Title	Certain stochastic differential equations with a singular drift
Sub Title	
Author	内藤, 昌宏(Naito, Yoshihiro)
Publisher	慶應義塾大学理工学部
Publication year	1995
Jtitle	Keio Science and Technology Reports Vol.48, No.1 (1995.) ,p.1- 10
JaLC DOI	
Abstract	
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00480001-0001

KEIO SCIENCE AND TECHNOLOGY REPORTS
VOL. 48, NO. 1, pp. 1–10, 1995, ISSN: 02864215

CERTAIN STOCHASTIC DIFFERENTIAL EQUATIONS WITH A SINGULAR DRIFT

by

Yoshihiro NAITO

Department of Mathematics, Faculty of Science and Technology, Keio University, 3–14–1, Hiyoshi, Kohoku-ku, Yokohama, 223, JAPAN

(Received December 7, 1995)

1. Introduction

Let $B(t)$, $t \geq 0$, be a one-dimensional Brownian motion with $B(0)=0$ defined on a certain probability space $(\Omega, \mathscr{F}, P)$. Given a real valued Borel funcion $a(x)$, $x \in \boldsymbol{R}$, satisfying

$$|a(x)| \leq K, \qquad x \in \boldsymbol{R}, \tag{1.1}$$

we consider the stochastic differential equaion (SDE)

$$dX(t) = a(X(t))dB(t) + \frac{X(t)}{t}dt, \qquad X(0)=0, \tag{1.2}$$

or equivalently

$$X(t) = \int_0^t a(X(s))dB(s) + \int_0^t \frac{X(s)}{s}ds, \qquad t>0, \tag{1.3}$$

where a solution $X(t)$ is assumed to be non-anticipating, so the first term in the right hand side of (1.3) is the usual Itô integral. We also consider the SDE

$$dX(t) = a(X(t))d^+B(t) + \frac{X(t)}{t}dt, \qquad X(0)=0, \tag{1.4}$$

or equivalently

$$X(t) = \int_0^t a(X(s))d^+B(s) + \int_0^t \frac{X(s)}{s}ds, \qquad t>0, \tag{1.5}$$

where a solution is assumed to be backward non-anticipating in the sense as defined later and the first term in the right hand side of (1.5) is a backward stochastic integral defined as the limit of $\sum a(X(t_k))(B(t_k)-B(t_{k-1}))$ as $|\Delta| = \max(t_k - t_{k-1}) \to 0$, Δ being a partition of $[0, t]$: $0=t_0<t_1<t_2<\cdots<t_n=t$. The second term in the right hand side of (1.3) as well as of (1.5) equals $\lim_{\varepsilon \downarrow 0}\int_\varepsilon^t s^{-1}X(s)ds$ that is assumed to exist a.s.

When $a(x) \equiv 1$, (1.3) and (1.5) have the same form that makes sense even if $X(t)$

is neither non-anticipating nor backward non-anticipating. In this case Jeulin and Yor ([3][5]) proved that for any general solution $X(t)$ of (1.3) (=(1.5)) the limit of $X(t)/t$ as $t\to\infty$, denoted by U, exists a.s. and that $X(t)$ is represented as $X(t)=\beta(t)+tU$, where $\beta(t)=-t\int_t^\infty s^{-1}dB(s)$, which is again a Brownian motion. In particular if $X(t)$ is backward non-anticipating, then U is independent of $B(t)$, $t\geq 0$. However non-anticipating solutions do not exist (e.g. [1]). Domenig-Nagasawa [1] discussed the uniqueness and non-uniqueness of solutions of (1.2) with Skorohod's additional term on the right hand side to make $|X(t)|\leq R(t)$ where $R(t)$ is a given strictly increasing continuous function with $R(0)=0$.

