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REMARKS ON THE MEAN CONCENTRATION FUNCTION OF A RANDOM VARIABLE

by

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1. Introduction.

Let $F(x)$ and $f(x)$ be respectively the distribution function and the characteristic function of a random variable X. The function

$$(1.1)\qquad C(X,\ h)=C(h)=h^{-1}\left[\int_{-\infty}^{\infty}\{F(x+h)-F(x)\}^{2}dx\right]$$

$$=2\pi^{-1}\left[\int_{0}^{\infty}\{\sin^{2}(ht/2)/(ht^{2}/2)\}|f(t)|^{2}dt\right],\quad h>0$$

is called the mean concentration function of X [5] and is a counterpart of Lévy concentration function

$$(1.2)\qquad Q(X,\ h)=Q(h)=\max_{-\infty<x<\infty}[F(x+h+0)-F(x-0)],\quad h>0.$$

$C(h)$ is a nondecreasing function of $h>0$ such that $0\leqq C(h)\leqq 1$, as $Q(h)$ is. They are related as

$$(1.3)\qquad 1/2\, Q^{2}(h/2)\leqq C(h)\leqq Q(h),$$

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or more precisely for the left hand side,

(1.4) $$((2-\alpha)/2)\,Q^2(\alpha h) \leqq C(h)$$

for any $0<\alpha\leqq 1$ [1].

As to behaviors of $C(h)$ as $h \longrightarrow 0+$ and $h \longrightarrow \infty$, it is known that

(1.5) $$\lim_{h\to 0+} C(h) = \lim_{T\to\infty} T^{-1}\left[\int_0^T |f(t)|^2 dt\right],$$

and

(1.6) $$\lim_{h\to\infty} C(h) = 1.$$

As a matter of fact, (1.5) is a particular case of the well known Wiener formula (see [5]) and (1.6) is no more than a property of Fejér integral of $|f(t)|^2$.

Let X^s be the symmetrized random variable $X-X'$ of X where X' is a random variable independent of X and with the same distribution $F(x)$. Write by $F^s(x)$ the distribution function of X^s, namely, $F(x)*(1-F(-x+0))$ and write

(1.7) $$G^s(x) = F^s(x) - F^s(-x),\quad x>0,$$

$$G^s(0+) = P(X^s=0),$$

$$G^s(x) = 0,\quad x\leqq 0.$$

Write

(1.8) $$\boldsymbol{M}(\phi, x) = x^{-1}\left[\int_0^x \phi(t)dt\right],\quad x>0,$$

for any function $\phi(t)$, $t>0$. The function

(1.9) $$\boldsymbol{M}(|f(t)|^2, x) = x^{-1}\left[\int_0^x |f(t)|^2 dt\right]$$

plays some particular role in connection with the concentration function of sums of independent random variables [11], [3], [4], [10], [12], [13], [9], [2]. Denote also

$$(1.10) \qquad \boldsymbol{M}(\phi)=\lim_{x\to\infty}x^{-1}\left[\int_0^x \phi(t)dt\right]$$

when the limit on the right hand side exists, so that the quantity on the right hand side of (1.5) is written by $\boldsymbol{M}(|f(t)|^2)$.

Define also

$$(1.11) \qquad M(X,\ h)=M(h)=h^{-1}E[\min(|X|,\ h)]$$

$$=h^{-1}\int_{-h}^{h}|x|dF(x)+\int_{(|x|\geqq h)}dF(x).$$

$$(1.12) \qquad D^2(X,\ h)=D^2(h)=h^{-2}E[\min(|X|,\ h)]^2$$

$$=h^{-2}\int_{-h}^{h}x^2dF(x)+\int_{(|x|\geqq h)}dF(x).$$

These are important quantities as well.

$D(X,\ h)$ is a nonincreasing function of $h\geqq 0$. $D(h)=0$ for some $h>0$, if and only if $F(x)$ is degenerate at $x=0$.

The aim of this paper is to give basic properties of $C(h)$ particulary in connection with $\boldsymbol{M}(|f(t)|^2,\ 1/h)$ or $\boldsymbol{M}(|f(t)|^2)$. Remarks are given on the mean concentration function of sums of independent random variables. Some known results are also included for completeness.

