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ALGEBRAIC INDEPENDENCE OF CERTAIN NUMBERS DEFINED BY LINEAR RECURRENCES

by

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§1. Introduction

Let $\{a_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers defined by

$$a_{k+n}=c_1a_{k+n-1}+\cdots+c_na_k\,,\qquad k=0,1,2,\ldots, \tag{1}$$

where $a_0,\ldots,a_{n-1}$ are not all zero and $c_1,\ldots,c_n$ are nonnegative integers with $c_n\neq 0$. We put

$$\Phi(X)=X^n-c_1X^{n-1}-\cdots-c_n\,. \tag{2}$$

In 1929, Mahler [3] proved the following theorem: Let $\{a_k\}_{k\geq 0}$ be a linear recurrence which satisfies (1). Suppose that $\Phi(X)$ is irreducible over the rational number field $\boldsymbol{Q}$ and the roots $\rho_1,\ldots,\rho_n$ of $\Phi(X)$ satisfy $\rho_1>\max\{1,|\rho_2|,\ldots,|\rho_n|\}$. If α is an algebraic number with $0<|\alpha|<1$, then $\sum_{k=0}^{\infty}\alpha^{a_k}$ is transcendental.

In this paper, we establish the algebraic independence of certain numbers defined by linear recurrences with conditions on $\Phi(X)$ weaker than those of Mahler (see Remark below).

Theorem. *Let $\{a_k\}_{k\geq 0}$ be a linear recurrence which satisfies* (1). *Suppose that $\Phi(\pm 1)\neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Let α be an algebraic number with $0<|\alpha|<1$ and $\beta_1,\ldots,\beta_m$ nonzero distinct algebraic numbers. Then*

$$\left\{\sum_{k=0}^{\infty}k^l\beta_j^k\alpha^{a_k}\right\}_{1\leq j\leq m,\,l\geq 0}$$

are algebraically independent.

Remark. Since we do not assume that $\Phi(X)$ is irreducible over $\boldsymbol{Q}$, our assumption on $\Phi(X)$ is weaker than that of Mahler, because of the following fact: Suppose that the polynomial $\Phi(X)$ defined by (2) with $n\geq 2$ is irreducible over $\boldsymbol{Q}$. Then the roots $\rho_1,\ldots,\rho_n$ of $\Phi(X)$ satisfy the condition $\rho_1>\max\{1,|\rho_2|,\ldots,|\rho_n|\}$ if and only if none of ρ_i/ρ_j $(i\neq j)$ is a root of unity. (For the proof of this statement see [11].)

Corollary 1. *Let $\{a_k\}_{k\geq 0}$ be as in Theorem and let $\{b_k^{(i)}\}_{k\geq 0}$, $i=1,\dots,r$, be linearly independent linear recurrences of algebraic numbers. If α is an algebraic number with $0<|\alpha|<1$, then*

$$\left\{\sum_{k=0}^{\infty} b_k^{(i)}\alpha^{a_k}\right\}_{1\leq i\leq r}$$

are algebraically independent.

Corollary 2. *Let $\{a_k\}_{k\geq 0}$ be as in Theorem and let m be a positive integer. Define*

$$f_i(z)=\sum_{k=0}^{\infty} z^{a_{mk+i}}, \qquad i=0,\dots,m-1.$$

If α is an algebraic number with $0<|\alpha|<1$, then $\{f_i^{(l)}(\alpha)\}_{0\leq i\leq m-1, l\geq 0}$ are algebraically independent, where $f_i^{(l)}(z)$ denotes the l-th derivative of $f_i(z)$.

Corollary 2 enables us to treat numbers defined by linear recurrences of the form (1) with c_i's not necessarily nonnegative as the following example shows.

Example. Let $\{a_k^{(i)}\}_{k\geq 0}$, $i=0, 1$, be linear recurrences defined by

$$a_{k+2}^{(i)}=13a_{k+1}^{(i)}-36a_k^{(i)}, \qquad k=0,1,2,\dots \quad (i=0,1)$$

with $a_0^{(0)}=2$, $a_1^{(0)}=13$, $a_0^{(1)}=1$, $a_1^{(1)}=19$. Let

$$f_i(z)=\sum_{k=0}^{\infty} z^{a_k^{(i)}}, \qquad i=0,1.$$

If α is an algebraic number with $0<|\alpha|<1$, then $\{f_i^{(l)}(\alpha)\}_{i=0,1,l\geq 0}$ are algebraically independent. This follows from Corollary 2 with $\{a_k\}_{k\geq 0}$ defined by

$$a_0=2, \quad a_1=1, \quad a_{k+2}=a_{k+1}+6a_k, \qquad k=0,1,2,\dots,$$

since $a_k=3^k+(-2)^k$ $(k=0,1,2,\dots)$, and so $a_k^{(0)}=a_{2k}$, $a_k^{(1)}=a_{2k+1}$ $(k=0,1,2,\dots)$.

