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MOTION OF HULA-HOOP AND ITS STABILITY

by

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In this report, the dynamics of the Hula-Hoop whose rotational motion is kept stabilized by reciprocating motion of the waist in horizontal direction are treated. Considering a three dimensional dynamic system with a mathematical pendulum whose supported point was caused to move periodically along the fixed axis and not to move vertically, the horizontal and vertical motions of the pendulum were analyzed in detail. Coupled set of nonlinear ordinary differential equations of the second order which were led by applying Lagrange's equations to the system were solved approximately. The basic mechanism of the stable motion of the pendulum due to an oscillatory motion of the supported point is made clear on account of the consideration for the approximate solution of the equations. The influence of the motion of the supported point on these motions was clarified, while these motions depended closely on the movement of the supported point. In addition, the consideration for the stability of the system led analytically such conclusion as the stable motions were preserved on account of the vertical motion of the Hula-Hoop. To examine the validity of the theory, the experimental model was provided. The experimental results were shown to be in qualitative good agreement with the theoretical ones.

1. Introduction

In the past year an interesting toy such as the Hula-Hoop has been in fashion.

Considering the motions of the Hula-Hoop, it is found that the Hula-Hoop is kept on rotating by reciprocating motion of the waist.

The motion, in a horizontal plane, which are regarded as the parametric excitation of a mathematical pendulum whose supported point is caused to move periodically along a fixed axis, has been analyzed by Caughey T. K. in reference [1].

Suppose the motions of the pendulum, its vertical motion also can be accepted the parametric excitation on account of the periodical motion of the supported point.

Considering a three dimensional dynamic system with a two degrees of freedom mathematical pendulum whose supported point is caused to move periodically along a fixed axis and not to move vertically, the horizontal and vertical motions of the pendulum are analyzed in detail by using Struble method in reference [2]. The system is shown by the equations of two types of the parametric excitation. These equations have the coupled nonlinear terms which are yielded due to the consideration for the three dimensional system. The consideration for the influence of these terms on these motions lead analytically the conclusion how the stability of the system is preserved.

In addition, the influence of the motion of the supported point on the system is made clear.

Moreover, to examine the validity of the theory, the experimental model is provided.

2. Equations of Motion of the Pendulum

Figure 1 shows a three dimensional dynamic system with a mathematical pendulum of length l and mass m whose supported point P is caused to move periodically along a fixed axis X and not to move vertically. The pendulum corresponds to the Hula-Hoop and the motion of the supported point, to the motion of the waist. Then the motion of the supported point $X(t)$ is shown by

$$X(t)=X_0 \cos(\omega t), \tag{1}$$

where X_0 is its amplitude and ω is its frequency.

Suppose the position of the pendulum is defined by the angles of θ and ϕ

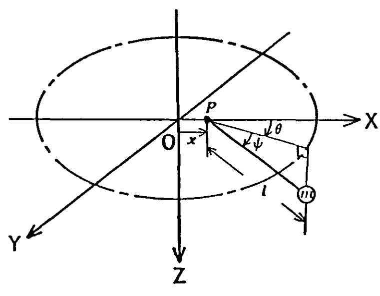

Figure 1. Pendulum used in theoretical study

respectively with X-Z and with X-Y planes. Applying Lagrange's equations to the system, and neglecting terms of order higher than the square of $\phi(t)$, the equations of motion of the system are derived as

$$\ddot{\theta}+(2\xi_1-2\phi\dot{\phi})\dot{\theta}+\frac{X_0\omega^2}{l}\cos(\omega t)\sin(\theta)=0\ , \tag{2}$$

$$\ddot{\phi}+2\xi_2\dot{\phi}+\left[\dot{\theta}^2+\frac{X_0\omega^2}{l}\cos(\omega t)\cos(\theta)\right]\phi-\frac{g}{l}\left(1-\frac{1}{2}\phi^2\right)=0\ , \tag{3}$$

where $2\xi_1=\frac{C_1}{ml^2}$, $2\xi_2=\frac{C_2}{ml^2}$, then C_1, and C_2 represent viscous damping coefficients respectively in θ- and ϕ-directions, and g is a gravitational acceleration.

