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ON THE STRUCTURE OF DYNAMICAL SYSTEMS SATISFYING A CERTAIN STABILITY CONDITION

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1. Introduction.

In this paper, we are going to classify in detail dynamical systems in locally compact spaces which have a global property called *characteristic 0^+*. Dynamical systems of characteristic 0^+ are those in which the positive prolongation of each point coincides with the closure of positive semi-trajectory through it. Actually we can define a dynamical system of characteristic α^+ for any ordinal number α by using the positive prolongation of order α, and the dynamical system of characteristic 0^+ is the simplest case of this definition.

In [1] Ahmad classified planar flows of characteristic 0^+ (0^-, $0^\pm$) in terms of their critical points. This property is really interesting and his results are almost complete. Though Ahmad [2] also deals with dynamical systems of characteristic 0^+ (0^-, $0^\pm$) in locally compact spaces, we work out this problem by using the method different from his adopted in [2]. Although our classification is not so complete as that of planar flows, it shows clearly the global structure of such flows and is more detailed than the results in [2]. In our process of the proof, first we show that the flow of characteristic 0^+ doesn't have any non-compact and non-trivial minimal sets. From this assertion, we can classify the flows of characteristic 0^+ in terms of the set F of all the points which belong to compact minimal sets. It is shown that if the set F is empty, then the flow is dispersive, if F is compact and X is connected, then F is globally positively asymptotically stable, and if F is non-compact, then $F=X$ or F is positively asymptotically stable. Furthermore, if X satisfies second countability axiom, the region of positive attraction $A^+(F)$ of F has a countable number of components. Those results are summarized as Theorem 3.24. In particular, the case when F is non-compact and $F\neq X$ is not treated in [2]. Finally, by making use of Knight's proof in [7], we give a necessary and sufficient condition for a flow in locally compact and connected spaces to be characteristic 0^+. Here the set F plays a very important role again. In short, each compact minimal set is positively stable and the set F is shown to be in one of three cases: $F=\phi$ and the flow is dispersive; F is compact and is a global positive attractor; F is non-compact and the flow restricted to X

$-A^+(F)$ is dispersive.

2. Definitions, Notations, and Preliminary Results.

Throughout this paper R, R^+, and R^- will denote the real numbers, non-negative real numbers, and non-positive real numbers, respectively. Given a topological space X and a mapping π from the product space $X\times R$ into X, the pair (X, π) is called to define a *dynamical system* or (*continuous*) *flow* if the following axioms are satisfied.

1. *Identity axiom:* $\pi(x, 0)=x$.
2. *Homomorphism axiom:* $\pi(\pi(x, t), s)=\pi(x, s+t)$ for each $x\in X$ and $s, t\in R$.
3. *Continuity axiom:* π is continuous on $X\times R$.

In this paper X will always be Hausdorff. For brevity, we denote $\pi(x, t)$ by xt. For each $x\in X, xR, xR^+$, and xR^- are called the *trajectory* (*or orbit*), *positive semi-trajectory*, and *negative semi-trajectory* through x and will be denoted by $C(x)$, $C^+(x)$, and $C^-(x)$, respectively. A point $x\in X$ is called a *critical point* or a *rest point* if $xR=x$. If x is not critical point and $xt=x$ for some $t>0$, then x is called *periodic*. A subset M of X is said to be *invariant* if $C(M)=M$, and *positively* (*negatively*) *invariant* if $C^+(M)=M(C^-(M)=M)$. A closed (positively) invariant set M is (*positively*) *minimal* if it has no proper subset which is closed and (positively) invariant. Negative minimality is defined similarly.

We denote the boundary, interior, and closure of a subset M of X by ∂M, $\operatorname{int} M$, and $\bar{M}$, respectipely. The sets $\overline{C(x)}$, $\overline{C^+(x)}$, and $\overline{C^-(x)}$ will be denoted by $K(x)$, $K^+(x)$, and $K^-(x)$, respectively. The *ω-limit set* and *α-limit set* of x are denoted by $L^+(x)$ and $L^-(x)$, respectively, i. e.,

$$L^+(x)=\{y\in X;\ xt_i\to y \text{ for some net } t_i\to+\infty\}$$
$$L^-(x)=\{y\in X;\ xt_i\to y \text{ for some net } t_i\to-\infty\}.$$

For each $x\in X$, the (*first*) *positive* (*negative*) *prolongation* $D^+(x)(D^-(x))$ *of* x is defined by

$$D^+(x)(D^-(x))=\{y\in X;\ x_it_i\to y \text{ for some net } x_i\to x \text{ and } t_i\in R^+(t_i\in R^-)\}.$$

