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INTERACTING PARTICLE SYSTEM AND BROWNIAN SHEET

by

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ABSTRACT

The algebraic method of cluster expansion is applied to the limit theorem related to the interacting particle system. The random field $X^L(t, s)$ is defined with respect to the Gibbs state by scaling the number of particles in (0, tL]×(0, sL]. It is proved that $X^L(t, s)$ converges to the Brownian sheet in law as $L\to\infty$ if the density of particles is low.

§ 1 Introduction

In statistical mechanics it is one of the most powerful techniques to expand observable quantities in power series by means of algebraic methods. Such expansions are called cluster expansion or polymer expansion. Employing these expansions we can take an infinite volume limit of the quantities and can define the thermodynamic functions. (See [1]~[4] for example). This makes it possible to study statistical mechanics mathematically. For instance a theory of phase transitions consists in studying the analyticity of the thermodynamic functions. In other words the occurrence of phase transitions is characterized as the singularity of the thermodynamic functions. Also these expansions have been applied to the various problems in statistical mechanics, including the study of phase separation and the phase diagram for lattice models [5]~[9], the study of the decay of correlations (see Ch. 4 of [1] for detail), and the limit theorem for Gibbs states [10]~[12].

In this article we shall apply the cluster expansion to the limit theorem for the interacting particle system on $\boldsymbol{R}^2$.

Consider the interacting particle system on the square V with side L. The configuration of particles is specified by the set of finite number of points $(x_1, \dots, x_n)$ in V. We associate the interaction energy to each configuration $(x_1, \dots, x_n)$

$$U(x_1, \dots, x_n)=\sum_{1\le i<j\le n}\Phi(x_i-x_j),$$

where $\Phi(\cdot)$ is the symmetric function on $\boldsymbol{R}^2$ which describes the interaction between two particles. We assume several conditions on this function Φ which will be stated precisely in Section 2. To describe the equilibrium state of the system the Gibbs state is introduced with the interaction Φ and the activity z. Here the activity z is the parameter which controls the density of particles. To be specific, the expectation $E(N_V)$ of the number of particles in V is the increasing function of z, so that the regime of small values of z corresponds to the state of low density.

Using this notion of Gibbs state we define a random field $X^L(t, s)$, $0 \leq t, s \leq 1$, by

$$X^L(t, s) = \frac{1}{\sigma L}\{N^L(t, s) - E(N^L(t, s))\},$$

where $N^L(t, s)$ is the number of particles in $(0, \mathrm{tL}] \times (0, \mathrm{sL}]$ and $E(\cdot)$ is the expectation with respect to the Gibbs state.

The purpose of this article is to prove that $X^L(t, s)$ converges to the Brownian sheet in law as $L \to \infty$ in the space D^2 with a suitably chosen function $\sigma = \sigma(z)$ when the activity z is sufficiently small. (See [14] and [15] for the definition of the Brownian sheet). Here D^2 is the space of all functions from $[0, 1]^2$ to $\boldsymbol{R}$ which are continuous from above and have limits from below. We endow this space with Bickel-Wichura's S-topology [16]. We will see that $\sigma^2(z)$ is a thermodynamic limit of the variance of N_V as $V \to \boldsymbol{R}^2$.

In Section 2 we formulate our problem and state our results precisely. In Section 3 we summarize the method of cluster expansion and properties of the Ursell functions for our system. The Section 4 is devoted to the proof of that the finite dimensional distribution of $X^L(t, s)$ converges to the corresponding distribution of the Brownian sheet as $L \to \infty$. The tightness of the distributions of $X^L(t, s)$ on D^2 is proved in Section 5. In Section 6 we restrict our argument to the system with a nonnegative interaction in which more detailed result is obtained. When an interaction is nonnegative it is proved that the Ursell function satisfies the alternating property. (See Ch. 4.5 of [1]). Using this property we prove that the radius of convergence $\mathcal{R}$ of $\sigma(z)$ is finite, and that the limit theorem mentioned above holds for all z with $0 < z < \mathcal{R}$.

§2 Statement of Results

Let V be a bounded subset of $\boldsymbol{R}^2$ and N be a nonnegative integer. We define configuration spaces $\Omega_{V,N}$ and Ω_V by

$$\Omega_{V,N} = \{\xi = \{x_1, x_2, \cdots, x_N\} : x_i \in V,\ x_i \neq x_j,\ 1 \leq i < j \leq N\},$$

and $\Omega_V = \bigcup_{N=0}^{\infty} \Omega_{V,N}$ respectively.

The σ-field $\mathcal{B}_{V,N}$ on $\Omega_{V,N}$ and $\mathcal{B}_V$ on Ω_V are defined as the smallest σ-field generated by the set $\{\xi \in \Omega_{V,N} ; N_A(\xi) = n\}$, $A \in \mathcal{B}(\boldsymbol{R}^2)$, $0 \leq n \leq N$, and the sets $\{\xi \in \Omega_V ; N_A(\xi) = n\}$, $A \in \mathcal{B}(\boldsymbol{R}^2)$, $n \geq 0$ respectively, where $N_A(\xi)$ is the number of particles in

A for the configuration ξ.

An alternative description of $\Omega_{V,N}$ is given by

(1) $$\Omega_{V,N}=(V^N)'/S_N,$$

where $(V^N)'=\{(x_1, x_2, \cdots, x_N)\in V^N ; x_i\neq x_j \text{ for } i<j\}$ and S_N is a symmetric group of order N.