In this paper some generalization of the result by Jeulin and Yor stated in the above will be given in the case of a variable coefficient $a(x)$. Before stating our results we introduce the following σ-fields on Ω.

$$\mathscr{F}(B)=\sigma\{B(t): t\geq 0\}\,, \qquad \mathscr{F}_t(B)=\sigma\{B(s): 0\leq s\leq t\}\,,$$
$$\mathscr{F}_t^{\,+}(dB)=\sigma\{B(s+t)-B(t): s\geq 0\}\,.$$

Here the notation $\sigma\{(*)\}$ stands for the smallest σ-field on Ω that makes $(*)$ measurable. For a process $\{X(t)\}$ and a random variable U we define $\mathscr{F}(X)$, $\mathscr{F}_t(X)$ and $\mathscr{F}(U)$ in a similar way and $\mathscr{F}_t^{\,+}(X, dB)=\sigma\{X(s+t), B(s+t)-B(t): s\geq 0\}$ for each $t\geq 0$. We say that $X(t)$ is non-anticipating if $\mathscr{F}_t(X)\vee\mathscr{F}_t(B)$ is independent of $\mathscr{F}_t^{\,+}(dB)$, and that $X(t)$ is backward non-anticipating if $\mathscr{F}_t^{\,+}(X, dB)$ is independent of $\mathscr{F}_t(B)$ for each $t\geq 0$. Let W be the space $C[0,\infty)$ of continuous functions. For an element w of W we denote by $w(t)$ the value of w at time t. On W we consider the σ-fields $\mathscr{B}(W)=\sigma\{w(t), t\geq 0\}$, $\mathscr{B}_t(W)=\sigma\{w(s), 0\leq s\leq t\}$ and $\mathscr{B}_t^{+}(W)=\sigma\{w(s+t)-w(t): s\geq 0\}$. The Brownian motion $\{B(t), t\geq 0\}$ can be regarded as a random variable taking values in W. When we take such a view-point we write $\boldsymbol{B}=\{B(t), t\geq 0\}$. Thus $\boldsymbol{B}$ is a random variable with values in W whose probability law is the Wiener measure.

Our results are the following.

Theorem 1. *Assume that $a(x)$ is Borel measurable in $\boldsymbol{R}$, continuous at $x=0$ and satisfies* (1.1). *Then there are no non-anticipating solutions of* (1.3) *provided that $a(0)\neq 0$.*

Theorem 2. *Assume that $a(x)$ satisfies* (1.1) *and is Lipschitz continuous.*

(i) *If $X(t)$ is a backward non-anticipating solution of* (1.5), *then the limit*

$$U=\lim_{t\to\infty}\frac{X(t)}{t} \tag{1.6}$$

exists almost surely and is independent of $\{B(t), t\geq 0\}$. The $X(t)$ satisfies

$$X(t)=-t\int_t^\infty\frac{a(X(s))}{s}\,d^+B(s)+tU\,, \qquad t>0\,. \tag{1.7}$$

Conversely, if $X(t)$ is a backward non-anticipating solution of (1.7) *for any given random variable U that is independent of $\{B(t), t\geq 0\}$, then $X(t)$ satisfies* (1.5).

(ii) *There exists a real valued function Φ defined on* $(0,\infty)\times\boldsymbol{R}\times W$ *and having the following properties.*

(1.8a) *For each $t>0$ the restriction of Φ on $[t,\infty)\times\boldsymbol{R}\times W$ is measurable with respect* *to* $\mathscr{B}([t,\infty))\otimes\mathscr{B}(\boldsymbol{R})\otimes\mathscr{B}_t^+(W)$.

(1.8b) *For each* $x\in\boldsymbol{R}$ *the set* $\{w\in W:\Phi(t,x,w)$ *is continuous in* $t\}$ *has the Wiener measure* 1.

(1.9) *Any backward non-anticipating solution* $X(t)$ *of* (1.5) *can be represented as*

$$X(t)=\Phi(t,U,\boldsymbol{B}),\qquad a.s.,$$

where U *is given by* (1.6).

Note that (1.8a) implies the measurability of Φ with respect to $\mathscr{B}((0,\infty))\otimes\mathscr{B}(\boldsymbol{R})\otimes\mathscr{B}(W)$.

2. Proof of Theorem 1

We begin by proving the following lemma.