In equations (2) and (3), dots denote differentiation with respect to time t. Equations (2) and (3) represent the balances of moment respectively with respect to the supported point and X-Y plane, and have coupled nonlinearities.

In equation (2), the second term exhibits a damping force, in θ-direction, which is yielded by the movement of the pendulum in θ- and ϕ-directions, and the third term exhibits a restoring force which is the component, in θ-direction, of the force which the supported point exerts on the pendulum. Equation (2) can be regarded as the equation with a periodic coefficient whose sign changes between positive and negative. Then $\theta(t)$ shows the motion of the pendulum in horizontal direction, so $\theta(t)$ can be considered that it increases or decreases monotonically if the pendulum is kept rotating stably. Moreover, in above condition, the analysis of reference [1] has brought us that $\theta(t)$ increases or decreases in proportion nearly to time t and the proportional constant is the frequency of the supported point, ω. Therefore, the third term can be considered to cause $\theta(t)$ to do as the above owing to the change of the sign of its spring constant between positive and negative.

According to the above, in the third term of equation (3) we can consider $\dot{\theta}^2\simeq\omega^2$, so within the limits of linearity and in the case $X_0/l<1$, equation (3) can be regarded as the equation with a periodic coefficient whose sign is usually positive. In equation (3), the second term exhibits a damping force, in ϕ-direction, which is yielded by the movement of the pendulum in ϕ-direction, the third term exhibits a restoring force which consists of a centrifugal force yielded by the rotational motion of the pendulum in θ-direction and the component, in the radial direction, of the force which the supported point exerts on the pendulum, and the fourth term exhibits a gravitational force.

Without the vertical motion of the pendulum for equations (2) and (3), the system considered here has corresponded to the system, which the pendulum can rotate only in a horizontal plane, as shown in reference [1].

3. Transform of Equations of Motion

Let

$$\theta(t)=\omega t+\phi(t) \tag{4}$$

as reference [1]. Substituting equation (4) into equations (2) and (3), using a non-dimensional independent variable $\tau=\omega t$, and retaining only terms up to the square of $\phi(\tau)$ and $\psi(\tau)$, we obtain

$$\ddot{\phi}+2\varepsilon^2\hat{\gamma}_1\dot{\phi}+\varepsilon\hat{l}\,[1+\cos(2\tau)]\,\phi=-2\varepsilon^2\hat{l}\hat{\gamma}_1-\varepsilon\hat{l}\,\sin(2\tau)+2\psi\dot{\psi}, \tag{5}$$

$$\ddot{\psi}+2\varepsilon^2\hat{\gamma}_2\dot{\psi}+[\omega_2^2+\varepsilon\hat{l}\,\cos(2\tau)+2\dot{\phi}]\psi=\varepsilon\hat{g}, \tag{6}$$

where $\varepsilon\hat{l}=\dfrac{X_0}{2l}$, $\varepsilon\hat{g}=\dfrac{g}{l\omega^2}$, $\varepsilon^2\hat{\gamma}=\dfrac{\xi_1}{\omega}$, $\varepsilon^2\hat{\gamma}_2=\dfrac{\xi_2}{\omega}$, and $\omega_2^2=1+\varepsilon\hat{l}$, then we assume $\varepsilon=O(1/10)$ and $\hat{l}=\hat{g}=\hat{\gamma}_1=\hat{\gamma}_2=O(\varepsilon^0)$.
In equations (5) and (6), dots denote differentiation with respect to τ.

Equations (5) and (6) exhibit respectively a phase difference between the horizontal motion of the pendulum and the motion of the supported point, and the vertical motion of the pendulum. These equations can be regarded as the equations with periodic coefficients and quadratic nonlinearities.

