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Elementary proof of some theorems on strongly q-additive or q-multiplicative functions

by

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1. Introduction

Let q be an integer greater than 1. Let $a(n)$ be a complex-valued arithmetical function. The function $a(n)$ is said to be *strongly q-additive* if

$$a(n) = \sum_{i \geq 0} a(b_i)$$

for any positive integer $n = \sum_{i \geq 0} b_i q^i$ with $b_i \in \{0, 1, \dots, q-1\}$. We define $a(0) = 0$. $a(n)$ is said to be *strongly q-multiplicative* if

$$a(n) = \prod_{i \geq 0} a(b_i)$$

for any positive integer n. We define $a(0) = 1$.The notion of q-additive functions and q-multiplicative functions were introduced by Gel'fond [2] and Delange [1] respectively. Recently, Toshimitsu [5] proved the following theorems on these arithmetical functions with distinct basis p and q.

Theorem 1 (Toshimitsu[5;Theorem 3]). *Let p and q be integers greater than* 1 *such that* $\log p / \log q$ *is irrational. Let $a(n)$ be strongly p-additive and also strongly q-additive function. Then $a(n)$ is identically zero.*

Theorem 2 (Toshimitsu[5;Theorem 4]). *Let p and q be integers greater than* 1 *such that* $\log p / \log q$ *is irrational. Let $a(n)$ be strongly p-multiplicative and also* *strongly q-multiplicative function. Then $a(n)$ $(n \geq 1)$ is identically zero or*

$$a(n) = \gamma^n$$

for all $n \geq 1$ and for some γ with $\gamma^{p-1} = \gamma^{q-1} = 1$.

His proofs based on the deep results in the transcendence theory of Mahler functions (cf. Nishioka [3], [4]). The purpose of this note is to give direct proofs of these theorems, which are rather involved, however completely elementary.

2. Proof of Theorem 1

Let p, q, and $a(n)$ be as in Theorem 1. We may assume $p < q$ and write

$$q = dp + r, \qquad r \in \{0, 1, \dots, p-1\}. \tag{1}$$

Then we have the following formulas.

Lemma 1.

$$a(d) = a(1) - a(r), \tag{2}$$

$$a(d+1) = 2a(1) - a(r). \tag{3}$$

Proof. (2) is obvious. We prove only (3). Since $a(n)$ is strongly p and q-additive, we have by (1)

$$a(q+p) = a((d+1)p + r) = a(d+1) + a(r),$$

and so

$$\begin{aligned} a(d+1) &= a(q+p) - a(r) \\ &= a(q) + a(p) - a(r) = 2a(1) - a(r). \end{aligned}$$

Lemma 2. *Assume that $r \neq 0$. Let k and l be nonnegative integers such that $0 \leq kp - lr < p$. Then*

$$a(kp - lr) = ka(1) - la(r).$$

Proof. This is true if $k + l = 0$. Let $k + l > 0$ and suppose that

$$a(k'p - l'r) = k'a(1) - l'a(r) \tag{4}$$

for any nonnegative integers k', l' such that $k'+l' < k+l$ and $0 \le k'p - l'r < p$. Since $0 \le kp - lr < p$, we have

$$r \le kp - (l-1)r < p + r.$$

First we consider the case in which k, l satisfy $p \le kp - (l-1)r < p + r$. Then $0 \le (k-1)p - (l-1)r < r$, and so we get

$$a(q + kp - lr) = a((d+1)p + (k-1)p - (l-1)r),$$

using (1). Since $a(n)$ is strongly p and q-additive, we have

$$\begin{aligned} a(1) + a(kp - lr) &= a(d+1) + a((k-1)p - (l-1)r) \\ &= 2a(1) - a(r) + (k-1)a(1) - (l-1)a(r) \end{aligned}$$

by (3) and (4). Therefore we obtain

$$a(kp - lr) = ka(1) - la(r).$$

Next we assume $r \le kp - (l-1)r < p$. Then we have

$$\begin{aligned} a(q + kp - lr) &= a(dp + kp - (l-1)r) \\ &= a(d) + a(kp - (l-1)r) \\ &= a(1) + ka(1) - la(r) \end{aligned}$$

by (2) and (4). Hence we get

$$a(kp - lr) = ka(1) - la(r),$$

since $a(q + kp - lr) = a(1) + a(kp - lr)$.

