Title	Further generalization of Stein estimators
Sub Title	
Author	赤井, 豊秋(Akai, Toyoaki)
Publisher	慶應義塾大学理工学部
Publication year	1987
Jtitle	Keio Science and Technology Reports Vol.40, No.4 (1987. 12) ,p.41- 54
JaLC DOI	
Abstract	Let $Y=(Y1,,Yp)'$ be a p-variate normal random vector with mean $\mu=(\mu1,,\mu p)'$ and the identity covariance matrix I. Stein estimator of μ is known to dominate the maximum likelihood estimator Y for P≥3. Estimators generalizing some of Tze and Wen's estimators, which dominate Stein estimator, are obtained. Further, estimators dominating some of Baranchik's estimators, which are better than the estimator Y and contain Stein estimator, are given.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00400004- 0041

慶應義塾大学学術情報リポジトリ(KOARA)に掲載されているコンテンツの著作権は、それぞれの著作者、学会または出版社/発行者に帰属し、その権利は著作権法によって 保護されています。引用にあたっては、著作権法を遵守してご利用ください。

The copyrights of content available on the KeiO Associated Repository of Academic resources (KOARA) belong to the respective authors, academic societies, or publishers/issuers, and these rights are protected by the Japanese Copyright Act. When quoting the content, please follow the Japanese copyright act.

FURTHER GENERALIZATION OF STEIN ESTIMATORS

by

Toyoaki AKAI

Department of Mathematics Faculty of Science and Technology, Keio University Hiyoshi, Yokohama 223, Japan

(Received 21 August 1987)

ABSTRACT

Let $Y = (Y_1, \ldots, Y_p)'$ be a *p*-variate normal random vector with mean $\mu = (\mu_1, \ldots, \mu_p)'$ and the identity covariance matrix *I*. Stein estimator of μ is known to dominate the maximum likelihood estimator *Y* for $p \ge 3$. Estimators generalizing some of Tze and Wen's estimators [4], which dominate Stein estimator, are obtained. Further, estimators dominating some of Baranchik's estimators [2], which are better than the estimator *Y* and contain Stein estimator, are given.

1. Introduction

Let $Y=(Y_1,\ldots,Y_p)'$ have a *p*-dimensional normal distribution with mean vector $\mu=(\mu_1,\ldots,\mu_p)'$ and the identity covariance matrix *I*. The estimation of $\mu=(\mu_1,\ldots,\mu_p)'$ by $\delta(Y)=(\delta_1(Y),\ldots,\delta_p(Y))'$ is evaluated by the risk

$$R(\delta(Y), \mu) = E_Y ||\delta(Y) - \mu||^2,$$

where $||\delta(Y) - \mu||^2 = \sum_{i=1}^{p} (\delta_i(Y) - \mu_i)^2$ and E_Y denotes the expectation with respect to Y.

James and Stein [3] showed that if $p \ge 3$, the estimator

(1.1)
$$\delta^{0}(Y) = (1 - (p-2)/S)Y,$$

where $S = ||Y||^2$, dominates the maximum likelihood estimator Y. Baranchik [1] showed that the estimator

(1.2)
$$\delta^{1}(Y) = (1 - \tau(S)/S)Y,$$

dominates the estimator Y if $0 < \tau(s) < 2(p-2)$ and $\tau(s)$ is nondecreasing. Efron and Morris [2] strengthened Baranchik's results: Baranchik's result holds under the condition that $s^{(p-2)}\tau(s)/(2(p-2)-\tau(s))$ is nondecreasing. Tze and Wen [4] showed that the estimator

(1.3)
$$\delta^2(Y) = \left(1 - \frac{p-2}{S} + \frac{d}{S^r}\right)Y,$$

dominates Stein estimator $\delta^0(Y)$ if 1 < r < (p+2)/4 and $0 < d \le 2^{r+1}(r-1)I'(p/2-r)/\Gamma(p/2-(2r-1))$.

In this paper estimators generalizing Tze and Wen's estimator (1.3) are obtained. Further estimators, dominating a special form

(1.4)
$$\delta^{1*}(Y) = \left(1 - \frac{b}{a+S}\right)Y, \quad a > 0 \quad \text{and} \quad 0 < b < 2(p-2),$$

of Baranchik's estimator (1.2), are given. These estimators are obtained under stronger conditions on p than $p \ge 3$.

