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ON THE SINGULARITY OF FUNDAMENTAL SOLUTIONS FOR DIFFERENCE-PARTIAL DIFFERENTIAL EQUATIONS OF THE TYPE $\frac{u_n(x)-u_{n-1}(x)}{h}=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}u_n(x)$ $(n=1, 2, \cdots, N, h=1/N)$ for $x\in \boldsymbol{R}^d$ $(d$: *odd*$)$

by

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1. Introduction

The purpose of this note is to show that the fundamental solution of the difference-partial differential equations of the type

$$\frac{u_n(x)-u_{n-1}(x)}{h}=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i{}^2}u_n(x)\ (n=1, 2, \cdots, N, h=1/N) \text{ for } x\in \boldsymbol{R}^d\ (d\text{: } odd) \quad (1)$$

has the singularity of type $|x|^{-d}$ at most, where N is a positive integer and $h=1/N$. Assuming that $u_n(x)$ $(n=1, 2, \cdots, N)$ are rapidly decreasing smooth functions, we take the Fourier transform $\wedge$ of the both sides of above equation (1) to obtain

$$\frac{\hat{u}_n(\xi)-\hat{u}_{n-1}(\xi)}{h}=-|\xi|^2\hat{u}_n(\xi)\ (n=1, 2, \cdots, N) \quad \text{for } \xi\in \boldsymbol{R}^d\ (d\text{: } odd) \qquad (2)$$

and hence

$$\hat{u}_n(\xi)=(1+h|\xi|^2)^{-n}\hat{u}_0(\xi) \quad \text{for } \xi\in \boldsymbol{R}^d \qquad (3)$$

where $\hat{u}(\xi)=\int_{\boldsymbol{R}^d} e^{-i\langle\xi, x\rangle}u(x)dx$ and $\langle\xi, x\rangle=\sum_{i=1}^{d}\xi_i x_i$, $|\xi|^2=\langle\xi, \xi\rangle$ for $\xi=(\xi_1, \cdots, \xi_d)$, $x=(x_1, \cdots, x_d)\in \boldsymbol{R}^d$. By taking the Fourier inverse transform of (3), we have

$$u_n(x)=\frac{1}{(2\pi)^d}\int_{\boldsymbol{R}^d}\left[\frac{1}{(1+h|\xi|^2)^n}\right]^{\wedge}(x-y)u_0(y)dy\,.$$

Now the following theorem holds:

Theorem. *Let d be a positive odd integer. Let N be a positive integer and put $h=1/N$. Then there exists a positive number C depending only on d such that*

$$\left|\left[\frac{1}{(1+h|\xi|^2)^n}\right]^{\wedge}(x)\right| \leq C|x|^{-d}$$

holds for any $x \in \boldsymbol{R}^d$ $(x \neq 0)$ and n $(n=1,2,\cdots,N)$.

Note that the above Fourier transform $\wedge$ should be interpreted in the distributional sense, since $\frac{1}{(1+\mathrm{h}|\xi|^2)^n}$ is not integrable for $1 \leq n \leq \frac{d+1}{2}-1$. For the proof of this theorem, we use the following well-known results:

$$\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)=(2\pi)^{d/2}|x|^{-(d-2)/2}\int_0^{\infty} J_{(d-2)/2}(|x|s)s^{d/2}ds \quad \left(n \geq \frac{d+1}{2}\right) \tag{4}$$

(see E. M. Stein and G. Weiss [10, page 155]),

$$\int_0^{\infty} \frac{s^{d/2}}{(1+s^2)^n} J_{(d-2)/2}(|x|s)ds = \frac{|x|^{n-1}K_{(d-2n)/2}(|x|)}{2^{n-1}(n-1)!} \quad \left(n \geq \frac{d+1}{2}\right) \tag{5}$$

(see [12, page 686])

and

$$K_{n+(1/2)}(|x|)=K_{-n-(1/2)}(|x|)=\left(\frac{\pi}{2|x|}\right)^{1/2} e^{-|x|}\sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!(2|x|)^r} \quad (n=0,1,2,\cdots) \tag{6}$$

(see [12, page 967]),

where J denotes Bessel function of the first kind and K denotes modified Bessel function.

