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Abstract	In this note, we shall propose a new measurement-axiom in nonrelativistic quantum theory, which can be considered a kind of generalized Copenhagen interpretation. This interpretation can assert rather radical statements, which may irritate our minds familiar with Copenhagen interpretation. We shall, under this interpretation, assert the following two statements: (i) the uncertainties (i.e. Δq and Δp) in Heisenberg's uncertainty relation "Δq· Δp≥■/2" can be characterized as (average) errors in the approximates simultaneous measurement, and so, Heisenberg's uncertainty relation can be clarified as the statement in physics for the first time, (ii) the (discrete) trajectory of a particle is enough significant (though this, of course, includes errors). Furthermore, concerning (ii) we shall show the numerical results of the trajectory of a particle in two slit experiment. (Though almost all parts of this note are composed of summaries of references [11], [12], [13] and [14], we add some discussions from other angles.)
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HEISENBERG'S UNCERTAINTY RELATION AND A NEW MEASUREMENT-AXIOM IN QUANTUM THEORY[(1)]

by

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ABSTRACT

In this note, we shall propose a new measurement-axiom in nonrelativistic quantum theory, which can be considered a kind of generalized Copenhagen interpretation. This interpretation can assert rather radical statements, which may irritate our minds familiar with Copenhagen interpretation. We shall, under this interpretation, assert the following two statements:

(i) the uncertainties (i.e. Δq and Δp) in Heisenberg's uncertainty relation "$\Delta q \cdot \Delta p \geq \hbar/2$" can be characterized as (average) errors in the approximates simultaneous measurement, and so, Heisenberg's uncertainty relation can be clarified as the statement in physics for the first time,

(ii) the (discrete) trajectory of a particle is enough significant (though this, of course, includes errors).

Furthermore, concerning (ii) we shall show the numerical results of the trajectory of a particle in two slit experiment.

(Though almost all parts of this note are composed of summaries of references [11], [12], [13] and [14], we add some discussions from other angles.)

1. Introduction

There are two kinds of uncertainty relations for a position and a momentum (see, for example, [19]). One is statistical uncertainty relation. That is, let Q and P be a position observable and a momentum observable respectively (i.e. Q and P are self-adjoint operators on a Hilbert space H satisfying that $QP-PQ=i\hbar$), and let u be a state (i.e. $u \in H$, $\|u\|=1$). By repeating the exact (i.e. the uncertainty $\Delta(q)=0$) measurements of the position q of particles with same states u, we can obtain its average value $\bar{q}$ and its variance $var(q)$. Also, by repeating

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the exact (i.e. the uncertainty $\Delta(p)=0$) measurements of the momentum p of the particles with same states u, we can similarly get its average value $\bar{p}$ and its variance $var(p)$. Of course we know that $\bar{q}=\langle u, Qu\rangle$, $\bar{p}=\langle u, Pu\rangle$, $var(q)=\|(Q-\bar{q})u\|^2$ and $var(p)=\|(P-\bar{p})u\|^2$. From this and a simple calculation, we can easily obtain the following uncertainty relation

$$[var(q)]^{1/2}\cdot[var(p)]^{1/2}\geq\frac{\hbar}{2}\,. \tag{1}$$

This is called the statistical uncertainty relation.

Another is the individualistic uncertainty relation, which was discovered by Heisenberg in 1927 using the famous thought experiment of γ-ray microscope. He asserted as follows:

(i) *The particle position q and momentum p can be measured "simultaneously", if the uncertainties $\Delta(q)$ and $\Delta(p)$ in determining the particle position and momentum are permitted to be non-zero. Moreover, for any $\varepsilon>0$, we can take the "simultaneous" measurement of the position q and momentum p such that $\Delta(q)<\varepsilon$(or $\Delta(p)<\varepsilon$).*

(ii) *However, the following Heisenberg's uncertainty relation holds:*

$$\Delta(q)\cdot\Delta(p)\geq\frac{\hbar}{2}\,, \tag{2}$$

for all "simultaneous" measurements of the particle position and momentum.

We shall call it Heisenberg's uncertainty relation in this note.

Most physicists seem to confuse unconsciously Heisenberg's uncertainty relation with statistical uncertainty relation. However, if they are asked about this difference formally, their answer should be as follows:

(1) *and* (2) *are clearly different statements. And none has yet given the theoretical foundation to Heisenberg's uncertainty relation* (2).

The purpose of this note is to propose an interpretation (new measurement-axiom) of quantum theory, which offers the foundation to Heisenberg's uncertainty relation (2). This note is constructed as follows[(2)]:

§2. Mathematical foundation of Heisenberg's uncertainty relations
§3. Discussions of the results obtained in §2 and some examples
§4. Proposal of new measurement-axiom of quantum theory
§5. An application of new measurement-axiom (the analysis of the trajectories of a particle)
§6. Its numerical result (two slit experiment)
§7. Conclusions
§8. Appendix

In §2, we shall give the mathematical (but temporary) foundation to Heisenberg's uncertainty relations (2) within Copenhagen interpretation. In §3, we discuss the results obtained in §2. And we conclude that the mathematical results

(2) In translation, we add §9 "Note added in translation".

obtained in § 2 are not able to be made completely clear in physics, as far as we are staying within Copenhagen interpretation. Section 4 is a main section in this note, in which we propose a new measurement-axiom (new interpretation) of quantum mechanics. The spirit of this axiom is summarized as follows:

By measurement, a "true" value is not only produced but also destroyed!

This can be considered a kind of generalizations of Copenhagen interpretation. Furthermore, we give the mathematical foundation to Heisenberg's uncertainty relations (2) within the new axiom (not within Copenhagen interpretation). And we conclude that the uncertainty $\Delta(q)$ (or $\Delta(p)$) in (2) can be characterized as the (average) error in measurement. This implies that Heisenberg's uncertainty relation (2) can, for the first time, be mentioned as the statement in physics. In § 5, we shall analyze the problem of the trajectories of a particle within our new axiom. In § 6, we shall exhibit the numerical results of the two slit experiment, by using the analysis developed in § 5. In § 7, the conclusions of this note will be mentioned. Thinking that physical plainness is prior to mathematical strictness, we write this note. So, we add some mathematical complements to § 8 Appendix. Also, § 2 and § 3 (resp. § 4, § 5 and § 8; resp. § 6) are chiefly due to [11] and [12] (resp. [13]; resp. [14]).

2. Mathematical foundations to Heisenberg's uncertainty relations

In this section, we shall give the mathematical (but temporary) foundation to Heisenberg's uncertainty relations (2) within Copenhagen interpretation. Note that giving the mathematical foundation to Heisenberg's uncertainty relations (2) is simultaneously equal to giving the physical foundation to Heisenberg's uncertainty relations (2). That is, giving the mathematical foundation to Heisenberg's uncertainty relations (2), we can make clear that his assertions in the previous § 1 is very ambiguous and can be interpretated in various senses, so Heisenberg's assertion is hard to be recognized as the statement in physics.

We begin the following definition.

Definition 1 (approximate simultaneous measurement in averge sense). *Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_H$. Let $A_0, A_1, \cdots, A_{N-1}$ be any physical quantities (i.e. self-adjoint operators) in a Hilbert space H. A triplet $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1, \dots, \hat{A}_{N-1}))$ is called an approximate simultaneous measurement (in average sense) of $\{A_k\}_{k=0}^{N-1}$ in H, if it satisfies the following conditions* (i)~(iii):

(i) *v is an element in a Hilbert space K such that $\|v\|_K=1$, and $\hat{A}_0, \hat{A}_1, \dots, \hat{A}_{N-1}$ are commutative self-adjoint operators in a tensor Hilbert space $H\otimes K$,*

(ii) *for each k, a set $D_v(\hat{A}_k)$ $(\equiv\{u\in H\colon u\otimes v\in D(\hat{A}_k)$, the domain of $A_k\})$ is a core of A_k, i.e. A_k is essentially telf-adjoint on $D_v(\hat{A}_k)$,*

(iii) *for each k, $\langle u, A_k u\rangle_H=\langle u\otimes v, \hat{A}_k(u\otimes v)\rangle_{H\otimes K}(u\in D_v(\hat{A}_k))$.*

Remark 1. In the following section, we shall consider the "approximate" simultaneous measurement $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1, \dots, \hat{A}_{N-1}))$ that is not satisfying the conditions (ii) and (iii). In this case, we shall call it an approximate simultaneous measurement in some sense.

Remark 2. Now we shall explain the meanings of the definition 1 (also see ref. [8], [10]). Assume that we hope to measure simultaneously the exact values $a_0, a_1, \ldots, a_{N-1}$ of observables $A_0, A_1, \ldots, A_{N-1}$ for a particle with the state u. It is natural to consider that its expectation $\bar{a}=(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-1})$ becomes to be $\bar{a}=(\langle u, A_0 u\rangle_H, \langle u, A_1 u\rangle_H, \ldots, \langle u, A_{N-1}u\rangle_H)$. However, it is impossible to obtain the exact simultaneous measurement-value $a=(a_0, a_1, \ldots, a_{N-1})$ of observables $(A_0, A_1, \ldots, A_{N-1})$ for a particle with the state u since we do not assume that observables $A_0, A_1, \ldots, A_{N-1}$ commute. Under Copenhagen interpretation (conventional quantum theory), it is nonsense to consider the exact simultaneous measurement-value $a=(a_0, a_1, \ldots, a_{N-1})$. So, we prepare another Hilbert space K and its unit vector v as in Definition 1. And we make the simultaneous measurement of commutative observables $(\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{N-1})$ for a state $u\otimes v$ in a tensor Hilbert space $H\otimes K$. Of course, it is possible for the commutativity of $(\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{N-1})$ (ref. [15, von Neumann]). When we get the simultaneous measurement-value $x=(x_0, x_1, \ldots, x_{N-1})$ of $(\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{N-1})$ for a state $u\otimes v$ in a tensor Hilbert space $H\otimes K$, we shall regard its simultaneous measurement-value $x=(x_0, x_1, \ldots, x_{N-1})$ as the substitute of $a=(a_0, a_1, \ldots, a_{N-1})$. Now we have, from (iii) in Definition 1, that the expectation $\bar{x}=(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1})=(\langle u\otimes v, \hat{A}_0(u\otimes v)\rangle_{H\otimes K}, \ldots, \langle u\otimes v, \hat{A}_{N-1}(u\otimes v)\rangle_{H\otimes K})=(\langle u, A_0 u\rangle_H, \langle u, A_1 u\rangle_H, \ldots, \langle u, A_{N-1}u\rangle_H)=\bar{a}$. This implies that the substitute $x=(x_0, x_1, \ldots, x_{N-1})$ is good in average sense. This is the meaning of Definition 1 (approximate simultaneous measurement in average sense).

