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ON A THEOREM ON QUASIANALYTICITY OF A WEAKLY STATIONARY PROCESS

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1. Introduction

Let $X(t, \omega)$, $t \in R^1$, $\omega \in \Omega$, Ω being a given complete probability space, be a measurable weakly stationary process with covariance function

$$(1.1) \quad \rho(u) = \int_{-\infty}^{\infty} e^{iu\lambda} dF(\lambda),$$

where $F(\lambda)$ is the spectral distribution function of $X(t, \omega)$.

Let $C(m_n, (a, b)) = C(m_n)$ be a class of complex valued infinitely many times differentiable functions $f(x)$ satisfying

$$(1.2) \quad \sup_{a < x < b} |f^{(n)}(x)| \leq AK^n m_n, \quad n = 0, 1, 2, \dots,$$

for some sequence $\{m_n, n = 0, 1, 2, \dots\}$ of positive numbers, ∞ being allowed, where A and K are constants independent of n . The class is called quasianalytic, if $f \in C(m_n)$ and $f^{(n)}(x_0) = 0$, $n = 0, 1, \dots$ for some $x_0 \in (a, b)$ implies that $f(x) = 0$ throughout (a, b) . We also consider the class $C_2(l_n, (a, b)) = C_2(l_n)$ of $f(x)$ which are infinitely many times differentiable and are such that

$$(1.3) \quad \sup_{a < x < b} |f^{(2n)}(x)| \leq AK^n l_n, \quad n = 0, 1, 2, \dots,$$

for some sequence $\{l_n, n = 0, 1, \dots\}$ of positive numbers. Writing $m_{2n} = l_n$, $m_{2n+1} = \infty$, $n = 0, 1, \dots$, $C_2(l_n)$ is identical with $C(m_n)$.

I. L. Ivanova [2] has given the theorem that if the covariance function $\rho(u)$ belongs to $C_2(l_n)$ which is supposed to be quasianalytic in R^1 , then $X(t, \omega)$ almost surely belongs to the class $C^{(1/2)}$ which is also quasianalytic.

She concludes this from the fact that

$$(1.4) \quad \sum_{n=0}^{\infty} P(|X^{(n)}(t, \omega)| \geq AK^n l_n^{1/2}) < \infty.$$

We here used slightly different notations from [2]. This derivation, however, involves a vague point, because the subset of Ω inside $P(\cdot)$ depends on t in general. The author has recently given a different proof [3] of the above theorem of Ivanova for the case of a periodic weakly stationary process. In this paper we shall give another proof for the general case, showing in place of (1.4) that

$$(1.5) \quad \sum_{n=0}^{\infty} P(\max_{|t| \leq B} |X_0^{(n)}(t, \omega)| \geq AK_1^n l_{n+1}^{1/2}) < \infty,$$

for every fixed constant B for some modification $X_0(t, \omega)$ of $X(t, \omega)$ and for some K_1 (depending on B).

2. Lemmas

Lemma 1. $C_2(l_n, (a, b))$ is quasianalytic if and only if $C(l_{n+1}^{1/2}, (a, b))$ is quasianalytic.

This was given in Lemma 7.2 [3].

Suppose without loss of generality that $EX(t, \omega) = 0, t \in R^1$. Write

$$(2.1) \quad Y(t, \omega) = (1+t^2)^{-1} X(t, \omega), \quad t \in R^1.$$

Since

$$\begin{aligned} E \int_{-\infty}^{\infty} |Y(t, \omega)| dt &= \int_{-\infty}^{\infty} (1+t^2)^{-1} E|X(t, \omega)| dt \leq \int_{-\infty}^{\infty} (1+t^2)^{-1} [E|X(t, \omega)|^2]^{1/2} dt \\ &= \rho^{1/2}(0) \int_{-\infty}^{\infty} (1+t^2)^{-1} dt, \end{aligned}$$

$Y(t, \omega) \in L^1(R^1)$ almost surely. Namely there is a subset Ω' of Ω with $P(\Omega') = 1$ such that $Y(t, \omega) \in L^1(R^1)$ for $\omega \in \Omega'$. For $\omega \in \Omega'$, we define the Fourier transform of $Y(t, \omega)$

$$(2.2) \quad \hat{Y}(t, \omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} Y(x, \omega) e^{-itx} dx,$$

which is measurable on $R^1 \times \Omega'$.

Lemma 2. If

$$(2.3) \quad \int_{-\infty}^{\infty} \lambda^{2(n+1)} dF(\lambda) < \infty,$$

for some nonnegative integer n , then there is a set $\Omega_n \in \Omega'$ with $P(\Omega_n) = 1$ such that

$$(2.4) \quad \int_{-\infty}^{\infty} |\hat{Y}(t, \omega)| |t|^n dt < \infty,$$

for $\omega \in \Omega_n$ and moreover

$$(2.5) \quad E \left(\int_{-\infty}^{\infty} |\hat{Y}(t, \omega)| |t|^n dt \right)^2 \leq C(2(n+1))! \rho(0) + C 2^n \int_{-\infty}^{\infty} (|\lambda| + 2)^{2(n+1)} dF(\lambda),$$

where C 's are absolute constants. Here and in what follows C 's may be different on each occurrence.