In Section 2, the estimator

(1.5)
$$\delta^{3}(Y) = \left(1 - \frac{b}{S} + \frac{g(S)}{S^{r}}\right)Y,$$

where g(s) is a real-valued function to be specified later, is discussed. It is shown that if r is an integer such that r>1 and p>2(2r-1) the estimator $\delta^3(Y)$ dominates the estimator

(1.6)
$$\delta^{0*}(Y) = \left(1 - \frac{b}{S}\right)Y$$
 and $0 < b < 2(p-2)$.

Some examples of this property are given.

In Section 3, the estimators

(1.7)
$$\delta^4(Y) = \left(1 - \frac{b}{a+S} + \frac{c}{S^2}\right)Y$$

and

(1.8)
$$\delta^{5}(Y) = \left(1 - \frac{b}{a+S} + \frac{h(S)}{(d_{1}+S)^{2}}\right)Y,$$

are discussed. The constants a, c and d_1 are suitably chosen and h(s) is a realvalued function to be specified later. It is shown that $\delta^4(Y)$ or $\delta^5(Y)$ dominates $\delta^{1*}(Y)$ if p>6. Further the estimator $\delta^{1*}(Y)$ with b=p-2 dominates Stein estimator $\delta^0(Y)$. Some examples of $\delta^5(Y)$ of the property are given.

In Section 4, special forms (4.1), (4.2) and (4.3) of $\delta^3(Y)$, $\delta^4(Y)$ and $\delta^5(Y)$ respectively, are discussed. The estimator $\delta^4(Y)$ dominates $\delta^3(Y)$ if p>8, and $\delta^5(Y)$ dominates $\delta^4(Y)$ if p>6.

Before stating the results, we need the following two lemmas which can be verified by integration by parts.

Lemma 1.1. Let Y_i be normally distributed with mean μ_i and variance 1. Let $f: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ be an absolutely continuous function and let f' denote the derivative of f. Then

$$E_{Y_i}\{f(Y_i)(Y_i - \mu_i)\} = E_{Y_i}\{f'(Y_i)\}.$$

Lemma 1.2. (Efron and Morris [2]) Let W be a chi-squared random variable with n degrees of freedom and let $f: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ be an absolutely continuous function.

$$E_{W}\{Wf(W)\} = nE_{W}\{f(W)\} + 2E_{W}\{Wf'(W)\},\$$

provided that all expectations exist and are finite.

2. Generalization of Tze and Wen's estimator

First Tze and Wen's estimator (1.3), which dominates Stein estimator $\delta^0(Y)$ of (1.1), is generalized. The following theorem and corollary give estimators which dominate $\delta^{0*}(Y)$ of (1.6). Throughout the paper, let W denote a chi-squared random variable with p+2k degrees of freedom.

Theorem 2.1. Suppose that r is an integer such that r>1 and that $p>2(2r-1)\ge 6$ in $\delta^{\mathfrak{d}}(Y)$. Then the risk of $\delta^{\mathfrak{d}}(Y)$ is uniformly smaller than that of $\delta^{\mathfrak{d}}(Y)$ if g(s) satisfies either condition (i) or (ii):

(i) $g(s)s^{-1/\alpha}$ is nonincreasing for $\alpha > 0$, and g(s) is nondecreasing and satisfies

 $0 < g(s) \le 2^r (b - (p - 2r + 2/\alpha))\Gamma(p/2 - r)/\Gamma(p/2 - (2r - 1))$ for $\alpha > 0$ and $b > p - 2r + 2/\alpha$. (ii) g(s) is nonincreasing and satisfies $0 > g(s) > 2^r (b - (p - 2r))\Gamma(p/2 - r)/\Gamma(p/2 - (2r - 1)))$ for b .

Proof.

$$\Delta(\mu) \equiv R(\delta^{0*}(Y), \ \mu) - R(\delta^{3}(Y), \ \mu)$$

$$= 2E_{Y}\sum_{i=1}^{p} \left\{ -(Y_{i} - \mu_{i}) \frac{g(S)Y_{i}}{S^{r}} + \frac{bg(S)Y_{i}^{2}}{S^{r+1}} \right\} - E_{Y} \left\{ \frac{g^{2}(S)}{S^{2r-1}} \right\}.$$

Lemma 1.1 is applied to the first term of the above expression to obtain

$$\begin{split} \mathcal{A}(\mu) &= E_{Y} \left\{ -\frac{4g'(S)}{S^{r-1}} + 2(b - (p - 2r))\frac{g(S)}{S^{r}} - \frac{g^{2}(S)}{S^{2r-1}} \right\} \\ &= e^{-1|\mu||^{2}/2} \sum_{k=0}^{\infty} \frac{(||\mu||^{2}/2)^{k}}{k!} E_{W} \left\{ -\frac{4g'(W)}{W^{r-1}} + 2(b - (p - 2r))\frac{g(W)}{W^{r}} - \frac{g^{2}(W)}{W^{2r-1}} \right\}. \end{split}$$