The third term, on the left-hand side of equation (5), which exhibits restoring force, has a case to vanish. This shows one of the characteristics of the horizontal motion of the pendulum given by equation (2). Then within the limits of linearity, $\phi(\tau)$ given by equation (5) is stable due to $\varepsilon\hat{l}<1$ and can be estimated that it is damply oscillated due to the second and third terms on left-hand side of equation (5), and that it is converged on the steady state given by the right-hand side of equation (5).

Within the limits of linearity, $\psi(\tau)$ given by equation (6) is stable. However, it is possible to become unstable if the amplitude of the component whose frequency is twice as large as that of the supported point is caused to increase, due to the term yielded by the second term on the right-hand side of equation (5) entering to $2\dot{\phi}$ in the third term of the left-hand side of equation (6). Then figure 2 shows the numerical solutions of equations (5) and (6). These solutions show the stable motions of the pendulum. Namely, $\psi(\tau)$, given by equation (6), that is, the vertical motion of the pendulum, can be estimated that it is damply oscillated due to the second and third terms on left-hand side of equation (6) and that it is converged on the steady state given by the right-hand side of equation (6). As the above the third term on the right-hand side of equation (5), which is called quadratic nonlinearity, which is equivalent to a coupled nonlinear term in the second term

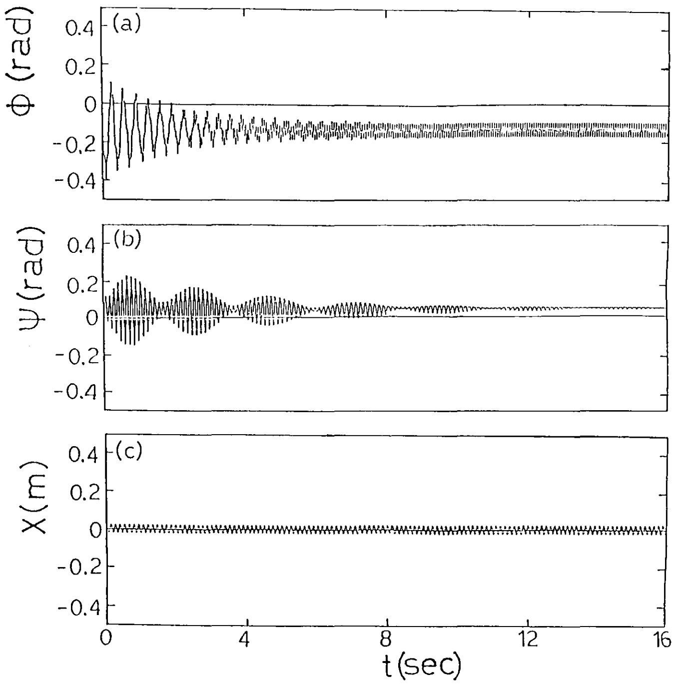

Figure 2. Response of the system for $X_0=0.023[\mathrm{m}]$, $l=0.076[\mathrm{m}]$, $2\xi_1=1.0[1/s]$, $2\xi_2=0.5[1/s]$, $\omega=50\,[\mathrm{rad}/s]$, and $g=9.8[m/s^2]$. (a), $\phi(t)$; (b), $\psi(t)$; (c), $X(t)$. Initial conditions: $\phi(0)=\dot{\phi}(0)=\psi(0)=\dot{\psi}(0)=0$.

on the left-hand side of equation (2) can be considered to have a very important effect upon the system. Moreover, figure 2 shows that $\psi(t)$ closely synchronizes with the motion of the supported point before it sets in steady state, and that $\phi(t)$ is converged on the steady state. Therefore the Hula-Hoop can be considered analytically to rotate as nearly to synchronize with the motion of the waist in horizontal direction.

