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ON THE GENERALIZED HILBERT TRANSFORMS OF FUNCTIONS OF TWO VARIABLES II

by

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ABSTRACTS

We shall reconstruct the spectral analysis of the generalized Hilbert transforms in $\boldsymbol{R}^2$, which was given in K. Matsuoka [6], under some weaker conditions.

1. Introduction

In [1], K. Anzai, S. Koizumi and K. Matsuoka extended the Wiener formula to the $\boldsymbol{R}^2$ case. On the basis of this Wiener formula, K. Matsuoka [5] constructed the generalized harmonic analysis (GHA) on $\boldsymbol{R}^2$. And, in [6], we showed the extension of Koizumi's theory ([2, 3]) of the spectral analysis of the generalized Hilbert transform (GHT)

$$\tilde{f}(x)=\lim_{\varepsilon\to 0}\frac{x+i}{\pi}\int_{0<\varepsilon\leq|x-t|}\frac{f(t)}{t+i}\frac{dt}{x-t}\quad (f\in L_c^2(\boldsymbol{R}))$$

(as for the notation "$L_c^2(\boldsymbol{R})$", see Section 2) to the $\boldsymbol{R}^2$ case by using the GHA on $\boldsymbol{R}^2$. In order to do this, we assumed several conditions ((M_1)-(M_7) in [6]).

The purpose of this paper is to show the same results as in [6] under some weaker conditions ($(M_2)_{\mathscr{W}^2}$, $(M_4)_{\mathscr{W}^2}$; $(M_2)_S$, $(M_4)_S$; $(M_2)_{S'}$, $(M_4)_{S'}$ in Section 3).

2. Preliminaries

Throughout this paper, all functions we consider will be complex valued and measurable on $\boldsymbol{R}$ or $\boldsymbol{R}^2$.

In this section, we list the notation, which will be used in what follows (see S. Koizumi [2, 3], P. Masani [4], K. Matsuoka [5, 6] and N. Wiener [7, 8]):

(a) $L_c^2(\boldsymbol{R})=\left\{g\in L_{loc}^2(\boldsymbol{R}):\ \int_{-\infty}^{\infty}|g(x)|^2dc(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|g(x)|^2}{1+x^2}dx<\infty\right\}$,

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$$L_c^2(\boldsymbol{R}^2)=\left\{f\in L_{loc}^2(\boldsymbol{R}^2):\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x_1,x_2)|^2dc(x_1,x_2)\right.$$
$$\left.=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(x_1,x_2)|^2}{(1+x_1^2)(1+x_2^2)}dx_1dx_2<\infty\right\};$$

(b) [Wiener's generalized Fourier transform (GFT)]

$$s(u;g)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_1^A+\int_{-A}^{-1}\right]g(t)\frac{e^{-iut}}{-it}dt$$
$$+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}g(t)\frac{e^{-iut}-1}{-it}dt\quad(g\in L_c^2(\boldsymbol{R})),$$
$$s(u,v;f)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\left[\int_1^A+\int_{-A}^{-1}\right]f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}}{-it}dsdt$$
$$+\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\int_{-1}^{1}f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}}{-it}dsdt$$
$$+\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-1}^{1}\left[\int_1^A+\int_{-A}^{-1}\right]f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}-1}{-it}dsdt$$
$$+\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}-1}{-it}dsdt\quad(f\in L_c^2(\boldsymbol{R}^2)),$$

where the notation "l.i.m." means the limit in $L^2(\boldsymbol{R})$ or $L^2(\boldsymbol{R}^2)$;

(c) $\Delta_\varepsilon s(u;g)=s(u+\varepsilon;g)-s(u-\varepsilon;g)$,

$$\Delta_{\varepsilon,\eta}s(u,v;f)=s(u+\varepsilon,v+\eta;f)-s(u-\varepsilon,v+\eta;f)$$
$$-s(u+\varepsilon,v-\eta;f)+s(u-\varepsilon,v-\eta;f);$$