2. Behavior of $C(h)$ for small h.

For the behavior of $C(h)$ as $h \longrightarrow 0+$, we begin with the following simple fact.

Theorem 1. (i) *The following five statements are equivalent to each other*:

(a) $h^{-1}C(h)$ *is bounded near the origin,*

(b) $\lim_{h\to 0+}h^{-1}C(h)$ *exists,*

(c) $f(t)\in L^2(-\infty,\ \infty)$,

(d) $F(x)$ *is absolutely continuous and its probability density* $p(x)$ *belongs to* $L^2(-\infty,\ \infty)$ *or the same thing as* $p(x)\in L^r(\infty,\ \infty)$ *for any* $1\leqq r\leqq 2$,

(e) $G^s(x)=O(x)$, *as* $x \longrightarrow 0+$.

(ii) *If either of the statments in* (i) *holds, then*

$$\lim_{h\to 0+} h^{-1}C(h) = 2\pi^{-1}\int_0^\infty |f(t)|^2 dt. \tag{2.1}$$

(a) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (e) has been given by Matsumoto [8]. We shall give the proof of Theorem 1 for completeness. The proof is quite simple. But before doing so, we shall state lemmas which are known. (2.3) follows from (2.2). They will be used later.

Lemma 1.

$$C(h) = h^{-1}\int_0^h G^s(x)dx \tag{2.2}$$

$$= \int_0^h dG^s(x) - h^{-1}\int_0^h x dG^s(x).$$

$$1 - C(h) = h^{-1}E(\min(|X^s|,\ h)) \tag{2.3}$$

$$= h^{-1}\int_0^h x dG^s(x) + \int_h^\infty dG^s(x).$$

Lemma 2. *For any* $r \geqq 0$, $\int_{-\infty}^\infty |x|^r dF(x) < \infty$ *is equivalent to* $\int_{-\infty}^\infty |x|^r dF^s(x) < \infty$.

(2.2) is known ([5], p. 446). The second equality is immediate from the first relation by the integration by parts. The second relation of (2.3) is also immediate from the second relation of (2.2). The authors think that Lemma 2 is also known.

We now go back to Theorem 1. From (1.1), we have, for any $A > 0$,

$$h^{-1}C(h) \geqq \pi^{-1}\int_0^A \{\sin^2(ht/2)/(ht/2)^2\}|f(t)|^2 dt$$

which implies

$$\liminf_{h\to 0+} h^{-1}C(h) \geqq \pi^{-1}\int_0^A |f(t)|^2 dt.$$

Since $A > 0$ is arbitrary, (a) $\Rightarrow$ $f(t) \in L^2(0,\ \infty)$ which is no more than $f(t) \in L^2(-\infty,\ \infty)$. Hence (a) $\Rightarrow$ (c). Suppose (c) is true. Divide both sides of the second relation of (1.1) by h and let h tend to $0+$. Then by the dominated convergence theorem, we have (b) and (2.1) of (ii) at the same time. Thus (c) ⇨ (b) and (c) ⇨ (2.1). (b) ⇨ (a) is trivial. (c) ⇨ (d) is well known. (e) ⇨ (a) is easily obtained from the first relation of (2.2). (a) ⇨ (e) is also simple to show, for $G^s(x)$ is nondecreasing and we see that from (2.2)

$$C(h)=h^{-1}\int_0^h G^s(x)dx \geqq h^{-1}\int_{h/2}^h G^s(x)dx \geqq 1/2\, G^s(h/2).$$

Thus the proof of Theorem 1 is complete.

We mentioned in Section 1, (1.5) that $C(h) \longrightarrow \boldsymbol{M}(|f(t)|^2)$ as $h \longrightarrow 0$. Now we shall give a result on the order of $C(h)-\boldsymbol{M}(|f(t)|^2)$ approaching zero when $h \longrightarrow 0$.

We introduce the quantity

(2.4)
$$\boldsymbol{R}(x)=x^{-1}\int_0^x |\boldsymbol{M}(|f(t)|^2,\ u)-\boldsymbol{M}(|f(t)|^2)|du$$

which obviously converges to zero as $x \longrightarrow \infty$.

