u_1, which is M-balanced from v_1 to x by Property 2.2.10 and Proposition 2.3.20. Therefore, by changing the roles of u_1 and v_1, without loss of generality, we obtain this claim. □

Therefore, hereafter let $x \in V(S;P)$ and let u_1Qx be an M-balanced path from u_1 to x with $v_1 \notin V(u_1Qx) \setminus \{u_1\}$. Since $x \in V(S;P)$ holds, Proposition 2.3.9 yields that there is an M-balanced path R from x to an internal vertex of P_1, say y, such that $V(R) \subseteq V(S;P)$ and $V(R) \cap V(P_1) = \{y\}$.

If u_1P_1y has an even number of edges, then $u_1Qx + xRy + yP_1u_1$ is an M-alternating circuit containing non-allowed edges, say, edges of $E(P_1) \cap \delta(u_1)$, a contradiction by Property 2.2.9.

Hence hereafter we assume u_1P_1y has an odd number of edges. By Proposition 2.3.4, there is a path L of G_1 which is M-saturated between v_1 and u_1 or M-balanced from v_1 to u_1. Trace L from v_1, and let z be the first vertex on u_1Qx; note that v_1Lz is an M-balanced path from v_1 to z by Property 2.2.10, since $u_1Qx \cap G_1$ consists of M-saturated paths. Also note that L is disjoint from $V(S;P)$, since $V(L) \subseteq V(G_1)$ holds and $V(S;P)$ is disjoint from $V(G_1)$. If u_1Qz has an odd number of edges, then $zQu_1 + P_1 + v_1Lz$ is an M-alternating circuit containing non-allowed edges, say, edges of $E(P_1)\cap\delta(G_1)$, a contradiction by Property 2.2.9. If u_1Qz has an even number of edges, then $v_1Lz + zQx + xRy + yP_1u_1$ is an M-alternating circuit, which is again a contradiction. Thus we get $G_1 = G_2$, and the theorem follows. $\square$

2.4 A Generalization of the Canonical Partition

For non-elementary graphs, the family of maximal barriers never gives a partition of its vertex set [30]. Therefore, to analyze the structures of general graphs with perfect matchings, we generalized the canonical partition based on Kotzig's way [24–26].

Definition 2.4.1. Let G be a factorizable graph. We define a binary relation $\sim_G$ on $V(G)$ as follows: For $u, v \in V(G)$, $u \sim_G v$ if

(i) u and v are contained in the same factor-connected component and

(ii) either u and v are identical, or $G - u - v$ is not factorizable.

Note the following fact, which is easy to see by Property 2.2.8:

Fact 2.4.2. *Let G be a factorizable graph, and M be a perfect matching of G. Let $u, v \in V(G)$ be vertices contained in the same factor-connected component of G. Then, $u \sim_G v$ if and only if there is no M-saturated path between u and v.*

Theorem 2.4.3. *For any factorizable graph G, the binary relation $\sim_G$ is an equivalence relation on $V(G)$.*

Proof. Since the reflexivity and the symmetry are obvious from the definition, we prove the transitivity. Let M be a perfect matching of G. Let $u, v, w \in V(H)$ be such that $u \sim_G v$ and $v \sim_G w$. If any two of them are identical, clearly the claim follows. Therefore it suffices to consider the case that they are mutually distinct. Suppose that the claim fails, that is, $u \not\sim_G w$. Then there is an M-saturated path P between u and w. By Proposition 2.3.4, there is an M-balanced path Q from v to u. Trace Q from v and let x be the first vertex we encounter that in $V(Q) \cap V(P)$. If uPx has an odd number of edges, $vQx + xPu$ is an M-saturated path between u and v, a contradiction. If uPx has an even number of edges, then xPw has an odd number of edges, and by the same argument we have a contradiction. □

If a graph G is elementary, then the family of equivalence classes by $\sim_G$, i.e., $V(G)/\sim_G$ coincides with Kotzig's *canonical partition* [24–26, 30] (see [19, 20]). Therefore, given a factorizable graph G, we call $V(G)/\sim_G$ the *generalized canonical partition*, and denote it by $\mathcal{P}(G)$. By the definition of $\sim_G$, each member of $\mathcal{P}(G)$ is contained in some factor-connected component.

Moreover our proof for Theorem 2.4.3 contains a short proof for Kotzig's Theorem on the canonical partition of elementary graphs (Theorem 1.3.5). Kotzig [24–26] takes three papers to prove it, thus to prove that $\sim$ is an equivalence relation "from scratch" is considered to be hard [30]. However, in fact, it can be shown in a simple way even without the premise of the Gallai-Edmonds structure theorem nor the notion of barriers.

Note also the following:

Fact 2.4.4. *Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$. Then, $\mathcal{P}_G(H)$ is a refinement of $\mathcal{P}(H) = \mathcal{P}_H(H)$; that is, if $u, v \in V(H)$ satisfies $u \sim_G v$, then $u \sim_H v$ holds.*

Proof. We prove the contrapositive. Let $u, v \in V(H)$ be such that $u \not\sim_H v$, which is equivalent to u and v satisfying $u \neq v$ and $H - u - v$ is factorizable. Let M be a perfect matching of $H - u - v$. Since $G - V(H)$ is also factorizable, by letting M' be a perfect matching of it, we can construct a perfect matching of $G - u - v$, namely, $M \cup M'$. Therefore, $u \not\sim_G v$.

2.5 Correlations between $\triangleleft$ and $\sim_G$

In this section we further analyze properties of factorizable graphs. Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$. We denote the upper bounds of H in the poset $(\mathcal{G}(G), \triangleleft)$ by $\mathcal{U}^*_G(H)$; that is, $\mathcal{U}^*_G(H) := \{H' \in \mathcal{G}(G) : H \triangleleft H'\}$. We define $\mathcal{U}_G(H) := \mathcal{U}^*_G(H) \setminus \{H\}$, and the vertices contained in $\mathcal{U}^*_G(H)$ (resp. $\mathcal{U}_G(H)$) as $U^*_G(H)$ (resp. $U_G(H)$); i.e., $U^*_G(H) := \bigcup_{H' \in \mathcal{U}^*_G(H)} V(H')$ and $U_G(H) := \bigcup_{H' \in \mathcal{U}_G(H)} V(H')$. We often omit the subscripts "G" if they are apparent from contexts.

Lemma 2.5.1. *Let G be a factorizable graph, M be a perfect matching of G, and $H \in \mathcal{G}(G)$. Let P be an M-ear relative to H with end vertices $u, v \in V(H)$. Then $u \sim_G v$.*

Proof. Suppose the claim fails, that is, $u \neq v$ and there is an M-saturated path Q between u and v. Trace Q from u and let x be the first vertex we encounter that is on $P - u$. If uPx has an even number of edges, $uQx + xPu$ is an M-alternating circuit containing non-allowed edges, a contradiction. Hence we suppose uPx has an odd number of edges. Let $I \in \mathcal{G}(G)$ be such that $x \in V(I)$. Then one of the components of $uQx + xPu - E(I)$ is an M-ear relative to I and through H, a contradiction by Theorem 2.3.13. □

Theorem 2.5.2. *Let G be a factorizable graph, and $G_0 \in \mathcal{G}(G)$. For each connected component K of $G[U(G_0)]$ there exists $T_K \in \mathcal{P}_G(G_0)$ such that $N(K) \cap V(G_0) \subseteq T_K$.*

Proof. Let M be a perfect matching of G.

Claim 2.5.3. *Let $H \in \mathcal{U}(G_0)$, and S and P be the shortest M-ear sequence from G_0 to H and associated M-ears which satisfy D1, D2 and D3. Then, there exists*

$T \in \mathcal{P}_G(G_0)$ such that for each factor-components H' that has common vertices with $V(S;P)$, $N(H') \cap V(G_0) \subseteq T$ holds.

Proof. Let us denote $S = (G_0 = H_0, \ldots, H_k = H)$, where $k \geq 1$, and $P = (P_1, \ldots, P_k)$. Let $u_1, v_1 \in V(G_0)$ be the end vertices of P_1. By Lemma 2.5.1, there exists $T \in \mathcal{P}_G(G_0)$ such that $u_1, v_1 \in T$.

Let $H' \in \mathcal{G}(G)$ be such that $V(H') \cap V(S;P) \neq \emptyset$. Suppose there exists $w \in N(H') \cap V(G_0)$ and let $z \in V(H')$ be such that $wz \in E(G)$. Take $x \in V(H') \cap V(S;P)$ arbitrarily. By Proposition 2.3.4, there exists a path Q which is M-balanced from z to x or M-saturated between z and x such that $V(Q) \subseteq V(H')$. Trace Q from z and let y be the first vertex we encounter that is in $V(S;P)$. Then, zQy is an M-balanced path from z to y with $V(zQy) \subseteq V(H')$ and $V(zQy) \cap V(S;P) = \{y\}$. By D3($S$, P), for either of $r \in \{u_1, v_1\}$, there is an M-balanced path R from y to r such that $V(R) \setminus \{r\} \subseteq V(S;P)$.

Therefore, $R + zQy + wz$ forms an M-ear relative to G_0, whose end vertices are r and w. By Lemma 2.5.1, therefore, $w \in T$ and the claim follows. $\square$

Immediately by Claim 2.5.3 we can see that for any $H \in \mathcal{U}(G_0)$ there exists $T \in \mathcal{P}_G(G_0)$ such that $N(H) \cap V(G_0) \subseteq T$. Hence for each $T \in \mathcal{P}_G(G_0)$ we can define

$$\mathcal{K}_T := \{H \in \mathcal{U}(G_0) : V(H) \subseteq V(K) \text{ and } N(H) \cap V(G_0) \subseteq T\}$$

and $V_T := \bigcup_{H \in \mathcal{K}_T} V(H)$. Note that $\bigcup_{T \in \mathcal{P}_G(G_0)} V_T = V(K)$.

We are going to prove the claim by showing that $|\{T \in \mathcal{P}_G(G_0) : V_T \neq \emptyset\}| = 1$. Suppose it fails; Then, since K is connected, there exist $T_1, T_2 \in \mathcal{P}_G(G_0)$ with $T_1 \neq T_2$ such that $E[V_{T_1}, V_{T_2}] \neq \emptyset$. Let $s_1 \in V_{T_1}$ and $s_2 \in V_{T_2}$ be such that $s_1 s_2 \in E[V_{T_1}, V_{T_2}]$.

Claim 2.5.4. *For each $i = 1, 2$, there is an M-balanced path L_i from s_i to a vertex in T_i, say r_i, such that $V(L_i) \setminus \{r_i\} \subseteq V_{T_i}$.*

Proof. Let $i \in \{1, 2\}$. Let $H \in \mathcal{G}(G)$ be such that $s_i \in V(H)$. Then, $V(H) \subseteq V_{T_i}$. Take an M-ear sequence $S = (G_0 = H_0, \ldots, H_k = H)$, where $k \geq 1$, from G_0 to

H and an associated M-ears $P = (P_1, \dots, P_k)$ which satisfy D1, D2 and D3; By Claim 2.5.3, $V(S;P) \subseteq V_{T_i}$. By D3, there is an M-balanced path L_i from s_i to either of the end vertices of P_1, say $r_i \in V(G_0)$ such that $V(L_i) \setminus \{r_i\} \subseteq V(S;P)$. Therefore, $V(L_i) \setminus \{r_i\} \subseteq V_{T_i}$. □

By Claim 2.5.4, $L_1 + s_1s_2 + L_2$ is an M-ear relative to G_0, whose end vertices are $r_1 \in T_1$ and $r_2 \in T_2$. By Lemma 2.5.1 this yields $T_1 = T_2$, a contradiction. Therefore, we can conclude that there exists $T \in \mathcal{P}_G(G_0)$ such that $V_T = V(K)$, namely the claim follows. □

By Theorem 2.5.2, we can see that upper bounds of a factor-component are each "attached" to an equivalence class of the generalized canonical partition.

Proposition 2.5.5. *Let G be a graph and M be a matching of G. Let $H_1, H_2 \subseteq G$ be factor-critical subgraphs of G such that there exists $v \in V(H_1) \cap V(H_2)$ and that for each $i = 1, 2$, M_{H_i} is a near-perfect matching of H_i exposing only v. Then, $H_1 \cup H_2$ is factor-critical.*

Proof. Apparently, $M_1 \cup M_2$ is a near-perfect matching of $H_1 \cup H_2$, exposing only v. Since H_1 and H_2 are both factor-critical, the claim follows by Property 2.2.1. □

Lemma 2.5.6. *Let G be a factorizable graph, and $H \in \mathcal{G}(G)$. Then, the graph $G[U^*(H)]/H$ is factor-critical.*

Proof. Let M be a perfect matching of G. Let $\mathcal{X} \subseteq 2^{V(G)}$ be the family of separable set for H. Then, by Theorem 2.3.29, $\bigcup_{X \in \mathcal{X}} X = U^*(H)$. On the other hand, $G[\bigcup_{X \in \mathcal{X}} X]/H$ is factor-critical by Proposition 2.5.5. Therefore, the claim follows. □

Theorem 2.5.7. *Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$ and $S \subseteq \mathcal{P}_G(H)$. Let $K_1, \dots, K_l$, where $l \geq 1$ be some connected components of $G[U(H)]$ such that $N(K_i) \cap V(H) \subseteq S$ for $i = 1, \dots, l$. Then, $G[V(K_1) \cup \cdots \cup V(K_l) \cup S]/S$ is factor-critical.*

Proof. First note that $G[U^*(H)]/H$ is factor-critical by Lemma 2.5.6. Let h be the contracted vertex of $G[U^*(H)]/H$. Note also that K is a connected component of $G[U(H)]$ if and only if there is a block $\hat{K}$ of $G[U^*(H)]/H$ such that $K = \hat{K} - h$. Therefore, by Property 2.2.2 the claim follows. □

Remark 2.5.8. There are factorizable graphs where $\lhd$ does not hold for any two factor-components, in other words, where all the factor-components are minimal in the poset. For example, we can see by Theorem 2.3.13 and Theorem 2.5.2 that bipartite factorizable graphs are such, which means Theorem 2.3.29 is not a generalization of the DM-decomposition, even though they have similar natures.

The following theorem shows that most of the factorizable graphs with $|\mathcal{G}(G)| \geq 2$, in a sense, have non-trivial structures as posets.

Theorem 2.5.9. *Let G be a factorizable graph, $G_1, G_2 \in \mathcal{G}(G)$ be factor-components for which $G_1 \lhd G_2$ does not hold, and let G_1 be minimal in the poset $(\mathcal{G}(G), \lhd)$. Then there are possibly identical complement edges e, f of G between G_1 and G_2 such that $\mathcal{G}(G + e + f) = \mathcal{G}(G)$ and $G_1 \lhd G_2$ in $(\mathcal{G}(G + e + f), \lhd)$.*

Proof. First we prove the case where there is an edge xy such that $x \in V(G_1)$ and $y \in V(G_2)$. Let M be a perfect matching of G. Choose a vertex $w \in V(G_2)$ such that $w \not\sim_{G_2} y$ in G_2, and let P be an M-saturated path between w and y. If $xw \in E(G)$, there is an M-ear $xy + P + wx$ relative to G_1 and through G_2, which means $G_1 \lhd G_2$ by Theorem 2.3.13. Thus $xw \notin E(G)$. Suppose $\mathcal{G}(G + xw) \neq \mathcal{G}(G)$. Then there is an M-alternating circuit C containing xw in $G + xw$. Give an orientation to C so that it becomes a dicircuit with the arc xx'. Trace C from x and let z be the first vertex we encounter that is in $V(G_2)$. Then $xy + xCz$ is an M-ear of G which is relative to G_2 and through G_1, which means $G_2 \lhd G_1$ by Theorem 2.3.13, a contradiction to the minimality of G_1. Thus $\mathcal{G}(G + xw) = \mathcal{G}(G)$ and we are done for this case.

Now we prove the other case, where there is no edge of G connecting G_1 and G_2. Choose any $x \in V(G_1)$ and $y \in V(G_2)$. If $\mathcal{G}(G + xy) = \mathcal{G}(G)$, we can reduce it to the first case and the claim follows. Therefore it suffices to consider the case that $\mathcal{G}(G+xy) \neq \mathcal{G}(G)$. Then, for any perfect matching M of G, there is an M-alternating circuit C in $G+xy$ containing xy. Give an orientation to C so that it becomes a dicircuit with the arc yy'. Trace C from y and let u be the first vertex of G_1, and let v be the first vertex in G_2 if we trace C from u in the opposite direction.

If $\mathcal{G}(G+uv) = \mathcal{G}(G)$, the claim follows by the same argument. Otherwise, that is, if $\mathcal{G}(G+uv) \neq \mathcal{G}(G)$, there is an M-alternating circuit D containing uv. Give an orientation to D so that it becomes a dicircuit with the arc uu'. If uDv is disjoint from the internal vertices of vCu, then $uDv+vCu$ forms an M-alternating circuit containing non-allowed edges, a contradiction. Otherwise, trace D from u and let w be the first vertex on $vCu - u$.

If wCu has an even number of edges, $wCu + uDw$ is an M-alternating circuit of G, a contradiction. Therefore, we assume wCu has an odd number of edges. Let $H \in \mathcal{G}(G)$ be such that $w \in V(H)$. Then $wCu + uDw - H$ leaves an M-ear in G which is relative to H and through G_1, contradicting the minimality of G_1. Thus this completes the proof. □

2.6 Examples

Example 2.6.1. Consider the graph G in Figure 2.1. The allowed edges of G are those indicated by bold lines in Figure 2.2. Hence, it has these six factor-connected components $G_1 = G[\{a,b,c,d\}]$, $G_2 = G[\{e,f\}]$, $G_3 = G[\{g,h\}]$, $G_4 = G[\{i,j\}]$, $G_5 = G[\{k,l,m,n\}]$, and $G_6 = G[\{o,p\}]$. They form the poset $(\mathcal{G}(G) = \{G_1,\ldots,G_6\}, \triangleleft)$, given by the Hasse diagram in Figure 2.3.

Example 2.6.2. Consider the factorizable graph G in Figure 2.1. Its generalized canonical partition is shown in Figure 2.4. Here the set of all the vertices is partitioned into fourteen parts $\{S_1,\ldots,S_{14}\} = \mathcal{P}(G)$, and $\{S_1,S_2,S_3\} = \mathcal{P}_G(G_1)$ forms a partition of $V(G_1)$, $\{S_4,S_5\} = \mathcal{P}_G(G_2)$ of $V(G_2)$, $\{S_6,S_7\} = \mathcal{P}_G(G_3)$ of $V(G_3)$, $\{S_8,S_9\} = \mathcal{P}_G(G_4)$ of $V(G_4)$, $\{S_{10},S_{11},S_{12}\} = \mathcal{P}_G(G_5)$ of $V(G_5)$, and $\{S_{13},S_{14}\} = \mathcal{P}_G(G_6)$ of $V(G_6)$.

