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# ON THE $L^1$ ESTIMATES FOR CERTAIN SINGULAR INTEGRALS

by

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**ABSTRACT**

In this note, we improve the $L^1$ estimates for singular integral operators which were studied by A. P. Calderon and A. Zygmund.

## §1. Introduction

Let $\boldsymbol{R}^n$ be an $n$-dimensional Euclidean space, and let us denote by $\Sigma$ a surface of the sphere of radius 1 with center at the origin. The kernel function $K$ has the form

$$K(x)=\frac{\Omega(x')}{|x|^n} \qquad \text{for} \quad x'=\frac{x}{|x|} ,$$

where $\Omega(x')$ is an integrable function defined on $\Sigma$ and satisfies the following two conditions (1.1) and (1.2):

$$\int_\Sigma \Omega(x')d\sigma=0 , \tag{1.1}$$

where $d\sigma$ is the area element on $\Sigma$,

$$|\Omega(x')-\Omega(y')|\leqq\omega(|x'-y'|) \qquad \text{for all} \quad x', y'\in\Sigma \tag{1.2}$$

and $\omega(t)$ is the increasing function such that $\omega(t)\geqq ct$ ($c$ is some positive constant) and

$$\int_0^1 \omega(t)\frac{dt}{t}=\int_1^\infty \omega\left(\frac{1}{t}\right)\frac{dt}{t}<\infty .$$

Now we define the operators $T_\lambda$, $T$ by

$$T_\lambda f(x)=\int_{\boldsymbol{R}^n} K_\lambda(x-y)f(y)dy ,$$

where

$$K_\lambda(x)=\begin{cases} K(x) & \text{for } |x|\geqq 1/\lambda\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$Tf(x)=\text{p.v.}\int_{R^n} K(x-y)f(y)dy=\lim_{\lambda\to\infty} T_\lambda f(x)\ .$$

In 1952 A. P. Calderon and A. Zygmund [1] have proved the following theorem.

**Theorem A.** *Let $f(x)$ be a function on $\boldsymbol{R}^n$ such that*

$$\int_{R^n} |f(x)|(1+\log^+|f(x)|+\log^+|x-x_0|)dx<\infty \quad (x_0\in \boldsymbol{R}^n)\ . \tag{1.3}$$

*Then for any $\lambda\geqq 1$ the function*

$$\tilde{T}_\lambda f(x)=T_\lambda f(x)-K_1(x-x_0)\int_{R^n} f(x)dx$$

*is integrable on the whole space $\boldsymbol{R}^n$ and*

$$\int_{R^n}|\tilde{T}_\lambda f(x)|dx\leqq C\int_{R^n}|f(x)|(1+\log^+|x-x_0|+\log^+|f(x)|)dx+D\,,$$

*where $C$ and $D$ are constants independent of $\lambda$, $f$ and $x_0$.*

By using the following inequality

$$\begin{aligned}\int_{R^n}&|f(x)|(1+\log^+|x-x_0|+\log^+|f(x)|)dx \\ &\leqq C'\int_{R^n}|f(x)|\log^+[(1+|x-x_0|^{n+1})|f(x)|]dx+D'\,,\end{aligned} \tag{1.4}$$

they have also the following corollary of Theorem A.

**Corollary A.** *The function $\tilde{T}_\lambda f(x)$ in Theorem A satisfies the following inequality*

$$\int_{R^n}|\tilde{T}_\lambda f(x)|dx\leqq C\int_{R^n}|f(x)|\log^+[(1+|x-x_0|^{n+1})|f(x)|]dx+D\,,$$

*where $C$ and $D$ are constants independent of $\lambda$, $f$ and $x_0$.*

Furthermore, by using this corollary, they have proved the following theorem.

**Theorem B.** *Let $f(x)$ be a function on $\boldsymbol{R}^n$ which satisfies the condition* (1.3). *Then $\tilde{T}_\lambda f(x)$ converges in the mean of order* 1 *to a function $\tilde{T}f(x)$ which is integrable on $\boldsymbol{R}^n$, that is,*

$$\lim_{\lambda\to\infty}\|\tilde{T}_\lambda f-\tilde{T}f\|_1=0\ .$$

**Remark 1.** It is well known that the $T_\lambda f$ or $Tf$ themselves can not be integrable on the whole space without the cancellation condition;

$$\int_{R^n} f(x)dx=0 .$$

For example, let us put

$$f(x)=\frac{1}{1+x^2} \quad (x \in \boldsymbol{R}^1) ,$$

and let us define $Tf$ as Hilbert transform;

$$Tf(x)=\text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-t}dt$$
$$=\frac{x}{1+x^2} .$$

Then we have $f \in L^1(\boldsymbol{R}^1)$ but $Tf \notin L^1(\boldsymbol{R}^1)$.