(d) The notations "$\mathscr{R}_1$-$\lim_{S,T\to\infty}$" and "$\mathscr{R}_2$-$\lim_{\varepsilon,\eta\to+0}$" mean that in each of them a limit exists and has the same limit for every positive constant C whenever S and T tend to infinity or ε and η tend to zero in such a way that $S=CT$ or $\eta=C\varepsilon$ respectively;

(e) $$W^2(\boldsymbol{R})=\left\{g\in L_{loc}^2(\boldsymbol{R}):\ \sup_{T>0}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2dt<\infty\right\},$$
$$W^2(\boldsymbol{R}^2)=\left\{f\in L_{loc}^2(\boldsymbol{R}^2):\ \sup_{S,T>0}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt<\infty\right\};$$

(f) $$\mathscr{W}^2(\boldsymbol{R})=\left\{g\in L_{loc}^2(\boldsymbol{R}):\ \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2dt\ \text{exists}\right\},$$
$$\mathscr{W}^2(\boldsymbol{R}^2)=\left\{f\in W^2(\boldsymbol{R}^2):\ \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt\ \text{exists}\right\};$$

(g) $S(\boldsymbol{R})=\{g\in L_{loc}^2(\boldsymbol{R}):\ \phi(x;g)$ exists for all $x\in\boldsymbol{R}\}$,

$S(\boldsymbol{R}^2)=\{f\in W^2(\boldsymbol{R}^2):\ \phi(x_1,x_2;f)$ exists for all $(x_1,x_2)\in\boldsymbol{R}^2\}$,

where

$$\phi(x;g)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(x+t)\overline{g(t)}dt$$

and

$$\phi(x_1, x_2; f)=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S} f(x_1+s, x_2+t)\overline{f(s, t)}dsdt ,$$

which are called the covariance functions of $g\in S(\boldsymbol{R})$ and $f\in S(\boldsymbol{R}^2)$ respectively;

(h) $S'(\boldsymbol{R})=\{g\in S(\boldsymbol{R}):\ \phi(x; g)$ is continuous on $\boldsymbol{R}\}$;
$S'(\boldsymbol{R}^2)=\{f\in S(\boldsymbol{R}^2):\ \phi(x_1, x_2; f)$ is continuous on $\boldsymbol{R}^2\}$;

(i) [The generalized Hilbert transform (GHT)]

$$(H^{(1)}f)(x_1, x_2)=\lim_{\varepsilon_1\to 0}\frac{x_1+i}{\pi}\int_{0<\varepsilon_1\le|x_1-s|}\frac{f(s, x_2)}{s+i}\frac{ds}{x_1-s}\quad (f\in L_c^2(\boldsymbol{R}^2)) ,$$

$$(H^{(2)}f)(x_1, x_2)=\lim_{\varepsilon_2\to 0}\frac{x_2+i}{\pi}\int_{0<\varepsilon_2\le|x_2-t|}\frac{f(x_1, t)}{t+i}\frac{dt}{x_2-t}\quad (f\in L_c^2(\boldsymbol{R}^2)) ,$$

$$\begin{aligned}(Hf)(x_1, x_2)&=(H^{(2)}H^{(1)}f)(x_1, x_2)=(H^{(1)}H^{(2)}f)(x_1, x_2)\\ &=\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{(x_1+i)(x_2+i)}{\pi^2}\iint_{\substack{0<\varepsilon_1\le|x_1-s|\\ 0<\varepsilon_2\le|x_2-t|}}\frac{f(s, t)}{(s+i)(t+i)}\frac{dsdt}{(x_1-s)(x_2-t)}\\ &\qquad (f\in L_c^2(\boldsymbol{R}^2)) .\end{aligned}$$

Note that

$$L_{loc}^2\supset L_c^2\supset W^2\supset \mathscr{W}^2\supset S\supset S'$$

on $\boldsymbol{R}$ or $\boldsymbol{R}^2$ (see Theorem 1 of K. Matsuoka [5]).