We are going to show

Theorem 2. *There exists an absolute constant A such that*

(2.5)
$$|C(h)-\boldsymbol{M}(|f(t)|^2)| \leqq A[\sup_{x\geqq 1/h}\boldsymbol{R}(x)]^{1/2},\ h>0.$$

Proof. For any $a>0$, we have

$$C(h)-\boldsymbol{M}(|f(t)|^2)=2\pi^{-1}\int_0^\infty [|f(2u/h)|^2-\boldsymbol{M}(|f(t)|^2)]\,(\sin u/u)^2 du$$

$$=2\pi^{-1}\int_0^a +2\pi^{-1}\int_a^\infty = I_1+I_2,$$

say. We note that $||f(2u/h)|^2-\boldsymbol{M}(|f(t)|^2)| \leqq 1$, for $|f(t)|^2$ and $\boldsymbol{M}(|f(t)|^2)$ are non-negative and not greater than 1. From this, we have

(2.6)
$$I_2 \leqq 2\pi^{-1}\int_a^\infty (\sin u/u)^2 du \leqq 2\pi^{-1}\int_a^\infty (1/u)^2 du = 2/(a\pi).$$

Integration by parts applied to I_1 gives us

$$I_1=(2\sin^2 a/\pi a)a^{-1}\int_0^a [|f(2v/h)|^2-\boldsymbol{M}(|f(t)|^2)]\,dv$$

$$-2\pi^{-1}\int_0^a udu(d(\sin^2 u/u^2)/du)u^{-1}\int_0^u [|f(2v/h)|^2-\boldsymbol{M}(|f(t)|^2)]\,dv$$

$$\leqq 2/(\pi a)+2\pi^{-1}\int_0^a u|d(\sin^2 u/u^2)/du|\,|\boldsymbol{M}(|f(t)|^2,\ 2u/h)-\boldsymbol{M}(|f(t)|^2)|du.$$

Since $u|d(\sin^2 u/u^2)/du|\leqq 4$, for all $u>0$, the last one is not greater than

$$2/(\pi a)+(8a/\pi)\boldsymbol{R}(2ah^{-1}). \tag{2.7}$$

Now choose $a=[\sup_{x\geqq 1/h}\boldsymbol{R}(x)]^{-1/2}$. Since $\boldsymbol{R}(x)\leqq 1$, $a>1$. Hence $2ah^{-1}>h^{-1}$ and

$$\boldsymbol{R}(2ah^{-1})\leqq \sup_{x\geqq 2a/h}\boldsymbol{R}(x)\leqq \sup_{x\geqq 1/h}\boldsymbol{R}(x).$$

From (2.7) we have

$$I_1\leqq 2/(\pi a)+8a\pi^{-1}[\sup_{x\geqq 1/h}\boldsymbol{R}(x)]=10\pi^{-1}[\sup_{x\geqq 1/h}\boldsymbol{R}(x)]^{1/2}.$$

This together with (2.6) by the same selection of a, gives us

$$|C(h)-\boldsymbol{M}(|f(t)|^2)|\leqq 12\pi^{-1}[\sup_{x\geqq 1/h}\boldsymbol{R}(x)]^{1/2}.$$

This proves the theorem.

From this theorem, (1.5) follows. The above proof, however, is an adaptation of a classical proof of Wiener formula ([5], p. 182).

3. Behavor of $C(h)$ for large h.

As was mentioned in section 1, $C(h)\longrightarrow 1$ as $h\longrightarrow\infty$. We here discuss the

behavior of $C(h)$ when $h \longrightarrow \infty$ a little bit more.
We shall begin with

Theorem 3. (i) *The following four statements are equivalent to each other*:

(a) $1-C(h)=O(h^{-1})$, *as* $h \longrightarrow \infty$,
(b) $\lim_{h\to\infty} h[1-C(h)]$ *exists*,
(c) $E|X|<\infty$,
(d) $\int_0^\infty x dG^s(x)<\infty$.