For any fixed n $(n=1,2,\cdots,N)$, putting

$$f(x)=(1+|x|^2)^{-n},$$

we have $\left[f\left(\frac{\xi}{\sqrt{N}}\right)\right]^{\wedge}(x)=N^{d/2}\hat{f}(\sqrt{N}x)$, by the change of variables $\eta=\xi/\sqrt{N}$. Hence for the proof of Theorem it is sufficient to prove that there exists a positive number C such that

$$\left|\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)\right| \leq C|x|^{-d} \tag{7}$$

holds for any $x \in \boldsymbol{R}^d$ $(x \neq 0)$ and any positive integer n. Therefore we have only to prove the following two Lemmas.

Lemma 1. *Let d be a positive odd integer. Then*

$$\left|\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)\right| \leq C\frac{1}{1+|x|^d}$$

holds for any $x \in \boldsymbol{R}^d$ *and any positive integer* $n \geq \dfrac{d+1}{2}$.

Proof. By using (4) and (5), we have

$$\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)=(2\pi)^{d/2}|x|^{-(d-2)/2}\int_0^\infty \frac{1}{(1+s^2)^n}J_{(d-2)/2}(|x|s)s^{d/2}ds$$
$$=(2\pi)^{d/2}|x|^{-d/2+1}\frac{|x|^{n-1}K_{(d-2n)/2}(|x|)}{2^{n-1}(n-1)!}$$
$$=(2\pi)^{d/2}\frac{|x|^{n-d/2}K_{(d-2n)/2}(|x|)}{2^{n-1}(n-1)!}.$$

Since d is an odd positive integer, we obtain from (6) that

$$\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)=\frac{(2\pi)^{d/2}|x|^{n-d/2}}{2^{n-1}(n-1)!}\left(\frac{\pi}{2|x|}\right)^{1/2}e^{-|x|}\sum_{r=0}^{n-(d+1)/2}\frac{\left(n-\frac{d+1}{2}+r\right)!}{r!\left(n-\frac{d+1}{2}-r\right)!(2|x|)^r}$$
$$=(2\pi)^{(d+1)/2}e^{-|x|}\sum_{r=0}^{n-(d+1)/2}\frac{\left(n-\frac{d+1}{2}+r\right)!|x|^{n-d/2-1/2-r}}{(n-1)!r!\left(n-\frac{d+1}{2}-r\right)!2^{n+r}}$$
$$=(2\pi)^{(d+1)/2}e^{-|x|}\sum_{k=0}^{n-(d+1)/2}\frac{(2n-d-1-k)!|x|^k}{(n-1)!\left(n-\frac{d+1}{2}-k\right)!k!2^{2n-d/2-1/2-k}},$$

where we have used the change of variables: $k=n-\dfrac{d+1}{2}-r$. Also, we see,

$$\left|\left[\frac{1}{(1+|\xi|^2)^n}\right]^{\wedge}(x)\right|\leq\int_{\boldsymbol{R}^d}\left[\frac{1}{(1+|\xi|^2)^n}\right]d\xi\leq\int_{\boldsymbol{R}^d}\left[\frac{1}{(1+|\xi|^2)^{(d+1)/2}}\right]d\xi<\infty$$

holds for any $x \in \boldsymbol{R}^d$ and any positive integer $n\geq\dfrac{d+1}{2}$. Hence, in order to prove Lemma, it is sufficient to show

$$\sum_{k=0}^{n-(d+1)/2}\frac{(2n-d-1-k)!|x|^{k+d}}{(n-1)!\left(n-\frac{d+1}{2}-k\right)!k!2^{2n-d/2-1/2-k}}\leq(2\pi)^{-(d+1)/2}Ce^{|x|} \tag{8}$$

for any $x \in \boldsymbol{R}^d$ and n sufficiently large, since the left side of (8) is a polynomial of degree $n+\dfrac{d}{2}-\dfrac{1}{2}$. Hence, by putting $n=m+\dfrac{d}{2}+\dfrac{1}{2}$, it is sufficient to show

$$\sum_{k=0}^{m}\frac{(2m-k)!|x|^{k+d}}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-k)!k!2^{2m-k}}=\sum_{k=0}^{m}\frac{(2m-k)!(k+1)(k+2)\cdots(k+d)2^k}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-k)!k!2^{2m}}\cdot\frac{|x|^{k+d}}{(k+d)!}$$
$$\leq\pi^{-(d+1)/2}Ce^{|x|}$$

for any $x\in\boldsymbol{R}^d$ and m sufficiently large.