Therefore it suffices to show that

(2.1)
$$J \equiv E_W \left\{ -\frac{4g'(W)}{W^{r-1}} + 2(b - (p - 2r)) \frac{g(W)}{W^r} - \frac{g^2(W)}{W^{2r-1}} \right\} \ge 0.$$

To evaluate J, we consider the following two cases: $b > p-2r+2/\alpha$ and b < p-2r. (i) $b > p-2r+2/\alpha$ and $\alpha > 0$. Notice that $g'(w) \le g(w)/\alpha w$ by the assumption

that $g(w)w^{-1/\alpha}$ is nonincreasing. Hence

(2.2)
$$J \ge E_W \left\{ 2(b - (p - 2r + 2/\alpha)) \frac{g(W)}{W^r} - \frac{g^2(W)}{W^{2r-1}} \right\}.$$

Lemma 1.2 is applied to the first term in (2.2) to obtain

$$J \ge E_{W} \left\{ 4(b - (p - 2r + 2/\alpha)) \frac{g'(W)}{W^{r}} + 2(b - (p - 2r + 2/\alpha))(p + 2k - 2(r + 1)) \frac{g(W)}{W^{r+1}} - \frac{g^{2}(W)}{W^{r+1}} \right\}$$

$$(2.3) \ge E_{W} \left\{ 2(b - (p - 2r + 2/\alpha))(p - 2(r + 1)) \frac{g(W)}{W^{r+1}} - \frac{g^{2}(W)}{W^{2r-1}} \right\}.$$

The last inequality of (2.3) is verified by the conditions that $b > p - 2r + 2/\alpha$ and g'(w) > 0. Furthermore Lemma 1.2 is applied to the first term in (2.3) r-2(>0) times in a similar way, then

$$J \ge E_W \left\{ 2^r (b - (p - 2r + 2/\alpha)) \frac{\Gamma(p/2 - r)}{\Gamma(p/2 - (2r - 1))} \quad \frac{g(W)}{W^{2r - 1}} - \frac{g^2(W)}{W^{2r - 1}} \right\},$$

which is nonnegative if $0 < g(w) \le 2^r (b - (p - 2r + 2/\alpha))\Gamma(p/2 - r)/\Gamma(p/2 - (2r - 1))$. (ii) b . Since <math>g(w) is nonincreasing, it follows that

(ii) b . Since <math>g(a) is nonincreasing, it follows that

(2.4)
$$J \ge E_W \left\{ 2(b - (p - 2r)) \frac{g(W)}{W^r} - \frac{g^2(W)}{W^{2r-1}} \right\}$$

Lemma 1.2 is applied to the first term in (2.4) in a similar way as the proof of (i) to obtain

$$J \ge E_{W} \left\{ 2^{r} (b - (p - 2r)) \frac{\Gamma(p/2 - r)}{\Gamma(p/2 - (2r - 1))} \frac{g(W)}{W^{2r - 1}} - \frac{g^{2}(W)}{W^{2r - 1}} \right\},$$

which is nonnegative if $0 > g(w) \ge 2^r (b - (p - 2r))\Gamma(p/2 - r)/\Gamma(p/2 - (2r - 1))$.

Setting g(s) equal to a constant d and b=p-2, we obtain Tze and Wen's estimator (1.3). The value b=p-2 is the midpoint of possible values of b in $\delta^{0*}(Y)$, whose risk is known to be minimum at this value. Actually r in Tze and Wen's estimator is a real number, but in $\delta^{3}(Y)$ of Theorem 2.1 an integer. When g(s)=d and r is real, we obtain the same results as Tze and Wen.

Corollary 2.1. Let g(s) in $\delta^{3}(Y)$ be equal to a constant d and let b equal to p-2. Then the risk of $\delta^{3}(Y)$ is uniformly smaller than that of $\delta^{0}(Y)$ if 1 < r < (p+2)/4 and $0 < d \le 2^{r+1}(r-1)\Gamma(p/2-r)/\Gamma(p/2-(2r-1))$.

Proof. From (2.1) in the proof of Theorem 2.1, it suffices to show that $E_W\{2(r-1)W^{-r}-dW^{-(2^r+1)}\}\geq 0$. It can be easily seen that the above inequality holds if the conditions on d and r are satisfied.