Lemma 1. *If* $X(t)$ *is a non-anticipating solution of* (1.3), *then the limit*

$$U=\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{X(s)}{s}ds=\lim_{t\to\infty}\frac{X(t)}{t}\tag{2.1}$$

exists almost surely and $X(t)$ *satisfies*

$$X(t)=-t\int_t^\infty\frac{a(X(s))}{s}dB(s)+tU,\qquad t>0.\tag{2.2}$$

Proof. If we put

$$U(t)=\frac{1}{t}\int_0^t\frac{X(s)}{s}ds,\qquad M(t)=\int_0^t a(X(s))dB(s),\tag{2.3}$$

then $X(t)=t\{tU(t)\}'=tU(t)+t^2U'(t)$ and (1.3) implies $X(t)=M(t)+tU(t)$. Therefore $U'(t)=t^{-2}M(t)$ and hence

$$U(s)-U(t)=\int_t^s M(r)r^{-2}dr,\qquad 0<t<s.\tag{2.4}$$

Since

$$E\left\{\int_t^\infty|M(r)|r^{-2}dr\right\}=\int_t^\infty E\{|M(r)|\}r^{-2}dr\le\int_t^\infty\sqrt{E\{|M(r)|\}^2}\,r^{-2}dr$$
$$=\int_t^\infty\left[E\left\{\int_0^r a(X(u))^2du\right\}\right]^{1/2}r^{-2}dr\le K\int_t^\infty r^{1/2}r^{-2}dr<\infty,$$

letting $s\uparrow\infty$ in (2.4) we see that $U=\lim_{s\uparrow\infty}U(s)$ exists (a.s.) and

$$U(t)=-\int_t^\infty M(s)s^{-2}ds+U,\tag{2.5}$$

or equivalently

$$\int_0^t \frac{X(s)}{s} ds = -t \int_t^\infty M(s) s^{-2} ds + tU, \tag{2.6}$$

and hence

$$\frac{X(t)}{t} = -\int_t^\infty \frac{M(s)}{s^2} ds + \frac{M(t)}{t} + U. \tag{2.7}$$

By putting $f(t, x) = x/t$ and applying Itô's formula to $df(t, M(t))$, we have

$$\frac{M(s)}{s} - \frac{M(t)}{t} = -\int_t^s \frac{M(u)}{u^2} du + \int_t^s \frac{a(X(u))}{u} dB(u).$$

Since $\lim_{s\uparrow\infty} s^{-1} M(s) = 0$ in probability which follows from $E|M(s)| \leq K\sqrt{s}$, we have

$$\frac{M(t)}{t} = \int_t^\infty \frac{M(s)}{s^2} ds - \int_t^\infty \frac{a(X(s))}{s} dB(s). \tag{2.8}$$

Comparing this with (2.7), we have (2.2). The second equality in (2.1) follows from (2.2). □

We now proceed to the proof of Theorem 1. Suppose there exists a non-anticipating solution $X(t)$ of (1.3). If we put $X_n(t) = \sqrt{n}\, X(t/n)$ and $a_n(x) = a(x/\sqrt{n})$ for $n \geq 1$, then $X(t) = n^{-1/2} X_n(nt)$ and $a_n(x) \to a(0)$ as $n \to \infty$. From (1.3) we have

$$X_n(t) = \sqrt{n} \int_0^{t/n} a(X(s)) dB(s) + \sqrt{n} \int_0^{t/n} \frac{X(s)}{s} ds. \tag{2.9}$$

Noting that

$$\sqrt{n} \int_0^{t/n} a(X(s)) dB(s) = \int_0^t a_n(X_n(s)) dB_n(s),$$

where $B_n(t) = \sqrt{n}\, B(t/n)$, which is again a Brownian motion, we see that $X_n(t)$ is a non-anticipating solution of

$$X_n(t) = \int_0^t a_n(X_n(s)) dB_n(s) + \int_0^t \frac{X_n(s)}{s} ds. \tag{2.10}$$

Therefore by Lemma 1 we have

$$X_n(t) = -t \int_t^\infty \frac{a_n(X_n(s))}{s} dB_n(s) + tU_n, \tag{2.11}$$

where

$$U_n = \lim_{t\to\infty} \frac{X_n(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{X_n(s)}{s} ds = \frac{1}{\sqrt{n}} U. \tag{2.12}$$

Let ν_n be the probability law of $\{U_n, \{B_n(t), t \geq 0\}, \{X_n(t), t > 0\}\}$ and put

$Y_n(t)=t^{-1}X_n(t)$. Making use of the representation (2.11), the bound $|a_n|\leq K$ and the Burkholder-Davis-Gundy inequalities, we have $E\{|Y_n(t)-Y_n(s)|^4\}\leq 3(Ks^{-1})^4|t-s|^2$, $0<s<t$. Therefore the sequence of probability laws of $\{Y_n(t), t>0\}$, $n\geq 1$ is tight, so is $\{\nu_n\}$. According to Skorohod's realization theorem of almost sure convergence (e.g. see [2], [4]), there exist $\tilde{U}_n$, $\{\tilde{B}_n(t), t\geq 0\}$ and $\{\tilde{X}_n(t), t>0\}$ satisfying the following conditions.

