A Study on
Even Embeddings of Graphs

March 2014

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主　論　文　要　旨

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主論文題目　：

A Study on Even Embeddings of Graphs

（偶角形分割グラフの研究）

（内容の要旨）

グラフの頂点彩色に関する研究はグラフ理論の中心的話題である。グラフの各頂点に、隣接した頂点対には異なる色を割り当てるという規則のもと色を割り当てるとき、必要な色数の最小数をグラフの染色数という。長い間未解決であったことで有名な「四色問題」は、任意の平面グラフの染色数は 4 以下であるかという問題である。四色問題は 1976 年に肯定的に解決されたが、多くの問題と研究の流れを生み出した。その主なものに「種数の高い閉曲面上のグラフの染色数」や、「染色数を抑えるためのグラフへの制限」、「特殊な条件を付加した彩色問題」などがある。本論文では偶角形分割となるグラフに焦点をあて、これらの問題を考える。

グラフを辺の交差なく閉曲面に描くことを、グラフの閉曲面への埋め込みという。閉曲面上のグラフの染色数の上限を決定するには、計算上求まる上界の値だけ色数を必要とするグラフがその閉曲面に埋め込めるかどうかが重要である。それには完全グラフの三角形分割や四角形分割埋め込みといった特徴的な埋め込みが存在することを示すことが鍵となっている。それらのグラフの埋め込みを構成する手法の一つに、current graph を用いるものがある。これはグラフの埋め込みと一対一に対応する rotation system を別のグラフから与える方法である。

本論文での主題である偶角形分割埋め込みとは、全ての面が偶角形となるようなグラフの閉曲面への埋め込みのことである。平面上の偶角形分割埋め込みは 2 部グラフの平面埋め込みと同値であるが、一般閉曲面上のグラフには非可縮なサイクルが存在するため、偶角形分割埋め込みに対して cycle parity という代数的不変量が定義される。本論文では、どのような current graph が偶角形分割グラフに対応し、また current graph のどのような性質が対応する偶角形分割の埋め込みに反映されるのかについて明らかにする。その結果として偶角形分割における帝国問題の染色数の上限を達成する例と、完全グラフの四角形分割埋め込みについて cycle parity のタイプ別の存在を示す。ここで帝国問題とは、飛び地を含む地図に対応するグラフの頂点彩色問題である。特に current graph から非可縮なサイクルの長さをコントロールするという手法は既存の定理にはない新しい手法である。

代表的な偶角形分割である四角形分割に関しては、上述の研究のほかに、多色彩色問題や、特別な彩色的性質をもつ三角形分割への拡張についても議論する。特に、一般閉曲面におけるそれらの彩色問題においては cycle parity やそれに類似する代数的不変量との関係を明らかにする。

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Title

A Study on Even Embeddings of Graphs

Abstract

The study of vertex colorings of graphs is one of the main topics in graph theory. The chromatic number of a graph is the number of colors to color vertices so that adjacent vertices receive different colors. Four Color Problem is the one asking whether the chromatic number of a plane graph is at most 4 or more. This problem was solved in 1976, and it produced a lot of problems and good streams of studies. The followings are examples: “Chromatic numbers in the case of the surfaces with higher genus,” “restrictions of graphs for reducing the chromatic number” and “coloring problems with special conditions.” In this thesis, we consider these problems while focusing on even embeddings of graphs.

To draw a graph on a surface without edge crossings is called an embedding of the graph on the surface. To determine the upper bound of chromatic numbers of graphs embedded on a surface, we need to show the existence of special embeddings, such as, a triangulation or a quadrangulation of the complete graph. One of the methods to construct such an embedding is to use a current graph, which gives a rotation system corresponding to the embedding.

An even embedding of a graph, which is the main theme of this thesis, is one in which each face has even length. On the sphere, an even embedding of a graph is equivalent to an embedding of a bipartite graph. But on general surfaces, there are essential cycles in graphs, and then an algebraic invariant which is called cycle parity is defined as the parities of the lengths of them. We study relations between current graphs and even embeddings of graphs. We construct even embeddings of empire graphs which achieve the upper bounds of the chromatic number. Here an empire graph is one which corresponds to a map having detached territories. We also construct quadrangulations of the complete graphs which have several types of cycle parities by using current graphs. Especially, we propose an entirely new method to control cycle parities by using current graphs.

About quadrangulations which are typical even embeddings, we also deal with polychromatic coloring problems and extension problems to a triangulation which has a special coloring. In these problems on general surfaces, we show relationships between algebraic invariants including cycle parities and these problems.

A Study on
Even Embeddings of Graphs

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I would like to express my thanks to all members of graph theorist who had mathematical contact with me. Finally, I thank my parents Kazuhiko Noguchi and Mihoko Noguchi and my sister Akiko Noguchi for their support.

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Preface

In this thesis we study even embeddings of graphs. We focus on current graphs and the graphs generated from them, in particular, we deal with even embeddings as those generated graphs. We also investigate extension and coloring problems of graphs even embedded on closed surfaces.

After an introductory chapter, the reader will find six chapters. First, terminology of graphs is found in Chapter 2. To represent embedded graphs on closed surfaces, we use two powerful methods; rotation systems and current graphs. We introduce them and how to generate another graph from a current graph in Chapter 3. In Chapter 4, we introduce an invariant of even embeddings of graphs, which is called a cycle parity. The empire problem is one of vertex-coloring problems of graphs. We discuss the empire problem in even embeddings of graphs in Chapters 5 and 6. Finally, we mention some related topics of even embeddings; extension and coloring problems, they are discussed in Chapter 7.

Papers underlying this thesis

[1] A. Nakamoto, K. Noguchi and K. Ozeki, Cyclic 4-colorings of graphs on surfaces, submitted.

[2] A. Nakamoto, K. Noguchi and K. Ozeki, Extension to even triangulations, submitted.

[3] K. Noguchi, Even embeddings of the complete graphs and their cycle parities, submitted.

[4] K. Noguchi, The empire problem in even embeddings on closed surfaces, J. Graph Theory 75 (2014), 20–30.

[5] K. Noguchi, The empire problem in even embeddings on closed surfaces with $\varepsilon \leq 0$, Discrete Math. 313 (2013), 1944–1951.

Contents

Chapter 1

Introduction

In the history of graph theory, one of the biggest problems is *Four Color Problem.* Four Color Problem is suggested by Francis Guthrie, in 1852. When we consider a map on the plane, we want to know how many colors we need to color the map so that the neighboring countries receive different colors. It is easy to see that there exists a map which needs four colors, but it is quite difficult to find a map which needs five or more colors. Then it was conjectured that four is the best possible. Many researchers intended to solve Four Color Problem by several ways, but no one could solve it for a long time. Finally, Appel and Haken solved the problem in 1976.

Theorem 1.1. (Appel and Haken [1]) *Every map M on the sphere can be colored with four colors.*

But their solution uses a computer program to check four colors are enough to color maps: fundamental 1405 cases. This problem is the origin of many variations of map color problems.

Map color problems are equivalent to vertex-coloring problems of graphs. When you take the dual of a map, you can get the graph corresponding to the map, see Figure 1.1. In this thesis after the introductory chapter, we deal with not map color problems but vertex-coloring problems of graphs.

First, we consider graphs on a closed surface F^2. The following theorem is called *Map Color Theorem.*

Theorem 1.2. (Ringel et al. [45]) *Let G be a graph on a closed surface F^2 with the Euler characteristic $\varepsilon < 2$. Then G can be colored with the Heawood number $H(\varepsilon)$ of*

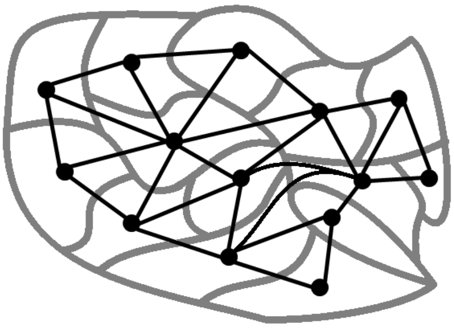

Figure 1.1: A map and its dual

colors, where

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{49 - 24\varepsilon}}{2} \right\rfloor.$$

This value is best possible unless F^2 is the Klein bottle. A graph on the Klein bottle can be colored with six colors, and this is best possible.

Showing that the Heawood number is the upper bound for the number of colors is a little bit easy. But it is quite difficult to show that the Heawood number is the lower bound. In order to show this, you must find a graph whose chromatic number is equal to the Heawood number in each closed surface. In particular, you may show that the complete graph on $H(\varepsilon)$ vertices can be embedded as a *minimum genus embedding.* We define the *genus* (resp. *nonorientable genus*) of a graph G as the minimum g such that G has an embedding into the closed surface $\mathbb{S}_g$ (resp. $\mathbb{N}_k$). A minimum genus embedding of G is such an embedding.

The problem of minimum genus embeddings of the complete graphs are solved separately by many researchers and gathered in one book by Ringel [45]. This problem is divided into twelve cases depending on the number of vertices of the complete graph.

Then how is a minimum genus embedding of the complete graph constructed? To construct it, *current graphs* are used, which we introduce in Chapter 3. For some cases, a minimum genus embedding of a complete graph G is a *triangulation*, that is, every face of G is triangular. A triangulation has some symmetries. Current graphs play an important role in making symmetric graphs.

Now we consider *even embeddings* of graphs. An even embedding of a graph G is an embedding of G where every face is bounded by an even closed walk. On the sphere, an even embedding of a graph is equivalent to an embedding of a bipartite one. As bipartite graphs have the chromatic number at most 2, we expect that the chromatic number of a graph even embedded on a surface F^2 is less than that of a graph on F^2. The following theorem is a kind of Map Color Theorem, which is one for even embedded graphs.

Theorem 1.3. (Hartsfield [16], Hutchinson [19]) *Let F^2 be a closed surface with the Euler characteristic $\varepsilon < 2$. Then for every even embedding of a graph G on F^2,*

$$\chi(G) \leq \left\lfloor \frac{5+\sqrt{25-16\varepsilon}}{2} \right\rfloor.$$

This value is best possible for any F^2 except for the Klein bottle $\mathbb{N}_2$ and the double torus $\mathbb{S}_2$. Every graph on $\mathbb{N}_2$ can be colored with four colors, one on $\mathbb{S}_2$ can be colored with five colors and these are best possible.

Theorem 1.3 is also proved by using current graphs to show the existence of *quadrangulations* of the complete graphs. A quadrangulation is an embedded graph where every face is quadrangular.

We can classify even embeddings of graphs on closed surfaces into several types by the parities of the lengths of essential closed walks. The concept of cycle parities is introduced in Chapter 4. The following theorem says that there are two types of cycle parities in each $\mathbb{S}_g$ and there are four types in each $\mathbb{N}_k$.

Theorem 1.4. (Nakamoto, Negami and Ota [36]) *Any non-trivial cycle parity ρ on $\mathbb{S}_g$ is congruent to $(1, 0, \ldots, 0)$. Any non-trivial cycle parity ρ on $\mathbb{N}_{2k+1}$ with $k \geq 1$ is congruent to exactly one of $A = (1, 0, 0, \ldots, 0, 0)$, $B = (1, 1, 0, \ldots, 0, 0)$ and $C = (0, 1, 0, \ldots, 0, 0)$. Any non-trivial cycle parity ρ on $\mathbb{N}_{2k}$ with $k \geq 2$ is congruent to exactly one of $D = (0, 0, 1, 0, \ldots, 0, 0)$, $E = (0, 1, 0, 0, \ldots, 0, 0)$ and $F = (1, 0, 0, 0, \ldots, 0, 0)$. For the Klein bottle $\mathbb{N}_2$, ρ is congruent to either $E = (0, 1)$ or $F = (1, 0)$.*

Furthermore, we deal with cycle parities by using current graphs. We show the following result which is the same as Theorem 4.2 in Section 4.1. This says that each quadrangulation of the complete graph on $\mathbb{N}_k$ can have each non-trivial type of the cycle parity.

Theorem 1.5. *For each pair $s \geq 1$ and $t \in \{1, 4\}$, there exists a quadrangulation of the complete graph on $8s + t$ vertices whose cycle parity is of each type A, B and C. For each pair $s \geq 1$ and $t \in \{0, 5\}$, there exists a quadrangulation of the complete graph on $8s + t$ vertices whose cycle parity is of each type D, E and F.*

The next problem is one of the variations of map color problems, called *the empire problem.* Let M be a map on a closed surface F^2 and suppose that each country of the map has at most r disjoint detached territories. Such a map is called an *r-pire map* on F^2. For each country, all disjoint detached territories have to be assigned by the same color. In 1890, Heawood proved that the countries of M can be properly colored as follows.

Theorem 1.6. (Heawood [17]) *Let M be an r-pire map on a closed surface F^2 with the Euler characteristic ε. Then M can be properly colored with $h_{\varepsilon,r}$ colors, where*

$$h_{\varepsilon,r} = \left\lfloor \frac{6r + 1 + \sqrt{(6r + 1)^2 - 24\varepsilon}}{2} \right\rfloor$$

except possibly in the case $\varepsilon = 2$ and $r = 1$.

In addition, he conjectured that this is best possible. Note that the case $\varepsilon = 2$ and $r = 1$ is Four Color Problem and $h_{2,1} = 4$, consequently, this case is also best possible. Taylor proved the conjecture for the case where F^2 is the torus. Note that $h_{0,r} = 6r + 1$.

Theorem 1.7. (Taylor [52]) *Every r-pire map on the torus is $(6r+1)$-colorable, and for each r, there is an r-pire map on the torus which is not $6r$-colorable.*

Jackson and Ringel proved it for the cases where F^2 is the projective plane and the sphere. Note that $h_{1,r} = h_{2,r} = 6r$.

Theorem 1.8. (Jackson and Ringel [21]) *Every r-pire map on the projective plane is $6r$-colorable, and for each r, there is an r-pire map on the projective plane which is not $(6r - 1)$-colorable.*

Theorem 1.9. (Jackson and Ringel [22]) *Every r-pire map on the sphere is $6r$-colorable, and for each $r \geq 2$, there is an r-pire map on the sphere which is not $(6r - 1)$-colorable.*

For the Klein bottle, Jackson and Ringel proved it when $r \geq 3$ and Borodin did it when $r = 2$.

Theorem 1.10. (Jackson and Ringel [20], Borodin [7]) *Every r-pire map on the Klein bottle is $(6r + 1)$-colorable, and for each $r \geq 2$, there is an r-pire map on the Klein bottle which is not $6r$-colorable.*

For $r = 1$, Franklin proved that 6 colors suffice to color any map on the Klein bottle.

Theorem 1.11. (Franklin [12]) *Every map on the Klein bottle is* 6*-colorable and there is a map which is not* 5*-colorable.*

This is the only known case where $h_{\varepsilon,r}$ is not best possible. On general closed surfaces, the lower bounds are determined in some cases, see Chapter 5.

The empire problem can be also considered as a graph coloring problem, see Section 2.7. This is also proved by using current graphs. Thus, we see that vertex-coloring problems of graphs is very related to current graphs.

In this thesis, we consider the empire problem in even embeddings of graphs. We can show the upper bound for the number of colors and the lower bounds for some cases in the same way as the above theorems. The followings are the same as Theorems 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6 in Section 6.1. Empire graphs are that corresponding to empire maps, defined in Section 2.7.

Theorem 1.12. *Let G be an r-pire graph such that G has an even embedding on a closed surface F^2 with $\varepsilon \leq 0$. Then, G is $n_{\varepsilon,r}$-colorable, where*

$$n_{\varepsilon,r} = \left\lfloor \frac{4r+1+\sqrt{(4r+1)^2-16\varepsilon}}{2} \right\rfloor.$$

Theorem 1.13. *Every r-pire graph which has an even embedding on $\mathbb{S}_0$ is $4r$-colorable. Moreover, for each $r \geq 2$, there is an r-pire graph even embedded on $\mathbb{S}_0$ which is not* $(4r-1)$*-colorable.*

Theorem 1.14. *Every r-pire graph which has an even embedding on $\mathbb{N}_1$ is $4r$-colorable. Moreover, for each r, there is an r-pire graph even embedded on $\mathbb{N}_1$ which is not* $(4r-1)$*-colorable.*

Theorem 1.15. *Every r-pire graph which has an even embedding on $\mathbb{S}_1$ is* $(4r+1)$*-colorable. Moreover, for each r, there is an r-pire graph even embedded on $\mathbb{S}_1$ which is not* $4r$*-colorable.*

Theorem 1.16. *Every r-pire graph which has an even embedding on $\mathbb{N}_2$ is* $(4r+1)$*-colorable. Moreover, for each $r \geq 2$, there is an r-pire graph even embedded on $\mathbb{N}_2$ which is not* $4r$*-colorable.*

Theorem 1.17. *The bound $n_{\varepsilon,r}$ in Theorem 1.12 is best possible if one of the following conditions is satisfied;*

(i) F^2 *is an orientable surface,* r *is even, and* $n_{\varepsilon,r}$ *is congruent to* 1 *modulo* 8*.*

(ii) F^2 *is an orientable surface,* r *is odd, and* $n_{\varepsilon,r}$ *is congruent to* 5 *modulo* 8*.*

(iii) F^2 *is a nonorientable surface and* $n_{\varepsilon,r}$ *is congruent to* 1 *modulo* 4 *except in the case* F^2 *is* $\mathbb{N}_2$ *and* $r = 1$*.*

Next, we focus on quadrangulations. An extension problem is to find, from a given graph G, a graph T with certain properties so that T is obtained from G by adding some edges. Now we deal with the problem of extending a quadrangulation to Eulerian triangulations, where a triangulation is Eulerian if all the vertices have even degree. This problem was first considered by Hoffmann and Kriegel in 1996 for the spherical case. Zhang and He improved the result for non-spherical orientable closed surfaces. We show that the result also holds for nonorientable cases. The following is the same as Theorem 7.3 in Section 7.1.

Theorem 1.18. *Let* G *be a quadrangulation on a closed surface* F^2*. Then* G *can be extended to an Eulerian triangulation.*

Finally, we consider the coloring problems again. A quadrangulation on the sphere is 2-colorable and an Eulerian triangulation extended from a quadrangulation on the sphere is 3-colorable. It is also natural to consider the following 4-colorings for a quadrangulation. A *cyclic coloring* of a graph G on a surface F^2 is a vertex-coloring of G such that any two vertices x and y receive different colors if x and y are incident with a common face of G. The problem of cyclic 4-colorings of quadrangulations on the sphere was first considered by Berman and Shank in 1979.

Theorem 1.19. (Berman and Shank [3]) *Let* G *be a quadrangulation on the sphere. Then* G *has a cyclic* 4*-coloring if and only if the edge set of the straight walk dual* $\widetilde{G}$ *of* G *has a proper* 3*-edge-coloring satisfying condition (C1).*

The straight walk dual $\widetilde{G}$ of a quadrangulation G is defined in Section 7.2 and condition (C1) is defined in the next theorem. We extend the above result in two directions, that is, considering graphs on a non-spherical surface and graphs called *mosaics* which may have some triangular faces. The following is the same as Theorem 7.4 in Section 7.2.

Theorem 1.20. *A mosaic* G *of a surface* F^2 *has a cyclic* 4*-coloring if and only if the straight walk dual* $\widetilde{G}$ *of* G *has a* 3*-edge-coloring* $c : E(\widetilde{G}) \to \{1, 2, 3\}$ *satisfying the following two conditions.*

(C1) *Any two edges of $\widetilde{G}$ that are pairwise crossing on F^2 receive different colors by c. (So, no edge intersects with itself.)*

(C2) *For every closed curve γ on F^2,*

$$|c^{-1}(1) \cap \gamma| \;\equiv\; |c^{-1}(2) \cap \gamma| \;\equiv\; |c^{-1}(3) \cap \gamma| \qquad \pmod 2.$$

Chapter 2

Definitions

In this chapter, we define some basic terminology of graph theory that are used throughout this thesis.

2.1 Graphs

A *graph* G consists of finite sets $V(G)$ and $E(G)$, where $V(G)$ is a nonempty set of elements called *vertices* and $E(G)$ is a set of unordered pairs of elements of $V(G)$ called *edges*. An edge $\{u, v\}$ is often represented as uv or vu. If there exists an edge xy where $x = y$, we call this edge a *loop*. If we allow at least two edges joining a pair of vertices, such edges are called *multiple edges*. and the graph is called a *multi-graph*. Graphs with no loops or multiple edges are called *simple graphs*. Let G and H be graphs. Note that $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. If $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then H is called a *subgraph* of G. A simple graph is *complete* if there exists $uv \in E(G)$ for every $u, v \in V(G)$ $(u \neq v)$. The complete graph on n vertices is denoted by K_n. The *neighbor* $N_G(v)$ of a vertex $v \in V(G)$ is the set of vertices of $G : N_G(v) := \{u \in V(G) | uv \in E(G)\}$. We define the *degree* $d(v)$ of a vertex $v \in V(G)$ as the number of the incident edges to v. In particular, we can denote $d(v) = |N_G(v)|$ for a simple graph G. We also define an average degree $\bar{d}(G)$ of a graph G as follows; $\bar{d}(G) = \sum_{v \in V(G)} d(v)/|V(G)|$.

A *walk* of length k in a graph G is a sequence $v_0e_0v_1e_1 \ldots e_{k-1}v_k$ of vertices and edges in G such that $e_i = v_iv_{i+1}$ for all $i < k$. We call a walk *closed* if $v_k = v_0$. A *path* of length k in a graph G is a walk W of length k of G such that every vertex v_i $(0 \leq i \leq k)$ in W are all distinct. A *cycle* of length k is a closed walk of length k with the set of distinct vertices v_i $(0 \leq i \leq k - 1)$. We say that a closed walk W is *even* (resp. *odd*) if the length of W is

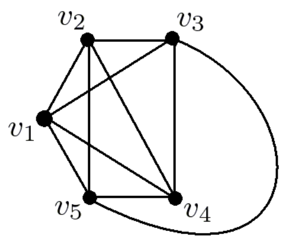

Figure 2.1: A graph

even (resp. odd). A graph G is *connected* if any two vertices of G are connected by a path in G. We call a connected graph G *Eulerian* if every vertex of G has even degree.

