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ON THE GALOIS GROUP OF $x^n-x^{n-1}-x^{n-2}-\cdots-x-1=0$

by

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1. In his doctoral thesis [4], Tamura obtained an interesting result on the irreducibility of certain polynomials: *Let* $k_1, k_2, \cdots, k_{n-1} (n>1)$ *be rational integers such that*

$$k_{n-1} \geq k_{n-2} \geq \cdots \geq k_2 \geq k_1 \geq 1 .$$

Then the polynomial

$$F(x)=x^n-k_{n-1}x^{n-1}-k_{n-2}x^{n-2}-\cdots-k_1x-1$$

is irreducible over the rational number field $\boldsymbol{Q}$. See [5], Lemma 10.

Consider now the following question: Is it possible to determine the Galois group of $F(x)=0$ over $\boldsymbol{Q}$? It seems very difficult to solve this problem completely. However, for the simplest case

$$k_1=k_2=\cdots=k_{n-1}=1 ,$$

we obtain

Theorem 1. *The Galois group of the equation*

$$x^n-x^{n-1}-x^{n-2}-\cdots-x-1=0$$

over $\boldsymbol{Q}$ *is the symmetric group* S_n *for every* $n>1$.

The purpose of this paper is to prove Theorem 1. We require a few theorems from algebraic number theory, including Minkowski's inequality on the discriminant of an algebraic number field.

2. Let α be a root of

$$f(x)=x^n-x^{n-1}-\cdots-x-1=0 . \tag{2.1}$$

Since

$$(x-1)f(x)=x^{n+1}-2x^n+1 , \tag{2.2}$$

we have

$$\alpha^{n+1}-2\alpha^n+1=0 . \tag{2.3}$$

Also, by (2.2),

$$(\alpha-1)f'(\alpha)=(n+1)\alpha^n-2n\alpha^{n-1} .$$

Hence

$$(1-\alpha)f'(\alpha)=\alpha^{n-1}\{2n-(n+1)\alpha\} . \tag{2.4}$$

3. For any $\xi\in\boldsymbol{Q}(\alpha)$, we denote by $N(\xi)$ its norm in $\boldsymbol{Q}(\alpha)$. For any $a\in\boldsymbol{Q}$, we have

$$N(a-\alpha)=f(a) . \tag{3.1}$$

Hence

$$N(1-\alpha)=f(1)=1-n . \tag{3.2}$$

Also, for any a, $b\in\boldsymbol{Q}(b\neq 0)$, we have

$$N(a-b\alpha)=b^nN\left(\frac{a}{b}-\alpha\right)=b^nf\left(\frac{a}{b}\right) . \tag{3.3}$$

Hence

$$N(2n-(n+1)\alpha)=(n+1)^nf\left(\frac{2n}{n+1}\right) .$$

Now, by (2.2),

$$\left(\frac{2n}{n+1}-1\right)f\left(\frac{2n}{n+1}\right)=\left(\frac{2n}{n+1}\right)^{n+1}-2\left(\frac{2n}{n+1}\right)^n+1 ,$$

$$\frac{n-1}{n+1}f\left(\frac{2n}{n+1}\right)=\left(\frac{1}{n+1}\right)^{n+1}((n+1)^{n+1}-2(2n)^n) .$$

Hence

$$N(2n-(n+1)\alpha)=\frac{(n+1)^{n+1}-2(2n)^n}{n-1} . \tag{3.4}$$

Now let

$$\delta=f'(\alpha) , \quad D=N(\delta) . \tag{3.5}$$

Then, from (2.4), (3.2) and (3.4), we obtain

$$\begin{aligned}(1-n)D&=N(\alpha^{n-1})\frac{(n+1)^{n+1}-2(2n)^n}{n-1}\\&=(-1)^{n-1}\frac{(n+1)^{n+1}-2(2n)^n}{n-1} ,\end{aligned}$$

since

(3.6) $$N(\alpha^{n-1})=(N(\alpha))^{n-1}=((-1)^{n+1})^{n-1}=(-1)^{n-1}\ .$$

Hence

(3.7) $$D=(-1)^{n-1}\frac{2(2n)^n-(n+1)^{n+1}}{(n-1)^2}\ .$$

4. Define the ring M by

$$M=[1,\alpha,\cdots,\alpha^{n-1}]$$
$$=\{a_0+a_1\alpha+\cdots+a_{n-1}\alpha^{n-1}|a_i\in \boldsymbol{Z}\}\ .$$