**Remark 2.** Note that for any $x_0, x_1 \in \boldsymbol{R}^n$, the following two conditions (1.5) and (1.6) are equivalent.

$$\int_{R^n}|f(x)|(1+\log^+|f(x)|+\log^+|x-x_0|)dx<\infty , \tag{1.5}$$

$$\int_{R^n}|f(x)|(1+\log^+|f(x)|+\log^+|x-x_1|)dx<\infty . \tag{1.6}$$

This is easily shown as follows: Let $A_0=\{x \in \boldsymbol{R}^n;\ |x-x_0|\leqq|x_0-x_1|\}$, $A_1=\{x \in \boldsymbol{R}^n;\ |x-x_0|>|x_0-x_1|\}$.

$$\int_{R^n}|f(x)|(1+\log^+|x-x_1|)dx \leqq \int_{R^n}|f(x)|[1+\log^+(|x-x_0|+|x_0-x_1|)]dx$$
$$=\int_{A_0}+\int_{A_1}$$
$$\leqq(1+\log^+2|x_0-x_1|+\log 2)\int_{R^n}|f(x)|+(1+\log^+|x-x_0|)dx .$$

Therefore (1.5) implies (1.6). The reverse relation is trivial by the symmetricity of $x_0$ and $x_1$. So (1.5) and (1.6) are equivalent.

**Remark 3.** Let $f(x)$ be a function on $\boldsymbol{R}^n$ and $x_0 \in \boldsymbol{R}^n$. Then the following (1.7)~(1.9) are equivalent conditions of (1.3).

$$\int_{R^n}|f(x)|[\log^+(1+|x-x_0|^{n+1})|f(x)|]dx<\infty , \tag{1.7}$$

$$\int_{R^n}|f(x)|[1+\log^+|f(x)|+\log^+\max(|x-x_0|,\ \|f\|_1^{-1})]dx<\infty \tag{1.8}$$

and

$$(1.9)\qquad \int_{R^n} |f(x)|[1+\log^+\{\max(1,\ |x-x_0|^{(n+1+\delta)},\ \|f\|_1^{-(n+1+\delta)})|f(x)|\}]dx<\infty\ ,$$

for some $\delta>0$.

This is shown as follows;

$$\int_{R^n} |f(x)|[\log^+(1+|x-x_0|^{n+1})|f(x)|dx$$
$$\leqq \int_{R^n} |f(x)|\log^+(1+|x-x_0|^{n+1})dx+\int_{R^n} |f(x)|\log^+|f(x)|dx$$
$$\leqq (n+1)\int_{R^n} |f(x)|(1+\log^+|f(x)|+\log^+|x-x_0|)dx\ .$$

In virtue of this and (1.4), we see that (1.3) and (1.7) are equivalent. By a similar argument, (1.8) and (1.9) are also equivalent. Next we see

$$\int_{R^n} |f(x)|[1+\log^+|f(x)|+\log^+\max(|x-x_0|,\ \|f\|_1^{-1})]dx$$
$$\leqq \int_{R^n} |f(x)|(1+\log^+|f(x)|)dx+\int_{R^n} |f(x)|(\log^+|x-x_0|+\log^+\|f\|_1^{-1})dx$$
$$\leqq \int_{R^n} |f(x)|(1+\log^+|f(x)|+\log^+|x-x_0|)dx+1/e\ .$$

Conversely, it is clear that (1.8) implies (1.3). Consequently (1.3) and (1.8) are equivalent.

In this note, we shall intend to exclude the constant term $D$ of Theorem A.

**Theorem 1.** *Let $f(x)$ be a function on $\boldsymbol{R}^n$ such that* (1.3) *holds. Then for any $\lambda\geqq 1$ $\widetilde{T}_\lambda f(x)$ is integrable on the whole space $\boldsymbol{R}^n$ and the following inequality holds.*

$$\int_{R^n} |\widetilde{T}_\lambda f(x)|dx\leqq C\int_{R^n} |f(x)|[1+\log^+|f(x)|+\log^+\max(|x-x_0|,\ \|f\|_1^{-1})]dx\ ,$$

*where $C$ is a constant independent of $\lambda, f$ and $x_0$.*

**Remark 4.** Since for any non-negative $a, b$ $\log^+\max(a, b)\leqq\log^+ a+\log^+ b\leqq 2\log^+\max(a, b)$, we can represent the consequence of Theorem 1 as follows;

$$\int_{R^n} |\widetilde{T}_\lambda f(x)|dx\leqq C\int_{R^n} |f(x)|[1+\log^+|f(x)|+\log^+|x-x_0|]dx+C\|f\|_1\log^+\|f\|_1^{-1}\ .,$$

where $C$ is a constant independent of $\lambda$, $f$ and $x_0$.