By means of the factorization (1) we introduce a measure μ on Ω_V such that

$$\begin{cases} \mu(\emptyset)=1, \\ \mu(A)=\dfrac{1}{N!}\displaystyle\int_{\hat{A}} dx_1 dx_2\cdots dx_N, & \text{if } A\subset\Omega_{V,N},\ N\geq 1, \end{cases}$$

where $\emptyset$ is the configuration of no particle and $\hat{A}$ is the inverse image of A with respect to the factorization (1).

Let Φ be a $\boldsymbol{R}$-valued measurable function on $\boldsymbol{R}^2$ satisfying $\Phi(-x)=\Phi(x)$ for all $x\in\boldsymbol{R}^2$. This function Φ is called the interaction function which describes the interaction between two particles. Several conditions are assumed on Φ to define the Gibbs state and to obtain their properties which will be used for the method of the cluster expansion.

First we assume that Φ is bounded from below and satisfies the following condition (I) called "regularity condition".

(I) $$C(\beta)=\int_{\boldsymbol{R}^2} dx|e^{-\beta\Phi(x)}-1|<\infty \quad \text{for some } \beta>0.$$

Remark. The condition (I) is equivalent to the following condition (I)'

(I)' $$C(\beta)<\infty \qquad \text{for all } \beta>0.$$

If the interaction function Φ has finite range then the condition (I) is satisfied.

Next we assume either one of the following conditions (II-1) and (II-2).

(II-1) Φ is nonnegative, i.e. $\Phi(\cdot)\geq 0$.

(II-2)

(i) There exists a positive number $d_0>0$ such that

$\Phi(x)=\infty$ for all $x\in\boldsymbol{R}^2$ with $0\leq|x|<d_0$ (Hard core condition)

and

(ii) There exists a nonnegative number $B\geq 0$ such that

$$\textstyle\sum_{1\leq i\leq n}\Phi(x_i)\geq -2B$$

for all n and all $x_1, \ldots, x_n\in\boldsymbol{R}^2$ satisfying $|x_i-x_j|\geq d_0$ for $i\neq j$.

Let us note that if the condition (II-1) is satisfied then the condition (ii) of (II-2) is automatically satisfied with $B=0$.

To each configuration $(x_1, x_2, \cdots, x_n)$ we associate an interaction energy

$$U(x_1, x_2, \cdots, x_n) = \sum_{1\le i<j\le n} \Phi(x_i - x_j).$$

The condition (ii) of (II-2) implies that $U(x_1, \ldots, x_n)$ is stable, i.e.

$$U(x_1, \ldots, x_n) \ge -nB.$$

Definition *The probability measure $P_V(\cdot)=P_{V,\beta,z}(\cdot)$ on Ω_V is called a Gibbs State if $P_V(\cdot)$ is absolutely continuous with respect to μ with the density function*

$$P_V(\xi|\beta, z) = \frac{1}{Z_V}\exp\{-\beta U(\xi) + N_V(\xi)\log z\},$$

where $N_V(\xi)$ is the number of particles of ξ in V and Z_V is the normalized constant called the partition function.

Put $V=(0, L]\times(0, L]$ and $V(t, s)=(0, tL]\times(0, sL]$, $0\le t, s\le 1$. For any $\xi\in\Omega_V$ we define the random field $X^L(t, s)(\xi)$ by

$$X^L(t, s)(\xi) = \frac{1}{\sigma L}\{N^L(t, s)(\xi) - E_V(N^L(t, s))\},$$

where $N^L(t, s)(\xi)$ is the number of particles of ξ in $V(t, s)$ and $E_V(\cdot)$ is the expectation with respect to the Gibbs state. Now we are in a position to state our first result.

Theorem 1. *For any $\beta>0$ there exists a positive constant $z_0(\beta)$ and a function $\sigma(z)$ which is positive for all $0<z<z_0(\beta)$. If $0<z<z_0(\beta)$, then*

$$X^L(t, s) \longrightarrow B(t, s), \qquad as \qquad L\longrightarrow\infty,$$

in the sense of finite dimensional distribution with $\sigma=\sigma(z)$, where $B(t, s)$ is the Brownian sheet.

Next we shall prove that $X^L(t, s)$ converges to the Brownian sheet in law in the space D^2, where D^2 is the space of all functions from $[0, 1]^2$ to $\boldsymbol{R}$ which are continuous from above and have limits from below. We endow this space with Bickel-Wichura's S-topology [16]. Let Λ be the set of all functions $\lambda(\cdot)$ which are strictly increasing and continuous from $[0, 1]$ to itself such that $\lambda(0)=0$ and $\lambda(1)=1$. Bickel-Wichura's S-topology is given by the following convergence: $w_n(t)\to w(t)$ in D^2 if and only if there exist λ_n and $\mu_n\in\Lambda$ such that

(1) $\lambda_n(t)$ converges to t uniformly as $n\to\infty$,

(2) $\mu_n(t)$ converges to t uniformly as $n\to\infty$, and

(3) $w_n(\lambda_n(t), \mu_n(s))$ converges to $w(t, s)$ uniformly as $n\to\infty$.

Now we shall state our main result.

Theorem 2. *If $0<z<z_0(\beta)$, then $X^L(t, s)$ converges to the Brownian Sheet in law with $\sigma=\sigma(z)$, as $L\to\infty$, in the space D^2, where $\sigma(z)$ is the function given in Theorem 1.*

§ 3 Algebraic method of cluster expansion

In this section we summarize the algebraic formalism of the cluster expansion in power of the activity z. (See [1] for detail).