(ii) *If either statement in* (i) *holds, then*

$$\lim_{h\to\infty} h[1-C(h)]=\int_0^\infty x dG^s(x)=\int_{-\infty}^\infty |x| dF^s(x). \tag{3.1}$$

Proof. From Lemma 1 (2.3), (a) $\Rightarrow$ (d) is immediate.
$\int_0^\infty x dG^s(x)=\int_{-\infty}^\infty |x| dF^s(x)$ which is the right equality of (3.1), is obvious by the definition of $G^s(x)$ and then (d) is equivalent to $\int_{-\infty}^\infty |x| dF^s(x)<\infty$ which is, in turn, equivalent to (c), because of Lemma 2. Hence (c) $\Longleftrightarrow$ (d). If (d) holds, then

$$h\int_h^\infty dG^s(x)\leqq \int_h^\infty x dG^s(x)$$

which converges to zero as $h \longrightarrow \infty$. Therefore from Lemma 1 (2.3), $h(1-C(h))$ converges to $\int_0^\infty x dG^s(x)$. Thus (d) $\Rightarrow$ (b) is proved. (b) $\Rightarrow$ (a) is trivial. The proof of (i) is now complete. The proof of (ii) is involved in the above proof of (d) $\Rightarrow$ (b).

We here give a proposition,

Proposition 1. *If X is nondegenerate at the origin, then $1-C(h)$ cannot be $o(h^{-1})$ as $h \longrightarrow \infty$.*

If $1-C(h)=o(h^{-1})$, then (b) holds and from (3.1), $\int_0^\infty |x| dF^s(x)=0$, which implies that X is degenerate at the origin.
We now take the quantity

$$1-\boldsymbol{M}(|f(t)|^2, x)=\boldsymbol{M}(1-|f(t)|^2, x), \tag{3.2}$$

which plays a great deal as well as $\boldsymbol{M}(|f(t)|^2, x)$.

We here consider the behavior of (3.2) with $x=h^{-1}$ when $h \longrightarrow \infty$. As a matter of fact there are some inequality relations between $1-\boldsymbol{M}(|f(t)|^2, h^{-1})$ and $1-C(h)$ as we shall see in Corollary 1 of Theorem 6 in the following section. Let X be nondegenerate. As we know (see for instance [16]), there are positive constants δ and C such that

$$|f(t)|\leqq 1-Ct^2, \text{ for } |t|<\delta. \tag{3.3}$$

Using this, we readily have

$$\begin{aligned}\boldsymbol{M}(1-|f(t)|^2, h^{-1})&=h\int_0^{1/h}(1-|f(t)|^2)dt\\ &\geqq Ch\int_0^{1/h} t^2dt\end{aligned}$$

for $h>\delta^{-1}$. Namely, for some constant $C>0$,

$$\boldsymbol{M}(1-|f(t)|^2, h^{-1})\geqq Ch^{-2}/3, \tag{3.4}$$

for $h>\delta^{-1}$.

On the other hand, we obtain

Theorem 4. *We have*

$$\boldsymbol{M}(1-|f(t)|^2, h^{-1})=O(h^{-2}), \tag{3.5}$$

for large h, if and only if $EX^2<\infty$.

We note that $EX^2<\infty$ is equivalent to

$$\int_0^\infty x^2dG^s(x)<\infty, \tag{3.6}$$

because of Lemma 2.

Proof. Since $|f(t)|^2=\int_0^\infty \cos xtdG^s(x)$,

$$\boldsymbol{M}(1-|f(t)|^2, h^{-1})=h\int_0^{1/h}(1-\int_0^\infty \cos xtdG^s(x))dt \tag{3.7}$$

$$=\int_0^\infty [1-(\sin (x/h))/(x/h)]dG^s(x)$$

$$=\int_0^h+\int_h^\infty .$$

Noting that

$$1-\sin u/u \geqq Au^2, \quad 0<u\leqq 1,$$

$$\geqq A, \qquad u>1,$$

for an absolute constant A, we see, from (3.7), that

$$\text{(3.8)} \qquad \boldsymbol{M}(1-|f(t)|^2,\ h^{-1})\geqq A[h^{-2}\int_0^h x^2dG^s(x)+\int_h^\infty dG^s(x)].$$

Hence $$A\int_0^h x^2dG^s(x)\leqq \boldsymbol{M}(1-|f(t)|^2,\ h^{-1})h^2=O(1)$$

if (3.5) holds. And then $EX^2<\infty$.