Some entire functions are known to have the following property: they, as well as their successive derivatives, take algebraically independent values at any given nonzero distinct algebraic numbers. Examples of such functions are $\sum_{k=0}^{\infty}\alpha^{k!}x^k$ and $\sum_{k=0}^{\infty}\alpha^{d^k}x^k$ (cf. Nishioka [8, 10]), where α is an algebraic number with $0<|\alpha|<1$ and d is an integer greater than 1.

Corollary 3. *Let $\{a_k\}_{k\geq 0}$ be as in Theorem. Let α be an algebraic number with $0<|\alpha|<1$ and define*

$$g(x)=\sum_{k=0}^{\infty}\alpha^{a_k}x^k.$$

Then $g(x)$ is an entire function and $\{g^{(l)}(\beta_j)\}_{1\leq j\leq m, l\geq 0}$ are algebraically independent for any nonzero distinct algebraic numbers $\beta_1,\dots,\beta_m$.

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§2. Preliminaries for the proof of Theorem

In this section, we prepare some notations and exhibit Mahler's proof of his theorem stated above as an introduction to our method in Section 4.

Let $\Omega=(\omega_{ij})$ be an $n\times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of Ω is itself an eigenvalue (cf. Gantmacher [2]). If $\boldsymbol{z}=(z_1,\dots,z_n)$ is a point of $\boldsymbol{C}^n$ with $\boldsymbol{C}$ the set of complex numbers, we define a transformation $\Omega: \boldsymbol{C}^n\to\boldsymbol{C}^n$ by

$$\Omega\boldsymbol{z}=\left(\prod_{j=1}^{n} z_j^{\omega_{1j}},\dots,\prod_{j=1}^{n} z_j^{\omega_{nj}}\right). \tag{3}$$

We suppose that the matrix Ω and an algebraic point $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$, where α_i are nonzero algebraic numbers, have the following four properties.

(I) Ω is non-singular and none of its eigenvalues is a root of unity, so that in particular $\rho>1$.

(II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.

(III) If we put $\Omega^k\boldsymbol{\alpha}=(\alpha_1^{(k)},\dots,\alpha_n^{(k)})$, then

$$\log|\alpha_i^{(k)}|\le -c\rho^k, \qquad i=1,\dots,n,$$

for all sufficiently large k, where c is a positive constant.

(IV) If $f(\boldsymbol{z})$ is any nonzero power series in n variables with complex coefficients which converges in some neighborhood of the origin, then there are infinitly many positive integers k such that $f(\Omega^k\boldsymbol{\alpha})\neq 0$.

We note that the property (II) is satisfied, if every eigenvalue of Ω of the absolute value ρ is a simple root of the minimal polynomial of Ω.

Let K be an algebraic number field and I_K the integer ring of K. We denote by $K[[z_1,\dots,z_n]]$ the ring of formal power series in variables $z_1,\dots,z_n$ with coefficients in K. Suppose that $f(\boldsymbol{z})\in K[[z_1,\dots,z_n]]$ converges in an n-polydisc U around the origin and satisfies the functional equation

$$f(\Omega\boldsymbol{z})=\frac{\sum_{i=0}^{m} a_i(\boldsymbol{z})f(\boldsymbol{z})^i}{\sum_{i=0}^{m} b_i(\boldsymbol{z})f(\boldsymbol{z})^i}, \tag{4}$$

where $1\le m<\rho$ and $a_i(\boldsymbol{z})$, $b_i(\boldsymbol{z})$ are polynomials of $z_1,\dots,z_n$ with coefficients in I_K. We denote by $\Delta(\boldsymbol{z})$ the resultant of polynomials $\sum_{i=0}^{m} a_i(\boldsymbol{z})u^i$ and $\sum_{i=0}^{m} b_i(\boldsymbol{z})u^i$ in u. If one of them is a constant $c(\boldsymbol{z})$ in u, we set $\Delta(\boldsymbol{z})=c(\boldsymbol{z})$. Then Mahler proved the following.