4. Method of Solution

We express the asymptotic solutions of equations (5) and (6) for small ε in the form

$$\phi(\tau)=a\cos[\varepsilon^{1/2}\hat{l}\tau+\beta_1]+\varepsilon^{1/2}\phi_1(\tau)+\varepsilon\phi_2(\tau)+\varepsilon^{3/2}\phi_3(\tau)+\cdots, \tag{7}$$

$$\psi(\tau)=\varepsilon^{1/2}b\cos[\omega_2\tau+\beta_2]+\varepsilon^{1/2}\psi_1(\tau)+\varepsilon\psi_2(\tau)+\varepsilon^{3/2}\psi_3(\tau)+\cdots, \tag{8}$$

where a, b, β_1, and β_2 are slowly varying function of time τ. Substituting equations (7) and (8) into equations (5) and (6), expanding, equating coefficients of same powers of ε, solving the resulting equations, and neglecting except the governable terms, we obtain the approximate solutions,

$$\phi(\tau)=-\frac{2\varepsilon\hat{\gamma}_1}{\hat{\iota}}+a\cos[\varepsilon^{1/2}\hat{l}\tau+\beta_1]+\frac{\varepsilon\hat{l}}{4-\varepsilon\hat{l}}\sin(2\tau)$$

$$+\frac{\varepsilon b^2}{4-3\varepsilon\hat{l}}\sin 2[\omega_2\tau+\beta_2]+O(\varepsilon^{3/2}), \tag{9}$$

$$\psi(\tau)=\frac{\varepsilon\hat{g}}{\omega_2^2}+\varepsilon^{1/2}b\cos[\omega_2\tau+\beta_2]-\frac{1}{2}\varepsilon^{1/2}b\cos[(2-\omega_2)\tau-\beta_2]$$

$$+\frac{2\varepsilon^2\hat{g}\hat{l}}{3+2\varepsilon\hat{l}}\cos(2\tau)+O(\varepsilon^{5/2}), \tag{10}$$

where

$$a(\tau)=a_0\exp(-\varepsilon^2\hat{\gamma}_1\tau), \tag{11}$$

$$\beta_1(\tau)=\frac{1}{16}\varepsilon^{3/2}\hat{l}\tau+\beta_{10}+O(\varepsilon^2), \tag{12}$$

$$b(\tau)=b_0\exp(-\varepsilon^2\hat{\gamma}_2\tau), \tag{13}$$

$$\beta_2(\tau)=-\frac{1}{4\omega_2}\varepsilon\hat{l}\tau+\frac{2}{\omega_2^2}\varepsilon^2\hat{g}\tau+\beta_{20}+O(\varepsilon^{5/2}), \tag{14}$$

then a_0, b_0, β_{10}, and β_{20} are constants of integration.

Equation (9) shows that the phase difference between the motion of the pendulum in θ-direction and the motion of the supported point is damply oscillated round the constant phase lag against the motion of the supported point due to the first and second terms, and that its motion is converged on steady state given by the first and third terms. In other words, it can be confirmed that the Hula-Hoop rotates as nearly to synchronize with the periodical motion of the waist in horizontal direction. The phase lag, which is yielded by a resistance against the rotational motion of the pendulum in θ-direction, which is led by the first term on the right-hand side of equation (5), becomes small on account of the increase of the inertia force of the pendulum in θ-direction if the amplitude of the supported point, X_0 or the frequency of one, ω is caused to increase. According to the third term of equation (9), in steady state the amplitude of $\phi(\tau)$ is independent of the frequency of the supported point and depends only on the amplitude of one, and its frequency is twice as large as the supported point. These can be understood on account of the consideration for the change of the force which the supported point exerts on the pendulum over one cycle of the movement of the supported point.