Definition 2 (uncertainty). *Let H be a Hilbert space with the inner product $\langle\cdot, \cdot\rangle_H$. Let $A_0, A_1, \ldots, A_{N-1}$ be any physical quantities (i.e. self-adjoint operators) in a Hilbert space H. Let a triplet $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{N-1}))$ be an approximate simultaneous measurement (in average sense) of $\{A_k\}_{k=0}^{N-1}$ in H. Then, the uncertainty $\{\Delta_{\boldsymbol{M}}(A_k, u)\colon k=0, \ldots, N-1\}$ of an approximate simultaneous measurement (in average sense) $\boldsymbol{M}$ for $\{A_k\}_{k=0}^{N-1}$ on a state u ($\|u\|_H=1$) is defined by*

$$\Delta_{\boldsymbol{M}}(A_k, u)\equiv\begin{cases}\|[\hat{A}_k-A_k\otimes I](u\otimes v)\|_{H\otimes K} & (\text{if } u\in D_v(\hat{A}_k))\\ \infty & (\text{if } u\in D(A_k)\backslash D_v(\hat{A}_k))\end{cases}. \tag{3}$$

Note that $D_v(\hat{A}_k)\subseteq D(A_k)$ (see [12]). Though the uncertainty $\Delta_{\boldsymbol{M}}(A_k, u)$ is not defined for $u\notin D(A_k)$, this problem will be naturally solved in § 4.

Remark 3. The problem "what is the uncertainty?" seems to happen. We shall give one answer in Remark 7. And another answer will be proposed in § 4. This will be one of main assertions in this note. In § 3, the uncertainty $\{\Delta_{\boldsymbol{M}}(A_k, u)\colon k=0, \ldots, N-1\}$ of an approximate simultaneous measurement in some sense $\boldsymbol{M}$ is also defined as in Definition 1.

Under these preparations, we have the following theorems.

Theorem 1 (existence). *Let $A_1, \ldots, A_n$ be self-adjoint operators in a Hilbert space H. Let $a_1, \ldots, a_n$ be any positive numbers such that $\sum_{i=1}^{n}(1+a_i^2)^{-1}=1$. Then, there exists an approximate simultaneous measurement (in average sense) $\boldsymbol{M}$ of*

$A_1, \ldots, A_n$ *such that*

$$\Delta_M(A_i, u)=a_i\|A_iu\|_H \quad (u \in D(A_i) \quad i=1, 2, \ldots, n)\ . \tag{4}$$

Remark 4. Theorem 1 without condition (4) was independently proved in [1] and [11]. Theorem 1 was proved in [12], which seems to correspond to the first part (i) of Heisenberg's assertion. That is, for any fixed k, we can take a_k sufficiently small. So, for this k, we can take an approximate simultaneous measurement (in average sense) $\boldsymbol{M}$ such that its uncertainty $\Delta_M(A_k, u)$ is sufficiently small.

Theorem 2 (uncertainty relations). *Let A_0 and A_1 be any self-adjoint operators in a Hilbert space H. Then, for any approximate simultaneous measurement (in average sense) $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1))$ of $(\hat{A}_0, \hat{A}_1)$ and for any $u \in D(A_0)\cap D(A_1)$ ($\|u\|_H=1$), the following inequalities hold:*

(i) (*approximate simultaneous uncertainty relation*)

$$[var(\hat{A}_0, u)]^{1/2}\cdot[var(\hat{A}_1, u)]^{1/2}\geq|\langle A_0u, A_1u\rangle-\langle A_1u, A_0u\rangle| \tag{5}$$

where $var[\hat{A}_k, u]=\|(\hat{A}_k-\langle u\otimes v, \hat{A}_k(u\otimes v)\rangle_{H\otimes K})(u\otimes v)\|^2_{H\otimes K} \quad (k=0, 1)$,

(ii) (*Heisenberg's uncertainty relation*)

$$\Delta_M(A_1, u)\cdot\Delta_M(A_2, u)\geq|\langle A_0u, A_1u\rangle-\langle A_1u, A_0u,\rangle|/2\ . \tag{6}$$

Remark 5. Of course, we have already another uncertainty relation that corresponds to the stochastic uncertainty relation mentioned in § 1 as (2): that is,

$$\|(A_0-\langle u, A_0u\rangle)u\|_H\cdot\|(A_1-\langle u, A_1u\rangle)u\|_H\geq|\langle A_0u, A_1u\rangle-\langle A_1u, A_0u\rangle|/2\ . \tag{7}$$

Therefore, we have now three uncertainty relations after all.

Remark 6. Inequality (5) means the uncertainty relation concerning the variances of measurement-values for the approximate simultaneous measurement (in average sense) $\boldsymbol{M}$. Also, inequality (6) seems to be the mathematical representation of Heisenberg's uncertainty relation. Of course, it is necessary to examine that (6) is just Heisenberg's uncertainty relation from various view-points. This is one of main themes of this note.

Remark 7. We mention the physical meaning of uncertainty $\Delta_M(A_k, u)$, though this is temporary and will be investigated deeply in § 4. For the approximate simultaneous measurement (in average sense) $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_{N-1}))$, the following equality holds (see [12]):

$$[var(A_k, u)]^2+|\Delta_M(A_k, u)|^2=[var(\hat{A}_k, u)]^2 \tag{8}$$

(note that this equality does not hold for the approximate simultaneous measurement (in some sense)). $[var(A_k, u)]^2$ (resp. $[var(\hat{A}_k, u)]^2$) can be known by repeating the exact measurements of the observable A_k (resp. $\hat{A}_k$) for the state u (resp. $u\otimes v$). So, if we regard the above equality (8) as the definition of uncertainty $\Delta_M(A_k, u)$, the physical meaning of uncertainty $\Delta_M(A_k, u)$ is significant. This is one answer of the physical meaning of uncertainty $\Delta_M(A_k, u)$. However, this answer is not applied to the approximate simultaneous measurement (in some sense). So, this physical meaning seems to be not enough. We shall propose another answer in § 4, which is one of main assertions of this note.

Remark 8. Inequality (5) in Theorem 2 was independently shown in [4] and [11]. Also, inequality (6) in Theorem 2 was independently shown in [12] and [16]. Examining the statement in the proof of (5) in [4] and [11], we can find that (5) is derived from (6), (8) and (7). Therefore, it may be said that (6) is shown in [4] and [11] though (6) is not mentioned explicitly. However, it seems that in [4] and [11] there is not the important recognition that (6) is just Heisenberg's uncertainty relation. We think that this recognition can be obtained without the observations in the following sections 3 and 4. Until now, we think that (6) is only the mathematical analogy of Heisenberg's uncertainty relation (2).

3. Discussions of the results obtained in § 2 and some examples

In this section, we shall firstly summarize the arguments in the previous sections. And next, we shall exhibit some examples for the preparation of the following section 4. Our purpose is to obtain simultaneously the exact measurement-values a_0, a_1 of observables A_0, A_1 for a particle with the state u in a Hilbert space H. However, it is impossible in general. So, we prepare another Hilbert space K and its unit vector v as in Definition 1. And we proceed the simultaneous measurement of commutative observables $(\hat{A}_0, \hat{A}_1)$ for a state $u \otimes v$ in a tensor Hilbert space $H \otimes K$. Of course, it is possible from the commutativity of $\hat{A}_0$ and $\hat{A}_1$. We call it an approximate simultaneous measurement (in average sense) $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1))$ of $\{A_k\}_{k=0}^1$. Furthermore, we define the uncertainty $\{\Delta_{\boldsymbol{M}}(A_k, u): k=0, 1\}$ of an approximate simultaneous measurement (in average sense) $\boldsymbol{M}$ for $\{A_k\}_{k=0}^1$ on a state $u(\|u\|_H=1)$ is defined by

$$\Delta_{\boldsymbol{M}}(A_k, u) \equiv \|[\hat{A}_k - A_k \otimes I](u \otimes v)\|_{H \otimes K} . \tag{3}$$

Then we have the Heisenberg's uncertainty relation

$$\Delta_{\boldsymbol{M}}(A_1, u) \cdot \Delta_{\boldsymbol{M}}(A_2, u) \geq |\langle A_0 u, A_1 u \rangle - \langle A_1 u, A_0 u \rangle|/2 . \tag{6}$$

The physical (but temporary) meaning of uncertainty $\Delta_{\boldsymbol{M}}(A_k, u)$ is given as the following equality:

$$[var(A_k, u)]^2 + |\Delta_{\boldsymbol{M}}(A_k, u)|^2 = [var(\hat{A}_k, u)]^2 \tag{8}$$

(note that this equality does not hold for the approximate simultaneous measurement (in some sense)). This is a summary of the previous section.

