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ON APPROXIMATE FOURIER SERIES OF A SECOND ORDER STOCHASTIC PROCESS

by

Tatsuo KAWATA*

Department of Mathematics
Faculty of Science and Technology, Keio University
Hiyoshi, Yokohama 223, Japan

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1. Introduction

The auther was informed by H. L. Hurd [1] that he had shown that a second order stochastic process (possibly discontinuous) $X(t, \omega)$, $-\infty<t<\infty$, ω being an element of a given probability space Ω, can be represented by the $(C, 1)$ sum of the Fourier-like series

$$\sum_{k=-\infty}^{\infty} a_k(t, \omega) \exp(2\pi ikt/T) , \tag{1.1}$$

under some conditions on the correlation function $EX(t, \omega)\overline{X(s, \omega)}$ and the natural condition on $X(t, \omega)$ at t, where $a_k(t, \omega)$ is given by

$$a_k(t, \omega)=\underset{A\to\infty}{\text{l.i.m.}} \frac{1}{T}\int_{-A}^{A} W_T(u-t) \exp(-2\pi iku/T)X(u, \omega)du \tag{1.2}$$

with

$$W_T(t)=\frac{\sin(\pi t/T)}{\pi t/T} \exp(-\pi it/T) \tag{1.3}$$

and T is any positive number. Here l.i.m. means the limit in $L^2(\Omega)$. We used a slightly different notation for W from Hurd's, for later convenience. (1.1) is not an ordinary Fourier series, since the coefficient $a_k(t, \omega)$ depends on the variable t.

Now we take t in $a_k(t, \omega)$ to be a fixed constant (independent of t) t_0 and write $a_k(\omega)=a_k(t_0, \omega)$ and consider the series

$$\sum_{k=-\infty}^{\infty} a_k(\omega) \exp(2\pi ikt/T) . \tag{1.4}$$

* Former Professor (Department of Mathematics).

For simplicity, we take $t_0=0$ in what follows, so that

$$a_k(\omega)=\underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{T}\int_{-A}^{A} W_T(u)\exp(-2\pi iku/T)X(u,\omega)du\ . \tag{1.5}$$

We then ask what will be said about (1.4). The problem will be treated in 4. Roughly speaking, we shall show that there is a T-periodic stochastic process $\hat{X}_T(t,\omega)$ which approximates $X(t,\omega)$ for each t when T is large and is such that Fourier series of $\hat{X}_T(t,\omega)$ is just (1.4) with (1.5). The precise statement will be given in 4.

In this sense (1.4) is called an *approximate Fourier series* of $X(t,\omega)$. Also we agree to call $\hat{X}_T(t,\omega)$ a *T-periodic approximate stochastic process.*

The auther [2], [3] has studied the approximate Fourier series of a weakly stationary process and a general linear process. In particular, the auther was interested in the almost sure absolute convergence of the approximate Fourier series, which enabled him to find sufficient conditions for the sample continuity of the given stochastic processes. In this paper, we shall make the similar investigation along this line, for a general second order process.

We throughout this paper suppose that $X(t,\omega)$ in consideration is a mesurable second order process with bounded second moment:

$$E|X(t,\omega)|^2\leq M\ ,\quad -\infty<t<\infty\ . \tag{1.6}$$

We here note that (1.6) implies

$$\int_{-A}^{A}|X(t,\omega)|^2dt<\infty\ ,\quad \textit{almost surely}\ , \tag{1.7}$$

for every finite A.

2. An approximate Fourier series

We now consider

$$V_T(t)=\left[\frac{\sin(\pi t/T)}{\pi t/T}\right]^2,\quad T>0 \tag{2.1}$$

in place of $W_T(t)$ in 1 and take up the series

$$\sum_{k=-\infty}^{\infty} c_k(\omega)\exp(2\pi ikt/T) \tag{2.2}$$

instead of (1.4) where

$$\begin{aligned} c_k(\omega)&=c_k(\omega,T)\\ &=\underset{A,B\to\infty}{\text{l.i.m.}}\frac{1}{T}\int_{-A}^{B} V_T(u)\exp(-2\pi iku/T)X(u,\omega)du\ ,\quad k=0,\pm1,\cdots\ . \end{aligned} \tag{2.3}$$

As we will see later, (2.2) is much easier to handle than (1.4). $c_k(\omega)$ is actually well defined. First we prove this.

Let $B'>B>0$. Using the Minkowski inequality, we have

$$E\left|\int_B^{B'} V_T(u)\exp(-2\pi iku/T)X(u,\omega)du\right|^2 \leq \left\{\int_B^{B'} V_T(u)[E|X(u,\omega)|^2]^{1/2}du\right\}^2$$
$$\leq M\left[\int_B^{B'} V_T(u)du\right]^2$$
$$\leq M\left[\int_B^{B'} \frac{du}{(\pi u/T)^2}\right]^2 \leq \frac{MT^4}{\pi^4 B^2}$$

which converges to zero as $B\to\infty$. For $\int_{-A'}^{-A}$ $(0<A<A')$ we have the similar estimate. Hence $c_k(\omega)=c_k(\omega, T)$ exists for every T and $E|c_k(\omega)|^2<\infty$.

Now for each t, we define

(2.4) $$\hat{X}_T(t,\omega)=\underset{K,L\to\infty}{\text{l.i.m.}} \sum_{i=-L}^{K} V_T(t+jT)X(t+jT,\omega)\,.$$

The right hand side really exists. For a fixed t, taking $K\geq 1$ so large that $KT>2|t|$, we have, using the Minkowski inequality, for $K'>K$,

(2.5) $$E\left|\sum_{j=K}^{K'} V_T(t+jT)X(t+jT,\omega)\right|^2 \leq \left\{\sum_{j=K}^{K'}[E|V_T(t+jT)X(t+jT,\omega)|^2]^{1/2}\right\}^2$$
$$\leq\left\{\sum_{j=K}^{K'}\left[\left(\frac{T}{\pi(t+jT)}\right)^4 E|X(t+jT,\omega)|^2\right]^{1/2}\right\}^2$$
$$\leq CM\left[\sum_{j=K}^{K'}\frac{1}{(j-1/2)^2}\right]^2$$
$$\leq CM/K^2\,,$$

where C's are absolute constants and may be different on each occurrence.

The similar estimate is obtained also for $\sum_{j=-L'}^{-L}$ $(1\leq L<L')$. Thus the right hand side of (2.4) exists.

Obviously we see

(2.6) $$E|\hat{X}_T(t,\omega)-\hat{X}_T(t+T,\omega)|^2=0$$

for every t, that is, $\hat{X}_T(t,\omega)$ is a T-periodic stochastic process.

Now we shall show, in the following section,

Theorem 1. *Let $X(t,\omega)$, $-\infty<t<\infty$, be a second order stochastic process satisfying (1.6). Then we have*

(i) *For $T\geq 2|t|$,*

(2.7) $$E|\hat{X}_T(t,\omega)|^2\leq CM\,,$$

and

(2.8) $$E|\hat{X}_T(t,\omega)-X(t,\omega)|^2\leq CM\cdot t^4/T^4\,.$$

(ii) *The Fourier series of $\hat{X}_T(t,\omega)$ is given by (2.2), where C is an absolute constant, and M is the constant in (1.6).*

If, for a given second order stochastic process $X(t, \omega)$, there exists a T-periodic second order process $\hat{X}_T(t, \omega)$, such that $E|\hat{X}_T(t, \omega)|^2$ is bounded and $\hat{X}_T(t, \omega)$ converges in $L^2(\Omega)$ to $X(t, \omega)$ for each t as $T\to\infty$, then we agree to call $\hat{X}_T(t, \omega)$ an *approximate T-periodic stochastic process* and the Fourier series of $\hat{X}_T(t, \omega)$ an *approximate Fourier series of $X(t, \omega)$*.

