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# FURTHER GENERALIZATION OF STEIN ESTIMATORS

by

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**ABSTRACT**

Let $Y=(Y_1, \dots, Y_p)'$ be a $p$-variate normal random vector with mean $\mu=(\mu_1, \dots, \mu_p)'$ and the identity covariance matrix $I$. Stein estimator of $\mu$ is known to dominate the maximum likelihood estimator $Y$ for $p \geq 3$. Estimators generalizing some of Tze and Wen's estimators [4], which dominate Stein estimator, are obtained. Further, estimators dominating some of Baranchik's estimators [2], which are better than the estimator $Y$ and contain Stein estimator, are given.

## 1. Introduction

Let $Y=(Y_1, \dots, Y_p)'$ have a $p$-dimensional normal distribution with mean vector $\mu=(\mu_1, \dots, \mu_p)'$ and the identity covariance matrix $I$. The estimation of $\mu=(\mu_1, \dots, \mu_p)'$ by $\delta(Y)=(\delta_1(Y), \dots, \delta_p(Y))'$ is evaluated by the risk

$$R(\delta(Y), \mu)=E_Y\|\delta(Y)-\mu\|^2,$$

where $\|\delta(Y)-\mu\|^2=\sum_{i=1}^{p}(\delta_i(Y)-\mu_i)^2$ and $E_Y$ denotes the expectation with respect to $Y$.

James and Stein [3] showed that if $p \geq 3$, the estimator

$$(1.1) \qquad \delta^0(Y)=(1-(p-2)/S)Y,$$

where $S=\|Y\|^2$, dominates the maximum likelihood estimator $Y$. Baranchik [1] showed that the estimator

(1.2) $$\delta^1(Y)=(1-\tau(S)/S)Y,$$

dominates the estimator $Y$ if $0<\tau(s)<2(p-2)$ and $\tau(s)$ is nondecreasing. Efron and Morris [2] strengthened Baranchik's results: Baranchik's result holds under the condition that $s^{(p-2)}\tau(s)/(2(p-2)-\tau(s))$ is nondecreasing. Tze and Wen [4] showed that the estimator

(1.3) $$\delta^2(Y)=\left(1-\frac{p-2}{S}+\frac{d}{S^r}\right)Y,$$

dominates Stein estimator $\delta^0(Y)$ if $1<r<(p+2)/4$ and $0<d\leq 2^{r+1}(r-1)\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$.

In this paper estimators generalizing Tze and Wen's estimator (1.3) are obtained. Further estimators, dominating a special form

(1.4) $$\delta^{1*}(Y)=\left(1-\frac{b}{a+S}\right)Y,\qquad a>0 \quad \text{and} \quad 0<b<2(p-2),$$

of Baranchik's estimator (1.2), are given. These estimators are obtained under stronger conditions on $p$ than $p\geq 3$.

In Section 2, the estimator

(1.5) $$\delta^3(Y)=\left(1-\frac{b}{S}+\frac{g(S)}{S^r}\right)Y,$$

where $g(s)$ is a real-valued function to be specified later, is discussed. It is shown that if $r$ is an integer such that $r>1$ and $p>2(2r-1)$ the estimator $\delta^3(Y)$ dominates the estimator

(1.6) $$\delta^{0*}(Y)=\left(1-\frac{b}{S}\right)Y \qquad \text{and} \qquad 0<b<2(p-2).$$

Some examples of this property are given.

In Section 3, the estimators

(1.7) $$\delta^4(Y)=\left(1-\frac{b}{a+S}+\frac{c}{S^2}\right)Y$$

and

$$\delta^5(Y)=\left(1-\frac{b}{a+S}+\frac{h(S)}{(d_1+S)^2}\right)Y, \tag{1.8}$$

are discussed. The constants $a$, $c$ and $d_1$ are suitably chosen and $h(s)$ is a real-valued function to be specified later. It is shown that $\delta^4(Y)$ or $\delta^5(Y)$ dominates $\delta^{1*}(Y)$ if $p>6$. Further the estimator $\delta^{1*}(Y)$ with $b=p-2$ dominates Stein estimator $\delta^0(Y)$. Some examples of $\delta^5(Y)$ of the property are given.