Now we shall discuss some examples $\boldsymbol{M}=(K, v, (\hat{A}_0, \hat{A}_1))$ of $\{A_k\}_{k=0}^1$, which are not approximate simultaneous measurements (in average sense) but approximate simultaneous measurements (in some sense) (see Remark 1 and 3). And we shall show that the uncertainty $\Delta_{\boldsymbol{M}}(A_k, u)$ is enough significant for even approximate simultaneous measurements (in some sense). And these arguments will be the preparations for the following section 4. Throughout this section, we shall consider the simple case, that is, $K=\boldsymbol{C}$ (complex field), $v=1$. Therefore, we use the following identifications:

$$H\otimes \boldsymbol{C}\Leftrightarrow H\,,\qquad u\otimes 1\Leftrightarrow u$$

and

$$\hat{A}_k:\ H\otimes \boldsymbol{C}\to H\otimes \boldsymbol{C}\Leftrightarrow \hat{A}_k:\ H\to H\,. \tag{9}$$

Example 1 (EPR and Heisenberg's uncertainty relation). Consider the classical two-particle system such that a particle S_0 and a particle S_1 move on one dimensional Euclidean space $\boldsymbol{R}$. Let $q_0(t)$ and $p_0(t)$ [resp. $q_1(t)$ and $p_1(t)$] be a position and a momentum of the particle S_0 [resp. particle S_1] at time t respectively. Assume that

$$q_0(t)-q_1(t)=a\,,\qquad p_0(t)-p_1(t)=b\quad (a \text{ and } b \text{ are constant})\,. \tag{10}$$

In [9], Einstein, Podolsky and Rosen investigated this simple system and pointed out that this system (or its extension to quantum system) has very interesting properties. Though the relation of this system and Heisenberg's uncertainty relation was not mentioned in their paper (Einstein did not believe Heisenberg's uncertainty relation?), many books of physics say concerning this relation as follows:

It is possible to measure the position $q_0(t)$ of the particle S_0 and $p_1(t)$ of the particle S_1. Then, we can know by (10) *that $p_0(t)=b-p_1(t)$. So, a position and a momentum of the particle S_0 at time t is $q_0(t)$ and $p_0(t)$ exactly. This result contradicts Heisenberg's uncertainty relation. This is a paradox!*

Though this paradox is very simple, it is hard to solve this contradiction. Now we shall show that this contradiction arises from the ambiguities of Heisenberg's uncertainty relation (2) (ref. [13]).

Let us extend this classic system to a quantum system as follows. We can put $H=L^2(\boldsymbol{R}^2)$ since EPR system is two-particles system. From the condition (10), the state $u(q_0, q_1)(\in H=L^2(\boldsymbol{R}^2))$ of this system is represented by

$$\frac{1}{\sqrt{\hbar\pi}}\exp\left[-\frac{(q_0-q_1-a)^2}{4\hbar\varepsilon^2}-\frac{\varepsilon^2(q_0+q_1-q)^2}{4\hbar}-\frac{aq_0-aq_1+bq_0+bq_1}{2\hbar i}\right] \tag{11}$$

where ε is sufficiently small. Our purpose is to measure simultaneously the following two observables

$$A_0=q_0\,,\qquad A_1=\frac{\hbar\partial}{i\partial q_0} \tag{12}$$

for the state u. Of course, it is impossible for the non-commutativity of A_0 and A_1. So, we consider the following approximate simultaneous measurement (in some sense) $\boldsymbol{M}=(\boldsymbol{C}, 1, (\hat{A}_0, A_1))$ under the identification as (9):

$$\hat{A}_0=q_0\,,\qquad \hat{A}_1=b-\frac{\hbar\partial}{i\partial q_1} \tag{13}$$

(note that this corresponds to the above classical case of EPR). Then, we have, from a simple calculation

$$\Delta_M(A_0, u) = \|A_0 u\| = 0 , \qquad \Delta_M(A_1, u) = \|A_1 u\| = \sqrt{2}\varepsilon . \tag{14}$$

Since $\varepsilon(>0)$ can be taken sufficiently small, the uncertainties $\Delta_M(A_0, u)$ and $\Delta_M(A_1, u)$ can be also taken sufficiently small.

This conclusion seems to be as same as that of the above classical EPR argument. And the Heisenberg's uncertainty relation seems not to hold. However, this result does not confuse us, since we now know the Heisenberg's uncertainty relation (6) (not (2)). That is, the above approximate simultaneous measurement (in some sense) $\boldsymbol{M}=(\boldsymbol{C}, 1, (\hat{A}_0, \hat{A}_1))$ is not an approximate simultaneous measurement (in average sense), so the above conclusion does not contradicts the Heisenberg's uncertainty relation (6). All confusions arises from the ambiguity of the statement (2). The reason to say in the first part of § 2, "Heisenberg's assertion is hard to be recognized as the statement in physics" is that the statement (2) is too ambiguous to be applied to such a simple example.

Remark 9. Of course, the problem "what is uncertainty?" is still unsolved. Saying more, this problem is now getting more important.

Example 2 (a very simple measurement that seems to break Heisenberg's uncertainty relation (2)). Let A_0, A_1 be non-commutative observables in a Hilbert space H. Consider the following approximate simultaneous measurement (in some sense) $\boldsymbol{M}$:

> *The measurement of the observable A_0 is not made, and we make it a rule that the measurement-value of the observable A_0 is always* 0. *And we measure exactly the observable A_1. Therefore, if the exact measurement-value of the observable A_1 is a_1, then the exact measurement-value of this approximate simultaneous measurement* (*in some sense*) $\boldsymbol{M}$ *is pointed* $(0, a_1)$.

That is, this approximate simultaneous measurement (in some sense) $\boldsymbol{M}=(\boldsymbol{C}, 1, (\hat{A}_0, \hat{A}_1))$ is, under the identification (9), represented as follows:

$$\hat{A}_0 = 0 , \qquad \hat{A}_1 = A_1 . \tag{15}$$

Then, we easily see that

$$\Delta_M(A_0, u) = \|A_0 u\| , \qquad \Delta_M(A_1, u) = 0 , \tag{16}$$

which clearly seems to contradicts Heisenberg's uncertainty relation (2). However, this is compatible with Heisenberg's uncertainty relation (6).

Remark 10. This example 2 seems to be interesting, though it is too simple. In this example 2, we give up the measurement of the observable A_0 (if we take the measurement of the obserqable A_0, we may get a_0 as the exact measurement-value of the observable A_0), and we always assign it 0. So, the error (concerning the observable A_0) of this approximate measurement is considered as [variance of $|a_0-0|]^{1/2}$. Clearly this is equal to the uncertainty $\Delta_M(A_0, u)=\|A_0 u\|$. This seems to show the possibility that the uncertainty $\Delta_M(A_0, u)$ can be characterized as the error, though these arguments are prohibited under the Copenhagen interpretation. In the following section, we devote ourselves to give the justification to these arguments.

4. Proposal of a new measurement-axiom of quantum mechanics

Heisenberg's uncertainty relation (2) has been connected with Copenhagen interpretation. We think that Bohr (and his school) was constructing Copenhagen interpretation based on Heisenberg's uncertainty relation (2). So, we think that the ambiguities of Heisenberg's uncertainty relation (2) is inherited to Copenhagen interpretation. The philosophical difficulties of Copenhagen interpretation seems to be caused by this process. If we set up the start point Heisenberg's uncertainty relation (6), all arguments become clear. For example, so called "complementary principle" is nothing but Heisenberg's uncertainty relation (6). Of course, we have still the problem: "what is uncertainty?".

In this section, we shall propose a new measurement axiom of quantum mechanics, which give the answer to the problem "what is uncertainty?". All arguments in this section are due to [13].

Now we shall prepare some definitions.

Let V be a Hilbert space. A projection valued probability space $(X, \mathscr{F}, F)$ in a Hilbert space V is defined such as it satisfies that

(i) $(X, \mathscr{F})$ is a measurable space,

(ii) for every $\Xi \in \mathscr{F}$, $F(\Xi)$ is a projection in V such that $F(\emptyset)=0$ and $F(X)=I$, where 0 is a 0-operator and I is an identity operator in V,

(iii) for any countable decomposition $\{\Xi_j\}_{j=1}^{\infty}$ of Ξ, $(\Xi_j, \Xi \in \mathscr{F})$, $F(\Xi)=\sum_{j=1}^{\infty} F(\Xi_j)$ holds where the series is weakly convergent

and

(iv) $\mathscr{F}$ can be generated by a countable family $\{\Xi_i^0 | \Xi_i^0 \subseteq X, i=1, 2, \dots\}$.

Remark 11. In this note, we use the projection valued probability spaces as the mathematical model of observables. Note that any self-adjoint operator A in V has the unique spectral representation $A=\int_{\boldsymbol{R}} \lambda E_A(d\lambda)$, then we sometimes consider the identification that $A \Leftrightarrow (\boldsymbol{R}, \mathscr{B}, E_A)$ (where $\mathscr{B}$ is a Borel field on $\boldsymbol{R}$).

Remark 12. Projection valued probability spaces (or more generally, positive operator valued probability spaces) are investigated in [7], in which Davies called a positive operator valued probability space an observable. We do not use a positive operator valued probability space in this note since we are interested in fundamental properties of quantum mechanics. However, it is easy to generalize our arguments to this direction.

As the preparation to mention our new measurement-axiom, we shall now study the conventional measurement-axiom.