Let $\mathcal{A}$ be the set of sequences ψ

$$\psi=\{\psi(x_n)\}_{n\geq 0},$$

where $\psi(x_n)$ is a bounded complex valued Lebesgue measurable function on $\boldsymbol{R}^{2n}$ and the 0th component $\psi(\emptyset)$ of ψ is a complex number. We denote by X the sequence $x_n=(x_1, x_2, \cdots, x_n)$ of finite number of particles in $\boldsymbol{R}^2$. We say (X_1, X_2) is a partition of X and write

$$X_1+X_2=X,$$

if X_1 is a subsequence of X and $X_2=X\backslash X_1$. We define a product in $\mathcal{A}$ by,

$$\psi_1 * \psi_2(X)=\sum_{X_1+X_2=X}\psi_1(X_1)\psi_2(X_2),$$

where the sum is taken over all partitions of X.

With this product $\mathcal{A}$ is a commutative algebra with unit element $\mathbf{1}$ defined by

$$\mathbf{1}(X)=\begin{cases}1, & \text{if } X=\emptyset\\ 0, & \text{otherwise}.\end{cases}$$

We define the subspaces $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ by

$$\mathcal{A}_0=\{\varphi\in\mathcal{A} : \varphi(\emptyset)=0\},$$

$$\text{and} \qquad \mathcal{A}_1=\{\psi\in\mathcal{A} : \psi(\emptyset)=1\} \qquad \text{respectively.}$$

The power series expansion of the exponential yields a well-defined mapping Exp from $\mathcal{A}_0$ to $\mathcal{A}_1$:

$$\operatorname{Exp}\varphi(X)=\mathbf{1}(X)+\sum_{n=1}^{\infty}\frac{\varphi^{*n}(X)}{n!}, \qquad \varphi\in\mathcal{A}_0.$$

Any element $\psi\in\mathcal{A}_1$ can be decomposed into the sum of the unit element $\mathbf{1}$ and $\psi_0\in\mathcal{A}_0$ in a unique way. We define the logarithm mapping Log from $\mathcal{A}_1$ to $\mathcal{A}_0$ by

$$\operatorname{Log}\psi(X)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\psi_0^{*n}(X), \qquad \psi\in\mathcal{A}_1.$$

We have the following relation between Exp and Log.

Lemma 3.1

(1) $$\operatorname{Log}\operatorname{Exp}\varphi=\varphi, \qquad \varphi\in\mathcal{A}_0.$$

(2) $$\operatorname{Exp}\operatorname{Log}\psi=\psi, \qquad \psi\in\mathcal{A}_1.$$

Let χ be a Lebsgue integrable function on $\boldsymbol{R}^2$. For any $\psi\in\mathcal{A}$ we associate the formal power series

$$\langle \chi, \psi\rangle (z)=\psi(\emptyset)+\sum_{n=1}^{\infty}\frac{z^n}{n\,!}\int_{\boldsymbol{R}^{2n}} dx_1dx_2\cdots dx_n\psi(x_1, x_2, \cdots, x_n)\chi(x_1)\chi(x_2)\cdots\chi(x_n)\,.$$

The following result plays an important role for the method of the cluster expansion.

Lemma 3.2 (1) *If* $\langle \chi, \psi_1\rangle(z)$ *and* $\langle \chi, \psi_2\rangle(z)$ *are absolutely convergent, then* $\langle \chi, \psi_1 * \psi_2\rangle(z)$ *is also absolutely convergent and the following equality holds*

$$\langle \chi, \psi_1 * \psi_2\rangle(z)=\langle \chi, \psi_1\rangle(z)\cdot\langle \chi, \psi_2\rangle(z)\,.$$

(2) *If* $\langle \chi, \mathrm{Log}\,\psi\rangle(z)$ *is absolutely convergent for* $\psi\in\mathcal{A}_1$, *then* $\langle \chi, \psi\rangle(z)$ *is also absolutely convergent and the following relation holds*

$$\langle \chi, \psi\rangle(z)=\exp\{\langle \chi, \mathrm{Log}\,\psi\rangle(z)\}.$$

We introduce an operator D_x: $\mathcal{A}\longrightarrow\mathcal{A}$ by

$$D_x\psi(y_1, y_2, \cdots, y_n)=\psi(x, y_1, \cdots, y_n).$$

For any $X=(x_1, x_2, \cdots, x_m)$, we define the mapping D_X: $\mathcal{A}\longrightarrow\mathcal{A}$ by

$$D_X\psi=D_{x_1}D_{x_2}\cdots D_{x_m}\psi\,.$$

It is clear that D_x and D_X are linear. Furthermore, D_x satisfies the following relations.

Lemma 3.3

(1) $$D_x(\psi_1 * \psi_2)=D_x\psi_1 * \psi_2+\psi_1 * D_x\psi_2, \quad \text{for} \quad \psi_1, \psi_2\in\mathcal{A},$$

(2) $$D_x \,\mathrm{Exp}\,\varphi=D_x\varphi * \mathrm{Exp}\,\varphi, \quad \text{for} \quad \varphi\in\mathcal{A}_0\,.$$

Now we introduce the function $\psi_\beta\in\mathcal{A}_1$ called the Boltzman's factor,

$$\psi_\beta(X)=\exp\{-\beta U(X)\}\,.$$

In terms of this function the partition function Z_V can be rewritten as

$$Z_V=\langle \chi_V, \psi_\beta\rangle(z)\,,$$

where $\chi_V(X)$ is given by

$$\chi_V(X)=\begin{cases}1 & \text{if } x\in V \text{ for all } x\in X\\ 0 & \text{otherwise.}\end{cases}$$