Let $\alpha_0=\alpha-1$. Then, by (2.3),

$$(\alpha_0+1)^{n+1}-2(\alpha_0+1)^n+1=0\ .$$

Hence

$$\alpha_0^{n+1}+b_n\alpha_0^n+\cdots+b_2\alpha_0^2+(1-n)\alpha_0=0\ ,$$

where $b_i\in \boldsymbol{Z}$. Hence

$$\frac{n-1}{\alpha_0}=\alpha_0^{n-1}+b_n\alpha_0^{n-2}+\cdots+b_2\in M\ .$$

By (3.2) we see that

(4.1) $$\frac{N(1-\alpha)}{1-\alpha}\in M\ .$$

Let $a\in \boldsymbol{Q}$, $a\neq 1$, and let $\beta=\alpha-a$. Then, by (2.3),

$$(\beta+a)^{n+1}-2(\beta+a)^n+1=0\ ,$$

and so

$$\beta^{n+1}+\{(n+1)a-2\}\beta^n+\cdots+(a^{n+1}-2a^n+1)=0\ .$$

On the other hand, by (2.2),

$$a^{n+1}-2a^n+1=(a-1)f(a)\ .$$

Hence

$$\beta^n+\{(n+1)a-2\}\beta^{n-1}+\cdots+\frac{(a-1)f(a)}{\beta}=0\ .$$

Now

$$\beta^n=(\alpha-a)^n=\alpha^n-na\alpha^{n-1}+\cdots+(-1)^na^n$$
$$=(\alpha^{n-1}+\cdots+1)-na\alpha^{n-1}+\cdots+(-1)^na^n$$
$$=(1-na)\alpha^{n-1}+\cdots+\{(-1)^na^n+1\}\ ,$$

$$\{(n+1)a-2\}\beta^{n-1}=\{(n+1)a-2\}(\alpha^{n-1}-\cdots+(-a)^{n-1})\ .$$

Hence

$$\frac{N(a-\alpha)}{a-\alpha}=\alpha^{n-1}+t_{n-2}\alpha^{n-2}+\cdots+t_1\alpha+t_0\,, \tag{4.2}$$

where $t_i\in\boldsymbol{Q}$. Now let $a, b\in\boldsymbol{Q}$, $a\neq b$, $b\neq 0$. Then

$$\frac{N(a-b\alpha)}{a-b\alpha}=b^{n-1}\cdot\frac{N\left(\frac{a}{b}-\alpha\right)}{\frac{a}{b}-\alpha}=b^{n-1}\alpha^{n-1}+s_{n-2}\alpha^{n-2}+\cdots+s_1\alpha+s_0\,, \tag{4.3}$$

where $s_i\in\boldsymbol{Q}$. From this we obtain

$$\frac{N(2n-(n+1)\alpha)}{2n-(n+1)\alpha}=(n+1)^{n-1}\alpha^{n-1}+c_{n-2}\alpha^{n-2}+\cdots+c_1\alpha+c_0\,, \tag{4.4}$$

where $c_i\in\boldsymbol{Q}$.

5. By Theorem 1 of [2] we see that $D/\delta\in M$. Let

$$D/\delta=x_0+x_1\alpha+\cdots+x_{n-1}\alpha^{n-1}\,,\quad x_i\in Z\,. \tag{5.1}$$

Let p denote a prime number such that

$$p|x_0,\ p|x_1,\cdots,p|x_{n-1}\,. \tag{5.2}$$

Then

$$\frac{D}{p\delta}\in M\,. \tag{5.3}$$

From (2.4) and (3.5) we obtain

$$\frac{N(1-\alpha)}{1-\alpha}\cdot\frac{D}{\delta}=\frac{N(\alpha^{n-1})}{\alpha^{n-1}}\cdot\frac{N(2n-(n+1)\alpha)}{2n-(n+1)\alpha}\,.$$

Hence, by (3.6),

$$\frac{N(2n-(n+1)\alpha)}{p(2n-(n+1)\alpha)}=(-1)^{n-1}\alpha^{n-1}\cdot\frac{N(1-\alpha)}{1-\alpha}\cdot\frac{D}{p\delta}\,.$$

Since M is a ring, it follows from (4.1) and (5.3) that

$$\frac{N(2n-(n+1)\alpha)}{p(2n-(n+1)\alpha)}\in M\,. \tag{5.4}$$

Since $1, \alpha, \cdots, \alpha^{n-1}$ are linearly independent over Q, by (4.4) and (5.4) we see that $n+1$ is divisible by p. On the other hand D is divisible by p ((5.1) and (5.2)). Hence, by (3.7), $p=2$. From Theorem 1 of [2] we obtain:

(5.5) *For every odd prime p, the discriminant d of $\boldsymbol{Q}(\alpha)$ is not divisible by p^2.*

6. Suppose that n is even. Then, by (3.7), D is odd. Since $d|D$, it follows

from (5.5) that d is square-free. Hence the Galois group of $f(x)=0$ is the symmetric group S_n ([3], Theorem 1).