Axiom 0 (Born's probabilistic interpretation and Copenhagen interpretation). *Let ψ be a state of a system S in a Hilbert space V (i.e. $\psi \in V$, $\|\psi\|_V=1$ and let $(X, \mathscr{F}, F)$ be an observable in V. Consider the measurement of the observable $(X, \mathscr{F}, F)$ for this system S. Then,*

(i) *the probability that $x_0(\in X)$, the measurement-value obtained by the measurement of the observable $(X, \mathscr{F}, F)$ for this system S, belongs to a set*

$\Xi(\in\mathscr{F})$ is given by $\langle\psi, F(\Xi)\psi\rangle_V$.

(ii) *When we get a measurement-value $x_0(\in X)$, for the observable $(Y', \mathscr{G}, G)$ such that $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ are ψ-commutative (i.e. $G(\Gamma)F(\Xi)\psi=F(\Xi)G(\Gamma)\psi$, $(\forall\Xi\in\mathscr{F}, \forall\Gamma\in\mathscr{G})$, the probability $\mu_\psi(x_0, \Gamma, (X, \mathscr{F}, F), (Y, \mathscr{G}, G))$ that the "true" value $\hat{y}_0(\in Y)$ of the observable $(Y, \mathscr{G}, G)$ for this system S belongs to $\Gamma(\in\mathscr{G})$ is given by*

$$\mu_\psi(x_0, \Gamma, (X, \mathscr{F}, F), (Y, \mathscr{G}, G))=\lim_{\substack{\Xi\to\{x_0\}\\ \mathscr{F}\ni\Xi\ni x_0}}\frac{\langle\psi, F(\Xi)G(\Gamma)\psi\rangle_V}{\langle\psi, F(\Xi)\psi\rangle_V}. \tag{17}$$

Moreover, we can not say anything, if $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ are not ψ-commutative.

Remark 13. We shall mention the mathematical definition of the conditional probability $\mu_\psi(x_0, \Gamma, (X, \mathscr{F}, F), (Y, \mathscr{G}, G))$ in §8 "Appendix". Since (17) is a symbolical representation, we will add some comments. If X is a finite set and $\langle\psi, F(\Xi)\psi\rangle_V\neq 0$ $(\forall\Xi(\in\mathscr{F}), \Xi\neq\emptyset)$, (17) is equal to

$$\mu_\psi(x_0, \Gamma, (X, \mathscr{F}, F), (Y, \mathscr{G}, G))=\frac{\langle\psi, F(\cap_{x_0\in\Xi\in\mathscr{F}}\Xi)G(\Gamma)\psi\rangle_V}{\langle\psi, F(\cap_{x_0\in\Xi\in\mathscr{F}}\Xi)\psi\rangle_V}.$$

All arguments after this can be understood, if the readers think so.

Remark 14. In this note, (i) and (ii) in Axiom 0 is called Born's probabilistic interpretation and Copenhagen interpretation respectively (though there may exist various "Copenhagen interpretations"). It should be noted that Copenhagen interpretation must be read as

By measurement, a "true" value is produced!

And it is prohibited to consider that a "true" value exists before measurement. This is a Copenhagen spirit. Therefore, a "true" value is not a true value in a classical mechanics. Also, it should be noted that Axiom 0 (ii) can be assured by experiment, that is, if we repeat the measurement of the observable $(Y, \mathscr{G}, G)$ after the measurement of the observable $(X, \mathscr{F}, F)$, we can examine Axiom 0 (ii). Saying contrarily, this is the physical meaning of "true" value.

Our purpose is to propose the new measurement-axiom (Axiom 1 (ii)), which is a natural extension of Axiom 0 (ii). For this, we shall consider the following example (EPR for spin).

Example 3 (EPR for spin). We consider a system of two particles (Particle 1 and particle 2) with singlet state (concerning z-axis). That is, $V=\boldsymbol{C}^2\otimes\boldsymbol{C}^2$, $\psi(\in V)$ is a singlet state, i.e. $\psi=\frac{1}{\sqrt{2}}(|\uparrow_{1z}\rangle\otimes|\downarrow_{2z}\rangle-|\downarrow_{1z}\rangle\otimes|\uparrow_{2z}\rangle)$, where $|\uparrow_{1z}\rangle=|\uparrow_{2z}\rangle=\begin{bmatrix}1\\0\end{bmatrix}(\in\boldsymbol{C}^2)$ and $|\downarrow_{1z}\rangle=|\downarrow_{2z}\rangle=\begin{bmatrix}0\\1\end{bmatrix}(\in\boldsymbol{C}^2)$.

Let $(Y, \mathscr{G}, G)$ be the observable concerning the z-axis spin of the particle 2. That is, $Y=\{\uparrow_{2z}, \downarrow_{2z}\}$, $\mathscr{G}=2^Y$ and

$$G(\{\uparrow_{1z}\})=\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&0\end{bmatrix}, \qquad E(\{\downarrow_{1z}\})=\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

We shall consider the following (I) and (II) cases. (I) is an easy exercise within Copenhagen interpretation (Axiom 0 (ii)), and (II) is an investigation without Axiom 0 (ii).

(I) Let $(X, \mathcal{F}, F)$ be the observable concerning the z-axis spin of the particle 1. That is, $X=\{\uparrow_{1z}, \downarrow_{1z}\}$, $\mathcal{F}=2^X$ and

$$F(\{\uparrow_{1z}\})=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad E(\{\downarrow_{1z}\})=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

We shall consider the following problem:

when we take a measurement of the observable $(X, \mathcal{F}, F)$ for the state ψ, how is the "true" value of $(Y, \mathcal{G}, G)$ produced?

Since $(X, \mathcal{F}, F)$ and $(Y, \mathcal{G}, G)$ are commutative, we can apply Axiom 0 (ii) to this problem and we obtain, by a simple calculation,

$$\begin{aligned}&\mu_\psi(\uparrow_{1z}, \{\uparrow_{2z}\})=\mu_\psi(\downarrow_{1z}, \{\downarrow_{2z}\})=0\\ &\mu_\psi(\downarrow_{1z}, \{\uparrow_{2z}\})=\mu_\psi(\uparrow_{1z}, \{\downarrow_{2z}\})=1\,.\end{aligned} \tag{18}$$

This simple example will be a preparation to consider the following problem.

(II) Let $(X', \mathcal{F}', F')$ be the observable concerning the x-axis spin of the particle 2. That is, $X'=\{\uparrow_{2x}, \downarrow_{2x}\}$, $\mathcal{F}=2^{X'}$ and

$$F'(\{\uparrow_{1z}\})=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}, \qquad E(\{\downarrow_{1z}\})=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix}.$$

Now we take a measurement of the z-axis spin of the particle 1 and a measurement of the x-axis spin of the particle 2. This is, of course, equal to a simultaneous measurement of the observable $(X, \mathcal{F}, F)$ and the observable $(X', \mathcal{F}', F')$ (for the commutativity, this is possible). Also, this is equal to the measurement of the observable $(\bar{X}, \bar{\mathcal{F}}, \bar{F})$, where $\bar{X}=X\times X'$, $\bar{\mathcal{F}}=2^{\bar{X}}$ and

$$\begin{gathered}\bar{\mathcal{F}}(\{(\omega_i, \omega'_j)\})=F(\{\omega_i\})F'(\{\omega'_j\}) \quad (i, j=1, 2)\\ (\text{where } \omega_1=\uparrow_{1z},\ \omega_2=\downarrow_{1z},\ \omega'_1=\uparrow_{2x},\ \omega'_2=\downarrow_{2x})\end{gathered}$$

We shall consider the following problem:

when we take a measurement of the observable $(\bar{X}, \bar{\mathcal{F}}, \bar{F})$ for the state ψ, how is the "true" value of $(Y, \mathcal{G}, G)$ produced?

Since $(\bar{X}, \bar{\mathcal{F}}, \bar{F})$ and $(Y, \mathcal{G}, G)$ are not ψ-commutative, we can not apply Axiom 0 (ii) to this problem. That is, this problem is nonsense within Copenhagen interpretation. However, we consider this in what follows. Since the measurement of the observable $(\bar{X}, \bar{\mathcal{F}}, \bar{F})$ is equivalent to the measurements of the z-axis spin of the particle 1 and the x-axis spin of the particle 2, we can take the measurement of the observable $(\bar{X}, \bar{\mathcal{F}}, \bar{F})$ as follows:

SPEP (i) *firstly (at time t_1), we take a measurement of the observable $(X, \mathcal{F}, F)$ for the singlet state ψ,* STEP (ii) *secondly (at time $t_2 (t_1 \wedge t_2)$), we take a measurement of the observable $(X', \mathscr{F}', F')$.*

Like this, we divide the measurement of the observable $(\bar{X}, \bar{\mathscr{F}}, \bar{F})$ into STEP (i) and STEP (ii). Note that STEP (i) is the same as the above Problem I. Therefore, we can say that the "true" value of $(Y, \mathscr{G}, G)$ for this singlet state ψ is, at time t_1, produced as (18). We think that this "true" value of $(Y, \mathscr{G}, G)$ is, at time t_2, destroyed by the measurement in STEP (ii). So, if we take the measurement of the observable $(Y, \mathscr{G}, G)$ after STEP (ii) (at time t_3 $(t_2 < t_3)$), we can not know the "true" value of $(Y, \mathscr{G}, G)$ produced in STEP (i). If the above investigation is accepted, the spirit of Copenhagen interpretation is extended as follows:

By measurement, a "true" value is not only produced but also destroyed!

This is all of our idea.

