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STRESS ANALYSIS OF TWO ELASTIC ANNULAR PLATES, CONNECTED EACH OTHER BY ELASTIC RODS—II

FUMIKI KITO*

Department of Mechanical Engineering, Keio University, Hiyoshi, Yokohama 223, Japan

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ABSTRACT

In the previous report, the author has given an analytical study of two elastic annular plates, which are placed concentrically in parallel each other, and which are interconnected each other by a number of elastic rods. These elastic rods were assumed to be arranged in a single row along a concentrical circle. The present paper in the outcome of continuation of the author's previous study, and gives the result of analytical study about the case in which two rows of rods are arranged along two different (but concentric) circles. Results of this analytical study were evaluated numerically for the case of the number of rods along each circular orbits are, N=6, 12, and 18. The rods (as done in the previous report) assumed to suffer a thermal expansion probably due to the external heating. The two (top and bottom) annular elastic plates aɹe assumed to be of different thickness. The author's intention is to supply some suggestion in connection with the structural design of once-through boilers and also of guide-rings of hydraulic turbines. The results of numerical evaluation are shown as graphs.

1. Introduction.

In the present paper, we consider, as before, the case of elastic plates, which are placed horizontally, keeping at a vertical distance l apart, with each other. These two elastic plates are taken to be of annular forms (with outer radius R and inner radius cR), and to be placed concentrically each other. These two annular elastic plates are taken to be connected each other by means of a number of vertical elastic rods of lengths l. Moreover, these elastic rods, in number N, and in two circular rows, are to be attached to the elastic plates, along two rows of points which lie on the circles of radii aR and bR ($c<b<a<1$). They are arranged each, equidistantly, by keeping at equal angular distances of $2\pi/N$.

In the author's previous report (Ref. (2)), we considered the case of only a

*) Prof. Emeritus, Keio Univ.

single row of these elastic rods, whereas in the present study, we consider the case of two rows of such elastic rods, while the other things are the same as in previous report. The arrangements of elastic plates and rods are, as sketched in Fig. 1.

Our purpose is, as before, to study analytically the state of deformations and stresses, which are caused when elastic bars exert transverse force to annular elastic plates. These transverse forces are thought to be generated by their thermal elongations due to probably the effect of external heating.

The mode of fixation of elastic plates are taken here to be the case of clamped edges, both along inner and outer boundary circumferences.

As to the practical use of our study, we may mention that it represents, in an idealized form, the problem of structural strengths of supporting bed-plates of once-through boilers. We may mention that it represents, also in an idealized form, the problem of strengths of guide-rings of hydraulic turbines.

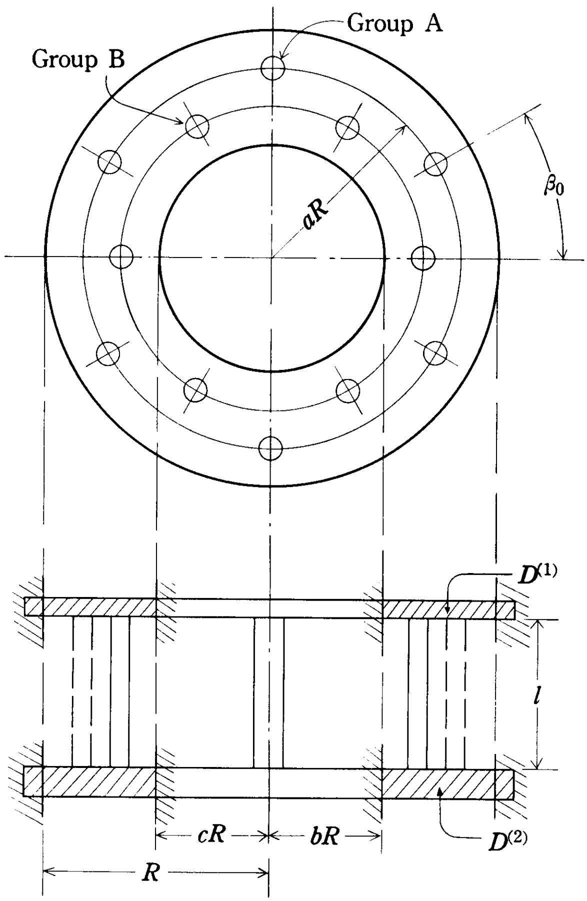

Fig. 1. Two Annular Elastic Plates Connected each other by several Elastic Bars. (The Figure shows us the Case of $N=6$, $\beta_0=\pi/6$)

The results of our analytical study were evaluated numerically, adopting a suitably chosen case of $a=0.8333$, $b=0.6666$, and $c=0.5000$. This numerical estimation was carried out for various values of the ratio $Z=D^{(1)}/D^{(2)}$ (the ratio of flexural rigidities of upper and bottom elastic plates).

The results of this numerical estimation are shown here as graphs (Fig. 2 to Fig. 7). In these graphs, there are shown variation of values of thrust forces imposed on elastic plates due to the action of elastic rods. Also, maximum bending moments caused thereby, on (inner and outer) edge-lines of annular elastic plates. There are shown mode of variation of such bending moments as functions of the argument $Z=D^{(1)}/D^{(2)}$, as mentioned above.

2. Notation.

The notations used here, being fundamentally the same as for previous report, are as follows; (r, θ)=polar coordinates of any point on the middle plane of our annular elastic plate; h=thickness (uniform) of the plate; (R, cR) outer and inner radius of the annular plate; (aR, bR)=radii of circular contour lines, along which concentrated loads (by means of elastic rods) are imposed. $(c<b<a<1)$, (E, ν)= Young's modulus and Poisson's ratio of the plate material; w=transverse displacement of middle plane of elastic plate (infinitesimally small, and taken as positive downwards); w_1=those values of w, which belong to the region of inside the circle of applications of concentric loads; $\nabla\equiv$two dimensional Laplace operator; $x=r/R$= no dimensional variable, expressing radial position of any point on the middle plane; P=transverse concentrated load, which is assumed to act on the plate (due to pushing action of elastic rod, and taken as positive downwards), $D=Eh^3/[12(1-\nu^2)]$=flexural rigidity of the elastic plate.

Numerical coefficients R_0, R_1, $\cdots$, S_0, S_1, $\cdots$ (these being functions of the variable x), are used to express the displacement w in form of infinite series. Indices, such as $D^{(k)}(k=1,2)$ are used in order to discriminate to which one of the elastic plate ($k=1$, upper; $k=2$, lower) it refers to.