Conversely suppose $EX^2<\infty$. Then (3.6) holds and

$$\text{(3.9)} \qquad \int_h^\infty dG^s(x)\leqq h^{-2}\int_h^\infty x^2dG^s(x)=o(h^{-2}), \text{ as } h \longrightarrow \infty.$$

Since $$0<1-\sin u/u\leqq Bu^2, \text{ for } 0<u\leqq 1,$$

$$\leqq B, \text{ for } u>1,$$

for some absolute constant B, we have, from (3.7)

$$\text{(3.10)} \qquad \boldsymbol{M}(1-|f(t)|^2,\ h^{-1})\leqq B\left[h^{-2}\int_0^h x^2dG^s(x)+\int_h^\infty dG^s(x)\right].$$

(3.6) and (3.9) then imply (3.5) which completes the proof.

4. Inequality relations between $\boldsymbol{M}(|f|(t)^2,\ h^{-1})$ and $C(h)$

We begin with the following theorem.

Theorem 5. *There exist two absolute constants A and B such that, for any $h>0$,*

$$AC(h)\leqq \boldsymbol{M}(|f(t)|^2,\ h^{-1})\leqq BC(h). \tag{4.1}$$

Similar inequalites were obtained for the concentration function $Q(h)$ by Esseen, from which (4.1) readily follows in view of (1.3). We give here a direct proof of (4.1) although it is no more than an adaptation of Esseen's, for the left inequality of (4.1).

Proof. Since
$$\sin^2(x/2)/(x/2)^2=\int_{-1}^{1}(1-|t|)\,e^{-itx}dt,$$

we have

$$\int_{-\infty}^{\infty}\{\sin^2(x/2h)/(x/2h)^2\}\,dF^s(x)=h\int_{-1/h}^{1/h}(1-h|t|)\,dt\int_{-\infty}^{\infty}e^{-tix}dF^s(x)$$

$$=h\int_{-1/h}^{1/h}(1-h|t|)|f(t)|^2dt$$

$$\leqq h\int_{-1/h}^{1/h}|f(t)|^2dt$$

$$=2\boldsymbol{M}(|f(t)|^2,\ h^{-1}). \tag{4.2}$$

On the other hand

$$\int_{-\infty}^{\infty}\{\sin^2(x/2h)/(x/2h)^2\}\,dF^s(x)\geqq\int_{-h}^{h}\geqq(2/\pi)^2\int_{-h}^{h}dF^s(x)$$

$$=(2/\pi)^2G^s(h),$$

which is, from (2.2) and monotoneness of $G^s(h)$, not less than $(2/\pi)^2C(h)$. Hence looking at (4.2), we have the left inequality of (4.1).

The right inequality of (4.1) is almost obvious. Since

$$(\pi/2)^2\sin^2(ht/2)/(ht/2)^2\geqq 1,\ \text{for}\ |t|\leqq h^{-1},$$

$$\boldsymbol{M}(|f(t)|^2,\ h^{-1})\leqq(\pi^2/2)\int_0^{1/h}[\sin^2(ht/2)/(ht^2/2)]|f(t)|^2dt$$

$$\leqq (\pi^2/2)\int_0^\infty = (\pi^3/4)\,C(h),$$

which completes the proof.

(4.1) is of particular interest when h is small. For large h, we will have Theorem 6 below.

Lemma 3.

$$2\,[1-C(h)] \geqq D^2(X^s,\ h) \geqq [1-C(h)]^2. \tag{4.3}$$

Proof. From Lemma 1 (2.3), we have by Schwarz inequality,

$$1-C(h) = h^{-1}E(\min(|X^s|,\ h)) \leqq h^{-1}\,[E(\min(|X^s|,\ h))^2]^{1/2}$$

$$= D(X^s,\ h),$$

which proves the right inequality of (4.3).

