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PROBABILITY CONTENTS INNER BOUNDARY OF INTERVAL-CENSORED DATA

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ABSTRACT

Structure of the boundary of the set of values of distribution functions at each censored point of interval-censored data is analyzed. The results are applied for many kinds of typical families of distributions to find sufficient conditions under which a maximum likelihood estimate from interval-censored data exists. The families are location parameter family, location-scale parameter family and so on.

1. Introduction.

Let X be a random variable and let the distribution of X belong to a family $\mathcal{P}=\{P_\theta;\theta\in\Theta\}$ of probability measures on $\bar{\mathcal{R}}=[-\infty,\infty]$, which are not degenerate at infinity. The parameter space Θ is an arbitrary nonempty set. Let $(X_1,\cdots,X_q)$ be a random sample from the distribution of X, and assume that we observe only the event that each X_i, $1\leq i\leq q$, lies in an interval C_i of $\mathcal{R}=(-\infty,\infty)$ with nonempty interior. The collection $\mathcal{C}=\{C_1,\cdots,C_q\}$ is called an interval-censored data of size q. Our problem is to find criteria which assure the existence of a maximum likelihood estimate (MLE) from the interval-censored data $\mathcal{C}$.

To solve this problem, structure of the boundary of the set of values of the distribution functions at each censored point of $\mathcal{C}$ plays an important role. This set is called the probability contents inner boundary (p. c. i. b.) of $\mathcal{C}$ (for $\mathcal{P}$), whose definition will be given in the next section. The notion of the p. c. i. b. of $\mathcal{C}$ was introduced in previous papers ([2], [3]), where, by means of this notion, a method of finding critria for the existence of an MLE was presented.

The purpose of this paper is to seek a general method of specifying the structure of the p. c. i. b. of $\mathcal{C}$ and to find the p. c. i. b. of $\mathcal{C}$ for many kinds of typical families of distributions.

2. Probability contents inner boundary

We begin with the definition of the probability contents inner boundary of $\mathcal{C}$ for $\mathcal{P}$. Throughout this paper we assume that:

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(2.1) For any C_k, $P_\theta(C_k)\not\equiv 0$ and 1 on Θ.

(2.2) For each k, $1\leq k\leq q$, there exist two points a_k and b_k of $\bar{\mathscr{R}}$ such that $P_\theta(C_k) \equiv P_\theta([a_k, b_k))$ on Θ.

Let $F(x,\theta)=P_\theta([-\infty, x))$, $x\in\bar{\mathscr{R}}$; $\theta\in\Theta$ and write $\mathscr{F}=\{F(x,\theta)\,;\,\theta\in\Theta\}$ corresponding to $\mathscr{P}=\{P_\theta\,;\,\theta\in\Theta\}$. Then intervals $[a_k, b_k)$ generate a covering $\{[x_i, x_{i+1})\,;\, i=0,\cdots,m\}$ such that

(i) $-\infty=x_0<\cdots<x_{m+1}=\infty$,

(ii) x_i and x_j $(i\neq j)$ are not equivalent (with respect to the family $\mathscr{P}$), i. e., there exists $\theta\in\Theta$ such that $F(x_i,\theta)\neq F(x_j,\theta)$,

(iii) each of a_k and b_k, $1\leq k\leq q$, is equivalent to some x_i, $0\leq i\leq m+1$ and each x_i, $1\leq i\leq m$, is equivalent to some a_k or b_k, $1\leq k\leq q$.

Each x_i, $1\leq i\leq m$, is called a censored point of C. Define the mapping $\boldsymbol{F}\colon \Theta\to\mathscr{R}^m$ (Euclidean m-space) by

$$\boldsymbol{F}(\theta)=(F(x_1,\theta),\cdots,F(x_m,\theta))\,,$$

and with the difference between the image $\boldsymbol{F}(\Theta)$ and the closure $\overline{\boldsymbol{F}(\Theta)}$ by

$$\partial\boldsymbol{F}(\Theta)=\overline{\boldsymbol{F}(\Theta)}-\boldsymbol{F}(\Theta)\,.$$

The set $\partial\boldsymbol{F}(\Theta)$ is said to be the probability contents inner boundary(p. c. i. b.) of C.

Let ψ be a mapping from Θ into some set Θ', T be a function from the set $\{x_1,\cdots,x_m\}$ into $\mathscr{R}$, and $\mathscr{G}=\{G(x,\theta')\,;\,\theta'\in\Theta'\}$ be a family of distributions (d. f.'s) on $\mathscr{R}$. We say that $\mathscr{G}$ is equivalent to $\mathscr{F}$ with respect to the pair (ψ, T) if

(i) $\psi(\Theta)=\Theta'$,

(ii) T is independent of $\theta\in\Theta$ and of $\theta'\in\Theta'$,

(iii) $G(T(x_i),\psi(\theta))=F(x_i,\theta)$ for all $\theta\in\Theta$ and for all $i=1,\cdots,m$.

To emphasize that the p. c. i. b. $\partial\boldsymbol{F}(\Theta)$ depends on the set $\{x_i\}$, we often write $\partial\boldsymbol{F}_{\{x_i\}}(\Theta)$ instead of $\partial\boldsymbol{F}(\Theta)$.

This equivalence means the following fact.

Proposition 2.1. Let $\mathscr{G}=\{G(x,\theta')\,;\,\theta'\in\Theta'\}$ be a family of d. f.'s on $\mathscr{R}$ which is equivalent to $\mathscr{P}$ with respect to the pair (ψ, T). Then $\partial\boldsymbol{F}_{\{x_i\}}(\Theta)=\partial\boldsymbol{G}_{\{T(x_i)\}}(\psi(\Theta))$.

In order to determine the structure of $\partial\boldsymbol{F}(\Theta)$, put

$$F_x^{-1}([u,u'])=\{\theta\in\Theta\,;\, u\leq F(x,\theta)\leq u'\}$$

for each $x\in\mathscr{R}$ and for each pair (u,u') with $0\leq u\leq u'\leq 1$. Let p be an integer with $1\leq p\leq m$, let $\mathscr{I}(p)$ denote the set of all p-tuples $(i_1,\cdots,i_p)$ of integers with $1\leq i_1<\cdots<i_p\leq m$ and put $\mathscr{D}(p)=\{(d_1,\cdots,d_p)\in\mathscr{I}(p)\,;\, d_p=m\}$. Each $(d_1,\cdots,d_p)\in\mathscr{D}(p)$ can be regarded as the division of the set $\{1,\cdots,m\}$ into p parts $\{1,\cdots,d_1\}$, $\cdots$, $\{d_{p-1}+1,\cdots,d_p\}$. To emphasize this fact we write $\boldsymbol{d}=\langle d_1,\cdots,d_p\rangle$ instead of $\boldsymbol{d}=(d_1,\cdots,d_p)$. For each integer p, $1\leq p\leq m$, and for each $\boldsymbol{d}=\langle d_1,\cdots,d_p\rangle\in\mathscr{D}(p)$, define the sets

$$\mathcal{A}(\boldsymbol{d})=\{(z_1,\cdots,z_m)\in\mathcal{R}^m\,;\ 0\le z_1=\cdots=z_{d_1}\le\cdots\le z_{d_{p-1}+1}=\cdots=z_{d_p}\le 1\},$$
$$\mathcal{A}_0(\boldsymbol{d})=\begin{cases}\{(0,\cdots,0)\}, & p=1,\\ \{(z_1,\cdots,z_m)\in\mathcal{R}^m\,;\ 0=z_1=\cdots=z_{d_1}<\cdots<z_{d_{p-1}+1}=\cdots=z_{d_p}\le 1\}, & p\ge 2,\end{cases}$$
$$\mathcal{A}_1(\boldsymbol{d})=\begin{cases}\{(1,\cdots,1)\}, & p=1,\\ \{(z_1,\cdots,z_m)\in\mathcal{R}^m\,;\ 0\le z_1=\cdots=z_{d_1}<\cdots<z_{d_{p-1}+1}=\cdots=z_{d_p}=1\}, & p\ge 2.\end{cases}$$

For convenience, we put $\mathcal{D}(p)=\mathcal{A}(\boldsymbol{d})=\mathcal{A}_0(\boldsymbol{d})=\mathcal{A}_1(\boldsymbol{d})=\emptyset$ for $p=0$ or $p\ge m+1$. For each integer p (≥ 1) put

$$\mathcal{A}_p=(\cup_{\boldsymbol{d}\in\mathcal{D}(p-1)}\mathcal{A}(\boldsymbol{d}))\cup(\cup_{\boldsymbol{d}\in\mathcal{D}(p)}\mathcal{A}_0(\boldsymbol{d})\cup\mathcal{A}_1(\boldsymbol{d}))\cup(\cup_{\boldsymbol{d}\in\mathcal{D}(p+1)}\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d}))\,.$$

We use the convention that the union over a null index set is the empty set $\emptyset$.