In Section 4, special forms (4.1), (4.2) and (4.3) of $\delta^3(Y)$, $\delta^4(Y)$ and $\delta^5(Y)$ respectively, are discussed. The estimator $\delta^4(Y)$ dominates $\delta^3(Y)$ if $p>8$, and $\delta^5(Y)$ dominates $\delta^4(Y)$ if $p>6$.

Before stating the results, we need the following two lemmas which can be verified by integration by parts.

**Lemma 1.1.** Let $Y_i$ be normally distributed with mean $\mu_i$ and variance 1. Let $f$: $R^1 \longrightarrow R^1$ be an absolutely continuous function and let $f'$ denote the derivative of $f$. Then

$$E_{Y_i}\{f(Y_i)(Y_i-\mu_i)\}=E_{Y_i}\{f'(Y_i)\}.$$

**Lemma 1.2.** (Efron and Morris [2]) Let $W$ be a chi-squared random variable with $n$ degrees of freedom and let $f$: $R^1 \longrightarrow R^1$ be an absolutely continuous function.

$$E_W\{Wf(W)\}=nE_W\{f(W)\}+2E_W\{Wf'(W)\},$$

provided that all expectations exist and are finite.

## 2. Generalization of Tze and Wen's estimator

First Tze and Wen's estimator (1.3), which dominates Stein estimator $\delta^0(Y)$ of (1.1), is generalized. The following theorem and corollary give estimators which dominate $\delta^{0*}(Y)$ of (1.6). Throughout the paper, let $W$ denote a chi-squared random variable with $p+2k$ degrees of freedom.

**Theorem 2.1.** Suppose that $r$ is an integer such that $r>1$ and that $p>2(2r-1)\geq 6$ in $\delta^3(Y)$. Then the risk of $\delta^3(Y)$ is uniformly smaller than that of $\delta^{0*}(Y)$ if $g(s)$ satisfies either condition (i) or (ii):

( i ) $g(s)s^{-1/\alpha}$ is nonincreasing for $\alpha>0$, and $g(s)$ is nondecreasing and satisfies

$0<g(s)\leq 2^r(b-(p-2r+2/\alpha))\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$ for $\alpha>0$ and $b>p-2r+2/\alpha$.
(ii) $g(s)$ is nonincreasing and satisfies $0>g(s)>2^r(b-(p-2r))\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$ for $b<p-2r$.

*Proof.*

$$\Delta(\mu)\equiv R(\delta^{0*}(Y),\ \mu)-R(\delta^{3}(Y),\ \mu)$$

$$=2E_Y\sum_{i=1}^{p}\left\{-(Y_i-\mu_i)\frac{g(S)Y_i}{S^r}+\frac{bg(S)Y_i^2}{S^{r+1}}\right\}-E_Y\left\{\frac{g^2(S)}{S^{2r-1}}\right\}.$$

Lemma 1.1 is applied to the first term of the above expression to obtain

$$\Delta(\mu)=E_Y\left\{-\frac{4g'(S)}{S^{r-1}}+2(b-(p-2r))\frac{g(S)}{S^r}-\frac{g^2(S)}{S^{2r-1}}\right\}$$

$$=e^{-||\mu||^2/2}\sum_{k=0}^{\infty}\frac{(||\mu||^2/2)^k}{k\,!}E_W\left\{-\frac{4g'(W)}{W^{r-1}}+2(b-(p-2r))\frac{g(W)}{W^r}-\frac{g^2(W)}{W^{2r-1}}\right\}.$$

Therefore it suffices to show that

$$J\equiv E_W\left\{-\frac{4g'(W)}{W^{r-1}}+2(b-(p-2r))\frac{g(W)}{W^r}-\frac{g^2(W)}{W^{2r-1}}\right\}\geq 0. \tag{2.1}$$

To evaluate $J$, we consider the following two cases: $b>p-2r+2/\alpha$ and $b<p-2r$.

