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CENTRAL FUNCTIONS ON SU(2) WITH NONNEGATIVE FOURIER COEFFICIENTS

by

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§ 1. Introduction

In the early 1950's N. Wiener proved the following theorem (cf. [2]).

Theorem A. If $\sum c_n e^{int}$ is the Fourier series of a function $f \in L^1(-\pi, \pi)$ with $c_n \geqq 0$ for all n and the restriction of f to a neighborhood $(-\delta, \delta)$ of the origin belongs to $L^2(-\delta, \delta)$, then f belongs to $L^2(-\pi, \pi)$.

The question whether we can replace L^2 by L^p with $1<p\leqq\infty$ was shown negatively for $1<p<2$ by S. Wainger [7] and for $p>2$ except even integers and ∞ by S. Shapiro [5]. If p is an even integer and ∞, it is easy to see that the answer is affirmative. Moreover, these results have been extended to compact abelian groups by M. Rains [3].

In 1987 J. M. Ash, M. Rains and S. Vági interpreted the conclusion of Wiener's theorem equivalently as "$\sum c_n^2<\infty$" and obtained the following theorem (see [1]).

Theorem B. Let $1<p<2$, $q=p/(p-1)$. If $\sum c_n e^{int}$ is the Fourier series of a function $f \in L^1(-\pi, \pi)$ with $c_n \geqq 0$ for all n and the restriction of f to a neighborhood $(-\delta, \delta)$ of the origin belongs to $L^p(-\delta, \delta)$, then $\sum c_n^q<\infty$.

In this paper we shall prove that an analogous result holds for central functions on SU(2).

§ 2. Statement of the result

Let $G=\mathrm{SU}(2)$ and $H=\{h(\theta)=\mathrm{diag}(e^{i\theta}, e^{-i\theta})\,;\,-\pi\leqq\theta\leqq\pi\}$ a maximal torus group of G. Then each element g of G is conjugate to an element h of H. Therefore, if f is a central function on G; $f(uvu^{-1})=f(v)$ for all $u, v \in G$, then f is completely determined by the restriction of f to the subgroup H of G. Here we let $f(\theta)=$

$f(h(\theta))$. Since $h(\theta)$ is conjugate to $h(-\theta)$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = h(-\theta),$$

f can be identified with an even function $f(\theta)$ of $-\pi \leqq \theta \leqq \pi$.

If f is an integrable central function on G, the normalized Haar integral of f on G is given by the integral of $f(\theta)$ on $[0, \pi]$ as

$$(1) \qquad \int_G f(g)dg = \frac{2}{\pi}\int_0^\pi f(\theta)\sin^2\theta d\theta$$

(cf. [6], p. 53). Then for $p \geqq 1$ we define

$$L^p(G, H) = \left\{ f;\ f \text{ is a central function on } G \text{ and } \|f\|_p = \left(\frac{2}{\pi}\int_0^\pi |f(\theta)|^p \sin^2\theta d\theta\right)^{1/p} < \infty \right\},$$

and for $0 < \delta < \pi$,

$$L^p(G, H_\delta) = \left\{ f;\ f \text{ is a central function on } G \text{ and } \int_0^\delta |f(\theta)|^p \sin^2\theta d\theta < \infty \right\}.$$

Let $\boldsymbol{N}$ be the set of nonnegative integers and for each $n \in \boldsymbol{N}$ χ_n the character of a $(n+1)$-dimensional irreducible unitary representation of G. Actually, χ_n is the central function on G given by

$$(2) \qquad \chi_n(\theta) = \begin{cases} n+1 & (\theta = 0), \\ \dfrac{\sin(n+1)\theta}{\sin\theta} & (\theta \neq 0, \ \pm\pi), \\ (-1)^n(n+1) & (\theta = \pm\pi) \end{cases}$$

and $\{\chi_n; n \in \boldsymbol{N}\}$ forms a complete orthonormal system of $L^2(G, H)$ (cf. [6], p. 49).