Remark 15. In the above investigation, we, for convenience' sake, use the concept of time. However, axiom 0 does not include the concept of time. Therefore, we do not touch the problem at what time the measurement is taken or how long the measurement is taken. Our standing point in this note is that these problems are not within quantum theory (at least quantum theory that we know at present).

Now we shall propose the new measurement-axiom in nonrelativistic quantum theory, which can be considered to be a kind of generalized Copenhagen interpretation.

Let ψ be a state of a system S in a Hilbert space V and let $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ be observables in V. Put $P_{(F,\psi)} \equiv$"the projection on a smallest closed subspace that contains $\{F(\Xi)\psi | \Xi \in \mathscr{F}\}$". Put $\mathscr{F}^{\psi}_{(Y,\mathscr{G},G)} = \{\Xi \in \mathscr{F} | G(\Gamma)F(\Xi)P_{(F,\psi)} = F(\Xi)G(\Gamma)P_{(F,\psi)} (\forall \Gamma \in \mathscr{G})\}$[3]. It is clear that $\mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}$ is a σ-subfield of $\mathscr{F}$ and $(X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, F)$ and $(Y, \mathscr{G}, G)$ commute with respect to ψ. Also, it is clear that $\emptyset \in \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}$ and $X \in \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}$, so $\mathscr{F}^{\psi}_{(Y,\mathscr{G},G)} \neq \emptyset$.

Now we have the following main Axiom. We think that it is sufficient in physics, but we will add the mathematical complement in § 8.

Axiom 1 (Born's probabilistic interpretation and generalized Copenhagen interpretation). *Let ψ be a state of a system S in a Hilbert space V and let $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ be observables in V. Consider the measurement of the observable $(X, \mathscr{F}, F)$ for this system S. Then,*

(i) *the probability that $x_0 (\in X)$, the measurement-value obtained by the measurement of the observable $(X, \mathscr{F}, F)$ for this system S, belongs to a set $\Xi (\in \mathscr{F})$ is given by $\langle \psi, F(\Xi)\psi \rangle_V$.*

(ii) *When we get a measurement-value $x_0 (\in X)$ by the measurement of the observable $(X, \mathscr{F}, F)$ for this system S, the probability that $y_0 (\in Y)$, the "true" value of the observable $(Y, \mathscr{G}, G)$ for this system S, belongs to a set $\Gamma (\in \mathscr{G})$ is given by $\mu_\psi(x_0, \Gamma, (X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G)))$, where*

(3) In § 9 "Note added in translation", we shall show another version of $\mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}$.

$$\mu_\psi(x_0, \Gamma, (X, \mathscr{F}^\psi_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G)) = \lim_{\substack{\Xi \to \{x_0\} \\ \mathscr{F}^\psi_{(Y,\mathscr{G},G)} \ni \Xi \ni x_0}} \frac{\langle \psi, F(\Xi)G(\Gamma)\psi \rangle_V}{\langle \psi, F(\Xi)\psi \rangle_V}. \quad (19)$$

Remark 16. We think that the readers can easily understand the meaning of this axiom, if the investigation in Example 3 (EPR for spin) is accepted. Repeatedly saying, our fundamental spirit is

By measurement, a "true" value is not only produced but also destroyed!

And the "true" value of $(Y, \mathscr{G}, G)$ for this singlet state ψ, produced by measurement, is represented by (19). Since this "true" value of $(Y, \mathscr{G}, G)$ is also destroyed by this measurement of $(X, \mathscr{F}, F)$, we have no device to know it in general.

Remark 17. We think that Axiom 1 (ii) must satisfy the following two conditions (Ci) and (Cii), in order that it is widely accepted.

(Ci) *there exists no experiment that denies Axiom* 1 (ii)
(Cii) *more events are able to be explained under Axiom* 1 (ii) *than under Axiom* 0 (ii).

We have the "proof" of (Ci), if Axiom 0 (ii) is accepted. We have no definition of "true" value of $(Y, \mathscr{G}, G)$ in the case that $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ are not ψ-commutative. So, if we regard Axiom 1 (ii) as the definition of "true" value of $(Y, \mathscr{G}, G)$, (Ci) has no contradiction. Of course, we are never satisfied with this "proof". And we believe that Axiom 1 (ii) would be assured by experiment. However, there may be another possibility that no experiment exists that assures Axiom 1 (ii).

Remark 18. Our purpose from now is to examine how Axiom 1 (ii) satisfies the condition (Cii). As mentioned in ABSTRACT, we can assert as follows:

(i) *the uncertainties* (*i.e.* Δq *and* Δp) *in Heisenberg's uncertainty relation* "$\Delta q \cdot \Delta p \geq \hbar/2$" *can be characterized as* (*average*) *errors in the approximate simultaneous measurement, so, Heisenberg's uncertainty relation can be firstly understood as the statement in physics,*
(ii) *the* (*discrete*) *trajectory of a particle is enough significant* (*though this, of course, includes errors*).

Of course, we should examine the condition (Cii) from various view points. However, we, in this note, conclude from the above (i) and (ii) that Axiom 1 (ii) satisfies the condition (Cii) enough.

Definition 3 (approximate simultaneous measurement in average sense (or in some sense)). *Let H be a Hilbert space with the inner product* $\langle \cdot, \cdot \rangle_H$. *Let* $A_0, A_1, \ldots, A_{N-1}$ *be any physical quantities* (*i.e. self-adjoint operators*) *in a Hilbert space H. A quartet* $\boldsymbol{M} = (K, v, (X, \mathscr{F}, F), f = (f_1, \ldots, f_n))$ *is called an approximate simultaneous measurement* (*in average sense*) *of* $\{A_k\}_{k=0}^{N-1}$ *in H, if it satisfies the following conditions* (i)~(iii):

(i) *v is an element in a Hilbert space K such that* $\|v\|_K = 1$, *and* $(X, \mathscr{F}, F)$

is a projection valued probability space in a tensor Hilbert space $H\otimes K$. *And* f: $X\to \boldsymbol{R}^N$ *is measurable.*

(ii) *Put* $\hat{A}_k=\int_X f_k(x)F(dx)$ $(k=1, 2, \ldots, N-1)$. *Then, for each* k, *a set* $D_v(\hat{A}_k)$ $(\equiv\{u\in H\colon u\otimes v\in D(\hat{A}_k)$, *the domain of* $\hat{A}_k\})$ *is a core of* A_k, *i.e.* A_k *is essentially self-adjoint on* $D_v(\hat{A}_k)$,

(iii) *for each* k, $\langle u, A_k u\rangle_H=\langle u\otimes v, \hat{A}_k(u\otimes v)\rangle_{H\otimes K}(u\in D_v(\hat{A}_k))$.

Also, $\boldsymbol{M}=(K, v, (X, \mathscr{F}, F), f=(f_1, \ldots, f_n))$ *is called an approximate simultaneous measurement* (*in some sense*) *of* $\{A_k\}_{k=0}^{N-1}$ *in* H, *if it satisfies the condition* (i).

Remark 19. Of course, this is essentially equivalent to Definition 1. For sake to read the following Definition 4, we shall explain the meaning of this definition. Our purpose is to measure simultaneously the exact values $a_0, a_1, \ldots, a_{N-1}$ of observables $A_0, A_1, \ldots, A_{N-1}$ for a particle with the state u. It is natural to consider that its expectation $\bar{a}=(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-1})$ becomes to be $\bar{a}=(\langle u, A_0 u\rangle_H, \langle u, A_1 u\rangle_H, \ldots, \langle u, A_{N-1}u\rangle_H)$. However, it is impossible to measure the exact simultaneous measurement-value $a=(a_0, a_1, \ldots, a_{N-1})$ of observables $(A_0, A_1, \ldots, A_{N-1})$ for a particle with the state u since we do not assume that observables $A_0, A_1, \ldots, A_{N-1}$ commute in general, Under the Copenhagen interpretation, it is nonsense to consider the exact simultaneous measurement-value $a=(a_0, a_1, \ldots, a_{N-1})$. So, we prepare another Hilbert space K and its unit vector v as in Definition 3. And we proceed the measurement of $(X, \mathscr{F}, F)$ for a state $u\otimes v$ in a tensor Hilbert space $H\otimes K$. When we get the simultaneous measurement-value $x(\in X)$ of the observable $(X, \mathscr{F}, F)$ for a state $u\otimes v$ in a tensor Hilbert space $H\otimes K$, we shall regard $f(x)=(f_0(x), f_1(x), \ldots, f_{N-1}(x))$ as the substitute of $a=(a_0, a_1, \ldots, a_{N1})$. Since the probability that x belongs to $\Xi(\in\mathscr{F})$ is $\langle u\otimes v, F(\Xi)(u\otimes v)\rangle$, the expectation $\bar{f}=(\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{N-1})$ of $f(x)=(f_0(x), f_1(x), \ldots, f_{N-1}(x))$ is represented by $\bar{f}_k=\int_X f_k(x)\langle u\otimes w, F(dx)(u\otimes v)\rangle=\langle u\otimes v, \hat{A}_k(u\otimes v)\rangle_{H\otimes K}$. This implies, from (iii), that $\bar{a}_k=\langle u, A_k u\rangle_H=\langle u\otimes v, \hat{A}_k(u\otimes v)\rangle_{H\otimes K}=\bar{f}_k$. By this mean, $\boldsymbol{M}=(K, v, (X, \mathscr{F}, F), f=(f_1, \ldots, f_n))$ is called an approximate simultaneous measurement (in average sense) of $\{A_k\}_{k=0}^{N-1}$ in H.

If Axiom 1 is accepted, the following definition seems to be natural.