3. The spectral analysis of the generalized Hilbert transform (GHT)

In K. Matsuoka [6], under several conditions (M_1)-(M_7), we determined the spectral relation between a given function on $\boldsymbol{R}^2$ and its GHT's, and saw that the properties of the given function are reflected on those of its GHT's. In this section, we shall show the same results as in the above under some weaker conditions.

First, we state the theorems concerning the mean total power of the GHT.

Theorem 1. *Suppose $f\in\mathscr{W}^2(\boldsymbol{R}^2)$, and it satisfies that*

(M_1)
$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\pi\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\varepsilon}^{\varepsilon}|\Delta_{\varepsilon,\eta}s(u, v; f)|^2dudv=0$$

and that
$(M_2)_{,2}$ *there exists a function $k_1^f(x_2)\in\mathscr{W}^2(\boldsymbol{R})$ such that*

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\pi\varepsilon\eta}\int_{-\infty}^{\infty}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s, t)}{s+i}\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt-\sqrt{\frac{\pi}{2}}\cdot\Delta_\eta s(v; k_1^f)\right|^2dudv=0 . \tag{3.1}$$

Then $H^{(1)}f\in \mathscr{W}^2(\boldsymbol{R}^2)$ *and*

$$\text{(3.2)}\qquad \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(H^{(1)}f)(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|k_1^f(t)|^2dt\,.$$

Theorem 1′. *Suppose* $f\in\mathscr{W}^2(\boldsymbol{R}^2)$, *and it satisfies that*

$$\text{(M}_3\text{)}\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\pi\varepsilon\eta}\int_{-\eta}^{\eta}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v;f)|^2dudv=0$$

and that

$(\mathrm{M}_4)_{\mathscr{W}^2}$ *there exists a function* $k_2^f(x_1)\in\mathscr{W}^2(\boldsymbol{R})$ *such that*

$$\text{(3.3)}\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{8\pi\varepsilon\eta}\int_{0}^{2\eta}\int_{-\infty}^{\infty}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{t+i}\frac{2\sin\varepsilon s}{s}e^{-i(us+vt)}dsdt\right.$$
$$\left.-\sqrt{\frac{\pi}{2}}\cdot\Delta_\varepsilon s(u;k_2^f)\right|^2dudv=0\,.$$

Then $H^{(2)}f\in\mathscr{W}^2(\boldsymbol{R}^2)$ *and*

$$\text{(3.4)}\qquad \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(H^{(2)}f)(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt+\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}|k_2^f(s)|^2ds\,.$$

Theorem 2. *Suppose* $f\in\mathscr{W}^2(\boldsymbol{R}^2)$, *and it satisfies that* (M_1), $(\mathrm{M}_2)_{\mathscr{W}^2}$, (M_3), $(\mathrm{M}_4)_{\mathscr{W}^2}$ *and, in addition, that*

$$\text{(M}_5\text{)}\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\pi\varepsilon\eta}\int_{-\eta}^{\eta}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt\right|^2dudv=0\,,$$

$$\text{(M}_6\text{)}\qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\pi\varepsilon\eta}\int_{0}^{2\eta}\int_{-\varepsilon}^{\varepsilon}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{t+i}\frac{2\sin\varepsilon s}{s}e^{-i(us+vt)}dsdt\right|^2dudv=0$$

and

(M_7) *there exists a constant* k_3^f *such that*

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\pi\varepsilon\eta}\int_{0}^{2\eta}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}\right.$$
$$\left.\cdot\, e^{-i(us+vt)}dsdt-\frac{\pi}{2}k_3^f\right|^2dudv=0\,.$$

Then $Hf\in\mathscr{W}^2(\boldsymbol{R}^2)$ *and*

$$\text{(3.5)}\qquad \mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(Hf)(s,t)|^2dsdt$$
$$=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2dsdt$$

$$+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|k_1^f(t)|^2dt+\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}|k_2^f(s)|^2ds+|k_3^f|^2 .$$

By Theorem 22 of N. Wiener [8] and Theorem 3 of K. Matsuoka [5], in order to prove Theorems 1, 1′ and 2, it suffices to verify the following three lemmas, respectively.