The following result gives a method of specifying the structure of the p.c.i.b. of C.

Proposition 2.2 (cf. [2; Theorem 3.3]). The relation $\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_p$ holds if the following condition is satisfied for a positive integer p $(\le m)$:

(F. 1) For every set of pairs (u_j, u_j'), $1\le j\le p$, with $0<u_j\le u_j'<u_{j+1}<1$ and $(i_1,\cdots,i_p)\in\mathcal{I}(p)$, $\overline{\boldsymbol{F}(\cap_{j=1}^{p}F_{x_j}^{-1}([u_j,u_j']))}\subset\boldsymbol{F}(\Theta)$.

For each integer p (≥ 1), define

$$\mathcal{D}^*(p)=\{\langle d_1,\cdots,d_p\rangle\in\mathcal{D}(p)\,;\ d_{p-1}\le d_1+p-1\},$$
$$\mathcal{D}^{**}(p)=\begin{cases}\emptyset, & p=1,\\ \{\langle d_1,\cdots,d_p\rangle\in\mathcal{D}(p)\,;\ d_{p-1}=d_1+p-2\}, & p\ge 2,\end{cases}$$

$$\begin{aligned}\mathcal{A}_p^*=&(\cup_{\boldsymbol{d}\in\mathcal{D}(p-1)}\mathcal{A}(\boldsymbol{d}))\cup\mathcal{A}_0(\boldsymbol{d}')\cup\mathcal{A}_1(\boldsymbol{d}'')\\ &\cup(\cup_{\boldsymbol{d}\in\mathcal{D}^*(p)}\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d}))\cup(\cup_{\boldsymbol{d}\in\mathcal{D}^{**}(p+1)}\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d}))\,,\end{aligned}\tag{2.1}$$

where $\boldsymbol{d}'=\boldsymbol{d}''=\langle m\rangle$ for $p=1$, and $\boldsymbol{d}'=\langle m-p+1,\cdots,m\rangle$ and $\boldsymbol{d}''=\langle 1,\cdots,p-1,m\rangle$ for $p\ge 2$. It should be noted that $\mathcal{A}_1^*=\mathcal{A}_1$ and $\mathcal{A}_p^*\subset\mathcal{A}_p$ for $p\ge 2$.

Proposition 2.3. The relation $\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_p^*$ holds if condition (F. 1) and the following condition are satisfied:

(F. 2) For every set of a p-tuple $(i,i+1,\cdots,i+p-1)\in\mathcal{I}(p)$ and a set $\{v_0,\cdots,v_{p-1}\}$ with $0<v_k<1$ for each k, there exist a pair (u,u') with $0<u<u'<1$ and a set $\{\delta_0,\cdots,\delta_{p-1}\}$ such that $0<\delta_k<\min(v_k,1-v_k)$ for each k and $\cap_{k=0}^{p-1}F_{x_{i+k}}^{-1}([v_k-\delta_k,v_k+\delta_k])\subset F_{x_1}^{-1}([u,u'])\cap F_{x_m}^{-1}([u,u'])$.

Proof. In case $p=1$, the relation $\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_1^*$ follows from the definition of $\mathcal{A}_1^*$ and Theorem 2.1. Consider the case $p\ge 2$. We show the following fact:

(2.2) For every $\boldsymbol{z}=(z_1,\cdots,z_m)\in\overline{\boldsymbol{F}(\Theta)}$ with $z_1=0$ or $z_m=1$, there is no p-tuple $(i,i+1,\cdots,i+p-1)\in\mathcal{I}(p)$ such that $0<z_i\le z_{i+p-1}<1$.

In fact, assume the contrary. Then there exist $\boldsymbol{z}=(z_1,\cdots,z_m)\in\overline{\boldsymbol{F}(\Theta)}$ with $z_1=0$ or $z_m=1$ and a p-tuple $(i,i+1,\cdots,i+p-1)\in\mathcal{I}(p)$ such that $0<z_i\le z_{i+p-1}<1$. Since $\boldsymbol{z}\in\overline{\boldsymbol{F}(\Theta)}$, there is a sequence $\{\theta_n\}$ in Θ such that $\lim_n\boldsymbol{F}(\theta_n)=\boldsymbol{z}$. Because of (F. 2), there exists a pair (u,u') with $0<u<u'<1$ such that $\theta_n\in F_{x_1}^{-1}([u,u'])\cap F_{x_m}^{-1}([u,u'])$

for infinitely many n. This yields that $u\leq z_1=\lim_n F(x_1,\theta_n)\leq z_m=\lim_n F(x_m,\theta_n)\leq u'$, which is a contradiction. We prove

$$(2.3)\qquad (\cup_{\boldsymbol{d}\in\mathcal{D}(p)}\mathcal{A}_0(\boldsymbol{d}))\cap\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_0(\boldsymbol{d}')\cup(\cup_{\boldsymbol{d}\in\mathcal{D}^*(p)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d}))\,.$$

Let $\boldsymbol{d}=\langle d_1,\cdots,d_p\rangle\in\mathcal{D}(p)$ and $\boldsymbol{z}=(z_1,\cdots,z_m)\in\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})^c\cap\partial\boldsymbol{F}(\Theta)$, where $\mathcal{A}_1(\boldsymbol{d})^c$ denotes the complement of $\mathcal{A}_1(\boldsymbol{d})$. Then $z_{d_1}=0$ and $0<z_m<1$. Since $\boldsymbol{d}\in\mathcal{D}(p)$, $d_1\leq m-p+1$. Assume $d_1<m-p+1$. Then we can find $(i,i+1,\cdots,i+p-1)\in\mathcal{J}(p)$ with $i=d_1+1$. This contradicts the statement (2.2). Thus $d_1=m-p+1$, i.e., $\boldsymbol{d}=\boldsymbol{d}'$. Namely $\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})^c\cap\partial\boldsymbol{F}(\Theta)\neq\emptyset$ implies $\boldsymbol{d}=\boldsymbol{d}'$. Next let $\boldsymbol{z}=(z_1,\cdots,z_m)\in\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})\cap\partial\boldsymbol{F}(\Theta)$. Then $0=z_1=\cdots=z_{d_1}<z_{d_1+1}=\cdots=z_{d_2}<\cdots<z_{d_{p-1}+1}=\cdots=z_m=1$. By (2.2), we see that $d_{p-1}\leq d_1+p-1$, i.e., $\boldsymbol{d}\in\mathcal{D}^*(p)$. Namely $\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})\cap\partial\boldsymbol{F}(\Theta)\neq\emptyset$ implies $\boldsymbol{d}\in\mathcal{D}^*(p)$. Thus $\boldsymbol{z}\in\mathcal{A}_0(d)\cap\partial\boldsymbol{F}(\Theta)$ implies that $\boldsymbol{z}\in\mathcal{A}_0(\boldsymbol{d}')$ or $\boldsymbol{z}\in\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d})$ with $\boldsymbol{d}\in\mathcal{D}^*(p)$ and hence the relation (2.3) holds. Similarly we can prove that

$$(2.4)\qquad (\cup_{\boldsymbol{d}\in\mathcal{D}(p)}\mathcal{A}_1(\boldsymbol{d}))\cap\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_1(\boldsymbol{d}'')\cup(\cup_{\boldsymbol{d}\in\mathcal{D}^*(p)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d}))\,,$$
$$(2.5)\qquad (\cup_{\boldsymbol{d}\in\mathcal{D}(p+1)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d})\cap\partial\boldsymbol{F}(\Theta)\subset\cup_{\boldsymbol{d}\in\mathcal{D}^{**}(p+1)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d})\,.$$

Hence we have $\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_p^*$ by Proposition 2.2 and (2.3)-(2.5).

