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CERTAIN STOCHASTIC DIFFERENTIAL EQUATIONS WITH A SINGULAR DRIFT

by

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1. Introduction

Let \( B(t), t \geq 0, \) be a one-dimensional Brownian motion with \( B(0)=0 \) defined on a certain probability space \( (\Omega, \mathcal{F}, P) \). Given a real valued Borel function \( a(x), x \in \mathbb{R} \), satisfying

\[
|a(x)| \leq K, \quad x \in \mathbb{R},
\]

we consider the stochastic differential equation (SDE)

\[
dX(t) = a(X(t))dB(t) + \frac{X(t)}{t} dt, \quad X(0)=0,
\]

or equivalently

\[
X(t) = \int_0^t a(X(s))dB(s) + \int_0^t \frac{X(s)}{s} ds, \quad t>0,
\]

where a solution \( X(t) \) is assumed to be non-anticipating, so the first term in the right hand side of (1.3) is the usual Itô integral. We also consider the SDE

\[
dX(t) = a(X(t))d^+ B(t) + \frac{X(t)}{t} dt, \quad X(0)=0,
\]

or equivalently

\[
X(t) = \int_0^t a(X(s))d^+ B(s) + \int_0^t \frac{X(s)}{s} ds, \quad t>0,
\]

where a solution is assumed to be backward non-anticipating in the sense as defined later and the first term in the right hand side of (1.5) is a backward stochastic integral defined as the limit of \( \sum a(X(t_k))(B(t_k) - B(t_{k-1})) \) as \( |A| = \max(t_k - t_{k-1}) \to 0 \), \( A \) being a partition of \([0, t]: 0 = t_0 < t_1 < t_2 < \cdots < t_n = t\). The second term in the right hand side of (1.3) as well as of (1.5) equals \( \lim_{t \to 0} \int_0^t s^{-1} X(s)ds \) that is assumed to exist a.s.

When \( a(x) \equiv 1 \), (1.3) and (1.5) have the same form that makes sense even if \( X(t) \)
is neither non-anticipating nor backward non-anticipating. In this case Jeulin and Yor ([3],[5]) proved that for any general solution $X(t)$ of (1.3) ($= (1.5)$) the limit of $X(t)/t$ as $t \to \infty$, denoted by $U$, exists a.s. and that $X(t)$ is represented as $X(t) = \beta(t) + tU$, where $\beta(t) = -\int_t^\infty s^{-1} dB(s)$, which is again a Brownian motion. In particular if $X(t)$ is backward non-anticipating, then $U$ is independent of $B(t)$, $t \geq 0$. However non-anticipating solutions do not exist (e.g. [1]). Domenig-Nagasawa [1] discussed the uniqueness and non-uniqueness of solutions of (1.2) with Skorohod’s additional term on the right hand side to make $|X(t)| \leq R(t)$ where $R(t)$ is a given strictly increasing continuous function with $R(0) = 0$.

In this paper some generalization of the result by Jeulin and Yor stated in the above will be given in the case of a variable coefficient $a(x)$. Before stating our results we introduce the following $\sigma$-fields on $\Omega$.

\[
\mathcal{F}(B) = \sigma\{ B(t), t \geq 0 \}, \quad \mathcal{F}(B) = \sigma\{ B(s), 0 \leq s \leq t \},
\]
\[
\mathcal{F}^+_1 (dB) = \sigma\{ B(s + t) - B(t), s \geq 0 \}.
\]

Here the notation $\sigma\{ (*) \}$ stands for the smallest $\sigma$-field on $\Omega$ that makes $(*)$ measurable. For a process $\{ X(t) \}$ and a random variable $U$ we define $\mathcal{F}(X)$, $\mathcal{F}(X)$ and $\mathcal{F}(U)$ in a similar way and $\mathcal{F}^+_1 (X, dB) = \sigma\{ X(s + t), B(s + t) - B(t), s \geq 0 \}$ for each $t \geq 0$. We say that $X(t)$ is non-anticipating if $\mathcal{F}^+_1 (X)$ is independent of $\mathcal{F}_t^+(dB)$, and that $X(t)$ is backward non-anticipating if $\mathcal{F}^+_1 (X, dB)$ is independent of $\mathcal{F}_t^+(B)$ for each $t \geq 0$. Let $W$ be the space $C[0, \infty)$ of continuous functions. For an element $w$ of $W$ we denote by $w(t)$ the value of $w(t)$ at time $t$. On $W$ we consider the $\sigma$-fields $\mathcal{B}(W) = \sigma\{ w(t), t \geq 0 \}$, $\mathcal{B}(W) = \sigma\{ w(s), 0 \leq s \leq t \}$ and $\mathcal{B}^+(W) = \sigma\{ w(s + t) - w(t), s \geq 0 \}$. The Brownian motion $\{ B(t), t \geq 0 \}$ can be regarded as a random variable taking values in $W$. When we take such a view-point we write $B = \{ B(t), t \geq 0 \}$. Thus $B$ is a random variable with values in $W$ whose probability law is the Wiener measure.