3. Proof of Theorem 1

We shall give the proof of Theorem 1. In what follows, in this section and also even throughout this paper, C's denote absolute constants which may differ from each other. M is the constant in (1.6) all the way.

We begin with

Lemma 1. *For $2|t|\le T$,*

$$E\left|\underset{K\to\infty, L\to\infty}{\text{l.i.m.}} \sum_{j=-L, j\neq 0}^{K} V_T(t+jT)X(t+jT, \omega)\right|^2 \le CM\sin^4(\pi t/T)\,. \tag{3.1}$$

Proof. The left hand side of (3.1) is not greater than

$$2E\left|\underset{K\to\infty}{\text{l.i.m.}} \sum_{j=1}^{K}\right|^2 + 2E\left|\underset{L\to\infty}{\text{l.i.m.}} \sum_{j=-L}^{-1}\right|^2 . \tag{3.2}$$

The first term is

$$\begin{aligned} 2E&\left|\underset{K\to\infty}{\text{l.i.m.}} \sum_{j=1}^{K} \frac{\sin^2[\pi(t+jT)/T]}{\pi^2(t/T+j)^2} X(t+jT, \omega)\right|^2 \\ &= 2\,\underset{K\to\infty}{\text{l.i.m.}}\, E\left|\sum_{j=1}^{K} \frac{\sin^2 \pi t/T}{\pi^2(t/T+j)^2} X(t+jT, \omega)\right|^2 \\ &= 2\pi^{-4}\sin^4(\pi t/T)\,\underset{K\to\infty}{\text{l.i.m.}}\, E\left|\sum_{j=1}^{K} \frac{X(t+jT, \omega)}{(t/T+j)^2}\right|^2 \end{aligned}$$

which is, as in (2.5)

$$\le CM\sin^4(\pi t/T)\,.$$

The same estimate is obtained also for the second term of (3.2) and hence the lemma is proved.

We are now going to prove Theorem 1.

(i) From the definition (2.4) of $\hat{X}_T(T, \omega)$, we see

$$\hat{X}_T(T, \omega) = V_T(t)X(t, \omega) + \underset{K\to\infty}{\text{l.i.m.}} \sum_{j=-K, j\neq 0}^{K} V_T(t+jT)X(t+jT, \omega) \tag{3.3}$$

and hence

$$E|\hat{X}_T(t, \omega)|^2 \le 2E|V_T(t)X(t, \omega)|^2 + 2E\left|\underset{K\to\infty}{\text{l.i.m.}} \sum_{j=-K, j\neq 0}^{K} V_T(t+jT)X(t+jT, \omega)\right|^2 .$$

Noting $0\le V_T(t)\le 1$ $(-\infty<t<\infty)$, we have from Lemma 1, that the last one is

$$\leq 2M+CM=CM.$$

This proves (2.7) of Theorem 1, (i).

From (3.3), we have

$$E|\hat{X}_T(t,\omega)-V_T(t)X(t,\omega)|^2=E\left|\underset{K\to\infty}{\text{l.i.m.}}\sum_{j=-K,j\neq 0}^{K} V_T(t+jT)X(t+jT,\omega)\right|^2$$

which is, by Lemma 1,

$$\leq CM\sin^4(\pi t/T)\leq CMt^4/T^4\,. \tag{3.4}$$

Thus, for $2|t|\leq T$

$$E|\hat{X}_T(t,\omega)-X(t,\omega)|^2\leq 2E|\hat{X}_T(t,\omega)-V_T(t)X(t,\omega)|^2+2(1-V_T(t))^2E|X(t,\omega)|^2\,.$$

Since $0\leq 1-V_T(t)\leq Ct^2/T^2$, we have, using (3.4), that the last one is

$$\leq CMt^4/T^4\,.$$

This proves (2.8) of (i).

(ii) From (2.3), for $k=0,\pm 1,\cdots$,

$$\begin{aligned}c_k(\omega)&=\underset{K\to\infty}{\text{l.i.m.}}\frac{1}{T}\sum_{j=-K}^{K}\int_{jT-T/2}^{jT+T/2}\exp(-2k\pi iu/T)V_T(u)X(u,\omega)du\\&=\underset{K\to\infty}{\text{l.i.m.}}\frac{1}{T}\sum_{j=-K}^{K}\int_{-T/2}^{T/2}\exp(-2k\pi iu/T)V_T(u+jT)X(u+jT,\omega)du\\&=\frac{1}{T}\int_{-T/2}^{T/2}\exp(-2k\pi iu/T)\underset{K\to\infty}{\text{l.i.m.}}\sum_{j=-K}^{K}V_T(u+jT)X(u+jT,\omega)du\\&=\frac{1}{T}\int_{-T/2}^{T/2}\exp(-2k\pi iu/T)\hat{X}_T(u,\omega)du\,,\end{aligned}$$

that is, $c_k(\omega)=c_k(\omega,T)$ is the Fourier coefficient of $\hat{X}_T(t,\omega)$. This proves (ii) of Theorem 1.

4. On the series (1.4)

We shall in this section consider the series (1.4) with $a_k(\omega)$ given by (1.5), $k=0,\pm 1,\cdots$ and show that it is also an approximate Fourier series of $X(t,\omega)$ in the sense similar to Theorem 1. However for the existence of $a_k(\omega)$ and for the proof of the theorem corresponding to Theorem 1, we need more conditions than in Theorem 1.

Write the correlation function of $X(t,\omega)$ by

$$\rho(t,s)=EX(t,\omega)\overline{X(s,\omega)}\,. \tag{4.1}$$

Consider the following

Condition A. *There is a nonnegative bounded function* $r(t)$, $-\infty<t<\infty$ *with the following properties:*

(i) *$r(t)$ and $r(-t)$ are nonincreasing for $t>0$,*
(ii)

$$|\rho(t, s)| \leq r(t-s) \tag{4.2}$$

and

$$\int_{-\infty}^{\infty} r(t)dt < \infty . \tag{4.3}$$

We here remark that Hurd assumed the conditions (ii) in proving his result stated in the beginning of 1 that (1.1) represents in $(C, 1)$ sense the given stochastic process $X(t, \omega)$ under some natural condition on $X(t, \omega)$ at t.

We also remark that, in the following discussions, the condition (i) can be replaced by the condition that $r(|t|)$ is monotone for large $|t|$, but just for simplicity we assume (i).

We first give a simple lemma.

Lemma 2. *If $r(t)$ and $r(-t)$ are nonnegative and nonincreasing for $t>0$ and* (4.2) *is satisfied, then*

$$\sum_{k=-\infty, k\neq 0}^{\infty} r(kT) \leq \frac{1}{T} \int_{|t|\geq T/2} r(t)dt \tag{4.4}$$

for every $T>0$.