We prove

Theorem 2.1. Let $p\geq 2$, Θ be a Hausdorff space and $\boldsymbol{F}(\theta)$ be continuous on Θ. Then $\partial\boldsymbol{F}(\Theta)=\mathcal{A}_p^*$ if condition (F. 2) and the following conditions are satisfied:

(F. 3) For every set of pairs (u_j,u_j'), $1\leq j\leq p$, with $0<u_j\leq u_j'<u_{j+1}<1$ and $(i_1,\cdots,i_p)\in\mathcal{J}(p)$, the set $\cap_{j=1}^{p}F_{i_j}^{-1}([u_j,u_j'])$ is compact.

(F. 4) For each $\theta\in\Theta$, there exists $(i_1,\cdots,i_p)\in\mathcal{J}(p)$ such that $0<F(x_{i_1},\theta)<\cdots<F(x_{i_p},\theta)<1$.

(F. 5) For every set of pairs (u_j,u_j'), $1\leq j\leq p$, with $0<u_j\leq u_j'<u_{j+1}<1$ and $(i_1,\cdots,i_p)\in\mathcal{J}(p)$, the set $\cap_{j=1}^{p}F_{x_{i_j}}^{-1}([u_j,u_j'])$ is nonempty.

(F. 6) For every set of pairs (u_j,u_j'), $1\leq j\leq p$, with $0<u_j<u_j'<u_{j+1}<1$ and $\langle d_1,\cdots,d_{p-1}\rangle\in\mathcal{D}(p-1)$, the set $\cap_{j=1}^{p-1}(\cap\{F_{x_k}^{-1}([u_j,u_j']);\ d_{j-1}<k\leq d_j\})$ is noempty.

Proof. The relation $\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_p^*$ follows from Proposition 2.3 and Remark 2.1 (see below). To show the converse relation, we prove $\cup_{\boldsymbol{d}\in\mathcal{D}^{**}(p+1)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d})\subset\partial\boldsymbol{F}(\Theta)$. Let $\boldsymbol{z}=(z_1,\cdots,z_m)\in\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})$ $(\boldsymbol{d}=\langle d_1,\cdots,d_{p+1}\rangle\in\mathcal{D}^{**}(p+1))$. Then $0=z_{d_1}<z_{d_2}<\cdots<z_{d_p}<z_m=1$. Take an integer n_0 so that $2n_0^{-1}<\min(z_{d_2},z_{d_3}-z_{d_2},\cdots,z_{d_p}-z_{d_{p-1}},1-z_{d_p})$. Because of (F. 6), the set $\mathcal{S}_n=F_{x_{d_1}}^{-1}([(1+n)^{-1},n^{-1}])\cap(\cap_{k=2}^{p}F_{x_{d_k}}^{-1}([z_{d_k}-n^{-1},z_{d_k}+n^{-1}]))$ is nonempty whenever $n\geq n_0$. Choose an arbitrary $\theta_n\in\mathcal{S}_n(n\geq n_0)$. Without loss of generality, we may assume that $\{\boldsymbol{F}(\theta_n)\}$ is a convergent sequence with limit $\boldsymbol{z}'=(z_1',\cdots,z_m')$. Since $d_j=d_1+j-1$, $1\leq j\leq p$, $z_j=z_j'$ for all $j=1,\cdots,d_p$. The statement (2.2) yields that $z_j'=1$ for all $j>d_p$. Hence $\boldsymbol{z}=\boldsymbol{z}'$. Since $\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})\cap\boldsymbol{F}(\Theta)=\emptyset$ by (F. 4), we can see $\boldsymbol{z}\in\partial\boldsymbol{F}(\Theta)$. Thus $\cap_{\boldsymbol{d}\in\mathcal{D}^{**}(p+1)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d})\subset\partial\boldsymbol{F}(\Theta)$. Similarly we can prove that $\mathcal{A}_0(\boldsymbol{d}')\subset\partial\boldsymbol{F}(\Theta)$, $\mathcal{A}_1(\boldsymbol{d}'')\subset\partial\boldsymbol{F}(\Theta)$ and $\cup_{\boldsymbol{d}\in\mathcal{D}^*(p)}\cap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d})\subset\partial\boldsymbol{F}(\Theta)$. Finally we show $\cup_{\boldsymbol{d}\in\mathcal{D}(p-1)}\mathcal{A}(\boldsymbol{d})\subset\partial\boldsymbol{F}(\Theta)$. Let $\boldsymbol{z}=(z_1,\cdots,z_m)\in\mathcal{A}(\boldsymbol{d})$ $(\boldsymbol{d}=\langle d_1,\cdots,d_{p-1}\rangle\in\mathcal{D}(p-1))$. We first prove that $\boldsymbol{z}\in\partial\boldsymbol{F}(\Theta)$ in case $0<z_{d_1}<\cdots<z_{d_{p-1}}<1$. Take an integer $n_1>0$ so that $2n_1^{-1}<\min(z_{d_1},z_{d_2}-z_{d_1},\cdots,z_{d_{p-1}}-z_{d_{p-2}},1-z_{d_{p-1}})$.

Because of (F. 6), the set $\mathcal{S}'_n=\bigcap_{k=1}^{p-1}(\bigcap\{F_{x_j}^{-1}([z_{d_k}-n^{-1}, z_{d_k}+n^{-1}])\,;\ d_{k-1}<j\leq d_k\})$ is nonempty whenever $n\geq n_1$. Choose an arbitrary $\theta_n\in\mathcal{S}'_n$ $(n\geq n_1)$. Note that $\lim_n \boldsymbol{F}(\theta_n)=\boldsymbol{z}$. Since $\mathcal{A}(\boldsymbol{d})\cap\boldsymbol{F}(\Theta)=\emptyset$ by (F. 4), we can see $\boldsymbol{z}\in\partial\boldsymbol{F}(\Theta)$. We prove $\boldsymbol{z}\in\partial\boldsymbol{F}(\Theta)$ in the general case. There is a sequence $\{\boldsymbol{z}_n=(z_{n1},\cdots,z_{nm})\}$ in $\mathcal{A}(\boldsymbol{d})$ such that $0<z_{nd_1}<\cdots<z_{nd_{p-1}}<1$ and $\lim_n \boldsymbol{z}_n=\boldsymbol{z}$. Since $\boldsymbol{z}_n\in\partial\boldsymbol{F}(\Theta)$ for all n, we can choose a sequence $\{\theta_n\}$ in Θ such that $\|\boldsymbol{z}_n-\boldsymbol{F}(\theta_n)\|<1/(2n)$, where $\|\ \ \|$ denotes the usual distance on $\mathcal{R}^m$. This yields $\lim_n \boldsymbol{F}(\theta_n)=\boldsymbol{z}$. Hence $\boldsymbol{z}\in\partial\boldsymbol{F}(\Theta)$. Now the converse relation $\mathcal{A}_p^*\subset\partial\boldsymbol{F}(\Theta)$ is proved. This comletes the proof.

Remark 2.1. If Θ is a Hausdorff space and if $\boldsymbol{F}(\theta)$ is continuous on Θ, then condition (F. 3) implies condition (F. 1).

Theorem 2.2. Let Θ be a Hausdorff space, $\boldsymbol{F}(\theta)$ be continuous on Θ and conditions (F. 2)-(F. 5) with $p=1$ be satisfied. If $\sup\{\max_{1\leq k\leq m-1}(F(x_{k+1},\theta)-F(x_k,\theta))\,;\ \theta\in\Theta\}<1$, then $\partial\boldsymbol{F}(\Theta)=\{\boldsymbol{0},\boldsymbol{1}\}$.

