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ON DISCRIMINANTS AND CERTAIN MATRICES

by

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0. Introduction

Let K be an algebraic number field of degree $n > 1$, and let α be an integer of K. In this paper we discuss the $n \times n$ matrix $C(\alpha) = (\ Tr\ (\alpha^{(i-1)+(j-1)})\ )$ and its minors. Certain minors of $C(\alpha)$ are closely related to the ramification of primes in $K/\boldsymbol{Q}$. For example: If the greatest common divisor of all the minors of order $(n-1)$ of the matrix $C(\alpha)$ is equal to 1, then the discriminant of K is square-free, and K has a very simple and explicit integral basis (§4). Therefore it seems important to study $C(\alpha)$ and its minors in relation to the discriminant and the ring of integers of K. In this paper we prove two theorems on the minors of order $(n-1)$, together with a few elementary results on the minors of order $i \leq n-1$.

1. The matrix $C(\alpha)$ and its minors of order $n-1$.

The main purpose of the present paper is to prove the following theorem.

Theorem 1. *Let K be an algebraic number field of degree $n > 1$. Let p be a prime number, and let $k \in \boldsymbol{Z}$, $k > 0$. Suppose that the discriminant of K is divisible by p^{2k}. Then, for any integer α of K, every minor of order $(n-1)$ of the $n \times n$ matrix*

$$C(\alpha) = \begin{pmatrix} Tr(1) & Tr(\alpha) & \dots & Tr(\alpha^{n-1}) \\ Tr(\alpha) & Tr(\alpha^2) & \dots & Tr(\alpha^n) \\ & & \dots & \\ Tr(\alpha^{n-1}) & Tr(\alpha^n) & \dots & Tr(\alpha^{2n-2}) \end{pmatrix}$$

is divisible by p^k, where $Tr(\xi)$ means the trace of ξ in $K/\boldsymbol{Q}$.

Proof. Let $\alpha^{(1)}, \cdots, \alpha^{(n)}$ denote the conjugates of α in $K/\boldsymbol{Q}$. Then

(1.1)

$$C(\alpha) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha^{(1)} & \alpha^{(2)} & \dots & \alpha^{(n)} \\ & \dots & & \\ \alpha^{(1)n-1} & \alpha^{(2)n-1} & \dots & \alpha^{(n)n-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha^{(1)} & \dots & \alpha^{(1)n-1} \\ 1 & \alpha^{(2)} & \dots & \alpha^{(2)n-1} \\ & & \dots & \\ 1 & \alpha^{(n)} & \dots & \alpha^{(n)n-1} \end{pmatrix}.$$

Suppose first that $K \neq \boldsymbol{Q}(\alpha)$. If $n > 2$, then

$$\text{rank}\begin{pmatrix} 1 & \alpha^{(1)} & \dots & \alpha^{(1)n-1} \\ & & \dots & \\ 1 & \alpha^{(n)} & \dots & \alpha^{(n)n-1} \end{pmatrix} \leq \frac{n}{2} < n-1. \tag{1.2}$$

By (1.1) we see that $\text{rank}C(\alpha) < n-1$; every minor of order $(n-1)$ is equal to 0. If $n = 2$, then $\alpha \in \boldsymbol{Z}$, $k = 1$ and $p = 2$; every entry of the matrix $C(\alpha)$ is divisible by $p^k = 2$. In any case, every minor of order $n-1$ of the matrix $C(\alpha)$ is divisible by p^k.

From now on, we assume that $K = \boldsymbol{Q}(\alpha)$. Let

$$\begin{aligned} f(x) &= (x-\alpha^{(1)})(x-\alpha^{(2)})\cdots(x-\alpha^{(n)}) \\ &= x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n. \end{aligned} \tag{1.3}$$

Then the coefficients a_i are rational integers, and $f(x)$ is irreducible over $\boldsymbol{Q}$.