3. Statement of our Problem, and Fundamental Equations which Represent it

In the present paper, we take up, as was done in the previous paper, the case of two annular elastic plates, as shown in Fig. 1, and which are built up in following manner; Each one of them has outer radius R and inner radius cR, and are concentrically located, keeping at a vertical distance of l apart, between them. These two elastic plates (No. 1 upper, and No. 2 lower) are interconnected each other by a number N of elastic rods of lengths l. These elastic rods are taken to be fixed (at both ends) to plates at N points of circumferences of radii aR and bR $(c<b<a<1)$. Thus, there are to be two circular rows of elastic rods, each in number N, whereas in the previous paper we have considered only a single row (circular) of elastic rods. Each bar is to be placed at equal distance respectively.

To fix our ideas, we consider the case in which inner and outer boundary edges of each annular elastic plate are fixed to rigid wall (in state of so-called clamped edges).

Among the various cases possible to occur, the author has confined himself to the case in which the connecting elastic rods suffer a kind of thermal expansion, thus generating axial forces which act upon two (upper and lower) annular elastic plates.

The fundamental differential equqtion of equilibrium of an elastic plate can be written (Ref. (1));

$$\Delta^2 w=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 w=0 \tag{1}$$

This equqtion may also be rewritten in the following form,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)^2 w=0 \tag{2}$$

by using the non-dimensional variable $x=r/R$. The equations (1) and (2) are given with regard to infinitesimally small displacement w, under the assumption that no external load act upon the surface of elastic plate in consideration. The stress resultant, force and moment, caused by the deformation w, are given by;

$$Q_x=-\frac{D}{R}\frac{\partial}{\partial x}(\Delta w), \qquad Q_t=-\frac{D}{R}\frac{\partial}{x\partial\theta}(\Delta w)$$

$$M_r=-\frac{D}{R^2}\left[\frac{\partial^2 w}{\partial x^2}+\nu\left(\frac{1}{x}\frac{\partial w}{\partial x}+\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]$$

$$M_t=-\frac{D}{R^2}\left[\frac{1}{x}\frac{\partial w}{\partial x}+\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}+\nu\frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{rt}=(1-\nu)\frac{D}{R^2}\left[\frac{1}{x}\frac{\partial^2 w}{\partial x\partial\theta}-\frac{1}{x^2}\frac{\partial^2 w}{\partial\theta^2}\right] \tag{9}$$

In order to obtain the solution of our problem, we assume the following form for the displacement

$$w=R_0+\sum_{m=1}^{\infty}R_m\cos m\theta \tag{3}$$

where R_0, R_1, $\cdots$, are functions of x. The boundary conditions to be satisfied by the displacement w are;

(a) Along $r=R(x=1)$, which is fixed-edge line,

$$w=0, \qquad \partial w/\partial x=0$$

(b) Along $r=cR(x=c)$, which also is fixed edge line,

$$w_1=0, \qquad \partial w_1/\partial x=0$$

It is to be noted that, we denote by w_1 the value of w which belongs to the region of $cR\leq\pi\leq bR$ $(c\leq x\leq b)$.

(c) Along the circle $r=bR$ $(x=b)$, two function of x (namely w and w_1) must be continuous each other, up to derivatives of second order, thus; $w=w_1$, $\partial w/\partial x=\partial w_1/\partial x$, $\partial^2 w/\partial x^2=\partial^2 w_1/\partial x^2$

(d) Assuming that, a concentrated load of amount P acts transversally on a point $(x=b,\ \theta=0)$, we have,

$$D\left|\frac{\partial}{\partial r}(\Delta w)\right|^{x=b}-D\left|\frac{\partial}{\partial r}(\Delta w_1)\right|^{x=b}$$

$$=\frac{P}{\pi bR}\left[\frac{1}{2}+\sum_{m=1}^{\infty}\cos m\theta\right]$$

It is to be understood that the r.h.s. of this eq. is being used in the sense of distribution (generalized function) of L. Schwartz.

In the above conditions (a) to (d), we are referring to the case of a single transverse load P which act on only an isolated point $(b,\ 0)$ on the face of a single elastic plate. On the ground of solution of this simplest case, we deduce by means of linear superposition, the solution for the case in which a number N of loads P, are arranged along a circular line of radius bR $(x=b)$. Moreover, by further deduction, we obtain the solution in which two circular rows of transverse loads $P_A^{(k)}$, $P_B^{(k)}$ are arranged along two circular lines of radii aR and bR $(x=a$ and $x=b)$.

4. Summary of Analytical Solution obtained in the Previous Paper.

The general solution of our fundamental problem was given in form of the infinite series (3), where R_0, R_1, $\cdots$ are functions of x. For convenience, we divide the whole region $c\leq x\leq 1$ into outer part $b\leq x\leq 1$ and inner part $c\leq x\leq b$.

(A) For outer part $b\leq x\leq 1$, we have

$$w = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta \tag{3a}$$

with

$$R_0 = A_0 + B_0 x^2 + C_0 \log x + D_0 x^2 \log x$$

$$R_1 = A_1 x + B_1 x^3 + C_1 x^{-1} + D_1 x \log x$$

$$R_m = A_m x^m + B_m x^{-m} + C_m x^{m+2} + D_m x^{-m+2}$$

for $2 \leq m$. A_i, B_i, C_i and D_i ($i = 0, 1, 2, \cdots$) are numerical constants which are determined by above mentioned boumdary conditions (a), (b), (c) and (d).

(B) For inner part, $c \leq x \leq b$,

$$w_1 = S_0 + \sum_{m=1}^{\infty} S_m \cos m\theta \tag{4}$$

with

$$S_0 = E_0 + F_0 x^2 + G_0 \log x + H_0 x^2 \log x$$

$$S_1 = E_1 x + F_1 x^3 + G_1 x^{-1} + H_1 x \log x$$

$$S_m = E_m x^m + F_m x^{-m} + G_m x^{m+2} + H_m x^{-m+2}$$

for $2 \leq m$. E_i, F_i, G_i and H_i are numerical constants to be determined by the boundary condition. The values of these numerical constants thus obtained, are as follows:

(A) For the case of $m = 0$,

$$A_0 = \frac{1}{2}(C_0 + D_0), \qquad B_0 = -A_0,$$

$$C_0 = (\Delta_C/\Delta) H_0, \qquad D_0 = (\Delta_D/\Delta) H_0,$$

$$G_0 = (\Delta_G/\Delta) H_0, \qquad F_0 = -\frac{1}{2c^2} G_0 - \left(\log c + \frac{1}{2}\right) H_0,$$

$$(D_0 - H_0) = PR^2/(8\pi D),$$

$$E_0 = \left(\frac{1}{2} - \log c\right) G_0 + \frac{1}{2} c^2 H_0$$

in which we have,

$$\Delta=\begin{vmatrix}(1,1) & (2,1) & (3,1)\\ (1,2) & (2,2) & (3,2)\\ (1,3) & (2,3) & (3,3)\end{vmatrix},\qquad \Delta_c=\begin{vmatrix}X & (2,1) & (3,1)\\ Y & (2,2) & (3,2)\\ Z & (2,3) & (3,3)\end{vmatrix}$$

$$\Delta_D=\begin{vmatrix}(1,1) & X & (3,1)\\ (1,2) & Y & (3,2)\\ (1,3) & Z & (3,3)\end{vmatrix},\qquad \Delta_G=\begin{vmatrix}(1,1) & (2,1) & X\\ (1,2) & (2,2) & Y\\ (1,3) & (2,3) & Z\end{vmatrix}$$

Values of elements (1, 1), etc., are as follows,

$$(1,\ 1)=\frac{1}{2}(1-b^2)+\log b\,,$$

$$(2,\ 1)=\frac{1}{2}(1-b^2)+b^2\log b$$

$$(3,\ 1)=-\frac{1}{2}\left(1-\frac{b^2}{c^2}\right)-\log\frac{b}{c}\,,$$

$$(1,\ 2)=\frac{1}{b}-b,\qquad (2,\ 2)=2b\log \mathrm{b},$$

$$(3,\ 2)=-\frac{1}{b}+\frac{b}{c^2},\qquad (1,\ 3)=-\left(1+\frac{1}{b^2}\right),$$

$$(2,\ 3)=2(1+\log b)\,,\qquad (3,\ 3)=\frac{1}{b^2}+\frac{1}{c^2}$$

$$X=\frac{1}{2}(c^2-b^2)+b^2\log\frac{b}{c}$$

$$Y=2b\log\frac{b}{c}$$

$$Z=2\left(1+\log\frac{b}{c}\right)$$

(2) For the case of $m=1$,

$$A_1=-2C_1+\frac{1}{2}D_1, \qquad B_1=C_1-\frac{1}{2}D_1,$$

$$C_1=(\Delta_c/\Delta)H_1, \qquad D_1=(\Delta_D/\Delta)H_1$$

$$G_1=(\Delta_G/\Delta)H_1, \qquad F_1=\frac{1}{c^4}G_1-\frac{1}{2c^3}H_1$$

Δ, Δ_c, Δ_D and Δ_G are determinants, as mentioned obove, but their elements have following values,

$$(1,\ 1)=\frac{1}{b}(b^4-2b^2+1), \qquad (2,\ 1)=\frac{1}{b}\left(\frac{1}{2}-\frac{1}{2}b^4+b^2\log b\right)$$

$$(3,\ 1)=-\frac{1}{b}\left(-2\frac{b^2}{c^2}+\frac{b^4}{c^4}+1\right), \qquad (1,\ 2)=\frac{1}{b^2}(3b^4-2b^2-1)$$

$$(2,\ 2)=\frac{3}{2}-\frac{3}{2}b^2+\log b, \qquad (3,\ 2)=-\frac{1}{b^2}\left(-1+3\frac{b^4}{c^4}-2\frac{b^2}{c^2}\right)$$

$$(1,\ 3)=\frac{2}{b^3}(3b^4+1), \qquad (2,\ 3)=\frac{1}{b}(1-3b^2)$$

$$(3,\ 3)=-\frac{2}{b^3}\left(3\frac{b^4}{c^4}+1\right), \qquad X=\frac{1}{b}\left(1-\frac{b^2}{c^2}\right)+b\log\frac{b}{c}$$

$$Y=\frac{3}{2}\left(1-\frac{b^2}{c^2}\right)+\log\frac{b}{c}, \qquad Z=-\frac{3b}{c^2}+\frac{1}{b}$$

Also, we have,

$$(B_1-F_1)-\frac{2}{b^2}(D_1-H_1)=PR^2/(8\pi bD)$$

(3) For the case of $2\leq m$,

$$A_m=-\frac{m+1}{m}C_m-\frac{1}{m}D_m, \qquad B_m=\frac{1}{m}C_m-\frac{m-1}{m}D_m$$

$$F_m=\frac{1}{m}c^{2m+2}G_m-\frac{m-1}{m}c^2H_m, \qquad E_m=-\frac{m+1}{m}c^2G_m-\frac{1}{m}c^{-2m+2}H_m$$

$$C_m=(\Delta_c/\Delta)H_m, \qquad D_m=(\Delta_D/\Delta)H_m, \qquad G_m=(\Delta_G/\Delta)H_m$$

The elements in determinants Δ, Δ_c, $\cdots$, have following values,

$$(1,\ 1)=-\frac{m+1}{m}b^m+\frac{1}{m}b^{-m}+b^{m+2}$$

$$(2,\ 1)=-\frac{1}{m}b^m-\frac{m-1}{m}b^{-m}+b^{-m+2}$$

$$(3,\ 1)=\frac{m+1}{m}b^mc^2-\frac{1}{m}c^{2m+2}b^{-m}-b^{m+2}$$

$$(1,\ 2)=-(m+1)b^{m-1}-b^{-m-1}+(m+2)b^{m+1}$$

$$(2,\ 2)=-b^{m-1}+(m-1)b^{-m-1}-(m-2)b^{-m+1}$$

$$(3,\ 2)=(m+1)c^2b^{m-1}+c^{2m+2}b^{-m-1}-(m+2)b^{m+1}$$

$$(1,\ 3)=-(m-1)(m+1)b^{m-2}+(m+1)b^{-m-2}+(m+2)(m-1)b^m$$

$$(2,\ 3)=-(m-1)b^{m-2}-(m+1)(m-1)b^{-m-2}+(m-2)(m-1)b^{-m}$$

$$(3,\ 3)=(m-1)(m+1)b^{m-2}c^2-(m+1)b^{-m-2}c^{2m+2}-(m+2)(m+1)b^m$$

$$X=-\frac{1}{m}c^{-2m+2}b^m-\frac{m-1}{m}c^2b^{-m}+b^{-m+2}$$

$$Y=-b^{m-1}c^{-2m+2}+(m-1)b^{-m-1}c^2-(m-2)b^{-m+1}$$

$$Z=-(m-1)b^{m-2}c^{-2m+2}-(m+1)(m-1)b^{-m-2}c^2+(m-2)(m-1)b^{-m}$$

Also, we have

$$4m(m+1)b^{m-1}(C_m-G_m)+4m(m-1)b^{-m-1}(D_m-H_m)=R^2P/(\pi bD)$$

Bending moment M_r is given by the general formula

$$M_r=-\frac{D}{R^2}\left[\frac{\partial^2w}{\partial x^2}+\nu\left(\frac{1}{x}\frac{\partial w}{\partial x}+\frac{1}{x^2}\frac{\partial^2w}{\partial\theta^2}\right)\right] \tag{5}$$

but we note that the term containing the factor ν in it, vanishes at both boundaries. Thus we have following expressions for the values of bending moments M_r, which act along (outer and inner) edge-lines. Here they are given for respective component terms of $m=0, 1, 2, \cdots$.