We define the Ursell function $\varphi_\beta(X)$ by

$$\varphi_\beta(X)=\mathrm{Log}\,\psi_\beta(X)\,.$$

An alternative description of $\varphi_\beta(x_1, x_2, \cdots, x_n)$ is given by

$$\varphi_\beta(x_1, x_2, \cdots, x_n)=\sum_{\Gamma}\prod_{i<j}^{\Gamma}(e^{-\beta\Phi(x_i-x_j)}-1)\,,$$

where the sum extends over all connected graphs γ with vertices $1, \cdots, n$ and the product takes over all bonds (i, j) of the graph γ. To obtain the estimate for the radius of convergence of the power series $\langle \chi_V, \varphi_\beta \rangle(z)$ the following lemma plays an important role.

Lemma 3.4

$$\int_{\boldsymbol{R}^{2(n-1)}} dx_2 \cdots dx_n |\varphi_\beta(0, x_2, x_3, \cdots, x_n)| \leq (n-1)!\ C(\beta)\{e^{2\beta B+1}C(\beta)\}^{n-1}$$

This lemma implies that the power series $\langle \chi_V, \varphi_\beta \rangle(z)$ converges for

$$|z| < \{C(\beta)e^{2\beta B+1}\}^{-1},$$

and that the partition function Z_V can be rewritten as

$$Z_V = \exp\{\langle \chi_V, \varphi_\beta \rangle(z)\}.$$

Put $z_0(\beta) = \{C(\beta)e^{2\beta B+1}\}^{-1}$. Applying Lemma 3.2(2), we get the convergence of the following thermodynamic function

$$f_\beta(z) = \lim_{V \to \boldsymbol{R}^2} \frac{1}{|V|} \log Z_V,$$

for any $|z| < z_0(\beta)$. Using the Ursell function φ_β, we have the explicit form for $f_\beta(z)$ as follows

$$f_\beta(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\boldsymbol{R}^{2(n-1)}} dx_2 \cdots dx_n \varphi_\beta(0, x_2, \cdots, x_n).$$

It follows from Lemma 3.4 that the radius of convergence $\mathcal{R}$ of $f_\beta(z)$ is equal or greater than $z_0(\beta)$.

Let $N_V(\xi)$ be a number of particles of the configuration ξ in V. From Lemma 3.2 and 3.4 it follows that

$$E_V(e^{iyN_V}) = \exp\left\{\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{V^n} d\underline{x}_n (e^{iyN_V(\underline{x}_n)} - 1)\varphi_\beta(\underline{x}_n)\right\},$$

if $|z| < z_0(\beta)$. Using this relation we can express the expectation and the variance of N_V as follows

$$E_V(N_V) = \sum_{n=1}^{\infty} \frac{z^n}{(n-1)!} \int_{V^n} \varphi_\beta(\underline{x}_n) d\underline{x}_n, \tag{3.1}$$

$$E_V(N_V^2) - E_V(N_V)^2 = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} z^n \int_{V^n} \varphi_\beta(\underline{x}_n) d\underline{x}_n. \tag{3.2}$$

Using the method which will be stated in the next section we can prove the convergence of the following thermodynamic functions

(3.3)
$$\lim_{V\to\boldsymbol{R}^2}\frac{1}{|V|}E_V(N_V)=\rho(z)\,,$$

(3.4)
$$\lim_{V\to\boldsymbol{R}^2}\frac{1}{|V|}\{E_V(N_V{}^2)-E_V(N_V)^2\}=z\rho'(z)\,,$$

uniformly on every compact set contained in $\{z\,;\ |z|<z_0(\beta)\}$, where $\rho(z)=zf'_\beta(z)$. This function is called the density function and has the following explicit form:

(3.5)
$$\rho(z)=\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}\int_{\boldsymbol{R}^{2(n-1)}}\varphi_\beta(0,\ x_2,\ \cdots,\ x_n)dx_2\ldots dx_n\,.$$

We use this function in the next section for the proof of theorems.

§ 4 Proof of Theorem 1

Throughout this section, V will be a square with side L. Let us consider the characteristic function $\theta_L(y_1,\cdots,y_m)$ of the random vector $(X^L(t_1,\ s_1),\cdots,$ $X^L(t_m,\ s_m))$ with respect to the probability measure $P_V(\cdot)$

$$\theta_L(y_1,\ \cdots,\ y_m)=E_V\left[\exp\left\{i\sum_{k=1}^{m}y_kX^L(t_k,\ s_k)\right\}\right].$$

To prove Theorem 1 it is sufficient to find some function $\sigma=\sigma(z)$ and prove,

(4.1)
$$\lim_{L\to\infty}\theta_L(y_1,\ \cdots,\ y_m)=\exp\left\{-\frac{1}{2}\sum_{j,k=1}^{m}y_jy_k(t_j\wedge t_k)(s_j\wedge s_k)\right\},$$

where the right hand side is the corresponding characteristic function of the Brownian sheet.