$K = \boldsymbol{Q}(\alpha)$ is a vector space over $\boldsymbol{Q}$. We fix its basis:$1, \alpha, \dots, \alpha^{n-1}$. An element $\xi = c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1} (c_i \in \boldsymbol{Q})$ of K is then represented by a column vector $(c_0, \dots, c_{n-1})^T$, where T denotes transposition. The linear transformation $\xi \longmapsto \alpha\xi$ is determined by the $n \times n$ matrix

$$A = (\boldsymbol{e}_2\boldsymbol{e}_3 \dots \boldsymbol{e}_n\boldsymbol{a}_1), \tag{1.4}$$

where

$$\boldsymbol{a}_1 = (-a_n, -a_{n-1}, \dots, -a_2, -a_1)^T; \tag{1.5}$$

$\boldsymbol{e}_j$ denotes the j-th column of the identity matrix I_n. We define $\boldsymbol{a}_2, \boldsymbol{a}_3, \dots$ inductively:

$$\boldsymbol{a}_j = A\boldsymbol{a}_{j-1}, \tag{1.6}$$

where $j \geq 2$. Clearly,

$$\boldsymbol{a}_1 = A\boldsymbol{e}_n. \tag{1.7}$$

By induction on j, we see that

$$A^j = (\boldsymbol{e}_{j+1}\boldsymbol{e}_{j+2}\cdots\boldsymbol{e}_n\boldsymbol{a}_1\cdots\boldsymbol{a}_j) \tag{1.8}$$

for $j = 1, 2, \dots, n-1$.

Now let

$$g(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in \boldsymbol{Q}[x], \tag{1.9}$$

and let $\boldsymbol{g}_j$ denote the j-th column of the matrix $g(A)$:

$$g(A) = c_0I_n + c_1A + \cdots + c_{n-1}A^{n-1}, \tag{1.10}$$

$$g(A) = (\boldsymbol{g}_1\boldsymbol{g}_2\cdots\boldsymbol{g}_n). \tag{1.11}$$

Then

$$\boldsymbol{g}_j = g(A)\boldsymbol{e}_j \tag{1.12}$$

for $j = 1, 2, \dots, n$. The matrix $g(A)$ determines a linear transformation $\xi \longmapsto g(\alpha)\xi$. By (1.12) we see that the column vector $\boldsymbol{g}_j$ represents $g(\alpha)\alpha^{j-1}$ in K. Since

$$g(\alpha)\alpha^{j-1} = \alpha^{j-1}g(\alpha),$$

it follows from (1.9) that

$$\boldsymbol{g}_j = A^{j-1}\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} \tag{1.13}$$

for $j = 1, 2, \ldots, n$. Hence

$$\boldsymbol{g}_1 = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}, \quad \boldsymbol{g}_j = A\boldsymbol{g}_{j-1}, \tag{1.14}$$

where $2 \leq j \leq n$.

The eigenvalues of the matrix A are the conjugates of α in $K/\boldsymbol{Q}$; $f(x)$ is the minimum polynomial of the matrix A. For any $h(x) \in \boldsymbol{Q}[x]$, the element $h(\alpha)$ of the field K is represented by the matrix $h(A)$:

$$h(\alpha) \longleftrightarrow h(A). \tag{1.15}$$

The norm $N(h(\alpha))$ of $h(\alpha)$ in $K/\boldsymbol{Q}$ is equal to the determinant of $h(A)$:

$$N(h(\alpha)) = \det h(A). \tag{1.16}$$

Now let $\boldsymbol{b}_j$ denote the j-th column of the matrix $B = f'(A)$:

$$B = f'(A) = nA^{n-1} + (n-1)a_1A^{n-2} + \cdots + a_{n-1}I_n, \tag{1.17}$$

$$B = (\boldsymbol{b}_1\boldsymbol{b}_2 \ldots \boldsymbol{b}_n). \tag{1.18}$$

Then it follows from (1.14) that

$$\boldsymbol{b}_1 = \begin{pmatrix} a_{n-1} \\ 2a_{n-2} \\ \vdots \\ (n-1)a_1 \\ n \end{pmatrix}, \quad \boldsymbol{b}_j = A\boldsymbol{b}_{j-1}, \tag{1.19}$$

where $2 \leq j \leq n$.