Let A be a group. A graph with a *weight function* $w : E(G) \to A$ is called a *weighted graph*. In a weighted graph, we say that each edge e has a weight $w(e)$. A *digraph* G is a multi-graph whose edges have orientations, and we call edges *arcs*. The *indegree* of $v \in V(G)$ is the number of incoming arcs to v, the *outdegree* of $v \in V(G)$ is the number of outgoing arcs to v and we denote them by $\deg^-(v)$ and $\deg^+(v)$, respectively. We see that $d(v) = \deg^-(v) + \deg^+(v)$ for every vertex v of a digraph G.

2.2 Colorings of graphs

A *vertex-coloring* of a graph G is a mapping $c : V(G) \to \{1, 2, \ldots\}$ such that $c(x) \neq c(y)$ whenever $xy \in E(G)$. We say that a graph G is *k-colorable* if there exists a vertex-coloring $c : V(G) \to \{1, 2, \ldots, k\}$. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is k-colorable.

An *edge-coloring* of a graph G is a mapping $c : E(G) \to \{1, 2, \ldots\}$ such that $c(e_1) \neq c(e_2)$ whenever edges e_1 and e_2 share a vertex of G. A graph with loops has no vertex-colorings or edge-colorings.

2.3 Closed surfaces

We call a connected compact 2-dimensional manifold without boundaries a *closed surface*. There are two classes of closed surfaces, *orientable* ones and *nonorientable* ones. Let $\mathbb{S}_g$ and $\mathbb{N}_k$ denote the orientable closed surface of genus g and the nonorientable closed surface of genus (or crosscap number) k, respectively. See Figures 2.2 and 2.3.

Theorem 2.1. (The classification theorem of closed surfaces) *Any connected closed surface F^2 is homeomorphic to one of the following surfaces.*

- *The sphere $\mathbb{S}_0$.*
- *The orientable closed surface $\mathbb{S}_g$ of genus g.*
- *The nonorientable closed surface $\mathbb{N}_k$ of genus k.*

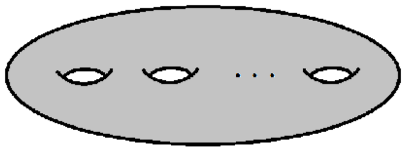

Figure 2.2: An orientable closed surface $\mathbb{S}_g$

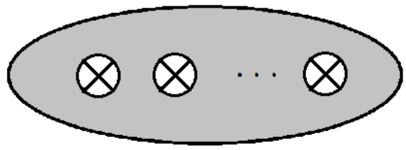

Figure 2.3: A nonorientable closed surface $\mathbb{N}_k$

Let $\varepsilon(F^2)$ denote the *Euler characteristic* of a closed surface F^2. Note that $\varepsilon(\mathbb{S}_g) = 2 - 2g$ and $\varepsilon(\mathbb{N}_k) = 2 - k$.

A simple closed curve l on F^2 is called *essential* if l does not bound a 2-cell. We say that l is *one-sided* if the tubular neighborhood of l is homeomorphic to a Möbius band, and l is *two-sided* otherwise.

2.4 Embeddings of graphs

If a graph G can be drawn on a closed surface F^2 without edge crossings, we say that G has an *embedding* on F^2, we also call an embedding of G an *embedded graph* on F^2. An embedding of a graph G on a closed surface F^2 is also regarded as an injective continuous map $f : G \to F^2$. Throughout this thesis, we denote a fixed embedding of G on F^2 by $G \to F^2$. An embedding $G \to F^2$ is called a 2-*cell embedding* if any connected component of $F^2 \backslash G$ is homeomorphic to an open disc. We only consider 2-cell embeddings of graphs and we often write "an embedding" instead of "a 2-cell embedding" in this thesis. Suppose that a graph G is embedded on a closed surface F^2, we denote the set of faces of G by $F(G)$. We call a graph G *planar* if there exists an embedding $G \to \mathbb{S}_0$, and the embedding a *plane graph* G. We call an embedded graph G a *triangulation* if every face of the embedding of G is triangular.

2.5 Locally planar graphs

In this section, we consider graphs on surfaces with small chromatic number. Even if the genus is much higher, the chromatic number of a graph is small if its representativity is high depending on the genus.

By Theorem 1.2, we see that the upper bound of the chromatic number of a graph on a closed surface F^2 depends on the genus of F^2. Roughly speaking, $\chi(G) = O(\sqrt{g})$. On the other hand, there exist infinitely many graphs which have a small chromatic number. It is known that a graph G with high *representativity* on a closed surface F^2 can be colored by only a few colors even if the genus of F^2 is high. Representativity is the measure of the density of embeddings [47]. Let G be a graph embedded on a closed surface F^2. The representativity $r(G)$ of G is defined as

$$r(G) := \min\{|G \cap l|\colon\ l \text{ is an essential simple closed curve on } F^2\}.$$

A graph with high representativity is sometimes called a *locally planar graph.* Chromatic numbers of locally planar graphs are much less than general graphs, see the following.

Theorem 2.2. (Thomassen [53]) *For any closed surface F^2 except for the sphere, there is a number $N = N(F^2)$ such that every graph G on F^2 with representativity $r(G) \geq N$ is* 5*-colorable.*

2.6 Even embeddings of graphs

For a plane graph G, chromatic number of G is two if and only if G is *bipartite.* A bipartite plane graph is also considered as an *even embedded graph.* An *even embedding* of a graph G on a closed surface F^2 is a 2-cell embedding such that each face of G is bounded by an even closed walk. Even if an embedding is not 2-cell, we also call it an even embedding if we can add some edges to get a 2-cell even embedding. In particular, we call it *quadrangulation* if every face is a quadrilateral.

In general surface, an even embedded graph may not be bipartite because essential closed walks may have odd lengths. It means that we do not know whether chromatic number is two of more. The followings are the results of chromatic numbers of locally planar graphs even embedded on a closed surface F^2.

Theorem 2.3. (Fisk and Mohar [11]) *For any closed surface F^2 except for the sphere,* *there is a number $N = N(F^2)$ such that every even embedded graph G on F^2 with representativity $r(G) \geq N$ is 4-colorable.*

Theorem 2.4. (Hutchinson [19]) *For any orientable closed surface $\mathbb{S}_g$ except for the sphere, there is a number $N = N(F^2)$ such that every even embedded graph G on $\mathbb{S}_g$ with representativity $r(G) \geq N$ is 3-colorable.*

2.7 Empire graphs

We define an *empire map* to be a map on a closed surface divided into regions, where the set of regions is partitioned into disjoint subsets which we call *empires.* We call an empire with exactly r regions an *r-pire.* An empire map where each empire has no more than r regions is called an *r-pire map.* Each empire map can be associated with a simple graph G, which is the dual of the map. The vertex set of G is the set of regions of the map and two vertices are adjacent if the corresponding regions share a common boundary edge. We call the graph obtained from the empire map an *underlying graph.*

We define an *empire graph* $(G, \mathcal{P})$ as follows. Suppose that G is a simple graph and $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ is a partition of $V(G)$, that is, $\bigcup_i P_i = V(G)$ and $P_i \cap P_j = \emptyset$ if $i \neq j$ such that each P_i is a set of mutually nonadjacent vertices. We call each P_i an empire. If $|P_i| \leq r$ for all i, we call $(G, \mathcal{P})$ an *r-pire graph.*

We define a proper vertex-coloring of an empire graph $(G, \mathcal{P})$ to be a mapping $c : V(G) \to \{1, 2, \ldots\}$ such that for $x \in P_i$ and $y \in P_j$, $c(x) = c(y)$ if $i = j$ and $c(x) \neq c(y)$ if $i \neq j$ and $xy \in E(G)$. We say that $(G, \mathcal{P})$ is *k-colorable* if there exists a proper vertex-coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the empire graph $(G, \mathcal{P})$. We simply call it a coloring instead of a proper vertex-coloring. A coloring of an empire graph $(G, \mathcal{P})$ is essentially equivalent to a coloring of the graph $G_{\mathcal{P}}$, which is obtained from G by identifying all vertices in each P_i of $\mathcal{P}$ into a vertex p_i, that is, $V(G_{\mathcal{P}}) = \{p_1, p_2, \ldots, p_t\}$ and $E(G_{\mathcal{P}}) = \{p_i p_j | \text{there exists an edge } xy \in E(G) \text{ where } x \in P_i \text{ and } y \in P_j \ (i \neq j)\}$.

Chapter 3

Embedding methods and current graphs

In this chapter, we study how to represent embeddings of graphs. Embedding methods have been widely studied in the literature. See [14] and [30] for the foundation of topological graph theory.

In each of Theorems 1.2 and 1.3, the proof of the sharpness is based on the fact that $K_{H(\varepsilon)}$ and $K_{\lfloor \frac{5+\sqrt{25-16\varepsilon}}{2} \rfloor}$ can be embedded on F^2, respectively. These are minimum genus embeddings of the complete graphs. To construct them, we use the notion of *rotation systems*. In Section 3.1, we introduce rotation systems. In Section 3.2, we introduce current graphs to get suitable rotation systems.

3.1 Rotation systems

We construct an (empire) graph by using the rotation systems. Let G be a connected multi-graph with at least one edge. Suppose that we have a cyclic permutation π_v of the edges incident with the vertex v for each $v \in V(G)$. We call such π_v a *rotation*. A *rotation system* π is the collection of π_v for all $v \in V(G)$. If G is simple, we may assume that a rotation π_v is a cyclic permutation of the neighbors of the vertex $v \in V(G)$. In this thesis, we often represent rotations by cyclic permutations of vertices. A closed walk $v_1e_1v_2e_2v_3\dots v_ke_kv_1$ is called a *π-polygon*, if π_{v_i} is $\dots, v_{i-1}, v_{i+1}, \dots$ for every i where indices are taken modulo k. Each edge e in G is contained twice in π-polygons. We see that all π-polygons construct a closed orientable surface and rotation systems represent embeddings of graphs on closed orientable surfaces.

To apply this method to nonorientable cases, in addition to the rotations, we use a mapping $\lambda : E(G) \to \{1, -1\}$, which is called a *signature*. We call an edge e *twisted* if $\lambda(e) = -1$. We define an embedding of G on a surface F^2 so that the clockwise orderings at u and v do not agree in the disk on F^2 around the edge uv if and only if $\lambda(uv) = -1$. By this fact, we see the following. Let u and v be vertices joined by a twisted (resp. non-twisted) edge in a graph G, π_u be a rotation $\ldots, w, v, x, \ldots$ and π_v be a rotation $\ldots, y, u, z, \ldots$. Then some π-polygon of G is $\ldots, w, wu, u, uv, v, vy, y, \ldots$ (resp. $\ldots, w, wu, u, uv, v, vz, z, \ldots$). For more details, see pp.91–94 in [30].

3.2 Current graphs

We explain how to obtain the rotation system. We use a powerful method of current graphs. A *current graph* G with a group A is a weighted digraph (it can have loops and multiple edges) such that each vertex v in G has a rotation σ_v which is a cyclic permutation of the neighbors of v. In this thesis, we only consider the case $A = \mathbb{Z}_n$; the cyclic group of order n. We define a weight function $\beta : \{uv, (uv)^{-1} | uv \in E(G)\} \to \mathbb{Z}_n \backslash \{0\}$ satisfying $\beta((uv)^{-1}) = -\beta(uv)$, which is called a *current*. Here $(uv)^{-1}$ represents the opposite direction of the edge uv. Let

$$\begin{aligned}
W_1 &= v_1^1, v_1^1 v_2^1, v_2^1, \ldots, v_{k_1}^1, v_{k_1}^1 v_1^1, v_1^1, \\
W_2 &= v_1^2, v_1^2 v_2^2, v_2^2, \ldots, v_{k_2}^2, v_{k_2}^2 v_1^2, v_1^2, \\
&\cdots \\
W_r &= v_1^r, v_1^r v_2^r, v_2^r, \ldots, v_{k_r}^r, v_{k_r}^r v_1^r, v_1^r
\end{aligned}$$

be all σ-polygons of G, where σ is the rotation system consisting of the collection of σ_v for all $v \in V(G)$.

We construct a new graph G_β, which is called a *generated graph*, with a vertex set

$$\mathbb{Z}_n \times \{1, 2, \ldots, r\} = \big\{ i_s | i \in \mathbb{Z}_n, s \in \{1, 2, \ldots, r\} \big\}$$

and a set of permutations $\pi = \{\pi_0, \pi_1, \ldots, \pi_{n-1}\}$ defined by G, σ and β as follows. Each value is an element of the cyclic group of order n.

$$
\begin{aligned}
\pi_0 &= \big(\beta(v_1^1v_2^1), \beta(v_2^1v_3^1), \ldots, \beta(v_{k_1}^1v_1^1)\big)\big(\beta(v_1^2v_2^2), \ldots, \beta(v_{k_2}^2v_1^2)\big) \\
&\quad \cdots \big(\beta(v_1^rv_2^r), \ldots, \beta(v_{k_r}^rv_1^r)\big) \\
\pi_1 &= \big(\beta(v_1^1v_2^1)+1, \beta(v_2^1v_3^1)+1, \ldots, \beta(v_{k_1}^1v_1^1)+1\big)\big(\beta(v_1^2v_2^2)+1, \ldots, \beta(v_{k_2}^2v_1^2)+1\big) \\
&\quad \cdots \big(\beta(v_1^rv_2^r)+1, \ldots, \beta(v_{k_r}^rv_1^r)+1\big) \\
\cdots \\
\pi_i &= \big(\beta(v_1^1v_2^1)+i, \beta(v_2^1v_3^1)+i, \ldots, \beta(v_{k_1}^1v_1^1)+i\big)\big(\beta(v_1^2v_2^2)+i, \ldots, \beta(v_{k_2}^2v_1^2)+i\big) \\
&\quad \cdots \big(\beta(v_1^rv_2^r)+i, \ldots, \beta(v_{k_r}^rv_1^r)+i\big) \\
\cdots \\
\pi_{n-1} &= \big(\beta(v_1^1v_2^1)+n-1, \beta(v_2^1v_3^1)+n-1, \ldots, \beta(v_{k_1}^1v_1^1)+n-1\big) \\
&\quad \cdot\big(\beta(v_1^2v_2^2)+n-1, \ldots, \beta(v_{k_2}^2v_1^2)+n-1\big) \\
&\quad \cdots \big(\beta(v_1^rv_2^r)+n-1, \ldots, \beta(v_{k_r}^rv_1^r)+n-1\big)
\end{aligned}
$$

The elements of $\mathbb{Z}_n$ correspond to the empires, and r represents the number of vertices in an empire. Each π_i has r cyclic permutations, each of which can be regarded as the rotation of a vertex of the empire i. The sth cyclic permutation of π_i is the rotation π_{i_s} of the vertex i_s. (Note that the cyclic permutations in π_i is not a permutation of $\mathbb{Z}_n \times \{1, 2, \ldots, r\}$, but of $\mathbb{Z}_n$. If i is contained in sth cyclic permutation of π_j, then $j \in \mathbb{Z}_n$ in π_i is regarded as $j_s \in \mathbb{Z}_n \times \{1, 2, \ldots, r\}$.)

We show an example. A graph in Figure 3.1 is a weighted digraph with the cyclic group of order 21. Each black vertex v_i ($1 \leq i \leq 4$) has the neighbors with a clockwise order as a rotation σ_i and the white vertex v_5 has the neighbors with a counterclockwise order as a rotation σ_5.

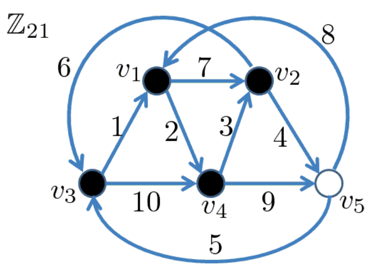

Figure 3.1: An example of a current graph

Now we take σ-polygons, we may start v_1 first. Trace the arc v_1v_3 in the opposite direction of the arrow, and reach the black vertex v_3, turn left. Trace the arc v_3v_4 in the direction of the arrow, and reach the black vertex v_4, turn left. Trace the arc v_4v_1 in the opposite direction of the arrow, and reach the black vertex v_1, turn left. Then the first σ-polygon is

$$W_1 = v_1, v_3v_1^{-1}, v_3, v_3v_4, v_4, v_1v_4^{-1}, v_1.$$

Similarly, the other σ-polygons are

$$\begin{aligned} W_2 &= v_1, v_1v_4, v_4, v_4v_2, v_2, v_1v_2^{-1}, v_1 \quad \text{and} \\ W_3 &= v_1, v_5v_1^{-1}, v_5, v_2v_5^{-1}, \ldots, v_3v_1, v_1. \end{aligned}$$

Thus, $r = 3$ and

$$\begin{aligned} \pi_0 &= \big(\beta((v_3v_1)^{-1}), \beta(v_3v_4), \beta((v_1v_4)^{-1})\big) \\ &\quad \cdot \big(\beta(v_1v_4), \beta(v_4v_2), \beta((v_1v_2)^{-1})\big)\big(\beta((v_5v_1)^{-1}), \beta((v_2v_5)^{-1}), \ldots, \beta(v_3v_1)\big) \\ &= (-1, 10, -2)(2, 3, -7)(-8, -4, -3, 9, 5, -6, 4, -9, -10, -5, 8, 7, 6, 1) \\ &= (20, 10, 19)(2, 3, 14)(13, 17, 18, 9, 5, 15, 4, 12, 11, 16, 8, 7, 6, 1). \end{aligned}$$

We can get the other permutations π_i $(1 \le i \le n-1)$ using the method mentioned before. We assume each empire i has three vertices i_1, i_2 and i_3. Then we get rotations

$$\begin{aligned} &\pi_{0_1} = \big(20_3, 10_3, 19_2\big), \quad \pi_{0_2} = \big(2_1, 3_3, 14_3\big), \quad \pi_{0_3} = \big(13_3, 17_3, \ldots, 1_1\big) \\ &\pi_{1_1} = \big(\ 0_3, 11_3, 20_2\big), \quad \pi_{1_2} = \big(3_1, 4_3, 15_3\big), \quad \pi_{1_3} = \big(14_3, 18_3, \ldots, 2_1\big) \\ &\ldots \\ &\pi_{20_1} = \big(19_3,\ 9_3, 18_2\big), \quad \pi_{20_2} = \big(1_1, 2_3, 13_3\big), \quad \pi_{20_3} = \big(12_3, 16_3, \ldots, 0_1\big). \end{aligned}$$

Then we can get the the generated graph G_β induced by the rotation system π, which is the collection of π_{i_s} for all $i_s \in \mathbb{Z}_{21} \times \{1, 2, 3\}$. We see that the current graph in Figure 3.1 generates a 3-pire graph with 21 mutually adjacent 3-pires on an orientable closed surface.

We also apply this method to nonorientable cases. We refer pp.137–139 in [45]. To construct a graph embedded on a nonorientable closed surface, we add the new idea which is called *broken arcs* to the current graphs. We call the current graph involving some broken arcs a *cascade*.

We show an example of a cascade. A graph in Figure 3.2 is a weighted digraph with the cyclic group of order 17. Each black vertex $v_i (i \in \{1, 2, 4\})$ has the neighbors with a clockwise order as a rotation σ_i and the white vertex v_3 has the neighbors with a counter-clockwise order as a rotation σ_3. It has three broken arcs v_1v_1, v_3v_4 and v_4v_4. Broken arcs work as follows. If we trace the broken arc, then the orientations of all arrows change, and the black vertex changes to the white vertex and vice versa.

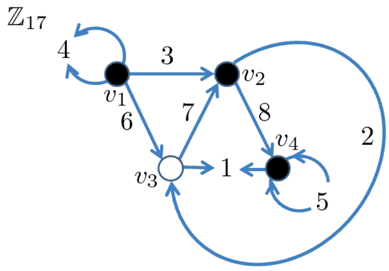

Figure 3.2: An example of a cascade

Now we take σ-polygons. Suppose that we may start v_1. Trace the arc v_1v_2 in the direction of the arrow, and reach the black vertex v_2, turn left. Trace the arc v_2v_3 in the direction of the arrow, and reach the white vertex v_3, turn right. Trace the arc v_3v_4 in the direction of the arrow, and this arc is broken, then reach the **white** vertex v_4, turn right. Trace the arc v_4v_4 in the **direction of the arrow**, and this arc is broken, then reach the black vertex v_4, turn left. Trace the arc v_4v_4 in the opposite direction of the arrow, and so on. Then we get one σ-polygon

$$W_1 = v_1, v_1v_2, v_2, v_2v_3, v_3, v_3v_4, v_4, v_4v_4, v_4, (v_4v_4)^{-1}, v_4, \dots, v_1.$$

Since $r = 1$, we see that

$$\begin{aligned}
\pi_0 &= \big(\beta(v_1v_2), \beta(v_2v_3), \beta(v_3v_4), \beta(v_4v_4), \beta((v_4v_4)^{-1}), \dots, \beta((v_1v_1)^{-1})\big) \\
&= (3, 2, 1, 5, -5, 8, -2, 6, -3, 7, -1, -8, -7, -6, 4, -4) \\
&= (3, 2, 1, 5, 12, 8, 15, 6, 14, 7, 16, 9, 10, 11, 4, 13).
\end{aligned}$$

Now we see that the signature λ of the generated graph is determined as follows. If an arc e is traced twice in the same direction on the cascade, then we define $\lambda(e) = 1$, and if an arc e is traced twice in both direction on the cascade, then we define $\lambda(e) = -1$.

We use the following lemma.

Lemma 3.1. (Ringel [45], p.145) *If a cascade with at least one broken arc defines a graph embedded on an orientable surface, then the following two properties must hold.*

(F1) *The order n of the group is even.*

(F2) *If the current of an arc (broken or not) is odd then the arc is traced twice in the same direction. If the current is even then the arc is traced in both directions.*

By Lemma 3.1, we see that the cascade in Figure 3.2 generates K_{17} on a nonorientable closed surface, since the order of the group $n = 17$ is odd.

Let G be a current graph or a cascade with a rotation system π, and $\beta : \{uv, (uv)^{-1} | uv \in E(G)\} \to \mathbb{Z}_n \backslash \{0\}$ be a weight function. If we suppose that broken arcs are twisted arcs, G has a 2-cell embedding on some closed surface S. Then the current graph or the cascade is denoted $\langle G \to S, \beta\rangle_n$ and the generated embedded graph is denoted $G_\beta \to S_\beta$. If we get the graph G_β from the current graph G in this way, we say "*G generates G_β*". For more details, see [14, 45].