(i) $b>p-2r+2/\alpha$ and $\alpha>0$. Notice that $g'(w)\leq g(w)/\alpha w$ by the assumption that $g(w)w^{-1/\alpha}$ is nonincreasing. Hence

$$J\geq E_W\left\{2(b-(p-2r+2/\alpha))\frac{g(W)}{W^r}-\frac{g^2(W)}{W^{2r-1}}\right\}. \tag{2.2}$$

Lemma 1.2 is applied to the first term in (2.2) to obtain

$$J\geq E_W\left\{4(b-(p-2r+2/\alpha))\frac{g'(W)}{W^r}\right.$$

$$\left.+2(b-(p-2r+2/\alpha))(p+2k-2(r+1))\frac{g(W)}{W^{r+1}}-\frac{g^2(W)}{W^{2r-1}}\right\}$$

$$\geq E_W\left\{2(b-(p-2r+2/\alpha))(p-2(r+1))\frac{g(W)}{W^{r+1}}-\frac{g^2(W)}{W^{2r-1}}\right\}. \tag{2.3}$$

The last inequality of (2.3) is verified by the conditions that $b>p-2r+2/\alpha$ and $g'(w)>0$. Furthermore Lemma 1.2 is applied to the first term in (2.3) $r-2(>0)$ times in a similar way, then

$$J\geq E_W\left\{2^r(b-(p-2r+2/\alpha))\frac{\Gamma(p/2-r)}{\Gamma(p/2-(2r-1))}\frac{g(W)}{W^{2r-1}}-\frac{g^2(W)}{W^{2r-1}}\right\},$$

which is nonnegative if $0<g(w)\leq 2^r(b-(p-2r+2/\alpha))\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$.

(ii) $b<p-2r$. Since $g(w)$ is nonincreasing, it follows that

$$J\geq E_W\left\{2(b-(p-2r))\cdot\frac{g(W)}{W^r}-\frac{g^2(W)}{W^{2r-1}}\right\}. \tag{2.4}$$

Lemma 1.2 is applied to the first term in (2.4) in a similar way as the proof of (i) to obtain

$$J\geq E_W\left\{2^r(b-(p-2r))\frac{\Gamma(p/2-r)}{\Gamma(p/2-(2r-1))}\frac{g(W)}{W^{2r-1}}-\frac{g^2(W)}{W^{2r-1}}\right\},$$

which is nonnegative if $0>g(w)\geq 2^r(b-(p-2r))\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$.

Setting $g(s)$ equal to a constant $d$ and $b=p-2$, we obtain Tze and Wen's estimator (1.3). The value $b=p-2$ is the midpoint of possible values of $b$ in $\delta^{0*}(Y)$, whose risk is known to be minimum at this value. Actually $r$ in Tze and Wen's estimator is a real number, but in $\delta^3(Y)$ of Theorem 2.1 an integer. When $g(s)=d$ and $r$ is real, we obtain the same results as Tze and Wen.

**Corollary 2.1.** Let $g(s)$ in $\delta^3(Y)$ be equal to a constant $d$ and let $b$ equal to $p-2$. Then the risk of $\delta^3(Y)$ is uniformly smaller than that of $\delta^0(Y)$ if $1<r<(p+2)/4$ and $0<d\leq 2^{r+1}(r-1)\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$.

*Proof.* From (2.1) in the proof of Theorem 2.1, it suffices to show that $E_W\{2(r-1)W^{-r}-dW^{-(2r+1)}\}\geq 0$. It can be easily seen that the above inequality holds if the conditions on $d$ and $r$ are satisfied.

From Theorem 2.1, some examples of this property are given.

**Example 2.1.** For $b=p-2$, setting $g(s)=c_2s^{1/\alpha}(c_1+s)^{-1/\alpha}$, we obtain the estimator

$$\delta^{3A}(Y)=\left(1-\frac{p-2}{S}+\frac{c_2S^{1/\alpha-2}}{(c_1+S)^{1/\alpha}}\right)Y.$$

If $\alpha>1$, $c_1>0$ and $0<c_2\leq 4(1-1/\alpha)(p-6)$, then $g(s)$ satisfies condition (i) of Theorem 2.1.

