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A REMARKABLE BANACH FUNCTION SPACE ON THE THEORY OF INTERPOLATION AND EXTRAPOLATION OF OPERATIONS

by

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ABSTRACT

The estimation which was obtained by the author, developes to the fine and useful formula by using the notion of Orlicz space. This enables us to construct the global theory in several fields of analysis.

1. About twenty years ago, the author [8] established the following theorem:

Theorem A. *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces with* σ*-finite measure respectively. Let* T *be a quasi-linear operation which transforms the measurable function* f *on* X *to* Tf *on* Y, *and weak type* (1, 1) *and type* (p, p) *for some* $p>1$ *respectively, that is*

$$\text{(i)} \qquad \nu(E_r[Tf]) \leq \frac{M}{r}\|f\|_{L^1_\mu}$$

for all positive real number r *and*

$$\text{(ii)} \qquad \|Tf\|_{L^p_\nu} \leq M\|f\|_{L^p_\mu} ,$$

where $E_r[Tf]=\{y \in Y: |Tf(y)|>r\}$ *and* M *is a constant independent on* f.
Then we have

$$(1.1) \qquad \int_{|Tf|\leq 1} |Tf|^p d\nu + \int_{|Tf|>1} |Tf| d\nu \leq A\left\{\int_{|f|\leq 1} |f|^p d\mu + \int_{|f|>1} |f|(1+\log^+|f|)d\mu\right\} ,$$

where A *is the constant depending only on* p *and not on* f.

As it is well known, by the theorem of interpolation due to J. Marcinkiewicz [10, 19], we have under the same hypothesis as Theorem A,

$$(1.2) \qquad \|Tf\|_{L^{p'}_\nu} \leq A'\|f\|_{L^{p'}_\mu} \quad (1<p'\leq p) ,$$

where A' is the constant depending on p' and not on the f. Nevertheless the operation T contains several important cases but as for $p'=1$, it breaks off.

The motivation of the study of this paper was to obtain the estimation of $|Tf|$ on the whole spase as well as the previous papers [8]. Recently the author has been aware that the formula (1.1) has the more fine and useful form by using the norm of Orlicz space. The study of Mr. T. Miyamoto [11] is one of the contribution in this field. The author is also interested in the papers of H. P. Heinig [6] and A. Torchinsky [16].

2. Let us put for some $p>1$,

$$\varphi(u)=\begin{cases} u^p & (0\leq u\leq 1)\ , \\ u & (1<u<\infty)\ , \end{cases} \tag{2.1}$$

and

$$\varphi^*(u)=\begin{cases} u^p & (0\leq u\leq 1)\ , \\ u(1+\log^+ u) & (1<u<\infty)\ . \end{cases} \tag{2.2}$$

Here we should pointed out that functions $\varphi(u)$ and $\varphi^*(u)$ are not convex at the neighborhood of $u=1$. It should be revised in the last stage.

Then the formula (1.1) could be rewritten to the following form:

$$\int_Y \varphi(|Tf|)d\nu\leq A\int_X \varphi^*(|f|)d\mu\ , \tag{2.3}$$

where A is the same constant as the formula (1.1). From now on we may assume that the constant $A\geq 1$.

Let us denote for any positive real number r,

$$E_{\varphi^*,r}[f]=\left\{\lambda>0:\ \int_X \varphi^*\left(\frac{|f|}{\lambda}\right)d\mu<r\right\}\ , \tag{2.4}$$

and for any positive real number s,

$$E_{\varphi,s}[Tf]=\left\{\lambda>0:\ \int_Y \varphi\left(\frac{|Tf|}{\lambda}\right)d\nu<s\right\}\ . \tag{2.5}$$

Then we have

$$E_{\varphi^*,r}[f]\subset E_{\varphi,Ar}[Tf]\ . \tag{2.6}$$

Because, if we put f/λ in (2.3) instead of f with positive real number λ, then since $\left|T\left(\frac{f}{\lambda}\right)\right|=\frac{|Tf|}{\lambda}$, we have

$$\int_Y \varphi\left(\frac{|Tf|}{\lambda}\right)d\nu\leq A\int_X \varphi^*\left(\frac{|f|}{\lambda}\right)d\nu \tag{2.3$'$}$$

and so we obtain (2.6). In particular if we put $r=1$ we obtain

$$E_{\varphi^*,1}[f]\subset E_{\varphi,A}[Tf]\ . \tag{2.6$'$}$$

Next we need the following lemma of which property is essential for our purposes.