Proof. For $k \geq 1$

$$\frac{1}{T} \int_{kT}^{(k+1)T} r(t)dt \geq r((k+1)T) \geq \frac{1}{2} r((k+1)T) ,$$

$$\frac{1}{T} \int_{T/2}^{T} r(t)dt \geq \frac{1}{2} r(T) .$$

Hence

$$\sum_{k=1}^{\infty} r(kT) \leq \frac{2}{T} \int_{T/2}^{\infty} r(t)dt .$$

Similarly

$$\sum_{k=-\infty}^{-1} r(kT) \leq \frac{2}{T} \int_{-\infty}^{-T/2} r(t)dt .$$

These two inequalities give us (4.4).

Now assuming Condition A we define as a counterpart of (2.4)

$$\tilde{X}_T(t, \omega) = \underset{K, L\to\infty}{\text{l.i.m.}} \sum_{j=-L}^{K} W_T(t+jT) X(t+jT, \omega) . \tag{4.5}$$

In fact the right hand side of this exists, if Condition A is satisfied. We prove this.

Note that

$$W_T(t+jT)=\sin(\pi t/T)\exp(-\pi it/T)\frac{T}{\pi(t+jT)}.$$

For $1\leq K\leq K'$, we then have

$$(4.6)\qquad E\left|\sum_{j=K}^{K'}W_T(t+jT)X(t+jT,\omega)\right|^2$$

$$=[T\cdot\pi^{-1}\sin(\pi t/T)]^2E\left|\sum_{j=K}^{K'}\frac{X(t+jT,\omega)}{t+jT}\right|^2$$

$$\leq T^2\pi^{-2}E\sum_{j=K}^{K'}\sum_{k=K}^{K'}\frac{X(t+jT,\omega)\overline{X(t+kT,\omega)}}{(t+jT)(t+kT)}$$

$$\leq T^2\pi^{-2}\sum_{j=K}^{K'}\sum_{k=K}^{K'}\frac{|\rho(t+jT,t+kT)|}{|(t+jT)(t+kT)|}$$

$$\leq T^2\pi^{-2}\sum_{j=K}^{\infty}\sum_{k=K}^{\infty}\frac{r((j-k)T)}{|(t+jT)(t+kT)|}$$

$$=T^2\pi^{-2}\left[\sum_{j=K}^{\infty}\sum_{k=j}^{\infty}+\sum_{k=K}^{\infty}\sum_{j=k+1}^{\infty}\right]$$

in which

$$\sum_{j=K}^{\infty}\sum_{k=j}^{\infty}=\sum_{j=K}^{\infty}\frac{1}{|t+jT|}\sum_{l=0}^{\infty}\frac{r(-lT)}{|t+(j+l)T|}.$$

Taking K so large that $t+KT>0$, we have the last one to be

$$\leq\sum_{j=K}^{\infty}\frac{1}{(t+jT)^2}\sum_{l=0}^{\infty}r(-lT)$$

which is from (4.4) of Lemma 2

$$(4.7)\qquad \leq\frac{C}{KT-|t|}\left[r(0)+\frac{1}{T}\int_{|u|>T/2}r(u)du\right]$$

and this converges to zero as $K\to\infty$.

Similarly $\sum_{k=K}^{\infty}\sum_{j=k+1}^{\infty}$ also converges to zero as $K\to\infty$. Hence

$$\underset{\substack{K\to\infty\\K'>K}}{\text{l.i.m.}}\ E\left|\sum_{j=K}^{K'}W_T(t+jT)X(t+jT,\omega)\right|^2=0.$$

In the same way

$$\underset{\substack{L\to\infty\\L'>L}}{\text{l.i.m.}}\ E\left|\sum_{j=-L'}^{-L}W_T(t+jT)X(t+jT,\omega)\right|^2=0.$$

Thus (4.5) is well defined.

From the definition of $\tilde{X}_T(t, \omega)$, it is obvious that $\tilde{X}_T(t, \omega)$ is T-periodic as $\hat{X}_T(t, \omega)$ is.

We mention the following lemma which corresponds to Lemma 1.

Lemma 3. *For* $2|t|\leq T$

$$E\left|\underset{K\to\infty, L\to\infty}{\text{l.i.m.}} \sum_{j=-L, j\neq 0}^{K} W_T(t+jT)X(t+jT, \omega)\right|^2 \leq C\left[r(0)+\frac{1}{T}\int_{|u|>t/2} r(u)du\right]\sin^2(\pi t/T) . \tag{4.8}$$

The proof can be carried out just in the similar way to that of Lemma 1, if one notes (4.6) and (4.7) with $K=1$.

As a counterpart of Theorem 1, the following theorem holds.

Theorem 2. *Let* $X(t, \omega)$, $-\infty<t<\infty$ *be a second order stochastic process satisfying Condition A. Then we have*

(i) *For* $T\geq 2|t|$

$$E|\tilde{X}_T(t, \omega)|^2\leq C\left[r(0)+\frac{1}{T}\int_{|u|>t/2} r(u)du\right] \tag{4.9}$$

and

$$E|\tilde{X}_T(t, \omega)-X(t, \omega)|^2\leq C\left[r(0)+\frac{1}{T}\int_{|u|>t/2} r(u)du\right]t^2/T^2 . \tag{4.10}$$

(ii) *The Fourier series of* $\tilde{X}_T(t, w)$ *is given by (1.4) where* $a_k(w)$ *is defined by (1.5).*

Proof. The proof is carried out just in the way similar to that of Theorem 1.

(i) We have only to replace $V_T(t)$ by $W_T(t)$ in the proof of Theorem 1 and note that

$$|1-W_T(t)|=\left|\frac{\sin(\pi t/T)}{\pi t/T}\exp(-i\pi t/T)-1\right|\leq C|t|/T .$$

And use Lemma 3 instead of Lemma 1.

(ii) The proof is also the same as in Theorem 1 if one replaces $V_T(t)$ by $W_T(t)$.

Thus the series in (4.5) is an approximate Fourier series of $X(t, \omega)$.

5. Absolute convergence of an approximate Fourier series

We are now interested in the almost sure absolute convergence of an approximate Fourier series. We know [2], [3] that this problem played interesting roles particularly in the study of sample continuity of weakly stationary processes, general linear processes and periodic processes. This suggests that the similar situation may come out also for the general stochastic processes. We will show that it actually is.

For simplicity, we only consider the approximate Fourier series

(5.1) $$\sum_{k=-\infty}^{\infty} c_k(\omega, T) \exp(2k\pi it/T)$$

of a second order stochastic process $X(t, \omega)$ satisfying (1.6), considered in **2**, and we shall investigate the almost sure absolute convergence of (5.1).

Before doing this, we begin with mentioning a certain known result on the periodic process.

Let $X(t, \omega)$ be a 2π-periodic process for which $\int_{-\pi}^{\pi} E|X(t, \omega)|^2 dt<\infty$. Write

(5.2) $$M_{2\pi}(\delta, X)=\sup_{|h|\leq\delta}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} E|X(t+h, \omega)-X(t, \omega)|^2 dt\right]^{1/2}.$$

Then we have the following result which we state as

Lemma 4. *If*

(5.3) $$\sum_{n=1}^{\infty} n^{-1/2}M_{2\pi}(1/n, X)<\infty ,$$

then the Fourier series of $X(t, \omega)$ converges absolutely almost surely.

In fact this is an analogue of the well known Bernstein theorem on absolute convergence of an ordinary Fourier series [6] p. 240 and is a particular case of Theorem 3.1 in [4].

Now we go back to our main stream.

For a second order stochastic process $X(t, W)$ satisfying (1.6), we introduce a kind of continuity modulus

(5.4) $$N_p(\delta, X)=\left[\sup_{|h|\leq\delta}\int_{-\infty}^{\infty} \frac{E|X(t+h, \omega)-X(t, \omega)|^2}{1+|t|^{1+p}}dt\right]^{1/2} \quad (p>0) .$$

Obviously this is well defined.

