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Author	石川, 史郎(Ishikawa, Shiro) Kikuchi, Norio
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ON THE SINGULARITY OF FUNDAMENTAL SOLUTIONS FOR DIFFERENCE-PARTIAL DIFFERENTIAL EQUATIONS OF THE TYPE

$$\frac{u_n(x)-u_{n-1}(x)}{h}=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}u_n(x)\ (n=1,2,\cdots,N,\ h=1/N)\ \text{for}\ x\in\mathbf{R}^d\ (d:\ odd)$$

by

Shiro Ishikawa and Norio Kikuchi*

Department of Mathematics
Faculty of Science and Technology, Keio University
Hiyoshi, Yokohama 223, Japan

1. Introduction

The purpose of this note is to show that the fundamental solution of the difference-partial differential equations of the type

$$\frac{u_n(x)-u_{n-1}(x)}{h}=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}u_n(x)\ (n=1,2,\cdots,N,\ h=1/N)\ \text{for}\ x\in\mathbf{R}^d\ (d:\ odd)\quad(1)$$

has the singularity of type $|x|^{-d}$ at most, where N is a positive integer and $h=1/N$. Assuming that $u_n(x)\ (n=1,2,\cdots,N)$ are rapidly decreasing smooth functions, we take the Fourier transform $\wedge$ of the both sides of above equation (1) to obtain

$$\frac{\hat{u}_n(\xi)-\hat{u}_{n-1}(\xi)}{h}=-|\xi|^2\hat{u}_n(\xi)\ (n=1,2,\cdots,N)\quad\text{for}\ \xi\in\mathbf{R}^d\ (d:\ odd)\qquad(2)$$

and hence

$$\hat{u}_n(\xi)=(1+h|\xi|^2)^{-n}\hat{u}_0(\xi)\quad\text{for}\ \xi\in\mathbf{R}^d\qquad(3)$$

where $\hat{u}(\xi)=\displaystyle\int_{\mathbf{R}^d}e^{-i\langle\xi,x\rangle}u(x)dx$ and $\langle\xi,x\rangle=\sum_{i=1}^{d}\xi_i x_i,\ |\xi|^2=\langle\xi,\xi\rangle$ for $\xi=(\xi_1,\cdots,\xi_d),\ x=(x_1,\cdots,x_d)\in\mathbf{R}^d$. By taking the Fourier inverse transform of (3), we have

$$u_n(x)=\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\left[\frac{1}{(1+h|\xi|^2)^n}\right]^{\wedge}(x-y)u_0(y)dy\ .$$

Now the following theorem holds:

Theorem. *Let d be a positive odd integer. Let N be a positive integer and put $h=1/N$. Then there exists a positive number C depending only on d such that*

$$\left| \left[\frac{1}{(1+h|\xi|^2)^n} \right]^\wedge (x) \right| \le C|x|^{-d}$$

holds for any $x \in \mathbf{R}^d$ $(x \ne 0)$ and n $(n=1, 2, \cdots, N)$.

Note that the above Fourier transform $\wedge$ should be interpreted in the distributional sense, since $\dfrac{1}{(1+h|\xi|^2)^n}$ is not integrable for $1 \le n \le \dfrac{d+1}{2}-1$. For the proof of this theorem, we use the following well-known results:

$$\left[\frac{1}{(1+|\xi|^2)^n} \right]^{'}(x)=(2\pi)^{d/2}|x|^{-(d-2)/2}\int_0^\infty J_{(d-2)/2}(|x|s)s^{d/2}ds \quad \left(n \ge \frac{d+1}{2} \right) \quad (4)$$

(see E. M. Stein and G. Weiss [10, page 155]),

$$\int_0^\infty \frac{s^{d/2}}{(1+s^2)^n}J_{(d-2)/2}(|x|s)ds = \frac{|x|^{n-1}K_{(d-2n)/2}(|x|)}{2^{n-1}(n-1)!} \quad \left(n \ge \frac{d+1}{2} \right) \quad (5)$$

(see [12, page 686])

and

$$K_{n+(1/2)}(|x|)=K_{-n-(1/2)}(|x|)=\left(\frac{\pi}{2|x|} \right)^{1/2}e^{-|x|}\sum_{r=0}^n \frac{(n+r)!}{r!(n-r)!(2|x|)^r} \quad (n=0, 1, 2, \cdots) \quad (6)$$

(see [12, page 967]),

where J denotes Bessel function of the first kind and K denotes modified Bessel function.