For $f \in L^1(G, H)$ the Fourier coefficient of f with respect to χ_n is defined by

$$(3) \qquad \begin{aligned} \hat{f}(n) &= \frac{2}{\pi}\int_0^\pi f(\theta)\bar{\chi}_n(\theta)\sin^2\theta d\theta \\ &= \frac{2}{\pi}\int_0^\pi f(\theta)\sin(n+1)\theta \sin\theta d\theta. \end{aligned}$$

Then the Fourier series of f is denoted as $\sum_{n \in \boldsymbol{N}} \hat{f}(n)\chi_n$. Here we let

$$\|\hat{f}\|_{\sharp, q} = \left(\sum_{n \in \boldsymbol{N}} (n+1)^{2-q} |\hat{f}(n)|^q\right)^{1/q}$$

for $2 \leqq q < \infty$.

Our theorem can be stated as follows.

Theorem. Let $1 < p \leqq 2$, $q = p/(p-1)$. If $f \in L^1(G, H)$ has nonnegative Fourier coefficients $\hat{f}(n)$ for all n and f belongs to $L^p(G, H_\delta)$, then $\|\hat{f}\|_{\sharp, q} < \infty$.

§ 3. Some lemmas

In order to prove Theorem we shall prepare some lemmas. The proof of the theorem will be given in § 4.

Lemma 1. Let $f \in L^1(G, H)$ and let g be a bounded central function on G. If f and g have nonnegative Fourier coefficients $\hat{f}(n)$ and $\hat{g}(n)$ for all $n \in \boldsymbol{N}$, then

$$(4) \qquad (fg)^\wedge(n) \geqq \hat{g}(0)\hat{f}(n) \qquad (n \in \boldsymbol{N}).$$

Proof. Since the Fourier series of f and g are given by $\sum_{n \in \boldsymbol{N}} \hat{f}(n)\chi_n$ and $\sum_{n \in \boldsymbol{N}} \hat{g}(n)\chi_n$ respectively, fg has an expansion like $\sum_{k,l \in \boldsymbol{N}} \hat{f}(k)\hat{g}(l)\chi_k\chi_l$. Then by using Clebsch-Goldan's law:

$$\chi_k\chi_l = \chi_{k+l} + \chi_{k+l-2} + \cdots + \chi_{|k-l|},$$

we can deduce that the Fourier coefficient of fg is given by

$$(fg)^\wedge(n) = \sum_{\substack{k,l \in \boldsymbol{N} \\ k+l=n}} \hat{f}(k)\hat{g}(l) + \sum_{\substack{k,l \in \mathbf{N} \\ K+l-2=n}} \hat{f}(k)\hat{g}(l) + \cdots + \sum_{\substack{k,l \in \boldsymbol{N} \\ |k-l|=n}} \hat{f}(k)\hat{g}(l).$$

Therefore, since $\hat{f}(k)$ and $\hat{g}(l)$ are nonnegative by the assumption, if we drop all nonnegative terms except $k=n$ and $l=0$, we can obtain the desired inequality.

Q. E. D.

Lemma 2. (the Hausdorff-Young theorem) Let $1 < p \leqq 2$, $q = p/(p-1)$. Then there exists a positive constant C_p such that

$$(5) \qquad \|\hat{f}\|_{\sharp, q} \leqq C_p \|f\|_p$$

for all $f \in L^p(G, H)$.

Proof. Let $f^\sim(n) = \hat{f}(n)/(n+1)$. We define

$$\|f^\sim\|_q = \Big(\sum_{n \in \boldsymbol{N}} (n+1)^2 |f^\sim(n)|^q\Big)^{1/q}$$

for $2 \leqq q < \infty$ and $\|f^\sim\|_\infty = \sup_{n \in \boldsymbol{N}} |f^\sim(n)|$. Then, since $|\chi_n/(n+1)|$ is bounded above by 1, it is easy to see that $\|f^\sim\|_\infty \leqq \|f\|_1$ for $f \in L^1(G, H)$. Moreover, since $\{\chi_n\}$ is a complete orthonormal system of $L^2(G, H)$, it follows that for $f \in L^2(G, H)$

$$\begin{aligned} \|f^\sim\|_2^2 &= \sum_{n \in \boldsymbol{N}} (n+1)^2 |f^\sim(n)|^2 \\ &= \sum_{n \in \boldsymbol{N}} |\hat{f}(n))|^2 = \|f\|_2^2. \end{aligned}$$