Definition 4 ((average) error). *Let* H *be a Hilbert space with the inner product* $\langle\cdot, \cdot\rangle_H$. *Let* $A_0, A_1, \ldots, A_{N-1}$ *be any physical quantities* (*i.e. self-adjoint operators*) *in a Hilbert space* H. *Let* $\boldsymbol{M}=(K, v, (X, \mathscr{F}, F), f=(f_1, \ldots, f_n))$ *be an approximate simultaneous measurement* (*in some sense*) *of* $\{A_k\}_{k=0}^{N-1}$ *in* H. *Then,*

(i) $\delta_{\boldsymbol{M}}(A_k, u; x)$, *the k-th error when we get* x *by the measurement* $\boldsymbol{M}$ (*i.e. the measurement of the observable* $(X, \mathscr{F}, F)$) *with respect to a state* $u(\in H)$, *is defined by*

$$\delta_{\boldsymbol{M}}(A_k, u;\ x)=\left[\int_{\boldsymbol{R}} |f_k(x)-\xi|^2\mu_{u\otimes v}(x, d\xi)\right]^{1/2} \tag{20}$$

where $\mu_{u\otimes v}(x, \Gamma)\in CP(u\otimes v;\ (X, \mathscr{F}^{u\otimes v}_{A_k\otimes I}, F), A_k\otimes I)$.

(ii) $\bar{\delta}_{\boldsymbol{M}}(A_k, u)$, *the k-th average error in the measurement* $\boldsymbol{M}$ *with respect to a state* $u(\in H)$, *is defined by*

$$\bar{\delta}_{\boldsymbol{M}}(A_k, u)=\left[\int_X |\delta_{\boldsymbol{M}}(A_k, u;\ x)|^2\langle u\otimes v,\ F(dx)(u\otimes v)\rangle\right]^{1/2}. \tag{21}$$

Also, $\{\bar{\bar{\delta}}_M(A_k, u), | k=0, 1, \ldots, N-1\}$ *is called an average error in the measurement* $\boldsymbol{M}$ *with respect to a state* u.

Under these preparations, we have the following theorem.

Theorem 3 (the relation between uncertainty and average error). *Let* $A_0, A_1, \ldots, A_{N-1}$ *be any physical quantities* (*i.e. self-adjoint operators*) *in a Hilbert space H. Let* $\boldsymbol{M}=(K, v, (X, \mathscr{F}, F), f=(f_0, f_1, \ldots, f_{N-1}))$ *be an approximate simultaneous measurement* (*in some sense*) *of* $A_0, A_1, \ldots, A_{N-1}$ *in* H. *Assume that* $\hat{A}_k(=\int_X f_k(x)F(dx))$ *and* $A_k \otimes I$ *commute for each k. Then, the following equality holds*:

$$\Delta_M(A_k, u)=\bar{\bar{\delta}}_M(A_k, u) \quad (k=0, 1, \ldots, N-1, \ \forall u \in H, \ \|u\|_H=1) .$$

Remark 20. This was proved in [13] for the approximate simultaneous measurement in average sense. For the approximate simultaneous measurement in some sense (see Remark 3), the proof in [13] can be applied automatically. From this Theorem, we can, under Axiom 1, characterize the uncertainty of measurement as its average error. If we, from this viewpoint, think Heisenberg's uncertainty relation and the examples in the previous section, all arguments (in particular, EPR in Example 1) seem to become clear.

5. An application of the new measurement-axiom (analysis of trajectories of a particle)

It is well-known that the concept of the trajectories of a particle is prohibited in conventional quantum theory. This fact is often stressed in contrast to classical mechanics. However, it takes time for the meaning of this fact to be registered in our mind. For example, the trajectories of "Wilson chamber" seem to be natural in our common sense. If the interpretation proposed in the previous section is accepted, the (discrete) trajectories of a particle are enough significant (though this, of course, includes errors). As a typical example, we shall analyze a trajectory of a particle under this interpretation. It will be done by developing the nice idea in [4] and [14]. This section is due to [13] and [14].

We shall consider a particle S in one dimensional real line $\boldsymbol{R}$, whose state function $u(t, \cdot)(\in H \equiv L^2(\boldsymbol{R}), -\infty<t<\infty)$ satisfies the following Schrödinger equation with a Hamiltonian $\mathscr{H}=-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}$:

$$i\hbar \frac{\partial u(t, x)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2 u(t, x)}{\partial x^2}=(\mathscr{H}u)(t) \quad (-\infty<t<\infty) , \tag{22}$$
$$u(0, x)=u(x) .$$

Put $\theta>0$ and $N \geq 2$ (integer). Let A be a position observable in H, that is, $(Au)(x)=xu(x)$.

Now we consider the approximate "simultaneous" measurement $\boldsymbol{M}$ of the positions of a particle S at time $t_k=\theta k$, $(k=0, 1, 2, \ldots, N-1)$. Note that (22) is equivalent to the following Heisenberg's kinetic equation of the time evolution of the observable A_t $(-\infty<t<\infty)$ in a Hilbert space H with a Hamiltonian $\mathscr{H}$

$$-i\hbar \frac{dA_t}{dt} = \mathscr{H}A_t - A_t\mathscr{H} \quad (-\infty < t < \infty) , \qquad (23)$$
$$A_0 = A .$$

Therefore we can consider that the measurement $\boldsymbol{M}$ is equivalent to the approximate simultaneous measurement of self-adjoint operators $\{A_{\theta k}\}_{k=0}^{N-1}$ for a particle S with a state $u(x)=u(0, x)$. An easy calculation shows that

$$A_t = U_{-t}AU_t = U_{-t}xU_t = x + \frac{\hbar t}{im}\frac{d}{dx}$$

where one parameter unitary group $U_t(=e^{-i\hbar^{-1}\mathscr{H}t})$ is represented by

$$(U_t u)(x) = u(t, x) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[\frac{im}{2\hbar t}(x-\xi)^2\right] u(\xi) d\xi .$$

Here we see easily that

$$A_t A_s - A_s A_t = \frac{\hbar}{im}(t-s) \quad (t, s \in \boldsymbol{R}) . \qquad (24)$$

Let $V = H \otimes K = H \otimes (\otimes_{k=1}^{N-1} H) = \otimes_{k=0}^{N-1} H = L^2(\boldsymbol{R}^N)$ and $\hat{U}_t = \otimes_{k=0}^{N-1} U_t$, that is, for all $\phi \in L^2(\boldsymbol{R}^N)$,

$$\hat{U}_t \phi(x_0, x_1, \ldots, x_{N-1})$$
$$= \left(\frac{m}{2\pi i\hbar t}\right)^{N/2} \int_{\boldsymbol{R}^N} \exp\left[\frac{im}{2\hbar t} \sum_{k=0}^{N} |x_k - \xi_k|^2\right] \phi(\xi_0, \ldots, \xi_{N-1}) d\xi_0, \ldots, d\xi_{N-1} .$$

Let α_{kn} $(k, n=0, 1, \ldots, N-1)$ be real numbers such that $\sum_{n=0}^{N-1} \alpha_{kn}\alpha_{ln} = 0$ $(k \neq l)$ and $\alpha_{k0} = 1 (\forall k)$. Define self-adjoint operators $\hat{A}_{\theta k}$ $(k=0, 1, \ldots, N-1)$ in $V(\equiv L^2(\boldsymbol{R}^N))$ by

$$A_{\theta k} = \sum_{n=0}^{N-1} \alpha_{kn}\left(x_n + \frac{\hbar\theta k}{im}\frac{\partial}{\partial x_n}\right) . \qquad (25)$$

It is clear that $\hat{A}_{\theta k}$ $(k=0, 1, 2, \ldots, N-1)$ commute. Also, for each k $(k=0, 1, 2, \ldots, N-1)$, $\hat{A}_{\theta k}$ and $A_{\theta k} \otimes I\left(\equiv x_0 + \frac{\hbar\theta k}{im}\frac{\partial}{\partial x_0}\right)$ commute. We see, by (24), that

$$\hat{A}_{\theta k} = \hat{U}_{-\theta k}\left(\sum_{n=0}^{N-1} \alpha_{kn} x_n\right) \hat{U}_{\theta k} , \quad A_{\theta k} \otimes I = \hat{U}_{-\theta k} x_0 \hat{U}_{\theta k} . \qquad (26)$$

Then, the spectral measure $\hat{E}_{\theta k}$ of $\hat{A}_{\theta k}$ (*i.e.* $\hat{A}_{\theta k} = \int_{\boldsymbol{R}} \lambda \hat{E}_{\theta k}(d\lambda)$) is represented by

$$\hat{E}_{\theta k}(\Xi) = \hat{U}_{-\theta k} \chi\left(\Xi; \sum_{n=0}^{N-1} \alpha_{kn} x_n\right) \hat{U}_{\theta k} \quad (\forall \Xi \in \mathscr{B}) \qquad (27)$$

where $\chi(\Xi; y) = 1 (y \in \Xi)$, $=0$ $(y \notin \Xi)$, i.e. a characteristic function of Ξ. From the commutativity of $\{\hat{E}_{\theta k}\}_{k=0}^{N-1}$ (i.e. $\{\hat{A}_{\theta k}\}_{k=0}^{N-1}$), we can define an observable $(X, \mathscr{F}, F) = \boldsymbol{R}^N, \mathscr{B}^N, F)$ in V where

$$F(\Xi_0 \times \Xi_1 \times \cdots \times \Xi_{N-1}) = \prod_{k=0}^{N-1} \hat{E}_{\theta k}(\Xi_k) . \qquad (28)$$

Put $u(x_0)=u_0(x_0)$ and $v(x_1, \dots, x_{N-1})=v_1(x_1), \dots, v_{N-1}(x_{N-1}) \in L^2(\boldsymbol{R}^{N-1})(\equiv K)$ $(\|v\|_K=1)$ such that

$$\int_{\boldsymbol{R}} x_k|v_k(x_k)|^2 dx_k=\int_{\boldsymbol{R}} \bar{v}_k(x_k) \frac{dv(x_k)}{dx_k} dx_k=0 \quad (k=1, 2, \dots, N-1) . \tag{29}$$

Put f_k: $X(\equiv \boldsymbol{R}^N)\to \boldsymbol{R}$ $(k=0, 1, \dots, N-1)$ such that $f_k(x_0, \dots, x_{N-1})=x_k$. Note that $\hat{A}_{\theta k}=\int_X f_k(x)F(dx)$.