Lemma 3. *Under the hypotheses of Theorem 1,*

$$(3.6)\quad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;H^{(1)}f)|^2dudv$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;f)|^2dudv+\lim_{\eta\to+0}\frac{1}{4\pi\eta}\int_{-\infty}^{\infty}|\varDelta_\eta s(v;k_1^f)|^2dv .$$

Lemma 3′. *Under the hypotheses of Theorem 1′,*

$$(3.7)\quad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;H^{(2)}f)|^2dudv$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;f)|^2dudv+\lim_{\varepsilon\to+0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\varDelta_\varepsilon s(u;k_1^f)|^2du .$$

Lemma 4. *Under the hypotheses of Theorem 2,*

$$(3.8)\quad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;Hf)|^2dudv$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;f)|^2dudv$$
$$+\lim_{\eta\to+0}\frac{1}{4\pi\eta}\int_{-\infty}^{\infty}|\varDelta_\eta s(v;k_1^f)|^2dv+\lim_{\varepsilon\to+0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\varDelta_\varepsilon s(u;k_2^f)|^2du+|k_3^f|^2 .$$

Proof. Using Theorem 9 and Lemma 4 of K. Matsuoka [6], and the conditions (M_1), $(M_2)_{\mathscr{V}^2}$, (M_3), $(M_4)_{\mathscr{V}^2}$, (M_5)-(M_7), we have

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;Hf)|^2dudv$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{|v|>\eta}\int_{|u|>\varepsilon}|\varDelta_{\varepsilon,\eta}s(u,v;f)|^2dudv$$
$$+\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\pi^2\varepsilon\eta}\int_{|v|>\eta}\int_0^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{s+i}\frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdt\right|^2dudv$$
$$+\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{4\pi^2\varepsilon\eta}\int_0^{2\eta}\int_{|u|>\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{t+i}\frac{2\sin\varepsilon s}{s}e^{-i(us+vt)}dsdt\right|^2dudv$$
$$+\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{\pi^2\varepsilon\eta}\int_0^{2\eta}\int_0^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-B}^{B}\int_{-B}^{B}\frac{f(s,t)}{(s+i)(t+i)}e^{-i(us+vt)}dsdt\right|^2dudv$$
$$=\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to+0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u,v;f)|^2dudv$$
$$+\lim_{\eta\to+0}\frac{1}{4\pi\eta}\int_{-\infty}^{\infty}|\varDelta_\eta s(v;k_1^f)|^2dv+\lim_{\varepsilon\to+0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\varDelta_\varepsilon s(u;k_2^f)|^2du+|k_3^f|^2 .$$

Thus, (3.8) is proved. ■

Similarly, we prove Lemmas 3 and 3′ by Theorem 8 and Lemma 4 of K. Matsuoka [6].

Next, we show the theorems concerning the covariance function of the GHT.

Theorem 5. *Suppose* $f \in S(\boldsymbol{R}^2)$, *and it satisfies* (M_1) *and that* $(\mathrm{M}_2)_S$ *there exists a function* $k_1^f(x_2) \in S(\boldsymbol{R})$ *such that* (3.1) *holds. Then* $H^{(1)}f \in S(\boldsymbol{R}^2)$ *and*

$$\phi(x_1, x_2; H^{(1)}f) = \phi(x_1, x_2; f) + \phi(x_2; k_1^f) . \tag{3.9}$$

Theorem 5′. *Suppose* $f \in S(\boldsymbol{R}^2)$, *and it satisfies* (M_3) *and that* $(\mathrm{M}_4)_S$ *there exists a function* $k_2^f(x_1) \in S(\boldsymbol{R})$ *such that* (3.3) *holds. Then* $H^{(2)}f \in S(\boldsymbol{R}^2)$ *and*

$$\phi(x_1, x_2; H^{(2)}f) = \phi(x_1, x_2; f) + \phi(x_1; k_2^f) . \tag{3.10}$$

Theorems 5 and 5′ follow by the same argument that will be used in the proof of the following theorem.