**Example 2.2.** For $b=p-2$, setting $g(s)=c_2 s^{1/\alpha}(c_1+s^{1/\alpha})^{-1}$, we get the estimator

$$\delta^{3B}(Y)=\left(1-\frac{p-2}{S}+\frac{c_2 S^{1/\alpha-2}}{c_1+S^{1/\alpha}}\right)Y.$$

If $\alpha>1$, $c_1>0$ and $0<c_2\leq 4(1-1/\alpha)(p-6)$, then $g(s)$ satisfies condition (i) of Theorem 2.1.

## 3. Estimators dominating $\delta^{1*}(Y)$

It is shown that the estimator $\delta^4(Y)$ of (1.7) or $\delta^5(Y)$ of (1.8) dominates $\delta^{1*}(Y)$ of (1.4), a special form of Baranchik's estimator. The constant $a$ of $\delta^{1*}(Y)$ for $b=p-2$ is determined so that $\delta^{1*}(Y)$ dominates Stein estimator $\delta^0(Y)$.

**Theorem 3.1.** Assume that $p>6$. Then the risk of $\delta^4(Y)$ is uniformly smaller than that of $\delta^{1*}(Y)$ if the constants $a$ and $c$ satisfy the following conditions.
(i) For $b>p-4$,

$$0\leq a<(b-(p-4))(p-6)/(p-4),$$

and

$$0<c\leq 2\{(b-(p-4))(p-6)-a(p-4)\}\frac{p-6}{a+p}.$$

(ii) For $b<p-4$,

$$a\geq 0 \quad\text{and}\quad 0>c\geq 2\{(b-(p-4))(p-6)-a(p-4)\}\frac{p-6}{a+p}.$$

*Proof.* Applying Lemma 1.1 to a term of $R(\delta^{1*}(Y),\mu)-R(\delta^4(Y),\mu)$ in a similar way as the proof of Theorem 2.1, we see the theorem can be proved if

$$K\equiv cE_W\left\{-\frac{2a(p-4)}{(a+W)W^2}+\frac{2(b-(p-4))}{(a+W)W}-\frac{c}{W^3}\right\}\geq 0. \tag{3.1}$$

To evaluate $K$, we consider the following two cases: $b>p-4$ and $b<p-4$.
(i) $b>p-4$. Lemma 1.2 is applied to the second term in (3.1) to obtain

$$K=cE_W\left\{-\frac{2a(p-4)}{(a+W)W^2}+\frac{2(b-(p-4))(p+2k-4)}{(a+W)W^2}\right. \tag{3.2}$$

$$-\frac{4(b-(p-4))}{(a+W)^2W}-\frac{c}{W^3}\Big\}.$$

The conditions that $b>p-4$ and $1/w>1/(a+w)$ show that

$$K\geq cE_W\Big\{2\{-a(p-4)+(b-(p-4))(p-6)\}\frac{1}{(a+W)W^2}-\frac{c}{W^3}\Big\}. \tag{3.3}$$

Since $1/(a+w)$ and $1/w^2$ are decreasing, it follows that

$$E_W\Big\{\frac{1}{(a+W)W^2}\Big\}\geq E_W\Big\{\frac{1}{a+W}\Big\}E_W\Big\{\frac{1}{W^2}\Big\} \tag{3.4}$$

$$\geq\frac{1}{a+p+2k}E_W\Big\{\frac{1}{W^2}\Big\}=\frac{p+2k-6}{a+p+2k}E_W\Big\{\frac{1}{W^3}\Big\}.$$

The second inequality is due to Jensen's inequality, and the last equality to Lemma 1.2. From the condition on the constant $a$,

$$K\geq cE_W\Big\{2\{-a(p-4)+(b-(p-4))(p-6))\}\frac{p+2k-6}{a+p+2k}\frac{1}{W^3}-\frac{c}{W^3}\Big\}. \tag{3.5}$$

The first term in (3.5) is increasing in $k$. Hence $K$ is nonnegative if $0<c\leq 2\{(b-(p-4))(p-6)-a(p-4)\}(p-6)/(a+p)$.

(ii) $b<p-4$. It can be proved in a similar way as the proof of (i) and the proof is omitted.