To get the desired graph, we construct current graphs or cascades with the following properties.

Lemma 3.2. (Hartsfield [15], Jackson and Ringel [21]) *A current graph or a cascade $\langle G \to S, \beta\rangle_n$ where n is odd which satisfies (P1), (P2), (P3) and (P4) generates a graph G_β such that G_β is the complete graph on n vertices and its embedding is a quadrangulation.*

(P1) *Each vertex has degree 4.*

(P2) *Each element from 1 to $(n-1)/2$ of $\mathbb{Z}_n$ appears exactly once among the arcs of G as a current.*

(P3) *At each vertex the sum of the currents outgoing from the vertex is zero modulo n.*

(P4) *The number of σ-polygons is one.*

Lemma 3.3. *A cascade $\langle G \to S, \beta\rangle_n$ where n is odd which satisfies (P2) in Lemma 3.2 generates the empire graph $(G, \mathcal{P})$ such that $G_{\mathcal{P}}$ is K_n.*

Proof. We assume that broken arcs are twisted edges. Since a cascade G has a rotation system σ, it has a 2-cell embedding on some closed surface F^2. If F^2 is orientable, then we can take fixed orientations of the all faces of G. Then we trace the arc with the direction of the orientations, we see that each arc is traveled twice in both direction. If F^2 is nonorientable, we cut open F^2 into an orientable surface S, and we take fixed orientations of the all faces of G on S. We trace the arc with the direction of the orientations. When we trace the broken arc, we trace the following arcs in the opposite direction of the orientations and the orientations of the arrows change. Then if we take good orientations of the facial walks of the σ-polygons of G, each arc with current a is traced twice and it is appeared as a and $-a$ in π_0. Since π_0 has all elements $\pm 1, \ldots, \pm(n-1)/2$, we see that empire 0 is adjacent to any other empire in the generated empire graph G_β. Since the construction of each π_i, G_β has n mutually adjacent r-pires. $\square$

The following lemma is lead by the relation between current graphs and generated graphs (see [14]).

Lemma 3.4. *A current graph or a cascade $\langle G \to S, \beta \rangle_n$ where n is odd which satisfies (P1'), (P2), (P3') and (P4) generates a graph G_β such that G_β is the complete graph on n vertices and its embedding has one $2n$-gonal face and all other faces quadrangular.*

(P1') *One vertex has degree 2, the others has degree 4.*

(P2) *Each number from 1 to $(n-1)/2$ appears exactly once as a current.*

(P3') *At each vertex of degree 4 the sum of the currents outgoing from the vertex is zero modulo n, and at vertex of degree 2, the sum of those is relatively prime to n.*

(P4) *The number of σ-polygons is one.*

Corollary 3.5. *If there exists a current graph (resp. a cascade) satisfying the properties of Lemma 3.4, K_{n+1} can be embedded on a closed surface $\mathbb{S}_g$ (resp. $\mathbb{N}_k$) as a quadrangulation.*

Proof. Let G be an embedding of K_n with conditions in Lemma 3.4. Add a new vertex in the $2n$-gonal face, and join it to every other vertex of the face boundary. By (P3'), such all n vertices are distinct. Then we see that we have the desired quadrangulation of K_{n+1}. $\square$

Chapter 4

Cycle parity

In this chapter, we study cycle parities. It is an invariant of even embeddings of graphs. All the new results we prove in this chapter can be found in [43].

4.1 Motivation

Minimum genus embeddings of complete graphs have been studied by many researchers. It is known that the number of minimum genus embeddings of the complete graphs is at least exponential in n, see [28, 27, 25, 26].

From Theorem 1.3, the following holds.

Corollary 4.1. *K_n has a quadrangular embedding on some closed surface $\mathbb{S}_g$ if and only if $n \equiv 0, 5 \pmod 8$. K_n has a quadrangular embedding on some closed surface $\mathbb{N}_k$ if and only if $n \equiv 0, 1 \pmod 4$ and $n \neq 5$.*

It is also known that the number of minimum genus even embeddings of K_{8s+5} on $\mathbb{S}_g$ is at least exponential in n, see [28].

Now we consider several types of minimum genus even embeddings of complete graphs. It is known that there is an invariant of even embeddings of graphs, which is called a *cycle parity*. It divides non-bipartite even embeddings of graphs into three classes on a fixed nonorientable closed surface.

Then we expect that for all n, there are minimum genus even embeddings of the complete graph on n vertices with each type. One of our main results in this section is the following. The types from A to F of cycle parities are defined in Section 4.2.

Theorem 4.2. *For each pair $s \geq 1$ and $t \in \{1, 4\}$, there exists a minimum genus even embedding of the complete graph on $8s + t$ vertices whose cycle parity is of each type A, B and C. For each pair $s \geq 1$ and $t \in \{0, 5\}$, there exists a minimum genus even embedding of the complete graph on $8s + t$ vertices whose cycle parity is of each type D, E and F.*

Let us consider a transformation of triangulations on surfaces, called an *edge contraction*. Let G be a triangulation on a surface and let $e = xy$ be an edge of G. Let xyz and xyw be two faces in G. *Contraction* of e in G is to remove e, identify x and y and replace two pairs of multiple edges (xz and yz, xw and yw) with two single edges respectively. If the contraction of e transforms G into a simple triangulation on the same surface, then we say e is *contractible*. We say that G is *contractible* to a triangulation T if T is obtained from G by a sequence of edge contractions. A triangulation G is called *irreducible* if G has no contractible edge. It is known that every surface admits finitely many irreducible triangulations, up to homeomorphism [9, 13, 23, 39], and the complete lists of irreducible triangulations are known for $\mathbb{S}_0$ [49], $\mathbb{S}_1$ [29], $\mathbb{S}_2$ [50], and $\mathbb{N}_1$ [4], $\mathbb{N}_2$ [51], $\mathbb{N}_3$, $\mathbb{N}_4$ [50]. Irreducible triangulations have many applications, see [9].

Let F^2 be a closed surface which can be triangulated by some K_n. By definition, every triangulation on a fixed surface F^2 is contractible to an irreducible triangulation. Moreover, we know that every *complete triangulation* (i.e., a triangular embedding of K_n) on F^2 is irreducible. However, every triangulation on F^2 is not necessarily contractible to a complete triangulation, since there exists an irreducible triangulation which is not a complete triangulation. On the other hand, by a consequence of the proof of Negami's theorem [40] on diagonal flips in triangulations on surfaces, for any surface F^2, there is an integer $N(F^2)$ such that

> *any triangulation G on F^2 with at least $N(F^2)$ vertices can be transformed into a complete triangulation by edge contractions and diagonal flips.*

Let us consider whether a similar fact holds for quadrangulations on surfaces. To do so, we begin by introducing an important homological invariant for quadrangulations, called a cycle parity.

Let G be an even embedding on a closed surface F^2. It is easy to see that any two homotopic closed walks of G have a same length modulo 2. Hence, regarding each closed walk W of G as a closed curve on F^2, we can assign "0" or "1" to each element of $\pi_1(F^2)$

of F^2. Hence we can define a homomorphism $\rho_G : \pi_1(F^2) \to \mathbb{Z}_2$. It is easy to see that G is bipartite if and only if ρ_G is *trivial*, that is, $\rho(l) = 0$ for every simple closed curve l on F^2.

A *face contraction* of a face $f = wxyz$ in a quadrangulation on a closed surface F^2 is to identify an opposite pair w and y and replace the resulting two pairs of multiple edges (wx and yx, wz and yz) with two simple edges. Then it is easy to see that face contraction preserves a cycle parity of quadrangulations. We say f is *contractible* if the resulting graph obtained from G by a face contraction of f at one of the two diagonal pairs is simple. If G can be transformed into a quadrangulation H by a sequence of face contractions, then G is called *contractible* to H. We say G is *irreducible* if G has no contractible face, and the complete lists of irreducible quadrangulations on $\mathbb{S}_0$ [35], $\mathbb{S}_1$ [33] and $\mathbb{N}_1$ [35], $\mathbb{N}_2$ [32] have been determined so far. The finiteness of the number of irreducible quadrangulations is also known [39].

Suppose that K_n quadrangulates a closed surface F^2, then the corresponding quadrangulation must be irreducible, by definition. As in the triangulation case, since there is a non-complete irreducible quadrangulation, not every quadrangulation is contractible to a complete quadrangulation. On the other hand, Nakamoto [31] shows that for any F^2, there exists a positive integer $M(F^2)$ such that any two quadrangulations G_1 and G_2 of F^2 with $|V(G_1)| = |V(G_2)| \geq M(F^2)$ can be transformed each other by a sequence of diagonal slides and diagonal rotations (see [31] for their definitions) if and only if ρ_{G_1} is congruent to ρ_{G_2}. Analogous to this result, we try to get a positive answer for the problem:

> *Can every quadrangulation on F^2 with sufficiently large order be transformed into a complete quadrangulation by face contractions, diagonal slides and diagonal rotations?*

Then, we need to know that a complete quadrangulation of F^2 by K_n can have an arbitrary non-trivial cycle parity. (Note that a complete quadrangulation is not bipartite, and hence its cycle parity must be non-trivial.)

We restate our main result in this paper.

Theorem 4.2'. *Suppose that K_n quadrangulates a closed surface F^2. Let ρ be any non-trivial cycle parity over F^2. Then K_n has a quadrangulation whose cycle parity is ρ.*

By Theorem 4.2', using the result in [31], we have the following:

Corollary 4.3. *Suppose that K_n quadrangulates a closed surface F^2. Then there exists a positive integer $M(F^2)$ such that any non-bipartite quadrangulation on F^2 with order* *at least $M(F^2)$ can be transformed into a complete quadrangulation by edge contractions, diagonal slides and diagonal rotations.*

Theorem 4.2' has another application for the minor relation of embedded graphs. Let G and H be embeddings on the same surface F^2. We say that H is a *minor* of G if H is obtained from a subgraph of G by contracting edges. By the well-known result of Robertson and Seymour [46], every locally planar triangulation on F^2 is contractible to a complete triangulation if F^2 admits a complete triangulation.

Let us consider an analogy for even embeddings on F^2. Let G and H be connected graphs. Let $n = |V(H)|$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. We say that H is an *odd minor* of G if

(i) H is obtained from a subgraph of G by contracting edges, and

(ii) For every cycle C of H, the cycles in G corresponding to C have a same length modulo 2 as C.

For details, see [24]. Hence, if H is an odd minor of G, then G and H have a same cycle parity. Using Theorem 4.2' and the argument in [34], we can prove the following.

Corollary 4.4. *For any closed surface F^2 except for the sphere, there is a number $N = N(F^2)$ such that every non-bipartite even embedded graph G on F^2 with representativity $r(G) \geq N$ has an odd minor that is a complete quadrangulation if and only if F^2 admits a complete quadrangulation.*

In order to prove Theorem 4.2', we construct quadrangulations of complete graphs with particular parities of the lengths of cycles. We deal with the current graphs in the construction of quadrangulation but we clarify the structures of the current graph and the cycle parity of the generated quadrangulation. Moreover, we describe how to control cycle parities of the embeddings by using the current graphs.

4.2 Cycle parities

Let $\pi_1(F^2)$ be the fundamental group of a closed surface F^2. We call any homomorphism $\rho : \pi_1(F^2) \to \mathbb{Z}_2$ a *cycle parity* over F^2. A closed curve l is called even (resp. odd) under a cycle parity ρ if $\rho([l]) = 0$ (resp. $= 1$), where $[l]$ denotes the homotopy class corresponding to l. We often write $\rho(l)$ instead of $\rho([l])$. Two cycle parities ρ and ρ' are called *congruent* if there is a homeomorphism $h : F^2 \to F^2$ which induces an automorphism $h_* : \pi_1(F^2) \to \pi_1(F^2)$ with $\rho h_* = \rho'$.

To express cycle parities, we fix a system $\{a_1, b_1, \dots, a_g, b_g\}$ of simple closed curves on the orientable closed surface $\mathbb{S}_g$ like Figure 4.1, which is a generator of $\pi_1(\mathbb{S}_g)$. Note that a cycle parity can be represented as $\rho = (\rho(a_1),\ \rho(b_1),\ \dots,\ \rho(a_g),\ \rho(b_g))$. In particular, $\rho = (0, 0, \dots, 0, 0)$ is called *trivial*. For the nonorientable closed surface $\mathbb{N}_{2k+1}$, we fix a system $\{x, a_1, b_1, \dots, a_k, b_k\}$ of simple closed curves like Figure 4.2, which is a generator of $\pi_1(\mathbb{N}_{2k+1})$. Note that x is the only one-sided closed curve and the others are two-sided. Note that a cycle parity can be represented as $\rho = (\rho(x), \rho(a_1),\ \rho(b_1),\ \dots,\ \rho(a_k),\ \rho(b_k))$. For the nonorientable closed surface $\mathbb{N}_{2k}$, we fix a system $\{m, l, a_2, b_2, \dots, a_k, b_k\}$ of simple closed curves like Figure 4.3, which is a generator of $\pi_1(\mathbb{N}_{2k})$. Note that l is the only one-sided closed curve and the others are two-sided. Note that a cycle parity can be represented as $\rho = (\rho(m), \rho(l), \rho(a_2),\ \rho(b_2),\ \dots,\ \rho(a_k),\ \rho(b_k))$.

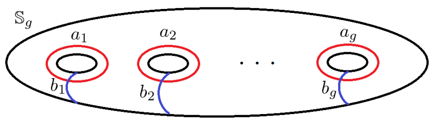

Figure 4.1: A set of generators for $\mathbb{S}_g$

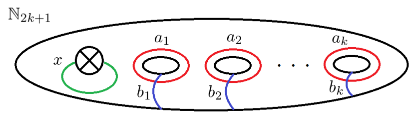

Figure 4.2: A set of generators for $\mathbb{N}_{2k-1}$

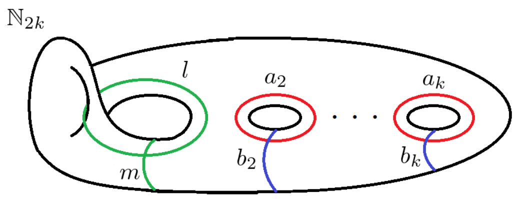

Figure 4.3: A set of generators for $\mathbb{N}_{2k}$

Another terminologies are referred to [36]. The following two theorems are shown.

Theorem 4.5. (Nakamoto, Negami and Ota [36]) *Any non-trivial cycle parity ρ on $\mathbb{N}_{2k+1}$ with $k \geq 1$ is congruent to exactly one of $A = (1, 0, 0, \dots, 0, 0)$, $B = (1, 1, 0, \dots, 0, 0)$ and $C = (0, 1, 0, \dots, 0, 0)$. Any non-trivial cycle parity ρ on $\mathbb{N}_{2k}$ with $k \geq 2$ is congruent to exactly one of $D = (0, 0, 1, 0, \dots, 0, 0)$, $E = (0, 1, 0, 0, \dots, 0, 0)$ and $F = (1, 0, 0, 0, \dots, 0, 0)$. For the Klein bottle $\mathbb{N}_2$, ρ is congruent to either $E = (0, 1)$ or $F = (1, 0)$.*

Theorem 4.6. (Nakamoto, Negami and Ota [36]) *There is a simple closed curve on the nonorientable closed surface $\mathbb{N}_k$ which is odd under a cycle parity ρ and which cuts open $\mathbb{N}_k$ into an orientable surface if and only if ρ is of either type A, B or F.*

4.3 Relations between cascades and cycle parities

In this section, we consider relations between cascades and generated graphs. We use the technique of Theorem 9 in [2].

Let G be a directed graph 2-cell embedded on $\mathbb{N}_k$. We see that $\mathbb{N}_k$ has a closed curve C which cuts open $\mathbb{N}_k$ into an orientable surface. Let W be a closed walk which is homotopic to C. Furthermore, let S be the orientable surface obtained from $\mathbb{N}_k$ by cutting it open along W and G' be the graph on S obtained from G cut by W. The resulting surface S has one (resp. two) boundary component(s) if W is one-sided (resp. two-sided). Since S is orientable, we can give an orientation of all faces of G' in which two faces incident to an edge $e \in E(G)$ are consistent to each other if e does not lie on W. That is, these orientations induce different directions of e.

Let $f \in F(G')$. We assign $+1$ (resp. -1) to each edge e on the boundary of f if its direction is the same as (resp. different from) the orientation of f. We denote this assignment by $\sigma_f(e)$. Let $\sigma_+(f)$ and $\sigma_-(f)$ be the number of edges on the boundary of f with $\sigma_f(e) = 1$ and -1, respectively. Define $\psi(f) = \sigma_+(f) - \sigma_-(f)$ for all faces f and consider their summation,

$$\psi(G') := \sum_{f \in F(G')} \psi(f) = \sum_{f \in F(G')} (\sigma_+(f) - \sigma_-(f)).$$

If an edge e does not lie on W in G', then $\sigma_f(e) = -\sigma_{f'}(e)$ for two faces f and f' incident to e in G. Then e contributes 0 to $\psi(G')$. If e lies on W, then $\sigma_f(e) = \sigma_{f'}(e)$ since $\mathbb{N}_k$ is nonorientable. Then $\psi(G')$ is the summation of $2\sigma_f(e)$ over all edges e lying on W.

Consequently, we can show the following corollary.

Corollary 4.7. *Let G be a directed graph 2-cell embedded on $\mathbb{N}_k$, C be a closed curve on $\mathbb{N}_k$ which cuts open $\mathbb{N}_k$ into an orientable surface and W be a closed walk which is homotopic to C. Furthermore, let S be the orientable surface obtained from $\mathbb{N}_k$ by cutting it open along W and G' be the graph on S obtained from G cut by W. If the length of W is odd, then $\psi(G') \equiv 2 \pmod 4$. If the length of W is even, then $\psi(G') \equiv 0 \pmod 4$.*

Now let $\langle G \to S, \beta\rangle_{2l+1}$ be a cascade and G_β be the generated graph. Let $\mathbb{Z}_{2l+1}$ be a vertex set of G_β. We define the orientations of all edges in G. Let $\pi_0 = (a_1, a_2, \ldots, a_{2l})$ be a rotation of vertex 0 in G_β. We determine an orientation of an arc $0a_i$ to $\{0, a_i\}$ if $a_i \leq l$, $\{a_i, 0\}$ if $a_i \geq l+1$. We also define an orientation of an edge st same as an edge $0(t-s)$ with indices taken modulo n.

For a graph G embedded on $\mathbb{N}_k$, we notice that we cannot define the value $\psi(f)$ but $|\psi(f)|$ of $f \in F(G)$. We also define $|\psi(G)| = \sum_{f\in F(G)} |\psi(f)|$.

Lemma 4.8. *Let $\langle G \to S, \beta\rangle_{2l+1}$ be an Eulerian cascade and G_β be the generated graph. Let $c(v)$ be the sum of the currents of $v \in V(G)$ and $r(v)$ be the order of $c(v)$ in $\mathbb{Z}_{2l+1}$. Then for the faces $f_1, \ldots, f_{(2l+1)/r(v)}$ of G_β corresponding to the vertex v, it holds that*

$$|\psi(f_i)| = r(v)|\deg^+(v) - \deg^-(v)|$$

for all $1 \leq i \leq (2l+1)/r(v)$.

Proof. See Theorem 4.4.1 of [14]. We see an example, Figures 4.4 and 4.5. Let v_1 and v_2 be neighbors of v such that $\sigma_v(v_1) = v_2$, $\beta(v_1v) = i$ and $\beta(vv_2) = j$. Since a σ-polygon goes through the arc vv_2 just after the arc v_1v, there exist vertices $0, i, j \in G_\beta$ such that $\pi_0(i) = j$. From defined orientations, we see that the arcs $0i$ and $0j$ outgo from 0. Then we see that an incoming edge and an outgoing edge from v are of different contribution for $|\psi(f)|$ in G_β. Then $|\psi(f)|$ is equal to the difference between $\deg^+(v)$ and $\deg^-(v)$.

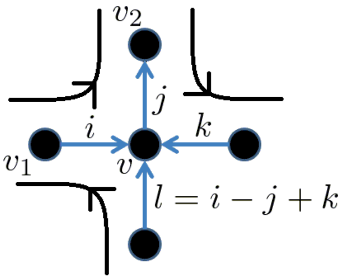

Figure 4.4: A part of a cascade

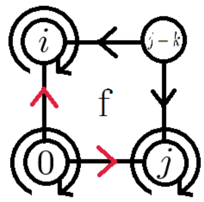

Figure 4.5: The generated graph

□

Lemma 4.9. *Let $\langle G \to S, \beta\rangle_{2l+1}$ be an Eulerian cascade and G_β be a generated graph. Then the parity of the number of broken arcs in G is equal to the parity of the number of faces in G_β which satisfy $|\psi(f)| \equiv 2 \pmod 4$.*

Proof. By Lemma 4.8,

$$\begin{aligned}\sum_{f\in F(G_\beta)}|\psi(f)| &= \sum_{v\in V(G)}\frac{2l+1}{r(v)}\cdot r(v)|\deg^+(v)-\deg^-(v)| \\ &= \sum_{v\in V(G)}(2l+1)|\deg^+(v)-\deg^-(v)|. \qquad (4.1)\end{aligned}$$

Now a non-broken arc contributes 1 to both $\deg^+$ and $\deg^-$ of its end vertices and a broken arc contributes 2 to either of the $\deg^+$ or $\deg^-$ of its end vertices. Since $2l+1$ is odd and (4.1), $|\psi(G_\beta)| \equiv 2\cdot\#\{e\in E(G)|e \text{ is a broken arc}\} \pmod 4$. This leads the desired result. □

Theorem 4.10. *Let $\langle G\to S,\beta\rangle_{2l+1}$ be an Eulerian cascade with $m\geq 1$ broken arcs. Then the generated graph G_β is an even embedding on a nonorientable closed surface $\mathbb{N}_k$. Moreover, if m is odd, then the cycle parity ρ of G_β is of type A, B or F. If m is even, then the cycle parity ρ of G_β is of type C, D or E.*

Proof. We easily see that G_β is an even embedding since G is Eulerian, the embedding is on $\mathbb{N}_k$ by Lemma 3.1 and its cycle parity ρ is non-trivial since $2l+1$ is odd. Then Theorem 4.10 holds by Theorem 4.6, Corollary 4.7 and Lemma 4.9. □

We present another way to distinguish cycle parities.