For any fixed n $(n=1, 2, \cdots, N)$, putting

$$f(x)=(1+|x|^2)^{-n},$$

we have $\left[f\left(\dfrac{\xi}{\sqrt{N}} \right) \right]^\wedge (x)=N^{d/2}\hat{f}(\sqrt{N}x)$, by the change of variables $\eta=\xi/\sqrt{N}$. Hence for the proof of Theorem it is sufficient to prove that there exists a positive number C such that

$$\left| \left[\frac{1}{(1+|\xi|^2)^n} \right]^{'}(x) \right| \le C|x|^{-d} \quad (7)$$

holds for any $x \in \mathbf{R}^d$ $(x \ne 0)$ and any positive integer n. Therefore we have only to prove the following two Lemmas.

Lemma 1. *Let d be a positive odd integer. Then*

$$\left| \left[\frac{1}{(1+|\xi|^2)^n} \right]^\wedge (x) \right| \le C\frac{1}{1+|x|^d}$$

holds for any $x \in \boldsymbol{R}^d$ *and any positive integer* $n \geq \dfrac{d+1}{2}$.

Proof. By using (4) and (5), we have

$$\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x) = (2\pi)^{d/2}|x|^{-(d-2)/2}\int_0^\infty \frac{1}{(1+s^2)^n}J_{(d-2)/2}(|x|s)s^{d/2}ds$$

$$= (2\pi)^{d/2}|x|^{-d/2+1}\frac{|x|^{n-1}K_{(d-2n)/2}(|x|)}{2^{n-1}(n-1)!}$$

$$= (2\pi)^{d/2}\frac{|x|^{n-d/2}K_{(d-2n)/2}(|x|)}{2^{n-1}(n-1)!} \ .$$

Since d is an odd positive integer, we obtain from (6) that

$$\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x) = \frac{(2\pi)^{d/2}|x|^{n-d/2}}{2^{n-1}(n-1)!}\left(\frac{\pi}{2|x|}\right)^{1/2}e^{-|x|}\sum_{r=0}^{n-(d+1)/2}\frac{\left(n-\dfrac{d+1}{2}+r\right)!}{r!\left(n-\dfrac{d+1}{2}-r\right)!(2|x|)^r}$$

$$= (2\pi)^{(d+1)/2}e^{-|x|}\sum_{r=0}^{n-(d+1)/2}\frac{\left(n-\dfrac{d+1}{2}+r\right)!|x|^{n-d/2-1/2-r}}{(n-1)!r!\left(n-\dfrac{d+1}{2}-r\right)!2^{n+r}}$$

$$= (2\pi)^{(d+1)/2}e^{-|x|}\sum_{k=0}^{n-(d+1)/2}\frac{(2n-d-1-k)!|x|^k}{(n-1)!\left(n-\dfrac{d+1}{2}-k\right)!k!2^{2n-d/2-1/2-k}} \ ,$$

where we have used the change of variables: $k = n - \dfrac{d+1}{2} - r$. Also, we see,

$$\left|\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)\right| \leq \int_{\boldsymbol{R}^d}\left[\frac{1}{(1+|\xi|^2)^n}\right]d\xi \leq \int_{\boldsymbol{R}^d}\left[\frac{1}{(1+|\xi|^2)^{(d+1)/2}}\right]d\xi < \infty$$

holds for any $x \in \boldsymbol{R}^d$ and any positive integer $n \geq \dfrac{d+1}{2}$. Hence, in order to prove Lemma, it is sufficient to show

$$\sum_{k=0}^{n-(d+1)/2}\frac{(2n-d-1-k)!|x|^{k+d}}{(n-1)!\left(n-\dfrac{d+1}{2}-k\right)!k!2^{2n-d/2-1/2-k}} \leq (2\pi)^{-(d+1)/2}Ce^{|x|} \tag{8}$$