Lemma 3. *If $r \neq 0$, then $a(n) = na(1)$ $(1 \le n \le d)$.*

Proof. Since $q > np$, we have by strongly p and q-additivity

$$a(1) + a(n) = a(q + np) = a((d+n)p + r) = a((d+n)p) + a(r),$$

and so

$$a(n) = a((d+n)p) - a(1) + a(r).$$

By (1) and Lemma 2, we have

$$\begin{aligned} a((d+n)p) &= a(dp + r + (n-1)p + p - r) \\ &= a(1) + a((n-1)p) + a(p-r) \\ &= a(n-1) + 2a(1) - a(r). \end{aligned}$$

Hence we get

$$a(n) = a(n-1) + a(1) = \cdots = na(1).$$

Lemma 4. *If $r \neq 0$, then $a(n) = 0$ $(n \geq 0)$.*

Proof. First we show that

$$a(n) = na(1) \quad (n \geq 0) \tag{5}$$

by induction on n. This holds for $n \leq d$, by Lemma 3. Let $n > d$ and assume that $a(k) = ka(1)$ for $0 \leq k < n$. Let k and l be integers such that $0 \leq np - kq < q$ and $0 \leq np - kq - lp < p$. Since $0 \leq k,\ l < n$, we have by strongly p and q-additivity

$$\begin{aligned} a(n) = a(np) &= a(kq + (np - kq)) \\ &= a(k) + a(lp + (np - kq - lp)) \\ &= a(k) + a(l) + a((n - l - dk)p - kr) \\ &= (k + l)a(1) + (n - l - dk)a(1) - ka(r) \end{aligned}$$

by Lemma 2 and the induction hypothesis, and so

$$\begin{aligned} a(n) &= na(1) + ka(1) - k(da(1) + a(r)) \\ &= na(1) + ka(1) - k(a(d) + a(r)) = na(1) \end{aligned}$$

by (1), (2) and Lemma 3. Hence (5) is proved.

Putting $n = p$ in (5), we have $a(1) = a(p) = pa(1)$, so that $a(1) = 0$. Therefore $a(n) = 0$ $(n \geq 0)$ again by (5).

Proof of Theorem 1. Since p and q are greater than 1 and $\log p / \log q$ is irrational, we have $p + q \geq 5$. If $p + q = 5$, then $p = 2$, $q = 3$, and so $r = 1$ in (1). Hence the theorem follows from Lemma 4 in this case. Assume that $p + q > 5$. We write q as in (1). In view of Lemma 4 we may assume $r = 0$, so that $q = dp$ with

$d \geq 2$. Let $n = \sum_{i \geq 0} b_i d^i$ be d-adic expansion of an integer $n \geq 1$. Then we have

$$a(n) = a(\sum_{i \geq 0} b_i d^i) = a(\sum_{i \geq 0} b_i d^i p) = a(\sum_{i \geq 1} b_i d^{i-1} q + b_0 p)$$
$$= a(\sum_{i \geq 1} b_i d^{i-1}) + a(b_0) = \cdots = \sum_{i \geq 0} a(b_i).$$

Hence $a(n)$ is strongly d-additive function with $\log d / \log p$ irrational, since

$$\frac{\log q}{\log p} = \frac{\log d}{\log p} + 1.$$

Noting $d + p < p + q$, we obtain $a(n) = 0$ $(n \geq 1)$, and the proof is completed.

3. Proof of Theorem 2

Let p, q, and $a(n)$ be as in Theorem 2. We may assume $p < q$.