Applying the method of cluster expansion developed in Section 3 the characteristic function $\theta_L(y_1,\cdots,y_m)$ can be rewritten as follows if $0<z<z_0(\beta)$:

(4.2)
$$\theta_L(y_1,\ \cdots,\ y_m)=\exp\left\{-\frac{i}{\sigma L}\sum_{k=1}^{m}y_kE_V[N^L(t_k,\ s_k)]\right\}$$
$$\cdot\exp\left\{\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n}d\boldsymbol{x}_n\left[\exp\left(\frac{i}{\sigma L}\sum_{k=1}^{m}y_kN^L(t_k,\ s_k)(\boldsymbol{x}_n)\right)-1\right]\varphi_\beta(\boldsymbol{x}_n)\right\}.$$

In the same way as above we get the alternative description of $E_V[N^L(t_k,\ s_k)]$ as follows

(4.3)
$$E_V[N^L(t_k,\ s_k)]=\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n}d\boldsymbol{x}_nN^L(t_k,\ s_k)(\boldsymbol{x}_n)\varphi_\beta(\boldsymbol{x}_n)\,.$$

Using the Taylar expansion and (4.3) we get

(4.4) $$\theta_L(y_1, \cdots, y_m)$$

$$=\exp\left\{-\frac{1}{2\sigma^2L^2}\sum_{n=1}^{\infty}\frac{z^n}{n!}\int_{V^n}d\underline{x}_n\Big(\sum_{k=1}^{m}y_kN^L(t_k, s_k)(\underline{x}_n)\Big)^2\varphi_\beta(\underline{x}_n)\right.$$

$$+\frac{1}{3!\,\sigma^3L^3}\sum_{n=1}^{\infty}\frac{z^n}{n!}\int_{V^n}d\underline{x}_n\Big(\sum_{k=1}^{m}y_kN^L(t_k, s_k)(\underline{x}_n)\Big)^3$$

$$\left.\cdot\,\epsilon\Big(\frac{i}{\sigma L}\sum_{k=1}^{m}y_kN^L(t_k, s_k)(\underline{x}_n)\Big)\,\varphi_\beta(\underline{x}_n)\right\},$$

where $|\epsilon(\cdot)|\leq 1$.

Put

$$I_1=-\frac{1}{2\sigma^2L^2}\sum_{n=1}^{\infty}\int_{V^n}d\underline{x}_n\Big(\sum_{k=1}^{m}y_kN^L(t_k, s_k)(\underline{x}_n)\Big)^2\varphi_\beta(\underline{x}_n),$$

and

$$I_2=\frac{1}{3!\,\sigma^3L^3}\sum_{n=1}^{\infty}\frac{z^n}{n!}\int_{V^n}d\underline{x}_n\Big(\sum_{k=1}^{m}y_kN^L(t_k, s_k)(\underline{x}_n)\Big)^3$$

$$\cdot\,\epsilon\Big(\frac{i}{\sigma L}\sum_{k=1}^{m}y_kN^L(t_k, s_k)(\underline{x}_n)\Big)\varphi_\beta(\underline{x}_n).$$

Setting $\sigma=(z\rho'(z))^{1/2}$, we shall prove that

(4.5) $$I_1\longrightarrow -\frac{1}{2}\sum_{j,k=1}^{m}y_jy_k(t_j\wedge t_k)(s_j\wedge s_k)\qquad\text{as}\qquad L\longrightarrow\infty$$

and

(4.6) $$I_2\longrightarrow 0\qquad\text{as}\qquad L'\longrightarrow\infty.$$

To prove the convergence of I_1 and I_2 we prepare some technical lemmas.

Let us consider the paths connecting all points $x_1, \cdots, x_n\in\boldsymbol{R}^2$. We define a minimal path $s(x_1, \cdots, x_n)$ in a certain natural way and denote by $|s(x_1, \cdots, x_n)|$ its length. The following lemma plays an important role for the proof of the convergence.

Lemma 4.1 If $0<z<z_0(\beta)$, then the following properties hold for all nonnegative integers p:

(1) $$\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{\substack{x_1\in V\\ s(x_1,\cdots,x_n)\cap\partial V\neq\emptyset}}dx_n|\varphi_\beta(x_n)|=o(L^2),\quad\text{as } L\longrightarrow\infty,$$

(2) $$\sum_{n=1}^{\infty}\frac{z^n}{n!}\int_{V^n} d\underline{x}_n\{N^L(t,s)(\underline{x}_n)\}^{2+p}\varphi_\beta(\underline{x}_n)=O(L^2),\ \text{as } L\longrightarrow\infty,\ \forall t,\ s\in[0,1].$$

Proof. Dividing the sum into two parts we get

$$\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{\substack{x_1\in V\\ s(x_1,\cdots,x_n)\cap\partial V\neq\emptyset}} d\underline{x}_n|\varphi_\beta(\underline{x}_n)|$$

$$\leq\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{\substack{x_1\in V\\ |s(x_1,\cdots,x_n)|\geq L}} d\underline{x}_n|\varphi_\beta(\underline{x}_n)|$$

$$+\sum_{u=p}^{\infty}\frac{z^n}{(n-p)!}\int_{\substack{s(x_1,\cdots,x_n)\cap\partial V\neq\emptyset\\ |s(x_1,\cdots,x_n)|<L}} d\underline{x}_n|\varphi_\beta(\underline{x}_n)|$$

$$\leq L^2\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{|s(0,x_2,\cdots,x_n)|\geq L} dx_2\cdots dx_n|\varphi_\beta(0,x_2,\cdots,x_n)|$$

$$+\sum_{k=1}^{L}\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{k-1\leq|s(0,y_2,\cdots,y_n)|<k} dy_2\cdots dy_n|\varphi_\beta(0,y_2,\cdots,y_n)|$$

$$\cdot\int_{s(y_1,y_1+y_2,\cdots,y_1+y_n)\cap\partial V\neq\emptyset} dy_1.$$

Introduce the function $\delta(k)$

$$\delta(k)=\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{k-1\leq|s(0,y_2,\cdots,y_n)|<k} dy_2\cdots dy_n|\varphi_\beta(0,y_2,\cdots,y_n)|.$$