When $X(t, \omega)$ is T-periodic, we write

(5.5) $$M_T(\delta, X)=\sup_{|h|\leq\delta}\left[\frac{1}{T}\int_{-T/2}^{T/2} E|X(t+h, \omega)-X(t, \omega)|^2 dt\right]^{1/2}.$$

Let $\hat{X}_T(t, \omega)$ be the T-periodic approximate process of $X(t, \omega)$ as before. We first give

Lemma 5. *For $0<\delta<T/3$, $1\geq p>0$, we have*

(5.6) $$M_T(\delta, \hat{X}_T)\leq CM^{1/2}T^{-1}\cdot\delta+CT^{p/2}N_p(\delta, X) .$$

M is the constant in (1.6).

We first note that for $|h|\leq\delta$, $T>2|t|$,

(5.7) $$\begin{aligned}|V_T(t+h+jT)-V_T(t+jT)|&\leq C|h|T^{-1}j^{-2} , && \textit{for } j\neq 0 ,\\ &\leq C|h|T^{-1} , && \textit{for } j=0 .\end{aligned}$$

Actually for $j\neq 0$,

$$(5.8)\qquad \frac{d}{du}V_T(u+jT)=\frac{d}{du}\left[\frac{\sin(\pi u/T)}{\pi(u/T+j)}\right]^2$$
$$=\frac{2}{\pi^2}\left[\frac{\sin(\pi u/T)}{u/T+j}\right]\frac{\pi(u/T+j)-\tan(\pi u/T)}{(u/T+j)^2}\cdot\cos(\pi u/T)\cdot\frac{1}{T}$$

and then

$$\left|\frac{d}{du}V_T(u+jT)\right|\leq\frac{C}{T}\frac{|j|+C(|u|/T)^3}{|u/T+j|^3}\,.$$

Hence for any $0<\theta<1$, from $||t+\theta h|/T+j|\geq|j|-(|t|+\delta)T^{-1}\geq|j|-5/6$, we see

$$\left|\left[\frac{d}{du}V_T(u+jT)\right]_{u=t+\theta h}\right|\leq CT^{-1}j^{-2}\,,$$

which gives us, because of the mean value theorem, that the left hand side of (5.7) is, for some $(0<\theta'<1)$,

$$|h|\cdot\left|\left[\frac{d}{du}V_T(u+jT)\right]_{u=t+\theta' h}\right|\leq C|h|T^{-1}j^{-2}\,.$$

For $j=0$, from (5.8), we easily see

$$\left|\frac{d}{du}V_T(u)\right|\leq CT^{-1}$$

and using this, we obtain the second inequality of (5.7).

Now we proceed to prove Lemma 5.

Proof of Lemma 5.

First we shall make the estimation of

$$S(t)=E|\hat{X}_T(t+h,\omega)-\hat{X}_T(t,\omega)|^2\,.$$

Putting the definition of $\hat{X}$ in the right hand side, we see

$$S(t)=E\left|\underset{K\to\infty}{\text{l.i.m.}}\sum_{j=-K}^{K}[V_T(t+h+jT)-V_T(t+jT)]X(t+h+jT,\omega)\right.$$
$$\left.+V_T(t+jT)[X(t+h+jT,\omega)-X(t+jT,\omega)]\right|^2$$
$$\leq 2\,\underset{K\to\infty}{\text{l.i.m.}}\,E\left|\sum_{j=-K}^{K}[V_T(t+h+jT)-V_T(t+jT)]X(t+h+jT,\omega)\right|^2$$
$$+2\,\underset{K\to\infty}{\text{l.i.m.}}\,E\left|\sum_{j=-K}^{K}V_T(t+jT)[X(t+h+jT,\omega)-X(t+jT,\omega)]\right|^2$$
$$=S_1(t)+S_2(t)\,,$$

say.

$$S_1(t)\leq CE[|V_T(t+h)-V_T(t)|^2|X(t+h,\omega)|^2]$$
$$+C\operatorname*{l.i.m.}_{K\to\infty} E\left|\sum_{j=-K,j\neq 0}^{K}[V_T(t+h+jT)-V_T(t+jT)]X(t+h+jT,\omega)\right|^2$$

which is, by (5.7) and the Minkowski inequality

$$\leq CMh^2T^{-2}$$
$$+C\left\{\sum_{j=-\infty,j\neq 0}^{\infty}[V_T(t+h+jT)-V_T(t+jT)][E|X(t+h+jT,\omega)|^2]^{1/2}\right\}^2$$
$$\leq CMh^2T^{-2}+CM|h|^2T^{-2}\left(\sum_{j=-\infty,j\neq 0}^{\infty}j^{-2}\right)^2$$

$$\leq CMh^2T^{-2}\,. \tag{5.9}$$

In the similar way,

$$S_2(t)\leq CV_T^2(t)E|X(t+h,\omega)-X(t,\omega)|^2$$
$$+C\left\{\sum_{j=-\infty,j\neq 0}^{\infty}V_T(t+jT)[E|X(t+h+jT,\omega)-X(t+jT,\omega)|^2]^{1/2}\right\}^2.$$

Using the Cauchy inequality to the last term, we see

$$S_2(t)\leq CV_T^2(t)E|X(t+h,\omega)-X(t,\omega)|^2$$
$$+C\sum_{j=-\infty,j\neq 0}^{\infty}V_T(t+jT)\sum_{j=-\infty,j\neq 0}^{\infty}V_T(t+jT)E|X(t+h+jT,\omega)-X(t+jT,\omega)|^2\,.$$

Since $V_T(t+jT)\leq Cj^{-2}$ $(j\neq 0)$ *for* $2|t|\leq T$, we have, replacing $V_T^2(t)$ in the first term of the last expression by the larger $V_T(t)$,

$$S_2(t)\leq CV_T(t)E|X(t+h,\omega)-X(t,\omega)|^2$$
$$+C\sum_{j=-\infty,j\neq 0}^{\infty}V_T(t+jT)E|X(t+h+jT,\omega)-X(t+jT,\omega)|^2\,.$$

Looking at (5.9),

$$\frac{1}{T}\int_{-T/2}^{T/2}S(t)dt\leq CMh^2T^{-2}+\frac{C}{T}\int_{-T/2}^{T/2}V_T(t)E|X(t+h,\omega)-X(t,\omega)|^2dt$$
$$+\frac{C}{T}\sum_{j=-\infty,j\neq 0}^{\infty}\int_{-T/2}^{T/2}V_T(t+jT)E|X(t+h+jT,\omega)-X(t+jT,\omega)|^2dt$$
$$\leq CMh^2T^{-2}+\frac{C}{T}\int_{-\infty}^{\infty}V_T(t)E|X(t+h,\omega)-X(t,\omega)|^2dt\,.$$

Noting that $V_T(t)\leq|\pi t/T|^{-(1+p)}\leq CT^{1+p}(1+|t|^{1+p})^{-1}$ *for* $|t|>T/2$, and $V_T(t)\leq 1\leq CT^{1+p}(1+|t|^{1+p})^{-1}$ *for* $|t|\leq T/2$, we finally get

$$\frac{1}{T}\int_{-T/2}^{T/2}S(t)dt\leq CMh^2T^{-2}+CT^p\int_{-\infty}^{\infty}\frac{E|X(t+h,\omega)-X(t,\omega)|^2}{1+|t|^{1+p}}dt$$

from which we have (5.6).