Our results are the following.

**Theorem 1.** Assume that $a(x)$ is Borel measurable in $R$, continuous at $x = 0$ and satisfies (1.1). Then there are no non-anticipating solutions of (1.3) provided that $a(0) \neq 0$.

**Theorem 2.** Assume that $a(x)$ satisfies (1.1) and is Lipschitz continuous.

(i) If $X(t)$ is a backward non-anticipating solution of (1.5), then the limit

\[
U = \lim_{t \to \infty} \frac{X(t)}{t}
\]

exists almost surely and is independent of $\{ B(t), t \geq 0 \}$. The $X(t)$ satisfies

\[
X(t) = -t \int_t^\infty \frac{a(X(s))}{s} d^+ B(s) + tU, \quad t > 0.
\]

Conversely, if $X(t)$ is a backward non-anticipating solution of (1.7) for any given random variable $U$ that is independent of $\{ B(t), t \geq 0 \}$, then $X(t)$ satisfies (1.5).

(ii) There exists a real valued function $\Phi$ defined on $(0, \infty) \times R \times W$ and having the following properties.

\[
\Phi(X, t) = \Phi(X, t) \text{ is measurable with respect to}
\]

(1.8a)
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to $\mathcal{B}([t, \infty)) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}^+(W)$.

(1.8b) For each $x \in \mathbb{R}$ the set $\{w \in W : \Phi(t, x, w) \text{ is continuous in } t\}$ has the Wiener measure 1.

(1.9) Any backward non-anticipating solution $X(t)$ of (1.5) can be represented as

$$X(t) = \Phi(t, U, \mathcal{B}), \quad \text{a.s.,}$$

where $U$ is given by (1.6).

Note that (1.8a) implies the measurability of $\Phi$ with respect to $\mathcal{B}((0, \infty)) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(W)$.

2. Proof of Theorem 1

We begin by proving the following lemma.

**Lemma 1.** If $X(t)$ is a non-anticipating solution of (1.3), then the limit

$$U = \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{X(s)}{s} ds = \lim_{t \to \infty} \frac{X(t)}{t}$$

exists almost surely and $X(t)$ satisfies

$$X(t) = -t \int_0^\infty \frac{a(X(s))}{s} dB(s) + t U, \quad t > 0.$$  

**Proof.** If we put

$$U(t) = \frac{1}{t} \int_0^t \frac{X(s)}{s} ds, \quad M(t) = \int_0^t a(X(s)) dB(s),$$

then $X(t) = t[U(t)]' = tU(t) + t^2 U'(t)$ and (1.3) implies $X(t) = M(t) + t U(t)$. Therefore $U'(t) = t^{-2} M(t)$ and hence

$$U(s) - U(t) = \int_t^s M(r)r^{-2} dr, \quad 0 < t < s.$$  

Since

$$E\left\{ \int_t^\infty |M(r)|r^{-2} dr \right\} = \int_t^\infty E\{|M(r)|r^{-2} dr \leq \int_t^\infty \sqrt{E\{|M(r)|^2} r^{-2} dr$$

$$= \int_t^\infty \left[ E\left\{ \int_0^r a(X(u))^2 du \right\} \right]^{1/2} r^{-2} dr \leq K \int_t^\infty r^{1/2} r^{-2} dr < \infty,$$

letting $s \uparrow \infty$ in (2.4) we see that $U = \lim_{s \uparrow \infty} U(s)$ exists (a.s.) and

$$U(t) = -\int_t^\infty M(s)s^{-2} ds + U,$$

or equivalently

$$U(t) = -\int_0^t M(s)s^{-2} ds.$$  

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and hence

\[ \frac{X(t)}{t} = -\int_{t}^{\infty} \frac{M(s)}{s^2} ds + \frac{M(t)}{t} + U. \]  