For $m=0$,

at outer boundary,

$$(M_r)_0 = -\frac{D}{R^2}[2B_0 - C_0 + 3D_0]$$

at inner boundary,

$$(M_r)_0 = -\frac{D}{R^2}\left[2F_0 - \frac{1}{c^2}G_0 + (3+2\log c)H_0\right]$$

For $m=1$,

at outer boundary,

$$(M_r)_1 = -\frac{D}{R^2}[6B_1 + 2C_1 + D_1]$$

at inner boundary,

$$(M_r)_1 = -\frac{D}{R^2}\left[6cF_1 + \frac{2}{c^3}G_1 + \frac{1}{c}H_1\right]$$

For $2 \leq m$,

at outer boundary,

$$-\frac{R^2}{D}(M_r)_m = m(m-1)A_m + m(m+1)B_m + (m+1)(m+2)C_m + (m-2)(m-1)D_m$$

at inner boundary

$$-\frac{R^2}{D}(M_r)_m = m(m-1)c^{m-2}E_m + m(m+1)c^{-m-2}F_m$$

$$+(m+1)(m+2)c^m G_m + (m-2)(m-1)c^{-m}H_m$$

The above results refer, as was already mentioned, to the case of an isolated single transverse load P, acting on our annular elastic plate.

From it we can deduce, by linear combination, the case of a number N of loads P which are arranged equidistantly along the circle $r=bR$. Namely we have, by superposition,

$$w_s=\sum_{i=1}^{N}\left[\sum_{m=1}^{\infty} R_m \cos\left\{m\left(\theta-\frac{2\pi i}{N}\right)\right\}\right] \tag{6}$$

which may be given in simplified form as follows;

$$w_s=NR_0+N\sum R_m \cos m\theta \tag{6a}$$

wherein we have to take $m=Ns$ $(s=1, 2, 3, \cdots)$

5. Application of previously acquired Result to the Solution of Present Problem.

As a next step, we proceed to the application of results of above mentioned fundamental case, to the study of our present case. The present case consist of two annular elastic plates, which are interconnected by a number N of elastic rods which are arranged in a circular row. Moreover, in the present instance, we are to take up two circular rows of elastic rods, each in number N. In order to express quantities such as, D, P, which represent individual part they belong to, we use following index notations; $w_A^{(k)}$ $(k=1,2)$; values of displacements of upper $(k=1)$ and bottom $(k=2)$ elastic plates, caused by thrusts $P_A^{(k)}$. Similarly, for $w_B^{(k)}$ $(k=1,2)$. $P_A^{(k)}$ $(k=1,2)$; values of thrust forces exerted by elastic rods, which belong to group on circle of radius $r=aR$ ($k=1$ acting on upper elastic plate, $k=2$ acting on bottom plate).

$P_B^{(k)}$ $(k=1,2)$; values of thrust forces exerted by elastic rods, which belong to group on circle of radius $r=bR$ ($k=1$ acting on upper elastic plate, $k=2$ acting on bottom elastic plate)

Naturally, we have

$$P_A^{(k)}=(-)^k|P_A|, \qquad P_B^{(k)}=(-)^k|P_B|$$

$D^{(k)}$ $(k=1,2)$; values of flexural rigidities of upper and bottom elastic plates, their values being given by

$$D^{(k)}=E(h^{(k)})^3/[12(1-\nu^2)] \tag{7}$$

In the same manner, we denote by $A_i^{(k)}$, $B_i^{(k)}$, $\cdots$, values of numerical coefficients A_i, B_i, $\cdots$, which relate to upper ($k=1$) and lower ($k=2$) elastic plates respectively.

Using these notations, and applying the analytical results for fundamental case, which were described in the previous section, we arrive at the following expressions; At first, we remark that displacements $w_B^{(k)}$ are represented respectively in forms of infinite series, as follows;

(a) For $aR \leq r \leq R$

$$w_A^{(k)} = R_{A0}^{(k)} + \sum_{m=1}^{\infty} R_{Am}^{(k)} \cos(m\theta_A)$$

(b) For $cR \leq r \leq aR$

$$w_{1A}^{(k)} = S_{A0}^{(k)} + \sum_{m=1}^{\infty} S_{Am}^{(k)} \cos(m\theta_A)$$

(c) For $bR \leq r \leq R$

$$w_B^{(k)} = R_{B0}^{(k)} + \sum_{m=1}^{\infty} R_{Bm} \cos(m\theta_B)$$

(d) For $cR \leq r \leq bR$

$$w_{1B}^{(k)} = S_{B0}^{(k)} + \sum_{m=1}^{\infty} S_{Bm} \cos(m\theta_B)$$

where θ_A and θ_B are (variable) angular coordinates. Here we take $\theta_A = \theta$, $\theta_B = \theta - \beta_0$, where θ denotes the intrinsic angular coordinate (independent variable), and β_0 is the off-set angle between the arrangement of circular rows of group A and that of circular group B (see Fig. 1).

