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ON THE MORSE FORMULA

by

V.Ya. BELOV*

Department of Mathematics
Faculty of Science and Technology, Keio University
Hiyoshi, Yokohama 223, Japan

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Let two systems of numbers $\{m_i\}_{i=1}^{p+1}$, $\{k_i\}_{i=1}^{p}$ satisfying the relations

$$m_1 > \cdots > m_p > m_{p+1} = 0, \qquad k_1 + k_2 + \cdots + k_p = n,$$

where p and n are integers.

We define the sequence of numbers $\{\eta_i\}_{i=0}^{p}$:

$$\eta_0 = 0, \qquad \eta_i = \sum_{e=1}^{i} k_e, \qquad i = 1, \cdots, p,$$

and introduce the spaces of functions:

$$M^0 = \overset{\circ}{H}{}^{m_1-1}(\Omega) \times \overset{k_1}{\cdots} \times \overset{\circ}{H}{}^{m_1-1}(\Omega) \times \cdots \times \overset{\circ}{H}{}^{m_p-1}(\Omega) \times \overset{k_p}{\cdots} \times H^{m_p-1}(\Omega),$$

$$M^1 = H^{m_1}(\Omega) \times \overset{k_1}{\cdots} \times H^{m_1}(\Omega) \times \cdots \times H^{m_p}(\Omega) \times \overset{k_p}{\cdots} \times H^{m_p}(\Omega),$$

where $\Omega = (0, 1)$, $\bar{\Omega} = [0, 1]$; and the inner products in M^0 and M^1 are denoted by $(\cdot, \cdot)_{M^0}$ and $(\cdot, \cdot)_{M^1}$, respectively.

Let M be the space

$$M = \{u \mid u \in M^1 \cap M^0\}.$$

Let us call the vector

$$\{(u_1^{(m_1-1)}, \cdots, u_{k_1}^{(m_1-1)}, \cdots, u_1^{(m_2)}, \cdots, u_{k_1}^{(m_2)}, \cdots, u_1, \cdots, u_n)(0),$$
$$(u_1^{(m_1-1)}, \cdots, u_{k_1}^{(m_1-1)}, \cdots, u_1^{(m_2)}, \cdots, u_{k_1}^{(m_2)}, \cdots, u_1, \cdots, u_n)(1)\}$$

trace of function u, which is denoted by $\bar{u}$. Notice that some components of vector $\bar{u}$ ($u \in M$) are equal to zero, i.e., $u_i^{(j)}(k) = 0$, if $u \in M$, $1 + \eta_e \leqq i \leqq \eta_{e+1}$, $0 \leqq j \leqq m_{e+1} - 1$, $e = 0, \cdots, p-1$; $k = 0, 1$.

Let C be the matrix of the following type:

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* Novosibirsk Institute of Electrical Engineering

$$C=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

where

$$C_i=\begin{pmatrix} \overbrace{C_{k_1,i}}^{r} \\ 0 \\ C_{k_2,i} \\ \vdots \\ C_{k_p,i} \\ 0 \end{pmatrix} \begin{matrix} \}\, k_1 \\ \}\, \eta_1(m_1-m_2) \\ \}\, k_2 \\ \vdots \\ \}\, k_p \\ \}\, \eta_p(m_p-1) \end{matrix} \qquad (i=1,2).$$

The rank of matrix C is equal to r, $0\leqq r\leqq 2n$. The matrices $C_{k_e,i}$ $(e=1,\cdots,p;i=1,2)$ have dimensions $k_e\times r$.