By putting \( f(t, x) = x/t \) and applying Itô's formula to \( df(t, M(t)) \), we have

\[ \frac{M(s)}{s} - \frac{M(t)}{t} = -\int_{t}^{s} \frac{M(u)}{u^2} du + \int_{t}^{s} \frac{a(X(u))}{u} dB(u). \]

Since \( \lim_{s \to \infty} s^{-1} M(s) = 0 \) in probability which follows from \( E|M(s)| \leq K \sqrt{s} \), we have

\[ \frac{M(t)}{t} = \int_{t}^{\infty} \frac{M(s)}{s^2} ds - \int_{t}^{\infty} \frac{a(X(s))}{s} dB(s). \]

Comparing this with (2.7), we have (2.2). The second equality in (2.1) follows from (2.2).

We now proceed to the proof of Theorem 1. Suppose there exists a non-anticipating solution \( X(t) \) of (1.3). If we put \( X_n(t) = \sqrt{n} X(t/n) \) and \( a_n(x) = a(x/n) \) for \( n \geq 1 \), then \( X(t) = n^{-1/2} X_n(nt) \) and \( a_n(x) \to a(0) \) as \( n \to \infty \). From (1.3) we have

\[ X_n(t) = \sqrt{n} \int_{0}^{t/n} a(X(s))dB(s) + \sqrt{n} \int_{0}^{t/n} \frac{X(s)}{s} ds. \]

Noting that

\[ \sqrt{n} \int_{0}^{t/n} a(X(s))dB(s) = \int_{0}^{t} a_n(X_n(s))dB_n(s), \]

where \( B_n(t) = \sqrt{n} B(t/n) \), which is again a Brownian motion, we see that \( X_n(t) \) is a non-anticipating solution of

\[ X_n(t) = \int_{0}^{t} a_n(X_n(s))dB_n(s) + \int_{0}^{t} \frac{X_n(s)}{s} ds. \]

Therefore by Lemma 1 we have

\[ X_n(t) = -t \int_{t}^{\infty} \frac{a_n(X_n(s))}{s} dB_n(s) + tU_n, \]

where

\[ U_n = \lim_{t \to \infty} \frac{X_n(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \frac{X_n(s)}{s} ds = \frac{1}{\sqrt{n}} U. \]

Let \( v_n \) be the probability law of \( \{U_n, \{B_n(t), t \geq 0\}, \{X_n(t), t > 0\}\} \) and put
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\[ Y_n(t) = t^{-1}X_n(t) \]. Making use of the representation (2.11), the bound \( |a_n| \leq K \) and the Burkholder-Davis-Gundy inequalities, we have \( E[|Y_n(t) - Y_n(s)|^4] \leq 3(Ks^{-1})^4(t-s)^2 \), \( 0 < s < t \). Therefore the sequence of probability laws of \( \{Y_n(t), t > 0\}, n \geq 1 \) is tight, so is \( \{v_n\} \). According to Skorohod’s realization theorem of almost sure convergence (e.g. see [2], [4]), there exist \( \tilde{U}_n, \{\tilde{B}_n(t), t \geq 0\} \) and \( \{\tilde{X}_n(t), t > 0\} \) satisfying the following conditions.

(i) \( \{\tilde{U}_n, \{\tilde{B}_n(t), t \geq 0\}, \{\tilde{X}_n(t), t > 0\}\} \) is identical in law to \( \{U_n, \{B_n(t), t \geq 0\}, \{X_n(t), t > 0\}\} \) for each \( n \geq 1 \).

(ii) There exist a Brownian motion \( \{\tilde{B}(t), t \geq 0\} \) and a process \( \{\tilde{X}(t), t > 0\} \) such that \( \tilde{B}_n(t) \to \tilde{B}(t) \) uniformly on each compact \( t \)-interval in \( (0, \infty) \) (a.s.), \( \tilde{X}_n(t) \to \tilde{X}(t) \) uniformly on each compact \( t \)-interval in \( (0, \infty) \) (a.s.), and \( \tilde{U}_n \to 0 \) a.s.