for any $x \in \boldsymbol{R}^d$ and n sufficiently large, since the left side of (8) is a polynomial of degree $n + \dfrac{d}{2} - \dfrac{1}{2}$. Hence, by putting $n = m + \dfrac{d}{2} + \dfrac{1}{2}$, it is sufficient to show

$$\sum_{k=0}^m\frac{(2m-k)!|x|^{k+d}}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-k)!k!2^{2m-k}} = \sum_{k=0}^m\frac{(2m-k)!(k+1)(k+2)\cdots(k+d)2^k}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-k)!k!2^{2m}}\cdot\frac{|x|^{k+d}}{(k+d)!}$$

$$\leq \pi^{-(d+1)/2}Ce^{|x|}$$

for any $x \in \mathbf{R}^d$ and m sufficiently large.

Here we set for any m sufficiently large and for k ($k=1,2,\cdots,m$)

$$a_k^m = \frac{(2m-k)!(k+1)(k+2)\cdots(k+d)2^k}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-k)!2^{2m}}$$

and successively estimate a_k^m for any fixed m. When we regard a_k^m as a function of k, we then have the following calculation:

$$a_{k+1}^m - a_k^m = \frac{(2m-(k+1))!((k+1)+1)((k+1)+2)\cdots((k+1)+d)2^{k+1}}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-(k+1))!2^{2m}}$$
$$- \frac{(2m-k)!(k+1)(k+2)\cdots(k+d)2^k}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-k)!2^{2m}}$$
$$= \frac{(2m-k-1)!(k+2)(k+3)\cdots(k+d)2^k}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-k)!2^{2m}}\{-k^2-(1+2d)k+2dm\}.$$

Hence, we can take a positive integer k_m such that

$$\left|\frac{-(1+2d)+\sqrt{(1+2d)^2+8dm}}{2}-k_m\right| \le 1$$

and

$$a_0^m < a_1^m < \cdots < a_{k_m-1}^m \le a_{k_m}^m \ge a_{k_m+1}^m \ge \cdots > a_{m-1}^m > a_m^m.$$

Consequently, it is sufficient to prove the following:

$$\limsup_{m\to\infty} a_{k_m}^m < \infty$$

to obtain the result of Theorem. We calculate a_k^m as follows:

$$a_{k_m}^m = \frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)2^{k_m}}{\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)!(m-k_m)!2^{2m}}$$
$$= \frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)2^{k_m}(m-k_m+1)(m-k_m+2)\cdots(m-k_m+k_m)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)}$$
$$= \frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)(2m-2k_m+2)(2m-2k_m+4)\cdots(2m-2k_m+2k_m)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)}$$
$$\le \frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)(2m-k_m+1)(2m-k_m+2)\cdots(2m-k_m+k_m)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)}$$

$$= \frac{(2m)!(k_m+1)(k_m+2)\cdots(k_m+d)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\dfrac{d}{2}-\dfrac{1}{2}\right)} \leq \frac{(2m)!}{(m!)^2 2^{2m}} \cdot \frac{(k_m+d)^d}{m^{(d/2)-(1/2)}}$$

$$\leq \frac{(2m)!}{(m!)^2 2^{2m}} \cdot \frac{\left(\dfrac{1+\sqrt{(1+d)^2+8dm}}{2}\right)^d}{m^{(d/2)-(1/2)}}$$

By the well-known Wallis formula, we have

$$\limsup_{m\to\infty} a_{k_m}^m \leq \frac{(2d)^{d/2}}{\sqrt{\pi}},$$

which completes the proof of Lemma.

Lemma 2. *Let d be a positive odd integer. Then*

$$\left| \left[\frac{1}{(1+|\xi|^2)^n} \right]^{\wedge}(x) \right| \leq C \max\left\{ \frac{1}{|x|}, \frac{1}{|x|^{d-2}} \right\} e^{-|x|}$$

holds for any $x \in \mathbf{R}^d$ $(x \neq 0)$ and $n = 1, 2, \cdots, \dfrac{d+1}{2}-1$.