Let D denote the norm of $\delta = f'(\alpha)$ in $K/\boldsymbol{Q}$:

$$\delta = f'(\alpha), \quad D = N(\delta). \tag{1.20}$$

Then (1.16) gives

$$D = \det B. \tag{1.21}$$

For $j = 1, 2, \ldots, n$, let

$$\alpha^{j-1}\delta = r_{1j} + r_{2j}\alpha + \cdots + r_{nj}\alpha^{n-1}, \tag{1.22}$$

where $r_{ij} \in \boldsymbol{Z}$. Then it follows from (1.15), (1.20) and (1.17) that

$$A^{j-1}B = r_{1j}I_n + r_{2j}A + \cdots + r_{nj}A^{n-1} \tag{1.23}$$

for $j = 1, 2, \cdots, n$. By (1.19) we see that the first column of $A^{j-1}B$ is $A^{j-1}\boldsymbol{b}_1 = \boldsymbol{b}_j$. Hence, by (1.14),

$$\boldsymbol{b}_j = (r_{1j}, r_{2j}, \ldots, r_{nj})^T. \tag{1.24}$$

Now let b_{ij} denote the (i, j)-entry of the matrix B:

$$B = \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ & \ldots & \\ b_{n1} & \ldots & b_{nn} \end{pmatrix}. \tag{1.25}$$

By (1.22) and (1.24) we see that

$$\alpha^{j-1}\delta = b_{1j} + b_{2j}\alpha + \cdots + b_{nj}\alpha^{n-1} \tag{1.26}$$

for $j = 1, 2, \ldots, n$. Let $\tilde{b}_{ij}$ denote the cofactor of the (i, j)-entry b_{ij}, and let

$$\alpha^{j-1}\frac{D}{\delta} = s_{1j} + s_{2j}\alpha + \cdots + s_{nj}\alpha^{n-1}, \tag{1.27}$$

where $s_{ij} \in \boldsymbol{Z}$, $1 \le j \le n$. From (1.15), (1.17), (1.20) and (1.21), we obtain

$$(\det B)B^{-1}A^{j-1} = s_{1j}I_n + s_{2j}A + \cdots + s_{nj}A^{n-1}. \tag{1.28}$$

By (1.8) we see that the first column of the matrix A^{j-1} is $\boldsymbol{e}_j$. From (1.14) we obtain

$$\begin{aligned} (s_{1j}, \ldots, s_{nj})^T &= (\det B)B^{-1}\boldsymbol{e}_j \\ &= (\tilde{b}_{j1}, \ldots, \tilde{b}_{jn})^T. \end{aligned}$$

Hence (1.27) becomes

$$\alpha^{j-1}\frac{D}{\delta} = \tilde{b}_{j1} + \tilde{b}_{j2}\alpha + \cdots + \tilde{b}_{jn}\alpha^{n-1} \tag{1.29}$$

for $j = 1, 2, \ldots, n$. In particular,

$$\frac{D}{\delta} = \tilde{b}_{11} + \tilde{b}_{12}\alpha + \cdots + \tilde{b}_{1n}\alpha^{n-1}. \tag{1.30}$$

It follows from (1.29) and (1.30) that every cofactor $\tilde{b}_{ij}$ is divisible by the greatest common divisor of $\tilde{b}_{11}, \ldots, \tilde{b}_{1n}$:

$$(\tilde{b}_{11}, \tilde{b}_{12}, \ldots, \tilde{b}_{1n}) \mid \tilde{b}_{ij}, \tag{1.31}$$

where $1 \leq i \leq n$, $1 \leq j \leq n$.

Clearly, the column vector

$$\boldsymbol{x} = (1, \alpha, \ldots, \alpha^{n-1})^T \tag{1.32}$$

is an eigenvector of the matrix A^T corresponding to the eigenvalue α:

$$A^T \boldsymbol{x} = \alpha \boldsymbol{x}, \quad \boldsymbol{x} \neq \mathbf{o}. \tag{1.33}$$

It is easily seen that an eigenvector of the matrix A corresponding to the eigenvalue α is given by $M\boldsymbol{x}$:

$$A(M\boldsymbol{x}) = \alpha M \boldsymbol{x}, \tag{1.34}$$

where

$$M = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & \\ \vdots & \vdots & \iddots & & \\ a_1 & 1 & & 0 & \\ 1 & & & & \end{pmatrix}. \tag{1.35}$$

Since $1, \alpha, \ldots, \alpha^{n-1}$ are linearly independent over $\boldsymbol{Q}$, it follows from (1.33) and (1.34) that

$$AM = MA^T. \tag{1.36}$$

Hence

$$A^j M = M(A^T)^j \tag{1.37}$$

for every $j \in \boldsymbol{Z}$.