Lemma 4.11. *Let $G\to\mathbb{N}_k$ be a non-trivial 2-cell even embeddings of a graph G. There exists one-sided even closed walk in G if and only if ρ of G is of either type B, C, D or F. There exists two-sided odd closed walk in G if and only if ρ of G is of either type B, C, D or F.*

Proof. We easily see that there is no even one-sided closed walk and odd two-sided closed walk in an even embedding of a graph whose cycle parity ρ is of either type A or E. Thus sufficiency is established. To establish necessity, we show the existence of closed walks. Suppose first that ρ of G is of type B. Take a closed walk C_1 of G which cuts open $\mathbb{N}_k$ into an orientable surface and let G' be the graph obtained from G cut by C_1. Note that C_1 is one-sided. By the definition of cycle parities, the length of C_1 is odd and G' is non-bipartite. Then we can find a two-sided odd closed walk C_2 in G'. Now $C_1\cup C_2$ is a one-sided even closed walk and C_2 is a two-sided odd closed walk as desired.

The other cases can be shown similarly. □

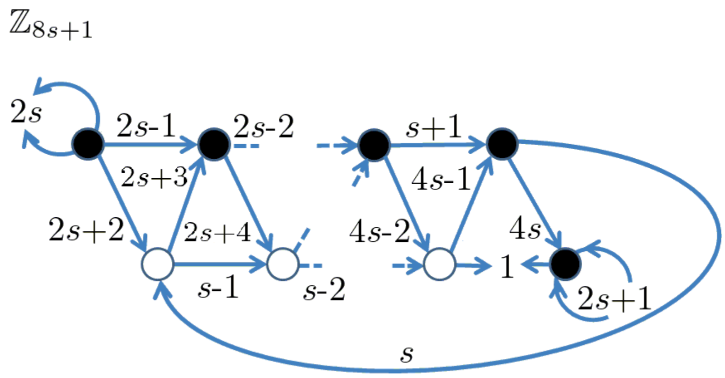

Figure 4.6: Type A of K_{8s+1}

Theorem 4.12. *Let $\langle G \to S, \beta \rangle_{2l+1}$ be an Eulerian cascade which generates an even embedding $G_\beta \to S_\beta = \mathbb{N}_k$. Then the cycle parity ρ of G_β is of either type A or E when all arcs in G are traced in the same direction on a σ-polygon and the cycle parity ρ of G_β is of type B, C, D or F when there exists an arc in G which is traced in both direction on a σ-polygon.*

Proof. We see that G_β is a non-trivial even embedding. An arc of $G_\beta \to S_\beta$ is twisted if and only if the corresponding arc of G is traced same directions on σ-polygons. If there is a non-twisted arc $0a$ in G_β, there exists a two-sided odd cycle $0, a, 2a, \ldots, -a, 0$ with indices taken modulo $2l+1$. By Lemma 4.11, the cycle parity ρ of G_β is of either type B, C, D or F. If all arcs of G_β are twisted, we see that there is no odd two-sided closed walk. By Lemma 4.11, the cycle parity ρ of G_β is of either type A or E. $\square$

4.4 Constructing cascades

In this section, we prove Theorems 4.2 and 4.2'. We show them by giving some cascades.

Proof. **Cases K_{8s+1} and K_{8s+5}.**
We construct cascades $\langle G \to S, \beta \rangle_n$ satisfying properties (P1)–(P4) in Lemma 3.2, and check their cycle parities using Theorems 4.10 and 4.12. First, we consider the cases $s \geq 2$.

A cascade in Figure 4.6 generates a quadrangulation of the complete graph on $8s+1$ vertices with type A because the cascade has 3 (odd) broken arcs and all arcs are traced same direction on the unique σ-polygon of G. A cascade in Figure 4.7 generates a quadrangulation of the complete graph on $8s+1$ vertices with type B because the cascade

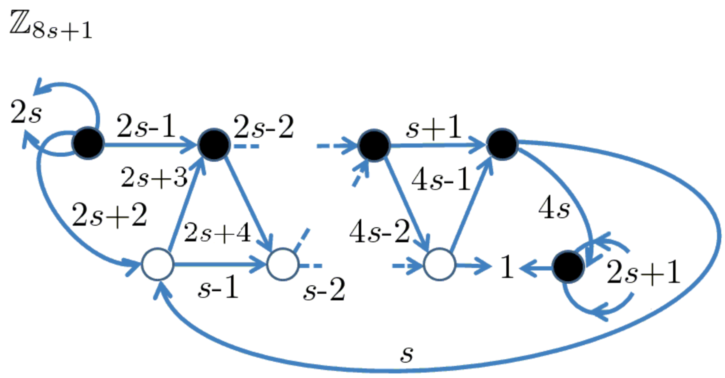

Figure 4.7: Type B of K_{8s+1}

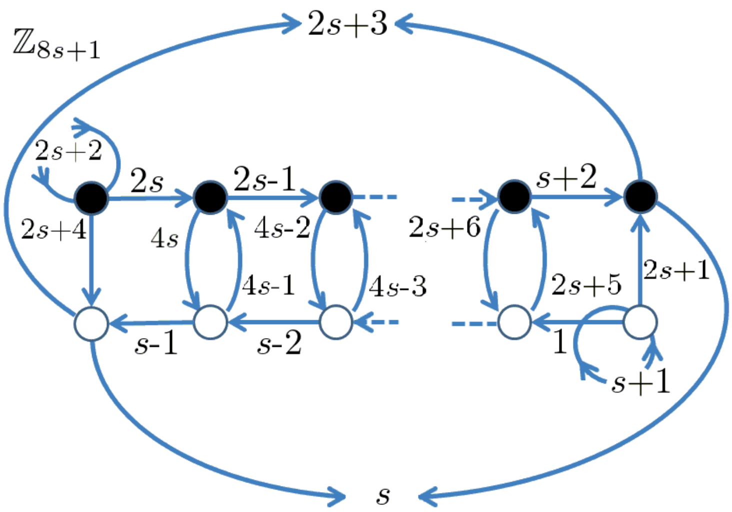

Figure 4.8: Type C of K_{8s+1}

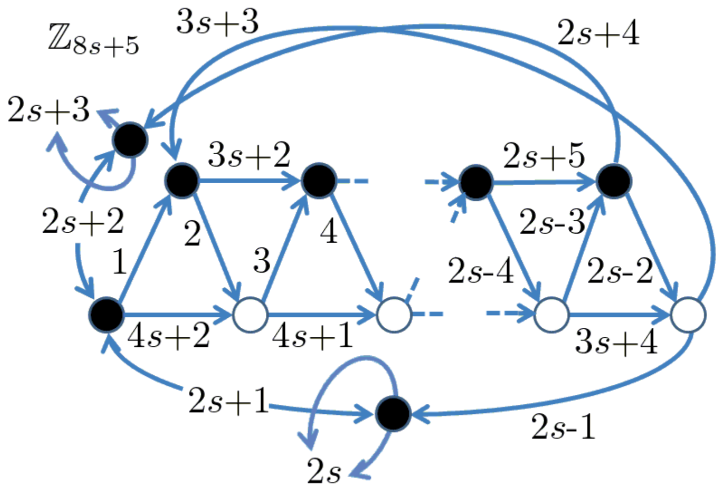

Figure 4.9: Type D of K_{8s+5}

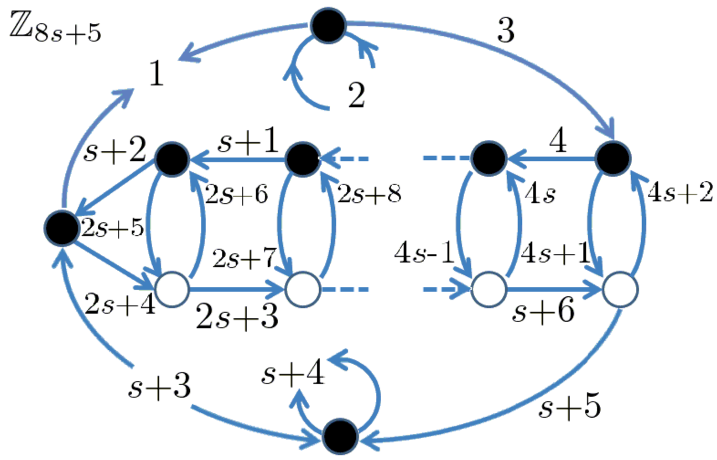

Figure 4.10: Type E of K_{8s+5}

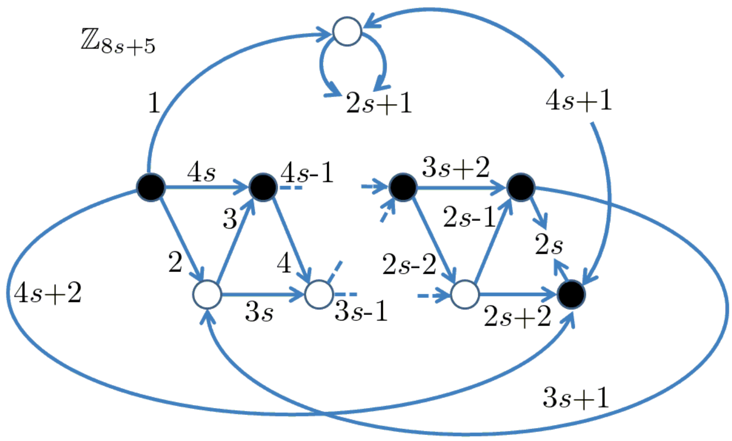

Figure 4.11: Type F of K_{8s+5}

has 3 (odd) broken arcs and arc with current $2s$ is traced both direction on the σ-polygon of G. A cascade in Figure 4.8 generates a quadrangulation of the complete graph on $8s+1$ vertices with type C because the cascade has 4 (even) broken arcs and arc with current $s+1$ is traced both direction on the σ-polygon of G.

A cascade in Figure 4.9 generates a quadrangulation of the complete graph on $8s+5$ vertices with type D because the cascade has 4 (even) broken arcs and arc with current 1 is traced same direction on the σ-polygon of G. A cascade in Figure 4.10 generates a quadrangulation of the complete graph on $8s+5$ vertices with type E because the cascade has 4 (even) broken arcs and all arcs are traced same direction on the σ-polygon of G. A cascade in Figure 4.11 generates a quadrangulation of the complete graph on $8s+5$ vertices with type F because the cascade has 3 (odd) broken arcs and arc with current 2 is traced both direction on the σ-polygon of G.

The case $s = 1$ with each type are generated from cascades in Figures 4.12 and 4.13. We can easily check that each cascade generates the complete graph with each type. We can also show that the existence of K_9 with type C, that is the case $s = 1$ and type C, see Section 4.5. But there is no cascade which generates such a graph.

Cases K_{8s+4} and K_{8s+8}.

We construct cascades $\langle G \to S, \beta\rangle_n$ satisfying properties (P1'), (P2), (P3') and (P4) in Lemma 3.4, and check their cycle parities using Theorems 4.10 and 4.12. First, we consider the cases $s \geq 1$.

A cascade in Figure 4.14 generates a quadrangulation of the complete graph on $8s+4$

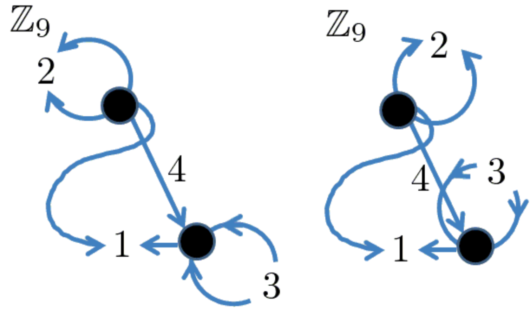

Figure 4.12: Types A and B of K_9

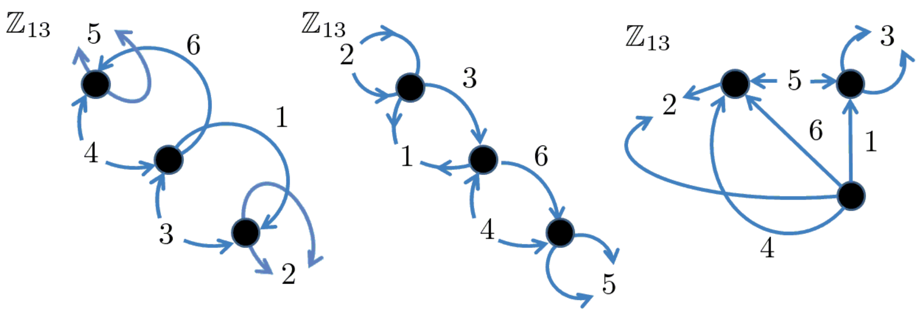

Figure 4.13: Types D, E and F of K_9

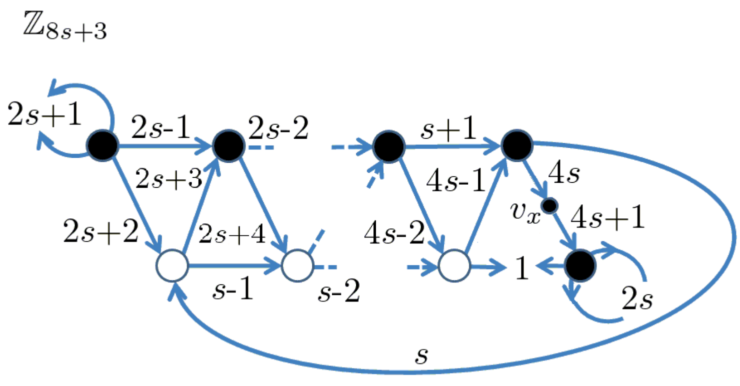

Figure 4.14: Type A of K_{8s+4}

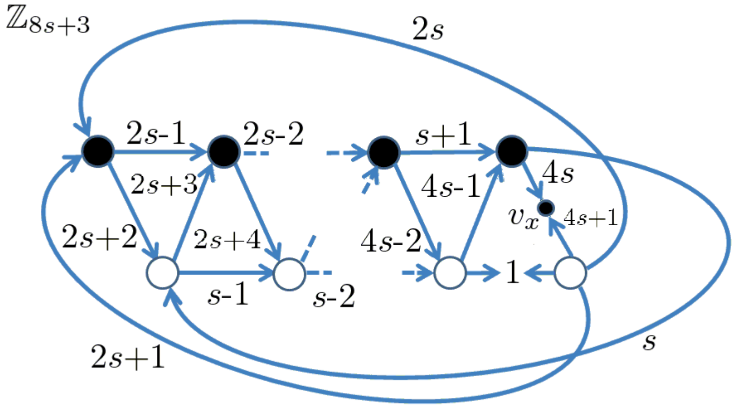

Figure 4.15: Type B of K_{8s+4}

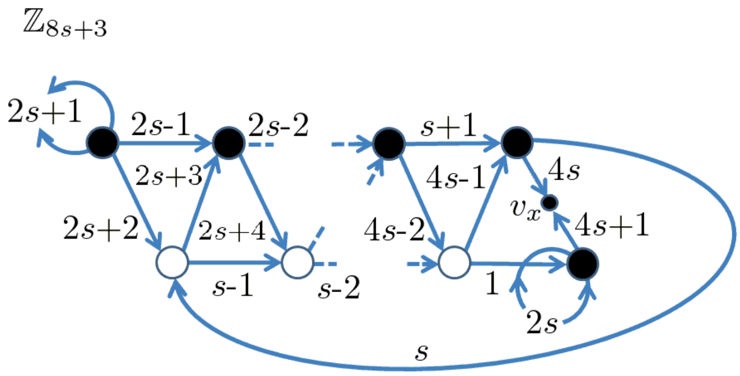

Figure 4.16: Type C of K_{8s+4}

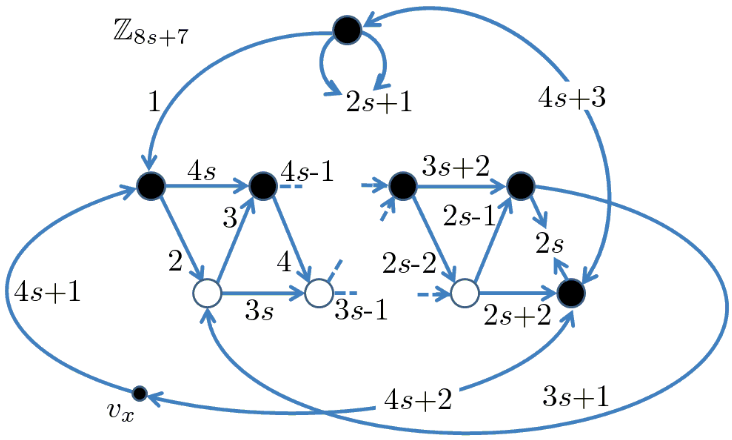

Figure 4.17: Type D of K_{8s+8}

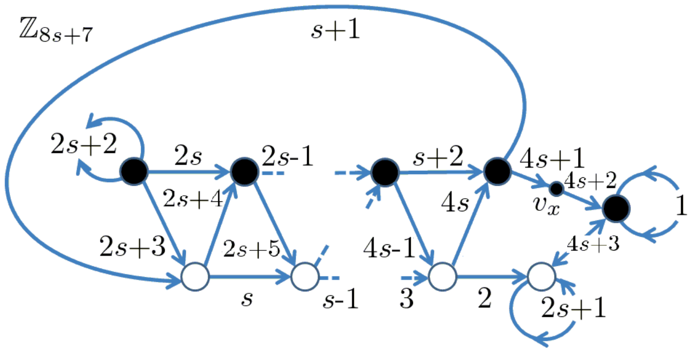

Figure 4.18: Type E of K_{8s+8}

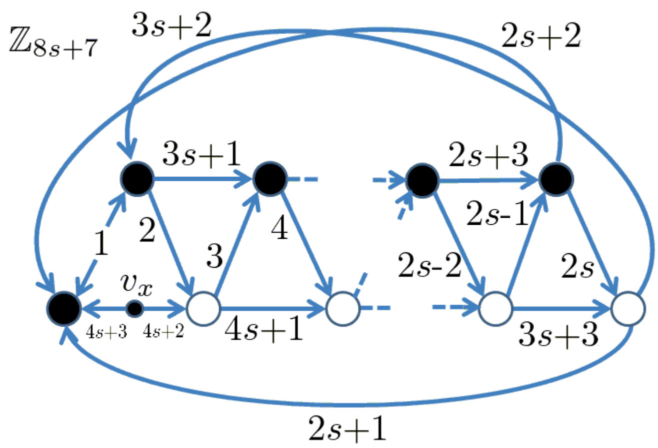

Figure 4.19: Type F of K_{8s+8}

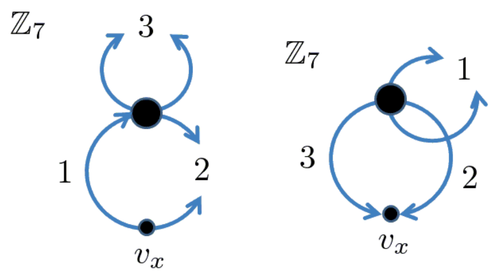

Figure 4.20: Types E and F of K_8

vertices with type A because the cascade has 3 (odd) broken arcs and all arcs are traced same direction on the σ-polygon of G. A cascade in Figure 4.15 generates a quadrangulation of the complete graph on $8s + 4$ vertices with type B because the cascade has 1 (odd) broken arcs and arc with current $2s$ is traced both direction on the σ-polygon of G. A cascade in Figure 4.16 generates a quadrangulation of the complete graph on $8s + 4$ vertices with type C because the cascade has 2 (even) broken arcs and arc with current $2s$ is traced both direction on the σ-polygon of G.

A cascade in Figure 4.17 generates a quadrangulation of the complete graph on $8s + 8$ vertices with type D because the cascade has 4 (even) broken arcs and arc with current 1 is traced same direction on the σ-polygon of G. A cascade in Figure 4.18 generates a quadrangulation of the complete graph on $8s + 8$ vertices with type E because the cascade has 4 (even) broken arcs and all arcs are traced same direction on the σ-polygon of G. A cascade in Figure 4.19 generates a quadrangulation of the complete graph on $8s + 8$ vertices with type F because the cascade has 1 (odd) broken arcs and arc with current 1 is traced both direction on the σ-polygon of G.

The case $s = 0$ with type E or F are generated from cascades in Figure 4.20. We can easily check that each cascade generates K_8 with each type. We can also show that the existence of K_8 with type D, that is the case $s = 0$ and type D, see Section 4.5. But there is no cascade which generates such a graph.

This and Corollary 3.5 complete the proof of Theorem 4.2. Theorem 4.2' also holds by Theorems 1.3, 1.4, 4.2 and Corollary 4.1.

□

We show that there are no cascades which generate K_9 with type C and K_8 with type D. The former case, by Theorem 4.10, if there exists such a cascade, it must have two vertices and four arcs which include two or four broken arcs. But we can easily check that property (P3) in Lemma 3.2 cannot hold in each case. The latter case, if there exists, it must have two vertices and three arcs which include two broken arcs. But we can easily check that such graphs with property (P4) in Lemma 3.2 generate only K_8 with type E. Then we see that there are no such cascades.

4.5 Exceptional cases

We give rotation systems corresponding to K_9 with type C and K_8 with type D. Each upper bar means a twisted arc.

K_9 **with type** C**.**

$$
\begin{aligned}
\pi_0 &= (1,8,3,\overline{4},\overline{7},\overline{5},6,\overline{2}) \\
\pi_1 &= (2,5,3,8,4,7,0,\overline{6}) \\
\pi_2 &= (3,5,4,8,7,1,\overline{0},6) \\
\pi_3 &= (\overline{4},\overline{5},0,1,2,7,8,6) \\
\pi_4 &= (5,\overline{0},1,2,6,8,\overline{3},7) \\
\pi_5 &= (6,2,1,7,\overline{3},4,\overline{0},8) \\
\pi_6 &= (7,2,\overline{1},0,\overline{8},3,4,5) \\
\pi_7 &= (8,1,\overline{0},4,5,2,3,6) \\
\pi_8 &= (0,7,5,\overline{6},4,3,2,1)
\end{aligned}
$$

There are a two-sided odd cycle $C_1 = 123$ and an even cycle $C_2 = 0163$ which cuts open $\mathbb{N}_{11}$ into an orientable surface.