Theorem A (Mahler [3]). *Assume that Ω and $\boldsymbol{\alpha}$ satisfy the properties* (I)$\sim$(IV) *and $f(\boldsymbol{z})$ is transcendental over the rational function field $K(z_1,\dots,z_n)$. If $\Omega^k\boldsymbol{\alpha}\in U$ and $\Delta(\Omega^k\boldsymbol{\alpha})\neq 0$ for any $k\ge 0$, then $f(\boldsymbol{\alpha})$ is transcendental.*

Let

$$P(\boldsymbol{z})=z_1^{a_{n-1}}\cdots z_n^{a_0}$$

be a monomial in $z_1, \ldots, z_n$, which will be denoted similarly to (3) by

$$P(\boldsymbol{z})=(a_{n-1}, \ldots, a_0)\boldsymbol{z}\,. \tag{5}$$

Put

$$\Omega=\begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{6}$$

By (3) and (5), we get

$$P(\Omega^k\boldsymbol{z})=z_1^{a_{k+n-1}}\cdots z_n^{a_k}\,, \qquad k=0, 1, 2, \ldots\,.$$

We then define a power series

$$f(\boldsymbol{z})=\sum_{k=0}^{\infty} P(\Omega^k\boldsymbol{z})\,, \tag{7}$$

which satisfies the functional equation

$$f(\boldsymbol{z})=f(\Omega\boldsymbol{z})+P(\boldsymbol{z})\,. \tag{8}$$

Let α be an algebraic number with $0<|\alpha|<1$ and set

$$\boldsymbol{\alpha}=(\underbrace{1, \ldots, 1}_{n-1}, \alpha)\,.$$

Then we have

$$f(\boldsymbol{\alpha})=\sum_{k=0}^{\infty} \alpha^{a_k}\,.$$

Therefore, to establish the transcendence of the number $\sum_{k=0}^{\infty}\alpha^{a_k}$, we may apply Theorem A to the function $f(\boldsymbol{z})$ defined by (7). However, it is necessary to verify that the matrix Ω defined by (6) and $\boldsymbol{\alpha}=(1, \ldots, 1, \alpha)$ satisfy the properties (I)$\sim$(IV). For this, Mahler used the following theorem.

Theorem B (Mahler [3]). *Suppose that the characteristic polynomial of Ω is irreducible over $\boldsymbol{Q}$ and Ω has an eigenvalue ρ (>1) which is greater than the absolute values of any other eigenvalues. We denote by A_{ij} the (i, j)-cofactor of the matrix $\Omega-\rho I$, where I is the identity matrix, so that $A_{i1}\neq 0$ $(1\leq i\leq n)$. If*

$$\sum_{i=1}^{n} |A_{i1}|\log|\alpha_i|<0\,, \tag{9}$$

then Ω and $\boldsymbol{\alpha}$ have the properties (I)$\sim$(IV).

We note that the inequality (9) holds if $|\alpha_i| \leq 1$ $(1 \leq i \leq n)$ and if at least one of $|\alpha_i|$ is less than 1.

By Theorem B, the matrix Ω defined by (6) and $\boldsymbol{\alpha}=(1, \ldots, 1, \alpha)$ satisfy the properties (I)$\sim$(IV), since the characteristic polynomial of Ω is $\Phi(X)$. If $f(\boldsymbol{z})$ is transcendental over the rational function field $\boldsymbol{C}(z_1, \ldots, z_n)$, we can apply Theorem A and the result of Mahler follows. To the contrary we assume that $f(\boldsymbol{z})$ is algebraic over $\boldsymbol{C}(z_1, \ldots, z_n)$. Then $f(1, \ldots, 1, z)=\sum_{k=0}^{\infty} z^{a_k}$ is algebraic over $\boldsymbol{C}(z)$. On the other hand, we can write

$$a_k = b\rho_1^k + o(\rho_1^k),$$

where $b \neq 0$ and $\rho_1 > 1$, so that $b > 0$, since $a_k \geq 0$ for any $k \geq 0$. Therefore $a_{k+1} - a_k \to \infty$ as k tends to infinity, which implies that $\sum_{k=0}^{\infty} z^{a_k}$ is transcendental over $\boldsymbol{C}(z)$ by Lemma 1 below, a contradiction.