Lemma 1. *As for the functions* φ *and* φ^* *which are defined by the formulas* (2.1) *and* (2.2), *they have the following property. That is, we have for any constant* $\rho\geq 1$,

$$\varphi(\rho u)\geq\rho\varphi(u) \quad (0\leq u<\infty)\ , \tag{2.7}$$

and

$$\varphi^*(\rho u)\geq\rho\varphi^*(u) \quad (0\leq u<\infty)\ , \tag{2.8}$$

Proof. We may assume that $\rho>1$. In the case of $\rho=1$, there is nothing to prove. In the first as for $0\leq u<\dfrac{1}{\rho}$, we have

$$\varphi(\rho u)=(\rho u)^p>\rho u^p=\rho\varphi(u)\ .$$

Next as for $\dfrac{1}{\rho}\leq u\leq 1$, we have

$$\varphi(\rho u)=\rho u\geq\rho u^p=\rho\varphi(u)\ .$$

In the last as for $1<u<\infty$, we have

$$\varphi(\rho u)=\rho u=\rho\varphi(u)\ .$$

Running on the same lines we shall prove the case of the φ^*.
Now if we denote

$$A^{-1}E_{\varphi,r}[Tf]=\{A^{-1}\lambda:\ \lambda\in E_{\varphi,r}[Tf]\}\ , \tag{2.9}$$

then we have

$$E_{\varphi,Ar}[Tf]\subset A^{-1}E_{\varphi,r}[Tf]\ . \tag{2.10}$$

Because, for any λ in $E_{\varphi,Ar}[Tf]$, if we apply the Lemma with $\rho=A$ and put $\lambda=A^{-1}\lambda'$, we have

$$Ar\geq\int_Y\varphi\left(\frac{|Tf|}{\lambda}\right)d\nu=\int_Y\varphi\left(A\frac{|Tf|}{\lambda'}\right)d\nu\geq A\int_Y\left(\frac{|Tf|}{\lambda'}\right)d\nu\ ,$$

so λ' is in $E_{\varphi,r}[Tf]$ and we obtain (2.10). In particular, if we put $r=1$, we have

$$E_{\varphi,A}[Tf]\subset A^{-1}E_{\varphi,1}[Tf]\ . \tag{2.10)'$$

Combining (2.6)′ and (2.10)′ we obtain

$$E_{\varphi^*,1}[f]\subset A^{-1}E_{\varphi,1}[Tf]\ . \tag{2.11}$$

According to the W. A. J. Luxemberg [9], the norm of the Orlicz space is defined as follows:

$$p.n.\|f\|_{L_\mu^{\varphi^*}}=\inf_{\lambda>0}E_{\varphi^*,1}[f]=\inf_{\lambda>0}\left\{\lambda:\ \int_X\varphi^*\left(\frac{|f|}{\lambda}\right)d\mu<1\right\}\ , \tag{2.12}$$

and

$$(2.13) \qquad p.n.\|Tf\|_{L^{\varphi}_{\nu}}=\inf_{\lambda>0} E_{\varphi,1}[Tf]=\inf_{\lambda>0}\left\{\lambda:\int_{Y}\varphi\left(\frac{|Tf|}{\lambda}\right)d\nu<1\right\}.$$

Therefore if we take the infimum of the set of the formula (2.11), we obtain the following inequality.

$$(2.14) \qquad p.n.\|Tf\|_{L^{\varphi}_{\nu}}\leq\max(A,1)p.n.\|f\|_{L^{\varphi^*}_{\mu}},$$

with the same constant A as in the formula (1.1).

We shall mean by $p.n.$ in the formulas (2.12)~(2.14), the pseudo-norm. Because the functions $\varphi(u)$ and $\varphi^*(u)$ are not convex at the neighborhood of $u=1$. So we shall revise it by replaceing the equivalent norm by which both sides of (2.14) become the Orlicz space.