**Theorem 3.2.** Suppose that $p>6$. Then the risk of $\delta^5(Y)$ is uniformly smaller than that of $\delta^{1*}(Y)$ if $h(s)$, $a$ and $d_1$ satisfy either condition (i) or (ii):
(i) $h(s)(d_1+s)^{-1/\alpha}$ is nonincreasing for $\alpha>0$ and $d_1>0$, and $h(s)$ is nondecreasing and satisfies $0<h(s)\leq 2\{(b-(p-4+2/\alpha))(p-6)-d_1p\}$ for $\alpha>0$, $b>p-4+2/\alpha$ and $a\leq d_1<(b-(p-4+2/\alpha))(p-6)/p$.
(ii) $h(s)$ is nonincreasing and satisfies $0>h(s)\geq 2\{(b-(p-4))(p-6)-d_1p\}$ for $a\geq d_1\geq 0$ and $b<p-4$.

*Proof.*

$$\Delta(\mu)\equiv R(\delta^{1*}(Y),\ \mu)-R(\delta^5(Y),\ \mu)$$

$$=2E_Y\sum_{i=1}^{p}\Big\{-(Y_i-\mu_i)\frac{h(S)Y_i}{(d_1+S)^2}+\frac{bh(S)Y_i^2}{(a+S)(d_1+S)^2}\Big\}-E_Y\Big\{\frac{h^2(S)S}{(d_1+S)^4}\Big\}.$$

Applying Lemma 1.1 to the first term of the above expression in a similar way as Theorem 2.1, we see that this theorem can be proved if we show that

$$L \equiv E_W\left\{-\frac{2ph(W)+4h'(W)W}{(d_1+W)^2}+\frac{8h(W)W}{(d_1+W)^3}+\frac{2bh(W)W}{(a+W)(d_1+W)^2}-\frac{h^2(W)W}{(d_1+W)^4}\right\}\geq 0.$$

To evaluate $L$, we consider the following two cases: $b>p-4+2/\alpha$ and $b<p-4$.

(i) $b>p-4+2/\alpha$ and $\alpha>0$. The condition that $\alpha\leq d_1$ gives

$$L\geq E_W\left\{-\frac{2ph(W)+4h'(W)W}{(d_1+W)^2}+\frac{2(b+4)h(W)W}{(d_1+W)^3}-\frac{h^2(W)W}{(d_1+W)^4}\right\}.$$

Notice that $h'(w)\leq h(w)/\alpha(d_1+w)$ by the assumption that $h(w)(d_1+w)^{-1/\alpha}$ is nonincreasing. Therefore

$$L\geq E_W\left\{-\frac{2d_1ph(W)}{(d_1+W)^3}+\frac{2(b-(p-4+2/\alpha))h(W)W}{(d_1+W)^3}-\frac{h^2(W)W}{(d_1+W)^4}\right\}. \tag{3.6}$$

Lemma 1.2 is applied to the second term in (3.6) to obtain

$$E_W\left\{\frac{h(W)W}{(d_1+W)^3}\right\}=E_W\left\{\frac{2h'(W)W+(p+2k-6)h(W)}{(d_1+W)^3}+\frac{6d_1h(W)}{(d_1+W)^4}\right\}. \tag{3.7}$$

Hence from the equality (3.7) and the conditions that $h'(w)\geq 0$ and $h(w)>0$,

$$L\geq E_W\left\{-\frac{2d_1ph(W)}{(d_1+W)^3}+2(b-(p-4+2/\alpha))(p-6)\frac{h(W)}{(d_1+W)^3}-\frac{h^2(W)W}{(d_1+W)^4}\right\}$$

$$\geq E_W\left\{\frac{h(W)}{(d_1+W)^3}\{-2d_1p+2(b-(p-4+2/\alpha))(p-6)-h(W)\}\right\},$$

which is nonnegative if $0\leq d_1<(b-(p-4+2/\alpha))(p-6)/p$ and $0<h(w)\leq 2\{(b-(p-4+2/\alpha))(p-6)-d_1p\}$.