§3. Lemmas

Lemma 1 (Mahler [6, p. 42, (21)]). *Let $\{a_k\}_{k \geq 0}$ be a sequence of nonnegative integers such that $a_{k+1} - a_k \to \infty$ as k tends to infinity and let $f(z)=\sum_{k=0}^{\infty} c_k z^{a_k}$ be a power series with nonzero complex coefficients c_k's and regular in a neighborhood of the origin. Then $f(z)$ is transcendental over the rational function field $\boldsymbol{C}(z)$.*

Let C be a field of characteristic 0, L the rational function field $C(z_1, \ldots, z_n)$, and M the quotient field of formal power series ring $C[[z_1, \ldots, z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries for which the property (I) holds. We define an endomorphism $\tau: M \to M$ by

$$f^{\tau}(\boldsymbol{z}) = f(\Omega \boldsymbol{z}) \quad (f(\boldsymbol{z}) \in M),$$

where $\Omega \boldsymbol{z}$ is defined as in Section 2.

Lemma 2 (Nishioka [9, 10]). *Suppose that $f_{ij} \in M$ $(i=1, \ldots, k, j=1, \ldots, n(i))$ satisfy the functional equation*

$$\begin{pmatrix} f_{i1} \\ \vdots \\ \vdots \\ f_{in(i)} \end{pmatrix} = \begin{pmatrix} a_i & 0 & \cdots & 0 \\ a_{21}^{(i)} & a_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n(i)1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \begin{pmatrix} f_{i1}^{\tau} \\ \vdots \\ \vdots \\ f_{in(i)}^{\tau} \end{pmatrix} + \begin{pmatrix} b_{i1} \\ \vdots \\ \vdots \\ b_{in(i)} \end{pmatrix},$$

where a_i, $a_{st}^{(i)} \in C$, $a_i \neq 0$, $a_{s\,s-1}^{(i)} \neq 0$, and $b_{ij} \in L$. If f_{ij} $(i=1, \ldots, k, j=1, \ldots, n(i))$ are algebraically dependent over L, then there exist a non-empty subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$ and nonzero elements $c_1, \ldots, c_r$ of C such that

$$a_{i_1} = \cdots = a_{i_r}, \qquad c_1 f_{i_1 1} + \cdots + c_r f_{i_r 1} \in L.$$

Lemma 3 (Nishioka [9, 10]). *Let K be an algebraic number field. Assume that $f_1(\boldsymbol{z}), \ldots, f_m(\boldsymbol{z}) \in K[[z_1, \ldots, z_n]]$ converge in an n-polydisc U around the origin and* *satisfy a functional equation of the form*

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A \begin{pmatrix} f_1(\Omega z) \\ \vdots \\ f_m(\Omega z) \end{pmatrix} + \begin{pmatrix} b_1(z) \\ \vdots \\ b_m(z) \end{pmatrix}, \tag{10}$$

where A is an $m \times m$ matrix with entries in K and $b_i(z)$ are rational functions of $z_1, \dots, z_n$ with coefficients in K. Suppose that the $n \times n$ matrix Ω and a point α whose components are nonzero algebraic numbers satisfy the properties (I) $\sim$ (IV), *and for all $k \geq 0$, $\Omega^k \alpha \in U$ and $b_i(z)$ are defined at $\Omega^k \alpha$. If $f_1(z), \dots, f_m(z)$ are algebraically independent over the rational function field $K(z_1, \dots, z_n)$, then $f_1(\alpha), \dots, f_m(\alpha)$ are algebraically independent.*

Lemma 4 (Tanaka [11]). *Let $\{a_k\}_{k \geq 0}$ be a linear recurrence which satisfies* (1). *Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. If α is an algebraic number with $0 < |\alpha| < 1$, then the matrix Ω defined by* (6) *and $\alpha = (1, \dots, 1, \alpha)$ satisfy the properties* (I) $\sim$ (IV). *Furthermore, we have $a_{k+1} - a_k \to \infty$ as k tends to infinity.*

We give the proof for completeness. The following Lemma 5 and Lemma 6 will be used only in the proof of this lemma. We denote by N_0 the set of nonnegative integers.