Let $\boldsymbol{c}_j$ denote the j-th column of the matrix $C(\alpha)$:

$$C(\alpha) = (\boldsymbol{c}_1 \boldsymbol{c}_2 \ldots \boldsymbol{c}_n). \tag{1.38}$$

By definition,

$$\boldsymbol{c}_j = (Tr(\alpha^{j-1}), Tr(\alpha^j), \ldots, Tr(\alpha^{j+n-2}))^T. \tag{1.39}$$

From (1.32), (1.33) and (1.39), we obtain

$$\boldsymbol{c}_j = A^T \boldsymbol{c}_{j-1} \tag{1.40}$$

for $j = 2, 3, \ldots, n$. From (1.19),

$$\boldsymbol{b}_2 = \begin{pmatrix} -na_n \\ -(n-1)a_{n-1} \\ \vdots \\ -2a_2 \\ -a_1 \end{pmatrix}. \tag{1.41}$$

Newton's formula gives

$$M\boldsymbol{c}_2 = \boldsymbol{b}_2. \tag{1.42}$$

From (1.19), (1.37), (1.40) and (1.42), we obtain the following formula (cf. [2], §10):

$$B = MC(\alpha). \tag{1.43}$$

Let m^2 ($m \in \boldsymbol{Z}$) denote the largest square dividing D. Then

$$\frac{D}{m\delta} \in O_K, \tag{1.44}$$

where O_K denotes the ring of integers of K ([4], Theorem 1). Let t denote the index of α:

$$t = (O_K : \boldsymbol{Z}[\alpha]). \tag{1.45}$$

Then

$$(-1)^{\frac{n(n-1)}{2}} D = d_K t^2, \tag{1.46}$$

where d_K denotes the discriminant of K. It follows from (1.30), (1.44) and (1.45) that

$$\frac{t\tilde{b}_{1j}}{m} \in \boldsymbol{Z} \tag{1.47}$$

for $j = 1, 2, \ldots, n$. By (1.46) we see that

$$\frac{D\tilde{b}_{1j}{}^2}{m^2 d_K} \in \boldsymbol{Z} \tag{1.48}$$

for $j = 1, 2, \ldots, n$. By hypothesis d_K is divisible by p^{2k}. Since D/m^2 is a square-free integer, $\tilde{b}_{1j}$ is divisible by p^k. From (1.31) we obtain

$$p^k \mid \tilde{b}_{ij} \tag{1.49}$$

for all i, j ($1 \le i \le n, 1 \le j \le n$).

By (1.35) we see that every entry of the inverse matrix of M is a rational integer:

$$M^{-1} \in M_n(\boldsymbol{Z}). \tag{1.50}$$

From (1.43),

$$C(\alpha) = M^{-1}B. \tag{1.51}$$

Hence the adjugate of $C(\alpha)$ satisfies

$$\mathrm{adj}C(\alpha) = \mathrm{adj}B \ \mathrm{adj}(M^{-1}). \tag{1.52}$$

It follows from (1.49), (1.50) and (1.52) that the entries of the matrix $\mathrm{adj}C(\alpha)$ are all divisible by p^k. Q.E.D.

Remark. It follows from (1.1) that, for any integer α of K, $\det C(\alpha)$ is equal to the discriminant of α in $K/\boldsymbol{Q}$, which is divisible by every prime factor p of the discriminant d_K of K. However, if d_K is not divisible by p^2, K may have an integer

α such that at least one minor of order $n-1$ of the matrix $C(\alpha)$ is not divisible by p. A simple example is

$$K = \boldsymbol{Q}(\alpha), \quad \alpha^2 - p = 0, \tag{1.53}$$

where p is an odd prime. The matrix

$$C(\alpha) = \begin{pmatrix} 2 & 0 \\ 0 & 2p \end{pmatrix} \tag{1.54}$$

has four minors of order one. One of them is not divisible by p, and the other three are all divisible by p.

2. The corner of order $n-1$.

In this section we prove a theorem on the corner of order $n-1$ (i.e. the cofactor of the (n,n)-entry) of the matrix $C(\alpha)$.