The (*first*) *positive* (*negative*) *prolongational limit set* $J^+(x)(J^-(x))$ *of* x is defined by

$$J^+(x)(J^-(x))=\{y\in X;\ x_it_i\to y \text{ for some net } x_i\to x \text{ and } t_i\to+\infty\ (t_i\to-\infty)\}.$$

Equivalently, $L^+(x)=\bigcap_{t\in R}\{K^+(xt)\}$ $(L^-(x)=\bigcap_{t\in R}\{K^-(xt)\})$ and $D^+(x)=\bigcap_{N\in\eta(x)}\{K^+(N)\}$ $(D^-(x)=\bigcap_{N\in\eta(x)}\{K^-(N)\})$ and $J^+(x)=\bigcap_{t\in R}\{D^+(xt)\}$ $(J^-(x)=\bigcap_{t\in R}\{D^-(xt)\})$, where $\eta(x)$ is the neighborhood filter of x. It follows that $L^+(x)\subset J^+(x)\subset D^+(x)$, $D^+(x)=C^+(x)\cup J^+(x)$ and $K^+(x)=C^+(x)\cup L^+(x)$ when X is Hausdorff.

A point $x\in X$ or the trajectory $C(x)$ is called *positively* (*negatively*) *receding*, if $L^+(x)(L^-(x))=\phi$; *receding*, if x is receding both positively and negatively; *positively* *(negatively) asymptotic*, if $L^+(x)(L^-(x))\neq\phi$ but $L^+(x)\cap C(x)=\phi(L^-(x)\cap C(x)=\phi)$; *positively (negatively) Poisson-stable*, if $L^+(x)\cap C(x)\neq\phi(L^-(x)\cap C(x)\neq\phi)$; *Poisson-stable*, if x is both positively and negatively Poisson-stable.

A set $M\subset X$ is said to be *positively orbitally stable* or simply *positively stable* if for every neighborhood of U of M, there exists a neighborhood V of M such that $C^+(V)\subset U$. Negative stability is defined similarly. M is said to be *bilaterally stable* if it is positively and negatively stable. A closed positively (negatively) invariant set M is said to be *positively (negatively) D-stable* if $D^+(M)=M(D^-(M)=M)$. The following theorem of Ura plays an important role in stability theory of compact sets.

Theorem 2.1. *Let (X,π) be a dynamical system on a locally compact space. Then a compact subset M is positively stable if and only if it is positively D-stable.*

The point $x\in X$ is said to be *attracted to M* if the net (xt) for $t\in R^+$ is ultimately in every neighborhood of M. The set of all such points x is called the *region of positive attraction of M* and will be denoted by $A^+(M)$. If $A^+(M)$ is a neighborhood of M, then M is called a *positive attractor*. M is said to be *positively asymptotically stable* if it is both positively stable and a positive attractor. M is said to be *globally positively asymptotically stable* if it is positively asymptotically stable and $A^+(M)=X$. We state some properties of $A^+(M)$ which will be used in this paper.

Lemma 2.2.

(1) *$A^+(M)$ is open if M is a positive attractor.*
(2) *If X is locally compact and M is compact, then it follows that $x\in A^+(M)$ if and only if $\phi\neq L^+(x)\subset M$.*

Also, in case of M being closed, (2) holds under suitable conditions.

Proposition 2.3. *Let (X,π) be a flow where X is normal and let M be a closed positively stable set with $\phi\neq L^+(x)$ for each $x\in M$. Then, $y\in A^+(M)$ if and only if $\phi\neq L^+(y)\subset M$.*

A point $x\in X$ is called *dispersive* if $J^+(x)=\phi$. A flow (X,π) is called *dispersive* if each point $x\in X$ is dispersive. A flow is called *parallelizable* if it is isomorphic to a parallel flow; that is, if there exists a set $S\subset X$ and a homeomorphism $h:X\to S\times R$ such that $SR=X$ and $h(xt)=(x,t)$ for every $x\in S$ and $t\in R$.

Theorem 2.4. *(X,π) on a locally compact separable metric space X is parallelizable if and only if it is dispersive.*

Now we give the definition of dynamical systems of characteristic 0^+.

Definition 2.5. A dynamical system (X,π) is said to have *characteristic 0^+ (0^-)*

if and only if $D^+(x)=K^+(x)$ $(D^-(x)=K^-(x))$ for all $x\in X$. It is said to have *characteristic* $0^\pm$ if and only if $D^+(x)=K^+(x)$ and $D^-(x)=K^-(x)$ for all $x\in X$, i.e., it has characteristic 0^+ and characteristic 0^-.

Actually, we can define a positive prolongation of order α, denoted by $D_\alpha^+(x)$, for any ordinal number α (see [3]), and (X,π) is said to have *characteristic* α^+ if and only if $D_\alpha^+(x)=D_{\alpha+1}^+(x)$ for all $x\in X$. (X,π) having characteristic 0^+ is the simplest case of this definition, since $D_0^+(x)=K^+(x)$ and $D_1^+(x)=D^+(x)$.