Now we can easily show that $\boldsymbol{M}=(K, v, (X, \mathscr{F}, F), f=(f_0, \dots, f_{N-1}))$ defined above is an approximate simultaneous measurement of $\{A_{\theta k}\}_{k=0}^{N-1}$ in H.

Note that the probability that the measurement-value $x=(x_0, \dots, x_{N-1})$ obtained by the measurement $\boldsymbol{M}$ belongs to a set $\Xi_0\times\Xi_1\times\cdots\times\Xi_{N-1}$ is given by

$$\langle u\otimes v, \textstyle\prod_{k=0}^{N-1} \hat{E}_{\theta k}(\Xi_k)(u\otimes v)\rangle_{H\otimes K} . \tag{30}$$

Of course, this measurement-value $x=(x_0, \dots, x_{N-1})$ is representing just the discrete trajectory of a particle S, though it includes errors.

Also, when we get x by this measurement $\boldsymbol{M}$ with respect to a state $u(\in H)$, the expectation $\bar{x}_k$ of "true" value $\hat{x}_k$ of $A_{\theta k}\otimes I$ $(k=0, 1, \dots, N-1)$ is given by

$$\bar{x}_k(x)=\int_{\boldsymbol{R}} \xi \mu_{u\otimes v}(x, d\xi, (X, \mathscr{F}^{u\otimes v}_{(Y,\mathscr{G},G)}, F), A_{\theta k}\otimes I) . \tag{31}$$

Also, its variance $\sigma^2=(\sigma_0^2, \dots, \sigma_{N-1}^2)$ is give by

$$\sigma_k^2(x)=\int_{\boldsymbol{R}} |\bar{x}_k-\xi|^2\mu_{u\otimes v}(x, d\xi, (X, \mathscr{F}^{u\otimes v}_{(Y,\mathscr{G},G)}, F), A_{\theta k}\otimes I). \tag{32}$$

6. Numerical results (two slit problem)

In this section, we shall apply the analysis of the previous section to two slit problem, and show numerical results. For simplicity, we put $\hbar/m=1$. The initial condition $u(0, x)$ is set up as follows:

$$u(0, x)=1/\sqrt{2}\,(x\in(-3/2, -1/2)\cup(1/2, 3/2)) , \quad =0 \text{ (otherwise)} . \tag{33}$$

Fix $T=\theta(N-1)=1/4$. We consider the case that $N=2$ (i.e. $\theta=1/4$). Put

$$v_1(x_1)=[\omega/\pi]^{1/4} \exp\left[-\frac{\omega|x_1|^2}{2}\right] . \tag{34}$$

Then, we see

$$\bar{\delta}_{\boldsymbol{M}}(A_0, u)=|\alpha_{01}|\left[\int_{\boldsymbol{R}} |x_1 v_1(x_1)|^2 dx_1\right]^{1/2}=|\alpha_{01}||2\omega|^{-1/2} \tag{35}$$

$$\begin{aligned}\bar{\delta}_{\boldsymbol{M}}(A_\theta, u)&=|\alpha_{11}|\left[\int_{\boldsymbol{R}} \left|\left(x_1-i\theta\frac{\partial}{\partial x_1}\right)v_1(x_1)\right|^2 dx_1\right]^{1/2}\\ &=|\alpha_{11}|\left|\frac{1}{2\omega}+\frac{\omega|\theta|^2}{2}\right|^{1/2} .\end{aligned} \tag{36}$$

We hope to make $\bar{\bar{\delta}}_M(A_0, u)$ small under the condition that $\bar{\bar{\delta}}_M(A_0, u)=\bar{\bar{\delta}}_M(A_\theta, u)$. From (24) and Heisenberg's uncertainty relation (6), this is roughly realized, if we put

$$\alpha_{00}=1\ , \quad \alpha_{10}=1\ , \quad \alpha_{01}=2\ , \quad \alpha_{11}=-1/2\ , \quad \omega=16\ ,$$

and so,

$$\bar{\bar{\delta}}_M(A_0, u)\approx\bar{\bar{\delta}}_M(A_\theta, u)\approx 0.359\cdots\approx 1/\sqrt{8}=0.353\cdots .$$

For each k ($k=0, 1$), we put

$$\Xi_k^i=[-3+(i-1)/4,\ -3+(i/4)]\ , \quad (i=1, 2, \ldots, 24)\ .$$

Figure 1 shows the numerical result of (30). In Figure 1, we connect Ξ_0^i and Ξ_1^j by $[r+(1/2)]$ lines, where $[\cdot]$ is Gauss symbol and

$$\langle u\otimes v_1, \textstyle\prod_{k=0}^{N-1} \hat{E}_{\theta k}(\Xi_k)(u\otimes v_1)\rangle_{H\otimes K}=r/100\ . \tag{39}$$

Therefore, Figure 1 represents the joint distribution of the measurement-values at $t=0$ and $t=1/4$. The readers should read this figure 1 under the following notices:

(i) *the average error between the measurement value and "true" value is about* 0.359. *Therefore, we have sometimes the measurement value outside of slit at* $t=0$.

(ii) *the "true" value is produced by measurement* $\boldsymbol{M}$. *So, this is not classical true value.*

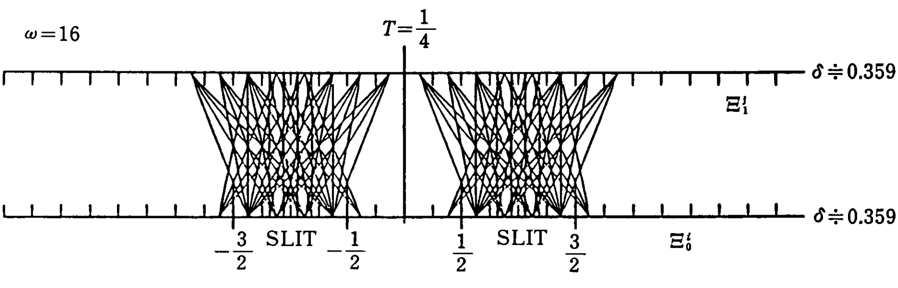

Figure 1

We can not only calculate (30) but also (31) and (32). These results are now in preparation. We shall show these results in [14].

7. Conclusions

In this note, we proposed a new measurement-axiom (Axiom 1 in § 4). The fundamental spirit of this axiom is summarized as follows:

By measurement, a "true" value is not only produced but also destroyed!

And a produced "true" value is given by (19). This axiom can be considered a kind of generalized Copenhagen interpretation. This can assert rather radical statements, which may irritate our minds familiar with Copenhagen interpretation. In this note, we assert the following two statements by this interpretation:

(i) *the uncertainties (i.e. Δq and Δp) in Heisenberg's uncertainty relation "$\Delta q \cdot \Delta p \geq \hbar/2$" can be characterized as (average) errors in the approximate simultaneous measurement, so, Heisenberg's uncertainty relation can be clarified as the statement in physics for the first time.*

(ii) *the (discrete) trajectory of a particle is enough significant. Though this, of course, includes errors, we can analyze trajectories of a particle numerically.*

Axiom 1 seems to be so powerful that it gives clear solutions to fundamental problems of quantum mechanics. And no experiment exists that denies Axiom 1 (see Remark 17). In contrast with conventional Copenhagen interpretation, we think that Axiom 1 is more consistent in theoretical aspect. Of course, to examine Axiom 1 by experiment is most important. We believe that it is possible, though we have no idea to do it at present. Another possibility is that Axiom 1 may be connected with philosophical problems as mentioned in Remark 17. However, it is not clever to form a hasty conclusion for these kinds of problems. We should be making efforts to examine Axiom 1 by experiment. For this, we must build its priority from theoretical aspects. In applied aspects, we should be giving actual results (for example, the proposal of measurement instruments that have special kinds of properrties under Axiom 1). In all cases discussions from various view points are necessary.