Lemma 5. *If $a(n)$ $(n \geq 1)$ is not identically zero, then $a(1) \neq 0$.*

Proof. Let N be the smallest positive integer n such that $a(n) \neq 0$. We have to prove that $N = 1$. Since $a(n)$ is strongly p-multiplicative, we have $1 \leq N < p$. First we note that there exists an integer $k \geq 1$ such that

$$Nq^k = \sum_{i=l_k}^{m_k} b_i p^i \quad (b_i \in \{0, 1, \cdots, p-1\}, \;\; b_{l_k} \neq 0, \; b_{m_k} \neq 0, \; m_k > l_k). \tag{6}$$

Indeed, if

$$Nq^k = b_{m_k} p^{m_k}$$

for any $k \geq 1$, there exist two integers k_1, k_2 $(1 \leq k_1 < k_2)$ and an integer b $(1 \leq b < p)$ such that

$$Nq^{k_i} = bp^{m_{k_i}} \qquad (i = 1, 2),$$

so that

$$q^{k_2 - k_1} = p^{m_{k_2} - m_{k_1}},$$

which contradicts the irrationality of $\log p / \log q$.

Since $a(Nq^k) = a(N) \neq 0$, we have

$$a(b_i) \neq 0 \qquad (l_k \leq i \leq m_k). \tag{7}$$

In particular, $b_{m_k} \geq N$. Hence

$$q^k > p^{m_k}. \tag{8}$$

In what follows, we put $l = l_k$ in (6) simply. We show that

$$b_i \leq p - N \qquad (l \leq i \leq m_k - 1). \tag{9}$$

We assume, to the contrary, that there exists j $(l \leq j \leq m_k - 1)$ such that $b_j > p - N$. We put $n = b_j + N - p$, so that $b_j + N = n + p$ and $0 < n < N$. Then we have by strongly p and q-multiplicativity

$$\begin{aligned} a(N)^2 = a(Nq^k + Np^j) &= a(\sum_{i=l}^{j-1} b_i p^i + (n+p)p^j + \sum_{i=j+1}^{m_k} b_i p^i) \\ &= a(\sum_{i=l}^{j-1} b_i p^i) a(n) a(\sum_{i=j+1}^{m_k} b_i p^i + p^{j+1}) = 0, \end{aligned}$$

which contradicts $a(N) \neq 0$.

Now we prove $N = 1$. Assume that $N > 1$. Let g be an integer such that $(g-1)b_l < p \leq g b_l$. Then we have

$$a((g-1)b_l) \neq 0. \tag{10}$$

Indeed, we can show that $a(hb_l) \neq 0$ for all $h = 1, \ldots, g-1$ by induction on h. This holds for $h = 1$ by (7). Suppose that $a((h-1)b_l) \neq 0$ for some $h \geq 2$. Then

$$a(Nq^k + (h-1)b_l p^l) = a(hb_l p^l + \sum_{i=l+1}^{m_k} b_i p^i),$$

and so

$$0 \neq a(N) a((h-1)b_l) = a(hb_l) \prod_{i=l+1}^{m_k} a(b_i).$$

Hence we get $a(hb_l) \neq 0$ by (7), and (10) follows.

We note that if $l + 1 < m_k$, then $b_{l+1} + 1 \leq p - 1$ by (9). We put $n = gb_l - p$, so that $gb_l = n + p$ and $0 \leq n < p$ by definition of g. Then we have

$$a(Nq^k + (g-1)b_l p^l) = a(np^l + (b_{l+1} + 1)p^{l+1} + \sum_{i=l+2}^{m_k} b_i p^i),$$

and so by (8) and (10)

$$0 \neq a(N)a((g-1)b_l) = a(np^l)a(b_{l+1}+1)\prod_{i=l+2}^{m_k} a(b_i p^i).$$

Hence we get

$$a(b_{l+1}+1) \neq 0.$$

At the same time we have

$$a(Nq^k + p^{l+1}) = a(b_l p^l + (b_{l+1}+1)p^{l+1} + \sum_{i=l+2}^{m_k} b_i p^i),$$

so that by (8)

$$a(N)a(1) = a(b_l p^l)a(b_{l+1}+1)\prod_{i=l+2}^{m_k} a(b_i p^i),$$

noting (8). The left-hand side above is zero, since we have assumed $N > 1$ and so $a(1) = 0$. Therefore we get $a(b_{l+1}+1) = 0$ by (7), a contradiction. The proof of Lemma 5 is now completed.