As a corollary of Theorem 1, we have

**Corollary 1.** *The function $\widetilde{T}_\lambda f(x)$ of Theorem* 1 *satisfies the following inequality,*

$$\int_{R^n}|\tilde{T}_\lambda f(x)|dx \leqq C\int_{R^n}|f(x)|[1+\log^+\{\max(1, |x-x_0|^{(n+1+\delta)}, \|f\|_1^{-(n+1+\delta)})|f(x)|\}]dx ,$$

*where $\delta$ is any positive number and $C$ is a constant independent of $\lambda$, $f$ and $x_0$.*

By Theorem B and Remark 3, we get the following corollaries.

**Corollary 2.** *In Theorem* 1 *and Corollary* 1, *we can replace $\tilde{T}_\lambda f$ by $\tilde{T}f$.*

**Corollary 3.** *In Theorem* 1 *and Corollaries* 1 *and* 2, *if we assume in addition the concellation condition*;

$$\int_{R^n} f(x)dx=0 ,$$

*then we can replace $\tilde{T}_\lambda f$ and $\tilde{T}f$ by $T_\lambda f$ and $Tf$ respectively.*

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## § 2. Proof of Theorem 1

Before we prove the Theorem 1, we shall need the following theorem due to A. P. Calderon and A. Zygmund [1].

**Theorem C.** *Let $f(x)$ be function on $R^n$ such that*

$$\int_{R^n}|f(x)|(1+\log^+|f(x)|)dx<\infty .$$

*Then $T_\lambda f(x)$ is integrable over any set $S$ of finite measure and*

$$\int_S |T_\lambda f(x)|dx \leqq C\int_{R^n}|f(x)|[1+\log^+(|S|^{(n+1)/n}|f(x)|)]dx+C|S|^{-1/n} ,$$

*where $C$ is a constant independent of $S$, $\lambda$, and $f$.*

It should be pointed out that starting the Theorem C they have proved Theorem A and others. The method of proof of the Theorem 1 is essentially the same as them but somewhat different.

**Proof of Theorem 1.** For the sake of notation we shall denote any constant by $C$. First we consider the case of $\|f\|_1>1$. For some positive integer $N$ such that $2^{N-1}<\|f\|_1 \leqq 2^N$, let

$$f_0(x)\equiv\begin{cases} f(x) & \text{for } |x-x_0|\leqq 2^N\|f\|_1^{-1} , \\ 0 & \text{otherwise} , \end{cases}$$

$$f_k(x)\equiv\begin{cases} f(x) & \text{for } 2^N\|f\|_1^{-1}2^{k-1}<|x-x_0|\leqq 2^N\|f\|_1^{-1}2^k , \\ 0 & \text{otherwise} , \quad k=1, 2, \dots \end{cases}$$

$$S_k\equiv\{x\in R^n;\ |x-x_0|\leqq 2^N\|f\|_1^{-1}2^{k+1}\} , \qquad k=-1, 0, 1, 2, \dots .$$

Then,

$$|S_k| = C(2^N \|f\|_1^{-1} 2^{k+1})^n .$$

First we shall estimate the integral of $|\widetilde{T}_\lambda f(x)|$ on the set $S_k$. Theorem C gives that for $k \geqq 1$

$$(2.1)\quad \int_{S_k} |T_\lambda f_k(x)|dx \leqq C \int_{R^n} |f_k(x)|[1+\log^+ C(2^N\|f\|_1^{-1}2^{k+1})^{n+1}|f_k(x)|]dx + C2^{-N}\|f\|_1 2^{-k-1}$$
$$\leqq C \int_{R^n} |f_k(x)|[1+\log^+|f_k(x)|+\log^+|x-x_0|]dx + C\|f\|_1 2^{-k-1} ,$$

and for $k=0$

$$(2.2)\quad \int_{S_0} |T_\lambda f_0(x)|dx \leqq C \int_{R^n} |f_0(x)|[1+\log^+ C(2^N\|f\|_1^{-1}2)^{n+1}|f_0(x)|]dx + C2^{-N}\|f\|_1 2^{-1}$$
$$\leqq C \int_{R^n} |f_0(x)|[1+\log^+|f_0(x)|]dx + C\|f\|_1 2^{-1} .$$