K_8 **with type** D**.**

$$
\begin{aligned}
\pi_0 &= (1,5,6,7,\overline{4},3,2) \\
\pi_1 &= (4,0,7,6,\overline{5},2,3) \\
\pi_2 &= (3,7,0,1,\overline{6},5,4) \\
\pi_3 &= (6,2,1,0,\overline{7},4,5) \\
\pi_4 &= (5,1,2,3,\overline{0},7,6) \\
\pi_5 &= (0,4,3,2,\overline{1},6,7) \\
\pi_6 &= (7,3,4,5,\overline{2},1,0) \\
\pi_7 &= (2,6,5,4,\overline{3},0,1)
\end{aligned}
$$

There are a two-sided odd cycle $C_3 = 012$ and an even cycle $C_4 = 025134$ which cuts open $\mathbb{N}_8$ into an orientable surface.

Chapter 5

A brief survey of the empire problem

In this chapter, we briefly survey the results of the empire problem. On closed surfaces $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{N}_1$ and $\mathbb{N}_2$, the problem is completely solved. On general closed surfaces, the problem is partially solved.

5.1 Known results of the empire problem

We introduce classical results of the empire problem. The following is the oldest result.

Theorem 5.1. (Heawood [17]) *Let $(G, \mathcal{P})$ be an r-pire graph such that G has an embedding on a closed surface F^2 with the Euler characteristic ε. Then $(G, \mathcal{P})$ is $h_{\varepsilon,r}$-colorable, where*

$$h_{\varepsilon,r} = \left\lfloor \frac{6r + 1 + \sqrt{(6r+1)^2 - 24\varepsilon}}{2} \right\rfloor.$$

except possibly in the case $\varepsilon = 2$ and $r = 1$.

This is the upper bound of chromatic numbers of empire graphs. A lot of studies on the lower bounds have done by many researchers. For the torus, $\varepsilon = 0$ and $h_{0,r} = 6r + 1$.

Theorem 5.2. (Taylor [52]) *Every r-pire graph $(G, \mathcal{P})$ such that G has an embedding on $\mathbb{S}_1$ is $(6r + 1)$-colorable. Moreover, for each r, there is an r-pire graph embedded on $\mathbb{S}_1$ which is not $6r$-colorable.*

For the projective plane, $\varepsilon = 1$ and $h_{1,r} = 6r$.

Theorem 5.3. (Jackson and Ringel [21]) *Every r-pire graph $(G, \mathcal{P})$ such that G has an embedding on $\mathbb{N}_1$ is $6r$-colorable. Moreover, for each r, there is an r-pire graph embedded on $\mathbb{N}_1$ which is not $(6r-1)$-colorable.*

For the sphere, $\varepsilon = 2$ and $h_{2,r} = 6r$.

Theorem 5.4. (Jackson and Ringel [22]) *Every r-pire graph $(G, \mathcal{P})$ such that G has an embedding on $\mathbb{S}_0$ is $6r$-colorable. Moreover, for each $r \geq 2$, there is an r-pire graph embedded on $\mathbb{S}_0$ which is not $(6r-1)$-colorable.*

The Klein bottle case is a special one of Theorem 5.6. But we state it as an independent theorem.

Theorem 5.5. (Jackson and Ringel [20]) *Every r-pire graph $(G, \mathcal{P})$ such that G has an embedding on $\mathbb{N}_2$ is $(6r+1)$-colorable. Moreover, for each $r \geq 2$, there is an r-pire graph embedded on $\mathbb{N}_2$ which is not $6r$-colorable.*

In general cases, the following holds.

Theorem 5.6. (Jackson and Ringel [20]) *The bound $h_{\varepsilon,r}$ in Theorem 5.1 is best possible if one of the following conditions is satisfied;*

(i) *F^2 is an orientable surface, r is even, and $h_{\varepsilon,r}$ is congruent to 1 modulo 12.*

(ii) *F^2 is an orientable surface, r is odd, and $h_{\varepsilon,r}$ is congruent to 4 or 7 modulo 12.*

(iii) *F^2 is a nonorientable surface and $h_{\varepsilon,r}$ is congruent to 1, 4 or 7 modulo 12 except in the case F^2 is $\mathbb{N}_2$ and $r = 1$.*

For the other cases, it is conjectured that $h_{\varepsilon,r}$ is best possible but it has not been solved yet. Table 5.1 is a list of above theorems.

Table 5.1: Lowest upper bounds of chromatic numbers of r-pire graphs $(G, \mathcal{P})$ for $r \geq 2$

Surfaces	Lowest upper bounds of $\chi(G_{\mathcal{P}})$	Theorems
$\mathbb{S}_0$	$6r$	Theorem 5.4
$\mathbb{N}_1$	$6r$	Theorem 5.3
$\mathbb{S}_1$	$6r+1$	Theorem 5.2
$\mathbb{N}_2$	$6r+1$	Theorem 5.5
F^2 with ε	$\leq \lfloor \frac{6r+1+\sqrt{(6r+1)^2-24\varepsilon}}{2} \rfloor$	Theorems 5.1 and 5.6

Chapter 6

The empire problem in even embeddings

In this chapter, we consider the empire problem whose underlying graphs are even embeddings. We give complete solutions where a closed surface is $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{N}_1$ and $\mathbb{N}_2$, and give partial solutions in general cases. All the new results we prove in this chapter can be found in [41, 42].

6.1 Main theorems

In this section, we introduce our results of the empire problem in even embeddings of graphs. First of all, we can easily obtain the upper bound of chromatic numbers in general cases.

Theorem 6.1. *Let $(G, \mathcal{P})$ be an r-pire graph such that G has an even embedding on a closed surface F^2 with $\varepsilon \leq 0$. Then, $(G, \mathcal{P})$ is $n_{\varepsilon,r}$-colorable, where*

$$n_{\varepsilon,r} = \left\lfloor \frac{4r + 1 + \sqrt{(4r+1)^2 - 16\varepsilon}}{2} \right\rfloor. \tag{6.1}$$

Secondly, we consider about the closed surfaces with nonnegative Euler characteristic. For the sphere, we get the following statement.

Theorem 6.2. *Every r-pire graph $(G, \mathcal{P})$ such that G has an even embedding on $\mathbb{S}_0$ is $4r$-colorable. Moreover, for each $r \geq 2$, there is an r-pire graph even embedded on $\mathbb{S}_0$ which is not $(4r-1)$-colorable.*

For the projective plane, we get a similar result.

Theorem 6.3. *Every r-pire graph $(G, \mathcal{P})$ such that G has an even embedding on $\mathbb{N}_1$ is $4r$-colorable. Moreover, for each r, there is an r-pire graph even embedded on $\mathbb{N}_1$ which is not $(4r-1)$-colorable.*

Thirdly, we consider the sharpness of $n_{\varepsilon,r}$ in Theorem 6.1. We get $n_{0,r} = 4r+1$ by (6.1). For the torus, $4r+1$ is sharp.

Theorem 6.4. *Every r-pire graph $(G, \mathcal{P})$ such that G has an even embedding on $\mathbb{S}_1$ is $(4r+1)$-colorable. Moreover, for each r, there is an r-pire graph even embedded on $\mathbb{S}_1$ which is not $4r$-colorable.*

For the Klein bottle, Hutchinson [19] shows that $n_{0,1} = 5$ is not sharp. We show that $4r+1$ is sharp when $r \geq 2$.

Theorem 6.5. *Every r-pire graph $(G, \mathcal{P})$ such that G has an even embedding on $\mathbb{N}_2$ is $(4r+1)$-colorable. Moreover, for each $r \geq 2$, there is an r-pire graph even embedded on $\mathbb{N}_2$ which is not $4r$-colorable.*

Furthermore, in general cases we show that $n_{\varepsilon,r}$ in Theorem 6.1 is sharp for the cases given in the following theorem.

Theorem 6.6. *The bound $n_{\varepsilon,r}$ in Theorem 6.1 is best possible if one of the following conditions is satisfied;*

(i) *F^2 is an orientable surface, r is even, and $n_{\varepsilon,r}$ is congruent to 1 modulo 8.*

(ii) *F^2 is an orientable surface, r is odd, and $n_{\varepsilon,r}$ is congruent to 5 modulo 8.*

(iii) *F^2 is a nonorientable surface and $n_{\varepsilon,r}$ is congruent to 1 modulo 4 except in the case F^2 is $\mathbb{N}_2$ and $r = 1$.*

Theorem 6.6 implies Theorem 6.5. Table 6.1 is a list of above theorems.

6.2 Proof of Theorem 6.1

We show two lemmas to prove Theorem 6.1.

Table 6.1: Lowest upper bounds of chromatic numbers of even embedded r-pire graphs $(G, \mathcal{P})$ for $r \geq 2$

Surfaces	Lowest upper bounds of $\chi(G_\mathcal{P})$ where G is an even embedding	Theorems
$\mathbb{S}_0$	$4r$	Theorem 6.2
$\mathbb{N}_1$	$4r$	Theorem 6.3
$\mathbb{S}_1$	$4r+1$	Theorem 6.4
$\mathbb{N}_2$	$4r+1$	Theorem 6.5
F^2 with ε	$\leq \lfloor \frac{4r+1+\sqrt{(4r+1)^2-16\varepsilon}}{2} \rfloor$	Theorem 6.1 and 6.6

Lemma 6.7. *Let $(G, \mathcal{P})$ be an empire graph. If there exists an integer λ such that for any subgraph $H \subset G_\mathcal{P}$, the average degree $\bar{d}(H)$ is less than λ, then $(G, \mathcal{P})$ can be colored with λ colors.*

Lemma 6.8. *Let G be a graph which has an even embedding on F^2 and $(G, \mathcal{P})$ be an r-pire graph with the number of empires $t = |\mathcal{P}|$. Then,*

$$\bar{d}(G_\mathcal{P}) \leq 4r - \frac{4\varepsilon(F^2)}{t}.$$

Proof of Lemma 6.7. We use induction on $t = |\mathcal{P}|$. If $t \leq \lambda$, the assertion is trivial. We assume that all empire graphs with less empires than $(G, \mathcal{P})$ can be colored with λ colors. Since the average degree of $G_\mathcal{P}$ is less than λ, there exists an empire P_1 such that the vertex p_1 in $(G_\mathcal{P})$ has the degree less than λ. By the induction hypothesis, there is a coloring c' of the empire graph $(G - P_1, \mathcal{P}\backslash\{P_1\})$ with colors $\{1, 2, \ldots, \lambda\}$. Then we can extend c' to a coloring of $(G, \mathcal{P})$ as follows. Let k be one of the colors in $\{1, 2, \ldots, \lambda\}$ which is not used in $N_{G_\mathcal{P}}(p_1)$. Such an integer k exists since the degree of p_1 in $(G_\mathcal{P})$ is less than λ. Let $c : V(G) \to \{1, 2, \ldots, \lambda\}$ be defined as $c(v) = c'(v)$ if $v \notin P_1$ and $c(v) = k$ if $v \in P_1$. □

Proof of Lemma 6.8. Note that $|V(G)| \leq rt$. Since G has an even embedding, we have $4|F(G)| \leq 2|E(G)|$. Together with Euler's formula $|V(G)| - |E(G)| + |F(G)| \geq \varepsilon(F^2)$, we obtain that $|E(G)| \leq 2rt - 2\varepsilon(F^2)$. Since $(G, \mathcal{P})$ consists of t empires and $|E(G)|$ edges, the average degree $\bar{d} = \bar{d}(G_\mathcal{P})$ is no more than $2|E(G)|/t$. So

$$\bar{d} \leq \frac{2|E(G)|}{t} \leq 4r - \frac{4\varepsilon(F^2)}{t}. \tag{6.2}$$

□

Proof of Theorem 6.1. Let t be the number of empires. We estimate the average degree $\bar{d} = \bar{d}(G_{\mathcal{P}})$. Note that

$$\bar{d} \leq t - 1. \tag{6.3}$$

Since $\varepsilon \leq 0$, by (6.2) and (6.3), we have $\bar{d} \leq 4r - 4\varepsilon/(\bar{d}+1)$, and hence $\bar{d}^2 - (4r-1)\bar{d} - 4r + 4\varepsilon \leq 0$. Thus,

$$\bar{d} \leq \frac{4r - 1 + \sqrt{(4r+1)^2 - 16\varepsilon}}{2}.$$

The same is true for all subgraphs H of $G_{\mathcal{P}}$. Therefore we obtain the formula (6.1) by Lemma 6.7. □

6.3 The spherical case

In this section, we consider empire graphs on $\mathbb{S}_0$.

Proof of Theorem 6.2. Let G be a graph which has an even embedding on $\mathbb{S}_0$ and $(G, \mathcal{P})$ be an r-pire graph. Note that $\varepsilon(\mathbb{S}_0) = 2$. By Lemma 6.8, for all $H \subset G_{\mathcal{P}}$, $\bar{d}(H) \leq 4r - 8/t < 4r$. By Lemma 6.7, we see that $4r$ colors are sufficient. We construct an r-pire graph $(G, \mathcal{P})$ such that G has an even embedding on $\mathbb{S}_0$ and the graph $G_{\mathcal{P}}$ is K_{4r}. The graph in Figure 6.1 is an example for the case $r = 2$. In the graph, the number in the circles represents each empire. Then it has eight empires $1, 2, \ldots, 8$ and any two empires are adjacent to each other. Next, we show the case $r = 3$. First, we put together the graph in Figure 6.1 and the graph in Figure 6.2. Let u_7 and u_8 be the vertices of the empires 7 and 8, respectively, which lie on the outer face of the graph in Figure 6.1, and let v_{10} and v_{12} be the vertices of the empires 10 and 12, respectively, which lie on the outer face of the graph in Figure 6.2. We add the edges u_7v_{10}, u_7v_{12}, u_8v_{10} and u_8v_{12}. Then we obtain the desired 3-pire graph.

Next, we show the case $r = 4$. We take two copies of the empire graph in Figure 6.1, say $(G_1, \mathcal{P}_1)$ and $(G_2, \mathcal{P}_2)$. Let $(G'_2, \mathcal{P}'_2)$ be the empire graph obtained from $(G_2, \mathcal{P}_2)$ by replacing the empires $1 \mapsto 9, 2 \mapsto 10, \ldots, 8 \mapsto 16$. Note that the empires $9, 16, 10, 15$ appear in the outer face boundary of G'_2 in this order. Let u_1, u_8, u_2 and u_7 be the vertices of the empires $1, 8, 2$ and 7, respectively, which lie on the outer face of G_1. Let v_9, v_{16}, v_{10} and v_{15} be the vertices of the empires $9, 16, 10$ and 15, respectively, which lie on the outer face of G'_2. We add the edges u_1v_{15}, u_8v_{10}, u_2v_{16} and u_7v_9. Then we obtain the empire graph $(G', \mathcal{P}')$. By the construction, G' also has an even embedding on $\mathbb{S}_0$. Next, we consider the

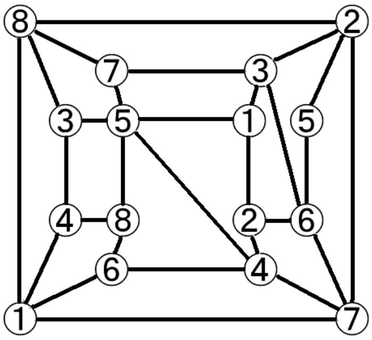

Figure 6.1: A 2-pire graph on $\mathbb{S}_0$

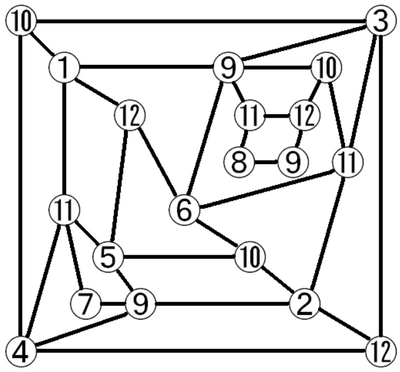

Figure 6.2: A 3-pire graph on $\mathbb{S}_0$

empire graph $(G'', \mathcal{P}'')$ in Figure 6.3, where the top and the bottom are identified. Let H be the graph obtained by putting together G' and G'', and $\mathcal{Q} = \{P'_1 \cup P''_1, \ldots, P'_{16} \cup P''_{16}\}$, where $\mathcal{P}' = \{P'_1, \ldots, P'_{16}\}$ and $\mathcal{P}'' = \{P''_1, \ldots, P''_{16}\}$. Note that in $(H, \mathcal{Q})$, any two empires are adjacent to each other, and the number of components is two. We have constructed the desired 4-pire graph.

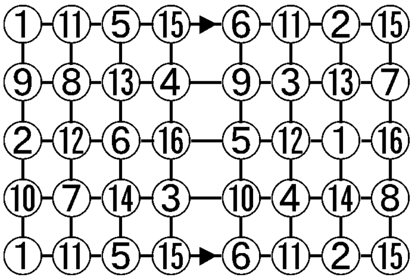

Figure 6.3: A part of a 4-pire graph on $\mathbb{S}_0$

To complete the proof of Theorem 6.2, we prove the following lemma by induction on r.

Lemma 6.9. *Let $r \geq 4$. Then there exists an r-pire graph $(G, \mathcal{P})$ where G has an even embedding on $\mathbb{S}_0$ such that*

(i) *G has two components each of which has a quadrangular outer face in which the four vertices are in different empires, and*

(ii) *$G_{\mathcal{P}}$ is K_{4r}.*

Proof. The case $r = 4$ is as mentioned before. Suppose $r \geq 5$. We assume that a desired $(r-1)$-pire graph $(G_{r-1}, \mathcal{P}_{r-1})$ exists. Let X and Y be the two components of G_{r-1}. Without loss of generality, we may assume that two nonadjacent vertices of the outer face belong to the empires 1 and 2 on X, 3 and 4 on Y, respectively. Next, we add four empires A, B, C, and D. Then we shall add one vertex which belongs to each empire from 1 to $4(r-1)$, and r vertices which belong to each of the empires A, B, C and D. We construct the graph in Figure 6.4 (the top and the bottom are identified), in which each empire in $\{A, B, C, D\}$ has $(r-2)$ vertices labeled same characters and there are $4(r-3)+2 = 4(r-1)-6$ other vertices. These $4(r-1)-6$ vertices correspond to one vertex of each empire from 7 to $4(r-1)$ respectively. We see that each empire from 7 to $4(r-1)$ is adjacent to all of the empires A, B, C and D.

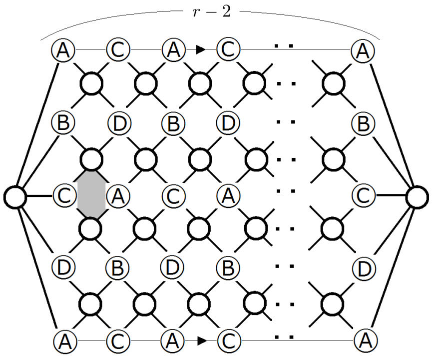

Figure 6.4: A part of G_r

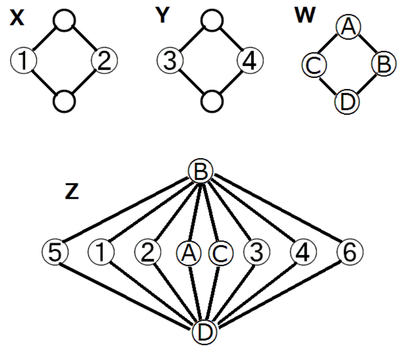

Figure 6.5: X,Y,Z and W

Let f_1 be a face of the graph in Figure 6.4 that has nonadjacent vertices a and c which belong to the empires A and C, respectively. For example, the shaded face in the figure. We construct the graph Z in Figure 6.5, and embed X, Y and Z in f_1. Let x_1 and x_2 be the vertices of the empires 1 and 2, respectively, which lie on the outer face of X, let y_3 and y_4 be the vertices of the empires 3 and 4, respectively, which lie on the outer face of Y, and let z_5 and z_6 be the vertices of the empires 5 and 6, respectively, which lie on the outer face of Z. We add the edges x_1a, x_1c, x_2a, x_2c, y_3a, y_3c, y_4a, y_4c, z_5a, z_5c, z_6a and z_6c. It is obvious that this operations can be done without creating edge crossing, and does not create odd faces. Finally, we construct the graph W in Figure 6.5. We obtain the desired r-pire graph $(G_r, \mathcal{P}_r)$ with two components such that one is the graph in Figure 6.4 with the described changes and the other is W. Each component has a quadrangular outer face in which the four vertices are in different empires. Then (i) holds. Clearly (ii) also holds. $\square$

The proof of Lemma 6.9 completes the proof of Theorem 6.2. $\square$

6.4 The projective planar case

In this section, we consider empire graphs on $\mathbb{N}_1$.

Proof of Theorem 6.3. Let $(G, \mathcal{P})$ be an r-pire graph such that G has an even embedding on $\mathbb{N}_1$. Note that $\varepsilon(\mathbb{N}_1) = 1$. By Lemma 6.8, for all $H \subset G_{\mathcal{P}}$, $\bar{d}(H) \leq 4r - 4/t < 4r$. By Lemma 6.7, we see that $4r$ colors are sufficient. Since every graph which has an embedding on $\mathbb{S}_0$ also has the one on $\mathbb{N}_1$, it is clear that for all $r \geq 2$, there exists some r-pire graphs that can be embedded on $\mathbb{N}_1$ which is not $(4r-1)$-colorable by Theorem 6.2. For the case $r = 1$, we see that K_4 can be embedded on $\mathbb{N}_1$ so that all faces are quadrilaterals. This completes the proof of Theorem 6.3. $\square$

6.5 The toroidal case

In this section, we consider empire graphs on $\mathbb{S}_1$.