Lemma 2.6. *Each of the following conditions is equivalent to (X,π) having characteristic 0^+:*

(1) *$L^+(x)=J^+(x)$ for all $x\in X$.*
(2) *Every closed positively invariant subset of X is positively D-stable.*

Thus, our main purpose is to study in detail the flows whose closed positively invariant set is always positively D-stable.

Ahmad [1] gives a complete classification of planar flows having characteristic 0^+, $0^\pm$. The results are summarized in the following two theorems. Hereafter, we denote by S, P the set of critical points and the set of periodic points of a given flow (R^2,π) respectively.

Theorem 2.7. *Let (R^2,π) be a dynamical system of characteristic 0^+. Then one of the following three assertions holds.*

(1) *$S=\phi$ and (R^2,π) is pararelizable.*
(2) *Compactness of S implies one of the following.*
   (a) *$S=\{s_0\}$ is a singleton and s_0 is a global Poincaré center.*
   (b) *$S=\{s_0\}$ is a singleton and s_0 is a local Poincaré center. Further, the set N consisting of s_0 and periodic orbits surrounding s_0, is a globally asymptotically positively stable simply connected continuum.*
   (c) *S is globally asymptotically stable and is a simply connected continuum.*
(3) *If S is unbounded, then either*
   (A) *$S=R^2$, or*
   (B) *the following hold.*
      (a) *R^2-S is unbounded.*
      (b) *S is positively asymptotically stable.*
      (c) *$A^+(S)$ has a countable number of components, each being homeomorphic to R^2 and unbounded.*
      (d) *S has a countable number of components, each being non-compact and simply connected. For each $s\in\partial S$, there is a regular point y with $L^+(y)=\{s\}$.*
      (e) *$A^+(S_0)$ is a component of $A^+(S)$ if and only if S_0 is a component of S.*
      (f) *For each $x\in R^2$, $L^+(x)$ is either empty or consists of a single critical point. Further, $L^+(x)=\phi$ for all $x\notin A^+(S)$ and $L^-(x)=\phi$ for all $x\in R^2-S$.*

Theorem 2.8. *Let* (R^2, π) *be a dynamical system of characteristic* $0^{\pm}$, *then one of the following holds.*

(1) $S=\phi$ *and* (R^2, π) *is parallelizable.*
(2) $S=R^2$.
(3) $S=\{s_0\}$ *is a singleton and* s_0 *is a global Poincaré center.*

Furthemore, for the dynamical systems of characteristic 0^+ on S^2, the following alternative holds.

(1) $S=S^2$.
(2) $S=\{x, y\}$, $P=S^2-\{x, y\}$, and x, y are both Poincaré centers.

Consequently, (S^2, π) has characteristic $0^{\pm}$ if it is of characteristic 0^+.

Successful results described in Theorem 2.7, 2.8 are due to the fact that phase space X is R^2 and in particular, to the validity of Jordan Curve Theorem. In more detail, they depend on the following two facts:

(1) There exists a critical point inside every periodic orbit.
(2) A point is positively (negatively) Poisson stable if and only if it is a critical point or a periodic point (see [9]).

However, for more general topological spaces, these two theorems are not always true. For example, consider the following flow illustrated in Fig. 1.

$$X=\{xy\text{-plane and } z\text{-axis}\}, \qquad S=\{(0,0,0)\}, \qquad P=\{(x, y, 0)\,;\ x^2+y^2\leq 1\}$$

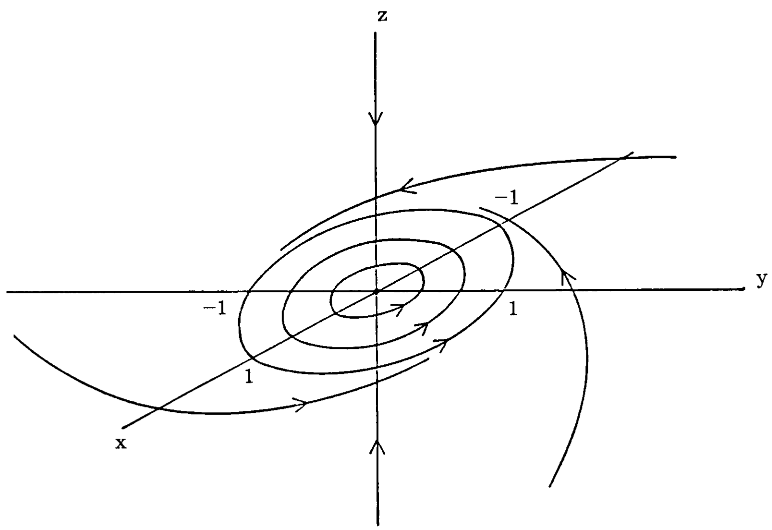

Fig. 1. Evidently, this flow has characteristic 0^+, but the critical point is neither positively asymptotically stable nor a Poincaré center.