Now we give

Theorem 3. *If $X(t, \omega)$ is a second order stochastic process satisfying (1.6) and*

$$\sum_{n=1}^{\infty} n^{-1/2} N_p(n^{-1}, X) < \infty , \tag{5.10}$$

for some $0 < p \leq 1$, then, for every fixed $T > 0$, the approximate T-periodic Fourier series (2.2) of $X(t, \omega)$ converges absolutely almost surely.

This follows from Lemmas 4 and 5. In fact, writing $\hat{X}_T(Tt/2\pi, \omega) = U(t, \omega)$, $U(t, \omega)$ is 2π-periodic and hence by Lemma 4, if

$$\sum_{n=1}^{\infty} n^{-1/2} M_{2\pi}(n^{-1}, U) < \infty , \tag{5.11}$$

then the Fourier series of $U(t, \omega)$ is almost surely absolutely convergent. As is easily seen the Fourier series of $U(t, \omega)$ is no more than the Fourier series of T-periodic $\hat{X}_T(t, \omega)$. Also we see that $M_{2\pi}(\delta, U) = M_T(\delta T/2\pi, \hat{X}_T)$ and (5.11) turns out to be

$$\sum_{n=1}^{\infty} n^{-1/2} M_T(an^{-1}, \hat{X}_T) < \infty , \tag{5.12}$$

where $a = T/2\pi$. As far as T is fixed, a is a positive constant and $M_T(\delta, \hat{X}_T)$ is nondecreasing with respect to δ, and then (5.12) is equivalent to

$$\sum_{n=1}^{\infty} n^{-1/2} M_T(n^{-1}, \hat{X}_T) < \infty . \tag{5.13}$$

This is seen in the following way. Letting $M(t)$ be nondecreasing for $t > 0$,

$$a^{-1/2} n^{-1/2} M(an^{-1}) \leq \int_{(n-1)a^{-1}}^{na^{-1}} y^{-1/2} M(y^{-1}) dy$$

and then

$$\begin{aligned} a^{-1/2} \sum_{n=[a]+2}^{\infty} n^{-1/2} M(an^{-1}) &\leq \sum_{n=[a]+2}^{\infty} \int_{(n-1)a^{-1}}^{na^{-1}} y^{-1/2} M(y^{-1}) dy \\ &\leq \int_1^{\infty} y^{-1/2} M(y^{-1}) dy \leq \sum_{n=1}^{\infty} \int_n^{n+1} y^{-1/2} M(y^{-1}) dy \\ &\leq \sum_{n=1}^{\infty} n^{-1/2} M(n^{-1}) . \end{aligned} \tag{5.14}$$

Hence the convergence of $\sum_{n=1}^{\infty} n^{-1/2} M(n^{-1})$ implies that of $\sum_{n=1}^{\infty} n^{-1/2} M(an^{-1})$. The converse is similarly shown.

Now from (5.6) with $\delta = n^{-1}$ and (5.10), (5.13) holds. And then the almost sure absolute convergence of the approximate Fourier series of $X(t, \omega)$ follows.

For later use, we give the following lemma.

Lemma 6. *We have, for $T > 4$*

(5.15) $$\sum_{|n|=[4T]}^{\infty} E|c_n(\omega, T)| \le CT^{1/2} \sum_{n=1}^{\infty} n^{-1/2} M_T(n^{-1}, \hat{X}_T) .$$

Proof. Let $U(t, \omega)=\hat{X}_T(Tt/(2\pi), \omega)$ as before. Let b be the largest integer such that $2^b \le 4T$. Write the Fourier coefficient of 2π-periodic $U(t, \omega)$ by $d_n(\omega)$. From the known argument in the proof of Bernstein's theorem [6] p. 240 or the argument in the proof of Theorem 3.1 in [4] with a minor trivial change, we have

$$\begin{aligned}\sum_{n=[4T]}^{\infty} E|d_n(\omega)| &\le \sum_{n=2^b}^{\infty} \le \sum_{n=2^{b-1}+1}^{\infty} \\ &= \sum_{n=b}^{\infty} \sum_{j=2^{n-1}+1}^{2^n} E|d_j(\omega)| \le C \sum_{n=2^{b-2}+1}^{\infty} n^{-1/2} M_{2\pi}(n^{-1}, U) \\ &\le C \sum_{n=2^{b-2}+1}^{\infty} n^{-1/2} M_T(an^{-1}, \hat{X}_T)\end{aligned}$$

where $a=T/(2\pi)$ as before and the last one, is from (5.14), not larger than

$$Ca^{1/2} \sum_{n=1}^{\infty} n^{-1/2} M_T(n^{-1}, \hat{X}_T) .$$

For $\sum_{n=-\infty}^{-[4T]} E|d_n(\omega)|$, we have the same estimate and noting $c_n(\omega, T)=d_n(\omega)$, we obtain (5.15).

We add a property of $\hat{X}_T(t, \omega)$ which we state as

Lemma 7. *If (5.10) is satisfied and $X(t, \omega)$ is stochastically continuous, then $\hat{X}_T(t, \omega)$ is sample continuous, namely there is a second order stochastic process $\hat{X}_T^{(0)}(t, \omega)$ which is a continuous function of t almost surely, and for each $-\infty<t<\infty$,*

(5.16) $$\hat{X}_T(t, \omega)=\hat{X}_T^{(0)}(t, \omega)$$

almost surely.

Note that for each t, (5.16) holds for $\omega \in \Omega_1$, $P(\Omega_1)=1$, where Ω_1 may depend on t. $\hat{X}_T^{(0)}(t, \omega)$ is a modification of $\hat{X}_T(t, \omega)$.

Because of (5.10), the approximate Fourier series $\sum_{n=-\infty}^{\infty} c_n(\omega)e^{2n\pi it/T}$ is almost surely absolutely convergent. Now define

$$\hat{X}_T^{(0)}(t, \omega)= \sum_{n=-\infty}^{\infty} c_n(\omega)e^{2n\pi it/T}$$

which is continuous for $-\infty<t<\infty$ almost surely.

If one notices that $\hat{X}_T(t, \omega)$ is stochastically continuous, then Lemma 7 is, as a matter of fact, substantially known. See the proof of Theorem 6.1 of [4]. Stochastic continuity of $\hat{X}_T(t, \omega)$ is easily shown from the fact that

$$\hat{X}_{T,K}(t, \omega)= \sum_{j=-K}^{K} V_T(t+jT)X(t+jT, \omega)$$

is, for each positive integer K, stochastically continuous and converges to $\hat{X}_T(t, \omega)$ as $K\to\infty$ in $L^2(\Omega)$.

6. Sample continuity of a second order stochastic process

The problem of sample continuity of a second order process was investigated by several authors. For the references, see [4] for instance. The author of the present paper has treated the problem for the periodic process by using the Fourier series and for the weakly stationally process or the general linear process using the approximate Fourier series [2], [3], [4] as he said in **1.**

For a general process, the Fourier series argument is also applicable if one considers the truncated process and its suitable periodic extention. In this section, however we shall show that the method which made use of the approximate Fourier series and was used in [2], [3], [4], enables us again to disscus the sample continuity.