Proof of Theorem 6.4. Let $(G, \mathcal{P})$ be an r-pire graph such that G has an even embedding on $\mathbb{S}_1$. Now $\varepsilon(\mathbb{S}_1) = 0$. By Lemma 6.8, for all $H \subset G_{\mathcal{P}}$, $\bar{d}(H) \leq 4r < 4r + 1$. By Lemma 6.7, we see that $4r + 1$ colors are sufficient. We construct an r-pire graph $(G, \mathcal{P})$ such that G has an even embedding on $\mathbb{S}_1$ and the graph $G_{\mathcal{P}}$ is K_{4r+1}. We also use induction on r.

(i) Cases $r = 1, 2, 3$. See the graph in Figure 6.6, one in Figure 6.7 and one in Figure 6.8 (in each graph, the top and the bottom, the left and the right are respectively identified with appropriate twists).

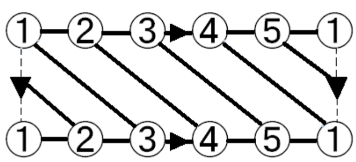

Figure 6.6: A 1-pire graph on $\mathbb{S}_1$

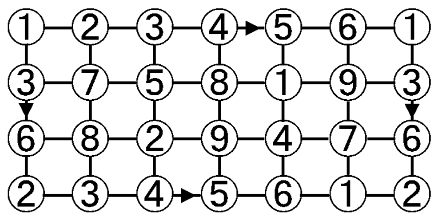

Figure 6.7: A 2-pire graph on $\mathbb{S}_1$

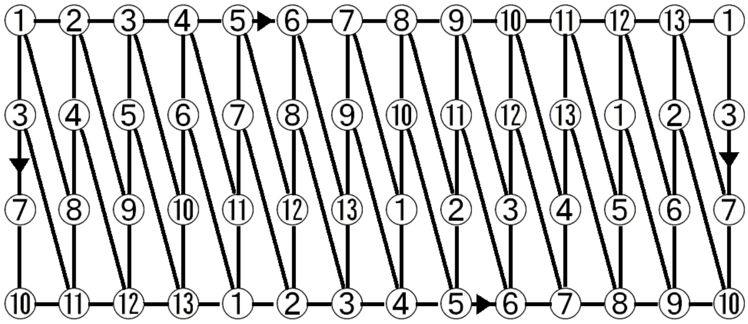

Figure 6.8: A 3-pire graph on $\mathbb{S}_1$

(ii) Case $r \geq 4$. Assume that a desired $(r-1)$-pire graph $(G_{r-1}, \mathcal{P}_{r-1})$ exists. Then we shall add one vertex which belongs to each empire from 1 to $4(r-1)+1$, and r vertices which belong to each of new empires A, B, C and D. We prepare the graph in Figure 6.4, in which each empire in $\{A, B, C, D\}$ has $r-2$ vertices and each of other vertices corresponds to a vertex of each empire from 8 to $4(r-1)+1$. Then A, B, C and D are adjacent to all empires from 8 to $4(r-1)+1$. We redraw it in the plane so that the outer face has nonadjacent vertices a and c which belong to the empires A and C, respectively. We name the graph G'. We have not used one vertex which belongs to the empires from 1 to 7 and two vertices which belong to the empires A, B, C and D yet. Take a face f_1 of G_{r-1} with nonadjacent vertices u_1 and u_2. We may assume that u_1 belongs to the empire 1 and u_2 belongs to 2. Then we embed G' in f_1, and add the edges u_1a, u_1c, u_2a and u_2c. We take a vertex u_7 in G_{r-1} which has neighbors u_3, u_4, u_5 and u_6 appearing consecutively in this order around u_7, such that u_7, u_3, u_4, u_5 and u_6 belong to pairwise distinct empires other than 1 or 2. We may assume that u_i ($i \in \{3, 4, 5, 6, 7\}$) belongs to the empire i. Then we embed two vertices from A, B, C and D into G_{r-1} as the graph in Figure 6.9. In this graph, all pairs (p, q) of empires are adjacent except for $p \in \{1, 2, 3, 4\}$ and $q \in \{B, D\}$, or $p \in \{5, 6\}$ and $q \in \{A, B\}$, or $p = 7$ and $q \in \{A, C\}$. Then we embed each vertex from 1 to 7 into G' as the graph in Figure 6.10, we get the desired r-pire graph.

□

6.6 The Klein bottlal case

In this section, we consider empire graphs on $\mathbb{N}_2$.

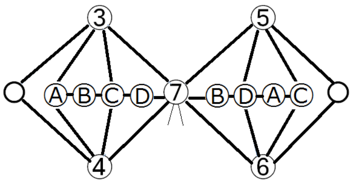

Figure 6.9: A part of G_{r-1}

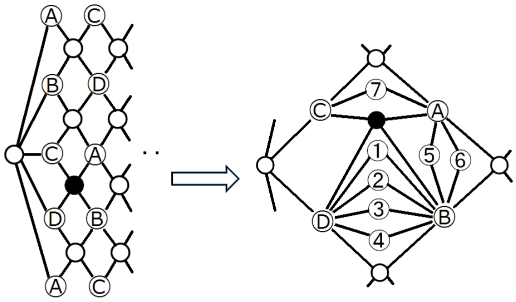

Figure 6.10: A part of G_r

Proof of Theorem 6.5. Let $(G, \mathcal{P})$ be an r-pire graph such that G has an even embedding on $\mathbb{N}_2$. Now $\varepsilon(\mathbb{N}_2) = 0$. By Lemma 6.8, for all $H \subset G_{\mathcal{P}}$, $\bar{d}(H) \leq 4r < 4r + 1$. By Lemma 6.7, we see that $4r + 1$ colors are sufficient. We construct an r-pire graph $(G, \mathcal{P})$ which has an even embedding on $\mathbb{N}_2$ such that the graph $G_{\mathcal{P}}$ is K_{4r+1}. Cases $r = 2, 3$ are the graph in Figure 6.11 and one in Figure 6.12 (the top and the bottom, the left and the right are respectively identified along with the arrow). For $r \geq 4$, we can apply a similar inductive construction as in the case of $\mathbb{S}_1$.

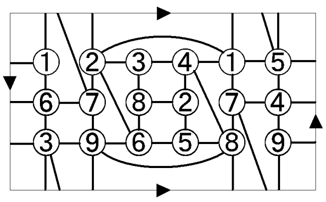

Figure 6.11: A 2-pire graph on $\mathbb{N}_2$

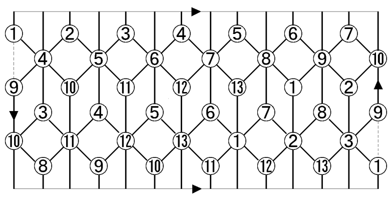

Figure 6.12: A 3-pire graph on $\mathbb{N}_2$

□

6.7 General cases

In this section, we consider general cases.

Lemma 6.10. *Let r and n be integers with $0 < 4r \leq n - 1$. If there exists a graph G which has a quadrangulation on a closed orientable (nonorientable) surface F^2 with the Euler characteristic ε and an r-pire graph $(G, \mathcal{P})$ with n mutually adjacent r-pires without extra adjacencies of vertices, then $n = n_{\varepsilon,r}$ and hence $n_{\varepsilon,r}$ in Theorem 6.1 is best possible for F^2 and r.*

Proof. By Theorems 6.2, 6.3 and the assumption $4r \leq n-1$, we see that we may only consider the cases $\varepsilon \leq 0$. Let r and n be given. We assume that there exists an r-pire graph $(G, \mathcal{P})$ in the assumption of Lemma 6.10 on a closed surface F^2 with the Euler characteristic ε. Here we have $|V(G)| = rn$, $|E(G)| = n(n-1)/2$ and $|F(G)| = n(n-1)/4$. By Euler's formula, we obtain $4\varepsilon = n(4r-n+1)$. Thus, we have $n = \frac{4r+1+\sqrt{(4r+1)^2-16\varepsilon}}{2} = n_{\varepsilon,r}$. This completes the proof of Lemma 6.10. □

We show Theorem 6.6.

Proof of Theorem 6.6. By Lemmas 3.2, 3.3 and 6.10, for all r, we only have to construct current graphs and cascades with properties (P1)–(P3) in Lemma 3.2 and $n = n_{\varepsilon,r}$. We may assume $4r \leq n-1$.

We divide the proof of Theorem 6.6 into three cases depending on the conditions.

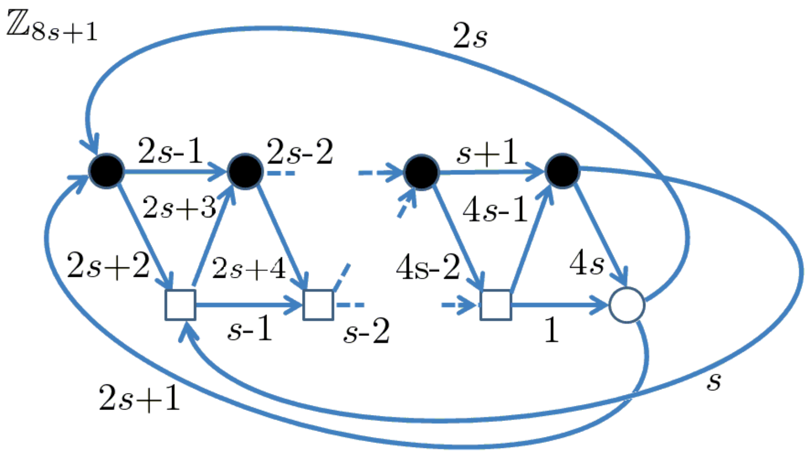

Figure 6.13: Case (i)

Case (i). F^2 is orientable, $n_{\varepsilon,r}$ is congruent to 1 modulo 8 and r is even. Let $n = 8s+1$. We use the current graph in Figure 6.13. The colors of the vertices represented by squares in the figure will be assigned to black or white depending on r. Label these vertices from left to right by $v_1, v_2, \ldots, v_{s-1}$. For every even integer r with $2 \leq r \leq 2s$, we choose the vertices $v_1, v_2, \ldots, v_{r/2-1}$ to be black and the vertices $v_{r/2}, v_{r/2+1}, \ldots, v_{s-1}$ to be white. Each vertex v in the current graph has a rotation σ_v which is a cyclic permutation of the neighbors of v depending on its color. Let σ be a rotation system consisting of the collection of σ_v for all v in the current graph. We consider the σ-polygons to take π_0.

If $r = 2$, then π_0 is

$$\big(2s-1,\ldots,s+1,s,s-1,\ldots,1,2s+1,-2s,-4s,\ldots,-(2s+2),-(2s+1),2s\big)$$
$$\cdot\big(-s,4s,-1,4s-1,-(s+1),4s-2,-2,\ldots,-(2s-1),2s+2\big).$$

Since there exists a σ-polygon W containing the arcs with the currents $2s$ and $-2s$, $2s$ is relatively prime to $8s+1$, W meets another σ-polygon on some arc, then the generated r-pire graph is connected.

If $r \neq 2$, then π_0 is

$$\big(2s-1,\ldots,s+1,s,-(2s+2),-(2s+1),2s\big)$$
$$\cdot\big(-2s,-4s,-(4s+1),\ldots,-(2s+r-1),s-r/2+1,1,2s+1\big)$$
$$\cdot\big(-s,4s,-1,\ldots,-(s-1)\big)$$
$$\cdot\big(2s+r-1,-(2s-r/2+1),2s+r-2\big)$$
$$\cdots$$
$$\cdot\big(2s+3,-(2s-1),2s+2\big).$$

Then the number of σ-polygons is r. Note that there exist σ-polygons W_1 containing the arcs with the currents $2s$, $-(2s+1)$ and s, W_2 containing the arcs with the currents $-2s$ and $2s+1$, and W_3 containing the arc with the current $-s$. Since $2s+(2s+1) = 4s+1$ is relatively prime to $8s+1$, we see that each vertex of the empires corresponding to W_1 and W_2 is connected. Since W_1 meets W_3 on the arc with the current s, and every other σ-polygon meets W_1 or W_3 on some arc, then the generated r-pire graph is connected. Therefore for each pair of $s \geq 1$ and even $r \leq 2s$ we obtain a connected r-pire graph of $8s+1$ mutually adjacent r-pires with a quadrangulation on an orientable closed surface. This completes the proof for Case (i).

Case (ii). F^2 is orientable and $n_{\varepsilon,r}$ is congruent to 5 modulo 8. Let $n = 8s+5$. We use the current graph in Figure 6.14. Label the square vertices from left to right by $v_1, v_2, \ldots, v_s$. For every odd integer r with $1 \leq r \leq 2s+1$, we choose the vertices $v_1, v_2, \ldots, v_{(r-1)/2}$ to be black and the vertices $v_{(r+1)/2}, v_{(r+3)/2}, \ldots, v_s$ to be white. As in the previous case, let σ be a rotation system of the current graph.

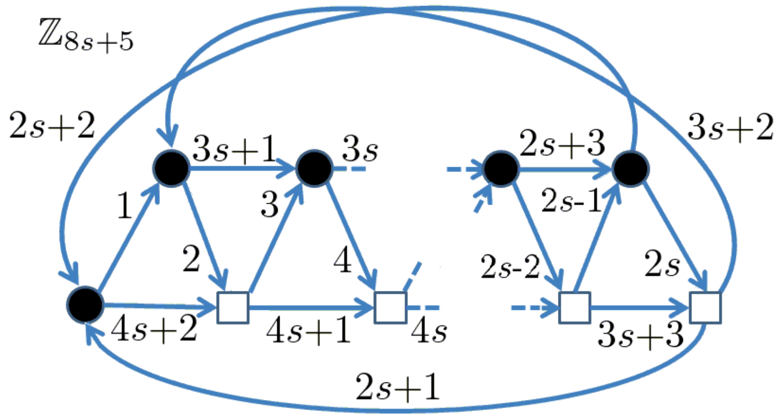

Figure 6.14: Case (ii)

If $r \neq 2s+1$, then π_0 is

$$\big(-(2s+1), 3s+2, 3s+1, \ldots, 2s+3, 2s+2, 1, -(3s+2), -2s, -(2s-1), \ldots,$$
$$-r, 4s-(r-5)/2, \ldots, 3s+3, 2s+1, -(2s+2), 2s, -(3s+3), 2s-1,$$
$$-(2s+3), \ldots, -(4s-(r-5)/2), -(4s+2)\big)$$
$$\cdot\big(r, -(3s-(r-5)/2), r-1\big)$$
$$\cdots$$
$$\cdot\big(2, -(4s+2), 1\big).$$

Then the number of σ-polygons is r. Since there exists a σ-polygon W containing the arcs with the currents $2s+1$ and $-(2s+1)$, $2s+1$ is relatively prime to $8s+5$, then we see that each vertex of the empire corresponding to W is connected. Since every other σ-polygon meets W on some arc, then the generated r-pire graph is connected.

If $r = 2s+1$, then π_0 is

$$\big(3s+2, 3s+1, \ldots, 2s+3, 2s+2, 1, -(3s+2), 2s+1, -(2s+2), 2s\big)$$
$$\cdot\big(-(2s+1), -(3s+3), -(3s+4), \ldots, -(4s+2)\big)$$
$$\cdot\big(-2s, -(2s-1), 3s+3\big)$$
$$\cdots$$
$$\cdot\big(2, -(4s+2), 1\big).$$

Then the number of σ-polygons is r. Note that there exist σ-polygons W_1 containing the

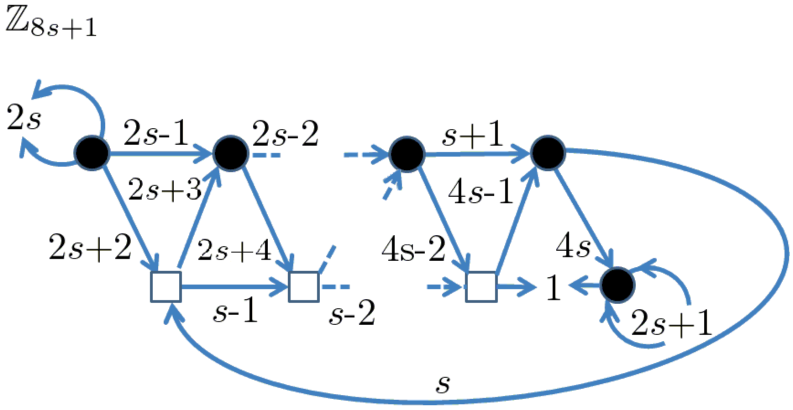

Figure 6.15: Case (iii)-1, r: odd

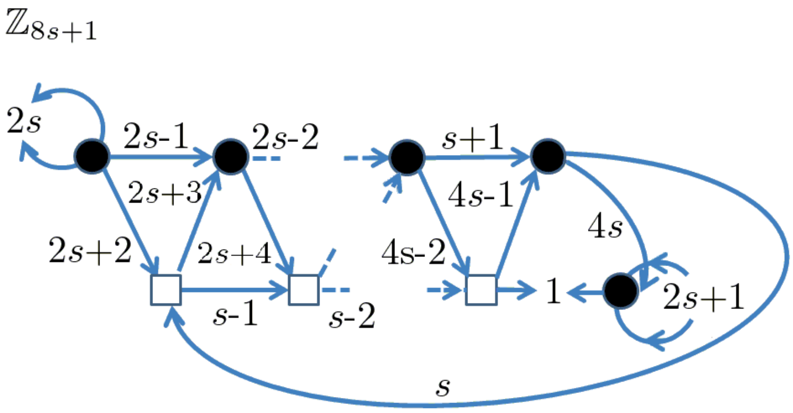

Figure 6.16: Case (iii)-1, r: even

arcs with the currents $3s+2$, $-(3s+2)$ and $2s+1$, and W_2 containing the arc with the current $-(2s+1)$. Since $3s+2$ is relatively prime to $8s+5$, then we see that each vertex of the empire corresponding to W_1 is connected. Since W_1 meets W_2 on the arc with the current $2s+1$, and every other σ-polygon meets W_1 or W_2 on some arc, then the generated r-pire graph is connected. Therefore for each pair of $s \geq 1$ and odd $r \leq 2s+1$ we obtain a connected r-pire graph of $8s+5$ mutually adjacent r-pires with a quadrangulation on an orientable closed surface. This completes the proof for Case (ii).

Case (iii)-1. F^2 is nonorientable and $n_{\varepsilon,r}$ is congruent to 1 modulo 8. Let $n = 8s+1$. For every $s \geq 1$ we use the cascade in Figure 6.15 or one in Figure 6.16 depending on the parity of r. There are three broken arcs with the currents 1, $2s$ and

$2s+1$ respectively. Label the square vertices from left to right by $v_1, v_2, \ldots, v_{s-1}$. For every odd integer r with $1 \le r \le 2s-1$, we choose the vertices $v_1, v_2, \ldots, v_{(r-1)/2}$ to be black and the vertices $v_{(r+1)/2}, v_{(r+3)/2}, \ldots, v_{s-1}$ to be white in the cascade in Figure 6.15. For every even integer r with $2 \le r \le 2s$, we choose the vertices $v_1, v_2, \ldots, v_{r/2-1}$ to be black and the vertices $v_{r/2}, v_{r/2+1}, \ldots, v_{s-1}$ to be white in the cascade in Figure 6.16. As in the previous cases, let σ be a rotation system of the current graph.

We check the nonorientability and the connectedness. Since the order of the cyclic group is odd, by Lemma 3.1, we see that the generated r-pire graphs are embedded on nonorientable closed surfaces. If $r = 1$, generated graph is the complete graph on $8s+1$ vertices.

If $r \neq 1$ and odd, then π_0 is

$$\begin{aligned}
&\big(2s, -2s, 2s-1, 2s-2, \ldots, s+1, s, -(2s+2)\big)\\
&\cdot\big(4s, -(2s+1), 2s+1, 1, 2, \ldots, s-(r-1)/2, -(2s+r), -(2s+r+1), \ldots,\\
&\quad -(4s-2), -(4s-1), -4s, -1, 4s-1, -(s+1), 4s-2, -2, \ldots, 2s+r+1,\\
&\quad -(s-(r-1)/2), -(s-(r-3)/2, \ldots, -(s-1), -s\big)\\
&\cdot\big(2s+r, -(2s-(r-1)/2, 2s+r-1\big)\\
&\cdots\\
&\cdot\big(2s+3, -(2s-1), 2s+2\big).
\end{aligned}$$

If $r = 2$, then π_0 is

$$\begin{aligned}
&\big(2s, -2s, 2s-1, 2s-2, \ldots, s+1, s, s-1, \ldots, 1, -(2s+1), -4s, -(4s-1), \ldots,\\
&\quad -(2s+2)\big)\\
&\cdot\big(4s, -(2s+1), -1, 4s-1, -(s+1), 4s-2, -2, 4s-3, -(s+2), \ldots, 2s+4,\\
&\quad -(s-1), 2s+3, -(2s-1), 2s+2, -s\big).
\end{aligned}$$

If $r \neq 2$ and even, then π_0 is

$$\begin{aligned}
&\big(2s, -2s, 2s-1, 2s-2, \ldots, s+1, s, \ldots, -(2s+2)\big)\\
&\cdot\big(4s, -(2s+1), -1, 4s-1, s+1, 4s-2, -2, \ldots, 2s+r, -(s-r/2+1),\\
&\quad -(s-r/2+2), \ldots, -s\big)\\
&\cdot\big(-4s, -(4s-1), \ldots, -(2s+r-1), s-r/2+1, s-r/2, \ldots, 1, 2s+1\big)\\
&\cdot\big(2s+r-1, -(2s-r/2+1), 2s+r-2\big)\\
&\cdots\\
&\cdot\big(2s+3, -(2s-1), 2s+2\big).
\end{aligned}$$

In each case, we see that the number of σ-polygons is r. Note that there exist σ-polygons W_1 containing the arcs with the currents $2s$, $-2s$ and s, and W_2 containing the arc with the current $-s$. Since $2s$ is relatively prime to $8s+1$, then we see that each vertex of the empire corresponding to W_1 is connected. Since W_1 meets W_2 on the arc with the current s and every other σ-polygon meets W_1 or W_2 on some arc, then the generated r-pire graph is connected. Therefore for each pair of $s \geq 1$ and $r \leq 2s$ we obtain a connected r-pire graph of $8s+1$ mutually adjacent r-pires with a quadrangulation on a nonorientable closed surface.