Theorem 2. *Let K be an algebraic number field of degree $n > 1$, and let α be an integer of K. Then for a prime number p to divide all the minors of order $n-1$ of the $n \times n$ matrix*

$$C(\alpha) = \begin{pmatrix} Tr(1) & Tr(\alpha) & \dots & Tr(\alpha^{n-1}) \\ Tr(\alpha) & Tr(\alpha^2) & \dots & Tr(\alpha^n) \\ & & \dots & \\ Tr(\alpha^{n-1}) & Tr(\alpha^n) & \dots & Tr(\alpha^{2n-2}) \end{pmatrix}$$

it is necessary and sufficient that the determinant of $C(\alpha)$ and its corner of order $n-1$ are both divisible by p.

To prove our theorem we require the following lemma.

Lemma 1. *Let F be a field, and let $S = (s_{ij})$ be a symmetric $n \times n$ matrix with (i,j)-entry $s_{ij} \in F$. Let $\tilde{s}_{ij}$ denote the cofactor of the entry s_{ij}. If $\det S = \tilde{s}_{nn} = 0$, then $\tilde{s}_{nj} = 0$ for $j = 1, 2, \dots, n$.*

Proof. By hypothesis,

$$S\boldsymbol{v} = \mathbf{o}, \tag{2.1}$$

where $\boldsymbol{v} = (\tilde{s}_{n1}, \tilde{s}_{n2}, \dots, \tilde{s}_{nn})^T$. For $j = 1, 2, \dots, n$, let S_j denote the $(n-1) \times (n-1)$ matrix obtained from S by deletion of the j-th row and the n-th column. Since $\tilde{s}_{nn} = 0$, it follows from (2.1) that

$$S_j \boldsymbol{v}_0 = \mathbf{o} \tag{2.2}$$

for $j = 1, 2, \dots, n$, where

$$\boldsymbol{v}_0 = (\tilde{s}_{n1}, \tilde{s}_{n2}, \dots, \tilde{s}_{n(n-1)})^T. \tag{2.3}$$

Suppose that $\tilde{s}_{nj} \neq 0$ for some $j < n$. Then $\boldsymbol{v}_0 \neq \mathbf{o}$, and so $\det S_j = 0$. This implies that $\tilde{s}_{jn} = \tilde{s}_{nj} = 0$, a contradiction. Hence $\tilde{s}_{nj} = 0$ for $j = 1, 2, \dots, n$.

Proof of Theorem. We may assume that $K = \boldsymbol{Q}(\alpha)$ (See the proof of Theorem 1).

Let $\tilde{c}_{ij}$ denote the cofactor of the (i,j)-entry c_{ij} of the matrix $C(\alpha)$. Let δ (resp. $d(\alpha)$) denote the different (resp. discriminant) of α in $K/\boldsymbol{Q}$. Then, from (1.30), (1.35) and (1.43),

$$\frac{d(\alpha)}{\delta} = \tilde{c}_{n1} + \tilde{c}_{n2}\alpha + \cdots + \tilde{c}_{nn}\alpha^{n-1}. \tag{2.4}$$

Let p denote a prime number such that $\det C(\alpha) \equiv \tilde{c}_{nn} \equiv 0 \pmod{p}$. Then Lemma 1 implies that $\tilde{c}_{nj} \equiv 0 \pmod{p}$ for $j = 1, 2, \ldots, n$. It follows from (1.31), (1.50) and (1.52) that $\tilde{c}_{ij} \equiv 0 \pmod{p}$ for all i, j.

3. Minors of order i.

In this section we discuss some elementary properties of the matrix $C(\alpha)$ and its minors.

Let K be an algebraic number field of degree $n > 1$, and let α be an integer of K. Let $i \in \boldsymbol{Z}$, $1 \leq i \leq n$. We denote by $\tilde{c}_i(\alpha)$ the greatest common divisor of all the minors of order i of the matrix $C(\alpha)$. Clearly, $\tilde{c}_i(\alpha)$ is divisible by $\tilde{c}_{i-1}(\alpha)$ for every $i > 1$.