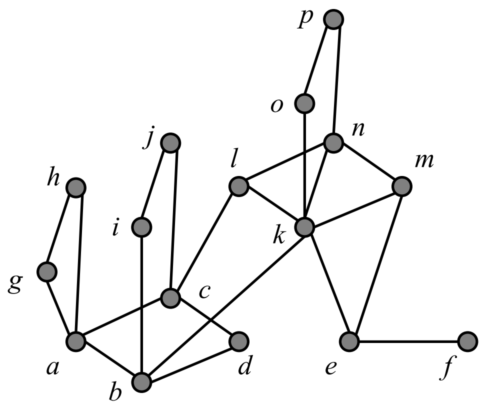

Figure 2.1: A factorizable graph G

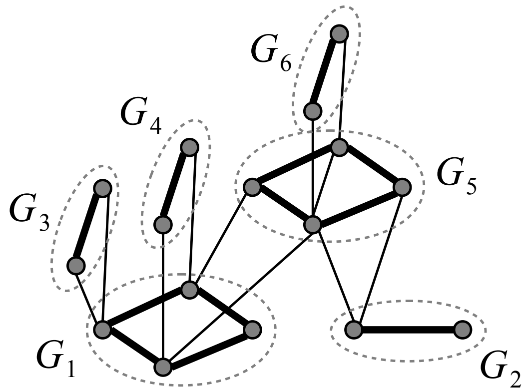

Figure 2.2: The factor-connected components of G

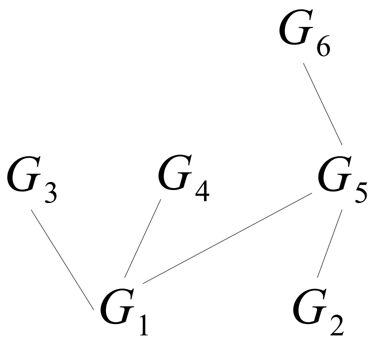

Figure 2.3: The Hasse diagram of $(\mathcal{G}(G), \triangleleft)$

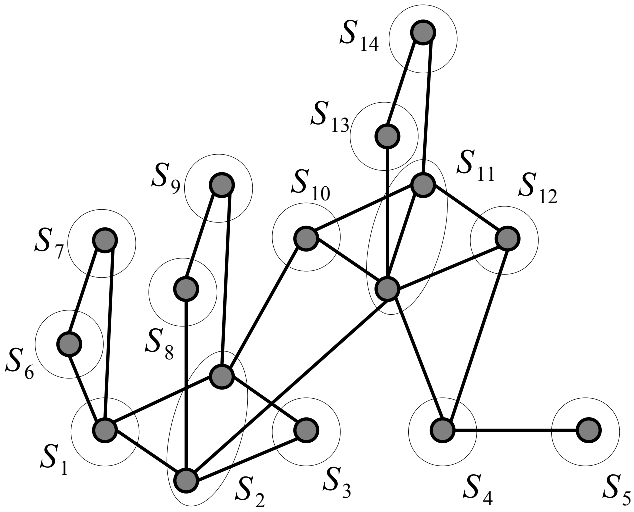

Figure 2.4: The generalized canonical partition of G

Example 2.6.3. Consider the elementary subgraph G_1, which is given as one of the factor-connected components in Figure 2.2. It is a circuit with four vertices $\{a, b, c, d\}$, and it is easy to see that its canonical partition, $\mathcal{P}(G_1)$, is composed of two sets: $\mathcal{P}(G_1) = \{\{a, d\}, \{b, c\}\}$. Therefore, compared with Figure 2.4, the partition $\mathcal{P}_G(G_1)$, which equals $\{\{a\}, \{b, c\}, \{d\}\}$, indeed gives a refinement (actually a proper refinement in this case) of $\mathcal{P}(G_1)$.

Example 2.6.4. Consider the factor-connected component G_1 in Figure 2.2. We have $\mathcal{U}^*(G_1) = \{G_1, G_3, G_4, G_5, G_6\}$ and $\mathcal{U}(G_1) = \{G_3, G_4, G_5, G_6\}$, while $U^*(G_1) = \{a, \ldots, d, g, \ldots p\}$ and $U(G_1) = \{g, \ldots, p\}$. Therefore, as indicated in Figure 2.5, $G[U(G_1)]$ has three connected components K_1, K_2, and K_3, and they satisfy $N(K_1) \cap V(G_1) \subseteq S_1$, $N(K_2) \cap V(G_1) \subseteq S_2$, and $N(K_3) \cap V(G_1) \subseteq S_2$.

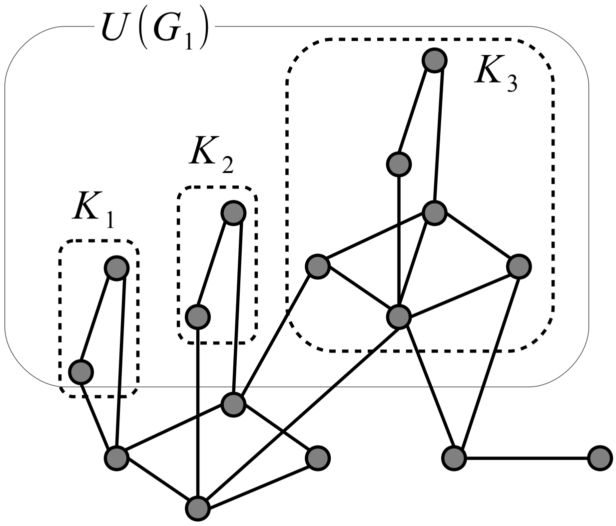

Figure 2.5: The strict upper bounds of G_1

2.7 Algorithmic Result

In this section, we discuss the algorithmic aspects of the partial order and the generalized canonical partition. We denote by n and m respectively the number of vertices and edges of input graphs. As we work on factorizable graphs and graphs with near-perfect matchings, we can assume $m = \Omega(n)$.

We start with some materials from Edmonds' maximum matching algorithm [7], referring mainly to [23, 30]. For a tree T with a specified root vertex r, we call a vertex $v \in V(T)$ *inner* (resp. *outer*) if the unique path in T from r to v has an odd (resp. even) number of edges. Let G be a graph and M be a matching of G. A tree $T \subseteq G$ is called M-*alternating* if exactly one vertex of it, the root, is exposed by M in G, and each inner vertex $v \in V(T)$ satisfies $|\delta(v) \cap E(T)| = 2$ and one of the edges of $\delta(v) \cap E(T)$ is contained in M.

A subgraph $S \subseteq G$ is called a *special blossom tree with respect to M* (M-*SBT*) if there is a partition $V(C_1)\dot\cup \cdots \dot\cup V(C_k) = V(S)$ such that

(i) $S' := S/C_1/\cdots/C_k$ is an M-alternating tree,

(ii) M_{C_i} is a near-perfect matching of C_i,

(iii) C_i is a maximal factor-critical subgraph of G if it corresponds to an outer vertex of S', and called an *outer blossom*, and

(iv) $|V(C_i)| > 1$ only if C_i is an outer blossom, for each $i = 1, \ldots, k$.

Edmonds' maximum matching algorithm tells us the following facts. Let G be a graph, M be a near-perfect matching of G, and $r \in V(G)$ be the vertex exposed by M. Then an M-SBT S, with root r, can be computed, if it is carefully implemented [11,37], in $O(m)$ time. Additionally, the set of vertices from which r can be reached by an M-balanced path is exactly the set of vertices contained in the outer blossoms of S.

Thus, due to an easy reduction of the above facts, the following proposition holds; they can be regarded as a folklore. See [1]. (In [1] they are presented as those for elementary graphs, but in fact, they can be applicable for general factorizable graphs.)

Proposition 2.7.1. *Let G be a factorizable graph, M be a perfect matching of G, and $u \in V(G)$.*

(i) *The set of vertices that can be reached from u by an M-saturated path can be computed in $O(m)$ time.*

(ii) *All the allowed edges adjacent to u can be computed in $O(m)$ time.*

(iii) *All the factor-components of G can be computed in $O(nm)$ time.*

Proposition 2.7.2. *Given a factorizable graph G, one of its perfect matchings M and $\mathcal{G}(G)$, we can compute the generalized canonical partition of G in $O(nm)$ time.*

Proof. For each $H \in \mathcal{G}(G)$, we can compute $\mathcal{P}_G(H)$ in a similar way to compute the canonical partition of an elementary graph [1]. That is, for each $v \in V(H)$, compute the set of vertices U that can be reached from v by an M-saturated path, and recognize $V(H) \setminus U$ as a member of $\mathcal{P}_G(H)$. This procedure is surely compatible by Theorem 2.4.3. Thus, the claim follows by Proposition 2.7.1. $\square$

Let G be a factorizable graph and M be a perfect matching of G. We say two distinct factor-components G_1, G_2 of G with $G_1 \triangleleft G_2$ are *non-refinable* if $G_1 \triangleleft H \triangleleft G_2$ yields $G_1 = H$ or $G_2 = H$ for any $H \in \mathcal{G}(G)$. Note that if G_1 and G_2 are non-refinable, then there is an M-ear relative to G_1 and through G_2 by Theorem 2.3.13. Note also that the converse of the above fact does not hold.

Lemma 2.7.3. *Let G be a factorizable graph, M be a perfect matching of G, and $H \in \mathcal{G}(G)$. Let S be a maximal M-SBT in G/H and let C be the blossom of T containing the contracted vertex h corresponding to H. Then any non-refinable upper bound of H in $(\mathcal{G}(G), \triangleleft)$ has common vertices with C. Additionally, if a factor-component $I \in \mathcal{G}(G)$ has some common vertices with C, then $H \triangleleft I$.*

Proof. For the former part, let H' be a non-refinable upper bound of H, and P be an M-ear relative to H and through H'. Since $P - C$ is a disjoint union of M-ears relative to C, we have $P \subseteq C$ by Theorem 2.2.5 and the maximality of the outer blossoms in M-SBT. Thus the former part of the claim follows.

For the latter part, by the definition of M-SBT and Proposition 2.2.6, there is an M-alternating odd ear-decomposition $\mathcal{P} = \{P_1, \dots, P_k\}$ of C. Let $I \in \mathcal{G}(G)$ be such that $V(I) \cap V(C) \neq \emptyset$ and that $V(P_j) \cap V(I) = \emptyset$ for $j = 1, \dots, i-1$ and $V(P_i) \cap V(I) \neq \emptyset$. We proceed by induction on i. If $i = 1$, the claim obviously follows. Let $i > 1$. $G_{i-1} := P_1 + \cdots + P_{i-1}$ is factor-critical by Theorem 2.2.5, and $M_{G_{i-1}}$ is a near-perfect matching of G_{i-1}. Moreover, P_i is an M-ear relative to G_{i-1}. Therefore, with the same technique as in the proof of Theorem 2.3.13, there exists $I' \in \mathcal{G}(G)$ such that $V(I') \cap V(C) \neq \emptyset$ and that there is an M-ear relative to I' and through I. Thus, by the induction hypothesis, the latter part of the claim follows. $\square$

Proposition 2.7.4. *Given a factorizable graph G, its perfect matching M, and $\mathcal{G}(G)$, we can compute the poset $(\mathcal{G}(G), \triangleleft)$ in $O(nm)$ time.*

Proof. It is sufficient to list all the non-refinable upper bounds for each factor-component of G by the following procedure:

$D := (\mathcal{G}(G), \emptyset)$; $A := \emptyset$;
for all $H \in \mathcal{G}(G)$ **do**
    compute a maximal M-SBT T; let C be the blossom of T corresponding to its root;
    for all $x \in V(C)$, which satisfies $x \in V(I)$ for some $I \in \mathcal{G}(G)$ **do**
        $A := A \cup \{(H, I)\}$;
    end for
end for
$D := (\mathcal{G}(G), A)$; STOP.

By Lemma 2.7.3, the partial order on $V(D)$ determined by the reachability corresponds to $\triangleleft$ after the above procedure. That is, if we define a binary relation $\prec$ on $V(D)$ so that $H' \triangleleft I'$ if there is a dipath from H' to I' in D, then $\prec$ and $\triangleleft$ coincide. For each $H \in \mathcal{G}(G)$, the above procedure costs $O(m)$ time, thus it costs $O(nm)$ time over the whole computation. $\square$

Remark 2.7.5. Given the digraph D after the procedure in Proposition 2.7.4, we can compute all the upper bounds of a factor-component in $O(n^2)$ time. Thus, an efficient data structure that represents the poset, for example, a boolean-valued matrix L where $L[i, j] =$ true if and only if $G_i \triangleleft G_j$, can be obtained in additional $O(n^2)$ time.

As a maximum matching of a graph can be computed in $O(\sqrt{n}m)$ time [31, 38], we have the following, combining Propositions 2.7.1, 2.7.2, and 2.7.4.

Theorem 2.7.6. *Let G be a factorizable graph. Then the poset $(\mathcal{G}(G), \triangleleft)$ and the generalized canonical partition $\mathcal{P}(G)$ can be computed in $O(nm)$ time.*

2.8 Concluding Remarks

Finally, we give remarks to conclude this chapter. We have investigated canonical structures of general factorizable graphs, noting the concept of factor-components, the fundamental building blocks of factorizable graphs. The key points are as follows. We show that the factor-components of a factorizable graph form a poset with respect to the canonical binary relation $\lhd$. On the other hand, the canonical partition by Kotzig [24–26] is generalized for general factorizable graphs, the *generalized canonical partition*, which induces a decomposition of each factor-components representing the overall structures of given graphs. These two notions are actually related as shown by Theorem 2.5.2. By this relationship, we can view these two as introducing a new canonical decomposition in which they, the partially ordered structure by $\lhd$ and the generalized canonical partition are unified. We name this canonical decomposition the *generalized cathedral structure* or just the *cathedral structure*, after the *cathedral theorem* for *saturated graphs* by Lovász [28]. As we will see in Chapter 4, this new canonical decomposition is closely related to the cathedral theorem.

Though we present the new canonical decomposition for factorizable graphs, actually we can also present it for non-factorizable graphs by combining it with the Gallai-Edmonds decomposition. Given a graph G, the Gallai-Edmonds decomposition outputs a factorizable subgraph, $G[C(G)]$; here this subgraph is treated as irreducible, and no further information is provided. By applying our new canonical decomposition to $G[C(G)]$, we can combine it with the Gallai-Edmonds decomposition and can formulate it as a decomposition for general graphs including those non-factorizable, giving further information about $G[C(G)]$. Accordingly, the new canonical decomposition provides a refinement of the Gallai-Edmonds decomposition.

As we note in Section 2.5, the partially ordered structure by $\lhd$ is not a generalization of the DM-decomposition. However, a generalization of the DM-decomposition is given using our new canonical decomposition [18]. Therefore, the new canonical decomposition provides a viewpoint to see the known canonical decompositions, i.e., the Gallai-Edmonds decomposition, the canonical partition the DM-decomposition, in a unified way.

Chapter 3

Barriers in General Graphs

3.1 Introduction

In this chapter, we show new results about structure of barriers, using the new canonical decomposition we introduced in Chapter 2. As we gave in Section 1.3.1, a barrier is a set of vertices determined by the Berge formula, a min-max formula characterizing the size of maximum matchings. The notion of barriers corresponds to a combinatorial interpretation of the dual optimal solutions of the maximum matching problem. Barriers also serve as effective tools in combinatorial arguments, taking on a role similar to canonical decompositions. (Inclusion-wise) maximal barriers seem especially important not only because they are easier to deal with but also because they are related to canonical decompositions and seem to have something intrinsic. However, investigating structures of barriers is not easy, and not so much has been known about barriers. The main difficulty may have lain lack of tools to see structures of factorizable graphs.

In this chapter, using the canonical decomposition for factorizable graphs introduced in Chapter 2, we reveal a structure of maximal barriers; actually we work on a wider notion, *odd-maximal* barriers (see Section 3.2). The main theorem (Theorem 3.4.1) here states that the equivalence classes of the generalized canonical partition are actually the "atoms" that constitute odd-maximal barriers; it also describes the structures of odd components associated with odd-maximal barriers.

There is another meaning of the main theorem. As we stated in Section 1.3.2, there are two formulation of the canonical partition of elementary graphs; Kotzig's formulation is by an equivalence relation [24–26], and Lovász's is based on barriers, stating that the family of maximal barriers forms a partition of the vertex set [27, 30]. In Chapter 2, we gave a generalization of the canonical partition based on Kotzig's formulation. On the other hand, in non-elementary graphs, the family of maximal barriers never forms a partition and there has not been known any generalization of Lovász canonical partition for general graphs. The structural description of odd-maximal barriers we give in the main theorem can be regarded as a generalization of the canonical partition based on Lovász formulation. Therefore, it is also a contribution of the theory of canonical decompositions.

The results in this chapter are also found in papers by the authoer [18,21]. The rest of this section is devoted to summarize the succeeding sections. In Section 3.2, we give the definition and some properties of odd-maximal barriers and explain why it is a reasonable notion to study, together with the observation that in order to know odd-maximal barriers in general graphs it suffices to work on factorizable graphs. In Section 3.3, we give more details about odd-maximal barriers and present proofs of some propositions in Section 3.2; readers familiar with might skip this section. In Section 3.4, we give the statement of the main theorem in this chapter, and then in Sections 3.5 and 3.6, we completes the proof of it.

3.2 Our Aim on Barriers

As we mention in Section 1.3.2, there is a structure of elementary graphs called the *canonical partition*; Kotzig first introduced it as the equivalence classes of a certain equivalence relation, and later Lovász reformulated it with the notion of barriers, stating that the family of maximal barriers forms a partition of the vertices in elementary graphs. This reformulation by Lovász has produced many fundamental properties in matching theory such as the *two ear theorem* [1,30], and the *brick decomposition* or the *tight cut decomposition*, and underlies polyhedral studies of matching theory; see the survey article [2].

However, in non-elementary graphs, the family of maximal barriers never forms a partition of the vertices, and there has not been known the counterpart structure of Lovász's canonical partition for general graphs.

The question remains: how all the maximal barriers exist in graphs and what is the counterpart in general graphs? Therefore, we are going to investigate it. Actually, we work on a wider notion: *odd-maximal barriers.* [1]

Definition 3.2.1. Let G be a graph. A barrier $X \subseteq V(G)$ is called an odd-maximal barrier if it is maximal with respect to $X \cup D_X$, i.e., no $Y \subseteq D_X$ with $Y \neq \emptyset$ satisfies that $X \cup Y$ is a barrier of G.

Odd-maximal barriers have some nice properties (see [15]):

- A maximal barrier is an odd-maximal barrier.
- For elementary graphs, the notion of maximal barriers and the notion of odd-maximal barriers coincide.

Hence, it seems reasonable to work on the odd-maximal barriers. The first one is easy to see by the definition. The second one is by the following proposition:

Proposition 3.2.2. *For an elementary graph G, if $X \subseteq V(G)$ is an odd-maximal barrier then it is also a maximal barrier.*

Since a maximal barrier is odd-maximal, two notions coincide for elementary graphs by Proposition 3.2.2. For the proof of Proposition 3.2.2, see the next section.

Actually, with the Gallai-Edmonds structure theorem and the following proposition by Király [15], it suffices to work on factorizable graphs:

Proposition 3.2.3 (Király [15])**.** *Let G be a graph. A set of vertices $S \subseteq V(G)$ is an odd-maximal barrier of G if and only if it is a disjoint union of $A(G)$ and*

[1] This is identical to those Király calls strong barriers [15], however we call it in the different way so as to avoid the confusion with the notion of strong end by Frank [9].

an odd-maximal barrier of the factorizable subgraph $G[C(G)]$. Now let S be an odd-maximal barrier. Then, the odd components of $G - S$ are the components of $G[D(G)]$ and the odd components of $G[C(G)] - (S \setminus A(G))$.