For this purpose, let us introduce the modified functions as follows:

$$(2.1)' \qquad \varphi_0(u)=\begin{cases}\dfrac{u^p}{p}, & (0\leq u\leq 1),\\ u-\left(1-\dfrac{1}{p}\right) & (1<u<\infty),\end{cases}$$

and

$$(2.2)' \qquad \varphi_0^*(u)=\begin{cases}\dfrac{u^p}{p}, & (0\leq u\leq 1),\\ \dfrac{1}{2}u(1+\log^+ u)-\left(\dfrac{1}{2}-\dfrac{1}{p}\right) & (1<u<\infty).\end{cases}$$

Now they are strictly increasing, continuously differentiable and convex functions, and we have

$$(2.15) \qquad p^{-1}\varphi(u)\leq\varphi_0(u)\leq\varphi(u) \quad (0\leq u<\infty),$$

and

$$(2.16) \qquad \max(p,2)^{-1}\varphi^*(u)\leq\varphi_0^*(u)\leq\varphi^*(u) \quad (0\leq u<\infty).$$

Therefore combining the formula (2.15) and Lemma, as before we have for any positive real number r,

$$(2.17) \qquad E_{\varphi,r}[Tf]\subset E_{\varphi_0,r}[Tf]\subset p^{-1}E_{\varphi,r}[Tf],$$

and we obtain by taking the infimum of the sets of (2.17) with $r=1$,

$$(2.18) \qquad p^{-1}p.n.\|Tf\|_{L^{\varphi}_{\nu}}\leq\|Tf\|_{L^{\varphi_0}_{\nu}}\leq p.n.\|Tf\|_{L^{\varphi}_{\nu}}.$$

By the same way as above arguments, we have

$$(2.19) \qquad E_{\varphi^*,r}[f]\subset E_{\varphi_0^*,r}[f]\subset\max(p,2)^{-1}E_{\varphi^*,r}[f],$$

and therefore we obtain

(2.20) $$\max(p, 2)^{-1} p.n.\|f\|_{L_\mu^{\varphi*}} \leq \|f\|_{L_\mu^{\varphi_0^*}} \leq p.n.\|f\|_{L_\mu^{\varphi*}} .$$

And we shall prove:

Theorem 1. *Under the same hypothesis as Theorem A, we have*

$$\|Tf\|_{L_\nu^{\varphi_0}} \leq qB\|f\|_{L_\mu^{\varphi_0^*}} ,$$

where $q=\max(p, 2)$ *and* $B=\max(A, 1)$ *with the same constant* A *as in the formula* (1.1).

3. Another results which the author proved in the paper [8, II] is as follows:

Theorem B. *Let us write* $\alpha_i=1/a_i$, $\beta_i=1/b_i$ $(i=1, 2)$. *Let* (α_1, β_1) *and* (α_2, β_2) *be two any points of the triangle*

$$\Delta:\ 0<\beta\leq\alpha\leq 1$$

such that $\alpha_2<\alpha_1$ *and* $\beta_2<\beta_1$.

Let us suppose that a quasi-linear operation T *is of weak type* $(1/\alpha_1, 1/\beta_1)$ *and of type* $(1/\alpha_2, 1/\beta_2)$ *with norms* M_1 *and* M_2 *respectivery.*

Then we have

(3.1) $$\begin{aligned}&\int_{|Tf|\leq 1}|Tf|^{b_2}d\nu+\int_{|Tf|>1}|Tf|^{b_1}d\nu\\ &\leq KM\left\{\left(\int_{|f|\leq 1}|f|^{a_2}du\right)^{b_2/a_2}+\left(\int_{|f|>1}|f|^{a_1}d\mu\right)^{b_2/a_2}\right.\\ &\left.+\left(\int_{|f|>1}|f|^{a_1}(1+b_1\log^+|f|)^{k_1}d\mu\right)^{b_1/a_1}\right\} ,\end{aligned}$$

where $k_1=a_1/b_1$ *and* K *is a constant depending only on* α_1, α_2, β_1, β_2 *and* κ *and not on* M_1, M_2 *and* f *and* $M=\max(M_1, M_2)$.