Therefore, applying the Riesz-Thorin interpolation theorem (cf. [4], p. 27), we see that there exists a positive constant C_p such that $\|f^\sim\|_q \leqq C_p \|f\|_p$ for $f \in L^p(G, H)$ $(1 < p < 2)$. Then the desired result follows from the definition of $f^\sim$. Q. E. D.

Lemma 3. Let $0<\delta<\pi$. Then there exists a central function g on G satisfying the following conditions.

(i) g is bounded,
(ii) supp $g \subset I_\delta$, where supp g is the support of the restriction of g to H and $I_\delta = [-\delta, \delta]$.
(iii) $\hat{g}(n) \geqq 0$ for all $n \in \boldsymbol{N}$ and $\hat{g}(0)>0$.

Proof. Let $g(\theta)$ be an even function of $-\pi \leqq \theta \leqq \pi$ defined by

$$g(\theta)=\begin{cases} 2\delta & (\theta=0) \\ \dfrac{(2\delta\theta-3\theta^2)}{\sin\theta} & (0<\theta\leqq\delta/2) \\ \dfrac{(\delta-\theta)^2}{\sin\theta} & (\delta/2\leqq\theta\leqq\delta) \\ 0 & (\delta\leqq\theta\leqq\pi)\,. \end{cases}$$

Then this function g on H can be uniquely extended to a central function on G, which we denote by the same letter. Obviously, g satisfies (i) and (ii). Moreover, a tedious calculation of the Fourier coefficient of g (see (3)) deduces that

$$\hat{g}(n)=4(1-\cos((n+1)\delta/2))^2/(n+1)^3\,.$$

Therefore, (iii) is also satisfied. Q. E. D.

Remark 4. Let $u(\theta)=g(\theta)\sin\theta$. Then for $\theta \geqq 0$ u is the convolution of the following two functions:

$$u_1(\theta)=\begin{cases} 2 & (0<\theta<\delta/2) \\ -2 & (-\delta/2<\theta<0) \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_2(\theta)=\begin{cases} \dfrac{\delta}{2}-|\theta| & (|\theta|<\delta/2) \\ 0 & \text{otherwise}\,. \end{cases}$$

§ 4. The proof of Theorem

Let g be the function obtained in Lemma 3 and let $F=gf$. Then F is a central function on G. Since $f \in L^p(G, H_\delta)$, it follows from Lemma 3 (i), (ii) and (1) that

$$\begin{aligned} \|F\|_p^p &= \frac{2}{\pi}\int_0^\pi |g(\theta)|^p |f(\theta)|^p \sin^2\theta d\theta \\ &\leqq \frac{2}{\pi}\|g\|_\infty^p \int_0^\delta |f(\theta)|^p \sin^2\theta d\theta < \infty\,. \end{aligned}$$

On the other hand, applying Lemma 3 (i), (iii) and Lemma 1, we see that for each $n \in \boldsymbol{N}$

$$\hat{f}(n) \leqq \hat{g}(0)^{-1} \hat{F}(n).$$

Then by Lemma 2 we can deduce that

$$\begin{aligned} \|\hat{f}\|_{\sharp,q} &= \Big(\sum_{n \in \boldsymbol{N}} (n+1)^{2-q} |\hat{f}(n)|^q\Big)^{1/q} \\ &\leqq \hat{g}(0)^{-1} \Big(\sum_{n \in \boldsymbol{N}} (n+1)^{2-q} |\hat{F}(n)|^q\Big)^{1/q} \\ &= \hat{g}(0)^{-1} \|\hat{F}\|_{\sharp,q} \\ &\leqq \hat{g}(0)^{-1} C_p \|F\|_p < \infty. \end{aligned}$$

This completes the proof of the theorem. Q. E. D.

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