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8. Appendix

In mathematics, we must prepare the following definition for conditional probability:

Definition 5. *Let ϕ be a state in a Hilbert space V. Let $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ be observables in V. Let $(X, \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}, F)$ be the observable as defined in* § 4. *A set $CP(\phi; (X, \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G))$ of all conditional probability μ_ϕ's ($\mu_\phi(x, \Gamma)$, or precisely $\mu_\phi(x, \Gamma: (X, \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G))$) is defined that satisfies the following conditions* (i), (ii) *and* (iii):

(i) *for each $\Gamma(\in \mathscr{G})$, $\mu_\phi(x, \Gamma)$* is *$\mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}$-measurable as a function of x and $0 \le \mu_\phi(x, \Gamma) \le 1$,*

(ii) *for each $x(\in X)$, $\mu_\phi(x, \cdot)$ is a probability measure on $(Y, \mathscr{G})$*

and

(iii) *for each $\mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}$-measurable function f: $X \to \boldsymbol{R}$ and each $\Gamma \in \mathscr{G}$,*

$$\int_X f(x)\langle\phi, F(dx)G(\Gamma)\phi\rangle_V = \int_X f(x)\mu_\phi(x, \Gamma)\langle\phi, F(dx)\phi\rangle_V \,. \tag{40}$$

Remark 21. This $\mu_\phi(x, \Gamma: (X, \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G))$ defined in Definition 5 was symbolically written as

$$\mu_\phi(x, \Gamma, (X, \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G)) = \lim_{\substack{\Xi \to \{x\} \\ \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)} \ni \Xi \ni x}} \frac{\langle\phi, F(\Xi)G(\Gamma)\phi\rangle_V}{\langle\phi, F(\Xi))\phi\rangle_V} \,. \tag{41}$$

Remark 22. Notice that the existence of μ_ϕ (i.e. $CP(\phi; (X, \mathscr{F}^{\phi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G)) \neq \emptyset$) is a well-known fact in probability theory under some conditions (for example, Y is a complete separable metric space and $\mathscr{G}$ is its Borel field (see, for example, Ash [6])).

Remark 23. Also, the uniqueness in the following sense is assured:

(24) *if* $\mu_1, \mu_2 \in CP(\psi;\ (X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G))$, *then there exists a null set* $N \in \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}$ *(i.e.* $\langle\psi, F(N)\psi\rangle_V = 0$*) such that* $\mu_1(x, \Gamma) = \mu_2(x, \Gamma)$ *for all* $x \in X - N$ *and* $\Gamma \in \mathscr{G}$.

For the proof, see [13]. Roughly speaking, the set $CP(\psi;\ (X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G))$ can be considered to be composed of "one" element. Note that Axiom 1 in § 4 has been mentioned on this premise.

Now we can mention Axiom 1 (in § 4) in mathematics as follows:

Axiom 1 (the mathematical representation of Axiom 1 in § 4). *Let* ψ *be a state of a system S in a Hilbert space V. Let* $(X, \mathscr{F}, F)$ *and* $(Y, \mathscr{G}, G)$ *be observables in V such that* $CP(\psi;\ (X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, F), (F, \mathscr{G}, G)) \neq \emptyset$. *Then,*

(i) *the probability that* $x_0(\in X)$, *the measurement-value obtained by the measurement of the observable* $(X, \mathscr{F}, F)$ *for this system S, belongs to a set* $\Xi(\in \mathscr{F})$ *is given by* $\langle\psi, F(\Xi)\psi\rangle_V$.

(ii) *there exists* $\mu_\psi \in CP(\psi;\ (X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, (Y, \mathscr{G}, G))$ *satisfying that, if we know that the "true" value of the observable* $(X, \mathscr{F}, F)$ *for this system S is* $x_0(\in X)$, *then the probability that* $y_0(\in Y)$, *the "true" value of an observable* $(Y, \mathscr{G}, G)$ *for this system S, belongs to a set* $\Gamma(\in \mathscr{G})$ *is given by* $\mu_\psi(x_0, \Gamma)$.

From this and Remark 23, we have the following Corollary.

Corollary. *Let* ψ *be a state of a system S in a Hilbert space V and let* $(X, \mathscr{F}, F)$ *and* $(Y, \mathscr{G}, G)$ *be observables in V. Let* μ_ψ *be any element in* $CP(\psi;\ (X, \mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}, F), (Y, \mathscr{G}, G))$. *Then,*

(i) *the probability that* $x_0(\in X)$, *the measurement-value obtained by the measurement of the observable* $(X, \mathscr{F}, F)$ *for this system S, belongs to a set* $\Xi(\in \mathscr{F})$ *is given by* $\langle\psi, F(\Xi)\psi\rangle_V$,

(ii) *the following statement is true almost surely in probability* $\langle\psi, F(\cdot)\psi\rangle_V$: *if we get* $x_0(\in X)$ *by the measurement of the observable* $(X, \mathscr{F}, F)$ *for this system S, then the probability that* $y_0(\in Y)$, *the "true" value of the observable* $(Y, \mathscr{G}, G)$ *for this system S, belongs to a set* $\Gamma(\in \mathscr{G})$ *is given by* $\mu_\psi(x_0, \Gamma)$.

9. Note added in translation

Concerning how to choose $\mathscr{F}^{\psi}_{(Y,\mathscr{G},G)}$ in Axiom 1, we have also another proposal except those mentioned in § 4. In this section, we shall consider this.

Let ψ be a state of a system S in a Hilbert space V (i.e. $\psi \in V$, $\|\psi\|_V = 1$) and let $(X, \mathscr{F}, F)$ and $(Y, \mathscr{G}, G)$ be observables in V.

Let $\mathscr{A} = \{\mathscr{F}^0 \mid \mathscr{F}^0$ is a σ-subfield of $\mathscr{F}$ such that $F(\Xi)G(\Gamma)\psi = G(\Gamma)F(\Xi)\psi\ (\forall \Xi \in \mathscr{F}^0, \forall \Gamma \in \mathscr{G}\}$. Clearly $\mathscr{A}$ is a semi-ordered set concerning inclusion relation. Let $(\Lambda, \leq)$ be a linear ordered set and let $\{\mathscr{F}^\lambda \mid \lambda \in \Lambda\} (\subseteq \mathscr{A})$ be a family in $\mathscr{A}$ with the index set Λ such that $\lambda_1 \leq \lambda_2 \Leftrightarrow \mathscr{F}^{\lambda_1} \subseteq \mathscr{F}^{\lambda_2}$. Put $\mathscr{F}^\Lambda = \{\Xi(\in \mathscr{F}) \mid \Xi \in \mathscr{F}^\lambda$ for some $\lambda \in \Lambda\}$. Clearly, $\mathscr{F}^\Lambda$ is a subfield (not necessarily σ-subfield) of $\mathscr{F}$. Moreover, we can easily show that $(X, \mathscr{F}^\Lambda, F)$ is the observable such that

$$F(\Xi)G(\Gamma)\psi = G(\Gamma)F(\Xi)\psi \quad (\forall \Xi \in \mathscr{F}^\Lambda, \forall \Gamma \in \mathscr{G})\ . \tag{42}$$

Now we shall show it. Let $\{\Xi_j\}_{j=1}^{\infty}$ be disjoint family in $\mathcal{F}^{\Lambda}$. And let $\phi\in V$. Then we see that, for any $\Gamma\in\mathcal{G}$,

$$\begin{aligned}\langle\phi, F(\cup_{i=1}^{\infty}\Xi_i)G(\Gamma)\phi\rangle_V &= \lim_{n\to\infty}\langle\phi, F(\cup_{i=1}^{n}\Xi_i)G(\Gamma)\phi\rangle_V\\ &= \lim_{n\to\infty}\langle\phi, G(\Gamma)F(\cup_{i=1}^{n}\Xi_i)\phi\rangle_V\\ &= \lim_{n\to\infty}\langle G(\Gamma)\phi, F(\cup_{i=1}^{n}\Xi_i)\phi\rangle_V\\ &= \langle\phi, G(\Gamma)F(\cup_{i=1}^{\infty}\Xi_i)\phi\rangle_V\end{aligned}$$

which implies that $\cup_{i=1}^{\infty}\Xi_i\in\mathcal{F}^{\Lambda}$ and $(X, \mathcal{F}^{\Lambda}, F)$ is the observable satisfying (42). So, $\{\mathcal{F}^{\lambda}|\lambda\in\Lambda\}$ has upper-bound $(X, \mathcal{F}^{\Lambda}, F)$ in $\mathcal{A}$. Hence, by Zorn's lemma, we see that $\mathcal{A}$ has a maximal element. Put $\mathcal{A}^{\max}=\{\mathcal{F}^{\max}(\subseteq\mathcal{F})|\mathcal{F}^{\max}$ is a maximal σ-subfield of $\mathcal{F}$ in $\mathcal{A}\}$. Then, we define that

$$\bar{\mathcal{F}}^{\phi}_{(Y,\mathcal{G},G)}=\{\Xi(\in\mathcal{F})|\Xi\in\mathcal{F}^{\max}(\forall\mathcal{F}^{\max}\in\mathcal{A}^{\max})\}=\bigcap_{\mathcal{F}^{\max}\in\mathcal{A}^{\max}}\mathcal{F}^{\max} \tag{43}$$

which is clearly σ-subfield of $\mathcal{F}$. Also, it is clear that $(X, \bar{\mathcal{F}}^{\phi}_{(Y,\mathcal{G},G)}, F)$ is the observable satisfying (42).

Using $\bar{\mathcal{F}}^{\phi}_{(Y,\mathcal{G},G)}$ instead of $\mathcal{F}^{\phi}_{(Y,\mathcal{G},G)}$ (in § 4), $\mu_{\phi}(x_0, \Gamma, (X, \bar{\mathcal{F}}^{\phi}_{(Y,\mathcal{G},G)}, F), (Y, \mathcal{G}, G))$ can be similarly defined by

$$\mu_{\phi}(x_0, \Gamma, (X, \bar{\mathcal{F}}^{\phi}_{(Y,\mathcal{G},G)}, F), (Y, \mathcal{G}, G))=\lim_{\substack{\Xi\to\{x_0\}\\ \bar{\mathcal{F}}^{\phi}_{(Y,\mathcal{G},G)}\ni\Xi\ni x_0}}\frac{\langle\phi, F(\Xi)G(\Gamma)\phi\rangle_V}{\langle\phi, F(\Xi)\phi\rangle_V}. \tag{19'}$$

And our Axiom 1 can be also mentioned similarly in this case. And all arguments in this paper are effective.