See the next section for a proof of Proposition 3.2.3. Given the above facts, in this chapter we give canonical structures of

odd-maximal barriers in general factorizable graphs

that can be regarded as a generalization of Lovász's canonical partition, aiming to contribute to the foundation of matching theory. Here, the new canonical decomposition, i.e., the generalized cathedral decomposition in Chapter 2 serves as a language to describe barriers. We also reveal structures of odd components associated with odd-maximal barriers.

3.3 More Details on Odd-maximal Barriers

3.3.1 Proof of Proposition 3.2.2

Here we give a proof of Proposition 3.2.2. This proposition is easy to see from known facts; readers familiar with matching theory might skip this section.

Proposition 3.3.1 (see [30] or [15]). *Let G be a graph and $X \subseteq V(G)$ be an odd-maximal barrier of G. Then, X is a maximal barrier if and only if $C_X = \emptyset$.*

Proof. The necessity part is obvious by the definition. For the sufficiency part, let $C_X \neq \emptyset$ and take $u \in C_X$ arbitrarily. Then $X \cup \{u\}$ is also a barrier of G, contradicting X being a maximal one. □

Proposition 3.3.2 (see [30] or [15]). *Let G be an elementary graph and X be a barrier of G. Then, $C_X = \emptyset$.*

Proof. If $C_X \neq \emptyset$, then since no the edges of $E[X, C_X]$ are allowed as stated in Proposition 3.5.1, we can see that G is not elementary, a contradiction. □

By combining Proposition 3.3.1 and Proposition 3.3.2, Proposition 3.2.2 is obtained.

3.3.2 Proof of Proposition 3.2.3

Here we give a proof of Proposition 3.2.3. Readers familiar with matching theory might skip this section.

Proposition 3.3.3 (folklore, see [30] or [15]). *Let G be a graph, $X \subseteq V(G)$ be a barrier of G, and $Y \subseteq V(G)$ be such that $X \subseteq Y$. Then, Y is a barrier of G if and only if $Y \setminus X$ is a union of barriers of some connected components of $G - X$.*

Additionally, Király shows that $A(G)$ is the minimum odd-maximal barriers in any graph G.

Theorem 3.3.4 (Kiráry [15]). *Let G be a graph, and $\mathcal{X} \subseteq 2^{V(G)}$ be the family of the odd-maximal barriers of G. Then, $\bigcap_{X \in \mathcal{X}} X = A(G)$.*

Therefore, combining up Theorem 1.3.4, Proposition 3.3.3, and Theorem 3.3.4, we obtain Proposition 3.2.3 immediately.

3.4 The Main Result in This Chapter

Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Based on Theorem 2.5.2, we denote the set of all the strict upper bounds of H "assigned" to S by $\mathcal{U}_G(S)$; that is to say, $H' \in \mathcal{U}_G(S)$ if and only if $H' \in \mathcal{U}(H)$ and there is a connected component K of $G[U(H)]$ such that $V(H') \subseteq V(K)$ and $N_G(K) \cap V(H) \subseteq S$. We define $U_G(S) := \bigcup_{H' \in \mathcal{U}_G(S)} V(H')$ and $U^*_G(S) := U_G(S) \cup S$. We often omit the subscripts "G" if they are apparent from contexts. Note that $\mathcal{U}(H) = \bigcup_{S \in \mathcal{P}_G(H)} \mathcal{U}(S)$. Our main result in this chapter is the following:

Theorem 3.4.1. *Let G be a factorizable graph, and $X \subseteq V(G)$ be an odd-maximal barrier of G. Then, X is a disjoint union of some members of $\mathcal{P}(G)$; namely, there exist $S_1, \ldots, S_k \in \mathcal{P}(G)$ such that $X = S_1 \dot\cup \cdots \dot\cup S_k$. Additionally, odd components of $G - X$ have structures as follows: $D_X = (U^*(G_1) \setminus U^*(S_1)) \dot\cup \cdots \dot\cup (U^*(G_k) \setminus U^*(S_k))$, where $G_i \in \mathcal{G}(G)$ is such that $S_i \in \mathcal{P}_G(G_i)$ for each $i \in \{1, \ldots, k\}$.*

This theorem states that in general graphs the equivalence classes of the generalized canonical partition are the "atoms" that constitute odd-maximal barriers, and that odd components associated to odd-maximal barriers are also described canonically by the generalized cathedral structure. As we see in previous chapters, among two formulations of the canonical partition of elementary graphs, the generalization of the canonical partition introduced in Chapter 2 is attained based on Kotzig's formulation; here we show it is as well a generalization based on Lovász's formulation.

Sketch of the Proof: This theorem is an immediate corollary of Theorem 3.6.11, and the rest of this chapter is to prove Theorem 3.6.11. We shall prove it by examining the reachability of alternating paths from two viewpoints— regarding odd-maximal barriers and regarding the generalized cathedral structure—and showing their equivalence.

Let us mention some additional propositions used later in this chapter.

Proposition 3.4.2 (Király [15])**.** *A barrier $X \subseteq V(G)$ of a graph G is odd-maximal if and only if all the odd components of $G - X$ are factor-critical.*

Proposition 3.4.3 (Dulmage & Mendelsohn [4–6, 32])**.** *Let $G = (A, B; E)$ be a bipartite factorizable graph, and M be a perfect matching of G. Let $G_1, G_2 \in \mathcal{G}(G)$, and let $u \in A \cap V(G_1)$, $v \in A \cap V(G_2)$, and $w \in B \cap V(G_2)$. Then there is an M-balanced path from u to v if and only if $G_1 \preceq_A G_2$; additionally, there is an M-saturated path between u to w if and only if $G_1 \preceq_A G_2$.*

3.5 Barriers vs. Alternating Paths

In this section we introduce some lemmas on the reachability of alternating paths regarding odd-maximal barriers. Given an odd-maximal barrier X of a factorizable graph G, we generate a bipartite graph, thus canonically decompose $X \cup D_X$ and state the reachability using the DM-decomposition as a language to describe the structures. This technique of generating a bipartite graph has been known [9, 30] and essences of ideas are found there. However, we first reveal it thoroughly to obtain Proposition 3.5.9 and Theorem 3.5.16.

The following proposition can be obtained as an immediate corollary of Proposition 1.3.2 and is for factorizable graphs.

Proposition 3.5.1. *Let G be a factorizable graph, and $X \subseteq V(G)$ be a barrier of G. Then for any perfect matching M of G,*

(i) *each vertex of X is matched to a vertex of D_X,*

(ii) *for each component K of $G[D_X]$, M_K is a near-perfect matching of K, accordingly $|\delta(K) \cap M| = 1$,*

(iii) *M contains a perfect matching of $G[C_X]$, and*

(iv) *no edge in $E[X, C_X]$ nor $E(G[X])$ is allowed.*

Proposition 3.5.2. *Let G be a factorizable graph, and M be a perfect matching of G. If $X \subseteq V(G)$ is a barrier, then for any $u, v \in X$ there is no M-saturated path between u and v.*

Proof. Suppose the claim fails, namely, there is an M-saturated path, say P, between $u \in X$ and $v \in X$. Then, $M \triangle E(P)$, i.e., $(M \setminus E(P)) \cup (E(P) \setminus M)$, forms a perfect matching of $G - u - v$; accordingly, $G - u - v$ is factorizable. Now recall that since X is a barrier of the factorizable graph G, $G - X$ has exactly $|X|$ odd components by the definition of barriers. Therefore, the graph $(G - u - v) - (X \setminus \{u, v\})$, which equals $G - X$, also has $|X|$ odd components; this means by the Berge formula that $G - u - v$ is not factorizable, a contradiction. $\square$

Proposition 3.5.3. *Let G be a factorizable graph, M be a perfect matching, and X be a barrier of G. Then, for any $x \in X$ and $y \in C_X$, there is no M-saturated path between x and y nor M-balanced path from x to y.*

Proof. Suppose otherwise, that is, for some $x \in X$ and $y \in C_X$ there is a path P which is M-saturated between x and y or M-balanced from x to y. Trace P from

x; let z be the first vertex we encounter that is in C_X, and let w be the last vertex in $X \cup D_X$ we encounter if we trace xPz from x. Since apparently $E[C_X, D_X] = \emptyset$ holds, we have $w \in X$ and $wz \in E(xPz)$, and by Proposition 3.5.1 (iv), $wz \notin M$ holds. Therefore, xPw is an M-saturated path between x and w, contradicting Proposition 3.5.2. □

Proposition 3.5.4. *Let G be a factorizable graph, M be a perfect matching, and $X \subseteq V(G)$ be an odd-maximal barrier. Then, for any $u \in X$ and $v \in X \cup C_X$ there is no M-saturated path between u and v.*

Proof. This is immediate by Proposition 3.5.2 and Proposition 3.5.3. □

Definition 3.5.5. Let G be a graph, $X \subseteq V(G)$ be a set of vertices, and $K_1, \dots, K_l$ be the odd components of $G - X$. We denote the bipartite graph resulting from deleting the even components of $G - X$, removing the edges whose vertices are all contained in X, and contracting each K_i, where $i = 1, \dots, l$, respectively into one vertex, as $H_G(X)$. Namely, $H_G(X) := (G - C_X - E(G[X]))/K_1/\cdots/K_l$.

Example 3.5.6. Figure 3.1 indicates a factorizable graph, say G, with an odd-maximal barrier (actually a maximal barrier) $X = \{b, f, h, i, j, r\}$. Actually, this graph is isomorphic to the one given in Figure 3.16. If you contract each of the six odd components of $G - X$ into one vertex and remove the two edges in $E(G[X])$, then you obtain the bipartite graph $H_G(X)$ as indicated in Figure 3.2. There are exactly two odd components with more than one vertex in $G - X$, say K and L; we denote the new vertices resulting from contracting K and L as v_K and v_L, respectively. On the other hand, there are four odd components of $G - X$ each composed of a single vertex, i.e., k, l, m, a, respectively; here in Figure 3.2 and some following figures, we call the vertices of $H_G(X)$ corresponding to them in just the same way: k, l, m, a. However, we identify edges of $H_G(X)$ with those of G naturally corresponding to them; e.g., we treat the edges bc and bv_L as equivalent.

The next proposition is easily seen by Propositions 3.5.1 and 3.4.2 and enables us to discuss Proposition 3.5.9 and so on.

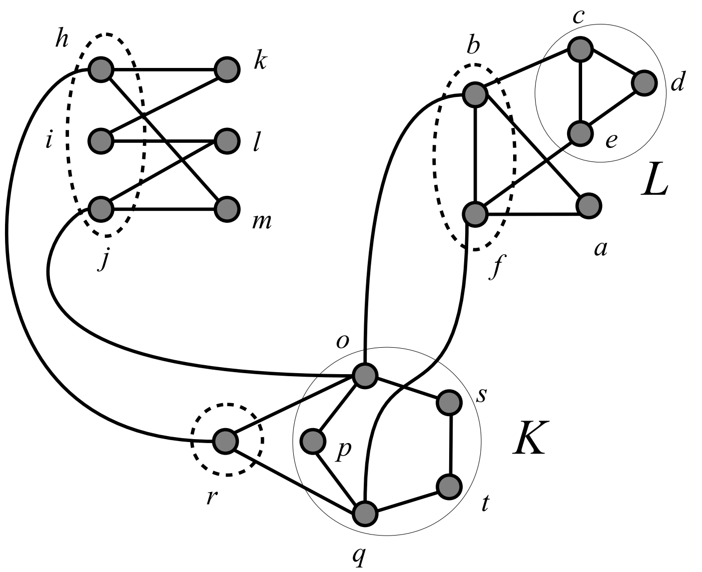

Figure 3.1: A factorizable graph G with an odd-maximal barrier X

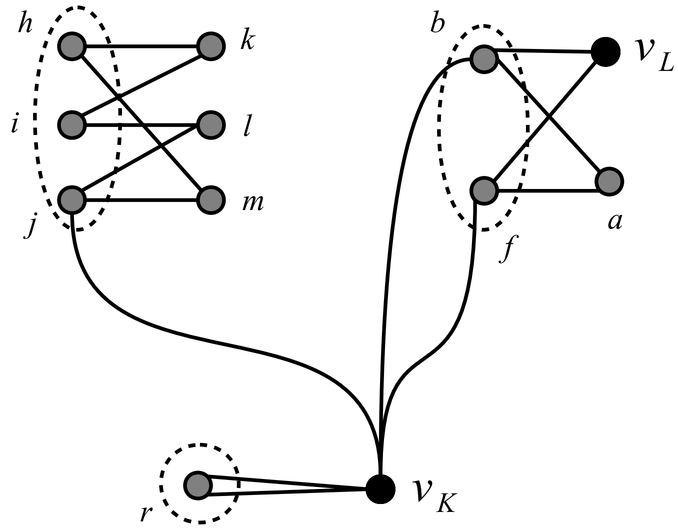

Figure 3.2: The bipartite graph $H_G(X)$

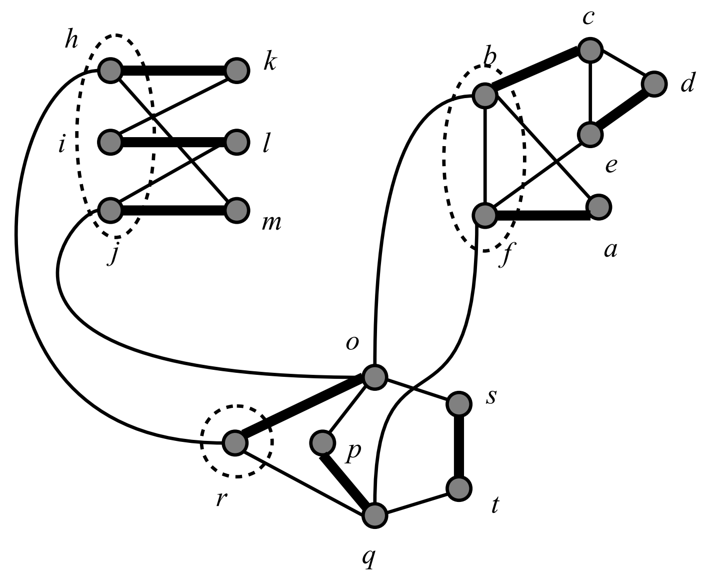

Figure 3.3: The graph G with a perfect matching M

Proposition 3.5.7. *Let G be a factorizable graph and X be an odd-maximal barrier of G. If $M \subseteq E(G)$ is a perfect matching of G, then $M \cap \delta(X)$ forms a perfect matching of $H_G(X)$. Conversely, if M' is a perfect matching of $H_G(X)$, there is a perfect matching M of G such that $M' = M \cap \delta(X)$.*

Proof. The first claim follows by Proposition 3.5.1. For the second claim, let M' be a given perfect matching of $H_G(X)$. Note that $G[C_X]$ is factorizable by Proposition 3.5.1 (iii), and let M'' be a perfect matching of $G[C_X]$. By Proposition 3.4.2, the odd components $K_1, \dots, K_l$ of $G - X$ are each factor-critical. For each $i = 1, \dots, l$, let M_i be a near-perfect matching of K_i exposing only the vertex covered by M'. Now by letting $M := M'' \cup M' \cup \bigcup_{i=1}^{l} M_i$, we obtain a desired perfect matching M of G. $\square$

Example 3.5.8. In Figure 3.3, the path of G induced by the vertices r, o, p, q, f, a, b, c, d, e is an M-saturated path from r to e; we call this path P. In $H_G(X)$,

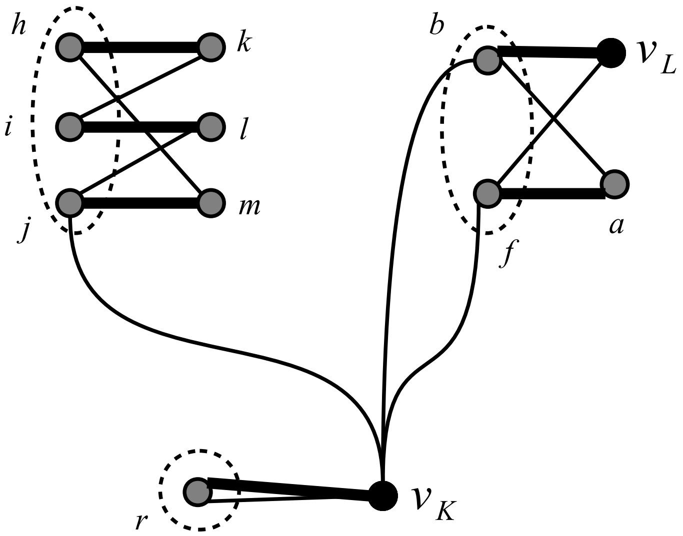

Figure 3.4: The graph $H_G(X)$ with the perfect matching M'

P corresponds to the M-saturated path P' induced by the vertices r, v_K, f, a, b, v_L. These paths P and P' illustrate the statement of Proposition 3.5.9.

The next proposition shows that the reachabilities of alternating paths are equivalent between G and $H_G(X)$, which, with Proposition 3.4.3, implies Theorem 3.5.16 immediately.

Proposition 3.5.9. *Let G be a factorizable graph, $X \subseteq V(G)$ be an odd-maximal barrier of G, and $\mathcal{K} := \{K_i : i = 1, \ldots, l\}$ be the family of odd components of $G - X$, where $l = |X|$. Let M be a perfect matching of G and M' be the perfect matching of $H_G(X)$ with $M' = M \cap \delta(X)$. Let $u, v \in X$ and $w \in V(K)$, where $K \in \mathcal{K}$, and let w_K be the contracted vertex of $H_G(X)$ corresponding to K.*

(i) *Then, for any M-balanced path (resp. M-saturated path) P of G from u to v (resp. between u and w), $P' := P/K_1/\cdots/K_l$ is an M'-balanced path (resp. M'-saturated path) of $H_G(X)$ from u to v (resp. between u and w_K).*

(ii) *Conversely, for any M'-balanced path (resp. M'-saturated path) P' from u*

to v in $H_G(X)$ (resp. between u and w_K), there is an M-balanced path (resp. M-saturated path) P from u to v in G (resp. between u and w) such that $P' = P/K_1/\cdots/K_l$.

Proof. **(i):** (See Figure 3.5 and Figure 3.6.) For (i), we first prove the case where P is an M-balanced path. Let $u = x_1, \ldots, x_q = v$ be the vertices of $X \cap V(P)$, and suppose, without loss of generality, they appear in this order if we trace P from u. For each $i = 1, \ldots, q$, let $L_i \in \mathcal{K}$ be such that $x_i' \in V(L_i)$, which is well-defined by Proposition 3.5.1 (i), and let z_i be the contracted vertex of $H_G(X)$ corresponding to L_i. Note that by Proposition 3.5.1 (ii),

Claim 3.5.10. *if $x_i \neq x_j$, then $L_i \neq L_j$ and accordingly $z_i \neq z_j$.*

We are going to prove a bit refined statement of (i):

Claim 3.5.11. *P' is an M'-balanced path from u to v, with $V(P') = \{x_i : i = 1, \ldots, q\} \cup \{z_i : i = 1, \ldots, q\} \setminus \{z_q\}$.*

Proof. We prove this claim by induction on q. If $q = 1$ (i.e., $u = v$), then P and P' are trivial paths and the claim is obviously true. Let $q \geq 2$, and suppose the claim is true for any case with a shorter path. Since $u' \in V(L_1)$ holds, the internal vertices of uPx_2 are contained in L_1. By Proposition 3.5.1 (ii), $\delta(L_i) \cap M = \{uu'\}$. Thus, if we trace uPx_2 from u then the last edge is not in M, which means

- uPx_2 is an M-balanced path from u to x_2. Accordingly,
- x_2Pv is also an M-balanced path from x_2 to v.