Equation (10) shows that in ϕ-direction the pendulum has a constant angle to X-Y plane, that is, a horizontal plane due to the first term and is damply oscillated with the frequency which is equal to that of the supported point due to the second and third terms, and that its motion is converged on steady state given by the first and fourth terms. The constant angle to a horizontal plane which is yielded by a gravitational force, which is led by the right-hand side of equation (6), becomes small on account of the increase of the centrifugal force yielded by the rotational motion of the pendulum in θ-direction if the frequency of the supported point, ω is caused to increase. And the change of, the amplitude of the supported point, X_0, has little influence on that of the constant angle to a horizontal plane in order that the force which the supported point exerts on the pendulum is much smaller than the centrifugal force yielded by the rotational motion of the pendulum in θ-direction. In steady state the frequency of $\phi(\tau)$ is also twice as large as the supported point due to the fourth term of equation (10). The consideration for the change of the force, which the supported point exerts on the pendulum over one cycle of the movement of the supported point, gives a clear explanation of the above.

Moreover, neglecting the fourth term of equation (10) in order that its amplitude is $O(\varepsilon^2)$, in steady state the system is considered to balance with the left-hand side of equation (5), the first and second terms on right-hand side of equation (5), the third term on the left-hand side of equation (6), and the right-hand side of equation (6). Therefore, neglecting terms of order higher than $O(\varepsilon)$, in steady state the Hula-Hoop can be considered analytically to have a constant angle to a horizontal plane, and to rotate as nearly to synchronize with the motion of the waist in horizontal direction.

5. Vertical Motion of the Pendulum

In this section we shall investigate the vertical motion of the pendulum, that is, the reason why it synchronizes closely with the motion of the supported point before it sets in steady state.

Figure 3 shows $F_1(\tau)=\cos(\tau)$ and $F_2(\tau)=\cos(2\tau)$. In this figure $\varepsilon\hat{\varepsilon}\cos(2\tau)$ and

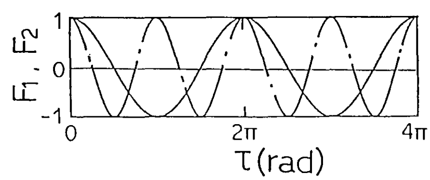

Figure 3. $F_1(\tau)=\cos(\tau)$ and $F_2(\tau)=\cos(2\tau)$.

$2\dot{\phi}$, and the motion of the supported point, are regarded respectively as $F_2(\tau)$ and $F_1(\tau)$. When the absolute value of the amplitude of the supported point has a maximum, that is, at the position of $F_1(\tau)=1$ and $F_1(\tau)=-1$, we can find $F_2(\tau)=1$, that is, the spring constant in ψ-direction has a maximum. And when the amplitude ot the supported point becomes zero, that is, at the position of $F_1(\tau)=0$, we can also find $F_2(\tau)=-1$, that is, the spring canstant in ψ-direction is minumum. In other words, these mean that the force which the supported point exerts on the pendulum is maximum at the position of $X(t)=X_0$ and that it is minimum at the position of $X(t)=0$. For example, if the pendulum is caused to move in ψ-direction at the position of $X(t)=X_0$, it will not change the direction to move until at the position $X(t)=-X_0$ where the restoring force in ψ-direction is the largest next, and at the position $X(t)=-X_0$ it will stop to move in that direction, then it will start to move in opposite direction at that position. In the result it repeats a series of these movements until its motion sets in steady state in which the gravitational force, the centrifugal force yielded by the rotational motion of the pendulum in θ-direction, and the force which the supported point exerts on the pendulum, are nearly balanced in ψ-direction.

6. Stability of the System

Considering the stability of the system, that is, that of the motion of the pendulum, the influence of the quadratic nonlinearities on the stability of the system are very important as previously stated.

If $2\psi\dot{\phi}$ which is the third term on the right-hand side of equation (5) has been neglected, we can find that $\psi(\tau)$ expressed by equation (6) has become unstable, that is, the system has become unstable. The mechanism which causes the system to become unstable can be explained as follows.