(i) $\{\tilde{U}_n, \{\tilde{B}_n(t), t\geq 0\}, \{\tilde{X}_n(t), t>0\}\}$ is identical in law to $\{U_n, \{B_n(t), t\geq 0\}, \{X_n(t), t>0\}\}$ for each $n\geq 1$.

(ii) There exist a Brownian motion $\{\tilde{B}(t), t\geq 0\}$ and a process $\{\tilde{X}(t), t>0\}$ such that $\tilde{B}_n(t)\to\tilde{B}(t)$ uniformly on each compact t-interval in $(0,\infty)$ (a.s.), $\tilde{X}_n(t)\to\tilde{X}(t)$ uniformly on each compact t-interval in $(0,\infty)$ (a.s.), and $\tilde{U}_n\to 0$ a.s.

The condition (i) implies that $\mathscr{F}_t(\tilde{X}_n)\vee\mathscr{F}_t(\tilde{B}_n)$ is independent of $\mathscr{F}_t^{+}(d\tilde{B}_n)$ and hence $\mathscr{F}_t(\tilde{X})\vee\mathscr{F}_t(\tilde{B})$ is also independent of $\mathscr{F}_t^{+}(d\tilde{B})$ for each $t\geq 0$; in other words, $\tilde{X}(t)$ is non-anticipating. Since the equation (2.11) also holds with $X_n(\cdot)$ and $B_n(\cdot)$ replaced by $\tilde{X}_n(\cdot)$ and $\tilde{B}_n(\cdot)$, respectively, by letting $n\uparrow\infty$ we have

$$\tilde{X}(t)=-a(0)t\int_t^{\infty}\frac{d\tilde{B}(s)}{s}. \tag{2.14}$$

It then follows that $\tilde{X}(t)$ is measurable with respect to $\mathscr{F}_t^{+}(d\tilde{B})$, which clearly contradicts the assertion that $\tilde{X}(t)$ is non-anticipating. □

3. Proof of Theorem 2

Let $X(t)$ be a backward non-anticipating solution of (1.5) and put $Y(t)=X(t)/t$, $\tilde{a}(t,x)=a(tx)/t$, $t>0$.

Lemma 2. (i) *The limit*

$$U=\lim_{t\to\infty}Y(t)=\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{X(s)}{s}\,ds \tag{3.1}$$

exists almost surely and is independent of $\{B(t), t\geq 0\}$.

(ii) *The process* $Y(t)$ *is a backward non-anticipating solution of*

$$Y(t)=-\int_t^{\infty}\tilde{a}(s, Y(s))d^{+}B(s)+U, \qquad t>0. \tag{3.2}$$

Moreover, for any backward non-anticipating solution $Y(t)$ *of* (3.2) *with a given* U *that is independent of* $\{B(t), t\geq 0\}$ *the process* $X(t)=tY(t)$ *is a backward non-anticipating solution of* (1.5).

Proof. If we put

$$U(t)=\frac{1}{t}\int_0^t\frac{X(s)}{s}\,ds, \qquad N(t)=\int_0^t a(X(s))d^{+}B(s), \tag{3.3}$$

then in the same way as we derived (2.7) we can prove the existence of the limit U and the formula

$$\frac{X(t)}{t} = -\int_t^\infty \frac{N(s)}{s^2}\,ds + \frac{N(t)}{t} + U\,. \tag{3.4}$$