When $k-1\leq|s(0,y_2,\cdots,y_n)|<k$ we have

$$\int_{s(y_1,y_1+y_2,\cdots,y_1+y_n)\cap\partial V\neq\emptyset} dy_1\leq 8kL.$$

Putting these estimates together we have

(4.7) $$\sum_{n=p}^{\infty}\frac{z^n}{(n-p)!}\int_{\substack{x_1\in V\\ s(x_1,\cdots,x_n)\cap\partial V\neq\emptyset}} d\underline{x}_n|\varphi_\beta(\underline{x}_n)|$$

$$\leq L^2\sum_{k=L}^{\infty}\delta(k)+8L\sum_{k=1}^{L}k\delta(k).$$

Since $\sum_{k=1}^{\infty}\delta(k)<\infty$ we have

$$\lim_{L\to\infty}\frac{1}{L}\sum_{k=1}^{L}k\delta(k)=0.$$

Hence the first assertion of this lemma has proved.

Using the estimate for $|\varphi(\underset{\sim}{x}_n)|$ we have,

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n}\{N^L(t,s)(\underset{\sim}{x}_n)\}^{2+p}\varphi_\beta(\underset{\sim}{x}_n)d\underset{\sim}{x}_n$$

$$\leq\sum_{n=0}^{\infty}\frac{z^n}{n!}\cdot n^{2+p}\int_{V^n}|\varphi_\beta(\underset{\sim}{x}_n)|d\underset{\sim}{x}_n$$

$$\leq L^2\sum_{n=0}^{\infty}\frac{z^n}{n!}n^{2+p}(n-1)!\,C(\beta)\{e^{2\beta B+1}C(\beta)\}^{n-1}$$

$$=L^2zC(\beta)\sum_{n=1}^{\infty}n^{1+p}\{ze^{2\beta B+1}C(\beta)\}^{n-1}.$$

Since $ze^{2\beta B+1}C(\beta)<1$, we have the convergence of the above sum

$$\sum_{n=1}^{\infty}n^{1+p}\{ze^{2\beta B+1}C(\beta)\}^{n-1}<\infty.$$

This completes the proof of (2).

From the second assertion of this lemma the convergence (4.6) is easily obtained.

To prove the convergence (4.5) it suffices to prove the following lemma.

Lemma 4.2 *If* $0\leq z<z_0(\beta)$, *then*

$$\lim_{L\to\infty}\frac{1}{L^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n}N^L(t_1,s_1)(\underset{\sim}{x}_n)N^L(t_2,s_2)(\underset{\sim}{x}_n)\varphi_\beta(\underset{\sim}{x}_n)d\underset{\sim}{x}_n$$

$$=(t_1\wedge t_2)(s_1\wedge s_2)z\rho'(z)$$

for $0\leq t_1, t_2, s_1, s_2\leq 1$.

Proof. Put $V(t_i, s_i)=(0, t_iL]\times(0, s_iL]$, $(i=1,2)$. Using the indicator functions of $V(t_i, s_i)$, $(i=1,2)$ we have

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{V^n}N^L(t_1,s_1)(\underset{\sim}{x}_n)N^L(t_2,s_2)(\underset{\sim}{x}_n)\varphi_\beta(\underset{\sim}{x}_n)d\underset{\sim}{x}_n \tag{4.8}$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{V^n}\chi_{V(t_1,s_1)}(x_i)\chi_{V(t_2,s_2)}(x_j)\varphi_\beta(\underset{\sim}{x}_n)d\underset{\sim}{x}_n$$

$$=\sum_{n=2}^{\infty}\frac{z^n}{(n-2)!}\int_{V(t_1,s_1)}dx_1\int_{V(t_2,s_2)}dx_2\int_{V^{n-2}}\varphi_\beta(x_1,x_2,\cdots,x_n)dx_3\cdots dx_n$$

$$+\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}\int_{V(t_1,s_1)\cap V(t_2,s_2)}dx_1\int_{V^{n-1}}\varphi_\beta(x_1,x_2,\cdots,x_n)dx_2\cdots dx_n$$

$$=\sum_{n=2}^{\infty}\frac{z^n}{(n-2)!}\int_{V(t_1,s_1)}dx_1\int_{V(t_2,s_2)}dx_2\int_{\boldsymbol{R}^{n(n-2)}}\varphi_\beta(x_1,x_2,\cdots,x_n)dx_3\cdots dx_n$$

$$+\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}\int_{V(t_1,s_1)\cap V(t_2,s_2)}dx_1\int_{\boldsymbol{R}^{2(n-1)}}\varphi_\beta(x_1,x_2,\cdots,x_n)dx_2\cdots dx_n$$

$$-\sum_{u=2}^{\infty}\frac{z^n}{(n-2)!}\int_{x_1\in V(t_1,s_1),x_2\in V(t_2,s_2),s(x_1,\cdots,x_n)\cap\partial V\neq\emptyset}d\underline{x}_n\varphi_\beta(\underline{x}_n)$$

$$-\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}\int_{x_1\in V(s_1,s_1)\cap V(t_2,s_2),s(x_1,\cdots,x_n)\cap\partial V\neq\emptyset}d\underline{x}_n\varphi_\beta(\underline{x}_n)$$

$$=I_1^1(L)+I_1^2(L)-I_1^3(L)-I_1^4(L).$$

From Lemma 4.1 (1) we have

$$\lim_{L\to\infty}\frac{1}{L^2}I_1^3(L)=0 \quad\text{and}\quad \lim_{L\to\infty}\frac{1}{L^2}I_1^4(L)=0. \tag{4.9}$$