(ii) $b>p-4$. Since $h(w)$ is nonincreasing and $\alpha\geq d_1$, it follows that

$$L\geq E_W\left\{-\frac{2d_1ph(W)}{(d_1+W)^3}+\frac{2(b-(p-4))h(W)W}{(d_1+W)^3}-\frac{h^2(W)W}{(d_1+W)^4}\right\}. \tag{3.8}$$

Applying Lemma 1.2 to the second term in (3.8) in a similar way as the proof of (i), we get

$$L \geq E_W\left\{\frac{h(W)}{(d_1+W)^3}\{-2d_1p+2(b-(p-4))(p-6)-h(W)\}\right\},$$

which is nonnegative if $0>h(w)\geq 2\{(b-(p-4))(p-6)-d_1p\}$.

**Theorem 3.3.** Let $p>6$ and let $b$ in $\delta^{1*}(Y)$ be equal to $p-2$. The risk of $\delta^{1*}(Y)$ is uniformly smaller than that of $\delta^0(Y)$ if $0<\alpha\leq 4(p-6)/(p-2)$.

*Proof.* Applying Lemma 1.1 to two terms of $R(\delta^0(Y),\mu)-R(\delta^{1*}(Y),\mu)$ in a similar way of Theorem 2.1 or 3.2, we see that it suffices to show that

$$M\equiv E_W\left\{\frac{-a(p-2)}{(a+W)^2W}+\frac{4}{(a+W)^2}\right\}\geq 0. \tag{3.9}$$

Lemma 1.2 is applied to the second term in (3.9) to obtain

$$\begin{aligned} M &= E_W\left\{\frac{-a(p-2)+4(p+2k-2)}{(a+W)^2W}-\frac{16}{(a+W)^3}\right\} \\ &\geq E_W\left\{\frac{(-a(p-2)+4(p-6))W+a(p-2)(-a+4)}{(a+W)^3W}\right\}, \end{aligned} \tag{3.10}$$

which is nonnegative if $0<a\leq 4(p-6)/(p-2)$.

Theorem 3.2 is used to obtain some other estimators.

**Example 3.1.** Setting $h(s)$ equal to a constant $d_2$, we get the estimator

$$\delta^{5A}(Y)=\left(1-\frac{b}{a+S}+\frac{d_2}{(d_1+S)^2}\right)Y.$$

If the constants $a$, $b$, $d_1$ and $d_2$ satisfy the following conditions, then the risk of $\delta^{5A}(Y)$ is uniformly smaller than that of $\delta^{1*}(Y)$.

( i ) For $b>p-4$,

$$a\leq d_1<(b-(p-4))(p-6)/p$$

and

$$0<d_2\le 2\{(b-(p-4))(p-6)-d_1 p\}.$$

(ii) For $b<p-4$,

$$a\ge d_1\ge 0 \quad \text{and} \quad 0>d_2\ge 2\{(b-(p-4))(p-6)-d_1 p\}.$$

**Example 3.2.** For $b=p-2$, setting $h(s)=d_2(d_1+s)^{1/\alpha}(d_3+s)^{-1/\alpha}$, we obtain the estimator

$$\delta^{5B}(Y)=\left(1-\frac{p-2}{a+S}+\frac{d_2(d_1+S)^{1/\alpha-2}}{(d_3+S)^{1/\alpha}}\right)Y.$$

If $\alpha>1$, $d_1\le d_3$, $a\le d_1<2(1-1/\alpha)(p-6)/p$ and $0<d_2\le 2\{2(1-1/\alpha)(p-6)-d_1 p\}$, then $h(s)$ satisfies condition (i) of Theorem 3.2.