Proof. Since the equality

$$\left[\frac{1}{(1+|\xi|^2)^{(d+1)/2}} \right]^{\wedge}(x) = \frac{\pi^{(d+1)/2}}{\left(\dfrac{d-1}{2}\right)!} e^{-|x|}$$

holds for any $x \in \mathbf{R}^d$, we have that, for $n = 1, 2, \cdots, \dfrac{d+1}{2}-1$,

$$\left[\frac{1}{(1+|\xi|^2)^n} \right]^{\wedge}(x) = \left[\frac{(1+|\xi|^2)^{(d+1)/2-n}}{(1+|\xi|^2)^{(d+1)/2}} \right]^{\wedge}(x)$$

$$= \frac{\pi^{(d+1)/2}}{\left(\dfrac{d-1}{2}\right)!} [(1-\varDelta)^{(d+1)2-n} e^{-|x|}],$$

where $\varDelta = \sum_{i=1}^{d} \dfrac{\partial^2}{\partial x_i^2}$. Therefore, it is sufficient to prove that there exists a positive number C such that

$$|(1-\varDelta)^m e^{-|x|}| \leq C \max\left\{ \frac{1}{|x|}, \frac{1}{|x|^{d-2}} \right\} e^{-|x|} \quad \left(m = 1, 2, \cdots, \frac{d-1}{2} \right)$$

for any $x \in \mathbf{R}^d$ $(x \neq 0)$. Using repeatedly the following equality

$$(1-\varDelta) \frac{e^{-|x|}}{|x|^k} = \left[(d-2-k)k \frac{1}{|x|^{k+2}} + (d-1-2k) \frac{1}{|x|^{k+1}} \right] e^{-|x|} \quad (k=1, 2, \cdots) \qquad (9)$$

for any $x \in \mathbf{R}^d$ $(x \neq 0)$, we obtain that, for any $x \in \mathbf{R}^d$ $(x \neq 0)$,

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$$(1-\varDelta)e^{-|x|} = \frac{C_1^1}{|x|}e^{-|x|} = \frac{d-1}{|x|}e^{-|x|}$$

$$(1-\varDelta)^2 e^{-|x|} = \left[\frac{C_1^2}{|x|} + \frac{C_2^2}{|x|^2}\right]\frac{e^{-|x|}}{|x|} = \left[\frac{(d-1)(d-3)}{|x|} + \frac{(d-1)(d-3)}{|x|^2}\right]\frac{e^{-|x|}}{|x|}$$

$$(1-\varDelta)^3 e^{-|x|} = \left[\frac{C_1^3}{|x|} + \frac{C_2^3}{|x|^2} + \frac{C_3^3}{|x|^3}\right]\frac{e^{-|x|}}{|x|^2}$$

$$= \left[\frac{(d-1)(d-3)(d-5)}{|x|} + \frac{3(d-1)(d-3)(d-5)}{|x|^2} + \frac{3(d-1)(d-3)(d-5)}{|x|^3}\right]\frac{e^{-|x|}}{|x|^2}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$(1-\varDelta)^m e^{-|x|} = \left[\frac{C_1^m}{|x|} + \frac{C_2^m}{|x|^2} + \cdots + \frac{C_m^m}{|x|^m}\right]\frac{e^{-|x|}}{|x|^{m-1}}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$(1-\varDelta)^{(d-1)/2} e^{-|x|} = \left[\frac{C_1^{(d-1)/2}}{|x|} + \frac{C_2^{(d-1)/2}}{|x|^2} + \cdots + \frac{C_{(d-1)/2}^{(d-1)/2}}{|x|^{(d-1)/2}}\right]\frac{e^{-|x|}}{|x|^{(d-1)/2-1}} ,$$

where $C_j^m \left(2 \leq m \leq \frac{d-1}{2}, 1 \leq j \leq m\right)$ is defined inductively as follows:

$$C_1^m = (d-2m+1)C_1^{m-1} ,$$
$$C_j^m = (d-2m-2j+3)C_j^{m-1} + (d-m-j+1)(m+j-3)C_{j-1}^{m-1} \quad (j=2, 3, \cdots, m-1)$$