Since $\varphi_\beta(\underline{x}_n)$ is shift invariant and is integrable with respect to $x_2,\cdots,x_n$ we have

$$\lim_{L\to\infty}\frac{1}{L^2}I_1^2(L)=(t_1\wedge t_2)\cdot(s_1\wedge s_2)\rho(z), \tag{4.10}$$

where

$$\rho(z)=\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}\int_{\boldsymbol{R}^{2(n-1)}}\varphi_\beta(0,x_2,x_3,\cdots,x_n)dx_2\cdots dx_n.$$

Introduce a measurable function $g(y)$ on $\boldsymbol{R}^2$

$$g(y)=\sum_{n=2}^{\infty}\frac{z^n}{(n-2)!}\int_{\boldsymbol{R}^{2(n-2)}}\varphi_\beta(0,y,x_3,\cdots,x_n)dx_3\cdots dx_n.$$

Let us note that

$$\int_{\boldsymbol{R}^2}g(y)dy=z\rho'(z)-\rho(z). \tag{4.11}$$

Finally we shall discuss the convergence of $I_1^1(L)/L^2$ as follows

$$\frac{1}{L^2}I_1^1(L) \tag{4.12}$$

$$=\frac{1}{L^2}\sum_{n=2}^{\infty}\frac{z^n}{(n-2)!}\int_{V(t_1,s_1)}dx_1\int_{V(t_2,s_2)}dx_2$$

$$\cdot \int_{\boldsymbol{R}^{2(n-2)}} \varphi_\beta(0, x_2-x_1, x_3, \cdots, x_n)dx_3 \cdots dx_n$$

$$=\frac{1}{L^2}\int_{V(t_1,s_1)} dx_1 \int_{V(t_2,s_2)} dx_2 g(x_2-x_1)$$

$$= \int_{[0,t_1)\times[0,s_1)} dx_1 \int_{\boldsymbol{R}^2} dy \chi_{V(t_2,s_2)}(Lx_1+y)g(y)$$

$$= \int_{[0,t_1)\times[0,s_1)} dx_1 \int_{\boldsymbol{R}^2} dy\, \chi_{[0,t_2)\times[0,s_2)} \left(x_1+\frac{y}{L}\right)g(y)$$

$$\longrightarrow \int_{[0,t_1)\times[0,s_1)} dx_1\, \chi_{[0,t_2)\times[0,s_2)}(x_1) \int_{\boldsymbol{R}^2} dy\, g(y) \quad \text{as } L \longrightarrow \infty$$

$$=(t_1\wedge t_2)\cdot(s_1\wedge s_2)(z\rho'(z)-\rho(z)).$$

Putting (4.9), (4.10) and (4.12) together we complete the proof.

In the remaining of this section we shall prove the positivity of the function $\sigma(z)=z\rho'(z)$ for all z with $0<z<z_0(\beta)$. We introduce the sequence $\{Q_n\}$ as follows

$$Q_n=\frac{z^n}{Z_V}\int_{V^n} d\underline{x}_n e^{-\beta U(\underline{x}_n)}.$$

In terms of Q_n the probability of $N_V=n$ is given as $Q_n/n!$. Also the expectation of N_V and $N_V{}^2$ are expressed as follows,

$$E_V(N_V)=\sum_{n=0}^{\infty}\frac{Q_{n+1}}{n!}$$

$$E_V(N_V{}^2)=\sum_{n=0}^{\infty}\frac{Q_{n+2}}{n!}+\sum_{n=0}^{\infty}\frac{Q_{n+1}}{n!}.$$

When the interaction Φ satisfies (I) and either (II-1) or (II-2), then the following inequality is proved by Ginibre [17]. (See also Proposition 3.4.9 in [1]).

Lemma 4.3

$$\frac{Q_{n+1}}{Q_n} \leq \frac{Q_{n+2}}{Q_{n+1}} + zC(\beta)e^{2\beta B}.$$

Using the Schwarz inequality and Lemma 4.3 we have

$$E_V(N_V)^2=\left(\sum_{n=0}^{\infty}\frac{Q_{n+1}}{n!}\right)^2=\left(\sum_{n=0}^{\infty}\frac{Q_n}{n!}\cdot\frac{Q_{n+1}}{Q_n}\right)^2$$

$$\leq\sum_{n=0}^{\infty}\frac{Q_n}{n!}\cdot\sum_{n=0}^{\infty}\frac{Q_n}{n!}\left(\frac{Q_{n+1}}{Q_n}\right)^2$$

$$=\sum_{n=0}^{\infty}\frac{Q_{n+1}}{n!}\frac{Q_{n+1}}{Q_n}$$

$$\leq\sum_{n=0}^{\infty}\frac{Q_{n+1}}{n!}\left(\frac{Q_{n+2}}{Q_{n+1}}+zC(\beta)e^{2\beta B}\right)$$

$$=E_V(N_V{}^2)-(1-zC(\beta)e^{2\beta B})E_V(N_V)$$

Hence we have the following lemma under the conditions on Φ mentioned above.

Lemma 4.4

$$E_V(N_V{}^2)-E_V(N_V)^2\geq(1-zC(\beta)e^{2\beta B})E_V(N_V).$$

From this lemma, (3.3) and (3.4), we have

$$\sigma(z)\geq(1-zC(\beta)e^{2\beta B})\rho(z)>0, \text{ if } 0<z<z_0(\beta).$$

§ 5 Proof of Theorem 2

In this section we prove Theorem 2 by checking the tightness condition.