## 4. Comparison of $\delta^3(Y)$, $\delta^4(Y)$ and $\delta^5(Y)$

The following three estimators

$$\delta^{3C}(Y)=\left(1-\frac{p-2}{S}+\frac{d}{S^2}\right)Y, \tag{4.1}$$

$$\delta^{4A}(Y)=\left(1-\frac{p-2}{a+S}+\frac{c}{S^2}\right)Y \tag{4.2}$$

and

$$\delta^{5C}(Y)=\left(1-\frac{p-2}{a+S}+\frac{d_2}{(d_1+S)^2}\right)Y, \tag{4.3}$$

are discussed. The estimator $\delta^{3C}(Y)$ is defined in Corollary 2.1, $\delta^{4A}(Y)$ and $\delta^{5C}(Y)$ are defined in Theorem 3.1 and Example 3.1, respectively. Conditions on the constants $a$, $c$ and $d$ are given such that $R(\delta^{4A}(Y),\mu)\le R(\delta^{3C}(Y),\mu)$. Further the constants $a$, $c$, $d_1$ and $d_2$ are given such that $R(\delta^{5C}(Y),\mu)\le R(\delta^{4A}(Y),\mu)$.

**Theorem4.1.** Suppose that $p>8$. Then the risk of $\delta^{4A}(Y)$ is uniformly smaller than that of $\delta^{3C}(Y)$ if the constants $a$, $c$ and $d$ satisfy the following conditions.
(i) For $c>d$,

$$0 \le a \le 4(p-6)/(p-2)$$

and

$$\{4(c-d)(p-6)-2a(c(p-4)+2d)+d^2-c^2\}(p-8)+a(d^2-c^2)\ge 0.$$

(ii) For $c \le d$,

$$d/(p-2) < a < 2 \text{ and } A(p-8)-4a^2 d \ge 0,$$

where $A=-a(a(p-2)^2+8d)+4(a(p-2)-d)(p-6)+2c((2-a)(p-4)-4)+d^2-c^2$.

*Proof.* Applying Lemma 1.1 to two terms of $\Delta(\mu)\equiv R(\delta^{3C}(Y),\mu)-R(\delta^{4A}(Y),\mu)$ in a similar way as the proof of Theorem 2.1, we have

$$\Delta(\mu)=E_Y\left\{-\frac{(p-2)^2}{S}-\frac{4d}{S^2}+\frac{d^2}{S^3}+\frac{2ap(p-2)+(p-2)^2 S}{(a+S)^2}-\frac{2c(p-4)}{S^2}\right.$$

$$\left.+\frac{2c(p-2)}{(a+S)S}-\frac{c^2}{S^3}\right\}.$$

Therefore it suffices to show that

$$N\equiv E_W\left\{\frac{-a^2(p-2)^2+4a(p-2)W}{(a+W)^2 W}-\frac{2(c(p-4)+2d)}{W^2}+\frac{2c(p-2)}{(a+W)W}+\frac{d^2-c^2}{W^3}\right\}\ge 0.$$

To evaluate $N$, we consider the following two cases: $c>d$ and $c\le d$.

(i) $c>d$.

$$N=E_W\left\{\frac{-a^2(p-2)^2+4a(p-2)W}{(a+W)^2 W}+\frac{-2a(c(p-4)+2d)+4(c-d)W}{(a+W)W^2}+\frac{d^2-c^2}{W^3}\right\}$$

$$\equiv N_1+N_2+N_3,$$

say. The proof of Theorem 3.3 shows that

$$N_1\ge a(p-2)E_W\left\{\frac{(-a(p-2)+4(p-6))W+a(p-2)(-a+4)}{(a+W)^3 W}\right\}. \tag{4.4}$$

From (3.2) in the proof of Theorem 3.1 and the condition that $c>d$,

$$(4.5)\qquad N_2+N_3\geq E_W\left\{\frac{4(c-d)(p-6)-2a(c(p-4)+2d)+d^2-c^2}{(a+W)W^2}+\frac{a(d^2-c^2)}{(a+W)W^3}\right\}.$$

Furthermore Lemma 1.2 is applied to the first term of (4.5) to obtain

$$(4.6)\qquad N_2+N_3\geq E_W\left\{\frac{\{4(c-d)(p-6)-2a(c(p-4)+2d)+d^2-c^2\}(p-8)+a(d^2-c^2)}{(a+W)W^3}\right\}.$$

Hence from the inequalities (4.4) and (4.6), $N$ is nonnegative if the constants $a$, $c$ and $d$ satisfy condition (i).