Here we set for any m sufficiently large and for k $(k=1, 2, \cdots, m)$

$$a_k^m=\frac{(2m-k)!(k+1)(k+2)\cdots(k+d)2^k}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-k)!2^{2m}}$$

and successively estimate a_k^m for any fixed m. When we regard a_k^m as a function of k, we then have the following calculation:

$$\begin{aligned}a_{k+1}^m-a_k^m&=\frac{(2m-(k+1))!((k+1)+1)((k+1)+2)\cdots((k+1)+d)2^{k+1}}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-(k+1))!2^{2m}}\\&\quad-\frac{(2m-k)!(k+1)(k+2)\cdots(k+d)2^k}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-k)!2^{2m}}\\&=\frac{(2m-k-1)!(k+2)(k+3)\cdots(k+d)2^k}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-k)!2^{2m}}\{-k^2-(1+2d)k+2dm\}.\end{aligned}$$

Hence, we can take a positive integer k_m such that

$$\left|\frac{-(1+2d)+\sqrt{(1+2d)^2+8dm}}{2}-k_m\right|\leq 1$$

and

$$a_0^m<a_1^m<\cdots<a_{k_m-1}^m\leq a_{k_m}^m\geq a_{k_m+1}^m\geq\cdots>a_{m-1}^m>a_m^m.$$

Consequently, it is sufficient to prove the following:

$$\limsup_{m\to\infty} a_{k_m}^m<\infty$$

to obtain the result of Theorem. We calculate a_k^m as follows:

$$\begin{aligned}a_{k_m}^m&=\frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)2^{k_m}}{\left(m+\frac{d}{2}-\frac{1}{2}\right)!(m-k_m)!2^{2m}}\\&=\frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)2^{k_m}(m-k_m+1)(m-k_m+2)\cdots(m-k_m+k_m)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\frac{d}{2}-\frac{1}{2}\right)}\\&=\frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)(2m-2k_m+2)(2m-2k_m+4)\cdots(2m-2k_m+2k_m)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\frac{d}{2}-\frac{1}{2}\right)}\\&\leq\frac{(2m-k_m)!(k_m+1)(k_m+2)\cdots(k_m+d)(2m-k_m+1)(2m-k_m+2)\cdots(2m-k_m+k_m)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\frac{d}{2}-\frac{1}{2}\right)}\end{aligned}$$

$$=\frac{(2m)!(k_m+1)(k_m+2)\cdots(k_m+d)}{(m!)^2 2^{2m}(m+1)(m+2)\cdots\left(m+\frac{d}{2}-\frac{1}{2}\right)}\leq\frac{(2m)!}{(m!)^2 2^{2m}}\cdot\frac{(k_m+d)^d}{m^{(d/2)-(1/2)}}$$

$$\leq\frac{(2m)!}{(m!)^2 2^{2m}}\cdot\frac{\left(\frac{1+\sqrt{(1+d)^2+8dm}}{2}\right)^d}{m^{(d/2)-(1/2)}}$$

By the well-known Wallis formula, we have

$$\limsup_{m\to\infty} a_{k_m}^m\leq\frac{(2d)^{d/2}}{\sqrt{\pi}},$$

which completes the proof of Lemma.

Lemma 2. *Let d be a positive odd integer. Then*

$$\left|\left[\frac{1}{(1+|\xi|^2)^n}\right]^\wedge(x)\right|\leq C\max\left\{\frac{1}{|x|},\frac{1}{|x|^{d-2}}\right\}e^{-|x|}$$

holds for any $x\in\boldsymbol{R}^d$ $(x\neq 0)$ and $n=1,2,\cdots,\frac{d+1}{2}-1$.