It is also easy to see that (3.4) with $N(t)$ given by (3.3) is equivalent to (1.5) under the assumption that $X(t)$ is backward non-anticipating. Fixing $T>0$, we put

$$M(t) = \int_{T-t}^{T} a(X(s))d^+B(s)\,, \qquad 0 \le t \le T\,.$$

Then $M(t)+N(T-t)=M(T)=N(T)$ holds and $\{M(t), 0\le t\le T\}$ is a $\mathscr{G}_t$-margingale where $\mathscr{G}_t = \mathscr{F}^+_{T-t}(X, dB)$, $0\le t\le T$. Moreover, if we put $\tilde{X}(t)=X(T-t)$ and $\tilde{B}(t)=B(T)-B(T-t)$ for $0\le t\le T$, then $\{\tilde{B}(t), 0\le t\le T\}$ is a $\mathscr{G}_t$-Brownian motion and $M(t)=\int_0^t a(\tilde{X}(s))d\tilde{B}(s)$ (Itô integral). Putting $f(t,x)=x/(T-t)$ and applying Itô's formula to $df(t, M(t))$, we have

$$\begin{aligned}
\frac{M(t)}{T-t} &= \int_0^t \frac{M(s)}{(T-s)^2}\,ds + \int_0^t \frac{dM(s)}{T-s} = \int_0^t \frac{M(s)}{(T-s)^2}\,ds + \int_0^t \frac{a(\tilde{X}(s))}{T-s}\,d\tilde{B}(s) \\
&= \int_0^t \frac{M(T)-N(T-s)}{(T-s)^2}\,ds + \int_{T-t}^{T} \frac{a(X(s))}{s}\,d^+B(s) \\
&= \frac{M(T)}{T-t} - \frac{M(T)}{T} - \int_{T-t}^{T} \frac{N(s)}{s^2}\,ds + \int_{T-t}^{T} \frac{a(X(s))}{s}\,d^+B(s)\,.
\end{aligned}$$

Hence

$$\frac{M(T)}{T} = \frac{N(T-t)}{T-t} - \int_{T-t}^{T} \frac{N(s)}{s^2}\,ds + \int_{T-t}^{T} \frac{a(X(s))}{s}\,d^+B(s)\,. \tag{3.5}$$

Since $\lim_{T\uparrow\infty} T^{-1}M(T)=0$ in probability which is a consequence of the inequality $E|M(T)|\le K\sqrt{T}$, after replacing $T-t$ by t and then by letting $T\uparrow\infty$ in (3.5), we have

$$\frac{N(t)}{t} - \int_t^\infty \frac{N(s)}{s^2}\,ds + \int_t^\infty \frac{a(X(s))}{s}\,d^+B(s) = 0\,. \tag{3.6}$$

Comparing this with (3.4), we have

$$\frac{X(t)}{t} = -\int_t^\infty \frac{a(X(s))}{s}\,d^+B(s) + U\,,$$

which proves (3.2). Note that U is indpendent of $\{B(t)\}$ since U is measurable with respect to $\sigma\{X(s): s\ge t\}$, which is independent of $\mathscr{F}_t(B)$, for any fixed t. To prove the latter half of (ii) of Lemma 2 we note that (3.6) with $N(t)$ given by (3.3) holds only under the assumption that $X(t)$ is backward non-anticipating. Therefore, if $Y(t)$ is a backward non-anticipating solution of (3.2), then $X(t)=tY(t)$ satisfies (3.4) and hence (1.5). $\square$

Lemma 3. *For each x let* $\{Y^x_T(t), t>0\}$ *be the unique backward non-anticipating solution of*

$$(3.7)\qquad \begin{cases} Y_T^x(t) = -\displaystyle\int_t^T \tilde{a}(s,\, Y_T^x(s))d^+B(s) + x\,, & 0<t\le T\,, \\ Y_T^x(t) = x\,, \quad t\ge T\,. \end{cases}$$

Then for any fixed t_0 *and* t_1 *with* $0<t_0<t_1$ *and for any* $\varepsilon>0$

$$(3.8)\qquad P\left\{\max_{t_0\le t\le t_1} |\, Y_T^x(t) - Y_{T'}^x(t)\,| > \varepsilon\right\} \to 0$$

uniformly with respect to x *as* $T,\ T'\to\infty$.