We suppose, as before, $X(t, \omega)$ satisfies (1.6) and (5.10). We take the modification of $\hat{X}_T(t, \omega)$

$$\hat{X}_T^{(0)}(t, \omega)=\sum_{n=-\infty}^{\infty} c_n(\omega, \hat{X}_T)e^{2n\pi it/T} . \tag{6.1}$$

Here in place of $c_n(\omega)=c_n(\omega, T)$ we have written as $c_n(\omega, \hat{X}_T)$ in order to make it clearer to be the Fourier coefficient of the T-periodic process $\hat{X}_T(t, \omega)$, so that if we write $c_n(\omega, \hat{X}_{2T})$ for example, then it means $\frac{1}{2T}\int_{-T}^{T} \hat{X}_{2T} \exp(-2n\pi it/(2T))dt$.

Now we set

$$S_T(t, \omega)=\hat{X}_{2T}^{(0)}(t, \omega)-\hat{X}_T^{(0)}(t, \omega) . \tag{6.2}$$

We have considered the similar quantity with $T=2^k$ in [2], [3], [4].

Note first that the Fourier series in (6.1) is absolute convergent almost surely in view of (5.10), and also that $\hat{X}_T^{(0)}(t, \omega)=\hat{X}_T(t, \omega)$ for almost all t, almost surely and then $c_n(\omega, \hat{X}_T)=c_n(\omega, \hat{X}_T^{(0)})$.

We have

$$\begin{aligned}
S_T(t, \omega)&=\sum_{n=-\infty}^{\infty} c_n(\omega, \hat{X}_{2T}) \exp(2n\pi it/(2T))-\sum_{n=-\infty}^{\infty} c_n(\omega, \hat{X}_T) \exp(2n\pi it/T)\\
&=\sum_{m=-\infty}^{\infty} c_{2m}(\omega, \hat{X}_{2T}) \exp(2m\pi it/T)\\
&\quad+\sum_{m=-\infty}^{\infty} c_{2m+1}(\omega, \hat{X}_{2T}) \exp((2m+1)\pi it/T)-\sum_{m=-\infty}^{\infty} c_m(\omega, \hat{X}_T) \exp(2m\pi it/T)\\
&=\sum_{m=-\infty}^{\infty} [c_{2m}(\omega, \hat{X}_{2T})+c_{2m+1}(\omega, \hat{X}_{2T})-c_m(\omega, \hat{X}_T)] \cdot \exp(2m\pi it/T)\\
&\quad+[\exp(\pi it/T)-1]\sum_{m=-\infty}^{\infty} c_{2m+1}(\omega, \hat{X}_{2T}) \exp(2m\pi it/T)\\
&=S_{T,1}(t, \omega)+S_{T,2}(t, \omega)
\end{aligned}$$

say.

Here

$$[S_{T,2}(t,\omega)|\le C|t|\cdot T^{-1}\sum_{m=-\infty}^{\infty}|c_{2m+1}(\omega,\hat{X}_{2T})|$$

$$\le C|t|\cdot T^{-1}\sum_{m=-\infty}^{\infty}|c_m(\omega,\hat{X}_{2T})|\,. \tag{6.3}$$

The inside of [ ] in the series $S_{T,1}(t,\omega)$ is

$$\frac{1}{2T}\underset{K\to\infty}{\text{l.i.m.}}\int_{-K}^{K}\exp\left(-2(2m)\pi iu/(2T)\right)V_{2T}(u)X(u,\omega)du \tag{6.4}$$
$$+\frac{1}{2T}\underset{K\to\infty}{\text{l.i.m.}}\int_{-K}^{K}\exp\left(-2(2m+1)\pi iu/(2T)\right)V_{2T}(u)X(u,\omega)du$$
$$-\frac{1}{T}\underset{K\to\infty}{\text{l.i.m.}}\int_{-K}^{K}\exp\left(-2m\pi iu/T\right)V_T(u)X(u,\omega)du\,.$$

Since

$$V_T(u)=V_{2T}(u)\cos^2(\pi u/(2T))$$

and

$$1+\exp(\pi iu/T)-2\cos^2(\pi u/(2T))=i\sin(\pi u/T)\,,$$

(6.4) is equal to

$$\frac{i}{2T}\underset{K\to\infty}{\text{l.i.m.}}\int_{-K}^{K}\exp\left(-2(2m)\pi iu/(2T)\right)V_{2T}(u)\sin(\pi u/T)X(u,\omega)du\,,$$

the existence of which is easily shown as the case of $c_n(\omega)$ in (2.3).

Writing

$$Y(t,\omega)=Y(t,T,\omega)=iX(t,\omega)\sin(\pi t/T) \tag{6.5}$$
$$\hat{Y}_{2T}(t,\omega)=\underset{K\to\infty}{\text{l.i.m.}}\sum_{j=-K}^{K}V_{2T}(t+2jT)iX(t+2jT)\sin(\pi t/T)\,.$$

The $2T$-periodic Fourier series of $\hat{Y}_{2T}(t,\omega)$ is thought of as a $2T$-periodic approximate Fourier series of $Y(t,\omega)$. $Y(t,\omega)$ depends also on T, but this gives no trouble at all. (6.4) is just $c_{2m}(\omega,\hat{Y}_{2T})$. Needless to say, $\hat{Y}_{2T}(t,\omega)$ is $2T$-periodic, and we can prove that $\sum|c_{2m}(\omega,\hat{Y}_{2T})|<\infty$ almost surely. (see the proof of Theorem 4 below, in particular (6.9) with Y for Z)

Hence we have, from (6.3) and the above,

$$|S_T(t,\omega)|\le\sum_{m=-\infty}^{\infty}|c_m(\omega,\hat{Y}_{2T})|+C|t|T^{-1}\sum_{m=-\infty}^{\infty}|c_m(\omega,\hat{X}_{2T})|\,. \tag{6.6}$$

Now we are going to prove

Theorem 4. *If $X(t,\omega)$ satisfies (1.6) and (5.10) for some $0<p<\infty$, then $X(t,\omega)$ is sample continuous on $(-\infty,\infty)$.*

Proof. In order to prove the theorem, it is sufficient to show that $X_1(t, \omega)=X(t, \omega)/(1+|t|)$ is sample continuous.

Obviously

$$E|X_1(t, \omega)|^2 \le M(1+|t|)^{-2} . \tag{6.7}$$

First we suppose $0<p<1$. Write $\hat{Z}_{2T}(t, \omega)$ for $\hat{Y}_{2T}(t, \omega)$ with $X_1(t, \omega)$ in place of $X(t, \omega)$. The Fourier series of $\hat{Z}_{2T}(t, \omega)$ is a $2T$-periodic approximate Fourier series of $Z(t, \omega)=iX_1(t, \omega)\sin(\pi t/T)$.