Theorem 1 becomes

Theorem 1a. *Let $s^2 (s \in \boldsymbol{Z})$ denote the largest square dividing the discriminant of an algebraic number field K of degree $n > 1$. Then, for any integer α of K, $\tilde{c}_{n-1}(\alpha)$ is divisible by s.*

Now we have

Proposition 1. *Let O_K denote the ring of integers of an algebraic number field K of degree $n > 1$, and let $j \in \boldsymbol{Z}$, $1 \leq j \leq n-1$. Let $\alpha \in O_K$, and let $c_0, \ldots, c_{j-1}$, m_0 $(m_0 \neq 0)$ be rational integers such that*

$$\frac{c_0 + c_1\alpha + \cdots + c_{j-1}\alpha^{j-1} + \alpha^j}{m_0} \in O_K. \tag{3.1}$$

Then $\tilde{c}_{j+1}(\alpha)$ is divisible by m_0.

Proof. Let $\boldsymbol{c}_k$ denote the k-th column of the matrix $C(\alpha)$:

$$\boldsymbol{c}_k = \begin{pmatrix} Tr(\alpha^{k-1}) \\ Tr(\alpha^k) \\ \vdots \\ Tr(\alpha^{k+n-2}) \end{pmatrix}. \tag{3.2}$$

By induction we see that

$$\alpha^{k-1} = s_{k0} + s_{k1}\alpha + \cdots + s_{k(j-1)}\alpha^{j-1} + m_0\xi_k \tag{3.3}$$

for $k = 1, 2, \ldots, n$, where $s_{kl} \in \boldsymbol{Z}$, $\xi_k \in O_K$. Hence

$$\boldsymbol{c}_k = s_{k0}\boldsymbol{c}_1 + s_{k1}\boldsymbol{c}_2 + \cdots + s_{k(j-1)}\boldsymbol{c}_j + m_0 \begin{pmatrix} Tr(\xi_k) \\ \vdots \\ Tr(\alpha^{n-1}\xi_k) \end{pmatrix} \tag{3.4}$$

for $k = 1, 2, \ldots, n$. Let $\boldsymbol{c}_{k_1}, \boldsymbol{c}_{k_2}, \ldots, \boldsymbol{c}_{k_{j+1}}$ be any $(j+1)$ columns of $C(\alpha)$, and let p be a prime number such that m_0 is exactly divisible by p^t $(t > 0)$. Then (3.4) implies that some $\boldsymbol{c}_{k_i}$ is a linear combination modulo p^t of the other j columns with integer coefficients. Hence every minor of order $(j+1)$ of the matrix $C(\alpha)$ is divisible by p^t, and so, by m_0. Hence $\tilde{c}_{j+1}(\alpha)$ is divisible by m_0.

It is well-known (e.g. [6], p.34) that an algebraic number field $K = \boldsymbol{Q}(\alpha)$ $(\alpha \in O_K)$ of degree $n > 1$ has an integral basis of the form

$$(3.5)\quad 1, \frac{c_{10} + \alpha}{m_1}, \frac{c_{20} + c_{21}\alpha + \alpha^2}{m_2}, \ldots, \frac{c_{(n-1)0} + \cdots + c_{(n-1)(n-2)}\alpha^{n-2} + \alpha^{n-1}}{m_{n-1}},$$

where $c_{ij}, m_j \in \boldsymbol{Z}$; m_j is divisible by m_{j-1} for every $j > 1$. By Proposition 1 we see that $\tilde{c}_{j+1}(\alpha)$ is divisible by m_j for every $j \leq n-1$.

Considering the elementary divisors of $C(\alpha)$, we obtain

Proposition 2. *Let K be an algebraic number field of degree $n > 1$, and let α be an integer of K such that $K = \boldsymbol{Q}(\alpha)$. Then $\tilde{c}_{i+1}(\alpha)/\tilde{c}_i(\alpha)$ is divisible by $\tilde{c}_i(\alpha)/\tilde{c}_{i-1}(\alpha)$ for every $i = 1, 2, \ldots, n-1$, where $\tilde{c}_0(\alpha) = 1$. Let p be a prime number such that $\tilde{c}_i(\alpha)$ is divisible by p^t $(t > 0)$. Then $\tilde{c}_{i+1}(\alpha)$ is divisible by p^{t+1}.*