Proof. Since the equality

$$\left[\frac{1}{(1+|\xi|^2)^{(d+1)/2}}\right](x)=\frac{\pi^{(d+1)/2}}{\left(\frac{d-1}{2}\right)!}e^{-|x|}$$

holds for any $x\in\boldsymbol{R}^d$, we have that, for $n=1,2,\cdots,\frac{d+1}{2}-1$,

$$\left[\frac{1}{(1+|\xi|^2)^n}\right]^\wedge(x)=\left[\frac{(1+|\xi|^2)^{(d+1)/2-n}}{(1+|\xi|^2)^{(d+1)/2}}\right]^\wedge(x)$$

$$=\frac{\pi^{(d+1)/2}}{\left(\frac{d-1}{2}\right)!}[(1-\Delta)^{(d+1)2-n}e^{-|x|}],$$

where $\Delta=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$. Therefore, it is sufficient to prove that there exists a positive number C such that

$$|(1-\Delta)^m e^{-|x|}|\leq C\max\left\{\frac{1}{|x|},\frac{1}{|x|^{d-2}}\right\}e^{-|x|}\quad\left(m=1,2,\cdots,\frac{d-1}{2}\right)$$

for any $x\in\boldsymbol{R}^d$ $(x\neq 0)$. Using repeatedly the following equality

$$(1-\Delta)\frac{e^{-|x|}}{|x|^k}=\left[(d-2-k)k\frac{1}{|x|^{k+2}}+(d-1-2k)\frac{1}{|x|^{k+1}}\right]e^{-|x|}\quad(k=1,2,\cdots)\qquad(9)$$

for any $x\in\boldsymbol{R}^d$ $(x\neq 0)$, we obtain that, for any $x\in\boldsymbol{R}^d$ $(x\neq 0)$,

$$(1-\Delta)e^{-|x|}=\frac{C_1^1}{|x|}e^{-|x|}=\frac{d-1}{|x|}e^{-|x|}$$

$$(1-\Delta)^2e^{-|x|}=\left[\frac{C_1^2}{|x|}+\frac{C_2^2}{|x|^2}\right]\frac{e^{-|x|}}{|x|}=\left[\frac{(d-1)(d-3)}{|x|}+\frac{(d-1)(d-3)}{|x|^2}\right]\frac{e^{-|x|}}{|x|}$$

$$(1-\Delta)^3e^{-|x|}=\left[\frac{C_1^3}{|x|}+\frac{C_2^3}{|x|^2}+\frac{C_3^3}{|x|^3}\right]\frac{e^{-|x|}}{|x|^2}$$

$$=\left[\frac{(d-1)(d-3)(d-5)}{|x|}+\frac{3(d-1)(d-3)(d-5)}{|x|^2}+\frac{3(d-1)(d-3)(d-5)}{|x|^3}\right]\frac{e^{-|x|}}{|x|^2}$$

....

$$(1-\Delta)^m e^{-|x|}=\left[\frac{C_1^m}{|x|}+\frac{C_2^m}{|x|^2}+\cdots+\frac{C_m^m}{|x|^m}\right]\frac{e^{-|x|}}{|x|^{m-1}}$$

....

$$(1-\Delta)^{(d-1)/2}e^{-|x|}=\left[\frac{C_1^{(d-1)/2}}{|x|}+\frac{C_2^{(d-1)/2}}{|x|^2}+\cdots+\frac{C_{(d-1)/2}^{(d-1)/2}}{|x|^{(d-1)/2}}\right]\frac{e^{-|x|}}{|x|^{(d-1)/2-1}},$$

where C_j^m $\left(2\le m\le\frac{d-1}{2}, 1\le j\le m\right)$ is defined inductively as follows:

$$C_1^m=(d-2m+1)C_1^{m-1},$$
$$C_j^m=(d-2m-2j+3)C_j^{m-1}+(d-m-j+1)(m+j-3)C_{j-1}^{m-1}\quad (j=2, 3, \cdots, m-1)$$

and

$$C_m^m=(d-2m+1)(2m-3)C_{m-1}^{m-1}.$$

Since C_j^m depends only on d, we can take positive numbers C such that

$$|(1-\Delta)^m e^{-|x|}|\le C\max\left\{\frac{1}{|x|}, \frac{1}{|x|^{d-2}}\right\}e^{-|x|}\quad \left(m=1, 2, \cdots, \frac{d-1}{2}\right)$$

holds for any $x\in\boldsymbol{R}^d$ $(x\neq 0)$, which completes the proof of Lemma.