3. Dynamical Systems of Characteristic 0^+ in Locally Compact Spaces.

Ahmad [2] studied flows of characteristic 0^+ ($0^\pm$) on locally compact spaces. The classification is based on three mutually exclusive and exhaustive cases; (I) $L^+(x)=\phi$ for all $x\in X$. (II) there exists a compact invariant subset of X which is isolated from positively minimal sets. (III) neither one of the above two cases occurs. His result can be stated as follows.

Theorem 3.1. *Let (X, π) be a dynamical system of characteristic 0^+ on a locally compact space X. Then one of the following three assertions (A), (B) and (C) holds.*

(A) *The flow is dispersive.*

(B) *There exists a compact invariant subset Q satisfying following two conditions (B1) and (B2). (B1). Q is positively asymptotically stable. Furthermore, it is globally asymptotically stable if X is connected; (B2). For each $x\in A^+(Q)-Q$, $L^-(x)=\phi$.*

(C) *There exists $x\in X$ such that $L^+(x)\neq\phi$. For each such x, $Q=L^+(x)$ is a compact minimal set and the following alternative holds; (C1). Q is bilaterally stable and has a neighborhood V consisting of Poisson-stable points such that $L^-(x)\cap Q=L^+(x)\cap Q=\phi$ for all $x\in V-Q$; (C2). Q is positively stable and negatively unstable. Furthermore, there exists $y\notin Q$ with $L^+(y)=Q$, and each neighborhood V of Q contains a complete trajectory contained in $V-Q$.*

As can be seen from the example shown in Fig. 1, it seems insufficient to notice only the set S of critical points for the classification of flows of characteristic 0^+ in general phase spaces. But, in this example, the union of critical points and periodic points is positively asymptotically stable. Therefore, from Seibert and Tulley's theorem, it seems natural to replace the set of critical points by the set of all the points which belong to compact minimal sets for the classification of flows having characteristic 0^+ in locally compact spaces. Adopting such viewpoint, we shall try to classify flows of characteristic 0^+ in locally compact spaces as minutely as possible following the method of Ahmad successfully developed in [1].

To begin with, we state some properties of dynamical systems of characteristic 0^+.

Lemma 3.2. *Let (X, π) be any flow. If $x\in X$ and $y_1, y_2\in L^+(x)$, then $y_1\in D^+(y_2)$ and $y_2\in D^+(y_1)$.*

Lemma 3.3. *Let (X, π) be a flow of characteristic 0^+. If $L^-(x)\neq\phi$, then x is negatively Poisson-stable.*

Proposition 3.4. *Let (X, π) be a flow of characteristic 0^+ on a connected locally compact space X. If M is a compact positively invariant subset of X and M is a positive attractor, then M is globally positively asymptotically stable.*

Also, the following properties of minimal sets are necessary for our discussion.

Definition 3.5. A minimal set is called *trivial* if it consists of only one trajectory. A minimal set which is not trivial is called *non-trivial.*

Therefore, a trivial minimal set is a receding orbit, or a periodic orbit, or a critical point. The structure of a compact minimal set is completely determined by Birkhoff [6] and recently Kono [8] discovered an intrinsic property of motions in a non-compact and non-trivial minimal set. The following proposition of non-compact and non-trivial minimal sets holds also in case of X being locally compact Hausdorff.

Proposition 3.6. *Let X be locally compact and $M\subset X$ be a non-compact and non-trivial minimal set. Then a trajectory in M is either Poisson-stable, or positively Poisson-stable and negatively receding, or negatively Poisson-stable and positively receding.*

Proof. See [8].

From the structure of non-compact and non-trivial minimal sets, we get the following theorem.

Theorem 3.7. *Let (X, π) be a dynamical system of characteristic 0^+ in a locally compact space. Then there do not exist any non-compact and non-trivial minimal sets.*

Before proving this theorem, we show some lemmas with respect to limit sets on dynamical systems of characteristic 0^+.

Lemma 3.8. *Let (X, π) be any dynamical system and $M\subset X$ be nonempty. Then the following assertions are equivalent.*

(1) *M is positively (negatively) minimal.*
(2) *$K^+(x)=M(K^-(x)=M)$ for all $x\in M$.*
(3) *$L^+(x)=M(L^-(x)=M)$ for all $x\in M$.*

Lemma 3.9. *Let (X, π) be a dynamical system of characteristic 0^+. If $L^+(x)$, $L^-(x)\neq\phi$, then they both positively minimal.*

Proof. Let y be any point of $L^+(x)$. If $z\in L^+(x)$, then $z\in D^+(y)=K^+(y)$ by Lemma 3.2. Hence $L^+(x)\subset K^+(y)$. Also, $K^+(y)\subset L^+(x)$ since $L^+(x)$ is a closed invariant set. Hence $K^+(y)=L^+(x)$ for all $y\in L^+(x)$. By (2) of Lemma 3.8, $L^+(x)$ is positively minimal. Similarly, by the dual result of Lemma 3.2, $L^-(x)$ is positively minimal.