Proof. By hypothesis, $\det C(\alpha) \neq 0$. The integers

$$e_1 = \frac{\tilde{c}_1(\alpha)}{\tilde{c}_0(\alpha)},\ e_2 = \frac{\tilde{c}_2(\alpha)}{\tilde{c}_1(\alpha)},\ \cdots,\ e_n = \frac{\tilde{c}_n(\alpha)}{\tilde{c}_{n-1}(\alpha)}$$

are the elementary divisors of $C(\alpha)$. Since e_{i+1} is divisible by e_i, it follows that $\tilde{c}_{i+1}(\alpha)/\tilde{c}_i(\alpha)$ is divisible by $\tilde{c}_i(\alpha)/\tilde{c}_{i-1}(\alpha)$. To prove the last assertion, suppose that $\tilde{c}_{i+1}(\alpha)$ is not divisible by p^{t+1}. Then $\tilde{c}_{i+1}(\alpha)$ is exactly divisible by p^t; $e_{i+1} = \tilde{c}_{i+1}(\alpha)/\tilde{c}_i(\alpha)$ is not divisible by p. On the other hand,

$$(3.6)\qquad \tilde{c}_{i+1}(\alpha) = e_1 e_2 \cdots e_{i+1}, \qquad e_j | e_{j+1}.$$

This implies that $\tilde{c}_{i+1}(\alpha)$ is not divisible by p, a contradiction.

4. Examples.

1) Consider now a cubic field:

$$(4.1)\qquad K = \boldsymbol{Q}(\alpha); \quad \alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 = 0, \quad a_i \in \boldsymbol{Z},$$

where $f(x) = x^3 + a_1x^2 + a_2x + a_3$ is irreducible. We obtain:

$$(4.2)\qquad A = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix};$$

$$(4.3)\qquad B = f'(A) = \begin{pmatrix} a_2 & -3a_3 & a_1a_3 \\ 2a_1 & -2a_2 & a_1a_2 - 3a_3 \\ 3 & -a_1 & {a_1}^2 - 2a_2 \end{pmatrix};$$

(4.4)

$$C(\alpha) = \begin{pmatrix} Tr(1) & Tr(\alpha) & Tr(\alpha^2) \\ Tr(\alpha) & Tr(\alpha^2) & Tr(\alpha^3) \\ Tr(\alpha^2) & Tr(\alpha^3) & Tr(\alpha^4) \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -a_1 & {a_1}^2 - 2a_2 \\ -a_1 & {a_1}^2 - 2a_2 & -{a_1}^3 + 3a_1a_2 - 3a_3 \\ {a_1}^2 - 2a_2 & -{a_1}^3 + 3a_1a_2 - 3a_3 & {a_1}^4 - 4{a_1}^2a_2 + 4a_1a_3 + 2{a_2}^2 \end{pmatrix}.$$

Let $\tilde{b}_{ij}$ (resp. $\tilde{c}_{ij}$) denote the cofactor of the (i,j)-entry of the matrix B (resp. $C(\alpha)$). Then

$$\begin{aligned} \tilde{c}_{31} &= -\tilde{b}_{11} = {a_1}^2a_2 - 4{a_2}^2 + 3a_1a_3, \\ \tilde{c}_{32} &= -\tilde{b}_{12} = 2{a_1}^3 - 7a_1a_2 + 9a_3, \\ \tilde{c}_{33} &= -\tilde{b}_{13} = 2({a_1}^2 - 3a_2). \end{aligned} \tag{4.5}$$

Let $d(\alpha)$ denote the discriminant of α. Then a classical formula

$$d(\alpha) = -4{a_1}^3a_3 + {a_1}^2{a_2}^2 + 18a_1a_2a_3 - 4{a_2}^3 - 27{a_3}^2 \tag{4.6}$$

follows from

$$d(\alpha) = -\det B = -(a_2\tilde{b}_{11} - 3a_3\tilde{b}_{12} + a_1a_3\tilde{b}_{13}). \tag{4.7}$$

Let p ($p \neq 2$) be a prime factor of $\tilde{c}_2(\alpha)$ (which we defined in §3). Then $\tilde{c}_{33}$ is divisible by p, and so

$${a_1}^2 \equiv 3a_2 \pmod{p}. \tag{4.8}$$

Since $d(\alpha) = \det C(\alpha)$ is divisible by p, it follows from (4.6) and (4.8) that

$$27d(\alpha) \equiv -({a_1}^3 - 3^3a_3)^2 \equiv 0 \pmod{p}. \tag{4.9}$$

Hence

$${a_1}^3 \equiv 3^3a_3 \pmod{p}. \tag{4.10}$$

Conversely, if p ($p \neq 3$) is a prime number which satisfies (4.8) and (4.10), then $\tilde{c}_{33}$ and $d(\alpha)$ are both divisible by p, and $\tilde{c}_2(\alpha)$ is also divisible by p (Theorem 2).