Theorem 6. *Suppose* $f \in S(\boldsymbol{R}^2)$, *and it satisfies* (M_1), $(\mathrm{M}_2)_S$, (M_3), $(\mathrm{M}_4)_S$ *and* (M_5)-(M_7). *Then* $Hf \in S(\boldsymbol{R}^2)$ *and*

$$\phi(x_1, x_2; Hf) = \phi(x_1, x_2; f) + \phi(x_2; k_1^f) + \phi(x_1; k_2^f) + |k_3^f|^2 . \tag{3.11}$$

Proof. By Theorem 9 and Lemma 4 of K. Matsuoka [6], the conditions (M_1), $(\mathrm{M}_2)_S$, (M_3), $(\mathrm{M}_4)_S$, (M_5)-(M_7), Theorem 27 of N. Wiener [8], and Theorem 6 of K. Matsuoka [5], we have immediately

$$\begin{aligned}
\phi(x_1, x_2; Hf) = \mathscr{R}_2\text{-} &\lim_{\varepsilon, \eta \to +0} \frac{1}{16\pi^2 \varepsilon\eta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(ux_1 + vx_2)} |\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv \\
&+ \lim_{\eta \to +0} \frac{1}{4\pi\eta} \int_{-\infty}^{\infty} e^{ivx_2} |\Delta_\eta s(v; k_1^f)|^2 dv \\
&+ \lim_{\varepsilon \to +0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux_1} |\Delta_\varepsilon s(u; k_2^f)|^2 du \\
&+ |k_3^f|^2 \\
= \phi(x_1, x_2; f) &+ \phi(x_2; k_1^f) + \phi(x_1; k_2^f) + |k_3^f|^2 . \quad \blacksquare
\end{aligned}$$

Finally, by Theorems 5, 5′ and 6, we obtain the results concerning the GHT of functions in the class $S'(\boldsymbol{R}^2)$.

Theorem 7. *Suppose* $f \in S'(\boldsymbol{R}^2)$, *and it satisfies* (M_1) *and that* $(\mathrm{M}_2)_{S'}$ *there exists a function* $k_1^f(x_2) \in S'(\boldsymbol{R})$ *such that* (3.1) *holds. Then* $H^{(1)}f \in S'(\boldsymbol{R}^2)$ *and* (3.9) *holds.*

Theorem 7′. *Suppose* $f \in S'(\boldsymbol{R}^2)$, *and it satisfies* (M_3) *and that* $(\mathrm{M}_4)_{S'}$ *there exists a function* $k_2^f(x_1) \in S'(\boldsymbol{R})$ *such that* (3.3) *holds. Then* $H^{(2)}f \in S'(\boldsymbol{R}^2)$ *and* (3.10) *holds.*

Theorem 8. *Suppose* $f \in S'(\boldsymbol{R}^2)$, *and it satisfies* (M_1), $(M_2)_{S'}$, (M_3), $(M_4)_{S'}$ *and* (M_5)-(M_7). *Then* $Hf \in S'(\boldsymbol{R}^2)$ *and* (3.11) *holds.*

Remark. Throughout this paper, the limit processes depend on the restricted limit processes $\mathscr{R}_1$-$\lim_{S,T\to\infty}$ and $\mathscr{R}_2$-$\lim_{\varepsilon,\eta\to+0}$ involved in the GHA on $\boldsymbol{R}^2$. On the other hand, the GHA on $\boldsymbol{R}^2$ is also established under the independent limit processes $\lim_{S,T\to\infty}$ and $\lim_{\varepsilon,\eta\to+0}$. Thus, the results in this paper also hold under the independent limit processes $\lim_{S,T\to\infty}$ and $\lim_{\varepsilon,\eta\to+0}$ instead of the restricted limit processes $\mathscr{R}_1$-$\lim_{S,T\to\infty}$ and $\mathscr{R}_2$-$\lim_{\varepsilon,\eta\to+0}$, respectively.

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