From Theorem 2.1, some examples of this property are given.

Example 2.1. For b=p-2, setting $g(s)=c_2s^{1/\alpha}(c_1+s)^{-1/\alpha}$, we obtain the estimator

$$\delta^{34}(Y) = \left(1 - \frac{p-2}{S} + \frac{c_2 S^{1/\alpha-2}}{(c_1 + S)^{1/\alpha}}\right) Y.$$

If $\alpha > 1$, $c_1 > 0$ and $0 < c_2 \le 4(1-1/\alpha)(p-6)$, then g(s) satisfies condition (i) of Theorem 2.1.

Example 2.2. For b=p-2, setting $g(s)=c_2s^{1/\alpha}(c_1+s^{1/\alpha})^{-1}$, we get the estimator

$$\delta^{_{3}B}(Y) = \left(1 - \frac{p-2}{S} + \frac{c_2 S^{_{1/\alpha-2}}}{c_1 + S^{_{1/\alpha}}}\right) Y.$$

If $\alpha > 1$, $c_1 > 0$ and $0 < c_2 \le 4(1-1/\alpha)(p-6)$, then g(s) satisfies condition (i) of Theorem 2.1.

3. Estimators dominating $\delta^{1*}(Y)$

It is shown that the estimator $\delta^4(Y)$ of (1.7) or $\delta^5(Y)$ of (1.8) dominates $\delta^{1*}(Y)$ of (1.4), a special form of Baranchik's estimator. The constant *a* of $\delta^{1*}(Y)$ for b=p-2 is determined so that $\delta^{1*}(Y)$ dominates Stein estimator $\delta^0(Y)$.

Theorem 3.1. Assume that p>6. Then the risk of $\delta^4(Y)$ is uniformly smaller than that of $\delta^{1*}(Y)$ if the constants a and c satisfy the following conditions. (i) For b>p-4,

$$0 \le a < (b - (p - 4))(p - 6)/(p - 4),$$

and

$$0 < c \le 2\{(b - (p-4))(p-6) - a(p-4)\}\frac{p-6}{a+p}.$$

(ii) For b < p-4,

$$a \ge 0$$
 and $0 > c \ge 2\{(b - (p-4))(p-6) - a(p-4)\}\frac{p-6}{a+p}$.

Proof. Applying Lemma 1.1 to a term of $R(\delta^{1*}(Y), \mu) - R(\delta^{4}(Y), \mu)$ in a similar way as the proof of Theorem 2.1, we see the theorem can be proved if

(3.1)
$$K \equiv c E_W \left\{ -\frac{2a(p-4)}{(a+W)W^2} + \frac{2(b-(p-4))}{(a+W)W_2} - \frac{c}{W^3} \right\} \ge 0.$$

To evaluate K, we consider the following two cases: b > p-4 and b < p-4. (i) b > p-4. Lemma 1.2 is applied to the second term in (3.1) to obtain

(3.2)
$$K = cE_W \left\{ -\frac{2a(p-4)}{(a+W)W^2} + \frac{2(b-(p-4))(p+2k-4)}{(a+W)W^2} \right\}$$

$$-\frac{4(b-(p-4))}{(a+W)^2W}-\frac{c}{W^3}\bigg].$$

The conditions that b > p-4 and 1/w > 1/(a+w) show that

(3.3)
$$K \ge c E_W \left\{ 2\{-a(p-4) + (b-(p-4))(p-6)\} \frac{1}{(a+W)W^2} - \frac{c}{W^3} \right\}.$$

Since 1/(a+w) and $1/w^2$ are decreasing, it follows that

(3.4)
$$E_{W}\left\{\frac{1}{(a+W)W^{2}}\right\} \ge E_{W}\left\{\frac{1}{a+W}\right\}E_{W}\left\{\frac{1}{W^{2}}\right\}$$
$$\ge \frac{1}{a+p+2k}E_{W}\left\{\frac{1}{W^{2}}\right\} = \frac{p+2k-6}{a+p+2k}E_{W}\left\{\frac{1}{W^{3}}\right\}$$

The second inequality is due to Jensen's inequality, and the last equality to Lemma 1.2. From the condition on the constant a,

(3.5)
$$K \ge c E_W \left\{ 2\{-a(p-4) + (b-(p-4))(p-6))\} \frac{p+2k-6}{a+p+2k} \quad \frac{1}{W^3} - \frac{c}{W^3} \right\}.$$

The first term in (3.5) is increasing in *k*. Hence *K* is nonnegative if $0 < c \le 2\{(b - (p-4))(p-6) - a(p-4)\}(p-6)/(a+p).$

(ii) b < p-4. It can be proved in a similar way as the proof of (i) and the proof is omitted.