$D^2(X^s,\ h)$ is from (1.12), written by

$$-h^{-2}\int_0^h x^2 d(1-G^s(x)) + \int_h^\infty dG^s(x)$$

which becomes, by integration by parts

$$2\,h^{-2}\int_0^h x(1-G^s(x))dx \leqq 2\,h^{-1}\int_0^h (1-G^s(x))dx,$$

$$= 2\,[1-C(h)]$$

in view of the first equality of Lemma 1 (2.2). This proves the left inequality of (4.3).

Theorem 6. *There exist absolute constants* $B > A > 0$ *such that*

$$AD^2(X^s,\ h) \leqq \boldsymbol{M}(1-|f(t)|^2,\ h^{-1}) \leqq BD^2(X^s,\ h). \tag{4.4}$$

This was already proved in (3.8) and (3.10). (4.4) combines with Lemma 3 to yield the following Corollary.

Corollary 1. *There exist absolute constants* A, $B>0$ *such that*

(4.5) $$A[1-C(h)]^2 \leqq \boldsymbol{M}(1-|f(t)|^2, h^{-1}) \leqq B[1-C(h)].$$

We remark that the left inequality is of a best possible type in the sense that the left hand side can not be replaced by $A[1-C(h)]^\alpha$, $0<\alpha<2$. Because if the left inequality in (4.5) holds with $A[1-C(h)]^\alpha$ in place of $A[1-C(h)]^2$, then by Theorem 4 we have $[1-C(h)]^\alpha=O(h^{-2})$ whenever $EX^2<\infty$, from which $1-C(h)=o(h^{-1})$. This contradicts Proposition 1.

5. Characteristic function and mean concentration function.

We know (3.3) for the behavior of the characteristic function near the origin. We here give another form of it in connection with mean concentration. What we are going to show is the following

Theorem 7. *For all* $t>0$, *we have*

(5.1) $$1-|f(t)|^2 \leqq (\pi^2/2)D^2(X^s, \pi/t).$$

If $G^s(x)$ *is concave on* $(0, \infty)$, *then we have*

(5.2) $$1-|f(t)|^2 \geqq (1/4)D^2(X^s, \pi/t).$$

Corollary 2. *For all* $t>0$, *we have*

(5.3) $$1-|f(t)|^2 \leqq \pi^2[1-C(\pi/t)]$$

and if $G^s(x)$ *is concave on* $(0, \infty)$, *then we have*

(5.4) $$1-|f(t)|^2 \geqq (1/4)[1-C(\pi/t)]^2.$$

Corollary 2 is immediate from Theorem 7 because of Lemma 3. We remark that (5.1) is sharper than the right inequality of (4.4), for integrating the both side of (5.1) over $(0, 1/h)$ and noting that for $0<t<1/h$,

$$D^2(X^s, \pi/t) \leqq D^2(X^s, h),$$

we have the right inequality of (4.4).

Proof of Theorem 7. The proof of (5.1) is quite simple. Actually

$$1-|f(t)|^2=\int_0^\infty(1-\cos tx)\,dG^s(x)$$

$$=2\int_0^\infty \sin^2(tx/2)\,dG^s(x)$$

$$\leqq(t^2/2)\int_0^{\pi/t}x^2dG^s(x)+\int_{\pi/t}^\infty dG^s(x)$$

$$\leqq(\pi^2/2)D^2(X^s,\ \pi/t).$$

We shall now show the second statement. As above,

$$1-|f(t)|^2=2\int_0^{\pi/t}\sin^2(tx/2)\,dG^s(x)+\int_{\pi/t}^\infty(1-\cos tx)\,dG^s(x)$$

$$\geqq 2\int_0^{\pi/t}\sin^2(tx/2)\,dG^s(x)+(1/4)\int_{\pi/t}^\infty(1-\cos tx)\,dG^s(x)$$

$$\geqq 2(t/\pi)^2\int_0^{\pi/t}x^2dG^s(x)+(1/4)\int_{\pi/t}^\infty dG^s(x)+ \tag{5.5}$$