The condition (i) implies that \( \mathcal{F}(\tilde{X}_n) \vee \mathcal{F}(\tilde{B}_n) \) is independent of \( \mathcal{F}_t^+(dB) \) and hence \( \mathcal{F}_t(\tilde{X}) \vee \mathcal{F}(\tilde{B}) \) is also independent of \( \mathcal{F}_t^+(dB) \) for each \( t \geq 0 \); in other words, \( \tilde{X}(t) \) is non-anticipating. Since the equation (2.11) also holds with \( X_n(\cdot) \) and \( B_n(\cdot) \) replaced by \( \tilde{X}_n(\cdot) \) and \( \tilde{B}_n(\cdot) \), respectively, by letting \( n \to \infty \) we have

\[ \tilde{X}(t) = -a(0) \int_0^t \frac{dB(s)}{s}. \]

It then follows that \( \tilde{X}(t) \) is measurable with respect to \( \mathcal{F}_t^+(dB) \), which clearly contradicts the assertion that \( \tilde{X}(t) \) is non-anticipating.

\[ \square \]

3. Proof of Theorem 2

Let \( X(t) \) be a backward non-anticipating solution of (1.5) and put \( Y(t) = X(t)/t, \tilde{a}(t, x) = a(tx)/t, t > 0 \).

**Lemma 2.** (i) The limit

\[ U = \lim_{t \to \infty} Y(t) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{X(s)}{s} ds \]

exists almost surely and is independent of \( \{B(t), t \geq 0\} \).

(ii) The process \( Y(t) \) is a backward non-anticipating solution of

\[ Y(t) = -\int_t^\infty \tilde{a}(s, Y(s)) dB(s) + U, \quad t > 0. \]

Moreover, for any backward non-anticipating solution \( Y(t) \) of (3.2) with a given \( U \) that is independent of \( \{B(t), t \geq 0\} \) the process \( X(t) = tY(t) \) is a backward non-anticipating solution of (1.5).

**Proof.** If we put

\[ U(t) = \frac{1}{t} \int_0^t \frac{X(s)}{s} ds, \quad N(t) = \int_0^t a(X(s)) dB(s), \]

then in the same way as we derived (2.7) we can prove the existence of the limit \( U \) and the formula

\[ 5 \]
It is also easy to see that (3.4) with \( N(t) \) given by (3.3) is equivalent to (1.5) under the assumption that \( X(t) \) is backward non-anticipating. Fixing \( T>0 \), we put

\[
M(t) = \int_{T-t}^{T} a(X(s)) d^+ B(s), \quad 0 \leq t \leq T.
\]

Then \( M(t) + N(T-t) = M(T) = N(T) \) holds and \( \{M(t), 0 \leq t \leq T\} \) is a \( \mathcal{G}_t \)-martingale where \( \mathcal{G}_t = \mathcal{F}_{T-t}(X, dB) \), \( 0 \leq t \leq T \). Moreover, if we put \( \tilde{X}(t) = X(T-t) \) and \( \tilde{B}(t) = B(T) - B(T-t) \) for \( 0 \leq t \leq T \), then \( \{ \tilde{B}(t), 0 \leq t \leq T\} \) is a \( \mathcal{G}_t \)-Brownian motion and \( M(t) = \int_{0}^{T-t} a(\tilde{X}(s)) d\tilde{B}(s) \) (Itô integral). Putting \( f(t, x) = x/(T-t) \) and applying Itô’s formula to \( df(t, M(t)) \), we have

\[
\frac{M(t)}{T-t} = \int_{0}^{t} \frac{M(s)}{(T-s)^2} ds + \int_{0}^{t} \frac{a(M(s))}{T-s} d\tilde{B}(s)
\]

\[
= \int_{0}^{t} \frac{M(T) - N(T-s)}{(T-s)^2} ds + \int_{T-t}^{T} \frac{a(X(s))}{s} d^+ B(s)
\]

\[
= \frac{M(T)}{T} - \frac{M(T)}{T-t} - \int_{T-t}^{T} \frac{N(s)}{s^2} ds + \int_{T-t}^{T} \frac{a(X(s))}{s} d^+ B(s).
\]

Hence

\[
\frac{M(T)}{T} = \frac{N(T-t)}{T-t} - \int_{T-t}^{T} \frac{N(s)}{s^2} ds + \int_{T-t}^{T} \frac{a(X(s))}{s} d^+ B(s).
\]