The resultant displacenent, due to action of two groups of thrust forces $P_A^{(k)}$ and $P_B^{(k)}$ combined, will be denoted by $w_s^{(k)}$, and thus we have

$$w_s^{(k)} = w_A^{(k)} + w_B^{(k)} \qquad (k=1,2) \tag{8}$$

Secondly we note that, formally we have

$$w_A^{(k)}(a) = K_A^{(k)} P_A^{(k)}, \qquad w_B^{(k)}(a) = H_A^{(k)} P_B^{(k)} \tag{9}$$

where $w_A^{(k)}(a)$ and $w_B^{(k)}(a)$ imply the values of displacements $w_A^{(k)}$ and $w_B^{(k)}$ at point of application of load $P_A^{(k)}$ (which at the radial position of $r = aR$). Herein, $K_A^{(k)}$ and $H_A^{(k)}$ are constants which are determined by dimensions and materials of our structure. The resultant dislacement will be given by

$$w_s^{(k)}(a) = K_A^{(k)} P_A^{(k)} + H_A^{(k)} P_B^{(k)} \tag{11}$$

Similarly, we shall have, for displacements about points of circle of radius $r=bR$;

$$w_s^{(k)}(b)=H_B^{(k)}P_A^{(k)}+K_B^{(k)}P_B^{(k)} \tag{12}$$

Thirdly, about the nature of thrust forces P_A and P_B, we take up the similar case, as we have considered in the previous paper, thus;

$$-P_A^{(1)}=\alpha+\beta\Delta w_s(a) \tag{13}$$

$$-P_B^{(1)}=\alpha+\beta\Delta w_s(b) \tag{14}$$

These equations are meant to express that thrust force of elastic rods are generated by two effects, namely, thermal expansion (term in α) and elastic deformation (term in β). Moreover, we have

$$\Delta w_s(a)=w_s^{(1)}(a)-w_s^{(2)}(a)$$

$$\Delta w_s(b)=w_s^{(1)}(b)-w_s^{(2)}(b)$$

$$w_A^{(2)}=-Zw_A^{(1)}, \qquad w_B^{(2)}=-Zw_B^{(1)}$$

in which we put $Z=D^{(1)}/D^{(2)}$

Combining these relations, we obtain;

$$(1+Z)K_AP_A^{(1)}+(1+Z)H_AP_B^{(1)}=\Delta w_s(a)$$

$$(1+Z)H_BP_A^{(1)}+(1+Z)K_BP_B^{(1)}=\Delta w_s(b)$$

Regarding these equations to be a linear algebraic equations about $P_A^{(1)}$ and $P_B^{(1)}$, and solving them, we obtain after some rearrangements, the following formula for thrust forces $P_A^{(1)}$ and $P_B^{(1)}$;

$$P_A^{(1)}=\frac{\alpha}{M}[1+(1+Z)\beta(K_B-H_A)] \tag{15}$$

$$P_B^{(1)}=\frac{\alpha}{M}[1+(1+Z)\beta(K_A-H_B)] \tag{16}$$

in which we put, for shortness,

$$M=1+\beta(1+Z)(K_A+K_B)+\beta^2(1+Z)^2(K_AK_B-H_AH_B)\,. \tag{17}$$

6. Explicit Form of Numerical Coefficients

Combining the above analytical results, we obtain following expressions for the numerical coefficients K_A, K_B, etc.

$$K_A = F_a \sum [K_{AAm}/S_{Am}]$$

$$K_B = F_b \sum [K_{BBm}/S_{Bm}]$$

$$H_A = F_b \sum [K_{BAm} \cos m\beta_0 / S_{Bm}]$$

$$H_B = F_a \sum [K_{ABm} \cos m\beta_0 / S_{Am}]$$

in which we have put

$$F_a = R^2/[\pi a D^{(1)}], \qquad F_b = R^2/[\pi b D^{(1)}].$$

The summations are to be made for $m=0, 1, 2, \cdots$, respectively. Furthermore, we have,

$$K_{BB0} = \left(\frac{\Delta_c}{\Delta}\right)_b \left(\frac{1}{2} - \frac{1}{2}b^2 + \log b\right)$$

$$K_{BB1} = \left(\frac{\Delta_c}{\Delta}\right)_b \left(-2b + b^3 + \frac{1}{b}\right) + \left(\frac{\Delta_D}{\Delta}\right)_b \left(\frac{1}{2}b - \frac{1}{2}b^3 + b \log b\right)$$

$$K_{BBm} = \left(\frac{\Delta_c}{\Delta}\right)_b \left(-\frac{m+1}{m}b^m + \frac{1}{m}b^{-m} + b^{m+2}\right)$$

$$+ \left(\frac{\Delta_D}{\Delta}\right)_b \left(-b^m - \frac{m-1}{m}b^{-m} + b^{-m+2}\right)$$

for $2 \leq m$.

Expressions for $K_{AAm} (m=0, 1, 2, \cdots)$ are to be obtained from that for K_{BBm}, by interchanging the figure b with the figure a.

$$K_{BA0} = \left(\frac{\Delta_c}{\Delta}\right)_a \left(\frac{1}{2} - \frac{1}{2}b^2 + \log b\right)$$

$$K_{BA1} = \left(\frac{\Delta_c}{\Delta}\right)_a \left(-2b + b^3 + \frac{1}{b}\right) + \left(\frac{\Delta_D}{\Delta}\right)_a \left(\frac{1}{2}b - \frac{1}{2}b^3 + b \log b\right)$$

$$K_{BAm}=\left(\frac{\Delta_c}{\Delta}\right)_a\left[-\frac{m+1}{m}b^m+\frac{1}{m}b^{-m}+b^{m+2}\right]$$

$$+\left(\frac{\Delta_D}{\Delta}\right)_a\left[-b^m-\frac{m-1}{m}b^{-m}+b^{-m+2}\right]$$

for $2\leq m$.

Here, $(\Delta_c/\Delta)_a$ and $(\Delta_D/\Delta)_a$ means those values of $(\Delta_c/\Delta)_b$ and $(\Delta_D/\Delta)_b$, wherein we interchange the figure b into the figure a.

Expressions for $K_{ABm}(m=0,\ 1,\ 2.\ \cdots)$ are to be obtained from K_{BAm}, by interchanging the figure a into the figure b, and b to a, respectively.

Lastly, we have

$$S_{A1}=\frac{8}{a}\left[\left(\frac{\Delta_D}{\Delta}\right)_a-1\right]$$

$$S_{A1}=8\left(\frac{\Delta_c}{\Delta}\right)_a-2\left(2+\frac{1}{a^2}\right)\left(\frac{\Delta_D}{\Delta}\right)_a-\frac{8}{c^4}\left(\frac{\Delta_G}{\Delta}\right)_a$$

$$S_{Am}=4m(m+1)a^{m-1}\left[\left(\frac{\Delta_c}{\Delta}\right)_a-\left(\frac{\Delta_G}{\Delta}\right)_a\right]$$

$$+4m(m-1)a^{-m-1}\left[\left(\frac{\Delta_D}{\Delta}\right)_a-1\right]$$

Expressions for S_{Bm} are to be formed from that for S_{Am}, by interchanging the figure a with b.