Proof. To prove $\partial\boldsymbol{F}(\Theta)\subset\{\boldsymbol{0},\boldsymbol{1}\}$ it suffices to show $\mathcal{S}=\partial\boldsymbol{F}(\Theta)\cap(\bigcup_{\boldsymbol{d}\in\mathcal{D}(2)}\bigcap_{k=0}^{1}\mathcal{A}_k(\boldsymbol{d}))=\emptyset$, since $\mathcal{A}_0(\langle m\rangle)=\{\boldsymbol{0}\}$, $\mathcal{A}_1(\langle m\rangle)=\{\boldsymbol{1}\}$ and $\partial\boldsymbol{F}(\Theta)\subset\mathcal{A}_1^*$. In case $m=1$, $\mathcal{D}(2)=\emptyset$ and hence $\mathcal{S}=\emptyset$. Let $m\geq2$ and suppose $\mathcal{S}\neq\emptyset$. Then there exists $\boldsymbol{d}=\langle i,m\rangle\in\mathcal{D}(2)$ such that $\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})\neq\emptyset$. Since $\mathcal{A}_0(\boldsymbol{d})\cap\mathcal{A}_1(\boldsymbol{d})=\{(\overbrace{0,\cdots,0}^{i},1,\cdots,1)\}$, we can find a sequence $\{\theta_n\}$ in Θ such that $\lim_n F(x_i,\theta_n)=0$ and $\lim_n F(x_{i+1},\theta_n)=1$. This implies that $\sup\{F(x_{i+1},\theta)-F(x_i,\theta)\,;\ \theta\in\Theta\}=1$, which is a contradiction. Next we show the converse inclusion. Put $u_n=(1+n)^{-1}$ and $u'_n=n^{-1}(n\geq2)$. From (F. 5), it follows that $F_{x_m}^{-1}([u_n,u'_n])\neq\emptyset$ for all $n\geq2$. Hence there exists a sequence $\{\theta_n\}$ in Θ such that $\lim_n \boldsymbol{F}(\theta_n)=\boldsymbol{0}$. Because of (F. 4), $\boldsymbol{0}\in\partial\boldsymbol{F}(\Theta)$. Similarly we can prove $\boldsymbol{1}\in\partial\boldsymbol{F}(\Theta)$. Hence $\{\boldsymbol{0},\boldsymbol{1}\}\subset\partial\boldsymbol{F}(\Theta)$. This gives the desired relation.

Choose a convergent subsequence of a sequence in Θ such that the induced sequence by $\boldsymbol{F}$ converges to a point of $\partial\boldsymbol{F}(\Theta)$, and by the continuity of $\boldsymbol{F}$, we have

Theorem 2.3. Let Θ be an interval of $\overline{\mathcal{R}}$ and $\boldsymbol{F}(\theta)$ be continuous on Θ. Then $\partial\boldsymbol{F}(\Theta)\subset\{\boldsymbol{z}\in\mathcal{R}^m\,;\ \boldsymbol{z}=\lim_n \boldsymbol{F}(\theta_n)$ for some sequence $\{\theta_n\}$ in Θ with its limit $\theta_0\in\overline{\Theta}-\Theta\}$.

3. Structure of the p. c. i. b.

In this section we shall determine the structure of the p. c. i. b. $\partial\boldsymbol{F}(\Theta)$ for typical families. For some typical families which does not appear in this section, structure of the p. c. i. b. are determined in Nakamura [4]. Throughout this paper we assume that $m\geq2$ and $F(x)$ is a distribution function (d. f.) on $\mathcal{R}$ such that $\{x\in\mathcal{R}\,;\ 0<F(x)<1\}=(a,b)$ with $-\infty\leq a<b\leq\infty$ and $F(x)$ is continuous on $\mathcal{R}$ and is strictly increasing on (a,b). Define $\boldsymbol{a}_k(u)=(\overbrace{0,\cdots,0}^{k-1},u,\overbrace{1,\cdots,1}^{m-k})$, $u\in[0,1]\,;\ 1\leq k\leq m$. Note $\boldsymbol{0}\equiv(0,\cdots,0)=\boldsymbol{a}_m(0)$ and $\boldsymbol{1}\equiv(1,\cdots,1)=\boldsymbol{a}_1(1)$. The proofs of results in this section will be given in the next section.

3.1. Location parameter family. Let $\mathcal{F}=\{F((x-\theta)/\sigma)\,;\ \theta\in\Theta\}$, where $\Theta=(-\infty,\infty)$ and σ is a positive number. Structure of the p. c. i. b. for all possible cases are summarized below.

Table 3.1.

$a=-\infty,\ b=\infty$	$a=-\infty,\ b<\infty$	$a>-\infty,\ b=\infty$	$a>-\infty,\ b<\infty$
$\{\mathbf{0},\ \mathbf{1}\}$	$\{\mathbf{0}\}$	$\{\mathbf{1}\}$	$\emptyset$

3.2. Scale parameter family. Let $\mathcal{F}=\{F((x-\mu)/\theta)\,;\ \theta\in\Theta\}$, where $\Theta=(0,\infty)$ and μ is a real number. Structure of the p. c. i. b. for all possible cases are summarized below with $u=F(0)$.

Table 3.2.

	$\mu<x_1$	$x_k<\mu<x_{k+1}$ for some k, $1\leq k\leq m-1$	$x_m<\mu$	$x_1=\mu$	$x_k=\mu$ for some k, $2\leq k\leq m-1$	$x_m=\mu$
$a=-\infty$ $b=\infty$	$\{\mathbf{1},\ u\mathbf{1}\}$	$\{\mathbf{a}_k(0),\ u\mathbf{1}\}$	$\{\mathbf{0},\ u\mathbf{1}\}$	$\{\mathbf{a}_k(u),\ u\mathbf{1}\}$		
$a=-\infty$ $0<b<\infty$	$\{u\mathbf{1}\}$	$\{\mathbf{a}_k(0),\ u\mathbf{1}\}$	$\{\mathbf{0},\ u\mathbf{1}\}$	$\{u\mathbf{1}\}$	$\{\mathbf{a}_k(u),\ u\mathbf{1}\}$	
$a=-\infty$ $b=0$			$\{\mathbf{0},\ \mathbf{1}\}$			
$a=-\infty$ $b<0$			$\{\mathbf{0}\}$			
$-\infty<a<0$ $b=\infty$	$\{\mathbf{1},\ u\mathbf{1}\}$	$\{\mathbf{a}_k(0),\ u\mathbf{1}\}$	$\{u\mathbf{1}\}$	$\{\mathbf{a}_k(u),\ u\mathbf{1}\}$		$\{u\mathbf{1}\}$
$-\infty<a<0$ $0<b<\infty$	$\{u\mathbf{1}\}$					
$-\infty<a<0$ $b=0$			$\{\mathbf{1}\}$			
$-\infty<a<0$ $b=0$			$\emptyset$			
$a=0$ $b=\infty$	$\{\mathbf{0},\ \mathbf{1}\}$					
$a=0$ $0<b<\infty$	$\{\mathbf{0}\}$					
$a>0$ $b=\infty$	$\{\mathbf{1}\}$					
$a>0$ $b<\infty$	$\emptyset$					

Remark 3.1. By Proposition 2.1 and the above result, we can show that structure of the p. c. i. b. for power parameter family $\{F(x^{\theta})\,;\ \theta\in\Theta\}$, where $0\leq a<b\leq\infty$ and $\Theta=(0,\infty)$, are summarized as in Table 3.2.

3.3. Truncation parameter family. Let $\mathcal{F}=\{F(x,\theta)\,;\ \theta\in\Theta\}$, where $\Theta=(a,b)$ and

$F(x, \theta)$ ($\theta \in \Theta$) is defind as follows: Left truncated case. $F(x, \theta)=0$ for $x \leq \theta$, $F(x, \theta) = (F(x)-F(\theta))/(1-F(\theta))$ for $\theta < x < b$ and $F(x, \theta)=1$ for $x \geq b$. Right truncated case. $F(x, \theta)=0$ for $x \leq a$, $F(x, \theta)=(F(\theta)-F(x))/F(\theta)$ for $a < x < \theta$ and $F(x, \theta)=1$ for $x \geq \theta$. Then $\partial\boldsymbol{F}(\Theta)=\{(F(x_1), \cdots, F(x_m))\}$ if $\mathcal{F}$ is left truncated and $\partial\boldsymbol{F}(\Theta)=\{(1-F(x_1), \cdots, 1-F(x_m))\}$ if $\mathcal{F}$ is right truncated.

3.4. Location parameter family with truncation. Let $\mathcal{F}=\{F(x, \theta);\ \theta \in \Theta\}$, where $\Theta=(b'-b, a'-a)$ with $a<a'<b'<b$, and $F(x, \theta)$ ($\theta \in \Theta$) is defined by $F(x, \theta)=0$ for $x \leq a'$, by $F(x, \theta)=(F(x-\theta)-F(a'-\theta))/(F(b'-\theta)-F(a'-\theta))$ for $a'<x<b'$ and by $F(x, \theta) =1$ for $x \geq b'$. Assume that:

(i) For each i, $1 \leq i \leq m$, $t_i=\lim_{\theta \to b'-b} F(x_i, \theta)$ and $s_i=\lim_{\theta \to a'-a} F(x_i, \theta)$ exist.
(ii) $F(x_i, \theta)$ is strictly monotone in θ for some i, $1 \leq i \leq m$.