$$+(1/4)\int_{\pi/(2t)}^{\pi/t}\cos tx dG^s(x)+(1/4)J,$$

say, where

$$J=-\int_{\pi/(2t)}^\infty \cos tx dG^s(x)$$

$$=-\sum_{k=1}^\infty\int_{(2k-1)\pi/(2t)}^{(2k+1)\pi/(2t)}\cos xt dG^s(x)$$

$$=-\sum_{k=1}^\infty(-1)^k\int_0^\pi \sin u\, dG^s((2k-1)\pi/(2t)+u/t)$$

$$=\sum_{k=1,3,5,\ldots}\int_0^\pi \sin u\, d\,[G^s((2k-1)\pi/(2t)+u/t)$$

$$-G^s((2k+1)\pi/(2t)+u/t)].$$

Since $G^s(x)$ is concave, $[G^s((2k-1)\pi/(2t)+u/t)-G^s((2k+1)\pi/(2t)+u/t)]$ is for every odd integer k, nondecreasing as a function of $0<u<\pi$ and every term of the last series is nonnegative. Thus J is nonnegative.

Therefore from (5.5)

$$1-|f(t)|^2 \geqq 2(t/\pi)^2 \int_0^{\pi/t} x^2 dG^s(x)+(1/4)\int_{\pi/t}^{\infty} dG^s(x)-$$

$$-(1/4)\int_{\pi/(2t)}^{\pi/t} dG^s(x).$$

Now

$$(1/4)\int_{\pi/(2t)}^{\pi/t} dG^s(x) \leqq (t/\pi)^2 \int_{\pi/(2t)}^{\pi/t} x^2 dG^s(x)$$

$$\leqq (t/\pi)^2 \int_0^{\pi/t} x^2 dG^s(x).$$

Putting this in the above, we have

$$1-|f(t)|^2 \geqq (t/\pi)^2 \int_0^{\pi/t} x^2 dG^s(x)+(1/4)\int_{\pi/t}^{\infty} dG^s(x)$$

$$\geqq (1/4) D^2(X^s, \pi/t),$$

which completes the proof of (5.2).

6. Remarks on the mean concentration of sums of independent random variables

Let

(6.1) $$X_1, X_2, X_3, \dots, X_n$$

be mutually independent random variables and write $S_n=\sum_{k=1}^n X_k$.

For the Lévy concentration function $Q(S_n, h)$ of S_n, the following Kolmogorov Rogozin inequality is known [7], [14], [15]:

(6.2) $$Q(S_n, h) \leqq Ah\,[\sum_{k=1}^n h_k^2(1-Q(X_k, h_k))]^{-1/2}$$

for any $0<h_k\leqq h$, $k=1, 2, 3, \ldots, n$ and for some absolute constant A. (6.2) was also proved by Esseen [3], [4] by an elegant characteristic function method. Kesten [6] gave the essential sharpening of (6.2) by using some combinatorial argument. By adopting the Kesten's idea, Miroshnikov and Rogozin [9] finally obtained the following inequality.

$$Q(S_n, h)\leqq Ah\,[\textstyle\sum_{k=1}^{n} h_k^2 D^2(X_k^s, h_k)Q^{-2}(X_k, h_k)]^{-1/2}, \tag{6.3}$$

for $0<h_k\leqq h$, $k=1, 2, 3, \ldots, n$ and for some absolute constant A.

On the other hand Postnikova and Yudin [13] have shown

$$Q(S_n, h)\leqq Ah\,[\textstyle\sum_{k=1}^{n} h_k^2(1-Q(X_k, h_k))Q^{-2}(X_k, h)]^{-1/2}, \tag{6.4}$$

for $0<h_k\leqq 2h$ and for some absolute constant A. Their proof depends on the following interesting lemma [2, lemma 2].

Lemma 4. *Let* $a=\min_{t\in[-h,h]}|f(t)|$, $\phi^*=\arccos(a)$.

If $0\leqq\phi\leqq\phi^*/4$ *and* $E_\phi=\{t\in[-h, h];\ |f(t)|\geqq\cos\phi\}$,
Then

$$mE_\phi\leqq(48\,\phi h/\phi^*)\boldsymbol{M}(|f(t)|^2, h), \tag{6.5}$$

where mE_ϕ *is the Lebesgue measure of the set* E_ϕ.