In applying the above mentioned expressions to numerical evaluation, we observed that; (a) The task of numerical evaluation to be rather complicated, (b) The infinite series to be estimated were seen to be rather slowly convergent (although they were absolutely convergent ones). From these view points, we felt it convenient to provide for practical use, some approximate formulae, which is fairly accurate, but supply convenient use for us. The approximate formula, which apply for the index number such that $M\leq m$ are listed up below;

$$(K_{AA}/S_A)_m\cong\frac{1}{2}a^3\left[\frac{1}{(m-1)(m+1)(2m+1)}\right]$$

$$(K_{BB}/S_B)_m\cong\frac{1}{b^3}\left[\frac{1}{(m-1)(m+1)(2m+1)}\right]$$

$$(K_{BA}/S_B)_m\cong\frac{b}{8m^2}\left(\frac{b}{a}\right)^m(a^2-b^2)$$

$$(K_{AB}/S_A)_m \cong \frac{a}{8m^2}\left(\frac{b}{a}\right)^m (a^2+b^2-2c^2)$$

for $M \leq m$. We may note that in the case of numerical examples which the author has taken up, we found appropriate to take $M=30$.

7. Numerical Examples

In order to make it clear the practical meaning of the analytical results thus obtained, we have made some numerical evaluation about these analytical results. The main dimensons of elastic annular plates and elastic rods, which we take up here, will be the same as that adopted in the author's previous report, except

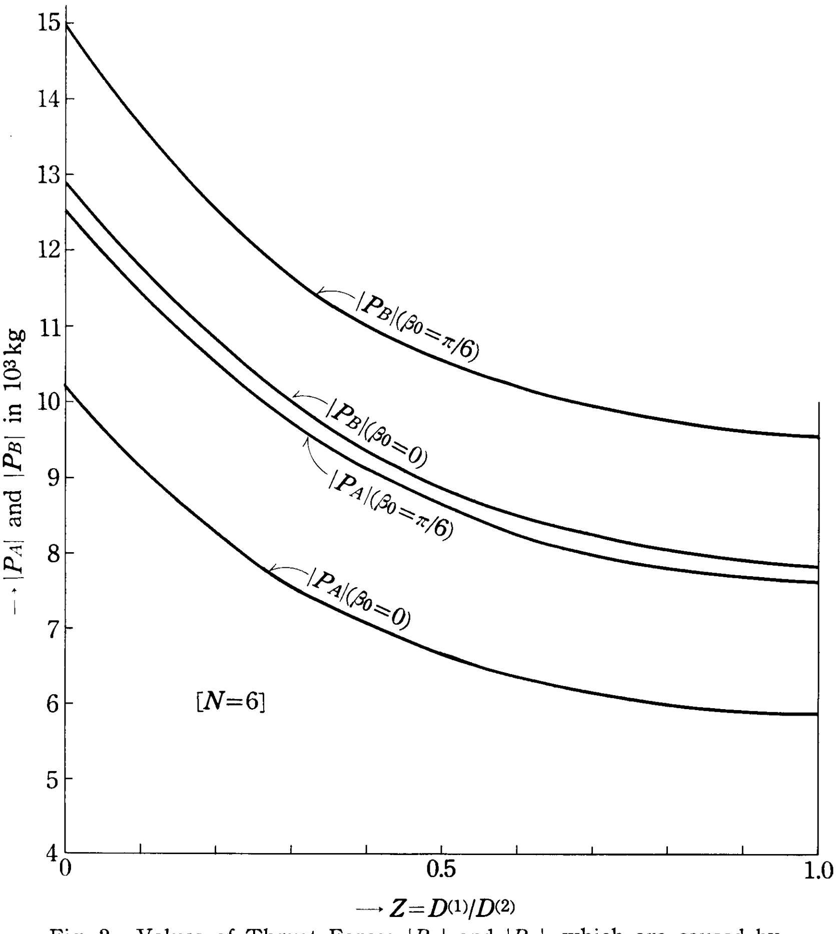

Fig. 2. Values of Thrust Forces $|P_A|$ and $|P_B|$, which are caused by Thermal Expansion of Connecting Rods. Case of $N=6$.

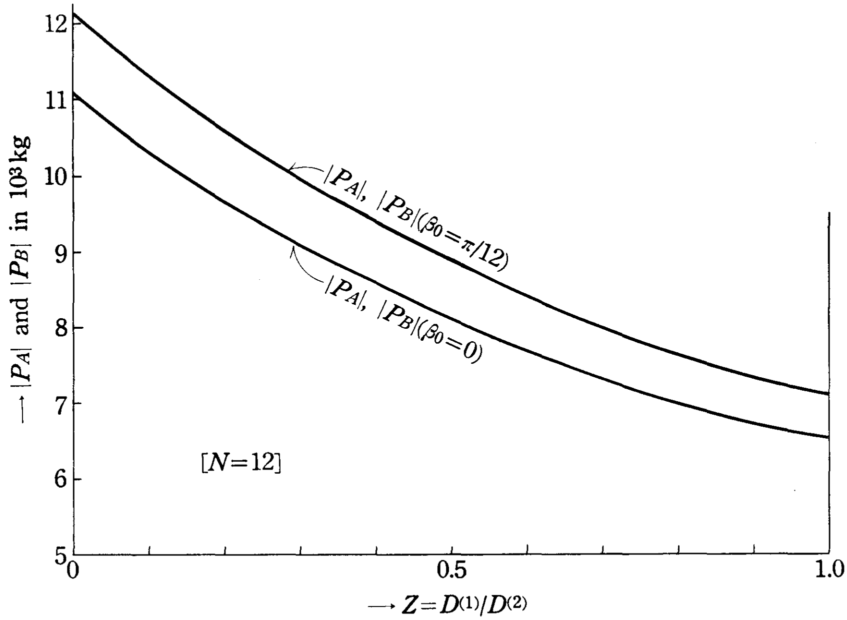

Fig. 3. Values of Thrust Forces $|P_A|$ and $|P_B|$, which are caused by Thermal Expansion of Connecting Rods. Case of $N=12$.

