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CENTRAL FUNCTIONS ON SU(2) WITH NONNEGATIVE FOURIER COEFFICIENTS

by

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§ 1. Introduction

In the early 1950's N. Wiener proved the following theorem (cf. [2]).

Theorem A. If $\sum c_n e^{\mathrm{int}}$ is the Fourier series of a function $f \in L^1(-\pi, \pi)$ with $c_n \geq 0$ for all n and the restriction of f to a neighborhood $(-\delta, \delta)$ of the origin belongs to $L^2(-\delta, \delta)$, then f belongs to $L^2(-\pi, \pi)$.

The question whether we can replace L^2 by L^p with $1 < p \leq \infty$ was shown negatively for $1 < p < 2$ by S. Wainger [7] and for $p > 2$ except even integers and ∞ by S. Shapiro [5]. If p is an even integer and ∞, it is easy to see that the answer is affirmative. Moreover, these results have been extended to compact abelian groups by M. Rains [3].

In 1987 J. M. Ash, M. Rains and S. Vági interpreted the conclusion of Wiener's theorem equivalently as " $\sum c_n^2 < \infty$ " and obtained the following theorem (see [1]).

Theorem B. Let $1 < p < 2$, $q = p/(p-1)$. If $\sum c_n e^{\mathrm{int}}$ is the Fourier series of a function $f \in L^1(-\pi, \pi)$ with $c_n \geq 0$ for all n and the restriction of f to a neighborhood $(-\delta, \delta)$ of the origin belongs to $L^p(-\delta, \delta)$, then $\sum c_n^q < \infty$.

In this paper we shall prove that an analogous result holds for central functions on SU(2).

§ 2. Statement of the result

Let $G = \mathrm{SU}(2)$ and $H = \{h(\theta) = \mathrm{diag}\,(e^{i\theta}, e^{-i\theta}); -\pi \leq \theta \leq \pi\}$ a maximal torus group of G. Then each element g of G is conjugate to an element h of H. Therefore, if f is a central function on G; $f(uvu^{-1}) = f(v)$ for all $u, v \in G$, then f is completely determined by the restriction of f to the subgroup H of G. Here we let $f(\theta) =$

$f(h(\theta))$. Since $h(\theta)$ is conjugate to $h(-\theta)$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} h(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = h(-\theta),$$

f can be identified with an even function $f(\theta)$ of $-\pi \leq \theta \leq \pi$.

If f is an integrable central function on G, the normalized Haar integral of f on G is given by the integral of $f(\theta)$ on $[0, \pi]$ as

(1) $$\int_G f(g)dg = \frac{2}{\pi} \int_0^\pi f(\theta) \sin^2 \theta d\theta$$

(cf. [6], p. 53). Then for $p \geq 1$ we define

$$L^p(G, H) = \left\{ f ; f \text{ is a central function on } G \text{ and} \right.$$

$$\left. \|f\|_p = \left(\frac{2}{\pi} \int_0^\pi |f(\theta)|^p \sin^2 \theta d\theta \right)^{1/p} < \infty \right\},$$

and for $0 < \delta < \pi$,

$$L^p(G, H_\delta) = \left\{ f ; f \text{ is a central function on } G \text{ and } \int_0^\delta |f(\theta)|^p \sin^2 \theta d\theta < \infty \right\}.$$

Let N be the set of nonnegative integers and for each $n \in N$ χ_n the character of a $(n+1)$-dimensional irreducible unitary representation of G. Actually, χ_n is the central function on G given by

(2) $$\chi_n(\theta) = \begin{cases} n+1 & (\theta = 0), \\ \dfrac{\sin(n+1)\theta}{\sin \theta} & (\theta \neq 0, \ \pm\pi), \\ (-1)^n(n+1) & (\theta = \pm\pi) \end{cases}$$

and $\{\chi_n ; n \in N\}$ forms a complete orthonormal system of $L^2(G, H)$ (cf. [6], p. 49).

For $f \in L^1(G, H)$ the Fourier coefficient of f with respect to χ_n is defined by

(3) $$\hat{f}(n) = \frac{2}{\pi} \int_0^\pi f(\theta) \bar{\chi}_n(\theta) \sin^2 \theta d\theta$$

$$= \frac{2}{\pi} \int_0^\pi f(\theta) \sin(n+1)\theta \sin \theta d\theta.$$

Then the Fourier series of f is denoted as $\sum_{n \in N} \hat{f}(n)\chi_n$. Here we let

$$\|\hat{f}\|_{*,q} = (\sum_{n \in N} (n+1)^{2-q} |\hat{f}(n)|^q)^{1/q}$$

for $2 \leq q < \infty$.

Our theorem can be stated as follows.

Theorem. Let $1 < p \leq 2$, $q = p/(p-1)$. If $f \in L^1(G, H)$ has nonnegative Fourier coefficients $\hat{f}(n)$ for all n and f belongs to $L^p(G, H_\delta)$, then $\|\hat{f}\|_{*,q} < \infty$.

§ 3. Some lemmas

In order to prove Theorem we shall prepare some lemmas. The proof of the theorem will be given in § 4.