And we see

$$\begin{aligned}
N_p(\delta, Z)&=\left[\sup_{|h|\le\delta}\int_{-\infty}^{\infty} E|\sin(\pi(t+h)/T)X_1(t+h, \omega)-\sin(\pi t/T)X_1(t, \omega)|^2/(1+|t|^{1+p})dt\right]^{1/2}\\
&\le\left[\sup_{|h|\le\delta}\int_{-\infty}^{\infty} |\sin(\pi(t+h)/T)-\sin(\pi t/T)|^2 E|X_1(t+h, \omega)|^2/(1+|t|^{1+p})dt\right]^{1/2}\\
&\quad+\left[\sup_{|h|\le\delta}\int_{-\infty}^{\infty} \sin^2(\pi t/T)E|X_1(t+h, \omega)-X_1(t, \omega)|^2/(1+|t|^{1+p})dt\right]^{1/2}\\
&\le C\delta M^{1/2}T^{-1}+C\sup_{|h|\le\delta}\left[\int_{-\infty}^{\infty} E|X_1(t+h, \omega)-X_1(t, \omega)|^2(t/T)^2/(1+|t|^{1+p})dt\right]^{1/2} .
\end{aligned}$$

The second term of the last expression is

$$\begin{aligned}
&\le C\sup_{|h|\le\delta}\left\{\int_{-\infty}^{\infty} E|X(t+h, \omega)|^2[(1+|t+h|)^{-1}-(1+|t|)^{-1}]^2(t/T)^2/(1+|t|^{1+p})dt\right\}^{1/2}\\
&\quad+C\sup_{|h|\le\delta}\left[\int_{-\infty}^{\infty} E|X(t+h, \omega)-X(t, \omega)|^2(1+|t|)^{-2}(t/T)^2/(1+|t|^{1+p})dt\right]^{1/2}
\end{aligned}$$

$$\le C\delta M^{1/2}T^{-1}+CT^{-1}N_p(\delta, X) . \tag{6.8}$$

Hence we have, from the condition (5.10)

$$\sum_{n=1}^{\infty} n^{-1/2}N_p(n^{-1}, Z)<\infty \tag{6.9}$$

which gives us the almost sure absolute convergence of the Fourier series of $\hat{Z}_{2T}(t, \omega)$ for each $T>0$.

Here we give a remark. We have

$$E|\hat{Z}_{2T}(t, \omega)-iX_1(t, \omega)\sin(\pi t/T)|^2 \le CMt^4T^{-6} . \tag{6.10}$$

This is shown in just the same way as proving (2.8). Write

$$\hat{X}^{(0)}_{1,T}(t, \omega)=\sum_{n=-\infty}^{\infty} c_n(\omega, \hat{X}_{1,T})e^{2n\pi it/T} ,$$

$$\hat{Z}^{(0)}_{2T}(t, \omega)=\sum_{n=-\infty}^{\infty} c_n(\omega, \hat{Z}_{2T})e^{2n\pi it/(2T)} ,$$

where $\hat{X}_{1,T}(t, \omega)$ is the T-periodic approximate process of $X_1(t, \omega)$. Then from (6.6), we have, writing

$$S'_T(t,\omega)=\hat{X}^{(0)}_{1,2T}(t,\omega)-\hat{X}^{(0)}_{1,T}(t,\omega)\,, \tag{6.11}$$
$$|S'_T(t,\omega)|\leq \sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{Z}_{2T})|+C|t|T^{-1}\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{X}_{1,2T})|\,.$$

Let A be any positive number and w shall prove that there is a modification $X_1^{(0)}(t,\omega)$ of $X_1(t,\omega)$, which is independent of A and continuous for $|t|<A$.

Now letting $\{\varepsilon_k, k=1,2,\cdots\}$ be a sequence of positive numbers and T so large that $A<T$, we have

$$\begin{aligned}
P\Big(\sup_{|t|\leq A}|S'_T(t,\omega)|>\varepsilon_k\Big)&\leq P\Big(\sup_{|t|\leq A}\Big[\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{Z}_{2T})|\\
&\qquad +C|t|T^{-1}\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{X}_{1,2T})|\Big]>\varepsilon_k\Big)\\
&\leq P\Big(\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{Z}_{2T})|>\varepsilon_k/2\Big)\\
&\qquad +P\Big(CAT^{-1}\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{X}_{1,2T})|>\varepsilon_k/2\Big)\\
&=J_1+J_2\,,
\end{aligned} \tag{6.12}$$

say. By the Markov inequality,

$$\begin{aligned}
J_1&\leq 2\varepsilon_k^{-1}E\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{Z}_{2T})|\\
&\leq 2\varepsilon_k^{-1}E\sum_{|n|\leq[4T]}|c_n(\omega,\hat{Z}_{2T})|+2\varepsilon_k^{-1}E\sum_{|n|\geq[4T]+1}|c_n(\omega,\hat{Z}_{2T})|\\
&=J_{11}+J_{12}\,,
\end{aligned} \tag{6.13}$$

say. By the Cauchy inequality,

$$\begin{aligned}
J_{11}&\leq C\varepsilon_k^{-1}E\Big[\sum_{|n|\leq[4T]}|c_n(\omega,\hat{Z}_{2T})|^2\Big]^{1/2}T^{1/2}\\
&\leq C\varepsilon_k^{-1}T^{1/2}E\Big[\sum_{n=-\infty}^{\infty}|c_n(\omega,\hat{Z}_{2T})|^2\Big]^{1/2}.
\end{aligned}$$

By the Parseval relation, the last one is

$$\begin{aligned}
&C\varepsilon_k^{-1}T^{1/2}E\Big[\frac{1}{2T}\int_{-T}^{T}|\hat{Z}_{2T}(t,\omega)|^2dt\Big]^{1/2}\\
&\qquad\leq C\varepsilon_k^{-1}T^{1/2}\Big[\frac{1}{2T}\int_{-T}^{T}E|\hat{Z}_{2T}(t,\omega)|^2dt\Big]^{1/2}
\end{aligned}$$

which is because of (6.7) and (6.10)

$$\begin{aligned}
&\leq C\varepsilon_k^{-1}T^{1/2}\Big[\frac{1}{2T}\int_{-T}^{T}\{E|\sin(\pi t/T)X_1(t,\omega)|^2+CMt^4T^{-6}\}dt\Big]^{1/2}\\
&\leq C\varepsilon_k^{-1}T^{-1/2}\Big[\frac{1}{2T}\int_{-T}^{T}|t|^2\cdot E|X_1(t,\omega)|^2dt+CM\Big]^{1/2}
\end{aligned} \tag{6.14}$$

$$\leq C\varepsilon_k^{-1}T^{-1/2}\left[\frac{1}{2T}\int_{-T}^{T} E|X(t,\omega)|^2dt+CM\right]^{1/2}$$
$$\leq C\varepsilon_k^{-1}M^{1/2}T^{-1/2}\,.$$

As to J_{12}, we have, from Lemma 6 with Z in place of X

$$J_{12}\leq C\varepsilon_k^{-1}T^{1/2}\sum_{n=1}^{\infty} n^{-1/2}M_{2T}(n^{-1},\hat{Z}_{2T})$$

which is, by Lemma 5 and (6.8),

$$\leq C\varepsilon_k^{-1}T^{1/2}\sum_{n=1}^{\infty} n^{-1/2}[CM^{1/2}T^{-1}n^{-1}+CT^{p/2}N_p(n^{-1},Z)]$$
$$\leq C\varepsilon_k^{-1}T^{-(1-p)/2}M^{1/2}+C\varepsilon_k^{-1}T^{(1+p)/2}\sum_{n=1}^{\infty} T^{-1}n^{-1/2}N_p(n^{-1},X)$$
$$\leq C\varepsilon_k^{-1}T^{-(1-p)/2}\left[M^{1/2}+\sum_{n=1}^{\infty} n^{-1/2}N_p(n^{-1},X)\right].$$

Putting the estimates of J_{11} and J_{12} just obtained into (6.11), we have

$$J_1\leq C\varepsilon_k^{-1}T^{-(1-p)/2}\left[M^{1/2}+\sum_{n=1}^{\infty} n^{-1/2}N_p(n^{-1},X)\right]. \tag{6.15}$$