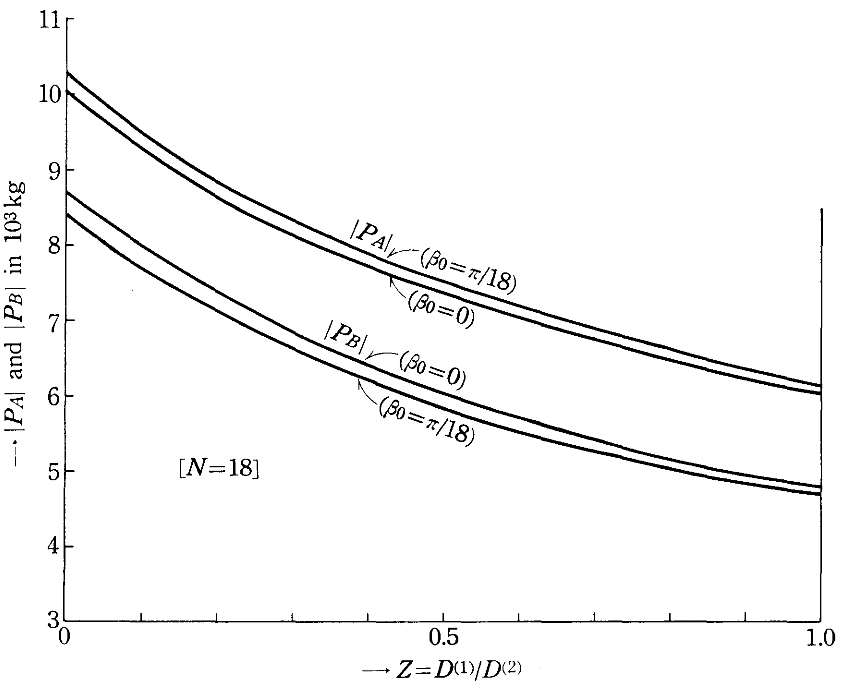

Fig. 4. Values of Thrust Forces $|P_A|$ and $|P_B|$, which are caused by Thermal Expansion of Connecting Rods. Case of $N=18$.

that here we take two circular rows of elastic bars, instead of a single row.

(1) About the annular elastic plates; outer radius $R=50$ cm, thickness of top plate $h^{(1)}=1.00$ cm, Young's modulus of its material $E=2.0\times10^6$ kg/cm^2.

(2) About the elastic rod; $A_b=4.805$ cm^2, $l=150$ cm, $E_b=2.0\times10^6$ kg/cm^2, coefficient of thermal expansion $\lambda=1.23\times10^{-5}/°C$, temperature rise $\vartheta=300°C$.

We take two cases, viz.,

Case C;	$a=0.83333$,	$b=0.6666$,
	$c=0.50000$,	$\beta_0=0.000$,
Case D;	$a=0.83333$,	$b=0.6666$,
	$c=0.50000$,	$\beta_0=\pi/N$,

For each cases, we take, for number N of circular group of bars, as $N=6$, 12 and 18. Here also we have as in the previous paper, $D^{(1)}=0.183150\times10^6$ kg/cm, $\alpha=35.400900\times10^3$, $\beta=6.40667\times10^4$ (in cm, °C units)

In carring out the required numerical estimation, we observed that for the

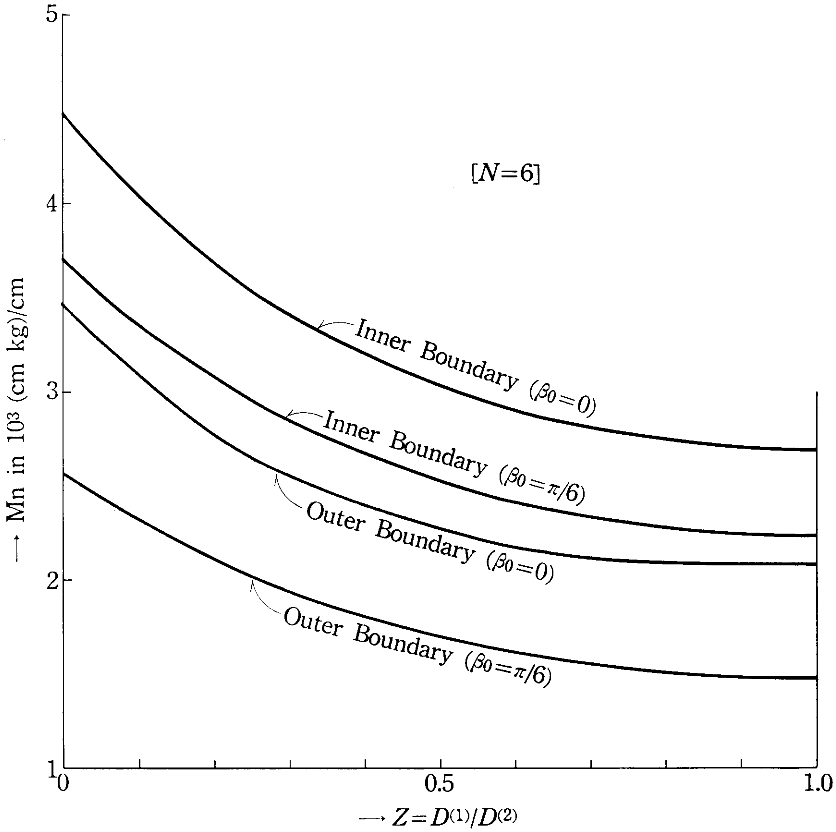

Fig. 5. Values of Bending Moments Mr, which are caused, along the Inner-and Outer-Boundaries of the Annular Plate. Case of $N=6$.

case of $m=30$, we have $(c^2/b^2)^m=1.0136\times10^{-12}$, $b^m=0.5215\times10^{-5}$, etc., (being of comparatively small values). From this view point, we have chosen as $M=30$. And therefore we made rigorous estimates about terms for which $0\leq m\leq M$, whereas we took approximate values for terms with $M\leq m$. The numerical results thus obtained are shown as graphs. (Figs. 2–7) wherein the ordinate is taken to be $Z=D^{(1)}/D^{(2)}$.

The values of thrust forces $|P_A|$ and $|P_B|$ which act on individual elastic rods are shown in Figs. 2 to 4. It may seem that these values of thrust forces are unreasonably large. This ows to the fact that we have chosen $h^{(1)}=1.00$ cm, which is done for sake of simplification. The values of $|P_A|$ and $|P_B|$ are seen to be proportionalt to $[h^{(1)}]^3$. Thus, if we had $h^{(1)}=1/2$ cm, the thrust values will be 1/8 of the values shown in these graphs. Looking at these figures, we observe that the effect of off-set angle β_0 is remarkable in the case of $N=6$ (that is, when the bars are relatively sparsely distributed). Whereas, in the case of $N=18$, there exist no remarkable difference between the cases of $\beta_0=0$, and that of $\beta_0=\pi/N$. It is thought that, this is due to the fact that in the case of $N=18$, bars are located very close each other.