For $C(S_n, h)$, Matsumoto [13] anounced that

$$C(S_n, h)\leqq Ah\,[\textstyle\sum_{k=1}^{n} h_k^2(1-C(X_k, h_k))^2C^{-2}(X_k, h)]^{-1/2}, \tag{6.6}$$

for $0<h_k\leqq 2h$ and for some absolute constant A.

This inequality when (6.1) are independently and identically distributed, has been shown by Ananevskii [2]. The following lemma is included in the proof of his result.

Lemma 5. *There exists an absolute constant* A *such that for all* $h>0$ *and any number* $r\geqq 1$,

$$\boldsymbol{M}(|f(t)|^r, h^{-1})\leqq AC(X, h)[r^{1/2}D(X^s, h)]^{-1}. \tag{6.7}$$

Ananevskii proved (6.7) with $D(X^s, \pi h)$ in place of $D(X^s, h)$. It is, however not substantially different.

We remark that once (6.7) holds for any positive integer r, it does for any $r \geqq 1$. Because supposing (6.7) holds for an positive integer n in place of r, then for any r with $n<r<n+1$,

$$\boldsymbol{M}(|f(t)|^r, h^{-1}) \leqq \boldsymbol{M}(|f(t)|^n, h^{-1})$$

$$\leqq AC(X, h)[n^{1/2}D(X^s, h)]^{-1}.$$

$$\leqq 2AC(X, h)[r^{1/2}D(X^s, h)]^{-1}.$$

Using Lemma 5, we prove

Theorem 8. *Let* $0<h_k \leqq h$, $k=1, 2, 3, \dots, n$. *Then there exists an absolute constant* A *such that*

$$C(S_n, h) \leqq Ah\left[\sum_{k=1}^{n} h_k^2 D^2(X_k^s, h_k) C^{-2}(X_k, h_k)\right]^{-1/2}. \tag{6.8}$$

This is an analogue of (6.3). Before going to prove this, we give a remark that the condition $0<h_k \leqq h$, $k=1, 2, 3, \dots, n$ can be replaced by $0<h_k \leqq ph$ for any $p \geqq 1$, but in this case A must be replaced by pA, because $C(S_n, h)$ is nondecreasing for h. Another remark is that (6.6) follows from (6.8) because of Lemma 3.

Proof. The proof is basically along the idea of Esseen proving (6.2). For $k=1, 2, 3, \dots, n$, write

$$p_k = h_k^2 D^2(X_k^s, h_k) C^{-2}(X_k, h_k), \quad P_k = p_k^{-1} \sum_{j=1}^{n} p_j.$$

We have

$$\sum_{k=1}^{n} P_k^{-1} = 1.$$

In what follows, A_1 and A_2 are absolute constants. By Theorem 5 we see that

$$C(S_n, h) \leqq A_1 \boldsymbol{M}\left(\prod_{k=1}^{n} |f_k(t)|^2, h^{-1}\right)$$

$$= A_1 h \int_0^{1/h} \prod_{k=1}^{n} |f_k(t)|^2 dt$$

which is, because of Hölder inequality, not greater than

$$A_1 h\, \Pi_{k=1}^{n} \left(\int_0^{1/h} |f_k(t)|^{2P_k}\, dt \right)^{1/P_k}$$

$$\leqq A_1 h\, \Pi_{k=1}^{n} \left(\int_0^{1/h_k} |f_k(t)|^{2P_k}\, dt \right)^{1/P_k},$$

$$= A_1 h\, \Pi_{k=1}^{n} [h_k^{-1} \boldsymbol{M}(|f_k(t)|^{2P_k},\ h_k^{-1})]^{1/P_k}.$$

Because of Lemma 5, the last one is not greater than

$$A_1 h\, \Pi_{k=1}^{n} \{h_k^{-1} A_2 C(X_k,\ h_k)[(2P_k)^{1/2} D(X_k^s,\ h_k)]^{-1}\}^{1/P_k}$$

$$= 2^{-1/2} A_1 A_2\, \Pi_{k=1}^{n} (\textstyle\sum_{j=1}^{n} p_j)^{-1/(2P_k)}$$

$$= 2^{-1/2} A_1 A_2 (\textstyle\sum_{j=1}^{n} p_j)^{-1/2}$$

which proves the theorem.

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