Case (iii)-2. F^2 is nonorientable and $n_{\varepsilon,r}$ is congruent to 1 modulo 8. Let $n = 8s+5$. For every $s \geq 1$ we use the cascade in Figure 6.17, one in Figure 6.18 or one in Figure 6.19. There are three broken arcs with the currents $s, s+1$ and $2s+1$ respectively. If s is odd, we put $(1,2), (3,4), \ldots, (s-2, s-1), (s+2, s+3), (s+4, s+5), \ldots, (2s-1, 2s)$ as the currents to the pairs of multiple edges in each figure from left to right to satisfy property (P3). If s is even, we exchange the currents $3s+1$ and $3s+3$, put the currents $s-1$ and $s+2$ to the rightmost multiple edges in each figure like Figure 6.20 and put $(1,2), (3,4), \ldots, (s-3, s-2), (s+3, s+4), (s+5, s+6), \ldots, (2s-1, 2s)$ as the currents to the pairs of multiple edges in each figure from left to right to satisfy property (P3). Label the square vertices from left to right by $v_1, v_2, \ldots, v_{s-1}$. For every odd integer r with $3 \leq r \leq 2s+1$, we choose the vertices $v_1, v_2, \ldots, v_{(r-3)/2}$ to be black and the vertices $v_{(r-1)/2}, v_{(r+3)/2}, \ldots, v_{s-1}$ to be white in the cascade in Figure 6.17. For $r = 1$ we use the cascade in Figure 6.18. For every even integer r with $2 \leq r \leq 2s$, we choose the vertices $v_1, v_2, \ldots, v_{r/2-1}$ to be black and the vertices $v_{r/2}, v_{r/2+1}, \ldots, v_{s-1}$ to be white in the cascade in Figure 6.19. As in the previous cases, let σ be a rotation system of the current graph.

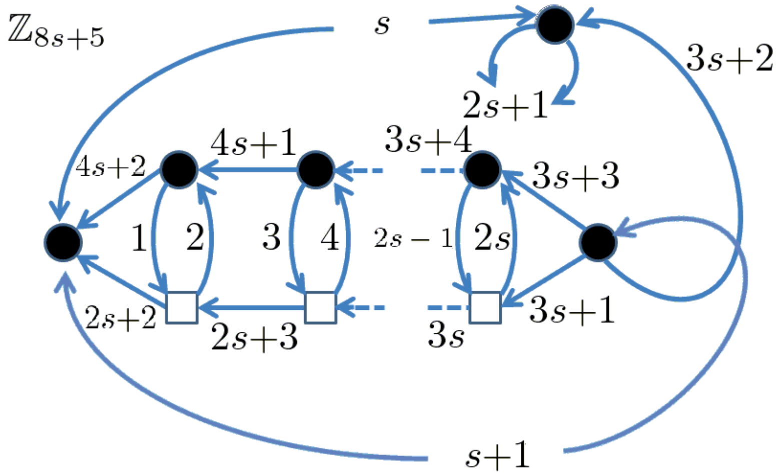

Figure 6.17: Case (iii)-2, $r \geq 3$: odd

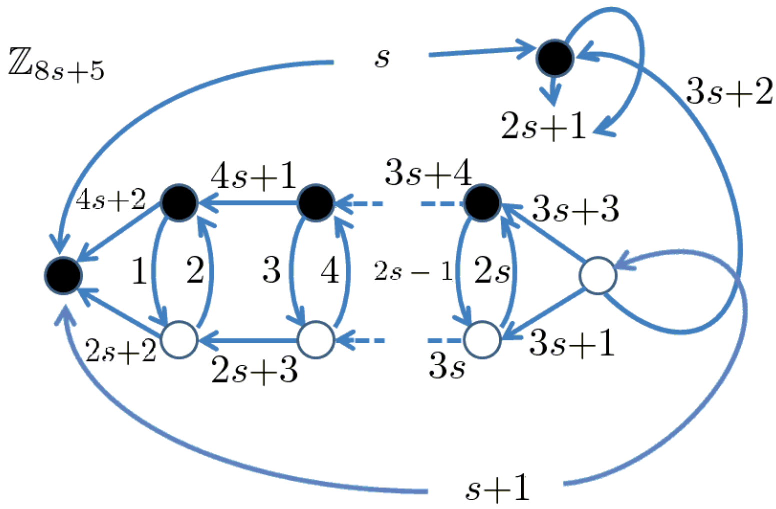

Figure 6.18: Case (iii)-2, $r = 1$

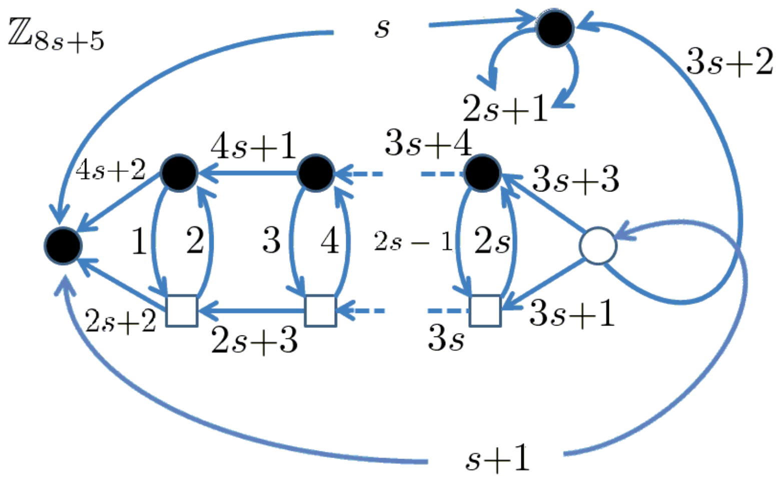

Figure 6.19: Case (iii)-2, $r \geq 2$: even

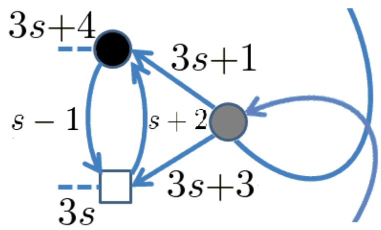

Figure 6.20: A part of Figure 6.19

We check the nonorientability and the connectedness. Since the order of the cyclic group is odd, by Lemma 3.1, we see that the generated r-pire graphs are embedded on nonorientable closed surfaces. If $r = 1$, generated graph is the complete graph on $8s + 5$ vertices.

If $r = 3$, then π_0 is

$$\begin{aligned}&\big(-s, -(2s+1), 2s+1, 3s+2, s+1\big)\\ &\cdot\big(-(4s+2), -(4s+1), \dots, -(3s+3), -(s+1), 2s+2, 1, 2, 2s+3, \dots, 3s+1,\\ &\quad -(3s+2), s\big)\\ &\cdot\big(-(2s+2), -(2s+3), \dots, -(3s+1), 3s+3, -2s, -(2s-1), 3s+4, \dots, 4s+2\big).\end{aligned}$$

If $r \neq 1, 3$ and odd, then π_0 is

$$\begin{aligned}&\big(-s, -(2s+1), 2s+1, 3s+2, s+1\big)\\ &\cdot\big(-(4s+2), -(4s+1), \dots, -(3s+3), -(s+1), 2s+2, \dots, 3s+1, -(3s+2), s\big)\\ &\cdots\\ &\cdot\big(1, 2\big)\\ &\cdot\big(4s+2, -(2s+2), -1\big).\end{aligned}$$

If $r = 2$, then π_0 is

$$\begin{aligned}&\big(-s, -(2s+1), 2s+1, 3s+2, -(3s+1), \dots, -(2s+2), 4s+2, -1, -2, 4s+1, \dots,\\ &\quad 3s+3, s+1\big)\\ &\cdot\big(-(4s+2), -(4s+1), \dots, -(3s+3), 3s+1, 2s, 2s-1, 3s, \dots, 2s+2, -(s+1),\\ &\quad -(3s+2), s\big).\end{aligned}$$

If $r \neq 2$ and even, then π_0 is

$$\begin{aligned}&\big(-s, -(2s+1), 2s+1, 3s+2, -(3s+1), \dots, -(2s+2), 4s+2, \dots, 3s+3, s+1\big)\\ &\cdot\big(-(4s+2), -(4s+1), \dots, -(3s+3), 3s+1, \dots, 2s+2, -(s+1), -(3s+2), s\big)\\ &\cdots\\ &\cdot\big(1, 2\big)\\ &\big(4s+2, -(2s+2), -1\big).\end{aligned}$$

In each case, we see that the number of σ-polygons is r. Note that there exist σ-polygons W_1 containing the arcs with the current $2s+1$, $-(2s+1)$ and $-s$, and W_2 containing the arc with the current s. Since $2s+1$ is relatively prime to $8s+5$, then we see that each vertex of the empire corresponding to W_1 is connected. Since W_1 meets W_2 on the arc with the current s and every other σ-polygon can trace to W_2 via some σ-polygons, then the generated r-pire graph is connected. Therefore for each pair of $s \geq 1$ and $r \leq 2s+1$ we obtain a connected r-pire graph of $8s+5$ mutually adjacent r-pires with a quadrangulation on a nonorientable closed surface.

Then this completes the proof for Case (iii). $\square$

Chapter 7

Topics of quadrangulations

In this chapter, we deal with topics related to quadrangulations. In this chapter, we always allow multiple edges and loops. All the new results we prove in this chapter can be found in [38, 37].

7.1 Extension to Eulerian triangulations

We can extend to a triangulation from a quadrangulation G by adding a diagonal edge in every face of G. We expect that there is an Eulerian triangulation in such triangulations. In 1996, Hoffmann and Kriegel showed that it is true for plane quadrangulations. In 2005, Zhang and He showed the orientable case.

Theorem 7.1. (Hoffmann and Kriegel [18]) *Let G be a quadrangulation on the sphere $\mathbb{S}_0$. Then G can be extended to an Eulerian triangulation.*

Theorem 7.2. (Zhang and He [55]) *Let G be a quadrangulation on an orientable closed surface $\mathbb{S}_g$. Then G can be extended to an Eulerian triangulation.*

Unfortunately, the proof in [55] does not work for the nonorientable case. Then we show the following theorem. Theorem 7.3 implies Theorems 7.1 and 7.2.

Theorem 7.3. *Let G be a quadrangulation on a closed surface F^2. Then G can be extended to an Eulerian triangulation.*

To prove Theorem 7.3, we need some definitions. For a graph G embedded on a closed surface, the *dual* of G is denoted by G^*. The *dual* edge e^* of an edge e of G is the one in G^* that corresponds to e in a natural way. We simply write G and e for $(G^*)^*$ and $(e^*)^*$, respectively, which are well-defined. Note that when G is a quadrangulation, every vertex of G^* has degree 4. For a vertex v with four incident edges e_1, e_2, e_3, e_4 in this cyclic order around v, we say that e_1 is the *opposite of* e_3 *at* v. In the same manner, e_2 is the opposite of e_4 at v. A walk W of G^* is a *straight walk*, or shortly an *S-walk*, of G^* if at each vertex, W passes through v from one edge to the opposite edge, and W does not use an edge twice or more. See Figure 7.1 for an example of S-walks. Note that possibly W might intersect with itself and the edge set of G^* is uniquely partitioned into S-walks. A set of S-walks of G^* is denoted by $\mathcal{S}(G^*) = \{S_1, \ldots, S_l\}$.

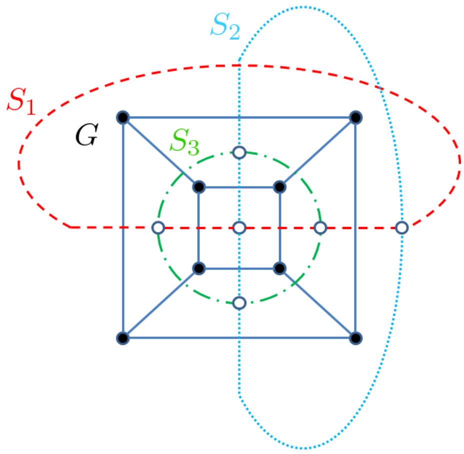

Figure 7.1: Three S-walks S_1, S_2 and S_3 of the dual G^* of a quadrangulation G

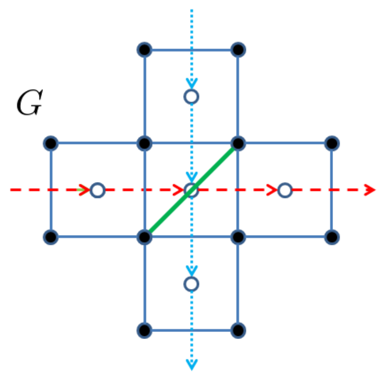

Figure 7.2: A primary diagonal

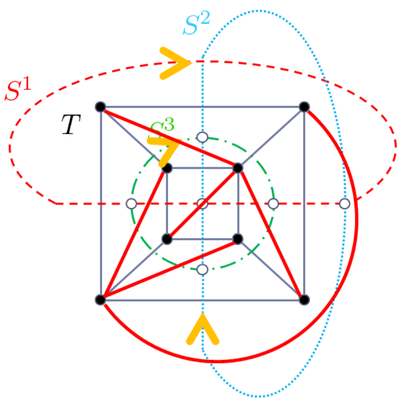

Figure 7.3: A triangulation T induced by the orientation $\mathcal{O}$

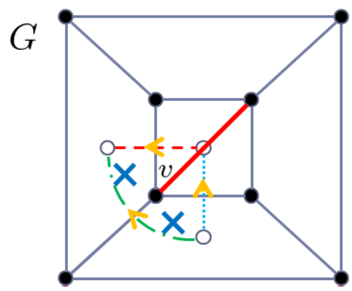

Figure 7.4: A $\deg_G(v)$-gon f_v

Proof of Theorem 7.3. Now we take an arbitrary direction of each S-walk of G^*, and we call the set of the direction an *orientation* of $\mathcal{S}(G^*)$. (In [55], they call it an *S-orientation.*) Let $\mathcal{O}$ be an orientation of the S-walks of G^*. Let f be a face of G. Note that f corresponds to a vertex of G^*, say v_f. Now we add a diagonal to f so that the two directed edges incoming to v_f are separated from the two directed edges outgoing from v_f, see Figure 7.2. Such diagonal is the *$\mathcal{O}$-primary diagonal* at f. (The other diagonal is called the *secondly diagonal* in [55], but we do not use it in this thesis). Adding the $\mathcal{O}$-primary diagonal to all faces of G, we obtain a triangulation T. We say that T is *induced* by the orientation $\mathcal{O}$, see Figure 7.3.

We show that T is an Eulerian triangulation. Let v be a vertex of G and $f_v = v_1v_2\cdots v_{\deg_G(v)}$ be the face of G^* corresponding to v. There are some vertices x of f_v such that both orientated S-walks incident to x go out from x, or come in x in f_v. For example, see Figure 7.4. Let $N(v)$ be the number of x's. It is obvious that $N(v)$ is even. Note that the number of primary diagonals which is incident to v is $\deg_G(v) - N(v)$. Then we see that

$$\deg_T(v) = 2\deg_G(v) - N(v) \equiv 0 \pmod 2$$

for every $v \in V(G)$, and T is an Eulerian triangulation.

□

7.2 Cyclic 4-colorings

Let G be a graph embedded on a surface F^2. A *cyclic coloring* of G is a vertex-coloring of G such that any two vertices x and y receive different colors if x and y are incident with a common face of G. Note that any cyclic coloring is a proper vertex-coloring, since any two adjacent vertices are incident with a common face. Ore and Plummer [44] defined the cyclic colorings of plane graphs and gave a conjecture on it, and many researchers have studied about cyclic colorings, for example, see [6, 10, 48].

It is clear that for a cyclic coloring of a graph G embedded on a surface, we need at least Δ^* colors, where Δ^* is the maximum size of faces of G. Now we deal with quadrangulations, we consider cyclic 4-colorings. For a cyclic 4-coloring, it is natural to consider not only quadrangulations but also *mosaics* G embedded on a surface F^2, where a mosaic is an embedded graph such that every face of G is triangular or quadrangular. Borodin [8, 5] proved that every plane mosaic has a cyclic 6-coloring.

To deal with mosaics, we modify the definition of S-walks. Let G be a mosaic on a surface F^2. A walk W of G^* is a *straight walk*, or shortly an *S-walk*, if W satisfies one of the followings;

(i) W connects vertices of degree 3 in G^*, and for every internal vertex v of W, v has degree 4 in G^* and W passes through v from one edge to the opposite edge at v.

(ii) W is a closed walk, and for every vertex v of W, v has degree 4 and W passes through v from one edge to the opposite edge at v.

In the same manner as the previous definition, note that possibly W might intersect with itself and the edge set of G^* is uniquely partitioned into S-walks.

Using the concept of straight walks, we define *the straight walk dual* $\widetilde{G}$ of a mosaic G as follows;

$$\begin{aligned} V(\widetilde{G}) &= \{F : F \text{ is a triangular face of } G\}, \quad \text{and} \\ E(\widetilde{G}) &= \{W : W \text{ is a straight walk of } G^*\}, \end{aligned}$$

where each straight walk W corresponds to an edge of $\widetilde{G}$ connecting two end vertices of W (if W satisfies (i)), or an edge having no vertex (if W satisfies (ii)). See Figure 7.5. (Black squares in the right side represent vertices of $\widetilde{G}$.) Note that $\widetilde{G}$ is 3-regular and might have multiple edges or loops. When G is a triangulation of a surface, then $\widetilde{G} = G^*$, and when G is a quadrangulation of a surface, then $\widetilde{G}$ has no vertices and consists of only edges. We can assume that $\widetilde{G}$ is drawn on the surface in the natural way as G does. Hence $\widetilde{G}$ might have crossing edges, and moreover, an edge of $\widetilde{G}$ might intersect with itself.

Throughout this chapter, we assume that a closed curve γ on a surface F^2 transversely intersects with a graph G drawn on F^2. For simplifying the arguments, we also assume that every closed curve γ on a surface F^2 passes through neither a vertex of G nor a crossing point of G, that is, γ intersects with G only at a point where exactly one edge of G is drawn. For a closed curve γ on F^2 and an edge set T of a graph G drawn on F^2, we denote by $T \cap \gamma$ the set of points on F^2 that are contained in both an edge in T and γ. Then we are ready to state our main theorem.

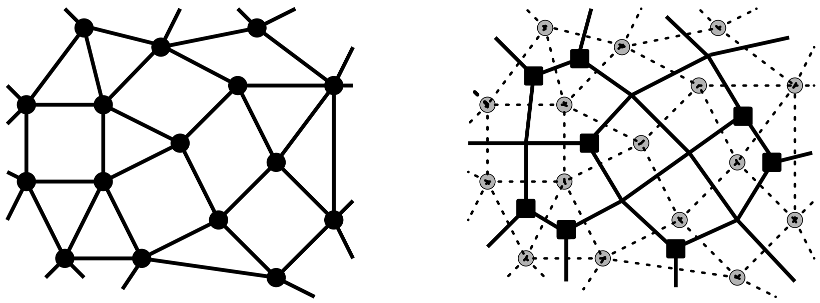

Figure 7.5: A mosaic G (the left side) and the straight walk dual $\widetilde{G}$ of G.

Theorem 7.4. *A mosaic G of a surface F^2 has a cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ of G has a 3-edge-coloring $c : E(\widetilde{G}) \to \{1,2,3\}$ satisfying the following two conditions.*

(C1) *Any two edges of $\widetilde{G}$ that are pairwise crossing on F^2 receive different colors by c. (So, no edge intersects with itself.)*

(C2) *For every closed curve γ on F^2,*

$$|c^{-1}(1) \cap \gamma| \equiv |c^{-1}(2) \cap \gamma| \equiv |c^{-1}(3) \cap \gamma| \pmod 2. \tag{7.1}$$

Recall that for $i = 1,2,3$, $c^{-1}(i)$ denotes the set of edges $\widetilde{e}$ of $\widetilde{G}$ such that $c(\widetilde{e}) = i$, and $c^{-1}(i) \cap \gamma$ denotes the set of points on F^2 that are contained in both an edge in $c^{-1}(i)$ and γ.

7.3 Properness of a 3-edge-coloring c

In Theorem 7.4, we do not require an edge-coloring c of $\widetilde{G}$ to be proper, but indeed, we need it. To be precisely, condition (C2) implies the properness of the 3-edge-coloring c.

Lemma 7.5. *Let $\widetilde{G}$ be a 3-regular graph drawn on a surface F^2. Suppose that $\widetilde{G}$ has a 3-edge-coloring $c : E(\widetilde{G}) \to \{1,2,3\}$. If c satisfies condition (C2), then c is a proper 3-edge-coloring of $\widetilde{G}$.*

Proof. Let v be any vertex of $\widetilde{G}$, and let $\widetilde{e_1}, \widetilde{e_2}$, and $\widetilde{e_3}$ be three edges that are incident with v. Let c be a 3-edge-coloring of $\widetilde{G}$, and suppose that c satisfies condition (C2). Let γ be a non-essential closed curve on F^2 that intersects with each of $\widetilde{e_1}, \widetilde{e_2}$ and $\widetilde{e_3}$ exactly once. It follows from equality (7.1) for γ that

$$|c^{-1}(1) \cap \gamma| \;\equiv\; |c^{-1}(2) \cap \gamma| \;\equiv\; |c^{-1}(3) \cap \gamma| \qquad (\text{mod } 2),$$

which directly implies that $\widetilde{e_1}, \widetilde{e_2}$, and $\widetilde{e_3}$ are colored by three distinct colors. Hence c is a proper 3-edge-coloring. $\square$

We point out that equality (7.1) for separating closed curves corresponds to a well-known lemma, called *Parity Lemma* on a proper 3-edge-coloring, see for example, P. 253 in [54]. Recall that an *edge-cut* of a graph G is an inclusionwise minimal set of edges whose removal makes G disconnected.

Lemma 7.6 (Parity Lemma). *Let H be a 3-regular graph with a proper 3-edge-coloring c by the colors $1, 2$ and 3. Then each edge-cut T of H satisfies $|c^{-1}(1) \cap T| \equiv |c^{-1}(2) \cap T| \equiv |c^{-1}(3) \cap T|$ (mod 2).*

We will briefly mention how equality (7.1) for separating closed curves is related to Lemma 7.6. For a closed curve γ on a surface F^2, let T_γ be the set of edges $\widetilde{e}$ of a graph $\widetilde{G}$ drawn on F^2 such that $\widetilde{e}$ intersects with γ odd number of times. It is easy to see that γ is separating if and only if T_γ is a disjoint union of edge-cuts of $\widetilde{G}$ or $T_\gamma = \emptyset$. In this sense, separating closed curves on F^2 correspond to edge-cuts of a graph $\widetilde{G}$, and we see the correspondence between equality (7.1) and the equality in Lemma 7.6. Indeed, if c is a proper 3-edge-coloring, then every separating closed curve γ on F^2 satisfies equality (7.1), that is, the converse of Proposition 7.5 also holds for separating closed curves on F^2.