A set A of $[0, 1]^2$ is called a block if it is given in the form $A=(s_1, s_2]\times(t_1, t_2]$. In a similar way to the one-dimensional case, we define the increment $w(A)$ of $w\in D^2$ for a block A by

$$w(A)=w(s_2, t_2)-w(s_1, t_2)-w(s_2, t_1)+w(s_1, t_1).$$

We say two blocks A and B are neighboring blocks if A and B are given in the following forms

$$A=(s, t]\times(a, b], \qquad B=(t, u]\times(a, b] \qquad (s<t<u)$$

or

$$A=(s, t]\times(a, b], \qquad B=(s, t]\times(b, c] \qquad (a<b<c).$$

In the previous section we proved that the finite dimensional distribution of $X^L(t, s)$ converges to the corresponding distribution of the Brownian sheet $B(t, s)$ as $L\to\infty$. To prove the convergence of the distributions $\{P_{X^L}(\cdot)\}$ of X^L to the distribution of $B(t, s)$, it suffices to prove the following lemma. (See Theorem 3 in [16] for detail).

Lemma 5.1 *There exist $\gamma\geq 0$ and $\delta>1$, such that*

$$P_{X^L}(|w(B)|\geq\lambda, |w(C)|\geq\lambda)\leq\lambda^{-\gamma}(\mu(B\cup C))^{\delta},$$

for all neighboring blocks B and C, where μ is the Lebesgue measure.

Proof of Lemma 5.1 Using the Chebyshev's inequality we have,

$$P_{X^L}(|w(B)|\geq\lambda,|w(C)|\geq\lambda)$$

$$\leq \frac{1}{\lambda_4} E_V(|X^L(B)|^2 |X^L(C)|^2)$$

$$\leq \frac{1}{2\lambda^4} \{E_V(|X^L(B)|^4) + E_V(|X^L(C)|^4)\}.$$

From the same argument developed in § 4 $E_V(X^L(B)^4)$ can be rewritten as,

$$E_V(X^L(B)^4) = \frac{1}{\sigma^4 L^4} \sum_{n=0}^{\infty} \frac{z^n}{n\,!} \int_{V^n} N^L(B)(\underline{x}_n)^4 \varphi_\beta(\underline{x}_n) d\underline{x}_n$$

$$+ \frac{3}{\sigma^4 L^4} \left(\sum_{n=0}^{\infty} \frac{z^n}{n\,!} \int_{V^n} N^L(B)(\underline{x}_n)^2 \varphi_\beta(\underline{x}_n) d\underline{x}_n \right)^2.$$

Using the properties of $\varphi_\beta(\underline{x}_n)$ we get easily that,

$$E_V(X^L(B)^4) \leq C_\beta(z) \mu(B)^2.$$

This completes the proof of the lemma.

§ 6 Nonnegative Interaction

In this section we restrict our argument to the system with a nonnegative pair interaction. When a pair interaction $\Phi(\cdot)$ is nonnegative the following facts are known. (See [1] for detail).

(1) The Ursell function $\varphi_\beta(x_1, \cdots, x_m)$ satisfies the alternating property

$$(-1)^{m-1} \varphi_\beta(x_1, \cdots, x_m) \geq 0.$$

(2) The activity expansion of $\rho(z)$ is an alternating series.

(3) The radius of convergence $\mathcal{R}$ of the expansion $\rho(z)$ satisfies

$$e^{-1} C(\beta)^{-1} \leq \mathcal{R} \leq C(\beta)^{-1}.$$

(4) $z(1+C(\beta)z)^{-1} \leq \rho(z) \leq z$ for all z with $0<z<\mathcal{R}$.

Using the alternating property of the Ursell function we can apply the method of cluster expansion for all z with $0<z<\mathcal{R}$, and then we get the same formula as (4.2) for the characteristic function $\theta_L(y_1, \cdots, y_m)$. This implies that the assertion of Theorem 1 and 2 holds for all z with $0<z<\mathcal{R}$.

Theorem 3 *If a pair interaction Φ is nonnegative, then the following limit theorem holds with $\sigma(z) = \rho'(z)^{\frac{1}{2}}$ for all z with $0<z<\mathcal{R}$*

$$X^L(t, s) \longrightarrow B(t, s) \quad \text{in } D^2 \text{ as } L \longrightarrow \infty,$$

where $B(t, s)$ is a Brownian sheet and $\mathcal{R}$ is the radius of convergence of $\rho(z)$.

From the property of (4) we see that if the density ρ satisfies

$$0 < \rho < (1+e)^{-1} C(\beta)^{-1}, \tag{6.1}$$

then the corresponding activity z determined by $\rho(z)=\rho$ satisfies

$$(6.2) \qquad 0<z<e^{-1}C(\beta)^{-1}.$$

Hence, if the density ρ satisfies the condition (6.1), then the limit theorem mentioned above holds.

In particular, if the interaction Φ has only hard core interaction of diameter r_0 and no further interaction, then $C(\beta)$ is the area of disc with radius r_0. Therefore, the limit theorem holds for all ρ with

$$(6.3) \qquad 0<\rho<(1+e)^{-1}\cdot(\pi r_0^2)^{-1}.$$

Here $(\pi r_0^2)^{-1}$ is the "close-packing" density of the system for particles having hard cores with radius r_0.

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