Let us suppose that

(3.2) $$\int_{|f|\leq 1}|f|^{a_2}d\mu+\int_{|f|>1}|f|^{a_2}(1+b_1\log^+|f|)^{k_1}d\mu<1 ,$$

then the formula (3.1) reads to the following formula

(3.3) $$\begin{aligned}&\int_{|Tf|\leq 1}|Tf|^{b_2}d\nu+\int_{|Tf|>1}|Tf|^{b_1}d\nu\\ &\leq 2KM\left\{\int_{|f|\leq 1}|f|^{a_2}d\mu+\int_{|f|>1}|f|^{a_1}(1+b_1\log^+|f|)^{k_1}d\mu\right\} .\end{aligned}$$

This formula is our starting point. From now on we may assume that $2KM\geq 1$.

Now as in section 2, let us put

(3.4) $$\phi(u)=\begin{cases}u^{b_2} & (0\leq u\leq 1) ,\\ u^{b_1} & (1<u<\infty) ,\end{cases}$$

and

(3.5) $$\psi^*(u)=\begin{cases} u^{a_2} & (0\le u\le 1)\ , \\ u^{a_1}(1+b_1\log^+ u)^{k_1} & (1<u<\infty)\ . \end{cases}$$

Then, the formula (3.3) can he rewritten as follows:

$$\int_Y \psi(|Tf|)d\nu \le 2KM\int_X \psi^*(|f|)d\mu\ ,$$

under the assumption $\int_X \psi^*(|f|)d\mu<1$, and therefore

(3.6)′ $$\int_Y \psi\left(\frac{|Tf|}{\lambda}\right)d\nu \le 2KM\int_X \psi^*\left(\frac{|f|}{\lambda}\right)d\mu\ ,$$

under the assumption $\int_X \psi^*\left(\frac{|f|}{\lambda}\right)d\mu<1$, with $\lambda>0$.

Lemma 2. *As for the functions ψ and ψ^* which are defined by the formulas* (3.4), (3.5.1) *and* (3.5.2) *have the following property. That is, for any constant $\rho\ge 1$, we have*

(3.7) $$\psi(\rho u)\ge \rho\psi(u)\quad (0\le u<\infty)\ ,$$

and

(3.8) $$\psi^*(\rho u)\ge \rho\psi^*(u)\quad (0\le u<\infty)\ .$$

Proof. We may assume that $\rho>1$. In the first case as for $0\le u<\frac{1}{\rho}$, we have

$$\psi^*(\rho u)=(\rho u)^{a_2}>\rho u^{a_2}=\rho\psi^*(u)\ .$$

Next as for $\frac{1}{\rho}\le u\le 1$, we have

$$\psi^*(\rho u)=(\rho u)^{a_1}(1+b_1\log^+\rho u)^{k_1}\ge \rho u^{a_2}=\rho\psi^*(u)\ .$$

In the last as for $1<u<\infty$, we have

$$\begin{aligned}\psi^*(\rho u)&=(\rho u)^{a_1}(1+b_1\log^+\rho u)^{k_1}\\ &>\rho u^{a_1}(1+b_1\log^+ u)^{k_1}=\rho\psi^*(u)\ .\end{aligned}$$

Running on the same lines, we shall prove the case of $\psi(u)$.

By the formula (3.6)′, we have

(3.9) $$E_{\psi^*,1}[f]\subset E_{\psi,2KM}[Tf]\ ,$$

and applying the Lemma 2, we have

(3.10) $$E_{\psi,2KM}[Tf]\subset (2KM)^{-1}E_{\psi,1}[Tf]\ ,$$

and therefore we obtain

(3.11) $$E_{\psi^*,1}[f]\subset (2KM)^{-1}E_{\psi,1}[Tf]\ .$$

Taking the infimum of the above sets, we get

$$p.n.\|Tf\|_{L^{\psi}_{\nu}} \leq \max(2KM, 1)p.n.\|f\|_{L^{\psi^*}_{\mu}} . \tag{3.12}$$