Since \( \lim_{T \to \infty} T^{-1} M(T) = 0 \) in probability which is a consequence of the inequality \( E|M(T)| \leq K\sqrt{T} \), after replacing \( T-t \) by \( t \) and then by letting \( T \uparrow \infty \) in (3.5), we have

\[
\frac{N(t)}{t} - \int_{t}^{\infty} \frac{N(s)}{s^2} ds + \int_{t}^{\infty} \frac{a(X(s))}{s} d^+ B(s) = 0.
\]

Comparing this with (3.4), we have

\[
\frac{X(t)}{t} = -\int_{t}^{\infty} \frac{a(X(s))}{s} d^+ B(s) + U,
\]

which proves (3.2). Note that \( U \) is independent of \( \{B(t)\} \) since \( U \) is measurable with respect to \( \sigma\{X(s) : s \geq t\} \), which is independent of \( \mathcal{F}_t(B) \), for any fixed \( t \). To prove the latter half of (ii) of Lemma 2 we note that (3.6) with \( N(t) \) given by (3.3) holds only under the assumption that \( X(t) \) is backward non-anticipating. Therefore, if \( Y(t) \) is a backward non-anticipating solution of (3.2), then \( X(t) = tY(t) \) satisfies (3.4) and hence (1.5).

**Lemma 3.** For each \( x \) let \( \{Y^x(t), t>0\} \) be the unique backward non-anticipating solution of
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\[ Y_T^x(t) = - \int_t^T \bar{a}(s, Y_T^x(s))d^+ B(s) + x, \quad 0 < t \leq T, \]

(3.7)

\[ Y_T^x(t) = x, \quad t \geq T. \]

Then for any fixed \( t_0 \) and \( t_1 \) with \( 0 < t_0 < t_1 \) and for any \( \varepsilon > 0 \)

(3.8) \[ P \left\{ \max_{t_0 \leq t \leq t_1} |Y_T^x(t) - Y_T^x(t)| > \varepsilon \right\} \to 0 \]

uniformly with respect to \( x \) as \( T, T' \to \infty. \)

**Proof.** Assume \( T \leq T' \). A comparison theorem (e.g. see Theorem 1.1, p. 437 of [2]) implies that \( Y_T^x(t) \leq Y_T^x(t) \leq Y_T^{x+\delta}(t) \) holds for \( t \leq T \) whenever \( x - \delta \leq Y_T^{x+\delta}(T) \leq x + \delta, \) \( \delta \) being an arbitrary but fixed constant. Therefore for any \( \varepsilon > 0 \) and for \( 0 < t_0 < t_1 \leq T \) we have

\[
P \left\{ \max_{t_0 \leq t \leq t_1} |Y_T^x(t) - Y_T^x(t)| > \varepsilon \right\} 
\leq P \left\{ \max_{t_0 \leq t \leq t_1} (Y_T^{x+\delta}(t) - Y_T^{x-\delta}(t)) > \varepsilon \right\} 
+ P \{ |Y_T^x(T) - x| > \delta \}
\leq \frac{1}{\varepsilon} E \left\{ Y_T^{x+\delta}(t_0) - Y_T^{x-\delta}(t_0) \right\} + \frac{1}{\delta^2} E \left\{ \int_0^{T'} \bar{a}(s, Y_T^{x+\delta}(s))^2 ds \right\}
\leq 2\delta \varepsilon^{-1} + K^2 T^{-1} \delta^{-2},
\]

from which the assertion of the lemma follows. \( \square \)

**Lemma 4.** There exists a real valued function \( \Psi \) defined on \( (0, \infty) \times R \times W \) and having the following properties.

(3.9a) For each \( t > 0 \) the restriction of \( \Psi \) on \( [t, \infty) \times R \times W \) is measurable with respect to \( \mathcal{B}([t, \infty)) \otimes \mathcal{B}(R) \otimes \mathcal{B}(W) \).

(3.9b) For each \( x \in R \) the set \( \{ w \in W : \Psi(t, x, w) \) \) is continuous in \( t \} \) has the Wiener measure 1.

(3.10) For any \( U \) independent of \( \{ B(t), t \geq 0 \} \), \( \Psi(t, U, B) \) is the unique backward non-anticipating solution of (3.2).