The term, whose frequency is twice as large as that of the supported point, which is yielded by the second term on right-hand side of equation (5), that is, the third term of equation (9),

$$\frac{\varepsilon\hat{l}}{4-\varepsilon\hat{l}}\sin(2\tau),$$

enters $2\dot{\phi}$ which is in the third term on the left-hand side of equation (6), where $2\dot{\phi}$ is expressed as

$$\frac{4\varepsilon\hat{l}}{4-\varepsilon\hat{l}}\cos(2\tau).$$

The above expression causes the amplitude of the term, whose frequency is twice as large as that of the supported point, to increase, and consequently the system has been caused to become unstable.

Namely, the stability of the system is kept in order that $2\phi\dot{\varphi}$ which is the third term on the right-hand side of equation (5) causes the amplitude of the term, whose frequency is twice as large as that of the supported point, that is, that of the second term on the right-hand side of equation (5), to decrease. The above can be explained to see the previous section and confirmed due to the fourth term of equation (9). In other words, $2\phi\dot{\varphi}$ can be considered to act to prevent the amplitude of the pendulum in φ-direction from increasing due to the function of the decrease of the centrifugal force, in θ-direction, which is yielded by the force that the supported point exerts on the pendulum.

In the result, the Hula-Hoop can be considered analytically to keep its stability due to its movement in vertical direction.

Moreover, we shall investigate the stability of the system which depends on the physical parameters.

Let

$$\theta(t)=\theta_0+\eta_1(t)=\omega t+\phi_0+\eta_1(t), \tag{15}$$

$$\varphi(t)=\varphi_0+\eta_2(t), \tag{16}$$

where ϕ_0 and φ_0 are constants, $\eta_1(t)$ and $\eta_2(t)$ are small perturbations. Substituting equatins (15) and (16) into equations (2) and (3), neglecting terms of order higher than the square of ϕ_0, φ_0, η_1, and η_2, using a non-dimensional independent variable $\tau=\omega t$, applying the equations which are caused $\phi(\tau)$ and $\varphi(\tau)$ to become constants for equations (5) and (6), and letting

$$2z=2\tau+\phi_0, \tag{17}$$

we obtain

$$\ddot{\eta}_1+2\varepsilon^2\hat{\gamma}_1\dot{\eta}_1+\varepsilon\hat{l}\,[\cos(\phi_0)+\cos(2z)]\eta_1=0, \tag{18}$$

$$\ddot{\eta}_2+2\varepsilon^2\hat{\gamma}_2\eta_2+\{1+\varepsilon\hat{l}\,[\cos(\phi_0)+\cos(2z)]\}\eta_2=0. \tag{19}$$

In equations (18) and (19), dots denote differentiation with respect to z. These equations can be regarded as Mathieu equations. Let

$$\delta_1=\varepsilon\hat{l}\cos(\phi_0), \tag{20}$$

$$\delta_2=1+\varepsilon\hat{l}\cos(\phi_0), \tag{21}$$

$$2q=\varepsilon\hat{l}, \tag{22}$$

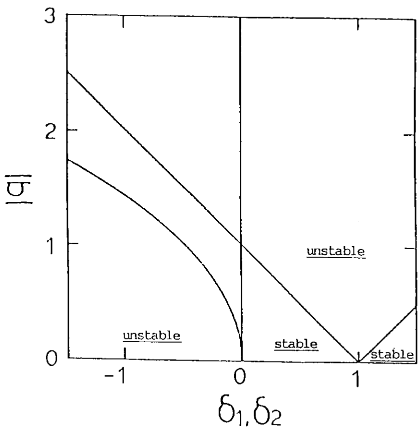

Figure 4. Stability chart of the system for $\omega=50$[rad/s], $2\xi_1=1.0$[1/s], and $2\xi_2=0.5$[1/s].

we obtain figure 4 which shows the stability chart of the system. In the case of $2\varepsilon^2\hat{\gamma}_1 \ll 1$ and $2\varepsilon^2\hat{\gamma}_2 \ll 1$, the stability of the system which depends on the physical parameters has been nearly equal to one as shown in reference [1].