Thus we have proved the following result: *For a prime number p ($p \neq 2, 3$) to divide all the minors of order two of the matrix $C(\alpha)$ it is necessary and sufficient that ${a_1}^2 \equiv 3a_2 \pmod{p}$ and ${a_1}^3 \equiv 3^3a_3 \pmod{p}$.*

2) Consider now a cubic field (4.1) satisfying $a_2 \equiv a_3 \equiv 0 \pmod{3}$, $a_1 \not\equiv 0 \pmod{3}$. Then by (4.5) and (4.6) we see that both $\tilde{c}_{31}$ and $d(\alpha) = \det C(\alpha)$ are divisible by 3, but $\tilde{c}_{33}$ is not divisible by 3 (cf. Theorem 2, Lemma 1). Suppose that $a_1 \equiv a_3 \equiv 1$, $a_2 \equiv -1 \pmod{4}$. Consider the prime $p = 2$. By (4.5) and (4.6) we see that both $\tilde{c}_{33}$ and $\det C(\alpha)(= d(\alpha))$ are divisible by p^2, but $\tilde{c}_{31}$ is not divisible by p^2 (cf. Theorem 2).

3) The converse of Theorem 1 is not true. Let $k = 1, p = 2$, and let K be a cubic field with odd discriminant d_K such that, for every integer α of K, the discriminant $d(\alpha)$ of α is even (Dedekind[3]). Then, for any integer α of K, $\det C(\alpha) = d(\alpha)$ is

divisible by $p = 2$; it follows from Theorem 2 and (4.5) that every minor of order two of the matrix $C(\alpha)$ is divisible by p, but d_K is not divisible by p^2.

4) Let O_K denote the ring of integers of an algebraic number field K of degree $n > 1$, and let $\alpha \in O_K$ such that $K = \boldsymbol{Q}(\alpha)$. Let δ (resp. $d(\alpha)$) denote the different (resp. discriminant) of α in $K/\boldsymbol{Q}$, and let $m^2 (m \in \boldsymbol{Z})$ denote the largest square dividing $d(\alpha)$. By (1.44) we see that

$$\frac{d(\alpha)}{m\delta} \in O_K. \tag{4.11}$$

From (2.4),

$$\frac{d(\alpha)}{m\delta} = \frac{\tilde{c}_{n1} + \tilde{c}_{n2}\alpha + \cdots + \tilde{c}_{nn}\alpha^{n-1}}{m}, \tag{4.12}$$

where $\tilde{c}_{ij}$ denotes the cofactor of the (i, j)-entry of the matrix $C(\alpha)$.

Now suppose that $\tilde{c}_{n-1}(\alpha) = 1$. Then K has a very simple integral basis (cf. [1],[4],[6]). By Theorem 2 we see that m is prime to $\tilde{c}_{nn}$. Let $a, b \in \boldsymbol{Z}$ such that

$$a\tilde{c}_{nn} + bm = 1, \tag{4.13}$$

and define

$$\beta = \frac{ad(\alpha)}{m\delta} + b\alpha^{n-1} \in O_K. \tag{4.14}$$

Then $\{1, \alpha, \ldots, \alpha^{n-2}, \beta\}$ is an integral basis of K, since

$$\begin{vmatrix} 1 & \alpha^{(1)} & \ldots & \alpha^{(1)n-2} & \beta^{(1)} \\ & & \ldots & & \\ 1 & \alpha^{(n)} & \ldots & \alpha^{(n)n-2} & \beta^{(n)} \end{vmatrix}^2 = \frac{d(\alpha)}{m^2} \tag{4.15}$$

is square-free. The discriminant of K is

$$d_K = \frac{d(\alpha)}{m^2}. \tag{4.16}$$

Since d_K is square-free, it follows from [5] (Theorem 1) that the Galois group of $\bar{K}/\boldsymbol{Q}$ is isomorphic to the symmetric group S_n, where $\bar{K}$ denotes the Galois closure of $K/\boldsymbol{Q}$.

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