We write

$$q = dp + r, \qquad r \in \{0, 1, \ldots, p-1\}.$$

To prove the theorem, we may assume that $a(n)$ $(n \geq 1)$ is not identically zero. Then $a(1) \neq 0$ by Lemma 5, and so

$$a(r) \neq 0.$$

We have the following formulas.

Lemma 6. $a(1) = a(d)a(r), \quad a(d+1) = \dfrac{a(1)^2}{a(r)}.$

Lemma 7. *Assume that $r \neq 0$. Let k and l be nonnegative integers such that $0 \leq kp - lr < p$. Then*

$$a(kp - lr) = \frac{a(1)^k}{a(r)^l}.$$

Lemma 8. *If* $r \neq 0$, *then* $a(n) = a(1)^n$ $(1 \leq n \leq d)$.

Lemma 9. *If* $r \neq 0$, *then*

$$a(n) = \gamma^n \qquad (n \geq 1),$$

where $\gamma^{p-1} = \gamma^{q-1} = 1$.

These lemmas can be proved by transforming the arguments in the preceding section in terms of q-multiplicativity. So we give only the proof of Lemma 9.

Proof of Lemma 9. First we show that

$$a(n) = a(1)^n \qquad (n \geq 1) \tag{11}$$

by induction on n. This holds for $n \leq d$, by Lemma 8. Let $n > d$ and assume that $a(k) = a(1)^k$ for $1 \leq k < n$. Let k and l be integers such that $0 \leq np - kq < q$ and $0 \leq np - kq - lp < p$. Since $0 \leq k,\ l < n$, we have by strongly p and q-multiplicativity

$$\begin{aligned} a(n) = a(np) &= a(kq + (np - kq)) \\ &= a(k)a(lp + (np - kq - lp)) \\ &= a(k)a(l)a((n - l - dk)p - kr) \\ &= a(1)^{k+l} \frac{a(1)^{n-l-dk}}{a(r)^k} \end{aligned}$$

by Lemma 7 and the induction hypothesis, and so

$$a(n) = \frac{a(1)^n a(1)^k}{(a(1)^d a(r))^k} = \frac{a(1)^n a(1)^k}{(a(d)a(r))^k} = a(1)^n$$

by Lemma 6 and 8. Hence (11) is proved.

Putting $n = p$ in (11), we have $a(1) = a(p) = a(1)^p$, so that $a(1) = 0$ or $a(1)^{p-1} = 1$. Similarly, we have $a(1) = 0$ or $a(1)^{q-1} = 1$ putting $n = q$. Since $a(1) \neq 0$, we have by (11)

$$a(n) = \gamma^n \qquad (n \geq 1),$$

for some γ with $\gamma^{p-1} = \gamma^{q-1} = 1$, and the lemma is proved.

Proof of Theorem 2. By induction on $p+q$. If $r \neq 0$, then Theorem 2 follows by Lemma 5 and 9. We assume $r = 0$, so that $q = dp$ with $d \geq 2$ since $p < q$. Let $n = \sum_{i\geq 0} b_i d^i$ be d-adic expansion of an integer $n \geq 1$. Then we have

$$\begin{aligned} a(n) &= a(\sum_{i\geq 0} b_i d^i) = a(\sum_{i\geq 0} b_i d^i p) = a(\sum_{i\geq 1} b_i d^{i-1} q + b_0 p) \\ &= a(\sum_{i\geq 1} b_i d^{i-1}) a(b_0) = \cdots = \prod_{i\geq 0} a(b_i). \end{aligned}$$

Hence $a(n)$ is a strongly d-multiplicative function, where $\log d/\log p$ is irrational, since

$$\frac{\log q}{\log p} = \frac{\log d}{\log p} + 1.$$

Noting $d + p < p + q$, we have, by induction hypothesis, $a(n) = 0$ $(n \geq 1)$ or $a(n) = \gamma^n$ for some γ with $\gamma^{d-1} = \gamma^{p-1} = 1$. We get also $\gamma^{q-1} = 1$, since

$$\gamma^q = \gamma^{dp} = \gamma.$$

The proof is completed.

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