Thus we have finished the proof of Theorem.

Now it is very attractive to the authors to construct a regular solution (Morse flow) for non-linear parabolic partial differential equations corresponding to the following problems in the calculus of variations: For mappings $u\in H^{1,2}(\Omega, \boldsymbol{R}^{d'})$ ($H^{1,2}(\Omega, \boldsymbol{R}^{d'})$ is the usual Sovolev space and Ω is an open and bounded domain with smooth boundary in $\boldsymbol{R}^d$), we consider the following functional

$$I(u)=\int_\Omega A^{\alpha\beta}(x, u(x))D_\alpha u^i(x)D_\beta u^i(x)dx.$$

Here in the summation over repeated indices, the Greek indices run from 1 to d and the Latin ones from 1 to d'. We assume that the coefficients $A^{\alpha\beta}$ are bounded functions suitably smooth in $\bar{\Omega}\times\boldsymbol{R}^{d'}$ and satisfy the condition

$$A^{\alpha\beta}(x, u)\xi_\alpha\xi_\beta\ge\lambda|\xi|^2\quad\text{for } \xi\in\boldsymbol{R}^d \text{ and } (x, u)\in\Omega\times\boldsymbol{R}^{d'}$$

with a uniform positive constant λ.

It has been successfully treated by Douglas and Morrey ([7]) to find a regular minimum point in the case $d=2$. In general case $d\geq 3$, an excellent result was proposed by Giaquinta and Giusti ([2]) in 1980. This result says that minima of the functional I (with the coefficients $A^{\alpha\beta}$ whose smoothness is only required to be continuous) are Hölder-continuous in Ω under the so-called one sided condition proposed by Hildebrandt and Widman ([5]). It should be remarked that the one-sided condition does not impose the solutions any "smallness".

Since the result appeared, it has been conjectured that the parabolic flow for equations

$$\frac{\partial u}{\partial t}=D_\beta(A^{\alpha\beta}(x, u)D_\alpha u)-\frac{1}{2}\nabla_u A^{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i$$

in the "weak" sense conserves the regularity of the initial data under the one-sided condition. Here we mention interesting papers [4] and [11], which treat non-linear parabolic differential equations. One* of the authors in this note has taken up this problem for these years and approached it by considering the following functionals:

$$I_n(u)=\int_\Omega\left(A^{\alpha\beta}(x, u(x))D_\alpha u^i(x)D_\beta u^i(x)+\frac{1}{h}|u-u_{n-1}|^2\right)dx \quad (n=1, 2, \cdots, N),$$

where N is a positive integer and $h=1/N$ (for example) and u_0 is an initial datum for the problem. By taking u_n as a minimum of the functional $I_n(u)$ inductively, we obtain the following Euler-Lagrange equations of I_n:

$$\frac{u_n-u_{n-1}}{h}=D_\beta(A^{\alpha\beta}(x, u_n)D_\alpha u_n)-\frac{1}{2}\nabla_u A^{\alpha\beta}(x, u_n)D_\alpha u_n^i D_\beta u_n^i \qquad (*)$$

where we notice that any $u_n(x)$ is known to be smooth in Ω by virtue of the result [2]. By constructing a suitable function comparative to the minimum u_n, we are trying to obtain the so-called reverse Hölder inequality due to Gehring-Giaquinta-Modica ([1] and [3]), about which we expect to be able to write in another paper. To obtain the conjectured result stated above, we think that such properties for solutions of equations (*) as the estimates in this note and Harnack property of Moser's type ([8] and [9]) will play an essential role.

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