Theorem 3.2. Suppose that p>6. Then the risk of $\delta^5(Y)$ is uniformly smaller than that of $\delta^{1*}(Y)$ if h(s), a and d_1 satisfy either condition (i) or (ii): (i) $h(s)(d_1+s)^{-1/\alpha}$ is nonincreasing for $\alpha>0$ and $d_1>0$, and h(s) is nondecreasing and satisfies $0 < h(s) < 2\{(b-(p-4+2/\alpha))(p-6)-d_1p\}$ for $\alpha>0$, $b > p-4+2/\alpha$ and

and satisfies $0 < h(s) \le 2\{(b - (p - 4 + 2/\alpha))(p - 6) - d_1p\}$ for $\alpha > 0$, $b > p - 4 + 2/\alpha$ and $a \le d_1 < (b - (p - 4 + 2/\alpha))(p - 6)/p$. (ii) h(s) is nonincreasing and satisfies $0 > h(s) > 2!(b - (p - 4))(p - 6) - d_1p\}$ for

(ii) h(s) is nonincreasing and satisfies $0 > h(s) \ge 2\{(b-(p-4))(p-6)-d_1p\}$ for $a \ge d_1 \ge 0$ and b < p-4.

Proof.

$$\Delta(\mu) \equiv R(\delta^{1*}(Y), \ \mu) - R(\delta^{5}(Y), \ \mu)$$

$$= 2E_{Y} \sum_{i=1}^{p} \left\{ -(Y_{i} - \mu_{i}) \frac{h(S)Y_{i}}{(d_{1} + S)^{2}} + \frac{bh(S)Y_{i}^{2}}{(a + S)(d_{1} + S)^{2}} \right\} - E_{Y} \left\{ \frac{h^{2}(S)S}{(d_{1} + S)^{4}} \right\}.$$

Applying Lemma 1.1 to the first term of the above expression in a similar way as Theorem 2.1, we see that this theorem can be proved if we show that

$$L \equiv E_W \left\{ -\frac{2ph(W) + 4h'(W)W}{(d_1 + W)^2} + \frac{8h(W)W}{(d_1 + W)^3} + \frac{2bh(W)W}{(a + W)(d_1 + W)^2} - \frac{h^2(W)W}{(d_1 + W)^4} \right\} \ge 0.$$

To evaluate L, we consider the following two cases: $b > p-4+2/\alpha$ and b < p-4. (i) $b > p-4+2/\alpha$ and $\alpha > 0$. The condition that $\alpha \le d_1$ gives

$$L \ge E_W \left\{ -\frac{2ph(W) + 4h'(W)W}{(d_1 + W)^2} + \frac{2(b+4)h(W)W}{(d_1 + W)^3} - \frac{h^2(W)W}{(d_1 + W)^4} \right\}.$$

Notice that $h'(w) \le h(w)/\alpha(d_1+w)$ by the assumption that $h(w)(d_1+w)^{-1/\alpha}$ is nonincreasing. Therefore

(3.6)
$$L \ge E_W \left\{ -\frac{2d_1 ph(W)}{(d_1 + W)^3} + \frac{2(b - (p - 4 + 2/\alpha))h(W)W}{(d_1 + W)^3} - \frac{h^2(W)W}{(d_1 + W)^4} \right\}.$$

Lemma 1.2 is applied to the second term in (3.6) to obtain

(3.7)
$$E_{W}\left\{\frac{h(W)W}{(d_{1}+W)^{3}}\right\} = E_{W}\left\{\frac{2h'(W)W + (p+2k-6)h(W)}{(d_{1}+W)^{3}} + \frac{6d_{1}h(W)}{(d_{1}+W)^{4}}\right\}.$$

Hence from the equality (3.7) and the conditions that $h'(w) \ge 0$ and h(w) > 0,

$$\begin{split} L \ge & E_W \left\{ -\frac{2d_1 ph(W)}{(d_1 + W)^3} + 2(b - (p - 4 + 2/\alpha))(p - 6) \frac{h(W)}{(d_1 + W)^3} - \frac{h^2(W)W}{(d_1 + W)^4} \right\} \\ \ge & E_W \left\{ \frac{h(W)}{(d_1 + W)^3} \{ -2d_1 p + 2(b - (p - 4 + 2/\alpha))(p - 6) - h(W) \} \right\}, \end{split}$$

which is nonnegative if $0 \le d_1 < (b - (p - 4 + 2/\alpha))(p - 6)/p$ and $0 < h(w) \le 2\{(b - (p - 4 + 2/\alpha))(p - 6) - d_1p\}.$ (ii) b > p - 4. Since h(w) is nonincreasing and $\alpha \ge d_1$, it follows that