7. Experimental Results

To examine the validity of the theory, the experimental model was provided.

Figure 5 shows the experimental model. The ring of radius 0.0760[m], thickness 0.0035[m], and mass 0.0140[kg], which is made of vinyl chloride, is equivalent to the Hula-Hoop. Moreover, an aluminum sheet was attached to the ring in order that two noncontacting displacement meters were caused to catch the motion of the ring. And the radius of the ring is equal to the length of the pendulum. The shaft of diameter 0.0061[m] and length 0.0810[m], which is made of iron, is equivalent to the waist. The shaft was fixed on the acrylic board of 0.255[m]×0.150[m]×0.010[m]. And the shaft was caused to move periodically along the fixed axis owing to the periodic motion of the board by using three ball slide bearings and the rotational motion given by the electric motor. The shaft was taped in order that the frictional force between the ring and the shaft was caused to increase. Using the taped shaft, the contact point between the ring and the shaft was caused not to move vertically. That is, the ring is made rotate by the periodic motion of the shaft, and the motions of the ring and the shaft were measured by using three noncontacting displacement meters, so the measured values which were transformed into digital ones were analyzed by the digital computer. The analysis brought us the phase difference between the ring and the shaft and the displacement of the ring in vertical direction at the position which the absolute value of the amplitude of the shaft had a maximum, that is,

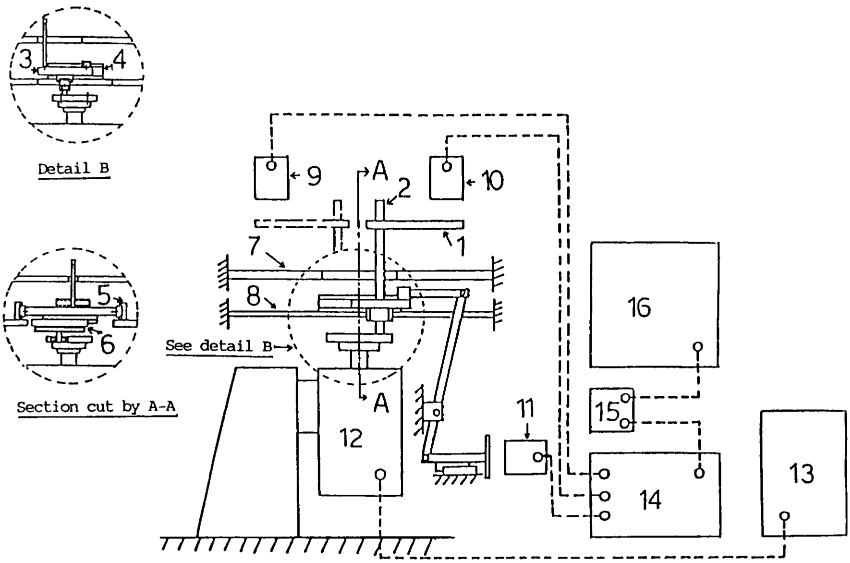

Figure 5. Experimental model. Legend: 1, Ring; 2, shaft; 3, acrylic board; 4, 5, 6, ball slide bearings; 7, acrylic stage; 8, foundation; 9, 10, 11, noncontacting displacement meters with sensors; 12, electric motor; 13, driving power control unit; 14, analog-to-digital converter; 15, data transmitter; 16, digital computer.