Lemma 5 (Skolem-Lech-Mahler's theorem, cf. Cassels [1] and Nishioka [9]). *Let C be a field of characteristic zero. Let $\rho_1, \dots, \rho_d \in C^\times$ be distinct and let $P_1(X), \dots, P_d(X) \in C[X]$ be nonzero. Then*

$$R = \left\{ k \in N_0 \,\middle|\, f(k) = \sum_{i=1}^{d} P_i(k) \rho_i^k = 0 \right\} \tag{11}$$

is the union of a finite set and finite number of arithmetic progressions. If R is an infinite set, then ρ_i / ρ_j is a root of unity for some distinct pair i and j.

Lemma 6 (Masser [7]). *Let Ω be an $n \times n$ matrix with nonnegative integer entries for which the property* (I) *holds. Let α be an n-dimensional vector whose components $\alpha_1, \dots, \alpha_n$ are nonzero algebraic numbers such that $\Omega^k \alpha \to (0, \dots, 0)$ as k tends to infinity. Then the negation of the property* (IV) *is equivalent to the following:*

There exist integers $i_1, \dots, i_n$ not all zero and positive integers a, b such that

$$(\alpha_1^{(k)})^{i_1} \cdots (\alpha_n^{(k)})^{i_n} = 1$$

for all $k = a + lb$ ($l = 0, 1, 2, \dots$).

Proof of Lemma 4. The property (I) is satisfied, since the characteristic polynomial of the matrix Ω defined by (6) is $\Phi(X)$. Let $\rho_1, \dots, \rho_t$ be the eigenvalues of Ω. Since every entry of Ω is nonnegative, we may assume $\rho_1 \geq \max\{|\rho_2|, \dots, |\rho_t|\}$ and then $\rho_1 > 1$. For each i $(0 \leq i \leq n-1)$, we define the sequence $\{a_k^{(i)}\}_{k \geq 0}$ by

$$a_{k+n}^{(i)} = c_1 a_{k+n-1}^{(i)} + \cdots + c_n a_k^{(i)}, \qquad k = 0, 1, 2, \dots$$

with

$$a_0^{(i)} = 0, \dots, a_{i-1}^{(i)} = 0, \quad a_i^{(i)} = 1, \quad a_{i+1}^{(i)} = 0, \dots, a_{n-1}^{(i)} = 0.$$

Then

$$\Omega^k = \begin{pmatrix} a_{k+n-1}^{(n-1)} & \cdots & a_k^{(n-1)} \\ \vdots & & \vdots \\ a_{k+n-1}^{(0)} & \cdots & a_k^{(0)} \end{pmatrix}, \qquad k=0, 1, 2, \ldots$$

holds. Since each $a_k^{(i)}$ can be expressed as an $f(k)$ in (11), the sequence $\{a_k^{(i)}\}_{k\geq 0}$ has only finitely many zeros by Lemma 5. Hence the entries of Ω^λ are positive for sufficiently large λ. By Perron's theorem (cf. Gantmacher [2, p. 53, Theorem 1]), it follows that ρ_1 is a simple root of $\Phi(X)$ and has the property $\rho_1 > \max\{|\rho_2|, \ldots, |\rho_t|\}$. Therefore the property (II) is satisfied. We can write

$$a_k^{(i)} = b^{(i)}\rho_1^k + o(\rho_1^k), \qquad i=0, \ldots, n-1, \tag{12}$$

where at least one of $b^{(i)}$ is not zero. Since $a_k^{(i)} \geq 0$ ($k=0, 1, 2, \ldots$), all the $b^{(i)}$ are nonnegative. Noting

$$\begin{pmatrix} a_{k+n}^{(n-1)} & \cdots & a_{k+1}^{(n-1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{k+n}^{(0)} & \cdots & a_{k+1}^{(0)} \end{pmatrix} = \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{k+n-1}^{(n-1)} & \cdots & a_k^{(n-1)} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{k+n-1}^{(0)} & \cdots & a_k^{(0)} \end{pmatrix},$$

we have

$$a_{k+1}^{(i)} = c_{n-i}a_k^{(n-1)} + a_k^{(i-1)}\ (1\leq i\leq n-1), \quad a_{k+1}^{(0)} = c_n a_k^{(n-1)}.$$

Thus

$$b^{(i)}\rho_1 = c_{n-i}b^{(n-1)} + b^{(i-1)}\ (1\leq i\leq n-1), \quad b^{(0)}\rho_1 = c_n b^{(n-1)},$$

so that

$$b^{(i)} \geq b^{(i-1)}/\rho_1\ (1\leq i\leq n-1), \quad b^{(0)} \geq b^{(n-1)}/\rho_1.$$