**Proof.** From Lemma 3 there exists an increasing sequence \( \{ T_n \} \) with \( T_n \to \infty \) such that

\[
P \left\{ \max_{1 \leq n \leq m} |Y_T^x(t) - Y_T^x(t)| > n^{-2} \right\} < n^{-2}, \quad 0 < t < T_n, \quad x \in R \]

holds for any \( T, T' \geq T_n \) and for any \( x \) \( (n = 1, 2, \cdots) \). If we take \( T = T_n \) and \( T' = T_{n+1} \), we have
From the Borel-Cantelli lemma,

\[
P\left\{ \max_{1/n \leq t \leq n} \left| Y_{T_n}^x(t) - Y_{T_{n+1}}^x(t) \right| \geq \frac{1}{n^2} \right\} < \frac{1}{n^2}, \quad x \in \mathbb{R}.
\]

which implies that \( Y_{T_n}^x(t) \) is uniformly convergent on each \( t \)-interval \([\varepsilon, 1/\varepsilon]\) almost surely for each fixed \( x \), \( \varepsilon \) being an arbitrary constant in \((0, 1)\). We now put

\[
Y^x(t, \omega) = \lim_{n \to \infty} Y_{T_n}^x(t, \omega).
\]

Then, for each \( x \), \( Y^x(t, \omega) \) is continuous in \( t \) (a.s.), \( \mathcal{F}_t^+ (dB) \)-measurable for each \( t > 0 \) and satisfies the SDE

\[
Y^x(t) = -\int_t^\infty \tilde{a}(s, Y^x(s)) d+ B(s) + x.
\]

We construct a function \( \Psi \) with the properties in Lemma 4 by considering a suitable modification of \( Y^x_T(t) \). Making use of a routine construction of \( Y^x_T(t) \) by iteration we can prove that there exists a function \( \Psi_T \) defined on \((0, \infty) \times \mathbb{R} \times \mathbb{W} \) and having the following properties (3.13a), (3.13b), (3.13c) and (3.14).

\[
(3.13a) \quad \Psi_T(t, x, w) = x \text{ for } t \geq T.
\]

\[
(3.13b) \quad \text{For each } t > 0 \text{ the restriction of } \Psi_T \text{ on } [t, \infty) \times \mathbb{R} \times \mathbb{W} \text{ is measurable with respect to } \mathcal{B}([t, \infty)) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}_1^+(\mathbb{W}).
\]

\[
(3.13c) \quad \text{For each } x \in \mathbb{R}, \Psi_T(t, x, B) \text{ is continuous in } t \text{ (a.s.).}
\]

\[
(3.14) \quad \text{For each } x \in \mathbb{R}, \Psi_T(t, x, B), 0 < t \leq T, \text{ satisfies the SDE (3.7) and hence } \Psi_T(t, x, B) = Y_T^x(t), t > 0, \text{ almost surely.}
\]

If we put \( \Psi^\infty(t, x, w) = \lim_{n \to \infty} \Psi_T(t, x, w) \), then \( \Psi^\infty \) inherits the properties (3.13b) and (3.13c). The property (3.14) together with (3.11) and (3.12) implies that \( \Psi^\infty(t, x, B) \) satisfies the SDE (3.12) for each \( x \). We now modify \( \Psi^\infty \):

\[
\Psi(t, x, w) = \lim_{n \to \infty} \Psi^\infty\left( \left\lfloor nt \right\rfloor + \frac{1}{n}, x, w \right).
\]

Then \( \Psi \) also inherits the properties (3.13b) and (3.13c) and \( \Psi(t, x, B) \) satisfies the SDE (3.12) for each \( x \). Consider the set

\[
\Gamma = \left\{ (x, w) : \Psi(t, x, w) \text{ is uniformly continuous in } t \right\},
\]

where \( \mathcal{Q} \) is the set of rational numbers. Then \( \Gamma \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{W}) \) and \( \Psi(t, x, w) \) is continuous in \( t \) if and only if \( (x, w) \in \Gamma \). Since \( \Psi(t, x, B) \) is continuous in \( t \) (a.s.) for each fixed \( x \), \( P\{(x, B) \in \Gamma\} = 1 \) for each \( x \), or equivalently, (3.9b) holds. Therefore, for any real random variable \( U \) that is independent of \( B \) we have \( P\{(U, B) \in \Gamma\} = 1 \) and this implies that \( \Psi(t, U, B) \) is continuous in \( t \) almost surely. We now put...
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\[ F(t, x, B) = \int_t^\infty \tilde{a}(s, \Psi(s, x, B))d^+ B(s) . \]