Here, we make use of the result due to Hájek concerning positively minimal sets.

Theorem 3.10. *A subset of a locally compact phase space is positively (negatively) minimal if and only if it is compact minimal.*

Consequently, if X is locally compact and $L^+(x)$, $L^-(x)$ are nonempty sets in a dynamical system of characteristic 0^+, they are compact minimal sets. Furthermore, either $L^-(x)=\phi$ or $L^-(x)=L^+(x)=K(x)$ by Lemma 3.3.

Proof of Theorem 3.7. Suppose that there exists a non-compact and non-trivial minimal set M. If there exists a positively Poisson-stable point x in M, then $L^+(x)=M$. But this contradicts to the non-compactness of M as $L^+(x)$ is compact minimal. Hence, from Proposition 3.6, every point in M is negatively Poisson-stable and positively receding. Then $L^-(x)=M$ for every $x\in M$. Hence by (3) of Lemma 3.8, $M=L^-(x)$ is negatively minimal, i. e., compact minimal, which is a contradiction. Therefore a dynamical system of characteristic 0^+ doesn't have any non-compact and non-trivial minimal sets.

From Theorem 3.7, we can classify the structure of flows of characteristic 0^+ only by noticing the set of all the points belonging to compact minimal sets. In this section, hereafter we assume that X is locally compact and (X, π) has characteristic 0^+.

Define a set F by;

$$F=\{x\in X: x\in M,\ M \text{ is any compact minimal set}\}$$

Theorem 3.11. *If $F=\phi$, then the flow (X, π) is dispersive. Furthermore, (X, π) is parallelizable if X is a separable metric space.*

Proof. $L^+(x)=\phi$ for all $x\in X$ as $F=\phi$. It is obvious from Lemma 2.6 that the flow is dispersive. Latter assertion follows from Theorem 2.4.

Theorem 3.12. *If F has a compact component F_0 which is isolated from $(F-F_0)$, then F_0 is positively asymptotically stable. Furthermore, F_0 is globally positively asymptotically stable and $F=F_0$ if X is connected.*

Proof. From the assumption, there exists a compact neighborhood U of F_0 such that $U\cap(F-F_0)=\phi$. Since $D^+(F_0)=F_0$, F_0 is positively stable. Then there exists a neighborhood V of F_0 such that $C^+(V)=V\subset U$. Since $\phi\neq L^+(x)$ for all $x\in V$, we get $L^+(x)\subset F_0\subset V$ because $L^+(x)$ is compact minimal. Therefore, F_0 is a positive attractor by Lemma 2.2. If X is connected, F_0 is globally positively asymptotically stable by Proposition 3.4. Hence $F=F_0$.

Theorem 3.13. *If F is compact, then F is positively asymptotically stable. Furthermore, it is globally positively asymptotically stable if X is connected.*

Theorem 3.14. *If F is non-compact, then the following assertions hold.*

(1) *Either* $F=X$, *or* F *is closed and* $\overline{X-F}$ *is non-compact.*
(2) *If* $F\neq X$, *then* F *is positively asymptotically stable.*
(3) $L^{\pm}(x)=\phi$ *for all* $x\in X-A^{+}(F)$.