Note that

> $P_1' := uPx_2/K_1/\ldots/K_l$, which equals uPx_2/L_1, is apparently an M-balanced path whose vertices are $\{u, z_1, x_2\}$,

since $E(P_1') = \{uz_1, z_1x_2\}$, $uz_1 \in M$, $z_1x_2 \notin M$ hold. Therefore, if $q = 2$ (i.e., $x_2 = v$) then the claim follows. Hence hereafter we prove the case where $q \geq 3$ (i.e., $x_2 \neq v$). Since x_2Pv is an M-balanced path from x_2 to v, the induction hypothesis yields that

$P_2' := x_2Pv/K_1/\ldots/K_l$ is an M'-balanced path of $H_G(X)$ from x_2 to v, whose vertices are $\{x_2, \ldots, x_q = v\} \cup \{z_2, \ldots, z_{q-1}\}$.

Thus, $V(P_1') \cap V(P_2') = \{x_2\}$ by Claim 3.5.10; accordingly, $P' = P_1' + P_2'$ is an M'-balanced path of $H_G(X)$ from u to v with $V(P') = \{x_i : i = 1, \ldots, q\} \cup \{z_i : i = 1, \ldots, q\} \setminus \{z_q\}$. Now we are done for this claim. □

Since we obtain Claim 3.5.11, we are done for this case of (i). The other case of (i) where the path P is an M-saturated path can be proved by similar arguments.

(ii): (See Figure 3.7 and Figure 3.8.) For (ii), we first prove the case where P' is an M'-balanced path of $H_G(X)$. Since it is apparently true if $u = v$, we prove the case where $u \neq v$. Let $u = x_0, y_0, \ldots, x_p, y_p, x_{p+1} = v$ be the vertices of P', and suppose they appear in this order if we trace P' from u. Note that

- $x_i \in X$ holds for each $i \in \{0, \ldots, p+1\}$,
- y_i is a contracted vertex corresponding to an odd component of $G - X$, say L_i, for each $i \in \{0, \ldots, p\}$, and
- $x_iy_i \in M'$ and $y_ix_{i+1} \notin M'$ hold for each $i \in \{0, \ldots, p\}$.

For each $i = 0, \ldots, p$, let $y_i^1, y_i^2 \in V(L_i)$ be such that G has edges $x_iy_i^1, y_i^2x_{i+1} \in E(G)$ that correspond to $x_iy_i, y_ix_{i+1} \in E(H_G(X))$, respectively. Since $x_iy_i \in M'$ and $y_ix_{i+1} \notin M'$, we have

$$x_iy_i^1 \in M \text{ and } y_i^2x_{i+1} \notin M.$$

Claim 3.5.12. *For each $i \in \{0, \ldots, p\}$, there is an M-balanced path Q_i from y_i^2 to y_i^1 whose vertices are contained in $V(L_i)$.*

Proof. The odd component L_i is factor-critical by Proposition 3.4.2, and M_{L_i} forms a near-perfect matching of L_i, which exposes y_i^1, by Proposition 3.5.1 (ii). Therefore, by Property 2.2.1, there is an M-balanced path Q_i of L_i from y_i^2 to y_i^1. □

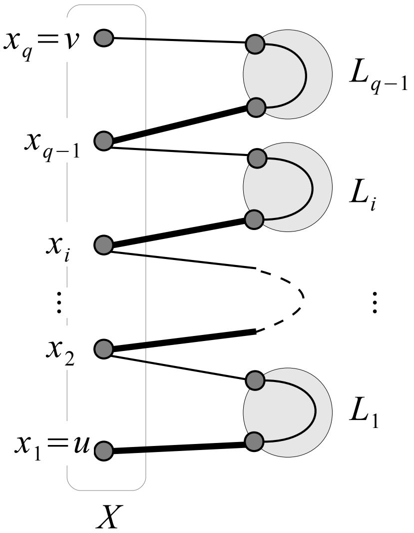

Figure 3.5: The path P in the proof of Proposition 3.5.9 (i)

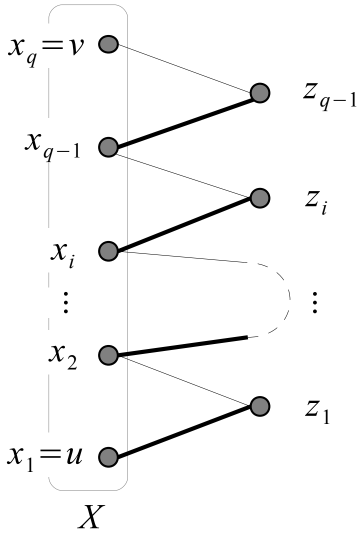

Figure 3.6: The path P' in the proof of Proposition 3.5.9 (i)

Thus, with Claim 3.5.12, by replacing each y_i by Q_i on P', we can get an M-balanced path P of G from u to v that satisfies $P' = P/K_1/\cdots/K_l$. Now we are done for this case of (ii). The other case of (ii) where the path P' is an M'-saturated path can be proved by similar arguments. $\square$

Given a factorizable graph G and an odd-maximal barrier X, we denote the DM-decomposition of $H_G(X)$ with respect to X as just the DM-decomposition of $H_G(X)$. In this case, we sometimes denote $\preceq_X$ as just $\preceq$, omitting the subscript "X".

Definition 3.5.13. Let G be a factorizable graph and X be an odd-maximal barrier of G. Let D be a DM-component of $H_G(X)$, and $K_1, \ldots, K_l$, where $l = |X \cap V(D)|$, be such that $V(D) \setminus X$ are the vertices resulting from contracting $K_1, \ldots, K_l$. We say $\hat{D}$ is the *expansion* of D if it is the subgraph of G induced by $(V(D) \cap X) \cup \bigcup_{i=1}^{l} V(K_i)$.

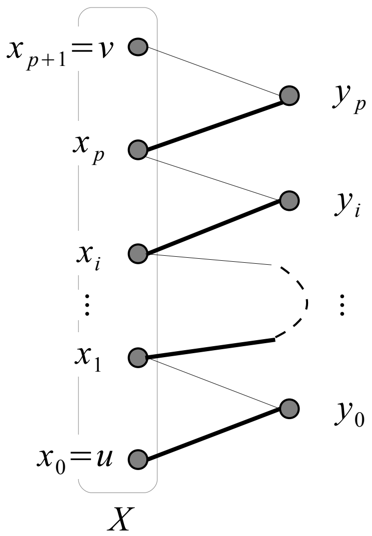

Figure 3.7: The path P' in the proof of Proposition 3.5.9 (ii)

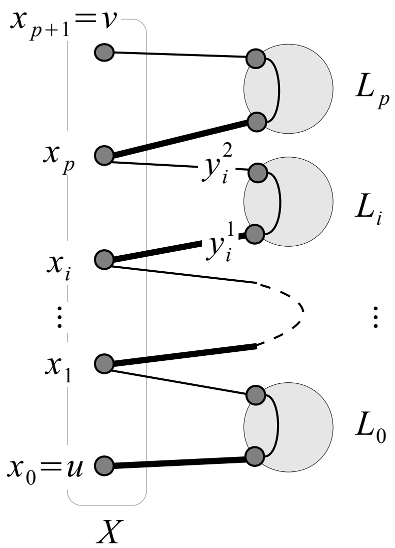

Figure 3.8: The path P is the proof of Proposition 3.5.9 (ii)

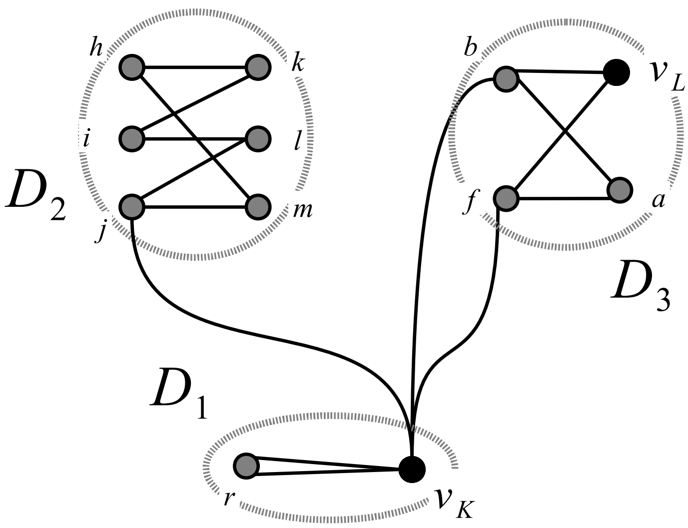

Figure 3.9: The DM-components of $H_G(X)$

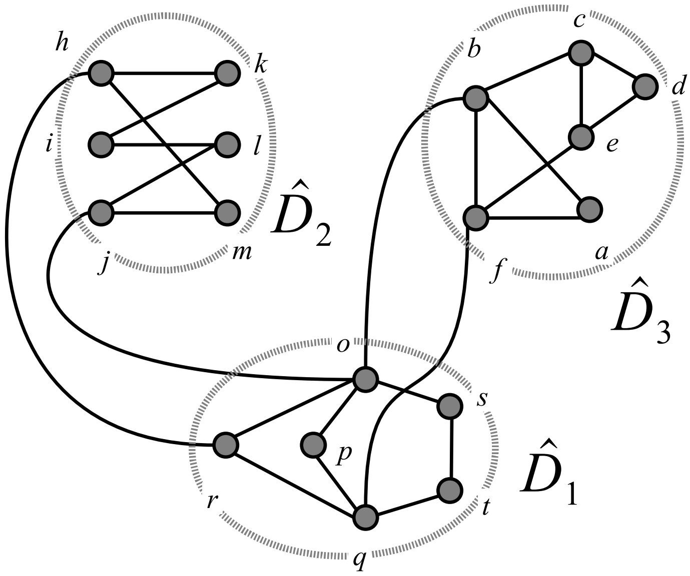

Figure 3.10: The expansions of the DM-components of $H_G(X)$

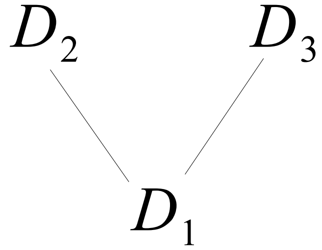

Figure 3.11: The Hasse diagram that indicates the DM-decomposition of $H_G(X)$

Example 3.5.14. Let G and X be the graph and the barrier given in Figure 3.1. The bipartite graph $H_G(X)$, indicated in Figure 3.2, has three DM-components shown by Figure 3.9: $D_1 = H_G(X)[\{r, v_K\}]$, $D_2 = H_G(X)[\{h, i, j, k, l, m\}]$, $D_3 = H_G(X)[\{b, f, v_L, a\}]$. Therefore, the expansions of D_1, D_2, D_3 are those indicated in Figure 3.10: $\hat{D}_1 = G[\{o, p, q, r, s, t\}]$, $\hat{D}_2 = G[\{h, i, j, k, l, m\}]$, $\hat{D}_3 = G[\{a, b, c, d, e, f\}]$, respectively. The DM-decomposition of $H_G(X)$ (with respect with X) has the poset structure $(\{D_1, D_2, D_3\}, \preceq)$, indicated by the Hasse diagram given in Figure 3.11.

The next proposition is a basic observation on expansions.

Proposition 3.5.15. *Let G be a factorizable graph, and X be an odd-maximal barrier of G. Let $D_1, \ldots, D_k$ be the DM-components of $H_G(X)$. For each $i = 1, \ldots, k$, let $\hat{D}_i$ be the expansion of D_i. Then,*

(i) *$\{V(\hat{D}_i)\}_{i=1}^k$ forms a partition of $X \cup D_X$,*

(ii) *$V(\hat{D}_i)$ is separating, accordingly $\hat{D}_i$ is factorizable,*

(iii) *$X \cap V(\hat{D}_i)$ is an odd-maximal barrier of $\hat{D}_i$, and*

(iv) *$H_{\hat{D}_i}(X \cap V(\hat{D}_i))$ is isomorphic to D_i, for each $i = 1, \ldots, k$.*

Proof. Since the DM-components of $H_G(X)$ give the partition of $V(H_G(X))$, (i) apparently follows from the definition of expansions. For the first half of (ii), suppose that $V(\hat{D}_i)$ is not separating, equivalently by Property 2.3.1, that there is a perfect matching M of G with $\delta(\hat{D}_i) \cap M \neq \emptyset$. Then, by Proposition 3.5.7, $M' := M \cap \delta(X)$ forms a perfect matching of $H_G(X)$ satisfying $\delta(D_i) \cap M' \neq \emptyset$; this is a contradiction, since of course $V(D_i)$ is a separating set. Therefore, $V(\hat{D}_i)$ is separating; accordingly, $\hat{D}_i$ is factorizable, and we are done for (ii).

Apparently by the definition of expansions, $\hat{D}_i \setminus X$ is composed of $|X \cap V(\hat{D}_i)|$ number of odd components; moreover, since X is an odd-maximal barrier, Proposition 3.4.2 yields that each of them are factor-critical. Therefore, $V(\hat{D}_i) \cap X$ is an odd-maximal barrier of $\hat{D}_i$ by the statement (ii) and Proposition 3.4.2 again. Thus, we are done for (iii). The statement (iv) is apparent from the definition. $\square$

Theorem 3.5.16. *Let G be a factorizable graph, X be an odd-maximal barrier, and M be a perfect matching of G. Let $u, v \in X$ and $w \in D_X$, and for each $\alpha \in \{u, v, w\}$ let D_α be the DM-component of $H_G(X)$ whose expansion $\hat{D}_\alpha$ satisfies $\alpha \in V(\hat{D}_\alpha)$. Then, there is an M-balanced path from u to v (resp. an M-saturated path from u to w) in G if and only if $D_u \preceq D_v$ (resp. $D_u \preceq D_w$).*

Proof. First note that $\hat{D}_\alpha$ is well-defined for each $\alpha \in \{u, v, w\}$ by Proposition 3.5.15 (i). Now the claim is immediate from Proposition 3.4.3 and Proposition 3.5.9. □

Lemma 3.5.17. *Let $G = (A, B; E)$ be a bipartite factorizable graph, M be a perfect matching of G, and D_1, D_2 be DM-components of G with $D_1 \preceq_A D_2$. Then, for any $u \in V(D_1) \cap A$ and $v \in V(D_2) \cap B$, any M-saturated path between u and v traverses $A \cap V(D_2)$.*

Proof. Let P be an M-saturated path between $u \in V(D_1) \cap A$ and $v \in V(D_2) \cap B$. Apparently $vv' \in E(P)$ holds, and since $V(D_2)$ is of course a separating set, $v' \in V(D_2) \cap A$ holds. Namely, P has a vertex in $V(D_2) \cap A$, i.e., v'; the claim follows. □

The following lemma is obtained by Propositions 3.5.9 and Lemma 3.5.17.

Lemma 3.5.18. *Let G be a factorizable graph, X be an odd-maximal barrier, and M be a perfect matching of G. Let $\hat{D}_1$ and $\hat{D}_2$ be the subgraphs of G which are respectively the expansions of DM-components D_1 and D_2 such that $D_1 \preceq D_2$. Then, for any $u \in X \cap V(\hat{D}_1)$ and $w \in V(\hat{D}_2) \setminus X$, any M-saturated path between u and w traverses $X \cap V(\hat{D}_2)$.*

Proof. Let P be an M-saturated path between $u \in X \cap V(\hat{D}_1)$ and $w \in V(\hat{D}_2) \setminus X$. Let $K_1, \ldots, K_l$, where $l = |X|$, be the odd components of $G - X$. By Proposition 3.5.9, $P' := P/K_1/\cdots/K_l$ is an M'-saturated path, where $M' = M \cap \delta(X)$, whose end vertices are respectively in $X \cap V(D_1)$ and $V(D_2) \setminus X$. Therefore, P' traverses $X \cap V(D_2)$ by Lemma 3.5.17, which means P traverses $X \cap V(\hat{D}_2)$. □

3.6 Canonical Structures of Odd-maximal Barriers

In this section we examine the reachability of alternating paths regarding the cathedral structure and show Theorem 3.4.1.

Proposition 3.6.1. *Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, $G[U^*(S)]/S$ is factor-critical.*

Proof. This is immediately obtained by Theorem 2.5.7. □

The next lemma is obtained by Proposition 3.6.1 and Property 2.2.1.

Lemma 3.6.2. *Let G be a factorizable graph and M be a perfect matching of G, and let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, for any $x \in U^*(S)$, there is an M-balanced path from x to some vertex $y \in S$, whose vertices except y are contained in $U(S)$.*

Proof. Since $U(S)$ is separating, $M_{U(S)}$ forms a perfect matching of $G[U(S)]$. Therefore, $M_{U(S)}$ forms a near-perfect matching of $G' := G[U^*(S)]/S$ exposing only the contracted vertex s corresponding to S. Additionally, by Proposition 3.6.1, G' is factor-critical. Therefore, by Property 2.2.1, in G' for any $x \in U^*(S)$ there is an $M_{U(S)}$-balanced path from x to s; this path corresponds to a desired path in G. Thus, the claim follows. □

Immediately by Theorem 2.4.3, we can see the next proposition:

Proposition 3.6.3. *Let G be a factorizable graph and M be a perfect matching of G, and let $H \in \mathcal{G}(G)$. A set of vertices $S \subseteq V(H)$ is a member of $\mathcal{P}_G(H)$ if and only if it is a maximal subset of $V(H)$ satisfying that there is no M-saturated path between any two vertices of it.*

Proof. This follows easily from Theorem 2.4.3 and Property 2.2.8. □

The next one is by Proposition 2.3.24 and Lemma 3.6.2.

Lemma 3.6.4. *Let G be a factorizable graph and M be a perfect matching of G, and let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, for any $s \in S$ and $x \in U(S)$, there is no M-saturated path between s and x nor M-balanced path from s to x.*

Proof. (See Figure 3.12.) Suppose the claim fails, that is, there is a path P that is M-balanced from s to x or M-saturated between s and x. Trace P from s and let y be the first vertex we encounter that is in $U(S)$. Trace sPy from y and let z be the first vertex we encounter that is in $V(H)$. Then, since $V(H)$ and $U(S)$ are separating, Property 2.3.1 yields $\delta(V(H)) \cap M = \emptyset$ and $\delta(U(S)) \cap M = \emptyset$; therefore,

zPy is an M-exposed path.

Consequently sPz is an M-saturated path between s and z, which means

$$z \notin S$$

by Proposition 3.6.3.

On the other hand, by Lemma 3.6.2, there is an M-balanced path Q from y to some vertex $t \in S$ whose vertices except t are contained in $U(S)$. Therefore, $zPy+yQt$ is an M-ear relative to H, whose end vertices are z and t; this contradicts Lemma 2.5.1 since $z \not\sim_G t$. □

The next one, Lemma 3.6.5, is rather easy to see by Proposition 2.3.24, and combining it with Lemma 3.6.2 we can obtain Lemma 3.6.6.