On the other hand,

$$\int_{S_k} |K_1(x-x_0)|dx \leqq C \log (2^N\|f\|_1^{-1}2^{k+1}) ,$$

so that for $k \geqq 1$

$$\left| \int_{S_k} K_1(x-x_0)dx \int_{R^n} f_k(y)dy \right| \leqq C \log (2^N\|f\|_1^{-1}2^{k+1}) \int_{R^n} |f_k(y)|dy$$
$$\leqq C \int_{R^n} |f_k(x)|(1+\log^+|x-x_0|)dx ,$$

and for $k=0$

$$\left| \int_{S_0} K_1(x-x_0)dx \int_{R^n} f_0(y)dy \right| \leqq C \log (2^N\|f\|_1^{-1}) \int_{R^n} |f_0(y)|dy$$
$$\leqq C \int_{R^n} |f_0(x)|dx .$$

This, together with the estimate for the integral of $|T_\lambda f_k(x)|$, gives

$$\int_{S_k} |\widetilde{T}_\lambda f_k(x)|dx \leqq C \int_{R^n} |f_k(x)|[1+\log^+|f_k(x)|+\log^+|x-x_0|]dx + C\|f\|_1 2^{-k-1} .$$

Next, we shall estimate the integral of $|\widetilde{T}_\lambda f(x)|$ on the complement of the set $S_k$. Since for $\lambda \geqq 1$ and $|x-x_0| \geqq 1$ we have $K_\lambda(x-x_0) = K(x-x_0)$, and since $f_k(x)$ vanishes outside $S_{k-1}$, we have for $x$ outside $S_k$

$$\widetilde{T}_\lambda f_k(x) = \int_{S_{k-1}} [K(x-y)-K(x-x_0)]f_k(y)dy .$$

Now, on account of the condition (1.2), for every $x$ outside $S_k$ and $y$ inside $S_{k-1}$ the following inequality holds:

$$|K(x-y)-K(x-x_0)|\leqq C\omega(C2^N\|f\|_1^{-1}2^{k+1}|x-x_0|^{-1})|x-x_0|^{-n} .$$

Thus, if $S_k^c$ denotes the complement of $S_k$, we obtain

$$\begin{aligned}(2.3)\qquad \int_{S_k^c}|\tilde{T}_\lambda f_k(x)|dx &\leqq \int_{S_k^c}dx\int_{R^n} C\omega(C2^N\|f\|_1^{-1}2^{k+1}|x-x_0|^{-1})|x-x_0|^{-n}|f_k(y)|dy \\ &\leqq C\int_{R^n}|f_k(y)|dy\int_1^\infty \omega\left(\frac{C}{r}\right)\frac{dr}{r} \\ &\leqq C\int_{R^n}|f_k(y)|dy\end{aligned}$$

and collecting the results we have

$$\int_{R^n}|\tilde{T}_\lambda f_k(x)|dx\leqq C\int_{R^n}|f_k(x)|[1+\log^+|f_k(x)|+\log^+|x-x_0|]dx+C\|f\|_1 2^{-k-1} .$$

Since $\tilde{T}_\lambda f(x)=\sum_{k=0}^{\infty}\tilde{T}_\lambda f_k(x)$, by adding the above inequalities we have

$$(2.4)\qquad \int_{R^n}|\tilde{T}_\lambda f(x)|dx\leqq C\int_{R^n}|f(x)|[1+\log^+|f(x)|+\log^+|x-x_0|]dx .$$

Next, we consider the case of $0<\|f\|_1<1$. Let,

$$f_0(x)\equiv\begin{cases} f(x) & \text{for } |x-x_0|\leqq\|f\|_1^{-1} ,\\ 0 & \text{otherwise} ,\end{cases}$$

$$f_k(x)\equiv\begin{cases} f(x) & \text{for } \|f\|_1^{-1}2^{k-1}\leqq|x-x_0|\leqq\|f\|_1^{-1}2^k ,\\ 0 & \text{otherwise} , \quad k=1, 2, \dots\end{cases}$$

$$S_k\equiv\{x\in \boldsymbol{R}^n;\ |x-x_0|\leqq\|f\|_1^{-1}2^{k+1}\} , \qquad k=-1, 0, 1, 2, \dots .$$