Proof. In statement (1), we first show that F is closed when $F\neq X$. If F is not closed, then there exists a point $x\in(\partial F-F)$. Here, for some $t>0$, we take the sequence of open neighborhoods (U_n) of $x(-t)$ such that $U_n\supset U_{n+1}$ for each n and $\bigcap_n U_n=x(-t)$. Since there exists a net (x_n) such that $x_n\in F$ for each n and $x_n\to x$, we can assume that $x_n(-t)\in U_n$ for each n by continuity of the flow. Further, for each fixed n, there exists a net (t_{n_k}) such that $t_{n_k}\geq 0$ and $x_nt_{n_k}\to x_n(-t)$ as $x_n(-t)\in F$. For $n=1$, we can select t_{1_k} such that $x_1t_{1_k}\in U_1$. Discarding the sequence $\{t_{1_j}; j<k\}$ from the original one, we may consider that $x_1t_{1_1}\in U_1$ without loss of generality. Also in case of $n=2$, we can select t_{2_2} so that $x_2t_{2_2}\in U_2$. Similarly we can assert that $x_nt_{n_n}\in U_n$ for each n. Then obviously $x_nt_{n_n}$ converges to $x(-t)$. Hence $x(-t)\in D^{+}(x)=K^{+}(x)$. If $x(-t)\in C^{+}(x)$, then $x(-t)=x(s)$ for some $s\geq 0$. Hence x is periodic because $x=x(s+t)$. And, if $x\in L^{+}(x)$, then x belongs to some compact minimal set. But each case contradicts the assumption that $x\notin F$. Therefore F is closed. Suppose that $F\neq X$ and $\overline{X-F}$ is compact. Then, for all $x\in X-F$, $L^{-}(x)=L^{+}(x)=K(x)$ is compact minimal, which is a contradiction. Therefore (1) holds. Since each compact minimal set $M\subset\partial F$ is positively stable, for all neighborhood U of F there exists a neighborhood V_M of M such that $C^{+}(V_M)\subset U$. Let a neighborhood V_F of F be defined to be $(\operatorname{int} F)\cup\{\bigcup_{M\subset\partial F}(V_M)\}$, then $C^{+}(V_F)\subset U$. Hence F is positively stable. For some compact neighborhood U_M of each $M\subset\partial F$, let W_M be a neighborhood of M such that $C^{+}(W_M)\subset U_M$. Then $\phi\neq L^{+}(x)\subset F$ for all $x\in W_M$. Therefore we select $(\operatorname{int} F)\cup\{\bigcup_{M\subset\partial F}(W_M)\}$ as a neighborhood W_F of F, then $\phi\neq L^{+}(x)\subset F$ for all $x\in W_F$. This implies the net (xt) must be ultimately in any neighborhood U of F for all $x\in W_F$. Since F is positively stable, this shows that F is positively asymptotically stable. The last assertion is obvious from Lemma 3.9 and Theorem 3.10. The proof is complete.

Corollary 3.15. *Let* X *be connected. If* $F\neq X$ *and* F *is non-compact, then an isolated compact component* F_0 *doesn't exist. Furthermore if* X *is normal, then each component of* F *is non-compact.*

Remark. F is not always globally asymptotically stable even if X is connected and F is connected.

Corollary 3.16. (Y,π_Y) *is dispersive where* $Y=X-A^{+}(F)$ *and* $\pi_Y=\pi|Y$.

Proof. Let $x\in Y$. $D_Y^{+}(x)=C_Y^{+}(x)\cup J_Y^{+}(x)\subset C_Y^{+}(x)\cup(J^{+}(x)\cap Y)=C_Y^{+}(x)\cup(L^{+}(x)\cap Y)$ $=C_Y^{+}(x)\cup L_Y^{+}(x)=K_Y^{+}(x)$. Hence (Y,π_Y) has characteristic 0^{+}. Since there doesn't exist a compact minimal set in Y, (Y,π_Y) is dispersive.

The following two theorems hold independent of the compactness of F.

Theorem 3.17. *Let* $\phi\neq F\subsetneq X$. *For each compact minimal set* $M\subset\partial F$, *there is a point* $y\notin F$ *with* $L^{+}(y)=M$.

Proof. We note that $L^-(x)=\phi$ for all $x\notin F$. Since F is positively asymptotically stable and $A^+(F)$ is open, there exists a compact neighborhood V of M such that $V\subset A^+(F)$. We can find nets (x_n) in X and (t_n) in R^- respectively such that $x_n \to M$, $x_n\in V-F$ and $x_nt_n\in\partial V-F$. Since ∂V is compact, we can find a subnet (x_{n_j}) of (x_n) and (t_{n_j}) of (t_n) and a point $y\in\partial V$ satisfying $x_{n_j}\to M$ and $x_{n_j}t_{n_j}\to y$. This implies that $y\in D^-(M)$. Thus there is a point $m\in M$ such that $y\in D^-(m)$ and we get $m\in D^+(y)=K^+(y)$. If $y\in F$, then there exists a compact minimal set M' such that $M'\cap M=\phi$ and $y\in M'$ as $y\in\partial V$. Hence $K^+(y)=M'$ and $m\in M'$, but this is a contradiction. Therefore $m\in L^+(y)$, i. e., $M=L^+(y)$ as $y\notin F$ and $m\in L^+(y)$.

Theorem 3.18. *If $\mathrm{int}\, F\neq\phi$, every compact minimal set $M\subset \mathrm{int}\, F$ is bilaterally stable.*

Proof. Suppose that M is not negatively stable. Then there exists a compact neighborhood V of M in $\mathrm{int}\, F$ and nets (x_n), (t_n) such that x_n converges to some point $m\in M$, $t_n\leq 0$ and $x_nt_n\in\partial V$. We may assume, if necessary by taking subnets, $x_nt_n\to y\in\partial V$. Hence $y\in D^-(m)$, or equivalently $m\in D^+(y)=K^+(y)$. This is obviously a contradiction since y belongs to a compact minimal set disjoint from M. Therefore M is bilaterally stable.