Proof. Assume $T\le T'$. A comparison theorem (e.g. see Theorem 1.1, p. 437 of [2]) implies that $Y_T^{x-\delta}(t)\le Y_{T'}^x(t)\le Y_T^{x+\delta}(t)$ holds for $t\le T$ whenever $x-\delta\le Y_{T'}^x(T)\le x+\delta$, δ being an arbitrary but fixed constant. Therefore for any $\varepsilon>0$ and for $0<t_0<t_1\le T$ we have

$$\begin{aligned} &P\left\{\max_{t_0\le t\le t_1} |\, Y_T^x(t) - Y_{T'}^x(t)\,| > \varepsilon\right\} \\ &\le P\left\{\left(\max_{t_0\le t\le t_1} |\, Y_T^x(t) - Y_{T'}^x(t)\,| > \varepsilon\right) \cap (x-\delta\le Y_{T'}^x(T)\le x+\delta)\right\} \\ &\quad + P\{|\, Y_{T'}^x(T) - x\,| > \delta\} \\ &\le P\left\{\max_{t_0\le t\le t_1} (Y_T^{x+\delta}(t) - Y_T^{x-\delta}(t)) > \varepsilon\right\} + P\{|\, Y_{T'}^x(T) - x\,| > \delta\} \\ &\le \frac{1}{\varepsilon} E\{Y_T^{x+\delta}(t_0) - Y_T^{x-\delta}(t_0)\} + \frac{1}{\delta^2} E\left\{\int_T^{T'} a(s,\, Y_{T'}^x(s))^2 ds\right\} \\ &\le 2\delta\varepsilon^{-1} + K^2T^{-1}\delta^{-2}\,, \end{aligned}$$

from which the assertion of the lemma follows. □

Lemma 4. *There exists a real valued function* Ψ *defined on* $(0,\infty)\times\boldsymbol{R}\times W$ *and having the following properties.*

(3.9a) *For each* $t>0$ *the restriction of* Ψ *on* $[t,\infty)\times\boldsymbol{R}\times W$ *is measurable with respect to* $\mathscr{B}([t,\infty))\otimes\mathscr{B}(\boldsymbol{R})\otimes\mathscr{B}_t^+(W)$.

(3.9b) *For each* $x\in\boldsymbol{R}$ *the set* $\{w\in W\colon \Psi(t,x,w)$ *is continuous in* $t\}$ *has the Wiener measure* 1.

(3.10) *For any* U *independent of* $\{B(t),\, t\ge 0\}$, $\Psi(t,U,\boldsymbol{B})$ *is the unique backward non-anticipating solution of* (3.2).

Proof. From Lemma 3 there exists an increasing sequence $\{T_n\}$ with $T_n\to\infty$ such that

$$P\left\{\max_{1/n\le t\le n} |\, Y_T^x(t) - Y_{T'}^x(t)\,| > n^{-2}\right\} < n^{-2}\,, \qquad 0<t<T_n\,,\quad x\in\boldsymbol{R}$$

holds for any $T,\ T'\ge T_n$ and for any x ($n=1,2,\cdots$). If we take $T=T_n$ and $T'=T_{n+1}$, we have

$$P\left\{\max_{1/n\le t\le n}\left|Y^x_{T_n}(t)-Y^x_{T_{n+1}}(t)\right|>\frac{1}{n^2}\right\}<\frac{1}{n^2}, \qquad x\in\boldsymbol{R}.$$

From the Borel-Cantelli lemma,

$$P\left\{\max_{1/n\le t\le n}\left|Y^x_{T_n}(t)-Y^x_{T_{n+1}}(t)\right|\le\frac{1}{n^2}\text{ for all sufficiently large }n\right\}=1,$$

which implies that $Y^x_{T_n}(t)$ is uniformly convergent on each t-interval $[\varepsilon, 1/\varepsilon]$ almost surely for each fixed x, ε being an arbitrary constant in $(0, 1)$. We now put

$$Y^x(t,\omega)=\overline{\lim_{n\to\infty}}\, Y^x_{T_n}(t,\omega). \tag{3.11}$$

Then, for each x, $Y^x(t,\omega)$ is continuous in t (a.s.), $\mathscr{F}_t^+(dB)$-measurable for each $t>0$ and satisfies the SDE

$$Y^x(t)=-\int_t^\infty \tilde{a}(s, Y^x(s))d^+B(s)+x. \tag{3.12}$$

We construct a function Ψ with the properties in Lemma 4 by considering a suitable modification of $Y^x_T(t)$. Making use of a routine construction of $Y^x_T(t)$ by iteration we can prove that there exists a function Ψ_T defined on $(0,\infty)\times\boldsymbol{R}\times W$ and having the following properties (3.13a), (3.13b), (3.13c) and (3.14).