Then
\[ (3.15) \quad \Psi(t, x, B) = -F(t, x, B) + x , \quad \text{a.s.}, \]
because \( \Psi(t, x, B) \) satisfies (3.12) for each \( x \). Approximating the stochastic integral \( F(t, x, B) \) by a suitable Riemann sum, we can prove that
\[ F(t, U, B) = \int_t^\infty \tilde{a}(s, \Psi(s, U, B))d^+ B(s) \quad \text{a.s.}, \]
which combined with (3.15) implies that \( \Phi(t, U, B) \) is a backward non-anticipating solution of (3.2).

To complete the proof of the lemma, we have to show that any solution \( Y(t) \) of (3.2) agrees with \( \Psi(t, U, B) \). We put \( Y^U(t) = \Psi(t, U, B) \) and \( U_T = -\int_T^\infty \tilde{a}(s, Y(s))d^+ B(s) + U \). Then \( Y(t) \) is the unique solution of
\[ Y(t) = -\int_t^T \tilde{a}(s, Y(s))d^+ B(s) + U_T , \quad 0 < t \leq T , \]
and hence by a comparison theorem we have for \( \varepsilon > 0 \)
\[ P \{ Y^{U-\varepsilon}(t) \leq Y(t) \leq Y^{U+\varepsilon}(t) \text{ for all } t \in (0, T) \} \]
\[ = P \{ Y^{U-\varepsilon}(T) \leq U_T \leq Y^{U+\varepsilon}(T) \} \]
\[ = 1 - P \{ U_T < Y^{U-\varepsilon}(T) \} - P \{ U_T > Y^{U+\varepsilon}(T) \} \]
\[ \geq 1 - P \left\{ \int_T^\infty (\tilde{a}(s, Y(s)) - \tilde{a}(s, Y^{U-\varepsilon}(s)))d^+ B(s) \bigg| > \varepsilon \right\} \]
\[ - P \left\{ \int_T^\infty (\tilde{a}(s, Y(s)) - \tilde{a}(s, Y^{U+\varepsilon}(s)))d^+ B(s) \bigg| > \varepsilon \right\} \]
\[ \geq 1 - \varepsilon^{-2} E \left\{ \int_T^\infty (\tilde{a}(s, Y(s)) - \tilde{a}(s, Y^{U-\varepsilon}(s)))d^+ B(s) \bigg| > \varepsilon \right\} \]
\[ - \varepsilon^{-2} E \left\{ \int_T^\infty (\tilde{a}(s, Y(s)) - \tilde{a}(s, Y^{U+\varepsilon}(s)))d^+ B(s) \bigg| > \varepsilon \right\} \]
\[ \geq 1 - 8K^2 \varepsilon^{-2} T^{-1} \rightarrow 1 \text{ as } T \rightarrow \infty . \]

Therefore
\[ Y^{U-\varepsilon}(t) \leq Y(t) \leq Y^{U+\varepsilon}(t) \text{ for all } t > 0 , \quad \text{a.s.} \]

On the other hand, by a comparison theorem we have \( \Psi(t, U - \varepsilon, B) \leq \Psi(t, U, B) \leq \Psi(t, U + \varepsilon, B) \), \( 0 < t \leq T \), a.s. and hence \( \Psi(t, U - \varepsilon, B) \leq \Psi(t, U, B) \leq \Psi(t, U + \varepsilon, B) \), \( t > 0 \), a.s. Consequently \( E[ Y^{U+\varepsilon}(t) - Y^{U-\varepsilon}(t) ] = E[ Y^{U+\varepsilon}(t) - Y^{U-\varepsilon}(t) ] = 2\varepsilon \), which implies \( Y^U(t) = Y(t) \), \( t > 0 \), almost surely. This completes the proof of Lemma 4. \( \square \)
Proof of Theorem 2. For any solution \( \{X(t)\} \) of (1.5), \( \{X(t)/t\} \) satisfies the SDE (3.2) by Lemma 3 and hence \( X(t)/t = \Psi(t, U, B) \) a.s., where \( U \) is given by (3.1). If we put \( \Phi(t, x, w) = t\Psi(t, x, w) \), then \( \Phi \) has all the properties stated in (ii) of Theorem 2. The proof of Theorem 2 is finished.

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References