Theorem 3.19. *If $F\neq X$ and F is non-compact, then the boundary of each component of $A^+(F)$ consists of trajectories such that $L^\pm(x)=\phi$. Furthermore, if X satisfies second countability axiom, then $A^+(F)$ has a countable number of components.*

Proof. Since F is positively asymptotically stable, $A^+(F)$ is open. Let K be any component of $A^+(F)$. Then, $\partial K\cap F=\phi$. Therefore, $L^\pm(x)=\phi$ for all $x\in\partial K$ because ∂K is a closed invariant set. Suppose that $\partial K=K$ for some component K of $A^+(F)$. $(X-C(x))$ is an open neighborhood of F as $x\in\partial K$. But this contradicts with $x\in K\subset A^+(F)$. Therefore $\mathrm{int}\, K\neq\phi$. The second assertion follows immediately from this.

Lemma 3.20. *Let (X,π) be a flow on a normal connected space. If a closed invariant subset M of X with $\phi\neq L^+(x)$ for all $x\in M$ is globally positively asmptotically stable, then M is connected.*

Theorem 3.21. *Let X be normal. If $F\neq X$ and F is non-compact, F_0 is a component of F if and only if $A^+(F_0)$ is a component of $A^+(F)$.*

Proof. Let K_0 be any component of $A^+(F)$. Let $F_0=F\cap K_0$. Then we shall show that F_0 is connected and hence is a component of F. Since $\partial K_0\cap F=\phi$ by Theorem 3.19, the component of F containg F_0 is contained in K_0. However, since F_0 is globally positively asymptotically stable in K_0, it follows from Lemma 3.20 that F_0 is connected. Conversely let F_0 be a component of F. Since F is positively asymptotically stable, it follows from the proof of Lemma 3.20 [1] that F_0 is positively asymptotically stable. Hence $A^+(F_0)$ is open. Suppose that $A^+(F_0)$ is disconnected. Then there exist two nonempty disjoint open sets A_1, A_2 such that

$A^+(F_0)=A_1 \cup A_2$. Thus F_0 must be contained only in one of A_i $(i=1,2)$ as F_0 is closed and connected, which is obviously a contradiction. The proof is complete.

The followings are immediate consequences of the above theorem and Corollary 3.15.

Corollary 3.22. *If X is normal and connected. $F \neq X$ and F is non-compact, then the closure of each component of $A^+(F)$ is non-compact.*

Corollary 3.23. *If X is normal and satisfies second countability axiom and $F \neq X$, then F has a countable number of components.*

Now we summarize the results.

Theorem 3.24. *Let (X, π) be of characteristic 0^+. Then one of the following properties holds.*

(1) *$F=\phi$ and the flow is dispersive.*
(2) *F has an isolated compact component F_0 and the following holds.*
  (a) *F_0 is positively asymptotically stable. Furthermore it is globally positively asymptotically stable and $F=F_0$ is X is connected.*

Moreover, in both cases we have the following two properties.

  (b1) *For each compact minimal set $M \subset \partial F$, there exists a point $y \notin F$ such that $L^+(y)=M$.*
  (b2) *If int $F \neq \phi$, then each compact minimal set $M \subset$ int F is bilaterally stable.*
(3) *F is non-compact and either $F=X$ or the following hold.*
  (a) *F is closed and $\overline{X-F}$ is non-compact.*
  (b) *F is positively asymptotically stable.*
  (c) *For each compact minimal set $M \subset \partial F$, there exist a point $y \notin F$ such that $L^+(y)=M$.*
  (d) *If int $F \neq \phi$, then each compact minimal set $M \subset$ int F is bilaterally stable.*
  (e) *For each $x \in X$, $L^+(x)$, or $L^-(x)$ is either empty or a compact minimal set. Further, $L^+(x)=\phi$ for all $x \in X-A^+(F)$ and $L^-(x)=\phi$ for all $x \notin F$.*
  (f) *The boundary of each component of $A^+(F)$ consists of trajectories of $C(x)$ such that $L^{\pm}(x)=\phi$.*

Adding further conditions to X, we get;

  (g) *If X satisfies second countability axiom, then $A^+(F)$ has a countable number of components.*

If X is normal, then

  (h1) *$A^+(F_0)$ is component of $A^+(F)$ if and only if F_0 is a component of F.*
  (h2) *If X is connected, each component of F is non-compact and the closure of $A^+(F)$ is non-compact.*
  (h3) *If X satisfies second countability axiom, F has a countable number of components.*

Clearly, this theorem is an extension of the results obtained by Ahmad in Theorem 2.7.