(3.13a) $\Psi_T(t,x,w)=x$ for $t\ge T$.

(3.13b) For each $t>0$ the restriction of Ψ_T on $[t,\infty)\times\boldsymbol{R}\times W$ is measurable with respect to $\mathscr{B}([t,\infty))\otimes\mathscr{B}(\boldsymbol{R})\otimes\mathscr{B}_t^+(W)$.

(3.13c) For each $x\in\boldsymbol{R}$, $\Psi_T(t,x,\boldsymbol{B})$ is continuous in t (a.s.).

(3.14) For each $x\in\boldsymbol{R}$, $\Psi_T(t,x,\boldsymbol{B})$, $0<t\le T$, satisfies the SDE (3.7) and hence $\Psi_T(t,x,\boldsymbol{B})=Y^x_T(t)$, $t>0$, almost surely.

If we put $\Psi_\infty(t,x,w)=\overline{\lim}_{n\to\infty}\Psi_{T_n}(t,x,w)$, then Ψ_∞ inherits the properties (3.13b) and (3.13c). The property (3.14) together with (3.11) and (3.12) implies that $\Psi_\infty(t,x,\boldsymbol{B})$ satisfies the SDE (3.12) for each x. We now modify Ψ_∞:

$$\Psi(t,x,w)=\overline{\lim_{n\to\infty}}\,\Psi_\infty\left(\frac{[nt]+1}{n}, x, w\right).$$

Then Ψ also inherits the properties (3.13b) and (3.13c) and $\Psi(t,x,\boldsymbol{B})$ satisfies the SDE (3.12) for each x. Consider the set

$$\Gamma=\left\{(x,w):\begin{array}{l}\Psi(t,x,w)\text{ is uniformly continuous in}\\ t\in[\varepsilon,1/\varepsilon]\cap\boldsymbol{Q}\text{ for any }\varepsilon\in(0,1)\end{array}\right\},$$

where $\boldsymbol{Q}$ is the set of rational numbers. Then $\Gamma\in\mathscr{B}(\boldsymbol{R})\otimes\mathscr{B}(W)$ and $\Psi(t,x,w)$ is continuous in t if and only if $(x,w)\in\Gamma$. Since $\Psi(t,x,\boldsymbol{B})$ is continuous in t (a.s.) for each fixed x, $P\{(x,\boldsymbol{B})\in\Gamma\}=1$ for each x, or equivalently, (3.9b) holds. Therefore, for any real random variable U that is independent of $\boldsymbol{B}$ we have $P\{(U,\boldsymbol{B})\in\Gamma\}=1$ and this implies that $\Psi(t,U,\boldsymbol{B})$ is continuous in t almost surely. We now put

$$F(t, x, \boldsymbol{B})=\int_t^{\infty} \tilde{a}(s, \Psi(s, x, \boldsymbol{B}))d^+B(s) .$$

Then

$$\Psi(t, x, \boldsymbol{B})=-F(t, x, \boldsymbol{B})+x , \qquad \text{a.s.} , \tag{3.15}$$

because $\Psi(t, x, \boldsymbol{B})$ satisfies (3.12) for each x. Approximating the stochastic integral $F(t, x, \boldsymbol{B})$ by a suitable Riemann sum, we can prove that

$$F(t, U, \boldsymbol{B})=\int_t^{\infty} \tilde{a}(s, \Psi(s, U, \boldsymbol{B}))d^+B(s) \qquad \text{a.s.} ,$$

which combined with (3.15) implies that $\Phi(t, U, \boldsymbol{B})$ is a backward non-anticipating solution of (3.2).