Next we shall introduce the modified functions as follows:

$$\psi_0(u)=\begin{cases} \frac{1}{b_2}u^{b_2} & (0\leq u\leq 1) , \\ \frac{1}{b_1}u^{b_1}-\left(\frac{1}{b_1}-\frac{1}{b_2}\right) & (1<u<\infty) , \end{cases} \tag{3.13}$$

and

$$\psi_0^*(u)=\begin{cases} \frac{1}{a_2}u^{a_2} & (0\leq u\leq 1) , \\ \frac{1}{2a_1}u^{a_1}(1+b_1\log^+ u)^{k_1}-\left(\frac{1}{2a_1}-\frac{1}{a_2}\right) & (1<u<\infty) . \end{cases} \tag{3.14}$$

They are strictly increasing, continuously differentiable and convex functions and satisfy the following properties:

$$b^{-1}\psi(u)\leq\psi_0(u)\leq\psi(u) \quad (0\leq u<\infty) , \tag{3.15}$$

where $\frac{1}{b}=\frac{1}{b_1}-\frac{1}{b_2}$, and

$$a^{-1}\psi^*(u)\leq\psi_0^*(u)\leq\psi^*(u) \quad (0\leq u<\infty) \tag{3.16}$$

where $a=\max(2a_1, a_2)$.

Then applying the Lemma 2, we have

$$E_{\psi,1}[Tf]\subset E_{\psi_0,1}[Tf]\subset b^{-1}E_{\psi,1}[Tf] \tag{3.17}$$

and

$$E_{\psi^*,1}[f]\subset E_{\psi_0^*,1}[f]\subset a^{-1}E_{\psi^*,1}[f] , \tag{3.18}$$

respectively. And by taking the infimum of the above sets, we obtain

$$b^{-1}p.n.\|Tf\|_{L^{\psi}_{\nu}}\leq\|Tf\|_{L^{\psi_0}_{\nu}}\leq p.n.\|Tf\|_{L^{\psi}_{\nu}} , \tag{3.19}$$

and

$$a^{-1}p.n.\|f\|_{L^{\psi^*}_{\mu}}\leq\|f\|_{L^{\psi_0^*}_{\mu}}\leq p.n.\|f\|_{L^{\psi^*}_{\mu}} , \tag{3.20}$$

respectively. And so we obtain the following theorem:

Theorem 2. *Under the same hypothesis as Theorem B, we have*

$$\|Tf\|_{L^{\psi_0}_{\nu}}\leq qB\|f\|_{L^{\psi_0^*}_{\mu}} ,$$

where $q=\max(2a_1, a_2)$, $B=\max(2KM, 1)$, *with the same constant as in the formula* (3.1).

The similar circumferences occure on the theory of extrapolation of operations. Mr. T. Sobukawa [15] intend to the extension of the extrapolation theorem of Yano [17] into that of measure spaces with σ-finite measure. He has obtained the similar results as theorems A and B in this paper and they can be rewrote into the formula by using the word of Orlicz space. He shall treat it on the other paper, which is published in Math. Japonica.

4. The method of proof in the previous section enables us to establish a criterion for the equivalence between norms of Orlicz space without any restricted condition.

We shall prove the following theorem:

Therem 3. *Let φ_0 and φ_1 are strictly increasing, vanish at $u=0$, continuous and convex functions and satisfy the following inequality:*

$$c_1\varphi_1(u)\le\varphi_0(u)\le c_2\varphi_1(u)\quad(0\le u<\infty)\ , \tag{4.1}$$

with constants $0<c_1\le 1\le c_2<\infty$. Then we have

$$c_1\|f\|_{L_\mu^{\varphi_1}}\le\|f\|_{L_\mu^{\varphi_0}}\le c_2\|f\|_{L_\mu^{\varphi_1}}\ . \tag{4.2}$$

Proof. Since φ_0 and φ_1 are convex and vanish at $u=0$, they have the following property.