We can introduce the space

$$\overline{M}=\{u|u\in M, \bar{u}=Cv, v\in \boldsymbol{R}^r\}.$$

Let the functional $J_1(u)$ on $\overline{M}$ be

$$J_1(u)=\int_0^1\sum_{e=1}^{p}\sum_{i,j=m_{e+1}}^{m_e}(P_{ij}^e u^{(i)}, u^{(j)})dx+(Bv, v),$$

where $u^{(i)}=(u_1^{(i)},\cdots,u_{\eta_e}^{(i)},0,\cdots,0)^T$, B be $r\times r$ symmetric matrix and $n\times n$ matrix P_{ij}^e $(1\leqq e\leqq p, m_e\geqq i, j\geqq m_{e+1})$ have form:

$$\eta_e\left\{\begin{pmatrix} \overbrace{\ast}^{\eta_e} & 0 \\ 0 & 0 \end{pmatrix}\right.=P_{ij}^e.$$

We must take into consideration the construction of the matrices $P_{m_em_e}^e$ $(e=1,\cdots,p)$:

$$P_{m_em_e}^e=\begin{pmatrix} \overbrace{\ast}^{\eta_{e-1}} & \overbrace{\ast\ast}^{k_e} & 0 \\ \ast\ast\ast & P_e & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{matrix} \}\,\eta_{e-1} \\ \}\,k_e \\ \end{matrix},$$

where $(P_e(x)\xi,\xi)\leqq\gamma(\xi,\xi)$, $0\leqq x\leqq 1$, $\xi\in\boldsymbol{R}^{k_e}$, $\gamma=\text{const}<0$.

Let the functional J_0 on M^0 be

$$J_0=\int_0^1\sum_{e=1}^{p}\sum_{i=m_{e+1}}^{m_e-1}(Q_i^e u^{(i)}, u^{(i)})dx,$$

where the $n\times n$ matrices Q_i^e $(e=1,\cdots,p;m_{e+1}\leqq i\leqq m_e-1)$ have the form:

$$Q_i^e = \left(\begin{array}{c|c} \overbrace{\ast}^{\eta_e} & 0 \\ \hline 0 & 0 \end{array} \right) \Big\} \eta_e$$

and the properties that $(Q_i^e(x)\xi, \xi) \geqq \chi(\xi, \xi)$, $0 \leqq x \leqq 1$, $\chi = \text{const} > 0$, $\xi \in \boldsymbol{R}^{\eta_e} \times \{0\}^{n-\eta_e}$. We shall investigate the functional

$$J(u) = J_1(u)/J_0(u)$$

on the space $\bar{M}$.

We calculate Gateau derivative of the functional $J(u)$ in the direction h for any $u, h \in \bar{C} \cap \bar{M}$, where

$$\bar{C} = C^{2m_1}(\bar{\Omega}) \times \overset{k_1}{\cdots} \times C^{2m_1}(\bar{\Omega}) \times \cdots \times C^{2m_p}(\bar{\Omega}) \times \overset{k_p}{\cdots} \times C^{2m_p}(\bar{\Omega}),$$

$$\bar{u} = Cv, \qquad \bar{h} = Cv_1.$$

$$E = \lim_{t \to 0} \partial J(u+ht)/\partial t = \Big\{ -\int_0^1 \sum_{e=1}^{p} \sum_{i=m_e+1}^{m_e-1} [(Q_i^e u^{(i)}, h^{(i)}) + (Q_i^{e*} u^{(i)}, h^{(i)})] dx \; J_1(u)$$

$$+ \Big[\int_0^1 \sum_{e=1}^{p} \sum_{i,j=m_e+1}^{m_e} [(P_{ij}^e u^{(i)}, h^{(j)}) + (P_{ij}^{e*} u^{(j)}, h^{(i)})] dx + 2(Bv, v) \Big] J_0(u) \Big\} J_0^{-2}(u).$$