To complete the proof of the lemma, we have to show that any solution $Y(t)$ of (3.2) agrees with $\Psi(t, U, \boldsymbol{B})$. We put $Y^U(t)=\Psi(t, U, \boldsymbol{B})$ and $U_T=-\int_T^{\infty} \tilde{a}(s, Y(s)) d^+B(s)+U$. Then $Y(t)$ is the unique solution of

$$Y(t)=-\int_t^T \tilde{a}(s, Y(s))d^+B(s)+U_T , \qquad 0<t\le T ,$$

and hence by a comparison theorem we have for $\varepsilon>0$

$$\begin{aligned}
&P\{Y^{U-\varepsilon}(t)\le Y(t)\le Y^{U+\varepsilon}(t) \text{ for all } t\in(0, T]\}\\
&=P\{Y^{U-\varepsilon}(T)\le U_T\le Y^{U+\varepsilon}(T)\}\\
&=1-P\{U_T< Y^{U-\varepsilon}(T)\}-P\{U_T> Y^{U+\varepsilon}(T)\}\\
&\ge 1-P\left\{\left|\int_T^{\infty}(\tilde{a}(s, Y(s))-\tilde{a}(s, Y^{U-\varepsilon}(s)))d^+B(s)\right|>\varepsilon\right\}\\
&\quad -P\left\{\left|\int_T^{\infty}(\tilde{a}(s, Y(s))-\tilde{a}(s, Y^{U+\varepsilon}(s)))d^+B(s)\right|>\varepsilon\right\}\\
&\ge 1-\varepsilon^{-2}E\left\{\left|\int_T^{\infty}(\tilde{a}(s, Y(s))-\tilde{a}(s, Y^{U-\varepsilon}(s)))d^+B(s)\right|^2\right\}\\
&\quad -\varepsilon^{-2}E\left\{\left|\int_T^{\infty}(\tilde{a}(s, Y(s))-\tilde{a}(s, Y^{U+\varepsilon}(s)))d^+B(s)\right|^2\right\}\\
&\ge 1-8K^2\varepsilon^{-2}T^{-1}\to 1 \quad \text{as} \quad T\to\infty .
\end{aligned}$$

Therefore

$$Y^{U-\varepsilon}(t)\le Y(t)\le Y^{U+\varepsilon}(t) \qquad \text{for all} \quad t>0 , \quad \text{a.s.}$$

On the other hand, by a comparison theorem we have $\Psi_T(t, U-\varepsilon, \boldsymbol{B})\le \Psi_T(t, U, \boldsymbol{B})\le \Psi_T(t, U+\varepsilon, \boldsymbol{B})$, $0<t\le T$, a.s. and hence $\Psi(t, U-\varepsilon, \boldsymbol{B})\le \Psi(t, U, \boldsymbol{B})\le \Psi(t, U+\varepsilon, \boldsymbol{B})$, $t>0$, a.s. Consequently $E[|Y^{U+\varepsilon}(t)-Y^{U-\varepsilon}(t)|]=E[Y^{U+\varepsilon}(t)-Y^{U-\varepsilon}(t)]=2\varepsilon$, which implies $Y^U(t)=Y(t)$, $t>0$, almost surely. This completes the proof of Lemma 4. □

Proof of Theorem 2. For any solution $\{X(t)\}$ of (1.5), $\{X(t)/t\}$ satisfies the SDE (3.2) by Lemma 3 and hence $X(t)/t = \Psi(t, U, \boldsymbol{B})$ a.s., where U is given by (3.1). If we put $\Phi(t, x, w) = t\Psi(t, x, w)$, then Φ has all the properties stated in (ii) of Theorem 2. The proof of Theorem 2 is finished. $\square$

Acknowledgment

The author wishes to express his gratitude to Professor H. Tanaka for advice and encouragement.

References

[1] Th. Domenig and M. Nagasawa: *Skorohod problem with singular drift and its application to the origin of universes*, Proc. Japan Acad. **70 Ser. A.** (1994).

[2] N. Ikeda and S. Watanabe: *Stachastic Differential Equations and Diffusion Processes*, 2nd Edition, North-Holland and Kodansha.

[3] Th. Jeulin and M. Yor: *Filtration des pons browniens et équations differentielles stchastiques linéaires*, Sem. de Probab. XXIV, Lecture Notes in Math. **1426** (1988–89), Springer-Verlag, 227–265.

[4] A. V. Skorohod: *Studies in the Theory of Random Process*, Addison-Wesley, Reading, Massachusetts, 1965.

[5] M. Yor: *Some Aspects of Brownian Motion, Part I: Some Special Functionals*, Birkhäuser Verlag, Basel-Boston-Berlin, 1992.