7.4 Checking condition (C2) and the fundamental group of F^2

In this section, we consider how to check condition (C2). In order to check condition (C2) in Theorem 7.4, we have to consider all closed curves on a surface F^2. But it is not necessary to do that, and we will mention that it is enough to check only $(2 - \varepsilon(F^2))$ appropriate non-separating closed curves on F^2, if we assume the properness of the 3-edge-coloring c of $\widetilde{G}$.

To see this, we first look at two situations (A) and (B) in Figure 7.6. Both situation represents a part of two closed curves γ and γ' on F^2, and assume that the remaining parts of γ and γ' are exactly same. Let c be a proper 3-edge-coloring of $\widetilde{G}$. In situation

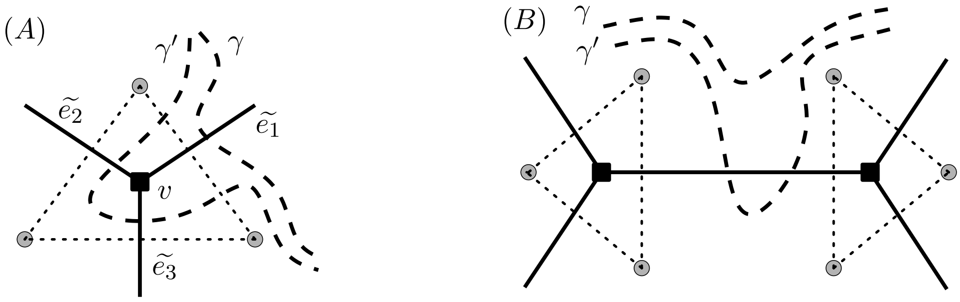

Figure 7.6: Two closed curves γ and γ' that are homotopic.

(A), there is a vertex v of degree 3 in $\widetilde{G}$, and let $\widetilde{e_1}, \widetilde{e_2}$, and $\widetilde{e_3}$ be three edges of $\widetilde{G}$ incident with v. Since c is a proper 3-edge-coloring, we may assume that by symmetry, $c(\widetilde{e_i}) = i$ for $i = 1, 2, 3$. Note that $|c^{-1}(1) \cap \gamma'| = |c^{-1}(1) \cap \gamma| - 1$, $|c^{-1}(2) \cap \gamma'| = |c^{-1}(2) \cap \gamma| + 1$, and $|c^{-1}(3) \cap \gamma'| = |c^{-1}(3) \cap \gamma| + 1$, which directly implies that if γ satisfies equality (7.1) then γ' also does. Similarly, in situation (B) in Figure 7.6, we can easily see that if γ satisfies equality (7.1) then γ' also does. These two facts imply that for any two closed curves γ and γ' with the same homotopy type on F^2, if γ satisfies equality (7.1), then γ' also does, since γ' can be obtained from γ by a sequence of homotopic shifts as in situation (A) or (B) in Figure 7.6.

On the other hand, let $[\gamma_1]$ and $[\gamma_2]$ be two generators of the fundamental group of F^2. (Note that $[\gamma_1]$ and $[\gamma_2]$ are two homotopy classes of the set of closed curves on F^2, and γ_1 and γ_2 are representatives of them, respectively.) It is easy to see that if both γ_1 and γ_2 satisfies equality (7.1), (and hence if every closed curve on F^2 homotopic to γ_1 or γ_2 satisfies equality (7.1),) then any closed curve γ contained in the homotopy class $[\gamma_1] * [\gamma_2]$ or $[\gamma_1]^{-1}$ also satisfies equality (7.1), where $[\gamma_1] * [\gamma_2]$ is the product of $[\gamma_1]$ and $[\gamma_2]$ on the fundamental group of F^2 and $[\gamma_1]^{-1}$ is the homotopy class containing γ_1^{-1}.

Since any homotopy class of the set of closed curves on F^2 is obtained by the products of generators of the fundamental group of F^2, these arguments, together with Proposition 7.5, imply that the following theorem is equivalent to Theorem 7.4. Indeed, since there are exactly $(2 - \varepsilon(F^2))$ generators in the fundamental group of F^2, it is enough to check only $(2 - \varepsilon(F^2))$ appropriate non-separating closed curves on F^2.

Theorem 7.7. *A mosaic G of a surface F^2 has a cyclic 4-coloring if and only if the straight walk dual $\widetilde{G}$ has a proper 3-edge-coloring $c : E(\widetilde{G}) \to \{1, 2, 3\}$ satisfying condition (C1) and the following condition;*

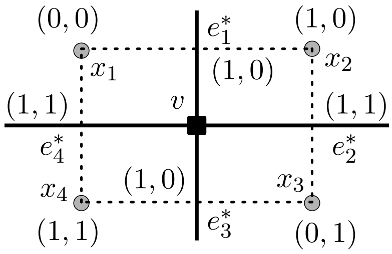

Figure 7.7: An example of the situation around v and the edge-coloring ∂f^* of G^*.

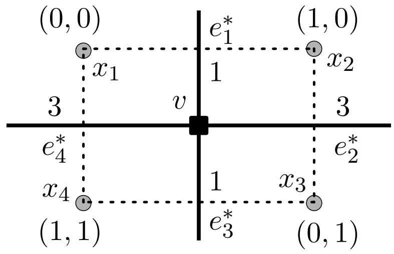

Figure 7.8: An example of the situation around v and the coloring f of G.

(C2') *for every generator* $[\gamma]$ *of the fundamental group of* F^2*, a representative* γ *of* $[\gamma]$ *satisfies equality (7.1).*

7.5 Proof of Theorem 7.4

First, we show the "only if" part of Theorem 7.4. Let G be a mosaic of a surface F^2 and suppose that G has a cyclic 4-coloring f. We regard the colors of f as the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, then we use the four colors $(0,0), (1,0), (0,1)$ and $(1,1)$. We construct the (not necessarily proper) edge-coloring ∂f of G as follows; for an edge $e = xy$ of G, define the color $\partial f(e) = f(x) + f(y)$, where $+$ means the sum on $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Since f is a proper coloring, every edge of G receives the color $(0,1), (1,0)$ or $(1,1)$ by ∂f. Let ∂f^* be the edge-coloring of G^* that is obtained from ∂f by a natural way; $\partial f^*(e^*) = \partial f(e)$ for any edge e^* of G^*. We show the following claim concerning the edge-coloring ∂f^*.

Claim 7.8. *Let* v *be a vertex of degree 4 in* G^**, and let* e_1^*, e_2^*, e_3^* *and* e_4^* *be the four edges incident with* v *in this cyclic order around* v*. Then* $\partial f^*(e_1^*) = \partial f^*(e_3^*) \neq \partial f^*(e_2^*) = \partial f^*(e_4^*)$.

Proof. Let $x_1x_2x_3x_4$ be the face of G corresponding to v, where x_i is a vertex of G for $i = 1, 2, 3, 4$. By symmetry, we may assume that $e_i = x_ix_{i+1}$ for $i = 1, 2, 3, 4$, where $x_5 = x_1$. Since f is a cyclic 4-coloring of G, the four vertices x_1, x_2, x_3 and x_4 receive four distinct colors by f. Figure 7.7 shows one example of a coloring of the four vertices x_1, x_2, x_3 and x_4 by f and the edge-coloring ∂f^* of e_1^*, e_2^*, e_3^* and e_4^*. It is easy to check that

$$\begin{aligned} & \partial f^*(e_1^*) = f(x_1) + f(x_2) = f(x_3) + f(x_4) = \partial f^*(e_3^*), \\ \text{and} \qquad & \partial f^*(e_1^*) = f(x_1) + f(x_2) \neq f(x_2) + f(x_3) = \partial f^*(e_2^*). \end{aligned}$$

The above (in)equalities show Claim 7.8 □

Now we construct the edge-coloring c of $\widetilde{G}$ as $c(W) = \partial f^*(e^*)$ for each edge W of $\widetilde{G}$, where e^* is an edge of G^* contained in the straight walk W. By Claim 7.8, this definition does not depend on the choice of an edge e^*, and hence that is well-defined. Moreover, the edge-coloring c of $\widetilde{G}$ satisfies condition (C1), by Claim 7.8.

To show that c satisfies condition (C2), we first need the fact that the spanning subgraph G_1^* of G^* induced by all edges colored by $(1,0)$ or $(1,1)$ by ∂f^* has a proper 2-face-coloring. The definition of G_1^*, together with the construction of the edge-coloring ∂f^*, implies that the dual edge e of an edge e^* of G_1^* connects two vertices of G, one of which has the color $(0,0)$ or $(0,1)$ by f and the other has the color $(1,0)$ or $(1,1)$. This means that a face of G_1^* on one side of the edge e^* contains faces of G^* corresponding to vertices of G with colors having 0 in the first coordinate by f, and that on the other side contains faces of G^* corresponding to vertices of G with a color having 1 in the first coordinate. Then depending on the first coordinate of the color by f, we can color each face of G_1^* by the two colors, 0 or 1.

Then each closed curve γ on F^2 has to pass through faces of G_1^* with color 0 and ones with color 1 alternatively. This directly implies that

$$\left|(\partial f^*)^{-1}\big((1,0)\big) \cap \gamma\right| + \left|(\partial f^*)^{-1}\big((1,1)\big) \cap \gamma\right| \equiv 0 \pmod 2.$$

By the definition of the edge-coloring c of $\widetilde{G}$, we have

$$\left|c^{-1}\big((1,0)\big) \cap \gamma\right| + \left|c^{-1}\big((1,1)\big) \cap \gamma\right| \equiv 0 \pmod 2. \tag{7.2}$$

Since we can use the same argument as above for the second coordinate of the colors of f, we also obtain

$$\left|c^{-1}\big((0,1)\big) \cap \gamma\right| + \left|c^{-1}\big((1,1)\big) \cap \gamma\right| \equiv 0 \pmod 2. \tag{7.3}$$

These equalities (7.2) and (7.3) imply that γ satisfies equality (7.1). Hence condition (C2) also holds, and this completes the proof of the "only if" part of Theorem 7.4.

Next, we show the "if" part of Theorem 7.4, by the almost inverse process of the proof of the "only if" part.

Let G be a mosaic of a surface F^2 and suppose that the straight walk dual $\widetilde{G}$ has a 3-edge-coloring c satisfying conditions (C1) and (C2). By Proposition 7.5, c is a proper 3-edge-coloring. Let c' be the edge-coloring of G^* such that $c'(e^*) = c(W)$ for each edge e^* of G^*, where W is the straight walk of G^* containing e^*.

First, we focus on the colors 1 and 3, and let G_1^* be the spanning subgraph of G^* induced by all edges in $(c')^{-1}(1) \cup (c')^{-1}(3)$. We will show that the dual G_1 of G_1^* is bipartite. Let S be a cycle of G_1, and let γ_S be the closed curve on F^2 corresponding to S. By Proposition condition (C2), γ_S satisfies equality (7.1), and hence

$$|E(S)| \;\equiv\; |c^{-1}(1) \cap \gamma_S| \;+\; |c^{-1}(3) \cap \gamma_S| \;\equiv\; 0 \qquad (\mathrm{mod}\ 2).$$

So, S has an even length. This implies that G_1 is bipartite. Hence G_1^* has a proper 2-face-coloring f_1 by the two colors, say $(0,0)$ and $(1,0)$. Let G_2^* be the spanning subgraph of G^* induced by all edges in $(c')^{-1}(2) \cup (c')^{-1}(3)$. By the same argument as above, G_2^* has a proper 2-face-coloring f_2 by the two colors, say $(0,0)$ and $(0,1)$.

Then we define the coloring f of G as follows; for each vertex x of G, $f(x) = f_1(F_1) + f_2(F_2)$, where F_i be the face of G_i^* that contains the face of G^* corresponding to x for $i = 1, 2$. Figure 7.8 shows an example of the coloring of f. We show that f is a cyclic 4-coloring of G.

For an edge $e = xy$ of G, e^* is contained in at least one of G_1^* and G_2^*, which implies that x and y receive the colors by f that are different value in the first and/or second coordinate. Thus, for any edge xy of G, x and y has distinct colors by f, that is, f is a proper 4-coloring of G. In particular, each triangular face of G receives three distinct colors by f. We show the following claim.

Claim 7.9. *For any quadrangular face of G, say $x_1x_2x_3x_4x_1$, we have $f(x_1) \neq f(x_3)$ and $f(x_2) \neq f(x_4)$.*

Proof. Let e_i be the edge of G connecting x_i and x_{i+1} for $i = 1, 2, 3, 4$, where $x_5 = x_1$. By condition (C1), $c'(e_1^*) \neq c'(e_2^*)$, and hence by symmetry, we may assume that $c'(e_1^*) \neq 3$, say $c'(e_1^*) = 1$. Note that $c'(e_3^*) = 1$, and e_1^* is contained in G_1^* but not in G_2^*. Let F_2 be the face of G_2^* containing the face of G^* corresponding to x_1. Since $c'(e_1^*) = 1$ and $c'(e_2^*) \neq 1$, F_2 contains the face of G_2^* corresponding to x_2, and does not contain the face of

G_2^* corresponding to x_3. Hence x_1 and x_3 receive distinct values in the second coordinate by f_2, and also by f. This implies that $f(x_1) \neq f(x_3)$, and similarly $f(x_2) \neq f(x_4)$. $\square$

Then for any quadrangular face of G, say $x_1x_2x_3x_4x_1$, $f(x_1) \neq f(x_2)$ and $f(x_1) \neq f(x_4)$ since f is a proper 4-coloring, and $f(x_1) \neq f(x_3)$ by Claim 7.9. Hence x_1 receives a color different from any of x_2, x_3 and x_4, and by symmetry, the four vertices in a quadrangular face receive four distinct colors by f. Thus, f is a cyclic 4-coloring of G. This completes the proof of the "if" part, and the proof of Theorem 7.4.

Bibliography

[1] K. Appel and W. Haken, Every planar map is four colorable, Bull. Amer. Math. Soc. 82 (1976), 711–712.

[2] D. Archdeacon, J. Hutchinson, A. Nakamoto, S. Negami and K. Ota, Chromatic numbers of quadrangulations on closed surfaces, J. Graph Theory 37 (2001), 100–114.

[3] K.A. Berman and H. Shank, Full 4-colorings of 4-regular maps, J. Graph Theory 3 (1979), 291–294.

[4] D. Barnette, Generating the triangulations of the projective plane, J. Combin. Theory Ser. B 33 (1982), 222–230.

[5] O.V. Borodin, A new proof of the 6 color theorem, J. Graph Theory 19 (1995), 507–521.

[6] O.V. Borodin, Cyclic coloring of plane graphs, Discrete Math. 100 (1992), 281–289.

[7] O.V. Borodin, Representing K_{13} as a 2-pire map on the Klein bottle, J. Reine Angew. Math. 393 (1989), 132–133.

[8] O.V. Borodin, Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs (in Russian), Metody Diskret. Analiz. 41 (1984), 12–26.

[9] A. Boulch, É. Colin de Verdière and A. Nakamoto, Irreducible triangulations of surfaces with boundary, Graphs Combin. 29 (2013), 1675–1688.

[10] H. Enomoto, M. Horňák, and S. Jendrol', Cyclic chromatic number of 3-connected plane graphs, SIAM J. Discrete Math. 14 (2001), 121–137.

[11] S. Fisk and B. Mohar, Coloring graphs without short non-bounding cycles, J. Combin. Theory Ser. B 60 (1994), 268–276.

[12] P.A. Franklin, Six-color problem, J. Math. Phys. 13 (1934), 363–369.

[13] Z. Gao, L.B. Richmond and C. Thomassen, Irreducible triangulations and triangular embeddings on a surface, Technical Report CORR 91-07, University of Waterloo, Canada, 1991.

[14] J.L. Gross and T.W. Tucker, Topological Graph Theory, Wiley Interscience, New York, 1987.

[15] N. Hartsfield, Minimal quadrangulations of orientable surfaces, J. Combin. Theory Ser. B 46 (1989), 84–95.

[16] N. Hartsfield, The quadrangular genus of complete graphs, preprint.

[17] P.J. Heawood, Map color theorem, Quart. J. Pure Appl. Math. 24 (1890), 332–338.

[18] F. Hoffmann and K. Kriegel, A graph-coloring result and its consequences for polygon-guarding problems, SIAM J. Discrete Math. 9 (1996), 210–224.

[19] J.P. Hutchinson, On coloring maps made from Eulerian graphs, Proceeding of 5th British Combinatorial Conference, 1975, 343–354.

[20] B. Jackson and G. Ringel, Heawood's empire problem, J. Combin. Theory Ser. B 38 (1985), 168–178.

[21] B. Jackson and G. Ringel, Maps of m-pires on the projective plane, Discrete Math. 46 (1983), 15–20.

[22] B. Jackson and G. Ringel, Solution of Heawood's empire problem in the plane, J. Reine. Angew. Math. 347 (1984), 146–153.

[23] G. Joret and D.R. Wood, Irreducible triangulations are small, J. Combin. Theory Ser. B 100 (2010), 446–455.

[24] K. Kawarabayashi and B. Mohar, Approximating the list-chromatic number and the chromatic number in minorclosed and odd-minor-closed classes of graphs, Proceedings of the 38th Annual ACM Symposium on Theory of Computing, ACM, New York, 2006, 401–416.

[25] V.P. Korzhik, Exponential many nonisomorphic orientable triangular embeddings of K_{12s}, Discrete Math. 308 (2008), 1046–1071.

[26] V.P. Korzhik, Exponential many nonisomorphic orientable triangular embeddings of K_{12s+3}, Discrete Math. 309 (2009), 852–866.

[27] V.P. Korzhik and H-J. Voss, Exponential families of nonisomorphic nonorientable genus embeddings of complete graphs, J. Combin. Theory Ser. B 91 (2004), 253–287.

[28] V.P. Korzhik and H-J. Voss, On the number of nonisomorphic orientable regular embeddings of complete graphs, J. Combin. Theory Ser. B 81 (2001), 58–76.

[29] S. Lawrencenko, Irreducible triangulations of the torus, Ukrain. Geom. Sb. 30 (1987), 52–62.

[30] B. Mohar and C. Thomassen, "Graphs on Surfaces", The Johns Hopkins University Press, Baltimore and London, 2001.

[31] A. Nakamoto, Diagonal transformations and cycle parities of quadrangulations on surfaces, J. Combin. Theory Ser. B 67 (1996), 202–211.

[32] A. Nakamoto, Irreducible quadrangulations of the Klein bottle, Yokohama Math. J. 43 (1995), 125–139.

[33] A. Nakamoto, Irreducible quadrangulations of the torus, J. Combin. Theory Ser. B 67 (1996), 183–201.

[34] A. Nakamoto, Odd minor of even embedding on surfaces, preprint.

[35] A. Nakamoto and S. Negami, Diagonal transformations of graphs on closed surfaces, Sci. Rep. Yokohama Nat. Univ., Sec. I No. 40 (1993), 71–97.

[36] A. Nakamoto, S. Negami and K. Ota, Chromatic numbers and cycle parities of quadrangulations on non orientable closed surfaces, Discrete Math. 285 (2004), 211–218.

[37] A. Nakamoto, K. Noguchi and K. Ozeki, Cyclic 4-colorings of graphs on surfaces, submitted.

[38] A. Nakamoto, K. Noguchi and K. Ozeki, Extension to even triangulations, submitted.

[39] A. Nakamoto and K. Ota, Note on irreducible triangulations of surfaces, J. Graph Theory 20 (1995), 227–233.

[40] S. Negami, Diagonal flips in triangulations of surfaces, Discrete Math. 135 (1994), 225–232.

[41] K. Noguchi, The empire problem in even embeddings on closed surfaces, J. Graph Theory 75 (2014), 20–30.

[42] K. Noguchi, The empire problem in even embeddings on closed surfaces with $\varepsilon \leq 0$, Discrete Math. 313 (2013), 1944–1951.

[43] K. Noguchi, Even embeddings of the complete graphs and their cycle parities, submitted.

[44] O. Ore, and M.D. Plummer, Cyclic coloration of plane graphs, in: W. T. Tutte (Ed.), Recent Progress in Combinatorics, (Proceedings of the Third Waterloo Conference on Combinatorics, May 1968), Academic Press, 1969, 287–293.

[45] G. Ringel, "Map Color Theorem", Band 209, Springer-Verlag, Berlin, 1974.

[46] N. Robertson and P. Seymour, Graph Minors. VII, Disjoint paths on a surface, J. Combin. Theory Ser. B 45 (1988), 212–254.

[47] N. Robertson and R. Vitray, Representativity of surface embeddings, in "Paths, Flows, and VLSI-Layout", Springer-Verlag, Heidelberg, 1990, 293–328.

[48] D.P. Sanders, and Y. Zhao, A new bound on the cyclic chromatic number, J. Combin. Theory Ser. B 83 (2001), 102–111.

[49] E. Steinitz and H. Rademacher, Vorlesungen über die Theorie der Polyeder. Springer, Berlin, 1934.

[50] T. Sulanke, Irreducible triangulations of low genus surfaces, arXiv:math/0606690.

[51] T. Sulanke, Note on the irreducible triangulations of the Klein bottle, J. Combin. Theory Ser. B 96 (2006), 964–972.

[52] H. Taylor, Synchronization patterns and related problems in combinatorial analysis and graph theory, Technical Report 509, University of Southern California Electronic Science Laboratory, 1981.

[53] C. Thomassen, Five-coloring maps on surfaces, J. Combin. Theory Ser. B 59 (1993), 89–105

[54] C.Q. Zhang, Circuit Double Cover of Graphs, Cambridge University Press, Cambridge, United Kingdom, 2012.

[55] H. Zhang and X. He, On even triangulations of 2-connected embedded graphs, SIAM J. Comput. 34 (2005), 683–696.