Then $\partial\boldsymbol{F}(\Theta)=\{(t_1, \cdots, t_m), (s_1, \cdots, s_m)\}$.

Remark 3.2. By Proposition 2.1 and the above result, we can determine the structure of the p. c. i. b. for scale parameter family with truncation $\{F(x, \theta);\ \theta \in \Theta\}$, where $\Theta=(b'/b, a'/a)$ with $0 \leq a<a'<b'<b$, and $F(x, \theta)$ ($\theta \in \Theta$) is defined by $F(x, \theta) =0$ for $x \leq a'$, by $F(x, \theta)=(F(x/\theta)-F(a'/\theta))/(F(b'/\theta)-F(a'/\theta))$ for $a'<x<b'$ and by $F(x, \theta)=1$ for $x \geq b'$.

3.5. Family with mean and variance. Let $\mathcal{F}=\{F(x, \theta);\ \theta \in \Theta\}$, where Θ is an interval or a discrete set of $\mathcal{R}$, and $F(x, \theta)$ ($\theta \in \Theta$) is a d. f. on $\mathcal{R}$ with the mean $\mu(\theta)$ and the variance $\sigma^2(\theta)$ and is continuous on Θ for every fixed $x \in \mathcal{R}$.

Let $\bar{\Theta}-\Theta \subset \{\alpha, \beta\}$ with $-\infty \leq \alpha < \beta \leq \infty$ and consider the following conditions:

(i) For each $\theta \in \Theta$, there exists i, $1 \leq i \leq m$, with $F(x_i, \theta)<1$.
(i)′ For each $\theta \in \Theta$, there exists i, $1 \leq i \leq m$, with $F(x_i, \theta)>0$.
(ii) $\lim_{\theta \to \alpha} \mu(\theta) < x_1$.
(ii)′ $x_m < \lim_{\theta \to \beta} \mu(\theta)$.
(iii) $\lim_{\theta \to \alpha} \sigma(\theta)^2/(\mu(\theta)-t)^2=0$ for every t with $\lim_{\theta \to \alpha} \mu(\theta)<t$.
(iii)′ $\lim_{\theta \to \beta} \sigma(\theta)^2/(\mu(\theta)-t)^2=0$ for every t with $t<\lim_{\theta \to \beta} \mu(\theta)$.

Then $\partial\boldsymbol{F}(\Theta)=\{\mathbf{0}, \mathbf{1}\}$ if conditions (i)-(iii) and (i)′-(iii)′ are satisfied, $\partial\boldsymbol{F}(\Theta)=\{\mathbf{0}\}$ if conditions (i)′-(iii)′ are satisfied and $\partial\boldsymbol{F}(\Theta)=\{\mathbf{1}\}$ if conditions (i)-(iii) are satisfied.

3.6. Family of Wald distributions. Let $\mathcal{F}=\{F(x, \theta);\ \theta \in \Theta\}$, where $\Theta=(0, \infty)$ and $F(x, \theta)$ ($\theta \in \Theta$) is defined by $F(x, \theta)=0$ for $x \leq 0$ and by

$$F(x, \theta)=\int_0^x (\theta/2\pi v^3)^{1/2} \exp(-\theta(v-1)^2/(2v))dv$$

for $x>0$. Structure of the p. c. i. b. for all possible cases are summarized below.

Table 3.3.

$1<x_1$	$x_k<1<x_{k+1}$ for some k, $1 \leq k \leq m-1$	$x_m<1$	$x_k=1$ for some k, $1 \leq k \leq m$
$\{\mathbf{1}\}$	$\{\boldsymbol{a}_k(0), \mathbf{1}\}$	$\{\mathbf{0}, \mathbf{1}\}$	$\{\boldsymbol{a}_k(1/2), \mathbf{1}\}$

3.7. Location-scale parameter family. Let $\mathcal{F}=\{F((x-\mu)/\sigma)\,;\,(\mu,\sigma)\in\Theta\}$, where $\Theta=\mathcal{R}\times(0,\infty)$. Structure of the p. c. i. b. for all possible cases are summarized below.

Table 3.4.

$a=-\infty,\ b=\infty$	$a=-\infty,\ b<\infty$	$a>-\infty,\ b=\infty$	$a>-\infty,\ b<\infty$
$\mathcal{A}_2^*$	$\mathcal{A}_2^*-\{\boldsymbol{a}_1(u)\,;\,0<u\leq 1\}$	$\mathcal{A}_2^*-\{\boldsymbol{a}_m(u)\,;\,0\leq u<1\}$	$\{u\boldsymbol{1}\,;\,0<u<1\}$

$\mathcal{A}_2^*=(\cup_{i=1}^m\{\boldsymbol{a}_i(u)\,;\,0\leq u\leq 1\})\cup\{u\boldsymbol{1}\,;\,0<u<1\}$.

Remark 3.3. By Proposition 2.1 (see Section 2) and the above result, we can show that structure of the p. c. i. b. for each of the following families are summarized as in Table 3.4:

(i) $\{F((g_1(x)-\mu)/\sigma)\,;\,(\mu,\sigma)\in\mathcal{R}\times(0,\infty)\}$, where $-\infty\leq a<b\leq\infty$.
(ii) $\{F(\beta g_1(x)+\alpha)\,;\,(\alpha,\beta)\in\mathcal{R}\times(0,\infty)\}$, where $-\infty\leq a<b\leq\infty$.
(iii) $\{F((g_2(x)/\alpha)^{1/\beta})\,;\,(\alpha,\beta)\in(0,\infty)\times(0,\infty)\}$, where $0\leq a<b\leq\infty$.
(iv) $\{F(\alpha g_2(x)^{1/\beta})\,;\,(\alpha,\beta)\in(0,\infty)\times(0,\infty)\}$, where $0\leq a<b\leq\infty$.

Here $g_1(x)$ and $g_2(x)$ are strictly increasing functions on an interval (a',b') of $\mathcal{R}$ such that $\{g_1(x)\,;\,x\in(a',b')\}=(-\infty,\infty)$ and $\{g_2(x)\,;\,x\in(a',b')\}=(0,\infty)$.

3.8. Scale-power-shift parameter family. Let $\mathcal{F}=\{F(\log((x-\lambda)/\alpha)^{1/\beta})\,;\,(\alpha,\beta,\lambda)\in\Theta\}$, where $\Theta=(0,\infty)\times(0,\infty)\times[\lambda_1,\lambda_2]$, $-\infty<\lambda_1\leq\lambda_2<x_1$ and $(a,b)=(-\infty,\infty)$. Then $\partial\boldsymbol{F}(\Theta)=\mathcal{A}_2^*$.

Remark 3.4. By Proposition 2.1 (see Section 2) and the above result, we can show that the structure of the p. c. i. b. for the family $\{F(\alpha(x-\lambda)^{1/\beta})\,;\,(\alpha,\beta,\lambda)\in(0,\infty)\times(0,\infty)\times[\lambda_1,\lambda_2]\}$ is $\mathcal{A}_2^*$. Here we assume that $(a,b)=(0,\infty)$ and $-\infty<\lambda_1<\lambda_2<x_1$.

3.9. Gamma distribution with shape, scale and shift parameters. Let $\mathcal{F}=\{F(x,\theta)\,;\,\theta\in\Theta\}$, where $\Theta=(0,\infty)\times(0,\infty)\times[\lambda_1,\lambda_2]$, $-\infty<\lambda_1\leq\lambda_2<x_1$ and $F(x,\theta)$ $(\theta=(\alpha,\beta,\lambda)\in\Theta)$ is defined by $F(x,\theta)=0$ for $x\leq\lambda$ and by

$$F(x,\theta)=\int_\lambda^x(\Gamma(\alpha)\beta^\alpha)^{-1}(v-\lambda)^{\alpha-1}\exp(-(v-\lambda)/\beta)dv$$

for $x>\lambda$. Then $\partial\boldsymbol{F}(\Theta)=\mathcal{A}_2^*$.