Lemma 3.6.5. *Let G be a factorizable graph and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$, and let $u, v \in V(H)$ be such that $u \not\sim_G v$. Let P be an M-saturated path between u and v such that $E(P) \setminus E(H) \neq \emptyset$, and let $P_1, \ldots, P_l$ be the connected components of $P - E(H)$. Let $S_0, S_{l+1} \in \mathcal{P}_G(H)$ be such that $u \in S_0$ and $v \in S_{l+1}$. Then,*

(i) *two end vertices of P_i belong to the same member of $\mathcal{P}_G(H)$, say S_i,*

(ii) *P_i is, except its end vertices, contained in $U(S_i)$ for each $i = 1, \ldots, l$, and*

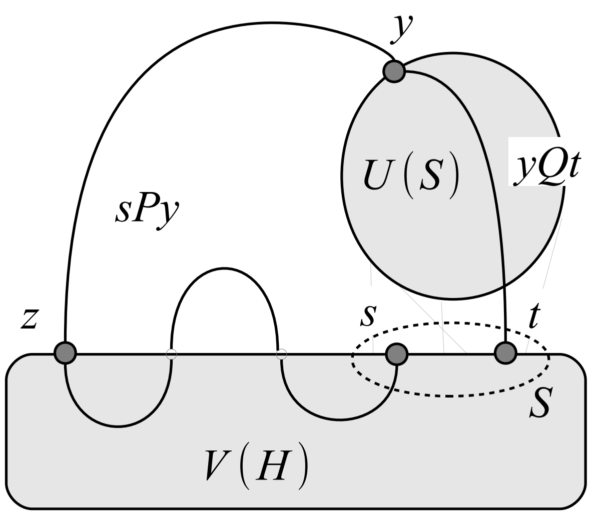

Figure 3.12: The proof of Lemma 3.6.4

(iii) $S_i \neq S_j$ *holds, for any* $i, j \in \{0, \ldots, l+1\}$ *with* $i \neq j$.

Proof. Since $V(H)$ is of course separating, P_i is an M-ear relative to H for each $i = 1, \ldots, l$; therefore, (i) follows by Lemma 2.5.1. Thus, the statement (ii) follows by Theorem 2.3.13. For proving (iii), let x_i and y_i be the end vertices of P_i for each $i = 1, \ldots, l$. Without loss of generality, we can assume that the vertices $u =: y_0, x_1, y_1, \ldots, x_l, y_l, x_{l+1} := v$ appear in this order if we trace P from u. Then, for any i, j with $0 \leq i < j \leq l+1$, y_iPx_j forms an M-saturated path between $y_i \in S_i$ and $x_j \in S_j$. Thus we have $S_i \neq S_j$ by Proposition 3.6.3; this means (iii), and we are done. □

Lemma 3.6.6. *Let G be a factorizable graph and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$, and let $S, T \in \mathcal{P}_G(H)$ be such that $S \neq T$. Then, for any $s \in S$ and $t \in U^*(T)$, there is an M-saturated path between s and t, whose vertices are contained in $U^*(H) \setminus U(S)$.*

Proof. (See Figure 3.13.) By Lemma 3.6.2, there is an M-balanced path P_1 from t to a vertex $x \in T$ whose vertices except x are contained in $U(T)$. By Proposition 3.6.3, there is an M-saturated path P_2 between s and x. By Lemma 3.6.5,

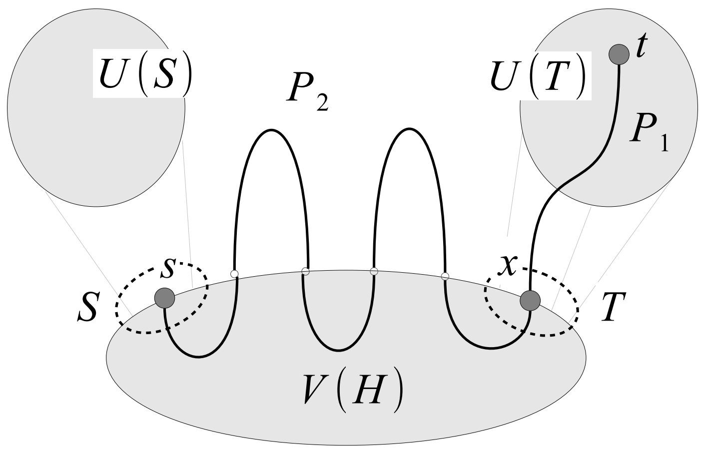

Figure 3.13: The proof of Lemma 3.6.6

$V(P_2)$ is contained in $U^*(H) \setminus U(S) \setminus U(T)$; accordingly, $V(P_1) \cap V(P_2) = \{x\}$. Hence, $P := P_1 + P_2$ is an M-saturated path between s and t, contained in $U^*(H) \setminus U(S)$. $\square$

Lemma 3.6.6 immediately yields the following: Lemma 3.6.7.

Lemma 3.6.7. *Let G be a factorizable graph and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$, and let $S, T \in \mathcal{P}_G(H)$ be such that $S \neq T$. Then, for any $s \in S$ and $t \in U^*(T)$, there is an M-saturated path P between s and t such that for any $u \in S$ and $v \in V(P) \setminus S$ there is an M-saturated path between u and v.*

Theorem 3.6.8. *Let G be a factorizable graph, M be a perfect matching of G, and $u, v \in V(G)$ be such that $G - u - v$ is not factorizable. If there are M-balanced paths respectively from u to v and from v to u, then u and v are in the same factor-component of G.*

Proof. Let P be an M-balanced path from u to v, and Q be an M-balanced path from v to u. Let $x_0, x_1, \dots$ be the sequence of vertices in $V(P) \cap V(Q)$ defined by the following procedure (see Figure 3.14):

1: $x_0 := v$; $i := 0$;
2: **while** $x_i \neq u$ **do**

trace x_iQu from x_i and let x_{i+1} be the first vertex we encounter that is in $V(uPx_i) \setminus \{x_i\}$;

$i++$.

end while

Note that this procedure surely stops in finite time (since at each repetition of the while-loop x_i draws nearer to u) and returns $v = x_0, \ldots, x_l = u$ for some $l \geq 0$. Note also the next claim, which is easy to see by the definition procedure.

Claim 3.6.9. (i) *Tracing P from u, we encounter $x_l, \ldots, x_0$ in this order.*

(ii) *For each $i = 0, \ldots, l-1$, uPx_i and x_iQx_{i+1} have only $\{x_i, x_{i+1}\}$ as common vertices.*

(iii) *For each $i = 0, \ldots, l$, uPx_i and vQx_i has only x_i as a common vertex.*

Proof. By the definition procedure, for each $i = 0, \ldots, l-1$, x_{i+1} is located on P nearer to u than x_i is; this yields (i). The statement (ii) is also apparent from the definition.

For (iii) note that $vQx_i = x_0Qx_1 + \cdots + x_{i-1}Qx_i$. Therefore it suffices to prove that for each $0 \leq j \leq i-1$ only x_i can be a common vertex of uPx_i and x_jQx_{j+1}; this holds true, since $V(uPx_j) \cap V(x_jQx_{j+1}) = \{x_j, x_{j+1}\}$ by (ii), and $V(uPx_i) \subseteq V(uPx_j) \setminus \{x_j, \ldots, x_{i-1}\}$ by (i). □

Claim 3.6.10. *For each $i = 0, \ldots, l-1$, x_iQx_{i+1} is an M-balanced path from x_i to x_{i+1}. For each $i = 0, \ldots, l$, uPx_i and vQx_i are M-balanced paths from u to x_i and from v to x_i, respectively.*

Proof. (See Figure 3.15.) We give it by the induction on i. If $i = 0$ then both of the claims are rather trivially true, and if $i = l$ then the second claim is trivially true. Therefore let $0 < i < l$ and suppose the claims are true for $i-1$, i.e.,

(a) $x_{i-1}Qx_i$ is an M-balanced path from x_{i-1} to x_i,

(b) uPx_{i-1} is an M-balanced path from u to x_{i-1}, and

(c) vQx_{i-1} is an M-balanced path from v to x_{i-1}.

Since $vQx_i = vQx_{i-1} + x_{i-1}Qx_i$ holds, the induction hypotheses (a) and (c) yield that

vQx_i is also an M-balanced path, from v to x_i.

Now suppose the remaining claim fails, namely suppose uPx_i is an M-saturated path between u and x_i. Note that $uPx_i + x_iQv$ forms a path, since they have only x_i as a common vertex by Claim 3.6.9 (iii). Thus, $uPx_i + x_iQv$ is an M-saturated path between u and v; this means $G - u - v$ is factorizable by Property 2.2.8, and we have a contradiction. Therefore,

uPx_i is an M-balanced path from u to x_i.

Hence, we have $x'_i \in V(x_iPv)$. Therefore, by the definition procedure of x_{i+1}, x_iQx_{i+1} is an M-balanced path from x_i to x_{i+1}, and we are done. □

Since

- Claim 3.6.10 states that uPx_i is an M-balanced path for each $i = 0, \ldots, l$, and
- Claim 3.6.9 (ii) yields $uPx_i = uPx_{i+1} + x_{i+1}Px_i$ for each $i = 0, \ldots, l-1$,

it follows that $x_{i+1}Px_i$ is an M-balanced path from x_{i+1} to x_i for each $i = 0, \ldots, l-1$. In addition to this fact, recall that

- x_iQx_{i+1} is an M-balanced path from x_i to x_{i+1} by Claim 3.6.10, and
- x_iQx_{i+1} and $x_{i+1}Px_i$ have only $\{x_i, x_{i+1}\}$ as common vertices by Claim 3.6.9 (ii),

for each $i = 0, \ldots, l-1$. Therefore, $x_iQx_{i+1} + x_{i+1}Px_i$ forms an M-alternating circuit. Therefore, by Property 2.2.9, x_i and x_{i+1} are contained in the same factor-component of G for each $i = 0, \ldots, l-1$. This yields that u and v are contained in the same factor-component. □

Now we are ready to prove the main theorem, combining up the results in this chapter.

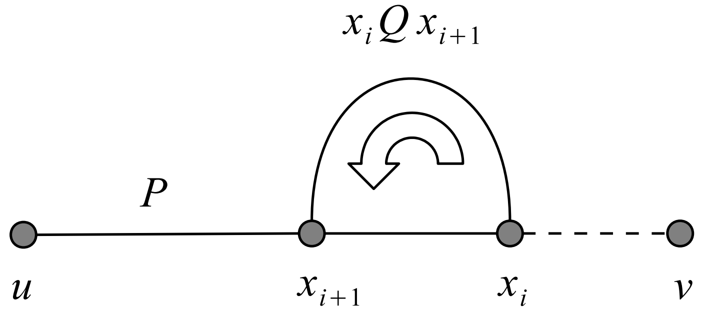

Figure 3.14: The definition procedure of $x_0, x_1, \ldots$ in the proof of Theorem 3.6.8

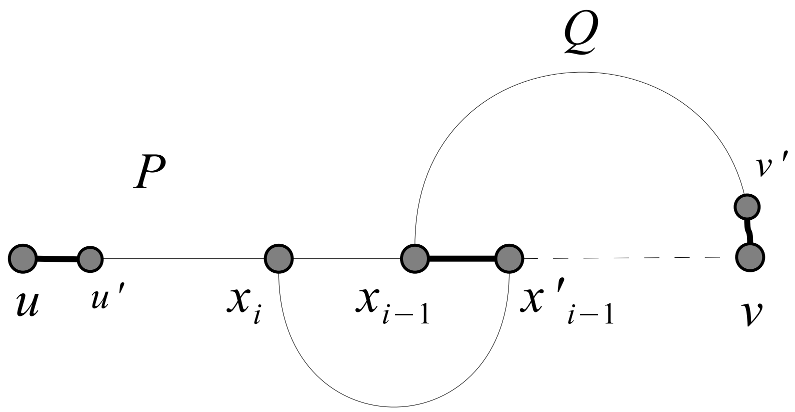

Figure 3.15: The proof of Claim 3.6.10 in Theorem 3.6.8

Theorem 3.6.11. *Let G be a factorizable graph, and X be an odd-maximal barrier of G. Let $D_1, \ldots, D_k$ be the DM-components of $H_G(X)$. Let $\hat{V}_1, \ldots, \hat{V}_k$ be the partition of $X \cup D_X$ such that for each $i = 1, \ldots, k$, $\hat{D}_i := G[\hat{V}_i]$ is the expansion of D_i. Then, for each $i = 1, \ldots, k$, $S_i := X \cap \hat{V}_i$ coincides with a member of $\mathcal{P}_G(H_i)$ for some $H_i \in \mathcal{G}(G)$, and $\hat{V}_i$ coincides with $U^*(H_i) \setminus U(S_i)$.*

Proof. Note that such a partition of $X \cup D_X$ surely exists by Proposition 3.5.15. Let M be a perfect matching of G. Let $i \in \{1, \ldots, k\}$.

Claim 3.6.12. *There is no M-saturated path between any two vertices of S_i.*

Proof. This is immediate from Proposition 3.5.4. □

Claim 3.6.13. *Any vertex of S_i is contained in the same factor-component of G, say H_i.*

Proof. Take $u, v \in S_i$ arbitrarily. Note first that there is no M-saturated path between u and v, by Claim 3.6.12. Additionally, there are M-balanced paths from u to v and from v to u respectively, which is immediate from Theorem 3.5.16 and Proposition 3.4.3. Therefore by Theorem 3.6.8, u and v are contained in the same factor-component. Thus, we have the claim. □

Since Proposition 3.5.15 says $\hat{V}_i$ is separating,

Claim 3.6.14. $V(H_i) \subseteq \hat{V}_i$.

Claim 3.6.15. *For any $u \in S_i$ and any $v \in \hat{V}_i \setminus S_i$, there is an M-saturated path between u and v whose vertices are contained in $\hat{V}_i$.*

Proof. Note that $M_{\hat{V}_i}$ is a perfect matching of $\hat{D}_i$, S_i is an odd-maximal barrier of $\hat{D}_i$, and $H_{\hat{D}_i}(S_i)$ is a factorizable bipartite graph with exactly one DM-component by Proposition 3.5.15. Thus, by applying Theorem 3.5.16 to $\hat{D}_i$, $M_{\hat{V}_i}$ and S_i, there is an M-saturated path of $\hat{D}_i$ between any $u \in S_i$ and any $v \in \hat{V}_i \setminus S_i$. Namely, the claim follows. □

By combining Claims 3.6.12, 3.6.13, 3.6.14, and 3.6.15, we obtain that S_i is a maximal subset of $V(H_i)$ such that there is no M-saturated path between any two vertices of it. Hence, by Proposition 3.6.3, $S_i \in \mathcal{P}_G(H_i)$ holds.

Claim 3.6.16. $\hat{V}_i \supseteq U^*(H_i) \setminus U(S_i)$.

Proof. Take $y \in U^*(H_i) \setminus U(S_i)$ arbitrarily; recall $U^*(H_i) \setminus U(S_i) = S_i \dot{\cup} (U^*(H_i) \setminus U^*(S_i))$. If $y \in S_i$, then of course $y \in \hat{V}_i$. Hence hereafter let $y \in U^*(H_i) \setminus U^*(S_i)$, and let $T \in \mathcal{P}_G(H_i) \setminus \{S_i\}$ be such that $y \in U^*(T)$.

Let $u \in S_i$. There is an M-saturated path P between u and y by Lemma 3.6.6. Hence, by Proposition 3.5.4, $y \in D_X$. Therefore, there exists $j \in \{1, \ldots, k\}$ such that $y \in \hat{V}_j$. By Theorem 3.5.16 and Proposition 3.4.3, $D_i \preceq D_j$.

If $i \neq j$, then by Lemma 3.5.18, P has some internal vertices which belong to S_j. However, by Proposition 3.5.4, there is no M-saturated path between any two vertices respectively in S_i and S_j, and of course $V(P) \cap S_j$ is disjoint from S_i. Namely, every M-saturated path between $u \in S_i$ and $y \in U^*(H_i) \setminus U^*(S_i)$ must traverse a vertex not in S_i, say p, such that there is no M-saturated path between y and p. This contradicts Lemma 3.6.7. Hence, we obtain $i = j$; accordingly, $U^*(H_i) \setminus U(S_i)$ is contained in $\hat{V}_i$. □

Claim 3.6.17. $\hat{V}_i \subseteq U^*(H_i) \setminus U(S_i)$.

Proof. Let $z \in \hat{V}_i \setminus V(H_i)$. By Claim 3.6.15, there is an M-saturated path P between z and some vertex of S_i which is contained in $\hat{V}_i$. Trace P from z and let w be the first vertex we encounter that is in $V(H_i)$. Since $V(H_i)$ is separating, zPw is an M-balanced path from z to w by Property 2.3.1. In $\hat{D}_i/H_i$, zPw corresponds to an M-balanced path from z to the contracted vertex h, corresponding to H_i. Obviously, M contains a near-perfect matching of $\hat{D}_i/H_i$ exposing only h.

Therefore, $\hat{D}_i/H_i$ is factor-critical by Property 2.2.1; accordingly, every factor-components with vertices in $\hat{V}_i$ is an upper bound of H_i with respect to $\triangleleft$ and therefore $\hat{V}_i$ is contained in $U^*(H_i)$. Additionally, by Claim 3.6.15 again and Lemma 3.6.4, we can see that $\hat{V}_i$ is disjoint from $U(S_i)$ and so $\hat{V}_i$ is contained in $U^*(H_i) \setminus U(S_i)$. □

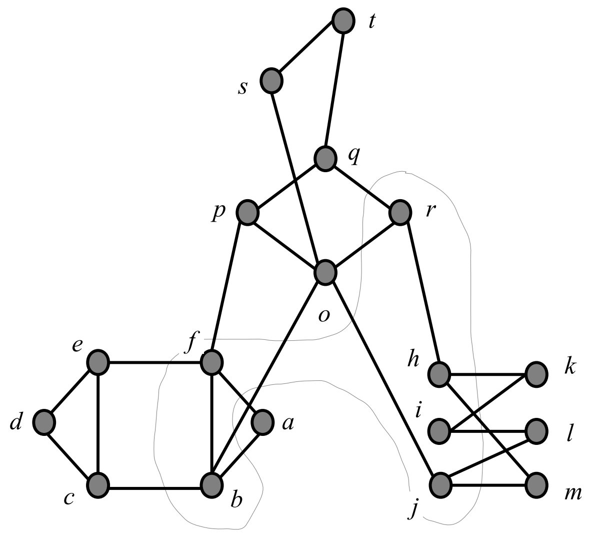

Figure 3.16: A factorizable graph G with an odd-maximal barrier $X = \{b, f, h, i, j, r\}$

Thus, by Claims 3.6.16 and 3.6.17, we have $\hat{V}_i = U^*(H_i) \setminus U(S_i)$. □

Example 3.6.18. The graph in Figure 3.16 is actually isomorphic to the graph in Figure 3.1; recall that the set of vertices $\{b, f, h, i, j, r\}$ is an odd-maximal barrier, X. Its factor-components, the poset $(\mathcal{G}(G), \triangleleft)$, and the generalized canonical partition are indicated in Figures 3.17, 3.18, and 3.19. Figure 3.20 shows that X is indeed the disjoint union of three equivalent classes of the generalized canonical partition; i.e., $X = S_2 \dot{\cup} S_5 \dot{\cup} S_8$. Additionally, the vertices in the odd components of $G - X$, namely D_X, can be also written canonically as the disjoint union of three sets; $D_X = {}^cU(S_2) \dot{\cup} {}^cU(S_5) \dot{\cup} {}^cU(S_8)$, where ${}^cU(S_2) = U^*(G_1) \setminus U^*(S_2)$, ${}^cU(S_5) = U^*(G_2) \setminus U^*(S_5)$, and ${}^cU(S_8) = U^*(G_3) \setminus U^*(S_8)$.