Lemma 1. Let $f \in L^1(G, H)$ and let g be a bounded central function on G. If f and g have nonnegative Fourier coefficients $\hat{f}(n)$ and $\hat{g}(n)$ for all $n \in N$, then

$$(4) \qquad (fg)^\wedge(n) \geq \hat{g}(0)\hat{f}(n) \qquad (n \in N).$$

Proof. Since the Fourier series of f and g are given by $\sum_{n \in N} \hat{f}(n)\chi_n$ and $\sum_{n \in N} \hat{g}(n)\chi_n$ respectively, fg has an expansion like $\sum_{k,l \in N} \hat{f}(k)\hat{g}(l)\chi_k\chi_l$. Then by using Clebsch-Goldan's law:

$$\chi_k\chi_l = \chi_{k+l} + \chi_{k+l-2} + \cdots + \chi_{|k-l|},$$

we can deduce that the Fourier coefficient of fg is given by

$$(fg)^\wedge(n) = \sum_{\substack{k,l \in N \\ k+l=n}} \hat{f}(k)\hat{g}(l) + \sum_{\substack{k,l \in N \\ K+l-2=n}} \hat{f}(k)\hat{g}(l) + \cdots + \sum_{\substack{k,l \in N \\ |k-l|=n}} \hat{f}(k)\hat{g}(l).$$

Therefore, since $\hat{f}(k)$ and $\hat{g}(l)$ are nonnegative by the assumption, if we drop all nonnegative terms except $k=n$ and $l=0$, we can obtain the desired inequality.

Q. E. D.

Lemma 2. (the Hausdorff-Young theorem) Let $1 < p \leq 2$, $q = p/(p-1)$. Then there exists a positive constant C_p such that

$$(5) \qquad \|\hat{f}\|_{\mathfrak{s},q} \leq C_p\|f\|_p$$

for all $f \in L^p(G, H)$.

Proof. Let $f^\sim(n) = \hat{f}(n)/(n+1)$. We define

$$\|f^\sim\|_q = (\sum_{n \in N} (n+1)^2 |f^\sim(n)|^q)^{1/q}$$

for $2 \leq q < \infty$ and $\|f^\sim\|_\infty = \sup_{n \in N} |f^\sim(n)|$. Then, since $|\chi_n/(n+1)|$ is bounded above by 1, it is easy to see that $\|f^\sim\|_\infty \leq \|f\|_1$ for $f \in L^1(G, H)$. Moreover, since $\{\chi_n\}$ is a complete orthonormal system of $L^2(G, H)$, it follows that for $f \in L^2(G, H)$

$$\|f^\sim\|_2^2 = \sum_{n \in N} (n+1)^2 |f^\sim(n)|^2$$
$$= \sum_{n \in N} |\hat{f}(n))|^2 = \|f\|_2^2.$$

Therefore, applying the Riesz-Thorin interpolation theorem (cf. [4], p. 27), we see that there exists a positive constant C_p such that $\|f^\sim\|_q \leq C_p\|f\|_p$ for $f \in L^p(G, H)$ $(1 < p < 2)$. Then the desired result follows from the definition of $f^\sim$. Q. E. D.

3

Lemma 3. Let $0<\delta<\pi$. Then there exists a central function g on G satisfying the following conditions.

(i) g is bounded,

(ii) supp $g\subset I_\delta$, where supp g is the support of the restriction of g to H and I_δ $=[-\delta,\delta]$.

(iii) $\hat{g}(n)\geq 0$ for all $n\in N$ and $\hat{g}(0)>0$.

Proof. Let $g(\theta)$ be an even function of $-\pi\leq\theta\leq\pi$ defined by

$$g(\theta)=\begin{cases} 2\delta & (\theta=0) \\ \dfrac{(2\delta\theta-3\theta^2)}{\sin\theta} & (0<\theta\leq\delta/2) \\ \dfrac{(\delta-\theta)^2}{\sin\theta} & (\delta/2\leq\theta\leq\delta) \\ 0 & (\delta\leq\theta\leq\pi). \end{cases}$$

Then this function g on H can be uniquely extended to a central function on G, which we denote by the same letter. Obviously, g satisfies (i) and (ii). Moreover, a tedious calculation of the Fourier coefficient of g (see (3)) deduces that

$$\hat{g}(n)=4(1-\cos((n+1)\delta/2))^2/(n+1)^3.$$

Therefore, (iii) is also satisfied.   Q. E. D.

Remark 4. Let $u(\theta)=g(\theta)\sin\theta$. Then for $\theta\geq 0$ u is the convolution of the following two functions:

$$u_1(\theta)=\begin{cases} 2 & (0<\theta<\delta/2) \\ -2 & (-\delta/2<\theta<0) \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_2(\theta)=\begin{cases} \dfrac{\delta}{2}-|\theta| & (|\theta|<\delta/2) \\ 0 & \text{otherwise}. \end{cases}$$

§4. The proof of Theorem

Let g be the function obtained in Lemma 3 and let $F=gf$. Then F is a central function on G. Since $f\in L^p(G,H_\delta)$, it follows from Lemma 3 (i), (ii) and (1) that

$$\|F\|_p^p=\frac{2}{\pi}\int_0^\pi |g(\theta)|^p|f(\theta)|^p\sin^2\theta d\theta$$

$$\leq\frac{2}{\pi}\|g\|_\infty^p\int_0^\delta |f(\theta)|^p\sin^2\theta d\theta<\infty.$$

On the other hand, applying Lemma 3 (i), (iii) and Lemma 1, we see that for each $n \in N$

$$\hat{f}(n) \leqq \hat{g}(0)^{-1}\hat{F}(n) \,.$$

Then by Lemma 2 we can deduce that

$$\begin{aligned}
||\hat{f}||_{\sharp,q} &= (\sum_{n \in N} (n+1)^{2-q}|\hat{f}(n)|^q)^{1/q} \\
&\leqq \hat{g}(0)^{-1}(\sum_{n \in N} (n+1)^{2-q}|\hat{F}(n)|^q)^{1/q} \\
&= \hat{g}(0)^{-1}||\hat{F}||_{\sharp,q} \\
&\leqq \hat{g}(0)^{-1}C_p||F||_p < \infty \,.
\end{aligned}$$

This completes the proof of the theorem.   Q. E. D.

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