Remark. In (B) of Theorem 3.1, a compact invariant set Q consists of compact minimal sets, because each point of Q belongs to some compact minimal set from Lemma 3.3. Comparing (B) of Theorem 3.1 with (2), our assertion is more detailed because (3)-(e) holds also in case of F having an isolated compact component.

Suppose that $\operatorname{int} F \neq \phi$. Then each compact minimal set M in $\operatorname{int} F$ is bilaterally stable, and by (2)-(b1) or (3)-(d) there exists a neighborhood V such that each point in $V-M$ belongs to some compact minimal set. Moreover, for a compact minimal set $M \subset \partial F$, we get (C2) of Theorem 3.1. Therefore our classification of flows of characteristic 0 is more detailed than Theorem 3.1.

We note that for a flow of characteristic $0^{\pm}$, $L^{+}(x)=L^{-}(x)$ if either $L^{-}(x) \neq \phi$ or $L^{+}(x) \neq \phi$. Thus we get immediately the following assertion from Theorem 3.24.

Theorem 3.25. *Let (X, π) be a flow of characteristic $0^{\pm}$, Let Y be any component of X, then one of the following holds.*

(1) *(Y, π_Y) is dispersive.*
(2) *$F_Y = Y$.*

4. Necessary and Sufficient Conditions for a Flow to Be of Characteristic 0^{+}.

The purpose of this section is to give a necessary and sufficient condition for a flow to be of characteristic 0^{+}. The condition given in Theorem 3.24 is necessary but not sufficient (see [7]).

The following theorem shows a necessary and sufficient condition for a flow to have characteristic 0^{+} in general spaces.

Theorem 4.1. *Let X be a locally compact and connected normal space. Then the flow (X, π) is of characteristic 0^{+} if and only if either (1) or (2) is satisfied.*

(1) *F is compact and one of the following holds.*
  (a) *(X, π) is dispersive.*
  (b) *Each compact minimal set is positively stable and F is a global positive attractor.*
(2) *F is non-compact and each of the following holds.*
  (a) *F is closed.*
  (b) *Each compact minimal set is positively stable.*
  (c) *Each $x \in \partial A^{+}(F)$ is dispersive.*
  (d) *The flow restricted to $X - A^{+}(F)$ is dispersive.*

Proof. The necessity of case (1) follows from Theorem 3.24. If the flow is dispersive, then $J^{+}(x)=L^{+}(x)=\phi$ for all $x \in X$. So (1)-(a) is sufficient. Next, suppose that (1)-(b) holds. Since there is a compact minimal set M such that $x\in M$ for each $x\in F$, we get $D^{+}(x)\subset D^{+}(M)=M=K^{+}(x)$. Furthermore, for each $x\in X-F=A^{+}(F)-F$, $\phi\neq L^{+}(x)\subset F$ as X is locally compact and F is compact. Therefore, there exists a compact minimal set M' such that $M'\subset L^{+}(x)$. Let $z\in M'$, then it follows from Lemma 3.2 that $y\in D^{+}(z)\subset D^{+}(M')=M'$ for all $y\in L^{+}(x)$. Hence $L^{+}(x)=M'$. Since $M'=L^{+}(x)\subset J^{+}(x)\subset J^{+}(z)\subset D^{+}(z)\subset D^{+}(M')=M'$ (see [4], 6.15), $J^{+}(x)=L^{+}(x)$. The proof of case (1) is complete.

The necessity of case (2) follows from Theorem 3.24. Assume that (2) holds. For each point $x\in F$, $D^{+}(x)\subset D^{+}(Q)=Q=K^{+}(x)$ where Q is a compact minimal set. $\partial A^{+}(F)\cap F=\phi$. Actually, if not, there is a compact minimal set $Q'\subset\partial A^{+}(F)$. Since there exists a point $x\in\partial A^{+}(F)$ such that $J^{+}(x)\neq\phi$, it contradicts the condition (c). Therefore F is a positive attractor by (a). Since F is closed and positively stable from (a), (b), $\phi\neq L^{+}(x)\subset F$ for each $x\in A^{+}(F)-F$ by Proposition 2.3. By an argument similar to the one used to prove the sufficiency of (1)-(b), we obtain $J^{+}(x)=L^{+}(x)$ for all $x\in A^{+}(F)-F$. For each $x\in\partial A^{+}(F)$, $D^{+}(x)=C^{+}(x)=K^{+}(x)$ as $J^{+}(x)=\phi$. For each $x\in X-\overline{A^{+}(F)}=Y$, $J_{Y}^{+}(x)=\phi$. However, since $J_{Y}^{+}(x)=J^{+}(x)$ for all $x\in X-\overline{A^{+}(F)}$, $J^{+}(x)=\phi$ for all $x\in X-\overline{A^{+}(F)}$. The proof of Theorem 4.1 is complete.

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