3.10. Polynomial distribution. Let $\mathcal{F}=\{F(\sum_{i=1}^r(\alpha_i x)^i)\,;\,(\alpha_1,\cdots,\alpha_r)\in\Theta\}$, where r is a positive integer, $\Theta=\{(\alpha_1,\cdots,\alpha_r)\in[0,\infty)^r\,;\,\sum_{i=1}^r\alpha_i\neq 0\}$ and $(a,b)=(0,\infty)$. Then $\partial\boldsymbol{F}(\Theta)=\{\boldsymbol{0},\boldsymbol{1}\}$.

3.11. Multinomial distribution. Let $\mathcal{F}=\{F(x,\theta)\,;\,\theta\in\Theta\}$, where $\Theta=\{(\alpha_1,\cdots,\alpha_m)\in(0,1)^m\,;\,\sum_{i=1}^m\alpha_i<1\}$ and $F(x,\theta)$ $(\theta=(\alpha_1,\cdots,\alpha_m)\in\Theta)$ is defined by $F(x,\theta)=0$ for $x\leq y_1$, by $F(x,\theta)=1$ for $x>y_{m+1}$ and by $F(x,\theta)=\sum_{k=1}^i\alpha_k$ for $y_i<x\leq y_{i+1}$, $1\leq i\leq m$. Here the y_i's are numbers such that $-\infty<y_1<x_1<\cdots<y_m<x_m<y_{m+1}<\infty$. Then $\partial\boldsymbol{F}(\Theta)$

$=\mathcal{A}_m^*$ (see (2.1) for $\mathcal{A}_m^*$).

3.12. Histogram distribution. Let $\mathcal{F}=\{F(x,\theta)\,;\,\theta\in\Theta\}$, where $\Theta=\{(\alpha_1,\cdots,\alpha_m)\in\mathcal{R}^m\,;\,0<\alpha_1<\cdots<\alpha_m<1\}$ and $F(x,\theta)$ $(\theta=(\alpha_1,\cdots,\alpha_m)\in\Theta)$ is the histogram d.f., that is, $F(x,\theta)=0$ for $x\leq y_0$, $F(x,\theta)=1$ for $x>y_{m+1}$ and $F(x,\theta)=(\alpha_{i-1}-\alpha_i)x/(y_{i+1}-y_i)+(\alpha_i y_{i+1}-\alpha_{i+1}y_i)/(y_{i+1}-y_i)$ for $y_i<x\leq y_{i+1}$, $0\leq i\leq m$. Here we assume that $-\infty<y_0<y_1=x_1<\cdots<y_m=x_m<y_{m+1}<\infty$. Then $\partial\boldsymbol{F}(\Theta)=\mathcal{A}_m^*$.

4. Proofs

In this section we shall give proofs of results in Section 3. Before giving these proofs, we state some definitions. We write P-lim $Z_n=Z$ if a sequence $\{Z_n\}$ of random variables converges in measure to a random variable Z, and P-lim $Z_n=\infty$ (resp. $-\infty$) if there exists an open interval $\mathcal{O}$ containing x_m (resp. x_1) such that for each $t\in\mathcal{O}$, $\lim_n \Pr(Z_n\leq t)=0$ (resp. $\lim_n \Pr(Z_n\geq t)=0$), where the symbol "$\lim_n$" denotes "$\lim_{n\to\infty}$". Note that P-lim $Z_n=\infty$ (resp. P-lim $Z_n=-\infty$) implies that $\lim_n \Pr(Z_n<x_i)=0$ (resp. $\lim_n \Pr(Z_n<x_i)=1$) for all i, $1\leq i\leq m$.

Proofs of 3.1–3.4. With simple calculation, structure of the p.c.i.b. are derived from Theorem 2.3.

Proof of 3.5. Consider the case where conditions (i)–(iii) and (i)′–(iii)′ are satisfied. Because of Theorem 2.3, it suffices to show that $\lim_{\theta\to\alpha}\boldsymbol{F}(\theta)=\boldsymbol{0}$ and $\lim_{\theta\to\beta}\boldsymbol{F}(\theta)=\boldsymbol{1}$. This follows from Lemma 4.1 below. Before stating Lemma 4.1, we prepare

Proposition 4.1 (cf. [1; Theorem 14.4]). Let Z be a random variable, $g(v)$ be a nonnegative, even, Borel measurable function on $\mathcal{R}$ which is nondecreasing on $(0,\infty)$, and $0<g(v)<\infty$ for all $v\in(0,\infty)$. Then $\Pr(|Z|\geq t)\leq E(g(Z))/g(t)$ for all $t\in(0,\infty)$.

Lemma 4.1. Let Y_n, $n\geq 1$, be a random variable which has the d.f. $F(x,\theta_n)$. If $\lim_n\theta_n=\alpha$ (resp. β), then P-lim$Y_n=-\infty$ (resp. ∞).

Proof. Let $t\in(\lim_{\theta\to\alpha}\mu(\theta),\lim_{\theta\to\beta}\mu(\theta))$ and $\{\theta_n\}$ be a sequence in Θ such that $\lim_n\theta_n=\alpha$. We may assume that $\mu(\theta_n)<t$ for all n. It is easy to see that $\{Y_n\geq t\}\subset\{|Y_n-\mu(\theta_n)|\geq t-\mu(\theta_n)\}$. By Proposition 4.1,

$$\Pr(Y_n\geq t)\leq\Pr(|Y_n-\mu(\theta_n)|\geq t-\mu(\theta_n))\leq\sigma(\theta_n)^2/(t-\mu(\sigma_n))^2\,.$$

From this and (iii) in 3.5, it follows that P-lim$Y_n=-\infty$. Similarly we can prove, for the case $\lim_n\theta_n=\beta$, that P-lim$Y_n=\infty$.

The argument in the proof of Lemma 4.1 also derives the desired structure of the p.c.i.b. for the rest cases.

Proof of 3.6. The structure of the p.c.i.b. follows from Theorem 2.3 and Lemmas 4.2 and 4.3 below.

Lemma 4.2. Let Y_n, $n \geq 1$, be a random variable which has the Wald d.f. $F(x, \theta_n)$. If $\lim_n \theta_n = 0$ (resp. ∞), then $\text{P-lim}\, Y_n = 0$ (resp. 1).

Proof. Note that $E(Y_n) = 1$ and $\text{Var}(Y_n) = \theta_n^{-1}$ (cf. [6]). Suppose $\lim_n \theta_n = \infty$ and let $t > 0$. By Proposition 4.1,

$$\Pr(|Y_n - 1| \geq t) \leq \text{Var}(Y_n)/t^2 = (t^2 \theta_n)^{-1} .$$

This implies $\text{P-lim}\, Y_n = 1$. Suppose that $\lim_n \theta_n = 0$ and $\theta_n < 1$ for all n. Let $t > 0$ and let $f(v, \theta)$ be the Wald density function, i.e., $f(v, \theta) = (\theta/2\pi v^3)^{1/2} \exp(-\theta(v-1)^2/2v)$. Since $v^{-3/2}$ is integrable on (t, ∞) and since $f(v, \theta_n) \leq (1/2\pi v^3)^{1/2}$, $\lim_n \Pr(|Y_n| \geq t) = \int_t^\infty \lim_n f(v, \theta_n) dv = 0$ by Lebesgue's dominated convergence theorem. Hence $\text{P-lim}\, Y_n = 0$.

Lemma 4.3. If $F(x, \theta)$ is the Wald d.f., then $\lim_{\theta \to \infty} F(1, \theta) = 1/2$.

Proof. The Wald d.f. can be expressed as

$$F(x, \theta) = G((\theta/x)^{1/2}(x-1)) + G(-(\theta/x)^{1/2}(x+1)) \exp(2\theta) ,$$

where G is the standard normal d.f. (cf. [5]). From this expression,

$$F(1, \theta) = 1/2 + G(-(4\theta)^{1/2}) \exp(2\theta) .$$

To establish our assertion, it suffices to show $\lim_{\theta \to \infty} G(-(2\theta)^{1/2}) \exp(\theta) = 0$. Since $dG(-(2\theta)^{1/2})/d\theta = -(2(\pi\theta)^{1/2} \exp(\theta))^{-1}$, $\lim_{\theta \to \infty} G(-(2\theta)^{1/2}) \exp(\theta) = \lim_{\theta \to \infty} G(-(2\theta)^{1/2})/\exp(-\theta) = \lim_{\theta \to \infty} (1/4\pi\theta)^{1/2} = 0$.