Then,

$$|S_k|=C(\|f\|_1^{-1}2^{k+1})^n .$$

As before, by Theorem C we have (2.1), (2.2). On the other hand,

$$\int_{S_k}|K_1(x-x_0)dx|\leqq C\log(\|f\|_1^{-1}2^{k+1}) ,$$

so that for $k\geqq 1$

$$\left|\int_{S_k}K_1(x-x_0)dx\int_{R^n}f_k(y)dy\right|\leqq C\int_{R^n}|f_k(x)|(1+\log^+|x-x_0|)dx ,$$

and for $k=0$

$$\left|\int_{S_0}K_1(x-x_0)dx\int_{R^n}f_0(y)dy\right|\leqq C\int_{R^n}|f_0(x)|(1+\log^+\|f\|_1^{-1})dx .$$

This, together with the estimate for the integral of $|T_\lambda f_k(x)|$, gives

$$\int_{S_k}|\tilde{T}_\lambda f_k(x)|dx\leqq C\int_{R^n}|f_k(x)|[1+\log^+|f_k(x)|+\log^+|x-x_0|]dx+C\|f\|_1 2^{-k-1}$$

for $k\geqq 1$. And for $k=0$

$$\int_{S_0}|\tilde{T}_\lambda f_0(x)|dx\leqq C\int_{R^n}|f_0(x)|[1+\log^+|f_0(x)|+\log^+\|f\|_1^{-1}]dx+C\|f\|_1 2^{-1}\ .$$

As before, (2.3) holds. So collecting the results we have

$$\int_{R^n}|\tilde{T}_\lambda f_k(x)|dx\leqq C\int_{R^n}|f_k(x)|[1+\log^+|f_k(x)|+\log^+|x-x_0|]dx+C\|f\|_1 2^{-k-1}$$

for $k\geqq 1$. And for $k=0$

$$\int_{R^n}|\tilde{T}_\lambda f_0(x)|dx\leqq C\int_{R^n}|f_0(x)|[1+\log^+|f_0(x)|+\log^+\|f\|_1^{-1}]dx+C\|f\|_1 2^{-1}\ .$$

By adding the above inequalities we have

$$\int_{R^n}|\tilde{T}_\lambda f(x)|dx\leqq C\int_{R^n}|f(x)|[1+\log^+|f(x)|+\chi_{E^c}(x)\log^+|x-x_0|+\chi_E(x)\log^+\|f\|_1^{-1}]dx\ ,$$

where $E=E(f)=\{x\in \boldsymbol{R}^n;\ |x-x_0|\leqq\|f\|_1^{-1}\}$. This and (2.4) complete the proof of theorem.

## § 3. Proof of Corollary 1

Since the functions

$$\Phi(s)=s\log^+\alpha s$$

and

$$\Psi(t)=\begin{cases}t\alpha^{-1} & \text{for } 0\leqq t<1\ ,\\ e^{t-1}\alpha^{-1} & \text{for } 1\leqq t\ ,\end{cases}$$

are conjugate in the sence of Young (cf. [1]). For any positive constant $\delta$ to be fixed, setting $s=|f(x)|$, $\alpha=\max(|x-x_0|^{n+1+\delta},\ \|f\|^{-(n+1+\delta)})$, $t=\delta\log^+\max(|x-x_0|,\ \|f\|_1^{-1})$, for $t\geqq 1$, Young's inequality gives

$$\begin{aligned}\delta|f(x)|\log^+\max(|x-x_0|,\ \|f\|_1^{-1})&\\ \leqq|f(x)|\log^+[\max(|x-x_0|^{n+1+\delta},\ \|f\|_1^{-(n+1+\delta)})|f(x)|]&\\ +[\max(|x-x_0|,\ \|f\|_1^{-1})]^\delta[\max(|x-x_0|^{n+1+\delta},\ \|f\|_1^{-(n+1+\delta)})]^{-1}&\\ \equiv F_1(x)+F_2(x)\ .&\end{aligned}$$

Then for $F_2(x)$ the following inequality holds.

$$\begin{aligned}\int_{R^n}F_2(x)dx&\leqq\int_E\|f\|_1^{n+1}dx+\int_{E^c}|x-x_0|^{-(n+1)}dx\\ &=C\|f\|_1\ ,\end{aligned}$$

where $E=E(f)=\{x\in \boldsymbol{R}^n;\ |x-x_0|\leqq\|f\|_1^{-1}\}$. Therefore this and Theorem 1 complete the proof of corollary.

**References**

[1] A. P. Calderon and A. Zygmund, *On tne existence of certain singular integrals,* Acta. Math. **88** (1952), 85-139.
[2] E. M. Stein, *Singlar Integrals and Differentiability Properties of Functions,* Princeton Univ. Press, Princeton N. J., 1970.
[3] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Space,* Princeton Univ. Press, Princeton N. J., 1971.