7. Suppose that n is odd. We require two lemmas.

Lemma 1. *Let d_K denote the discriminant of an algebraic number field K of degree $n>1$. Then $|d_K|>2^{n-1}$.*

Proof. From Minkowski's inequality ([1], §18) and Stirling's formula, we obtain

$$\begin{aligned}|d_K| &> \left(\frac{\pi}{4}\right)^n \left(\frac{n^n}{n!}\right)^2 \\ &> \left(\frac{\pi e^2}{4}\right)^n \frac{e^{-1/6n}}{2\pi n}\,.\end{aligned}$$

It is easily seen that

$$\begin{aligned}\log\left\{\left(\frac{\pi e^2}{4}\right)^n \frac{e^{-1/6n}}{2\pi n}\right\} &= n(\log\pi+2-2\log 2)-\frac{1}{6n}-\log(2\pi n) \\ &> (n-1)\log 2=\log 2^{n-1}\,.\end{aligned}$$

Hence we obtain

$$|d_K|>2^{n-1}\,.$$

Lemma 2. *For any odd integer $n>1$,*

$$D_n=\frac{2(2n)^n-(n+1)^{n+1}}{(n-1)^2}$$

is exactly divisible by 2^{n-1}.

Proof. Let $n=2m+1$, $m\in\boldsymbol{Z}$, $m\geq 1$. Then

$$(2m)^2 D_n=2^{n+1}(2m+1)^n-2^{n+1}(m+1)^{n+1}\,,$$

and so

$$m^2 D_n=2^{n-1}\{(2m+1)^n-(m+1)^{n+1}\}\,. \tag{7.1}$$

If m is odd, then both m^2 and $(2m+1)^n-(m+1)^{n+1}$ are odd, and D_n is exactly divisible by 2^{n-1}. Suppose that m is even. Now, by (7.1),

$$\begin{aligned}m^2 D_n &= 2^{n-1}\left\{\sum_{k=0}^{n-2} {}_nC_k(2m)^{n-k}+n(2m)+1-\sum_{k=0}^{n-1} {}_{n+1}C_k m^{n+1-k}-(n+1)m-1\right\} \\ &= 2^{n-1}m^2\left(\sum_{k=0}^{n-2} {}_nC_k 2^{n-k}m^{n-2-k}-\sum_{k=0}^{n-1} {}_{n+1}C_k m^{n-1-k}+2\right)\,.\end{aligned}$$

Hence D_n is divisible by 2^{n-1}. Since m is even,

$$\frac{D_n}{2^{n-1}} \equiv {}_{n+1}C_{n-1} = \frac{n(n+1)}{2}$$
$$= n(m+1) \equiv 1 \pmod 2 .$$

Hence D_n is exactly divisible by 2^{n-1}.

Now we prove our theorem for odd $n(n>1)$. It follows from Lemma 2 that D is exactly divisible by 2^{n-1}. Since D is divisible by the discriminant d of $\boldsymbol{Q}(\alpha)$, it follows from (5.5) that

$$|d| = 2^t b , \quad t \leq n-1 , \tag{7.2}$$

where b is a square-free odd integer. Lemma 1 implies that $b>1$; the discriminant d is exactly divisible by a prime number q. Hence the Galois group G of $f(x)=0$ over $\boldsymbol{Q}$, which is a transitive permutation group on the set $\{1, 2, \cdots, n\}$, contains a transposition ([6]). Suppose that $\boldsymbol{Q}(\alpha)$ has a subfield F such that

$$\boldsymbol{Q} \subset F \subset \boldsymbol{Q}(\alpha) , \quad F \neq \boldsymbol{Q} , \quad F \neq \boldsymbol{Q}(\alpha) .$$

Let d_F denote the discriminant of F, and let

$$m = [\boldsymbol{Q}(\alpha) : F] , \quad k = [F : \boldsymbol{Q}] .$$

Then d is divisible by d_F^m ([1], Satz 39). Since $m>1$, it follows from (7.2) that $|d_F|$ is a power of 2:

$$|d_F| = 2^s .$$

Since $k>1$, it follows from Lemma 1 that $s \geq k$. Hence d_F is divisible by 2^k, and d is divisible by $2^{km} = 2^n$. However, it follows from (7.2) that d is not divisible by 2^n. A contradiction proves that G is primitive ([7], Theorem 7.4). Hence we obtain $G = S_n$ ([7], Theorem 13.3).

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