Remark 3.6.19. If G in Theorem 3.6.11 is elementary, then $k = 1$ and $\hat{V}_1 = V(G)$, which follows by Propositions 3.5.1 and 3.5.15. Therefore, in this case, Theorem 3.6.11 claims that $\mathcal{P}(G)$ is the family of (odd-) maximal barriers; namely,

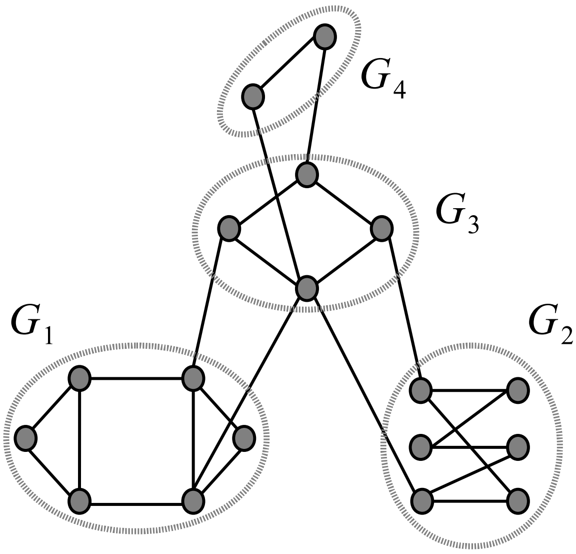

Figure 3.17: The factor-components of G

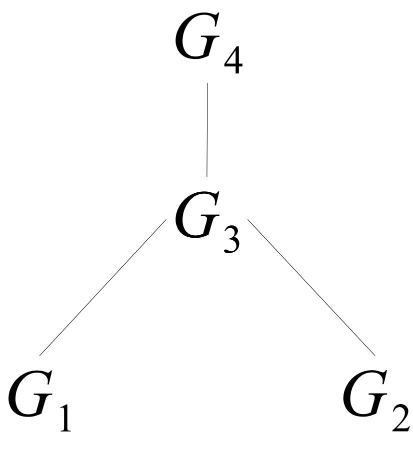

Figure 3.18: The Hasse diagram of the poset $(\mathcal{G}(G), \triangleleft)$

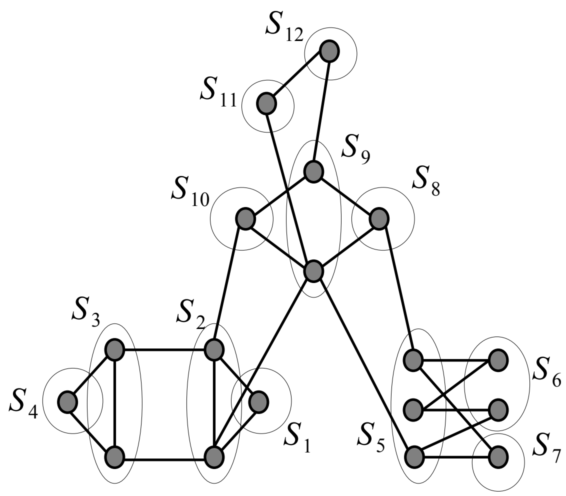

Figure 3.19: The generalized canonical partition of G

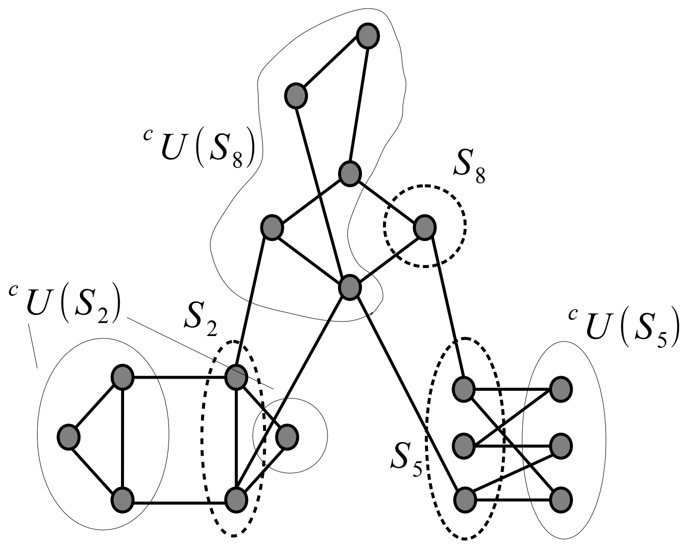

Figure 3.20: The canonical structure of the odd-maximal barrier $X = \{b, f, h, i, j, r\}$

Theorem 3.6.11 coincides with Theorem 1.3.5. Therefore, Theorem 3.6.11 can be regarded as a generalization of Theorem 1.3.5.

Remark 3.6.20. Let G be a factorizable graph. For an arbitrary vertex $x \in V(G)$, take a maximal barrier of $G - x$, say X. Then, $X \cup \{x\}$ is a maximal barrier of G; namely, for any vertex x there is an odd-maximal barrier that contains x. Therefore, for any $S \in \mathcal{P}(G)$, there exists an odd-maximal barrier that contains S.

Remark 3.6.21. With Király [15], if G is a non-factorizable graph, then $\{A(G)\} \cup \mathcal{P}(G[C(G)])$ are the "atoms" that constitute odd-maximal barriers. For each odd-maximal barrier X, the odd components of $G - X$ are the components of $G[D(G)]$ and the odd components of $G[C(G)] - (X \setminus A(G))$; here $G[C(G)]$ forms a factorizable graph and $X \setminus A(G)$ is an odd-maximal barrier.

Chapter 4

Lovász's Cathedral Theorem

4.1 Introduction

A factorizable graph G, with the edge set $E(G)$, is called *saturated* if $G+e$ has more perfect matchings than G for any edge $e \notin E(G)$. There is a constructive characterization of the saturated graphs known as the *cathedral theorem* [28, 30, 35, 36]. Counting the number of perfect matchings is one of the most fundamental enumeration problems, which has applications to physical science, and the cathedral theorem is known to be useful for such a counting problem. For a given factorizable graph, we can obtain a saturated graph which possesses the same family of perfect matchings by adding appropriate edges repeatedly. Many matching-theoretic structural properties are preserved by this procedure. Therefore, we can find several properties on perfect matchings of factorizable graphs using the cathedral theorem, such as relationships between the number of perfect matchings of a given factorizable graph and its structural properties such as its connectivity [30] or the numbers of vertices and edges [14].

The cathedral theorem was originally given by Lovász [28] (see also [30]), and later another proof was given by Szigeti [35, 36]. Lovász's proof is based on the Gallai-Edmonds structure theorem [30], which is one of the most powerful theorem in matching theory. The Gallai-Edmonds structure theorem tells non-trivial structures only for non-factorizable graphs, because it treats factorizable graphs as irreducible. Thus, Lovász proved the cathedral theorem by applying the Gallai-Edmonds structure theorem to non-factorizable subgraphs of saturated graphs.

Szigeti's proof is based on some results on the *optimal ear-decompositions* by Frank [9], which is also based on the Gallai-Edmonds structure theorem and is not a "matching-theory-closed" notion, while the cathedral theorem itself is closed.

The cathedral theorem is outlined as follows:

- There is a constructive characterization of the saturated graphs with an operation called the *cathedral construction*.
- A set of edges of a saturated graph is a perfect matching if and only if it is a disjoint union of perfect matchings of each "component part" of the cathedral construction that creates the saturated graph.
- For each saturated graph, the way to construct it by the cathedral construction uniquely exists.
- There is a relationship between the cathedral construction and the Gallai-Edmonds partition.

In Chapter 2, we introduced canonical structure theorems which tells non-trivial structures for general factorizable graphs. Based on these results, we provide yet another proof of the cathedral theorem in this chapter. The features of the new proof are the following: First, it is quite natural and provides new facts as by-products. The notion of "saturated" is defined by edge-maximality. By considering this edge-maximality over the canonical structures of factorizable graphs, we obtain the new proof in quite a natural way. Therefore, our proof reveals the essential structure that underlies the cathedral theorem, and provides a bit more refined or generalized statements from the point of view of the canonical structure of general factorizable graphs.

Second, it shows that the cathedral theorem can be proved without the Gallai-Edmonds structure theorem nor the notion of barriers, since our previous works, as well as the proofs presented in this chapter, are obtained without them. Even the portion of the statements of the cathedral theorem stating its relationship to the Gallai-Edmonds partition can be obtained without them.

The results in this chapter are also found in papers by the author [16, 22]. The rest of this section is to summarize the succeeding sections in this chapter. In Section 4.2, we show some rather well-known propositions about the Gallai-Edmonds partition, which are distinct from the Gallai-Edmonds structure theorem. In Section 4.3, we present an outline of how we give the new proof of the cathedral theorem. In Section 4.4, we further consider the theorems in Chapter 2 and show one of the new theorems, which later turns out to provide a generalized version of the part of the cathedral theorem regarding the Gallai-Edmonds partition. In Section 4.5, we complete the new proof of the cathedral theorem. Finally, in Section 4.6 we conclude this chapter.

4.2 The Gallai-Edmonds Partition

In this section, we present a proposition which shows another property of the Gallai-Edmonds partition that is different from the Gallai-Edmonds structure theorem. This proposition is a well-known fact that connects the Gallai-Edmonds structure theorem and Edmonds' maximum matching algorithm, and we can find it in [1, 23]. However, this proposition can be proved in an elementary way without using them, nor the notion of barriers. In the following we present it with a proof to confirm it. Note that Proposition 4.2.1 itself is NOT the Gallai-Edmonds structure theorem.

Proposition 4.2.1. *Let G be a graph, M be a maximum matching of G, and S be the set of vertices that are exposed by M. Then, the following hold:*

(i) *A vertex u is in $D(G)$ if and only if there exists $v \in S$ such that there is an M-balanced path from u to v.*

(ii) *A vertex u is in $A(G)$ if and only if there is no M-balanced path from u to any vertex of S, while there exists $v \in S$ such that there is an M-exposed path between u and v.*

(iii) *A vertex u is in $C(G)$ if and only if for any $v \in S$ there is neither an M-balanced path from u to v nor an M-exposed path between u and v.*

Proof. For the necessity part of (i), let P be the M-balanced path from u to v. Then, $M \triangle E(P)$ is a maximum matching of G that exposes u. Thus, $u \in D(G)$.

Now we move on to the sufficiency part of (i). If $u \in D(G) \cap S$, the trivial M-balanced path $(\{u\}, \emptyset)$ satisfies the property. Otherwise, that is, if $u \in D(G) \setminus S$, by the definition of $D(G)$ there is a maximum matching M' of G that exposes u. Then, $G.M \triangle M'$ has a connected component which is an M-balanced path from u to some vertex in S. Hence, we are done for (i). □

For (ii), we first prove the necessity part. Let P be the M-exposed path between u and v, and $w \in V(P)$ be such that $uw \in E(P)$. Then, $P - u$ is an M-balanced path from w to v, which means $w \in D(G)$ by (i). Then, we have $u \in A(G)$, since the first part of the condition on P yields $u \notin D(G)$ by (i).

Now we move on to the sufficiency part of (ii). Note that the first part of the conclusion follows by (i). By the definition of $A(G)$, there exists $w \in D(G)$ such that $wu \in E(G)$. By (i), there is an M-balanced path Q from w to a vertex $v \in S$. If $u \in V(Q)$, then since $u \notin D(G)$, the subpath of Q from v to u is an M-exposed path between v and u by (i). Thus, the claim follows. Otherwise, that is, if $u \notin V(Q)$, then $Q + wu$ forms an M-exposed path between v and u. Therefore, again the claim follows. Thus, we are done for (ii). □

Since we obtain (i) and (ii), consequently (iii) follows. □

The next proposition is also known (see [1]) and is easily obtained from Proposition 4.2.1.

Proposition 4.2.2. *Let G be a factorizable graph and M be a perfect matching of G. Then, for any $x \in V(G)$, the following hold:*

(i) *A vertex u is in $D(G-x)$ if and only if there is an M-saturated path between x and u.*

(ii) *A vertex u is in $A(G-x) \cup \{x\}$ if and only if there is no M-saturated path between x and u, while there is an M-balanced path from x to u.*

(iii) *A vertex u is in $C(G-x)$ if and only if there is neither an M-saturated path between u and x nor an M-balanced path from x to u.*

Proof. Let $x' \in V(G)$ be such that $xx' \in M$. Let $G' := G-x$ and $M' := M\setminus\{xx'\}$. Note that apparently

$$M' \text{ is a maximum matching of } G', \text{ exposing only } x'.$$

By Propositions 4.2.1, $u \in D(G')$ if and only if there is an M'-balanced path from u to x'. Additionally, the following apparently holds: there is an M'-balanced path from u to x' in G' if and only if there is an M-saturated path between u and x in G. Thus, we obtain (i). The other claims, (ii) and (iii), also follow by similar arguments. □

Proposition 4.2.2 associates factorizable graphs with the Gallai-Edmonds partition, and it will be used later in the proof of Theorem 4.4.1. Hence it will contribute to the new proof of the cathedral theorem.

4.3 Outline of the New Proof

Here we give an outline of how we give a new proof of the cathedral theorem together with backgrounds of the theorem. In Chapter 2, we introduced a new canonical decomposition of factorizable graphs. The key points of it are as follows.

(a) For a factorizable graph G, a partial order $\lhd$ can be defined on the factor-connected components $\mathcal{G}(G)$ (Theorem 2.3.29).

(b) An equivalence relation $\sim_G$ based on factor-connected components can be defined on $V(G)$ (Theorem 2.4.3). The equivalence classes by $\sim_G$ can be regarded as a generalization of Kotzig's canonical partition [24–26].

(c) These two notions $\lhd$ and $\sim_G$ are related each other in the sense that for $H \in \mathcal{G}(G)$ a relationship between H and its strict upper bounds in the poset $(\mathcal{G}(G), \lhd)$ can be described using $\sim_G$ (Theorem 2.5.2).

In Section 4.4, we begin to present new results in this paper. We further consider the structures given by (a) (b) (c) and show a relationship between the structures and the Gallai-Edmonds partition:

> If the poset $(\mathcal{G}(G), \triangleleft)$ of a factorizable graph G has the minimum element G_0, then $V(G_0) = V(G) \setminus \bigcup_{x \in V(G)} C(G - x)$ (Theorem 4.4.1).

This theorem later plays a crucial role in the new proof of the cathedral theorem.

In Section 4.5, we consider saturated graphs and present a new proof of the cathedral theorem. Given a saturated elementary graph and a family of saturated graphs satisfying a certain condition, we can define an operation, the *cathedral construction*, that creates a new graph obtained from the given graphs by adding new edges. Here the given graphs are called the *foundation* and the family of *towers*, respectively. We consider the canonical decomposition in Chapter 2 for saturated graphs and obtain the following:

> If G is a saturated graph, then the poset $(\mathcal{G}(G), \triangleleft)$ has the minimum element G_0 (Lemma 4.5.7).

Moreover, G_0 and all connected components of $G - V(G_0)$ are saturated and they are well-defined as a foundation and towers (Lemmas 4.5.9 and 4.5.11). We show that G is the graph obtained from them by the cathedral construction (Theorem 4.5.3).

> Conversely, if a graph G obtained by the cathedral construction from a foundation G_0 and some towers is saturated, and G_0 is the minimum element of the poset $(\mathcal{G}(G), \triangleleft)$ (Theorem 4.5.4).

By Theorems 4.5.3 and 4.5.4, the constructive characterization of the saturated graphs—the most important part of the cathedral theorem—is obtained. Additionally, the other parts of the cathedral theorem follow quite smoothly by Theorem 4.4.1 and the natures of the canonical decomposition in Chapter 2.

4.4 Factorizable Graphs through the Gallai-Edmonds Partition

In this section, we present a new result on a relationship between the Gallai-Edmonds partition and the canonical structures of factorizable graphs in Chapter 2. As we later see in Section 4.5, Theorem 4.4.1 can be regarded as a generalization of a part of the statements of the cathedral theorem.

Theorem 4.4.1. *Let G be a factorizable graph such that the poset $(\mathcal{G}(G), \triangleleft)$ has the minimum element G_0. Then, $V(G_0)$ is exactly the set of vertices that is disjoint from $C(G-x)$ for any $x \in V(G)$; that is, $V(G_0) = V(G) \setminus \bigcup_{x \in V(G)} C(G-x)$.*

To show Theorem 4.4.1, we give some lemmas.

In the following lemma, we present a structure of factorizable graphs, combining up some results in previous Chapters.

Lemma 4.4.2. *Let G be a factorizable graph, and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$, $S \in \mathcal{P}_G(H)$, and $T \in \mathcal{P}_G(H) \setminus \{S\}$.*

(i) *For any $u \in U^*(S)$, there is an M-balanced path from u to some vertex $v \in S$ whose vertices except v are in $U(S)$.*

(ii) *For any $u \in S$ and $v \in U^*(T)$, there is an M-saturated path between u and v whose vertices are all contained in $U^*(H) \setminus U(S)$.*

(iii) *For any $u \in S$ and $v \in U(S)$, there are neither M-saturated paths between u and v nor M-balanced paths from u to v.*

(iv) *For any $u, v \in S$, there is no M-saturated path between u and v, while there is an M-balanced path from u to v.*

Proof. The statements (i), (ii), and (iii) are stated in Lemma 3.6.2, Lemma 3.6.6, Lemma 3.6.4, respectively. The statement (iv) is immediately obtained by combining Fact 2.4.2 and Proposition 2.3.4. □

By Proposition 2.3.4 and Lemma 4.4.2, the next lemma follows.

Lemma 4.4.3. *Let G be a factorizable graph, and M be a perfect matching of G. Let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, the following hold:*

(i) *For any $u \in U(S)$ and $v \in U^*(H) \setminus U^*(S)$, there is an M-saturated path between u and v.*

(ii) *For any $u \in U(S)$ and $v \in S$, there is no M-saturated path between u and v; however, there is an M-balanced path from u to v.*

(iii) *For any $w \in S$ and $v \in U^*(H) \setminus U^*(S)$, there is an M-saturated path between w and v.*

(iv) *For any $w, v \in S$, there is no M-saturated path between w and v; however, there is an M-balanced path from w and v.*

(v) *For any $w \in S$ and $v \in U(S)$, there is neither an M-saturated path between w and v nor an M-balanced path from w to v.*

Proof. The statements (iii), (iv), and (v) are immediate from (ii), (iv), and (iii) of Lemma 4.4.2, respectively.

For (i), let P_1 be an M-balanced path from u to some vertex $x \in S$ such that $V(P_1) \setminus \{x\} \subseteq U(S)$, given by (i) of Lemma 4.4.2. By (ii) of Lemma 4.4.2, there is an M-saturated path P_2 between x and v such that $V(P_2) \subseteq U^*(H) \setminus U(S)$. Hence, the path obtained by adding P_1 and P_2 forms an M-saturated path between u and v, and (i) follows.

The first and the latter halves of (ii) are restatements of (iii) and (i) of Lemma 4.4.2, respectively. □

By comparing Proposition 4.2.2 and Lemma 4.4.3, the next lemma follows.