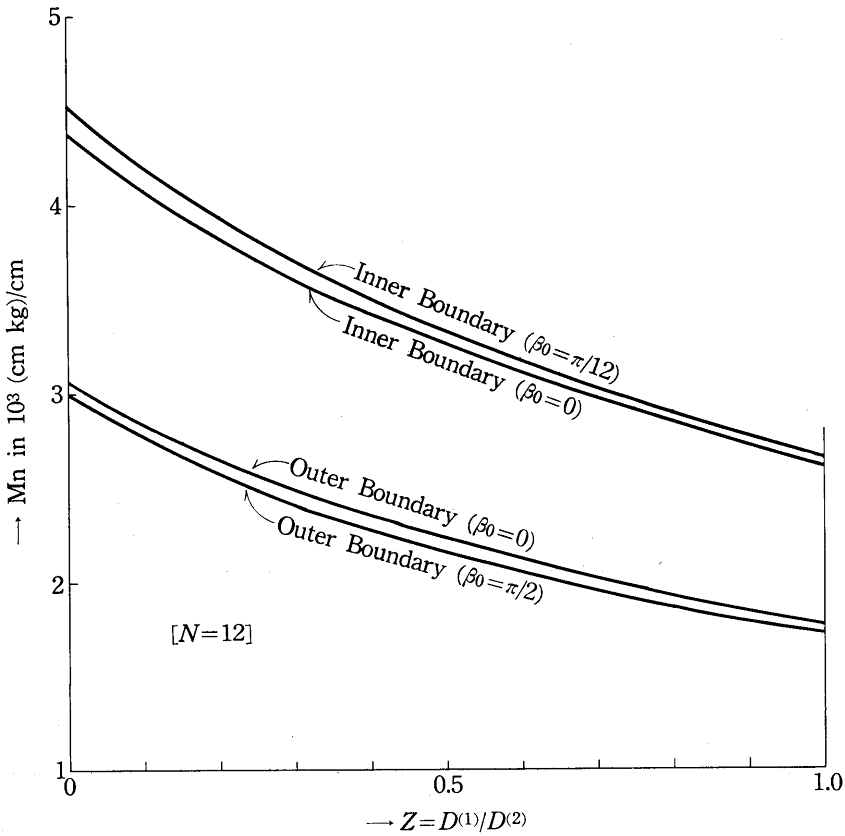

Fig. 6. Values of Bending Moments Mr, which are caused, along the Inner- and Outer-Boundaries of the Annular Plate. Case of $N=12$.

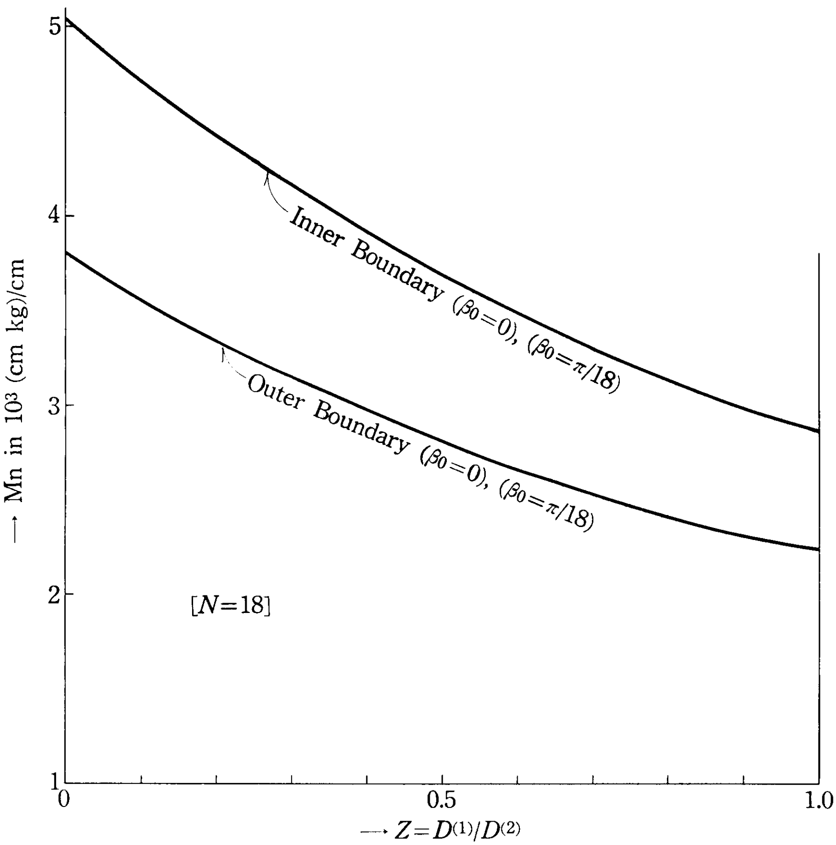

Fig. 7. Values of Bend'ng Moments Mr, which are caused, along the Inner- and Outer-Boundaries of the Annular Plate. Case of $N=18$.

The maximum values of bending moments M_r, which act along (inner and outer) boundary edge-lines are shown as graphs in Figs. 5 to 7. We see that the values of M_r at inner boundary are invariably larger than those at outer boundary.

Also, we observe that here again, the effect of off-set angle is remarkable for the case $N=6$, but that it is of comparatively small differene in the case of $N=$ 18, owing to the denseness of arrangements of elastic bars.

8. Concluding Remarks

In the present report, being the continuation of the author's previous study, under the same title, we have made analytical study about two elastic annular plates, which are interconnected each other by means of a number of elastic rods. In the case of this second report, we studied the case in which there exist two circular rows of these elastic rods, each arranged equidistantly. The elastic rods are assumed to be heated externally, thus causing a state of longitudinal expansion.

This expansion of elastic rods give rise to transverse deformations of both annular elastic plates.

In this case of two pairs of circular rows of elastic bars, two elastic plates and two circular rows of elastic bars, their deformation affect each other, giving rise to a kind of mutual interferences, and form a state of equilibrium.

In the present paper, the author has obtained expressions for this state of equilibrium, by solving a system of simultaneous linear (algebraic) equations. Furthemore, we have made some numerical estimation about them, with regard to suitably chosen practical cases.

The results of numerical evaluation are shown in graphs of Figs. 2 to 7, and demonstrate the effect of mutual interfernce between elastic rods and plates, giving, us the degree of the interferences as rods are distributed densely or sparsely.

In the above treatment, we confined ourselves to the case in which both elastic annular plates are held, along their respective inner and outer boundady edge lines, in state of so-called fixed (clamped) edge-lines. It may here be mentioned that, we may treat the case of other mode of fixation of edge lines, in quite similar manner as mentioned in the author's two reports. Furthermore, it may be noticed that we can treat the case in which there exist three or more circular rows of elastic rods, by the same method as given in the present report.

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