This implies that $b^{(i)}>0$ for any i, since at least one of $b^{(i)}$ is positive. Put $\Omega^k\boldsymbol{\alpha} = (\alpha_{n-1}^{(k)}, \ldots, \alpha_0^{(k)})$. Then

$$\alpha_i^{(k)} = (a_{n-1}^{(i)}, \ldots, a_0^{(i)})\Omega^k\boldsymbol{\alpha} = \boldsymbol{\alpha}^{a_k^{(i)}}, \qquad i=0, \ldots, n-1.$$

Hence the property (III) is satisfied. Assume that there exist integers $i_0, \ldots, i_{n-1}$ not all zero and positive integers a, b such that

$$(\alpha_{n-1}^{(k)})^{i_{n-1}} \cdots (\alpha_0^{(k)})^{i_0} = 1$$

for all $k=a+lb$ ($l=0, 1, 2, \ldots$). Let $\{a_k^*\}_{k\geq 0}$ be a linear recurrence defined by (1) with

$$a_0 = i_0, \ldots, a_{n-1} = i_{n-1}.$$

Then

$$\alpha^{a_k^*}=(i_{n-1},\dots,i_0)\Omega^k\boldsymbol{\alpha}=1\ ,$$

namely $a_k^*=0$ for all $k=a+lb$ $(l=0,1,2,\dots)$. Since $\{a_k^*\}_{k\geq 0}$ is nonzero linear recurrence, there are distinct i and j such that ρ_i/ρ_j is a root of unity by Lemma 5. This contradicts the assumption of the lemma. Therefore the property (IV) is satisfied. We can write

$$a_k=b\rho_1^k+o(\rho_1^k)\,,$$

where $b=\sum_{i=0}^{n-1}a_ib^{(i)}>0$, since $a_k=\sum_{i=0}^{n-1}a_ia_k^{(i)}$ and $a_0,\dots,a_{n-1}$ are not all zero. Hence $a_{k+1}-a_k\to\infty$ as k tends to infinity. This completes the proof of the lemma.

§4. Proofs of Theorem and Corollaries

Proof of Theorem. Let

$$P(\boldsymbol{z})=z_1^{a_{n-1}}\cdots z_n^{a_0}$$

and set

$$F(x,\boldsymbol{z})=\sum_{k=0}^{\infty}x^kP(\Omega^k\boldsymbol{z})\,,$$

where Ω is the matrix defined by (6). Then $F(x,\boldsymbol{z})$ satisfies the functional equation

$$F(x,\boldsymbol{z})=xF(x,\Omega\boldsymbol{z})+P(\boldsymbol{z})\,. \tag{13}$$

Letting $D_x=x\dfrac{\partial}{\partial x}$, we have

$$D_x^lF(x,\boldsymbol{z})=\sum_{q=0}^{l}\binom{l}{q}xD_x^qF(x,\Omega\boldsymbol{z}) \tag{14}$$

for $l\geq 1$, where $\binom{l}{q}$ denote the binomial coefficients. By (13) and (14), $\{D_x^lF(\beta_j,\boldsymbol{z})\}_{1\leq j\leq m,0\leq l\leq L}$ satisfy the functional equation of the form (10). We assert that $\{D_x^lF(\beta_j,\boldsymbol{\alpha})\}_{1\leq j\leq m,l\geq 0}$ are algebraically independent, where $\boldsymbol{\alpha}=(1,\dots,1,\alpha)$ as before. Then the theorem follows, since $D_x^lF(\beta_j,\boldsymbol{\alpha})=\sum_{k=0}^{\infty}k^l\beta_j^k\alpha^{a_k}$. To the contrary we assume that $\{D_x^lF(\beta_j,\boldsymbol{\alpha})\}_{1\leq j\leq m,0\leq l\leq L}$ are algebraically dependent. By Lemma 4, Ω and $\boldsymbol{\alpha}$ satisfy the properties (I)$\sim$(IV). Hence, by Lemma 3, $\{D_x^lF(\beta_j,\boldsymbol{z})\}_{1\leq j\leq m,0\leq l\leq L}$ are algebraically dependent over the rational function field $\bar{\boldsymbol{Q}}(z_1,\dots,z_n)$ with algebraic coefficients. Therefore $F(\beta_j,\boldsymbol{z})\in\bar{\boldsymbol{Q}}(z_1,\dots,z_n)$ for some j by Lemma 2 and so

$$F(\beta_j,\underbrace{1,\dots,1}_{n-1},z)=\sum_{k=0}^{\infty}\beta_j^kz^{a_k}\in\bar{\boldsymbol{Q}}(z)\,.$$

Since $a_{k+1}-a_k\to\infty$ as k tends to infinity, $F(\beta_j,1,\dots,1,z)$ is transcendental over $\boldsymbol{C}(z)$ by Lemma 1. This is a contradiction.