Lemma 4.4.4. *Let G be a factorizable graph such that the poset $(\mathcal{G}(G), \triangleleft)$ has the minimum element G_0. Let $S \in \mathcal{P}_G(G_0)$.*

(i) *If $x \in U(S)$, then $D(G - x) \supseteq U^*(G_0) \setminus U^*(S)$, $A(G - x) \cup \{x\} \supseteq S$, and $C(G - x) \subseteq U(S)$.*

(ii) *If* $x \in S$*, then* $D(G-x) = U^*(G_0) \setminus U^*(S)$*,* $A(G-x) \cup \{x\} = S$*, and* $C(G-x) = U(S)$*.*

Proof. The claims are all obtained by comparing the reachabilities of alternating paths regarding Proposition 4.2.2 and Lemma 4.4.3. Let $x \in U(S)$. By Proposition 4.2.2 (i) and Lemma 4.4.3 (i), we have $D(G-x) \supseteq U^*(G_0) \setminus U^*(S)$. It also follows that $A(G-x) \cup \{x\} \supseteq S$ by a similar argument, comparing Proposition 4.2.2 (ii) and Lemma 4.4.3 (ii). Therefore, since $V(G) = D(G) \dot\cup A(G) \dot\cup C(G) = (U^*(G_0) \setminus U^*(S)) \dot\cup S \dot\cup U(S)$, we have $C(G-x) \subseteq U(S)$, and we are done for (i). The statement (ii) also follows by similar arguments with Proposition 4.2.2 and Lemma 4.4.3 (iii) (iv) (v). □

Now we can prove Theorem 4.4.1 using Lemma 4.4.4.

Proof of Theorem 4.4.1.

Claim 4.4.5. *For any* $x \in V(G)$*,* $V(G_0) \cap C(G-x) = \emptyset$*.*

Proof. Let $u \in V(G_0)$ and let $S \in \mathcal{P}_G(G_0)$ be such that $u \in S$. By Lemma 4.4.4, if $x \in U^*(S)$ then $u \in A(G-x)$, and if $x \in U^*(G_0) \setminus U^*(S)$ then $u \in D(G-x)$. Thus, anyway we have $u \notin C(G-x)$, and the claim follows. □

Claim 4.4.6. *For any* $u \in V(G) \setminus V(G_0)$*, there exists* $x \in V(G)$ *such that* $u \in C(G-x)$*.*

Proof. Let $u \in V(G) \setminus V(G_0)$ and let $S \in \mathcal{P}_G(G_0)$ be such that $u \in U(S)$. Then, for any $x \in S$, we have $u \in C(G-x)$ by Lemma 4.4.4. Thus, we have the claim. □

By Claims 4.4.5 and 4.4.6, we obtain the theorem. □

We will obtain in Section 4.5 that if a graph is saturated then the poset by $\lhd$ has the minimum element. Thus, the above theorem, Theorem 4.4.1, will turn out to be regarded as a generalized version of the part of the cathedral theorem related to the Gallai-Edmonds partition.

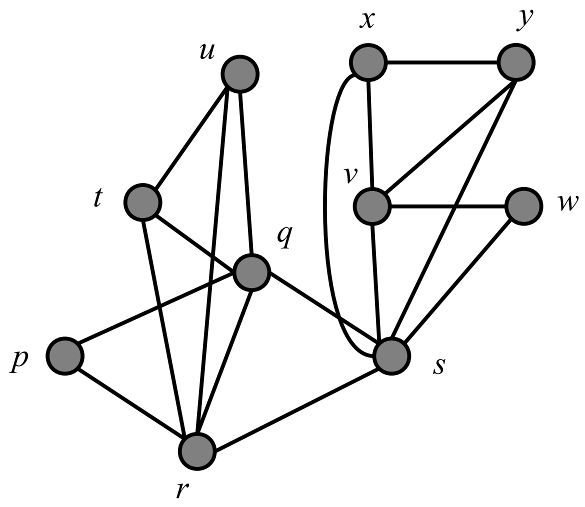

Figure 4.1: A saturated graph $\tilde{G}$

4.5 Another Proof of the Cathedral Theorem

4.5.1 The Cathedral Theorem

In this section, we give yet another proof as a consequence of the structures given in Chapter 2. For convenience, we treat empty graphs as factorizable and saturated.

Definition 4.5.1 (The Cathedral Construction)**.** Let G_0 be a saturated elementary graph and let $\{G_S\}_{S\in\mathcal{P}(G_0)}$ be a family of saturated graphs, some of which might be empty. For each $S \in \mathcal{P}(G_0)$, join every vertex in S and every vertex of G_S. We call this operation the *cathedral construction*. Here G_0 and $\{G_S\}_{S\in\mathcal{P}(G_0)}$ are respectively called the *foundation* and the family of *towers*.

Figures 4.1, 4.2, 4.3 show examples of the cathedral construction. In Figure 4.2, the graph G_0 is an elementary saturated graph with the canonical partition $\mathcal{P}(G_0) = \{S, T, R\}$, and the graphs G_S, G_T, G_R are saturated graphs such that G_S and G_R are respectively elementary and non-elementary while G_T is an empty graph. If we conduct the cathedral construction with the foundation G_0 and the family of towers $\mathcal{T} = \{G_S, G_T, G_R\}$, we obtain the saturated graph $\tilde{G}$

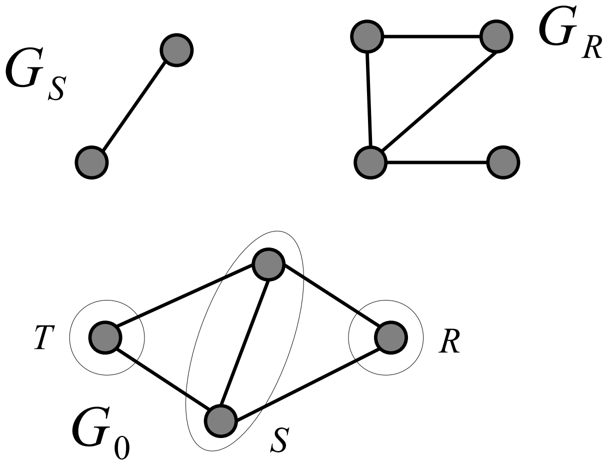

Figure 4.2: The foundation and the towers that create $\tilde{G}$

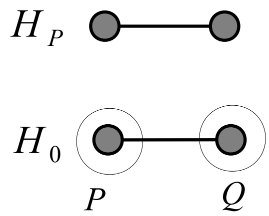

Figure 4.3: The foundation and the towers that create G_R

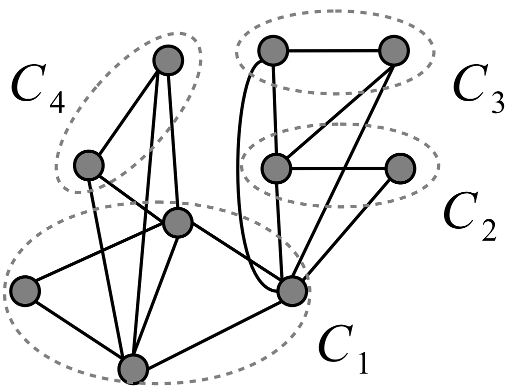

Figure 4.4: The factor-connected components of $\tilde{G}$

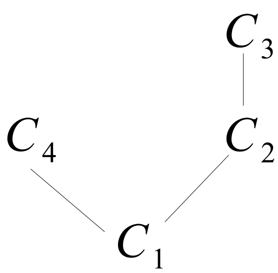

Figure 4.5: The Hasse diagram of $(\mathcal{G}(\tilde{G}), \triangleleft)$

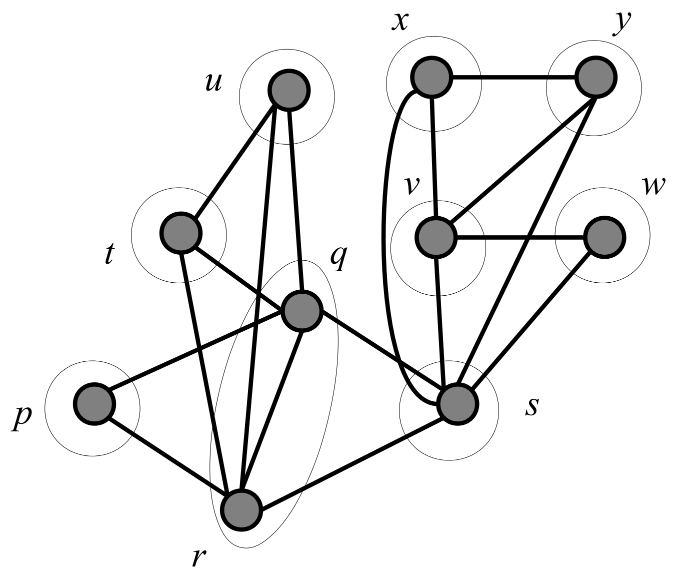

Figure 4.6: The generalized canonical partition of $\tilde{G}$

in Figure 4.1. Moreover, Figure 4.3 shows that if we conduct the cathedral construction with the foundation H_0 with $\mathcal{P}(H_0) = \{P, Q\}$ and the family of towers $\{H_P, H_Q\}$, where H_P is an elementary saturated graph and H_Q is an empty graph, then we obtain the saturated graph G_R. (Therefore, in other words, the graph $\tilde{G}$ is constructed by a repetition of the cathedral construction using the elementary saturated graphs H_0, H_P, G_0, and G_S as fundamental building blocks.)

Theorem 4.5.2 (The Cathedral Theorem [28,30])**.** *A factorizable graph G is saturated if and only if it is constructed from smaller saturated graphs by the cathedral construction. In other words, if a factorizable graph G is saturated, then there is a subgraph G_0 and a family of subgraphs $\mathcal{T}$ of G which are well-defined as a foundation and a family of towers, and G is the graph constructed from G_0 and $\mathcal{T}$ by the cathedral construction; conversely, if G is a graph obtained from a foundation and towers by the cathedral construction, then G is saturated.*

Additionally, if G is a saturated graph obtained from a foundation G_0 and a *family of towers* $\mathcal{T} = \{G_S\}_{S\in\mathcal{P}(G_0)}$ *by the cathedral construction, then,*

(i) $e \in E(G)$ *is allowed if and only if it is an allowed edge of* G_0 *or* G_S *for some* $S \in \mathcal{P}(G_0)$*,*

(ii) *such* G_0 *uniquely exists; that is, if* G *can be obtained from a foundation* G'_0 *and a family of towers* $\mathcal{T}'$ *by the cathedral construction, then* $V(G_0) = V(G'_0)$ *holds, and*

(iii) $V(G_0)$ *is exactly the set of vertices that is disjoint from* $C(G-x)$ *for any* $x \in V(G)$*.*

In the cathedral construction, each tower is saturated. Therefore, the first sentence of Theorem 4.5.2 reveals a nested or inductive structure and gives a constructive characterization of the saturated graphs by the cathedral construction. In this characterization, the elementary saturated graphs are the fundamental building blocks. Theorem 4.5.2 (i) tells that a set of edges in a saturated graph G is a perfect matching if and only if it is a disjoint union of perfect matchings of the foundation and the towers that create G. Theorem 4.5.2 (ii) tells that for each saturated graph, the way to construct it uniquely exists, and (iii) shows a relationship between the cathedral construction and the Gallai-Edmonds partition.

In the new proof, the following two theorems, Theorems 4.5.3 and 4.5.4, together with Theorem 4.4.1, will serve as nuclei, referring to the special features of the poset and the canonical partition for saturated graphs.

Theorem 4.5.3. *If a factorizable graph* G *is saturated, then the poset* $(\mathcal{G}(G), \lhd)$ *has the minimum element, say* G_0*, and it satisfies* $\mathcal{P}_G(G_0) = \mathcal{P}(G_0) =: \mathcal{P}_0$*. Additionally, for each* $S \in \mathcal{P}_0$*, the connected component* G_S *of* $G - V(G_0)$ *such that* $N_G(G_S) \subseteq S$ *exists uniquely or is an empty graph, and* G *is the graph obtained from the foundation* G_0 *and the family of towers* $\mathcal{T} := \{G_S\}_{S\in\mathcal{P}_0}$ *by the cathedral construction.*

Theorem 4.5.4. *Let* G_0 *be a saturated elementary graph, and* $\mathcal{T} := \{G_S\}_{S\in\mathcal{P}(G_0)}$ *be a family of saturated graphs. Let* G *be the graph obtained from the foundation*

G_0 and the family of towers $\mathcal{T}$ by the cathedral construction. Then, G is saturated, G_0 forms a factor-connected component of G, that is, $G[V(G_0)] \in \mathcal{G}(G)$, and it is the minimum element of the poset $(\mathcal{G}(G), \triangleleft)$.

In the remaining part of this chapter, we are going to prove Theorem 4.5.3 and Theorem 4.5.4 and then obtain Theorem 4.5.2. With Theorem 4.5.3 and Theorem 4.5.4, we obtain the constructive characterization of the saturated graphs. We also obtain a new characterization of foundations and families of towers, which gives a clear comprehension of saturated graphs by the canonical structures of factorizable graphs in Chapter 2. Thanks to this new characterization, the remaining statements of the cathedral theorem will be obtained quite smoothly.

4.5.2 Proof of Theorem 4.5.3

Here we show some lemmas etc. to show that any saturated graph is constructed by the cathedral construction and prove Theorem 4.5.3. Hereafter note the following properties, which will be used frequently, sometimes without explicitly mentioning it.

Property 4.5.5. *Let G be a factorizable graph, M be a perfect matching, and $x, y \in V(G)$ be such that $xy \notin E(G)$. Then, the following properties are equivalent:*

(i) *The complement edge xy creates a new perfect matching in $G + xy$.*

(ii) *The edge xy is allowed in $G + xy$.*

(iii) *There is an M-saturated path between x and y in G.*

Property 4.5.6. *Let G be a factorizable graph and M be a perfect matching of G. Let $X \subseteq V(G)$ be a separating set and P be an M-saturated path. Then,*

(i) *each connected component of $P[X]$ is an M-saturated path, and*

(ii) *any connected component of $P - E(G[X])$ that does not contain any end vertices of P is an M-ear relative to X.*

Lemma 4.5.7. *If a factorizable graph G is saturated, then the poset $(\mathcal{G}(G), \triangleleft)$ has the minimum element.*

Proof. Suppose the claim fails, that is, the poset has distinct minimal elements $G_1, G_2 \in \mathcal{G}(G)$. Then, by Theorem 2.5.9, there exist possibly identical complement edges e, f joining $V(G_1)$ and $V(G_2)$ such that $\mathcal{G}(G + e + f) = \mathcal{G}(G)$. This means that adding e or f to G does not create any new perfect matchings, which contradicts G being saturated. □

In order to obtain Theorem 4.5.3, by letting G be a saturated graph, we show in the following that the minimum element G_0 of the poset by $\triangleleft$ and the connected components of $G - V(G_0)$ are well-defined as a foundation and towers of the cathedral construction and G is the graph obtained by the cathedral construction with them.

The next fact is easy to see from Fact 2.4.2 and Property 4.5.5. We will use this fact in the proofs of Lemma 4.5.9 and Lemma 4.5.11 later.

Fact 4.5.8. *Let G be a saturated graph, and let $H \in \mathcal{G}(G)$. Then, for any $u, v \in V(H)$ with $u \sim_G v$, $uv \in E(G)$.*

Next, we give the following lemma, which will contribute to the proofs of both of Theorems 4.5.3 and 4.5.4, actually.

Lemma 4.5.9. *Let G be a saturated graph, and let $G_0 \in \mathcal{G}(G)$. Then, $\mathcal{P}_G(G_0) = \mathcal{P}(G_0)$.*

Proof. Since we know by Fact 2.4.4 that $\mathcal{P}_G(G_0)$ is a refinement of $\mathcal{P}(G_0)$, it suffices to prove that $\mathcal{P}(G_0)$ is a refinement of $\mathcal{P}_G(G_0)$, that is, if $u \sim_{G_0} v$, then $u \sim_G v$. We prove the contrapositive of this.

Let $u, v \in V(G_0)$ with $u \not\sim_G v$. Let M be a perfect matching of G. By Fact 2.4.2, there are M-saturated paths between u and v; let P be a shortest one. Suppose $E(P) \setminus E(G_0) \neq \emptyset$, and let Q be one of the connected components of $P - E(G_0)$, with end vertices x and y. Since Q is an M-ear relative to G_0 by Property 4.5.6, $x \sim_G y$ follows by Lemma 2.5.1. Therefore, $xy \in E(G)$ by

Fact 4.5.8, which means we can get a shorter M-saturated path between u and v by replacing Q by xy on P, a contradiction. Thus, we have $E(P) \setminus E(G_0) = \emptyset$; that is, P is a path of G_0. Accordingly, $u \not\sim_{G_0} v$ by Fact 2.4.2. □

As we mention in Fact 2.4.4, for a factorizable graph G and $H \in \mathcal{G}(G)$, $\mathcal{P}_G(H)$ is generally a refinement of $\mathcal{P}(H)$. However, the above lemma states that if G is a saturated graph then they coincide. Therefore this lemma associates the generalized canonical partition with the cathedral theorem.

Next, note the following fact, which we present to prove Lemma 4.5.11:

Fact 4.5.10. *If a factorizable graph G is saturated, G is connected.*

Proof. Suppose the claim fails, that is, G has two distinct connected components, K and L. Let $u \in V(K)$ and $v \in V(L)$, and let M be a perfect matching of G. By Property 4.5.5, there is an M-saturated path between u and v, contradicting the hypothesis that K and L are distinct. □

Before reading Lemma 4.5.11, note that if a factorizable graph G has the minimum element G_0 for the poset $(\mathcal{G}(G), \triangleleft)$, then for each connected component K of $G - V(G_0)$, $N_G(K) \subseteq V(G_0)$ holds.

Lemma 4.5.11. *Let G be a saturated graph, and G_0 be the minimum element of the poset $(\mathcal{G}(G), \triangleleft)$. Then, G_0 and the connected components of $G - V(G_0)$ are each saturated. Additionally, for each $S \in \mathcal{P}_G(G_0)$, a connected component K of $G - V(G_0)$ such that $N_G(K) \subseteq S$ exists uniquely or does not exist.*

Proof. We first prove that G_0 is saturated. Let $e = xy$ be a complement edge of G_0. By the contrapositive of Fact 4.5.8, $x \not\sim_G y$, which means $x \not\sim_{G_0} y$ by Lemma 4.5.9. Therefore, by Fact 2.4.2 and Property 4.5.5, the complement edge e creates a new perfect matching if it is added to G_0. Hence, G_0 is saturated.

Now we move on to the remaining claims. Take $S \in \mathcal{P}_G(G_0)$ arbitrarily, and let $K_1, \ldots, K_l$ be the connected components of $G - V(G_0)$ which satisfy $N_G(K_i) \subseteq S$ for each $i = 1, \ldots, l$. Let $\hat{K} := G[V(K_1) \dot\cup \cdots \dot\cup V(K_l)]$.

We are going to obtain the remaining claims by showing that $\hat{K}$ is saturated. Now let $e = xy$ be a complement edge of $\hat{K}$, i.e., $x, y \in V(\hat{K})$ and $xy \notin E(\hat{K})$. Let M be a perfect matching of G. With Property 4.5.5, in order to show that $\hat{K}$ is saturated it suffices to prove that there is an M-saturated path between x and y in $\hat{K}$. Since G is saturated, there is an M-saturated path P between x and y in G by Property 4.5.5.