Integrating by parts the integrals in the last sum we get:

$$E = \Big\{ \Big[\int_0^1 \Big(\Big(\sum_{e=1}^{p} \sum_{i,j=m_e+1}^{m_e} \Big((-1)^i \frac{d^i}{dx^i} \Big(P_{ij}^{e*} \frac{d^j}{dx^j} u \Big) + (-1)^j \frac{d^j}{dx^j} \Big(P_{ij}^e \frac{d^i}{dx^i} u \Big) \Big), h \Big) dx$$

$$+ 2(Bv, v) + \Big(\sum_{e=1}^{p} ((P_{m_e m_e}^{e*} + P_{m_e m_e}^e) u^{(m_e)} + (P_{m_e m_e-1}^{e*} + P_{m_e-1 m_e}^e) u^{(m_e-1)}), h^{(m_e-1)} \Big) \Big|_0^1 \Big] J_0(u)$$

$$- \Big[\int_0^1 \Big(\Big(\sum_{e=1}^{p} \sum_{i=m_e+1}^{m_e-1} (-1)^i \frac{d^i}{dx^i} \Big((Q_i^{e*} + Q_i^e) \frac{d^i}{dx^i} u \Big), h \Big) dx \Big] J_1(u) \Big\} J_0^{-2}(u).$$

If $E \equiv 0$ hold for all h such that $\bar{h} = 0$, then the vector-function u must satisfy the system of equations

$$(1) \quad L_1(u) \equiv \sum_{e=1}^{p} \sum_{i,j=m_e+1}^{m_e} \Big\{ (-1)^i \frac{d^i}{dx^i} \Big(\frac{1}{2} P_{ij}^{e*} \frac{d^j}{dx^j} u \Big) + (-1)^j \frac{d^j}{dx^j} \Big(\frac{1}{2} P_{ij}^e \frac{d^i}{dx^i} u \Big) \Big\}$$

$$= J(u) \sum_{e=1}^{p} \sum_{i=m_e+1}^{m_e-1} (-1)^i \frac{d^i}{dx^i} \Big(\Big(\frac{1}{2} Q_i^{e*} + \frac{1}{2} Q_i^e \Big) \frac{d^i}{dx^i} u \Big) \equiv J(u) L_2(u).$$

Conversely, let the vector-function u satisfy the system (1). Then $E \equiv 0$ holds for all $h \in \bar{C} \cap \bar{M}$ if and only if the vector-function u satisfy the boundary conditions:

$$(2) \quad \bar{u} = Cv,$$

$$(3) \quad \frac{1}{2} \sum_{e=1}^{p} \{ \bar{C}_{e,1}^{*} [(P_{m_e m_e}^{e*} + P_{m_e m_e}^e) u^{(m_e)} + (P_{m_e m_e-1}^{e*} + P_{m_e-1 m_e}^e) u^{(m_e-1)}](0)$$

$$- \bar{C}_{e,2}^{*} [(P_{m_e m_e}^{e*} + P_{m_e m_e}^e) u^{(m_e)} + (P_{m_e m_e-1}^{e*} + P_{m_e-1 m_e}^e) u^{(m_e-1)}](1) \} = Bv,$$

where $v\in\boldsymbol{R}^r$ and

$$\bar{C}_{e,j}=\overbrace{\left(\begin{array}{c} 0 \\ \hline C_{k_{e,j}} \\ \hline 0 \end{array}\right)}^{r}\begin{array}{l}\}\,\eta_{e-1} \\ \}\,k_e \\ \}\,n-\eta_e\end{array}\qquad (1\leqq e\leqq p\,;\, j=1,2)\,.$$

The conditions (2) are defined in the functional space $\overline{M}$ where we study the functional $J(u)$. The conditions (3) are the conditions of transversality.