(3.8)
$$L \ge E_W \left\{ -\frac{2d_1 ph(W)}{(d_1 + W)^3} + \frac{2(b - (p - 4))h(W)W}{(d_1 + W)^3} - \frac{h^2(W)W}{(d_1 + W)^4} \right\}.$$

Applying Lemma 1.2 to the second term in (3.8) in a similar way as the proof of (i), we get

$$L \ge E_W \left\{ \frac{h(W)}{(d_1 + W)^3} \left\{ -2d_1 p + 2(b - (p - 4))(p - 6) - h(W) \right\} \right\},$$

which is nonnegative if $0 > h(w) \ge 2\{(b - (p-4))(p-6) - d_1p\}$.

Theorem 3.3. Let p>6 and let b in $\partial^{1*}(Y)$ be equal to p-2. The risk of $\partial^{1*}(Y)$ is uniformly smaller than that of $\partial^{0}(Y)$ if $0 < \alpha \le 4(p-6)/(p-2)$.

Proof. Applying Lemma 1.1 to two terms of $R(\delta^0(Y), \mu) - R(\delta^{1*}(Y), \mu)$ in a similar way of Theorem 2.1 or 3.2, we see that it suffices to show that

(3.9)
$$M \equiv E_{W} \left\{ \frac{-a(p-2)}{(a+W)^2 W} + \frac{4}{(a+W)^2} \right\} \ge 0.$$

Lemma 1.2 is applied to the second term in (3.9) to obtain

(3.10)
$$M = E_{W} \left\{ \frac{-a(p-2) + 4(p+2k-2)}{(a+W)^{2}W} - \frac{16}{(a+W)^{3}} \right\}$$
$$\geq E_{W} \left\{ \frac{(-a(p-2) + 4(p-6))W + a(p-2)(-a+4)}{(a+W)^{3}W} \right\},$$

which is nonnegative if $0 < a \le 4(p-6)/(p-2)$.

Theorem 3.2 is used to obtain some other estimators.

Example 3.1. Setting h(s) equal to a constant d_2 , we get the estimator

$$\delta^{5A}(Y) = \left(1 - \frac{b}{a+S} + \frac{d_2}{(d_1+S)^2}\right)Y.$$

If the constants a, b, d_1 and d_2 satisfy the following conditions, then the risk of $\delta^{54}(Y)$ is uniformly smaller than that of $\delta^{1*}(Y)$. (i) For b > p-4,

 $a \le d_1 < (b - (p - 4))(p - 6)/p$

and

,£

$$0 < d_2 \le 2\{(b - (p - 4))(p - 6) - d_1p\}.$$

(ii) For b < p-4,

$$a \ge d_1 \ge 0$$
 and $0 > d_2 \ge 2\{(b - (p-4))(p-6) - d_1p\}$.

Example 3.2. For b=p-2, setting $h(s)=d_2(d_1+s)^{1/\alpha}(d_3+s)^{-1/\alpha}$, we obtain the estimator

$$\delta^{5B}(Y) = \left(1 - \frac{p-2}{a+S} + \frac{d_2(d_1+S)^{1/\alpha-2}}{(d_3+S)^{1/\alpha}}\right)Y.$$

If $\alpha > 1$, $d_1 \le d_3$, $a \le d_1 < 2(1-1/\alpha)(p-6)/p$ and $0 < d_2 \le 2\{2(1-1/\alpha)(p-6)-d_1p\}$, then h(s) satisfies condition (i) of Theorem 3.2.

4. Comparison of $\partial^{3}(Y)$, $\partial^{4}(Y)$ and $\partial^{5}(Y)$

The following three estimators

(4.1)
$$\delta^{3C}(Y) = \left(1 - \frac{p-2}{S} + \frac{d}{S^2}\right)Y,$$

(4.2)
$$\delta^{4}(Y) = \left(1 - \frac{p-2}{a+S} + \frac{c}{S^2}\right)Y$$

and

(4.3)
$$\delta^{5C}(Y) = \left(1 - \frac{p-2}{a+S} + \frac{d_2}{(d_1+S)^2}\right)Y,$$

are discussed. The estimator $\delta^{3C}(Y)$ is defined in Corollary 2.1, $\delta^{4A}(Y)$ and $\delta^{5C}(Y)$ are defined in Theorem 3.1 and Example 3.1, respectively. Conditions on the constants a, c and d are given such that $R(\delta^{4A}(Y), \mu) \leq R(\delta^{3C}(Y), \mu)$. Further the constants a, c, d_1 and d_2 are given such that $R(\delta^{5C}(Y), \mu) \leq R(\delta^{4A}(Y), \mu)$.