Obviously by the definition, $N_G(\hat{K}) \subseteq S$; on the other hand, $V(G) \setminus V(\hat{K})$ is of course a separating set. Therefore, if $E(P) \setminus E(\hat{K}) \neq \emptyset$, each connected component of $P - V(\hat{K})$ is an M-saturated path, both of whose end vertices are contained in S, by Property 4.5.6. This contradicts Fact 2.4.2. Hence, $E(P) \subseteq E(\hat{K})$, which means $\hat{K}$ is itself saturated. Thus, by Fact 4.5.10, it follows $\hat{K}$ is connected, which is equivalent to $l = 1$. This completes the proof. □

By Lemma 4.5.9 and Lemma 4.5.11, it follows that G_0 is well-defined as a foundation and the connected components of $G - V(G_0)$ are well-defined as towers (of course if indices out of $\mathcal{P}(G_0)$ are assigned to them appropriately).

Lemma 4.5.12. *Let G be a saturated graph, and G_0 be the minimum element of the poset $(\mathcal{G}(G), \triangleleft)$, and let K be a connected component of $G - V(G_0)$, whose neighbors are in $S \in \mathcal{P}_G(G_0)$. Then, for any $u \in V(K)$ and for any $v \in S$, $uv \in E(G)$.*

Proof. Suppose the claim fails, that is, there are $u \in V(K)$ and $v \in S$ such that $uv \notin E(G)$. Then, by Property 4.5.5, there is an M-saturated path between u and v, where M is an arbitrary perfect matching of G. By the definitions, $V(K) \subseteq U(S)$; therefore, $u \in U(S)$. Hence, this contradicts (iii) of Lemma 4.4.2, and we have the claim. □

Now we are ready to prove Theorem 4.5.3:

Proof of Theorem 4.5.3. The first sentence of Theorem 4.5.3 is immediate from Lemma 4.5.7 and Lemma 4.5.9. The former of the second sentence is also immediate by Lemma 4.5.11.

For the remaining claim, first note that by Lemma 4.5.11, G_0 and any G_S are saturated. Therefore, G_0 and $\mathcal{T} = \{G_S\}_{S \in \mathcal{P}_0}$ are well-defined as a foundation and a family of towers of the cathedral construction.

By the definition, for each $S \in \mathcal{P}_0$, it follows that $N_G(G_S) \subseteq S$. Additionally by Lemma 4.5.12 every vertex of $V(G_S)$ and every vertex of S are joined. Therefore, it follows that G has a saturated subgraph G' obtained from G_0 and $\mathcal{T}$ by the cathedral construction. Moreover, by Theorem 2.5.2, for each connected component K of $G - V(G_0)$ there exists $S \in \mathcal{P}_0$ such that $N(K) \subseteq S$; in other words, K denotes the same subgraph of G as G_S. Hence, $V(G) = V(G_0) \cup \bigcup_{S \in \mathcal{P}_0} V(G_S)$ holds and actually G' is G. Thus, G is the graph obtained from G_0 and $\mathcal{T}$ by the cathedral construction. □

4.5.3 Proof of Theorem 4.5.4

Next we consider the graphs obtained by the cathedral construction and show Theorem 4.5.4, which states that the foundations of them are the minimum elements of the posets by $\triangleleft$.

Since the necessity of the first claim of Theorem 4.5.2, the next proposition, is not so hard (see [30]), we here present it without a proof.

Proposition 4.5.13 (Lovász [28, 30]). *Let G_0 be a saturated elementary graph, and $\mathcal{T} = \{G_S\}_{S \in \mathcal{P}(G_0)}$ be a family of saturated graphs. Then, the graph G obtained from the foundation G_0 and the family of towers $\mathcal{T}$ by the cathedral construction is saturated.*

We give one more lemma:

Lemma 4.5.14. *Let G be a saturated graph, obtained from the foundation G_0 and the family of towers $\{G_S\}_{S \in \mathcal{P}(G_0)}$ by the cathedral construction. Then, $G' := G/V(G_0)$ is factor-critical.*

Proof. Let M^S be a perfect matching of G_S for each $S \in \mathcal{P}(G_0)$, and let $M := \bigcup_{S \in \mathcal{P}(G_0)} M^S$. Then, M forms a near-perfect matching of G', exposing only the contracted vertex g_0 corresponding to $V(G_0)$. Take $u \in V(G') \setminus \{g_0\}$ arbitrarily and let u' be the vertex such that $uu' \in M$. Since $uu' \in M \cap E(G')$ and $u'g_0 \in E(G') \setminus M$, there is an M-balanced path from u to g_0 in G', namely, the one with edges $\{uu', u'g_0\}$. Thus, by Property 2.2.1, G' is factor-critical. □

Now we shall prove Theorem 4.5.4:

Proof of Theorem 4.5.4. By Proposition 4.5.13, G is saturated. Since we have Lemma 4.5.14, in order to complete the proof, it suffices to prove $G_0 \in \mathcal{G}(G)$. Let p be the number of non-empty graphs in $\mathcal{T}$. We proceed by induction on p. If $p = 0$, the claim obviously follows. Let $p > 0$ and suppose the claim is true for $p-1$. Take a non-empty graph G_S from $\mathcal{T}$, and let $G' := G - V(G_S)$. Then, G' is the graph obtained by the cathedral construction with G_0 and $\mathcal{T} \setminus \{G_S\} \cup \{H_S\}$, where H_S is an empty graph. Therefore, Proposition 4.5.13 yields that G' is saturated, and the induction hypothesis yields that $G_0 \in \mathcal{G}(G')$ and G_0 is the minimum element of the poset $(\mathcal{G}(G'), \triangleleft)$. Thus, by Lemma 4.5.9,

Claim 4.5.15. $\mathcal{P}_{G'}(G_0) = \mathcal{P}(G_0)$.

Let M' be a perfect matching of G' and M^S be a perfect matching of G_S, and construct a perfect matching $M := M' \cup M^S$ of G.

Claim 4.5.16. *No edge of $E_G[S, V(G_S)]$ is allowed in G.*

Proof. Suppose the claim fails, that is, an edge $xy \in E_G[S, V(G_S)]$ is allowed in G. Then, there is an M-saturated path Q between x and y by Property 2.2.7, and $Q[V(G')]$ is an M-saturated path by Property 4.5.6. Moreover, since $N_G(G_S) \cap V(G') \subseteq S$, it follows that $Q[V(G')]$ is an M-saturated path of G' between two vertices in S. With Fact 2.4.2 this is a contradiction, because $S \in \mathcal{P}_{G'}(G_0)$ by Claim 4.5.15. Hence, we have the claim. □

By Claim 4.5.16, it follows that a set of edges is a perfect matching of G if and only if it is a disjoint union of a perfect matching of G' and G_S. Thus, G_0 forms a factor-connected component of G, and we are done. □

4.5.4 Proof of Theorem 4.5.2 and an Example

Now we can prove the cathedral theorem, combining Theorems 4.5.3, 4.5.4, and 4.4.1:

Proof of Theorem 4.5.2. By Proposition 4.5.13 and Theorem 4.5.3, the first claim of Theorem 4.5.2 is proved. The statement (i) is by Theorem 4.5.4, since it states that $G_0 \in \mathcal{G}(G)$. The statement (ii) is also by Theorem 4.5.4, since the poset $(\mathcal{G}(G), \triangleleft)$ is a canonical notion. The statement (iii) is by combining Theorem 4.5.4 and Theorem 4.4.1. $\square$

Example 4.5.17. The graph $\tilde{G}$ in Figure 4.1 consists of four factor-connected components, say $C_1, \dots, C_4$ in Figure 4.4, and Figure 4.5 shows the Hasse diagram of $(\mathcal{G}(\tilde{G}), \triangleleft)$, which has the minimum element C_1, as stated in Lemma 4.5.7. Figure 4.6 indicates the generalized canonical partition of $\tilde{G}$:

$$\mathcal{P}(\tilde{G}) = \{\{p\}, \{q, r\}, \{s\}, \{t\}, \{u\}, \{v\}, \{w\}, \{x\}, \{y\}\}.$$

Here we have $\mathcal{P}_{\tilde{G}}(C_i) = \mathcal{P}(C_i)$ for each $i = 1, \dots, 4$, as stated in Lemma 4.5.9. From these two figures we see examples for other statements on the saturated graphs in this section.

4.6 Concluding Remarks

Finally, we give some remarks.

Remark 4.6.1. Theorems 4.5.3 and 4.5.4 can be regarded as a refinement, and Theorem 4.4.1 as a generalization of Theorem 4.5.2, from the point of view of the canonical structures of Chapter 2.

Remark 4.6.2. The poset $(\mathcal{G}(G), \triangleleft)$ and $\mathcal{P}(G)$ can be computed in $O(|V(G)| \cdot |E(G)|)$ time (Theorem 2.7.6), where G is any factorizable graph. Therefore, given a saturated graph, we can also find how it is constructed by iterating the cathedral construction in the above time by computing the associated poset and the generalized canonical partition.

Remark 4.6.3. The canonical structures of general factorizable graphs in Chapter 2 can be obtained without the Gallai-Edmonds structure theorem nor the notion of barriers. The other properties we cite to prove the cathedral theorem are also obtained without them. Therefore, our proof shows that the cathedral theorem holds without assuming either of them.

With the whole proof, we can conclude that the structures in Chapter 2 is what essentially underlie the cathedral theorem. We see how a factorizable graph leads to a saturated graph having the same family of perfect matchings by sequentially adding complement edges. Our proof is quite a natural one because the cathedral theorem—a characterization of a class of graphs defined by a kind of edge-maximality "saturated"—is derived as a consequence of considering edge-maximality over the underlying general structure. We hope yet more would be found on the field of counting the number of prefect matchings with the results in this thesis.

Chapter 5

Computing the Cathedral Structure More Efficiently

5.1 Preliminaries

Hereafter we denote by n and m the number of vertices and edges (resp. arcs) of an input graph (resp. digraph), respectively. Note that factorizable graphs satisfy $m = \Omega(n)$ and accordingly $O(n+m) = O(m)$.

In Section 2.7, we show that the partial order $\triangleleft$ and the generalized canonical partition can be computed in $O(nm)$ time if an input graph is factorizable. The algorithm is composed of three stages, each of which is $O(n)$ times iteration of $O(m)$ time procedure of growing alternating trees. It first computes the factor-components, then computes $\triangleleft$ and $\mathcal{P}(G)$ respectively.

With the results in this thesis including Chapter 3, we present another $O(nm)$ time algorithm to compute them. The upper bound of its time complexity is the same as the known one, however the factor-components, $\triangleleft$, and $\mathcal{P}(G)$ are here computed simultaneously. Thus, it has some possibility of exhibiting a bit more efficiency.

Theorem 5.1.1 (Micali & Vazirani [31], Vazirani [38]). *A maximum matching of a graph can be computed in $O(\sqrt{n}m)$ time.*

Theorem 5.1.2 (Edmonds [7], Tarjan [37], Gabow & Tarjan [11]). *Let G be a graph with $m = \Omega(n)$ and suppose we are given a perfect matching of G. Then, $D(G)$, $A(G)$, and $C(G)$ can be computed in $O(m)$ time.*

Theorem 5.1.3 (Dulmage & Mendelsohn [4–6,32]). *For any bipartite factorizable graph G, the Dulmage-Mendelsohn decomposition of G can be computed in $O(m)$ time.*

Proposition 5.1.4 (folklore, see [32]). *Let D be a digraph, and $\mathcal{D}$ be the set of strongly-connected components of D. For $D_1, D_2 \in \mathcal{D}$ we say $D_1 \to D_2$ if for any $u \in V(D_1)$ and any $v \in V(D_2)$ there is a dipath from u to v. Then, $\to$ is a partial order on $\mathcal{D}$.*

Proposition 5.1.5 (see [34]). *For any digraph D, the strongly connected components of D can be computed in $O(n+m)$ time.*

5.2 A New Algorithm

Below is the new algorithm, Algorithm 1:

Require: a factorizable graph G

Ensure: the generalized canonical partition $\mathcal{P}(G)$ and the digraph $Aux(G)$ representing $(\mathcal{G}(G), \triangleleft)$

compute a perfect matching M of G;
$U := V(G)$; initialize $f : V(G) \to \{0,1\}$ by 0;
$A := \emptyset$; $\mathcal{P}(G) := \emptyset$;
while $U \neq \emptyset$ **do**
    choose $u \in U$;
    compute $X := A(G-u) \cup \{u\}$;
    compute the DM-decomposition of $H_G(X)$;
    for all DM-component D of $H_G(X)$ **do**
        let $S := X \cap V(D)$; choose arbitrary $v \in S$;
        if $f(v) = 0$ **then**

11: $\mathcal{P}(G) := \mathcal{P}(G) \cup \{S\}$;
12: let $\hat{D} \subseteq G$ be the expansion of D;
13: **for all** $x \in S$ **do**
14: **for all** $y \in V(\hat{D}) \setminus X$ **do**
15: $A := A \cup \{(x, y)\}$;
16: **end for**
17: $U := U \setminus \{x\}$; $f(x) := 1$;
18: **end for**
19: **end if**
20: **end for**
21: **end while**
22: output $\mathcal{P}(G)$;
23: $Aux(G) := (V(G), A)$; decompose $Aux(G)$ into its strongly-connected components and output it; STOP.

Proposition 5.2.1. *While Algorithm 1 is running,*

(i) $X = A(G - u) \cup \{u\}$ *of Line 6 is an odd-maximal barrier of* G,

(ii) S *defined at Line 9 coincides with a member of* $\mathcal{P}(G)$, *and*

(iii) $V(\hat{D}) \setminus X$ *at Line 14 coincides with* ${}^{c}U(S)$[1].

Proof. The statement (i) follows by a simple counting argument. Therefore, (ii) and (iii) follows by Theorem 3.6.11. □

Lemma 5.2.2. *Let* G *be a factorizable graph and* $Aux(G) = (V(G), A)$ *be the digraph obtained by inputting* G *to Algorithm 1. Let* $H_1, H_2 \in \mathcal{G}(G)$, $u \in V(H_1)$, *and* $v \in V(H_2)$.

(i) *If* $(u, v) \in A$, *then* $H_1 \triangleleft H_2$.

(ii) *If there exists a dipath from* u *to* v *in* $Aux(G)$, *then* $H_1 \triangleleft H_2$.

[1] Given $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$, we denote $U^*(H) \setminus U^*(S)$ as ${}^{c}U(S)$

Proof. The arc (u, v) is added to A only at Line 15 if $u \in X \cap V(\hat{D})$ and $v \in V(\hat{D}) \setminus X$. Thus (i) follows by Proposition 5.2.1. Hence (ii) follows by the transitivity of $\lhd$. □

Lemma 5.2.3. *Let G be a factorizable graph and $Aux(G) = (V(G), A)$ be the digraph obtained by inputting G to Algorithm 1. Let $H_1, H_2 \in \mathcal{G}(G)$ be such that $H_1 \lhd H_2$. Then, for any $u \in V(H_1)$ and $v \in V(H_2)$, there is a dipath from u to v in $Aux(G)$.*

Proof. Let $S \in \mathcal{P}_G(H_1)$ be such that $u \in S$. First suppose that $v \in {}^cU(S)$. Then, (u, v) is added to A at Line 15 when $X \cap V(D)$ of Line 13 coincides with S, which surely occurs by Proposition 5.2.1. Hence, the claim holds for this case.

Now suppose the other case that $v \in U^*(S)$. Take $T \in \mathcal{P}_G(H_1) \setminus \{S\}$ and $w \in T$ arbitrarily. The arc (u, w) is added to A at Line 15 when S coincides with $X \cap V(D)$ of Line 13, so is the arc (w, v) when T coincides with $X \cap V(D)$. Therefore the dipath $uw + wv$ satisfies the claim for this case, and we are done. □

Theorem 5.2.4. *Let G be a factorizable graph and $Aux(G) = (V(G), A)$ be the digraph obtained by inputting G to Algorithm 1. Then, $H \in \mathcal{G}(G)$ holds if and only if there is a strongly-connected component D of $Aux(G)$ with $V(H) = V(D)$. Additionally, for any $H_1, H_2 \in \mathcal{G}(G)$, $H_1 \lhd H_2$ holds if and only if $D_1 \to D_2$, where D_i is the strongly-connected component of $Aux(G)$ with $V(H_i) = V(D_i)$, for each $i = 1, 2$.*

Proof. Combining Lemmas 5.2.2 and 5.2.3, we immediately obtain the following claim:

Claim 5.2.5. *$H_1 \lhd H_2$ holds if and only if for any $u \in V(H_1)$ and any $v \in V(H_2)$ there is a dipath from u to v in $Aux(G)$.*

Therefore, we are done by Proposition 5.1.4. □

Theorem 5.2.6. *Given a factorizable graph G, the poset $(\mathcal{G}(G), \lhd)$ and the generalized canonical partition $\mathcal{P}(G)$ can be computed in $O(nm)$ time by Algorithm 1.*

Proof. The correctness follows by Proposition 5.2.1 and Theorem 5.2.4.

Hereafter we prove the complexity. Line 1 costs $O(\sqrt{n}m)$ time by Theorem 5.1.1. Line 2 costs $O(n)$ time, and Line 3 costs $O(1)$ time. Lines 4 to 7 cost $O(m)$ time per each iteration of the while-loop in Line 4. As the while-loop in Line 4 is repeated $O(n)$ times, they cost $O(nm)$ time over the whole algorithm.

Each operations in Lines 8 to 10 costs $O(1)$ time per iteration, and they are iterated $O(n^2)$ time over the whole computation; therefore, they cost $O(n^2)$ time.

Note that $f(v) = 0$ at Line 10 holds true for at most n times. Therefore, Lines 11 and 12 cost $O(n)$ time. The number of repetition of Lines 13 to 19 is bounded by $|A| = O(n^2)$. Therefore, the operations there costs $O(n^2)$ over the algorithm. $\square$

Chapter 6

Conclusion

In this chapter, we conclude this thesis. We investigated two central notions which supports the foundation of matching theory, i.e., canonical decompositions and barriers.

In Chapter 2, we gave a new canonical decomposition, the cathedral decomposition. While any other known canonical decompositions is not applicable substantially to general factorizable graphs, the new canonical decomposition is applicable non-trivially to and describe structures of general factorizable graphs. Although this result is given as those for factorizable graphs, it can be formulated as a canonical decomposition for general graphs including non-factorizable graphs by combining our results with the Gallai-Edmonds structure theorem. Additionally, it enables us to see all the known canonical decompositions in a unified way.

Thanks to this new canonical decomposition, we become able to see matching-theoretic properties we have not been able to see so far; the structure of barriers is one such example. In Chapter 3, we gave a canonical description of the structures of odd-maximal barriers in general graphs. Although the notion of barriers is important, not so much results has been known about barriers; actually, this is because of lack of effective tools to analyze general factorizable graphs. Considering the history of the study of barriers, this result is an explosive advance in the theory of barriers. Additionally, this result corresponds to a generalization of the canonical partition based on Lovász's formulation, among two formulation of the canonical partition, and therefore can be regarded as a piece of contribution to the theory of canonical decompositions.

We also contributed to the enumeration problem of perfect matchings. In Chapter 4, as a consequence of the new canonical decomposition, we gave another proof of Lovász's cathedral theorem [28, 30]. Our new proof reveals that the intrinsic structure behind the cathedral theorem is the new canonical decomposition in Chapter 2 and proves the cathedral theorem in quite a natural way providing more refined statements.

In Chapter 5, using results in Chapter 3, we propose more efficient algorithms to compute the new canonical decomposition than the one presented in Chapter 2.

Our results in thesis form a great step in the foundation of matching theory, and yet more consequences will be produced from our results. Considering the nature of matchings, we are sure that our results here will contribute to developing the heart of discrete mathematics and combinatorial optimization.

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