(ii) $c\leq d$.

$$N=E_W\left\{\frac{-a^2(p-2)^2+4a(p-2)W}{(a+W)^2W}-\frac{4d}{W^2}-\frac{2c(p-4)}{W^2}+\frac{2c(p-2)}{(a+W)W}+\frac{d^2-c^2}{W^3}\right\}$$

$$=E_W\left\{\frac{-4a^2d-a(a(p-2)^2+8d)W+4(a(p-2)-d)W^2}{(a+W)^2W^2}+\frac{-2ac(p-4)+4cW}{(a+W)W^2}\right.$$

$$\left.+\frac{d^2-c^2}{W^3}\right\}$$

$$\equiv N'_1+N'_2+N'_3,$$

say. To evaluate $N$ in a similar way as the proof of (i), we decompose $N$ as the above last equality. Applying Lemma 1.2 to the third term of $N'_1$ and the second term of $N'_2$ respectively, and using the condition that $d/(p-2)<a<2$, we have

$$(4.7)\qquad N'_1\geq E_W\left\{-\frac{4a^2d}{(a+W)^2W^2}+\frac{-a(a(p-2)^2+8d)+4(a(p-2)-d)(p-6)}{(a+W)^2W}\right\}$$

and

$$(4.8)\qquad N'_2\geq E_W\left\{\frac{2c((2-a)(p-4)-4)}{(a+W)^2W}\right\}.$$

The inequalities (4.7) and (4.8) give

$$N \geq E_W\left\{-\frac{4a^2d}{(a+W)^2W^2}+\frac{A}{(a+W)^2W}\right\}, \tag{4.9}$$

where $A$ is defined in condition (ii). Furthermore Lemma 1.2 is applied to the second term of (4.9) to obtain

$$N \geq E_W\left\{\frac{-4a^2d+A(p-8)}{(a+W)^2W^2}\right\},$$

which is nonnegative if $A(p-8)-4a^2d \geq 0$.

**Theorem 4.2.** Let $p>6$ and let $a$ be equal to $d_1$. Then the risk of $\delta^{5C}(Y)$ is uniformly smaller than that of $\delta^{4A}(Y)$ if $c<d_2$ and $4(d_2-c)(p-6)+2a(c(p-6)-d_2(p+2))+c^2-d_2^2 \geq 0$.

*Proof.* Applying Lemma 1.1 to a term of $\Delta(\mu)\equiv R(\delta^{4A}(Y), \mu)-R(\delta^{5C}(Y), \mu)$ in a similar way as the proof of Theorem 3.2, we have

$$\Delta(\mu)=E_W\left\{\frac{2c(p-4)}{S^2}-\frac{2c(p-2)}{(a+S)S}+\frac{c^2}{S^3}-\frac{2d_2p}{(d_1+S)^2}+\frac{8d_2S}{(d_1+S)^3}\right.$$

$$\left.+\frac{2d_2(p-2)S}{(a+S)(d_1+S)^2}-\frac{d_2^2S}{(d_1+S)^4}\right\}.$$

Therefore from the comdition that $a=d_1$, it suffices to show that

$$P \equiv E_W\left\{\frac{2ac(p-4)}{(a+W)W^2}+\frac{c^2}{W^3}+\frac{4(d_2-c)}{(a+W)^2}-\frac{4ac}{(a+W)^2W}-\frac{2ad_2(p+2)+d_2^2}{(a+W)^3}+\frac{ad_2^2}{(a+W)^4}\right\}.$$

From (3.10) in the proof of Theorem 3.3 and the condition that $c>d_2$,

$$P \geq E_W\left\{\frac{2ac(p-4)}{(a+W)W^2}+\frac{c^2}{W^3}+\frac{4((d_2-c)(p-2)-ac)}{(a+W)^2W}-\frac{2ad_1(p+2)+16(d_2-c)+d_2^2}{(a+W)^3}\right\}$$

$$\geq E_W\left[\frac{1}{(a+W)^3}\{4(d_2-c)(p-6)+2a(c(p-6)-d_2(p+2))+c^2-d_2^2\}\right],$$

which is nonnegative if the constants $a$, $c$ and $d_2$ satisfy the conditions.

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