Theorem4.1. Suppose that p>8. Then the risk of $\delta^{4d}(Y)$ is uniformly smaller than that of $\delta^{3C}(Y)$ if the constants a, c and d satisfy the following conditions. (i) For c>d,

$$0 \le a \le 4(p-6)/(p-2)$$

and

$$\{4(c-d)(p-6)-2a(c(p-4)+2d)+d^2-c^2\}(p-8)+a(d^2-c^2)\geq 0.$$

(ii) For $c \leq d$,

$$d/(p-2) < a < 2$$
 and $A(p-8) - 4a^2d \ge 0$,

where $A = -a(a(p-2)^2+8d) + 4(a(p-2)-d)(p-6) + 2c((2-a)(p-4)-4) + d^2 - c^2$.

Proof. Applying Lemma 1.1 to two terms of $\Delta(\mu) \equiv R(\delta^{3C}(Y), \mu) - R(\delta^{4A}(Y), \mu)$ in a similar way as the proof of Theorem 2.1, we have

$$\begin{aligned} \mathcal{\Delta}(\mu) = E_Y \left\{ -\frac{(p-2)^2}{S} - \frac{4d}{S^2} + \frac{d^2}{S^3} + \frac{2ap(p-2) + (p-2)^2S}{(a+S)^2} - \frac{2c(p-4)}{S^2} + \frac{2c(p-2)}{(a+S)S} - \frac{c^2}{S^3} \right\}. \end{aligned}$$

Therefore it suffices to show that

$$N \equiv E_W \left\{ \frac{-a^2(p-2)^2 + 4a(p-2)W}{(a+W)^2W} - \frac{2(c(p-4)+2d)}{W^2} + \frac{2c(p-2)}{(a+W)W} + \frac{d^2 - c^2}{W^3} \right\} \ge 0.$$

To evaluate N, we consider the following two cases: c > d and $c \le d$. (i) c > d.

$$N = E_W \left\{ \frac{-a^2(p-2)^2 + 4a(p-2)W}{(a+W)^2W} + \frac{-2a(c(p-4)+2d) + 4(c-d)W}{(a+W)W^2} + \frac{d^2 - c^2}{W^3} \right\}$$

$$\equiv N_1 + N_2 + N_3,$$

say. The proof of Theorem 3.3 shows that

(4.4)
$$N_1 \ge a(p-2)E_W \left\{ \frac{(-a(p-2)+4(p-6))W + a(p-2)(-a+4)}{(a+W)^3W} \right\}.$$

From (3.2) in the proof of Theorem 3.1 and the condition that c > d,

(4.5)
$$N_2 + N_3 \ge E_W \left\{ \frac{4(c-d)(p-6) - 2a(c(p-4) + 2d) + d^2 - c^2}{(a+W)W^2} + \frac{a(d^2 - c^2)}{(a+W)W^3} \right\}.$$

Furthermore Lemma 1.2 is applied to the first term of (4.5) to obtain

(4.6)
$$N_2 + N_3 \ge E_W \left\{ \frac{\{4(c-d)(p-6) - 2a(c(p-4) + 2d) + d^2 - c^2\}(p-8) + a(d^2 - c^2)}{(a+W)W^3} \right\}.$$

Hence from the inequalities (4.4) and (4.6), N is nonnegative if the constants a, c and d satisfy condition (i).