For any constant $\rho\ge 1$, we have

$$\varphi_i(\rho u)\ge\rho\varphi_i(u)\quad(0\le u<\infty,\ i=0,1)\ . \tag{4.3}$$

We may assume that $\rho>1$. In the case of $\rho=1$, there is nothing to prove. Because $u=\left(1-\frac{1}{\rho}\right)0+\frac{1}{\rho}(\rho u)$, we have

$$\begin{aligned}\varphi_i(u)&=\varphi_i\left(\left(1-\frac{1}{\rho}\right)0+\frac{1}{\rho}\varphi_i(\rho u)\right)\\&\le\left(1-\frac{1}{\rho}\right)\varphi_i(0)+\frac{1}{\rho}\varphi_i(\rho u)=\frac{1}{\rho}\varphi_i(\rho u)\ ,\end{aligned}$$

and so $\rho\varphi_i(u)\le\varphi_i(\rho u)$ $(0\le u<\infty,\ i=0,1)$.

Now by running the same lines as our previous section, we have

$$E_{\varphi_1,1}[f]\subset c_2^{-1}E_{\varphi_0,1}[f]\subset c_1c_2^{-1}E_{\varphi_1,1}[f]\ . \tag{4.4}$$

Then if we take the infimum of the above sets, we obtain

$$\|f\|_{L_\mu^{\varphi_1}}\ge c_2^{-1}\|f\|_{L_\mu^{\varphi_0}}\ge c_1c_2^{-1}\|f\|_{L_\mu^{\varphi_1}}\ ,$$

and so

$$c_1\|f\|_{L_\mu^{\varphi_1}}\le\|f\|_{L_\mu^{\varphi_0}}\le c_2\|f\|_{L_\mu^{\varphi_1}}\ . \tag{4.5}$$

We shall point out that they have also the following property.

For any constant $0<\rho\le 1$, we have

$$\varphi_i(\rho u)\le\rho\varphi_i(u)\quad(0\le u<\infty,\ i=0,1)\ . \tag{4.6}$$

5. Let $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ be points of the n-dimensional Eucledian space $\mathbb{R}^n$, A. P. Calderón–A. Zygmund [1, 2] studied the singular integral operator:

$$\tilde{f}(x)=(f*K)(x)=p.v.\int_{\mathbf{R}^n} K(x-y)f(y)dy \ , \tag{5.1}$$

where the kernel $K(x)$ has the form

$$K(x)=\frac{\Omega(x')}{|x|^n} \ , \qquad x'=\frac{x}{|x|} \ . \tag{5.2}$$

Let us denote by Σ, the unit sphere on which $\Omega(x')$ is defined. Let us denote by $\omega(\delta)$, the modulus of continuity of $\Omega(x')$:

$$|\Omega(x')-\Omega(y')|\leq\omega(x'-y') \ . \tag{5.3}$$

Let us suppose that

(i) $$\int_\Sigma \Omega(x')dx'=0 \ .$$

(ii) $\Omega(x')\in L^1(\Sigma)$ and its modulus of continuity $\omega(\delta)$ satisfy the Dini condition:

$$\int_0^1 \frac{\omega(\delta)}{\delta}d\delta<\infty \ .$$

As a special case, there are the Hilbert transform:

$$Hf(x)=p.v.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy \ , \tag{5.4}$$

and the Riesz transform [5];

$$R_j f(x)=p.v.\frac{1}{c_n}\int_{\mathbf{R}^n} \frac{x_j-y_j}{|x-y|^{n+1}}f(y)dy \ , \tag{5.5}$$

$j=1, \dots, n$ and

$$c_n=\frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \ .$$

The unified operator of the Hilbert transform and ergodic operator due to M. Cotlar [3] belongs to our category. The maximal operator due to G. H. Hardy and J. E. Littlewood [5] does too.

Another one is that of Hardy–Littlewood–Sobolev and they considered the singular integral operator of potential type:

$$\tilde{f}_\lambda(x)=\int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n\lambda}}dy \quad (0<\lambda<1) \ . \tag{5.6}$$

If we write $1<r<s<\infty$, $\frac{1}{r}-\frac{1}{s}=1-\lambda$, then it is proved that this operator is of type (r, s) in the one dimensional case by G. H. Hardy–J. E. Littlewood [4], in the n-dimensional case by S. L. Sobolev [14] respectively. A. Zygmund [19] also proved that it is of weak type $\left(1, \frac{1}{\lambda}\right)$ in the n-dimensional case. It is also refered to the book of C. Sadosky [13].

As for the theory of Orlicz space, it is refered to the book of W. Orlicz [12] and A. C. Zaanen [18].

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