We shall establish a formula for a number of positive eigenvalues of the spectral problem $L_1u=\lambda L_2u$ with the boundary conditions (2), (3). If $r=0$, then the conditions (2), (3) imply

$$\bar{u}=0\,. \tag{4}$$

It is well investigated the special case of problem (1)-(3): $p=m_1=1$. In this case we must find a formula for a number of positive eigenvalues of problem

$$Lu\equiv(Pu')'+Gu'-(G^*u)'+Qu=\lambda u \tag{1$'$}$$

with conditions

$$u(0)=C_1v\,,\qquad u(1)=C_2v\,, \tag{2$'$}$$

$$Bv=C_2^*P(1)u'(1)-C_1^*P(0)u'(0)+C_1^*G^*(0)u(0)-C_2^*G^*(1)u(1)\,, \tag{3$'$}$$

where B is a symmetric $r\times r$ matrix; P, Q, G are $n\times n$ matrices, P, Q are symmetric; $(P(x)\xi,\xi)\geqq\gamma(\xi,\xi)$, $\gamma=\text{const}>0$, $\xi\in\boldsymbol{R}^n$, $0\leqq x\leqq 1$; C_1, C_2 are $n\times n$ matrices such that the matrix

$$C=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

has rank r, $0\leqq r\leqq 2n$.

In the noncritical case (i.e., $Lu=0$, (2)$'$, (3)$'$ or $Lu=0$, $u(0)=u(1)=0$ does not have nontrivial solutions), works on this question have been done by M.G. Krein [2], G.D. Birkhoff and M.R. Hestenes [3], W.T. Reid [4, 5], M. Morse [6–8], K.S. Hu [9].

In the critical case (i.e., $Lu=0$, (2)$'$, (3)$'$ or $Lu=0$, $u(0)=u(1)=0$ may have nontrivial solutions) the problem (1)$'$ with the special conditions:

$$(G^*u-Pu'+Au)|_{x=0}=0\,,\qquad (G^*u-Pu'+B_0u)|_{x=1}=0\,, \tag{4$'$}$$

where A, B_0 are symmetric $n\times n$ matrices, was studied by T.I. Zelenyak (for example, [17]). By the aid of only n solutions of the system $Lu=0$ (i.e., solutions which satisfy boudary conditions (4)$'$ at zero) and of functional

$$J_2=\int_0^1\{-(Pu',u')+2(Gu',u)+(Qu,u)\}dx+(B_0u,u)|^1-(Au,u)|_0\,,$$

he constructed the quadratic form $R(e)$, $e\in\boldsymbol{R}^r$ and showed that the number of positive eigenvalues of problem (1)′, (4)′ is equal to the sum of positive indices of quadratic form $R(e)$ and the number of non-negative eigenvalues of the problem

$$Lu=\lambda u\,,\qquad (G^*u-Pu'+Au)|_{x=0}=0\,,\qquad u|_{x=1}=0\,.$$

In [10, 11] the present author established a formula for the number of positive eigenvalues of the problem (1)′-(3)′ in the critical case.

The problems of this kind (1)-(3) with arbitrary m_1, p arise through the investigation of stability of stationary solutions of hydrodynamical models [12-16], therefore we want to generalize the results of [10, 11] to a more general case (1)-(3).

We call the vector-function u belonging to $\overline{M}$ a generalized eigenfunction of the problem (1)-(3) for eigenvalue λ if the integral equality

$$(5)\qquad -W_1(u,u_1)\equiv\frac{1}{2}\int_0^1\sum_{e=1}^{p}\sum_{i,j=m_{e+1}}^{m_e}[(P^e_{ij}u_1^{(i)},u^{(j)})+(P^e_{ij}u^{(i)},u_1^{(j)})]dx+(Bv,v)$$
$$=\frac{1}{2}\lambda\int_0^1\sum_{e=1}^{p}\sum_{i=m_{e+1}}^{m_e-1}[(Q^e_iu_1^{(i)},u^{(i)})+(Q^e_iu^{(i)},u_1^{(i)})]dx\equiv\lambda W_2(u,u_1)$$

holds for all $u, u_1\in\overline{M}$.