Proof of Corollary 1. The linear recurrences $\{b_k^{(i)}\}_{k\geq 0}$, $i=1,\dots,r$, can be written in the form

$$b_k^{(i)}=\sum_{j=1}^{m} P_{ij}(k)\beta_j^k\,, \qquad i=1,\dots,r\,,$$

where $P_{ij}(X)$ are polynomials of X with algebraic coefficients and β_j are nonzero distinct algebraic numbers. Let

$$L=\max_{\substack{1\le i\le r\\ 1\le j\le m}} \deg P_{ij}(X)\,.$$

Then the numbers

$$\left\{\sum_{k=0}^{\infty} k^l\beta_j^k\alpha^{a_k}\right\}_{1\le j\le m,\,0\le l\le L}$$

are algebraically independent by the theorem. Let V be a vector space of linear recurrences over $\bar{Q}$ spanned by $\{\{k^l\beta_j^k\}_{k\ge 0}\}_{1\le j\le m,\,0\le l\le L}$. Then $\dim V=m(L+1)$. Since $\{b_k^{(i)}\}_{k\ge 0}$, $i=1,\dots,r$, are linearly independent elements in V, we can choose

$$\{b_k^{(r+1)}\}_{k\ge 0},\dots,\{b_k^{(m(L+1))}\}_{k\ge 0}\in V$$

such that $\{\{b_k^{(i)}\}_{k\ge 0}\}_{1\le i\le m(L+1)}$ are linearly independent. Then the numbers $\{\sum_{k=0}^{\infty} b_k^{(i)}\alpha^{a_k}\}_{1\le i\le m(L+1)}$ and $\{\sum_{k=0}^{\infty} k^l\beta_j^k\alpha^{a_k}\}_{1\le j\le m,\,0\le l\le L}$ are linearly equivalent over $\bar{Q}$. Hence $\{\sum_{k=0}^{\infty} b_k^{(i)}\alpha^{a_k}\}_{1\le i\le m(L+1)}$ are algebraically independent. This completes the proof of the corollary.

Proof of Corollary 2. Let $\{b_k^{(i)}\}_{k\ge 0}$, $i=0,\dots,m-1$, be linear recurrences defined by

$$b_{k+m}^{(i)}=b_k^{(i)}\,, \qquad k=0,1,2,\dots \quad (i=0,\dots,m-1)$$

with

$$b_0^{(i)}=0,\dots,b_{i-1}^{(i)}=0\,,\quad b_i^{(i)}=1\,,\quad b_{i+1}^{(i)}=0,\dots,b_{m-1}^{(i)}=0\,.$$

Then the linear recurrences $\{\{b_k^{(i)}(a_k)^l\}_{k\ge 0}\}_{0\le i\le m-1,\,l\ge 0}$ are linearly independent. Hence, by Corollary 1, the numbers

$$\left\{\sum_{k=0}^{\infty} b_k^{(i)}(a_k)^l\alpha^{a_k}\right\}_{0\le i\le m-1,\,l\ge 0}$$

are algebraically independent. We have here

$$\sum_{k=0}^{\infty} b_k^{(i)}(a_k)^l\alpha^{a_k}=\sum_{k=0}^{\infty}(a_{mk+i})^l\alpha^{a_{mk+i}}\,, \qquad i=0,\dots,m-1\,,\quad l\ge 0\,.$$

Since the numbers $\{\sum_{k=0}^{\infty}(a_{mk+i})^l\alpha^{a_{mk+i}}\}_{0\le i\le m-1,\,0\le l\le L}$ and $\{f_i^{(l)}(\alpha)\}_{0\le i\le m-1,\,0\le l\le L}$ are linearly equivalent over $\bar{Q}$ for any $L\ge 0$, we have the corollary.

Proof of Corollary 3. Since $a_{k+1}-a_k\to\infty$ as k tends to infinity, $g(x)$ is an entire function; and the corollary follows from Corollary 1.

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