J_2 is similarly handled. In fact J_2 is no more than J_1 with $C^{-1}A^{-1}\varepsilon_kT$ in place of ε_k and $\hat{X}_{1,2T}$ in place of $\hat{Z}_{2T}$, and using, in the course of getting (6.14), the estimate similar to (6.10),

$$E|\hat{X}_{1,2T}(t,\omega)-X_1(t,\omega)|^2\leq CMt^2T^{-4}\,,$$

which is obtained from (2.8) and (6.7), we have after some manipulation,

$$J_2\leq CA\varepsilon_k^{-1}T^{-(1-p)/2}\left[M^{1/2}+\sum_{n=1}^{\infty} n^{-1/2}N_p(n^{-1},X)\right].$$

Putting this estimate and (6.15) into (6.12), we have

$$P\left(\sup_{|t|\leq A}|S_T'(t,\omega)|>\varepsilon_k\right)\leq D\cdot\varepsilon_k^{-1}T^{-(1-p)/2}\,, \tag{6.16}$$

where D is a constant $CA[M^{1/2}+\sum_{n=1}^{\infty} n^{-1/2}N_p(n^{-1},X)]$ which is independent of ε_k and T.

We are now in the final stage of the proof of Theorem 4, for $0<p<1$.

Take $T=2^k$, $\varepsilon_k=2^{-\alpha k}$ where $\alpha=(1-p)/4$. Then from (6.16)

$$P\left(\sup_{|t|\leq A}|\hat{X}^{(0)}_{1,2^{k+1}}(t,\omega)-\hat{X}^{(0)}_{1,2^k}(t,\omega)|>2^{-\alpha k}\right)\leq D2^{-\alpha k}\,,$$

for $k\geq 2$. Since $\sum 2^{-\alpha k}<\infty$, by the Borel-Cantelli lemma

$$\sum_{k=1}^{\infty}|\hat{X}^{(0)}_{1,2^{k+1}}(t,\omega)-\hat{X}^{(0)}_{1,2^k}(t,\omega)|<\infty$$

uniformly for $|t|\leq A$, almost surely and hence as $k\to\infty$, $\hat{X}_{1,2^k}^{(0)}(t, \omega)$ converges uniformly for $|t|\leq A$, almost surely. Thus its limit $X_1^{(0)}(t, \omega)$ is continuous for $|t|\leq A$ since $\hat{X}_{1,2^k}^{(0)}(t, \omega)$ is continuous for $|t|\leq A$ almost surely.

On the other hand, from Theorem 1, (2.8) with $\hat{X}_{1,T}$ in place of $\hat{X}_T$, and X_1 in place of X, $\hat{X}_{1,2^k}(t, \omega)$ converges for $|t|\leq A$ to $X_1(t, \omega)$ in $L^2(\Omega)$. Therefore for each $t(|t|\leq A)$, $X_1(t, \omega)=X_1^{(0)}(t, \omega)$ almost surely. Namely $X_1(t, \omega)$ is sample continuous. Hence the theorem is proved.

Finally we show that the restriction $0<p<1$ can be removed and the theorem is true for $0<p<\infty$. Let $p\geq 1$. We consider $X_{p-q}(t, \omega)=X(t, \omega)/(1+|t|^{p-q})$ where q is any positive number less than 1. Then we can prove

$$N_q(\delta, X_{p-q})\leq C_{p,q}[M^{1/2}(\delta+\delta^{p-q+1/2})+N_p(\delta, X)]\ , \tag{6.17}$$

$C_{p,q}$ being a constant depending only on p and q and $0<\delta<1$. In fact

$$\begin{aligned}N_q^2(\delta, X_{p-q})&\leq 2\sup_{|h|\leq\delta}\int_{-\infty}^{\infty} E|X(t+h, \omega)|^2\Big(\frac{1}{1+|t+h|^{p-q}}-\frac{1}{1+|t|^{p-q}}\Big)^2\frac{dt}{1+|t|^{1+q}}\\ &\quad+2\sup_{|h|\leq\delta}\int_{-\infty}^{\infty} E|X(t+h, \omega)-X(t, \omega)|^2\frac{1}{(1+|t|^{p-q})^2}\frac{dt}{1+|t|^{1+q}}\\ &=2K_1+2K_2\ ,\end{aligned}$$

say. Noting that $(1+|t|^{p-q})^2(1+|t|^{1+q})\geq 1+|t|^{1+p}$ for all t, we have

$$K_2\leq N_p^2(\delta, X)\ . \tag{6.18}$$

Writing

$$P(t, h)=\frac{1}{1+|t+h|^{p-q}}-\frac{1}{1+|t|^{p-q}}\ ,$$

we have

$$\begin{aligned}K_1&\leq M\sup_{|h|\leq\delta}\int_{-\infty}^{\infty} P^2(t, h)\frac{dt}{1+|t|^{1+q}}\\ &=M\sup_{|h|\leq\delta}\Big(\int_{|t|<2|h|}+\int_{|t|\geq 2|h|}\Big)\\ &=M(K_{1,1}+K_{1,2})\end{aligned}$$

say. For $|t|<2|h|$, $|P(t, h)|\leq||t+h|^{p-q}-|t|^{p-q}|\leq C_{p,q}|h|^{p-q}$. Hence

$$K_{1,1}\leq\sup_{|h|\leq\delta}\int_{|t|<2|h|} P^2(t, h)dt\leq C_{p,q}\delta^{2(p-q)+1}\ . \tag{6.19}$$

Now as we easily see, for both cases, $p-q\geq 1$, and $p-q<1$,

$$||t+h|^{p-q}-|t|^{p-q}|\leq C_{p,q}|t|^{p-q-1}|h|\ ,$$

for $|t|\geq 2|h|$, and thus

$$K_{1,2}\leq C_{p,q}\sup_{|h|\leq\delta}h^2\int_{|t|\geq 2|h|}\frac{|t|^{2(p-q-1)}dt}{(1+|t+h|^{p-q})^2(1+|t|^{p-q})^2(1+|t|^{1+q})} \tag{6.20}$$

$$K_{1,2}\leq C_{p,q}\sup_{|h|\leq\delta}h^2\left[\int_{|t|>2}+\int_{2\geq|t|\geq2|h|}\right]$$
$$\leq C_{p,q}\delta^2\left[C_{p,q}+\sup_{|h|\leq\delta}\int_{2\geq|t|\geq2|h|}|t|^{2(p-q-1)}dt\right]$$
$$\leq C_{p,q}\delta^2(C_{p,q}+C+\delta^{2(p-q)-1})$$
$$\leq C_{p,q}(\delta^2+\delta^{2(p-q)+1}) .$$

From this and (6.19), we have

$$K_1\leq C_{p,q}M(\delta^2+\delta^{2(p-q)+1}) .$$

Noting (6.18), we have

$$N_q^2(\delta, X_{p-q})\leq C_{p,q}[M(\delta^2+\delta^{2(p-q)+1})+N_p^2(\delta, X)]$$

from which we get (6.17). Hence

$$\sum_{n=1}^{\infty} n^{-1/2}N_q(n^{-1}, X_{p-q})\leq C_{p,q}\left[M^{1/2}+\sum_{n=1}^{\infty} n^{-1/2}N_p(n^{-1}, X)\right].$$

So the series on the left hand side is convergent from (5.10). Thus from what we have shown above, $X_{p-q}(t, \omega)$ is sample continuous which implies the sample continuity of $X(t, \omega)$. Theorem is then shown.

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