By the well known method (see, for example, [1]) we can show that the bilinear form $W_2(u,u_1)$ defines the inner product $[\cdot,\cdot]_0$ on M^0 which is equivalent to $(\cdot,\cdot)_{M^0}$. We can also show that there is a positive number $N>0$ such that the form $W(u,u_1)=W_1(u,u_1)+NW_2(u,u_1)$ defines the inner product $[\cdot,\cdot]$ on $\overline{M}$ which is equivalent to $(\cdot,\cdot)_{M^1}$. We add the quantity $-NW_2(u,u_1)$ to both side of the identity (5) and write it in the form:

$$(6)\qquad -[u,u]=-W(u,u_1)=(\lambda-N)W_2(u,u_1)=(\lambda-N)[u,u_1]_0\,.$$

Lemma 1. There is a linear and bounded operator A defined on M^0 such that the equality

$$[u,u_1]_0=[Au,u_1]$$

holds for all $u_1\in\overline{M}$. The operator A is positive, self-adjoint and absolute continuous if we consider A as an operator on $\overline{M}$.

The proof of Lemma 1 is well known (see, for example, [1]). In view of Lemma 1, we can write (6) in the form of operator equation on $\overline{M}$:

$$-u=(\lambda-N)Au\,,\qquad u\in\overline{M}\,.$$

Thus the number λ is an eigenvalue of problem (1)-(3) and u is a corresponding generalized eigenfunction if and only if the number $N-\lambda$ is a characteristic value of the self-adjoint and absolutely continuous operator from $\overline{M}$ to $\overline{M}$ and u is a corresponding eigenfunction. Since the operator A is positive and has the inverse operator A^{-1}, the spectrum of problem (1)-(3) is discrete, semiconfined and does not have finite limit points and the system of generalized eigenfunctions is complete in $\overline{M}$. It means that any element f of $\overline{M}$ may be approximated by the series $\sum_{i=1}^{\infty} \alpha_i u_i$ in the norm $\|\cdot\|_{M^1}=(\cdot,\cdot)_{M^1}^{1/2}$, where $u_1, \cdots, u_n, \cdots$ are the generalized eigenfunctions of the problem (1)-(3). We can show (see, for example, [1]) that the generalized eigenfunctions of the problem (1)-(3) are the classical eigenfunctions. From solutions of the system $L_1 u=0$ we can choose the set of maximal dimension $u^1, \cdots, u^s$ such that the traces $\bar{u}^1, \cdots, \bar{u}^s$ of these vector-functions are linear combinations of the columns of the matrix C. By W we denote the linear vector space spanned by $u^1, \cdots, u^s$. In $\boldsymbol{R}^s$ the quadratic form $H(\alpha)$, $\alpha \in \boldsymbol{R}^s$, is introduced by the following way:

$$u=\sum_{i=1}^{s} \alpha_i u^i, \qquad J_1(u)=J_1\Big(\sum_{i=1}^{s} \alpha_i u^i\Big)=(F\alpha, \alpha)=H(\alpha),$$

where F is a symmetric $s\times s$ matrix.

By n^+ and n^0 we denote the numbers of positive and of zero eigenvalues of the matrix F, respectively. In case $L_1 u=0$, (2), (3) or $L_1 u=0$, $\bar{u}=0$ does not have nontrivial solutions we can repeat the proofs in [10, 11] for the problem (1)-(3). We have new results.

Theorem 1. If $L_1 u=0$, (2), (3) or $L_1 u=0$, $\bar{u}=0$ does not have nontrivial solutions, then the number of nonnegative eigenvalues of the problem (1)-(3) is equal to

$$n^0+n^++N+P,$$

wherer P is the number of solutions of the problem $L_1 u=0$, $\bar{u}=0$; N is the number of positive eigenvalues of (1), (4).

Theorem 2. If $L_1 u=0$, (2), (3) or $L_1 u=0$, $\bar{u}=0$ does not have nontrivial solutions, then the number of positive eigenvalues of the problem (1)-(3) is equal to

$$n^++N+(r-s).$$

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