Proof of 3.7. We show $\partial \boldsymbol{F}(\Theta) \subset \mathcal{A}_2^*$. Because of Proposition 2.3 and Remark 2.1, it suffices to show that conditions (F.2) and (F.3) with $p=2$ are satisfied. To see that condition (F.3) is satisfied, let $0 < u_1 \leq u_1' < u_2 \leq u_2' < 1$ and $1 \leq i \leq j \leq m$. It is easy to see that $(\mu, \sigma) \in F_{x_i}^{-1}([u_1, u_1']) \cap F_{x_j}^{-1}([u_2, u_2'])$ if and only if for some $(u, v) \in [u_1, u_1'] \times [u_2, u_2']$,

$$\sigma = \sigma_{ij}(u, v) \equiv (x_j - x_i)/(F^{-1}(v) - F^{-1}(u)) ,$$
$$\mu = \mu_{ij}(u, v) \equiv x_i - F^{-1}(u)\sigma_{ij}(u, v) ,$$

where $F^{-1}(u)$ is the inverse function of $F(x)$. This implies that condition (F.3) is satisfied, since $\sigma_{ij}(u, v)$ and $\mu_{ij}(u, v)$ are continuous on the compact set $[u_1, u_1'] \times [u_2, u_2']$.

To see that condition (F.2) is satisfied, let $0 < v_j < 1, j = 0, 1$, and let $1 \leq i \leq m$. In the case where $v_0 < v_1$ or $v_0 > v_1$, the assertion follows from the above argument. Assume $v_0 = v_1 = v$. Take u and u' so that $0 < u \leq v \leq u' < 1$ and $u \neq u'$. Then $(x_k - \mu_{i\,i+1}(u, u'))/\sigma_{i\,i+1}(u, u') = (x_k - x_i)(x_{i+1} - x_i)^{-1}(F^{-1}(u') - F^{-1}(u)) + F^{-1}(u)$. This implies that condition (F.2) is satisfied.

To determine presicely the structure of $\partial \boldsymbol{F}(\Theta)$, let $0 < u < 1$ and $1 \leq i \leq m$, and put $\theta(t) = (x_i - F^{-1}(u)t, t)$. Then $\lim_{t \to 0} \boldsymbol{F}(\theta(t)) = \boldsymbol{a}_i(u)$ and $\lim_{t \to \infty} \boldsymbol{F}(\theta(t)) = u\mathbf{1}$. Consider the case $a = -\infty$ and $b < \infty$. It can be easily seen that $\boldsymbol{a}_1(v) \in \boldsymbol{F}(\theta)$ for all $v \in (0, 1]$,

$\{\boldsymbol{a}_1(0), \cdots, \boldsymbol{a}_m(0), \boldsymbol{a}_2(u), \cdots, \boldsymbol{a}_m(u), u\boldsymbol{1}\} \cap \boldsymbol{F}(\Theta)=\emptyset$. Hence $\partial\boldsymbol{F}(\Theta)=\mathcal{A}_2^*-\{\boldsymbol{a}_1(v)\,;\ 0<v\leq 1\}$. The rest of the proof can be carried out by the same argument as above.

Proof of 3.8. Because of Theorem 2.1, it suffices to show that conditions (F. 2)-(F. 6) with $p=2$ are satisfied. Let $0<u_j\leq u_j'<1$, $j=1, 2$, let $(i_1, i_2)\in\mathcal{J}(2)$ (see Section 2 for $\mathcal{J}(2)$) and let $F^{-1}(u)$ be the inverse function of $F(x)$. Put $t_j=F^{-1}(u_j)$ and $t_j'=F^{-1}(u_j')$, $j=1, 2$. It is easy to see that $(\alpha, \beta, \lambda)\in\mathcal{S}\equiv\bigcap_{j=1}^{2} F_{x_{i_j}}^{-1}([u_j, u_j'])$ if and only if

$$\lambda=\lambda(t, t', \xi)\equiv\xi\,,$$
$$\beta=\beta(t, t', \xi)\equiv(t'-t)^{-1}\log((x_{i_2}-\xi)/(x_{i_1}-\xi))\,,$$
$$\alpha=\alpha(t, t', \xi)\equiv\exp[(t'-t)^{-1}(t'\log(x_{i_1}-\xi)-t\log(x_{i_2}-\xi))]$$

for some $(t, t', \xi)\in[t_1, t_1']\times[t_2, t_2']\times[\lambda_1, \lambda_2]$ with $t<t'$.

To see that condition (F. 2) is satisfied, put $f(x, \theta)=\log((x-\lambda)/\alpha)^{1/\beta}$ $(\theta=(\alpha, \beta, \lambda)\in\Theta)$ and $\theta(t, t', \xi)=(\alpha(t, t', \xi), \beta(t, t', \xi), \lambda(t, t', \xi))$. Then

$$f(x_h, \theta(t, t', \xi))=(1-\eta(\xi))t+\eta(\xi)t'\,,$$

where $\eta(\xi)=\log[(x_h-\xi)/(x_{i_1}-\xi)]/\log[(x_{i_2}-\xi)/(x_{i_1}-\xi)]$. Since $\eta(\xi)$ is continuous on $[\lambda_1, \lambda_2]$ and since t and t' are bounded, $f(x_h, \theta(t, t', \xi))$ is bounded on $[t_1, t_1']\times[t_2, t_2']\times[\lambda_1, \lambda_2]$ for all $h=1, \cdots, m$.

To see that condition (F. 3) is satisfied, let $u_1'<u_2$. Then $\mathcal{S}$ is compact, since $\theta(t, t', \xi)$ is continuous on $[t_1, t_1']\times[t_2, t_2']\times[\lambda_1, \lambda_2]$ and since $\mathcal{S}=\{\theta(t, t', \xi)\,;\ (t, t', \xi)\in[t_1, t_1']\times[t_2, t_2']\times[\lambda_1, \lambda_2]\}$.

The proof for condition (F. 2) also shows that condition (F. 5) is satisfied. Condition (F. 5) implies condition (F. 6). It is easy to see that condition (F. 4) is satisfied.

Proof of 3.9. Because of Theorem 2.1, it suffices to show that conditions (F. 2)-(F. 6) with $p=2$ are satisfied.

To do this we prepare

Lemma 4.4. Let Y_n, $n\geq 1$, be a random variable which has the gamma d. f. $F(x, \theta_n)$ with $\theta_n=(\alpha_n, \beta_n, \lambda_n)\in\Theta$, and let $\lim_n \alpha_n=\alpha$, $\lim_n \beta_n=\beta$, $\lim_n \lambda_n=\lambda$ and $\lim_n E(Y_n)=M$. Then:

(i) If $(\alpha, \beta)\notin(0, \infty)\times(0, \infty)\cup[0, \infty)\times\{\infty\}$, then P-$\lim Y_n=M$.
(ii) If $\alpha_n>1$ for all n and if $1\leq\alpha<\beta=\infty$, then P-$\lim Y_n=\infty$.
(iii) If $\alpha_n\leq 1$ for all n, if $\beta=\infty$ and if $\lim_n F(v', \theta_n)$ exists for some $v'\in(\lambda_2, \infty)$, then $\lim_n F(v, \theta_n)=\lim_n F(v', \theta_n)$ for all $v\in(\lambda_2, \infty)$.