and

$$C_m^m = (d-2m+1)(2m-3)C_{m-1}^{m-1} .$$

Since C_j^m depends only on d, we can take positive numbers C such that

$$|(1-\varDelta)^m e^{-|x|}| \leq C \max\left\{\frac{1}{|x|}, \frac{1}{|x|^{d-2}}\right\}e^{-|x|} \quad \left(m=1, 2, \cdots, \frac{d-1}{2}\right)$$

holds for any $x \in \boldsymbol{R}^d$ $(x \neq 0)$, which completes the proof of Lemma.

Thus we have finished the proof of Theorem.

Now it is very attractive to the authors to construct a regular solution (Morse flow) for non-linear parabolic partial differential equations corresponding to the following problems in the calculus of variations: For mappings $u \in H^{1,2}(\Omega, \boldsymbol{R}^{d'})$ ($H^{1,2}(\Omega, \boldsymbol{R}^{d'})$ is the usual Sovolev space and Ω is an open and bounded domain with smooth boundary in $\boldsymbol{R}^d$), we consider the following functional

$$I(u) = \int_\Omega A^{\alpha\beta}(x, u(x))D_\alpha u^i(x)D_\beta u^i(x)dx .$$

Here in the summation over repeated indices, the Greek indices run from 1 to d and the Latin ones from 1 to d'. We assume that the coefficients $A^{\alpha\beta}$ are bounded functions suitably smooth in $\bar{\Omega} \times \boldsymbol{R}^{d'}$ and satisfy the condition

$$A^{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \geq \lambda|\xi|^2 \quad \text{for } \xi \in \boldsymbol{R}^d \text{ and } (x, u) \in \Omega \times \boldsymbol{R}^{d'}$$

with a uniform positive constant λ.

It has been successfully treated by Douglas and Morrey ([7]) to find a regular

minimum point in the case $d=2$. In general case $d\geq3$, an excellent result was proposed by Giaquinta and Giusti ([2]) in 1980. This result says that minima of the functional I (with the coefficients $A^{\alpha\beta}$ whose smoothness is only required to be continuous) are Hölder-continuous in Ω under the so-called one sided condition proposed by Hildebrandt and Widman ([5]). It should be remarked that the one-sided condition does not impose the solutions any "smallness".

Since the result appeared, it has been conjectured that the parabolic flow for equations

$$\frac{\partial u}{\partial t}=D_\beta(A^{\alpha\beta}(x,u)D_\alpha u)-\frac{1}{2}\nabla_u A^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i$$

in the "weak" sense conserves the regularity of the initial data under the one-sided condition. Here we mention interesting papers [4] and [11], which treat non-linear parabolic differential equations. One* of the authors in this note has taken up this problem for these years and approached it by considering the following functionals:

$$I_n(u)=\int_\Omega \left(A^{\alpha\beta}(x,u(x))D_\alpha u^i(x)D_\beta u^i(x)+\frac{1}{h}|u-u_{n-1}|^2\right)dx \quad (n=1,2,\cdots,N),$$

where N is a positive integer and $h=1/N$ (for example) and u_0 is an initial datum for the problem. By taking u_n as a minimum of the functional $I_n(u)$ inductively, we obtain the following Euler-Lagrange equations of I_n:

$$\frac{u_n-u_{n-1}}{h}=D_\beta(A^{\alpha\beta}(x,u_n)D_\alpha u_n)-\frac{1}{2}\nabla_u A^{\alpha\beta}(x,u_n)D_\alpha u_n^i D_\beta u_n^i \qquad (*)$$

where we notice that any $u_n(x)$ is known to be smooth in Ω by virtue of the result [2]. By constructing a suitable function comparative to the minimum u_n, we are trying to obtain the so-called reverse Hölder inequality due to Gehring-Giaquinta-Modica ([1] and [3]), about which we expect to be able to write in another paper. To obtain the conjectured result stated above, we think that such properties for solutions of equations $(*)$ as the estimates in this note and Harnack property of Moser's type ([8] and [9]) will play an essential role.

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