(ii) $c \leq d$.

$$N = E_{W} \left\{ \frac{-a^{2}(p-2)^{2} + 4a(p-2)W}{(a+W)^{2}W} - \frac{4d}{W^{2}} - \frac{2c(p-4)}{W^{2}} + \frac{2c(p-2)}{(a+W)W} + \frac{d^{2}-c^{2}}{W^{3}} \right\}$$
$$= E_{W} \left\{ \frac{-4a^{2}d - a(a(p-2)^{2} + 8d)W + 4(a(p-2) - d)W^{2}}{(a+W)^{2}W^{2}} + \frac{-2ac(p-4) + 4cW}{(a+W)W^{2}} + \frac{d^{2}-c^{2}}{W^{3}} \right\}$$

 $\equiv N'_1 + N'_2 + N'_3$,

say. To evaluate N in a similar way as the proof of (i), we decompose N as the above last equality. Applying Lemma 1.2 to the third term of N'_1 and the second term of N'_2 respectively, and using the condition that d/(p-2) < a < 2, we have

(4.7)
$$N'_{1} \ge E_{W} \left\{ -\frac{4a^{2}d}{(a+W)^{2}W^{2}} + \frac{-a(a(p-2)^{2}+8d)+4(a(p-2)-d)(p-6)}{(a+W)^{2}W} \right\}$$

and

(4.8)
$$N'_{2} \ge E_{W} \left\{ \frac{2c((2-a)(p-4)-4)}{(a+W)^{2}W} \right\}.$$

The inequalities (4.7) and (4.8) give

(4.9)
$$N \ge E_W \left\{ -\frac{4a^2d}{(a+W)^2 W^2} + \frac{A}{(a+W)^2 W} \right\},$$

where A is defined in condition (ii). Furthermore Lemma 1.2 is applied to the second term of (4.9) to obtain

$$N \ge E_W \left\{ \frac{-4a^2d + A(p-8)}{(a+W)^2 W^2} \right\},$$

which is nonnegative if $A(p-8)-4a^2d \ge 0$.

Theorem 4.2. Let p>6 and let a be equal to d_1 . Then the risk of $\delta^{5C}(Y)$ is uniformly smaller than that of $\delta^{4A}(Y)$ if $c < d_2$ and $4(d_2-c)(p-6)+2a(c(p-6)-d_2(p+2))+c^2-d_2^2 \ge 0$.

Proof. Applying Lemma 1.1 to a term of $\Delta(\mu) \equiv R(\delta^{*4}(Y), \mu) - R(\delta^{50}(Y), \mu)$ in a similar way as the proof of Theorem 3.2, we have

$$\begin{split} \mathcal{A}(\mu) &= E_{W} \left\{ \frac{2c(p-4)}{S^{2}} - \frac{2c(p-2)}{(a+S)S} + \frac{c^{2}}{S^{3}} - \frac{2d_{2}p}{(d_{1}+S)^{2}} + \frac{8d_{2}S}{(d_{1}+S)^{3}} \right. \\ &\left. + \frac{2d_{2}(p-2)S}{(a+S)(d_{1}+S)^{2}} - \frac{d_{2}^{2}S}{(d_{1}+S)^{4}} \right\}. \end{split}$$

Therefore from the comdition that $a=d_1$, it suffices to show that

$$P \equiv E_W \left\{ \frac{2ac(p-4)}{(a+W)W^2} + \frac{c^2}{W^3} + \frac{4(d_2-c)}{(a+W)^2} - \frac{4ac}{(a+W)^2W} - \frac{2ad_2(p+2) + d_2^2}{(a+W)^3} + \frac{ad_2^2}{(a+W)^4} \right\}.$$

From (3.10) in the proof of Theorem 3.3 and the condition that $c > d_2$,

$$P \ge E_W \left\{ \frac{2ac(p-4)}{(a+W)W^2} + \frac{c^2}{W^3} + \frac{4((d_2-c)(p-2)-ac)}{(a+W)^2W} - \frac{2ad_1(p+2)+16(d_2-c)+d_2^2}{(a+W)^3} \right\}$$
$$\ge E_W \left[\frac{1}{(a+W)^3} \left\{ 4(d_2-c)(p-6) + 2a(c(p-6)-d_2(p+2)) + c^2 - d_2^2 \right\} \right],$$

which is nonnegative if the constants a, c and d_2 satisfy the conditions.

Acknowledgement

The author would like to thank Professor M. Sibuya of Keio University and Professor N. Shinozaki of Tokyo Keizai University for helpful suggestions and criticisms.

REFERENCES

- [1] Baranchik, A.J. (1970). A family of minimax estimators of the mean of a multivariate normal distribution, Ann. Math. Statist., 41, 517-525.
- [2] Efron, B., and Morris, C. (1973). Families of minimax estimators of the mean of a multivariate normal distribution, Ann. Statist., 4, 11-21.
- [3] James, W., and Stein, C. (1961). Estimation with quadratic loss, Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman, (ed.), University of California Press, 361-379.
- [4] Tze, F. L., and Wen, H. K. (1982). Generalized James-Stein estimators, Comm. Statist., 11(20), 2249-2257.