Proof. (i) Note that $E(Y_n)=\alpha_n\beta_n+\lambda_n$ and $\mathrm{Var}(Y_n)=\alpha_n\beta_n^2$. In order to prove the assertion (i), it suffices to show that $\lim_n Y_n=M$ in case $M<\infty$ and $\lim_n \mathrm{Var}(Y_n)=0$, and that P-$\lim Y_n=\infty$ in case $\alpha=M=\infty$. Let $t>0$. Then, by Proposition 4.1, $\lim_n \Pr(|Y_n-M|\geq t)\leq\lim_n \mathrm{Var}(Y_n)/(t-|E(Y_n)-M|)^2=0$ for the first case and $\lim_n \Pr(|Y_n|\leq t)\leq\lim_n \alpha^{-1}(E(Y_n)-\lambda_n)^2(E(Y_n)-t)^{-2}=0$ for the second case.
(ii) Put $f(x, \theta)=(\Gamma(\xi)\eta^{\xi})^{-1}(x-\nu)^{\xi-1}\exp(-(x-\nu)/\eta)$ with $\theta=(\xi, \eta, \nu)$ and let $t>0$. Noting that $f(\beta_n(\alpha_n-1)+\lambda_n, \theta_n)=\max\{f(x, \theta_n)\,;\ x\in(\lambda_2, \infty)\}$, we have

$$\begin{aligned}\Pr(|Y_n|\le t)&\le \Pr(\min(-t,\lambda_n)\le Y_n\le \max(t,\lambda_n))\\ &\le(\max(t,\lambda_n)-\min(-t,\lambda_n))f(\beta_n(\alpha_n-1)+\lambda_n,\theta_n)\\ &\le 2(t+|\lambda_n|)(\alpha_n-1)^{\alpha_n-1}(\Gamma(\alpha_n)\beta_n)^{-1}\exp(1-\alpha_n)\,.\end{aligned}$$

This leads to the assertion (ii).
(iii) Note that $\lim_n f(x,\theta_n)=0$ for all $x\in(\lambda_2,\infty)$ and that $f(x,\theta_n)$ is strictly decreasing on (λ_n,∞). Let $v\in(\lambda_2,\infty)$. Then

$$|F(v',\theta_n)-F(v,\theta_n)|=\left|\int_v^{v'} f(x,\theta_n)dx\right|\le|v'-v|(f(v',\theta_n)+f(v,\theta_n))\,.$$

This completes the proof.

Let $0<u_j\le u_j'<1$, $j=1,2$, let $v\in(\lambda_2,\infty)$ and let $(i_1,i_2)\in\mathcal{J}(2)$. Put $\mathcal{S}=\cap_{j=1}^2 F_{x_{i_j}}^{-1}([u_j,u_j'])$.

To see that condition (F. 2) is satisfied, it suffices to show that there is a pair (u,u') with $0<u<u'<1$ such that $u\le F(v,\theta)\le u'$ for all $\theta\in\mathcal{S}$. Suppose that there is a sequence $\{\theta_n=(\alpha_n,\beta_n,\lambda_n)\}$ in $\mathcal{S}$ such that $\lim_n F(v,\theta_n)=0$ or 1, and let Y_n be a random variable having the gamma d. f. $F(x,\theta_n)$. Without loss of generality we may assume that $\lim_n F(x_{i_1},\theta_n)=s_1$, $\lim_n\alpha_n=\alpha$, $\lim_n\beta_n=\beta$, $\lim_n\lambda_n=\lambda$ and $\lim_n(\alpha_n\beta_n+\lambda_n)=M$. Suppose $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$. Then $\lim_n F(v,\theta_n)=F(v,\theta)$ with $\theta=(\alpha,\beta,\lambda)$, which is a contradiction. Suppose $(\alpha,\beta)\in[0,\infty)\times\{\infty\}$. Then, by (ii) or (iii) in Lemma 4.4, we can derive a contradiction. Therefore P-lim$Y_n=M$ by (i) in Lemma 4.4. This is impossible since $\{\theta_n\}\subset\mathcal{S}$.

To see that condition (F. 3) is satisfied, let $u_1'<u_2$. Assume that $\{\theta_n=(\alpha_n,\beta_n,\lambda_n)\}$ is a sequence in $\mathcal{S}$ such that $\lim_n\alpha_n=\alpha$, $\lim_n\beta_n=\beta$, $\lim_n\lambda_n=\lambda$ and $\lim_n F(x_{i_1},\theta_n)=s_1$. By the same argument as above we see $(\alpha,\beta,\lambda)\in(0,\infty)\times(0,\infty)\times[\lambda_1,\lambda_2]$.

To see that condition (F. 5) is satisfied, let $0<u<1$, let $v<v'<\infty$ and fix $\alpha\in(0,\infty)$. Because of Lemma 4.4, $\lim_{\beta\to0}F(v,(\alpha,\beta,\lambda_2))=1$ and $\lim_{\beta\to\infty}F(v,(\alpha,\beta,\lambda_2))=0$. Hence for each $\alpha\in(0,\infty)$ there exists a number $\beta(\alpha)$ such that $F(v,(\alpha,\beta(\alpha),\lambda_2))=u$. We show $\{F(v',(\alpha,\beta(\alpha),\lambda_2))\,;\ \alpha\in(0,\infty)\}=(u,1)$. Choose a sequence $\{\alpha_n\}$ in $(0,\infty)$ such that $\lim_n\alpha_n=0$ and $\lim_n\beta(\alpha_n)=\beta$. Suppose $\beta<\infty$ and let Y_n be a random variable having the gamma d. f. $F(x,\theta_n)$ with $\theta_n=(\alpha_n,\beta_n,\lambda_2)$. Then, by (i) in Lemma 4.4, P-lim$Y_n=\lim_n(\alpha_n\beta(\alpha_n)+\lambda_2)=\lambda_2$. This implies that $\lim_n F(v,\theta_n)=1$, which is a contradiction. Hence $\beta=\infty$ and $\lim_n F(v,\theta_n)=u$ by (iii) in Lemma 4.4. Choose a sequence $\{\alpha_n\}$ in $(1,\infty)$ such that $\lim_n\alpha_n=\infty$ and that $\lim_n\beta(\alpha_n)=\beta$ and $\lim_n\alpha_n\beta(\alpha_n)+\lambda_2)=M$. By the same argument as above, we have $\beta=0$. Since $F(v,\theta_n)=u$ with $\theta_n=(\alpha_n,\beta(\alpha_n),\lambda_2)$, we conclude $M\le v$ by (i) in Lemma 4.4. This yields $\lim_n F(v',\theta_n)=1$. It is easy to see that condition (F. 5) implies condition (F. 6) and that condition (F. 4) is satisfied.

Proof of 3.10. Because of Theorem 2.2, is suffices to show that conditions (F. 2)-(F. 5) with $p=1$ are satisfied and that $s=\sup\{\max_{1\le k\le m-1}(F(\sum_{i=1}^r(\alpha_i x_{k+1})^i)-F(\sum_{i=1}^r(\alpha_i x_k)^i))\,;\ (\alpha_1,\cdots,\alpha_r)\in\Theta\}<1$. It is easy to see that condition (F. 4) is satisfied.

To see that conditions (F. 2) and (F. 3) are satisfied, let $0<u_1\le u_1'<1$ and let $1\le j\le m$. It can be easily seen that $(\alpha_1,\cdots,\alpha_r)\in F_{x_j}^{-1}([u_1,u_1'])$ if and only if $F^{-1}(u_1)\le\sum_{i=1}^r(\alpha_i x_j)^i\le F^{-1}(u_1')$, where $F^{-1}(u)$ is the inverse function of $F(x)$. This implies that $F_{x_j}^{-1}([u_1,u_1')]$ is nonempty and compact.

From $\lim_{(\alpha_1,\cdots,\alpha_r)\to(0,\cdots,0)} F(\sum_{i=1}^r (\alpha_i x_j)^i)=0$ and $\lim_{\alpha_1\to\infty} F(\sum_{i=1}^r (\alpha_i x_j)^i)=1$, it follows that condition (F. 5) is satisfied.

To see $s<1$, assume that there exist an integer k, $1\leq k\leq m-1$, and a sequence $\{(\alpha_{1n}, \cdots, \alpha_{rn})\}$ in Θ such that $\lim_n F(\sum_{i=1}^r (\alpha_{in} x_{k+1})^i)=1$ and $\lim_n F(\sum_{i=1}^r (\alpha_{in} x_k)^i)=0$. The former implies that at least one of sequences $\{\alpha_{1n}\}, \cdots, \{\alpha_{rn}\}$ is unbounded, and the latter implies that $\lim_n \alpha_{in}=0$ for all $i=1, \cdots, r$. This is impossible.

Proof of 3.11. Noting that $F(x_i, \theta)=\sum_{k=1}^i \alpha_k$ with $\theta=(\alpha_1, \cdots, \alpha_m)$, $i=1, \cdots, m$, we see that conditions (F. 2)-(F. 6) with $p=m$ are satisfied. By Theorem 2.1, $\partial \boldsymbol{F}(\Theta)=\mathcal{A}_m^*$.

Proof of 3.12. Noting that $F(x_i, \theta)=\alpha_i$ with $\theta=(\alpha_1, \cdots, \alpha_m)$, $i=1, \cdots, m$, we see that conditions (F. 2)-(F. 6) with $p=m$ are satisfied. By Theorem 2.1, $\partial \boldsymbol{F}(\Theta)=\mathcal{A}_m^*$.

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