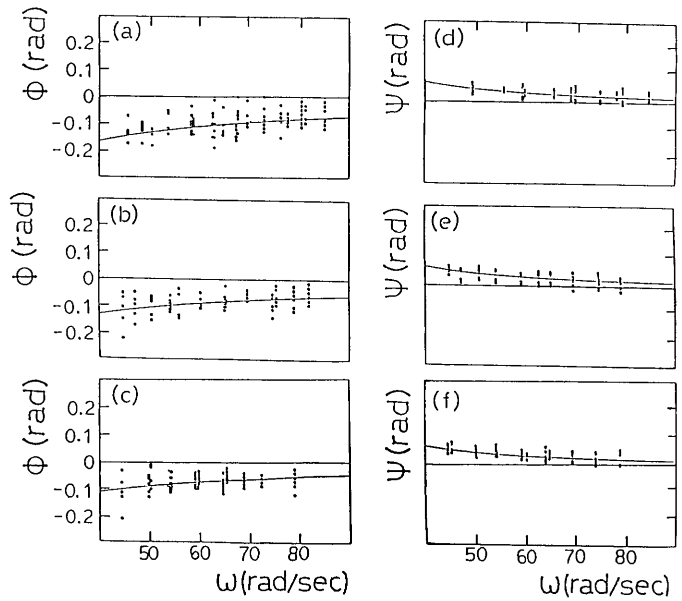

Figure 6. ϕ vs. ω and ψ vs. ω in steady state for $l=0.0760$[m], $2\xi_1=1.0[1/s]$, $g=9.8[m/s^2]$, and several values of X_0. (a) and (d) $X_0=0.0230$[m]; (b) and (e) $X_0=0.0290$[m]; (c) and (f) $X_0=0.0350$[m]. ∘, Experimental; ——, theoretical.

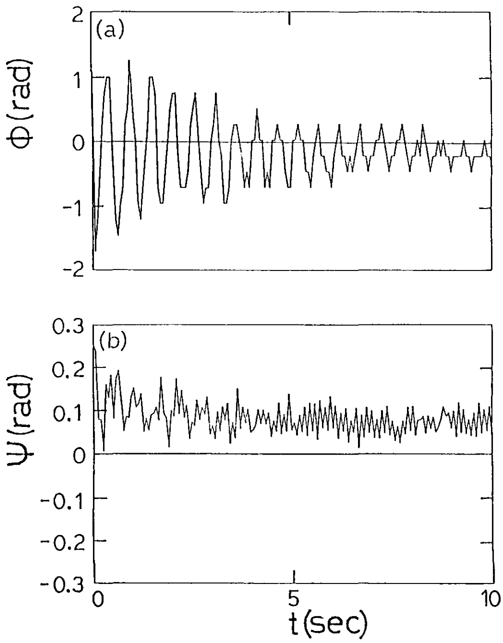

Figure 7. Ring motion for $X_0=0.0230$[m], $l=0.0760$[m], and $\omega=49.40$[rad/s].

$X(t)=X_0$ and $X(t)=-X_0$.

Figure 6 shows the phase difference between the ring and the shaft and the displacement of the ring in vertical direction with a horizontal plane, that is, respectively $\phi(t)$ and $\psi(t)$ in steady state at the position which the absolute value of the amplitude of the shaft had a maximum, that is, $X(t)=X_0$ and $X(t)=-X_0$. Figure 7 shows measurements of $\phi(t)$ and $\psi(t)$ in transient state at the same position as the above. This was obtained by means that the time of $t=0$ was defined as the start time of measurements, then initial conditions could not be measured.

The experimental results were shown to be in qualitative good agreement with the theoretical ones.

8. Conclusions

The foregoing analysis and experiment have clarified the mechanism of the stable motion of the Hula-Hoop as follows.

The Hula-Hoop can be considered to rotate in horizontal direction as nearly to synchronize with the periodical motion of the waist, and to oscillate in vertical direction as closely to synchronize with the supported point, before its motion sets in steady state in which the gravitational force, the centrifugal force yielded by its rotational motion in horizontal direction, and the force which the waist exerts on it, are nearly balanced. Therefore, in steady state the Hula-Hoop has a constant angle to a horizontal plane, and rotates in horizontal direction as nearly to synchronize with the periodical motion of the waist.

In transient state the Hula-Hoop can be considered analytically to keep its stability due to its movement in vertical direction.

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