Proof. It is sufficient to prove (2.5). The left hand side of (2.5) is, writing $E|X|^2 = \|X\|^2$,

On a theorem on quasianalyticity

$$\begin{aligned}
S &= \left\| \sum_{k=-\infty}^{\infty} \int_k^{k+1} |\hat{Y}(t, \omega)| |t|^n dt \right\|^2 \\
&\leq \left\| \sum_{k=-\infty}^{\infty} (|k|+1)^n \int_k^{k+1} |\hat{Y}(t, \omega)| dt \right\|^2 \\
&\leq \left\| \sum_{k=-\infty}^{\infty} (|k|+1)^n \left(\int_k^{k+1} |\hat{Y}(t, \omega)|^2 dt \right)^{1/2} \right\|^2 \\
&\leq E \sum_{k=-\infty}^{\infty} (|k|+1)^{-2} \cdot \sum_{k=-\infty}^{\infty} (|k|+1)^{2(n+1)} \int_k^{k+1} |\hat{Y}(t, \omega)|^2 dt \\
(2.6) \quad &= C \sum_{k=-\infty}^{\infty} (|k|+1)^{2(n+1)} \int_k^{k+1} \|\hat{Y}(t, \omega)\|^2 dt.
\end{aligned}$$

Now

$$\begin{aligned}
\|\hat{Y}(t, \omega)\|^2 &= (2\pi)^{-1} E \int_{-\infty}^{\infty} X(t, \omega) (1+x^2)^{-1} e^{-itx} dx \cdot \int_{-\infty}^{\infty} \overline{X(y, \omega)} (1+y^2)^{-1} e^{ity} dy \\
&= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+x^2)^{-1} (1+y^2)^{-1} e^{-it(x-y)} \rho(x-y) dx dy \\
&= (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+x^2)^{-1} (1+y^2)^{-1} e^{-it(x-y)} dx dy \cdot \int_{-\infty}^{\infty} e^{i(x-y)\lambda} dF(\lambda) \\
&= (2\pi)^{-1} \int_{-\infty}^{\infty} dF(\lambda) \left| \int_{-\infty}^{\infty} (1+x^2)^{-1} e^{-i(\lambda-t)x} dx \right|^2 \\
&= (\pi/2) \int_{-\infty}^{\infty} e^{-2|\lambda-t|} dF(\lambda).
\end{aligned}$$

Hence

$$\begin{aligned}
S &\leq C \sum_{k=-\infty}^{\infty} (|k|+1)^{2(n+1)} \int_{-\infty}^{\infty} dF(\lambda) \int_k^{k+1} e^{-2|\lambda-t|} dt \\
&\leq C \int_{-\infty}^{\infty} dF(\lambda) \sum_{k=-\infty}^{\infty} \int_k^{k+1} (|t|+2)^{2(n+1)} e^{-2|\lambda-t|} dt \\
&\leq C 2^{2n} \int_{-\infty}^{\infty} dF(\lambda) \int_0^{\infty} [u^{2(n+1)} + (|\lambda|+2)^{2(n+1)}] e^{-2u} du
\end{aligned}$$

in which

$$\int_0^{\infty} u^{2(n+1)} e^{-2u} du = 2^{-2n-3} (2(n+1))!.$$

Thus we get

$$S \leq C (2(n+1))! \rho(0) + C 2^{2n} \int_{-\infty}^{\infty} (|\lambda|+2)^{2(n+1)} dF(\lambda).$$

This is no more than (2.5).

Define for $\omega \in \Omega'$ and for $R > 0$,

$$(2.7) \quad \sigma_R(x, \omega) = \int_{-\infty}^{\infty} Y(y, \omega) K_R(x-y) dy,$$

where $K_R(u)$ is the Fejer kernel $\pi^{-1} \sin^2(Ru/2)/(Ru^2/2)$.

Lemma 3. For every x ,

$$(2.8) \quad \|\sigma_R(x, \omega) - Y(x, \omega)\| \rightarrow 0, \quad R \rightarrow \infty.$$

Proof. The quantity in (2.8) is

$$\begin{aligned} \|\sigma_R(x, \omega) - Y(x, \omega)\| &= \left\| \pi^{-1} \int_{-\infty}^{\infty} [Y(x+u, \omega) - Y(x, \omega)] K_R(u) du \right\| \\ &\leq \pi^{-1} \int_{-\infty}^{\infty} \|Y(x+u, \omega) - Y(x, \omega)\| K_R(u) du \end{aligned}$$

which converges to zero as $R \rightarrow \infty$, because it is easy to see that $Y(x, \omega)$ is continuous in $L^2(\Omega)$, if one notes that any measurable weakly stationary process is continuous in $L^2(\Omega)$ [1].

3. The theorem and the proof

We consider the class $C_2(l_n, R^1)$ which is broader than the class of analytic functions and suppose substantially without loss of generality that

$$(3.1) \quad l_n \geq (2n)! M^{-n}, \quad n=0, 1, \dots$$

for some constant $M > 0$.

The theorem we are going to prove is stated in the following form.

Theorem. Suppose $C_2(l_n, R^1)$ is a quasianalytic class and suppose (3.1). If the covariance function $\rho(u)$ of a measurable weakly stationary process $X(t, \omega)$ belongs to the class $C_2(l_n, R^1)$, then in every finite interval $(-B, B)$ there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ with the property that $X_0(t, \omega)$ almost surely belongs to the quasianalytic class $C(l_{n+1}^{1/2})$ on $(-B, B)$.

Note that $C(l_{n+1}^{1/2})$ is a quasianalytic class by Lemma 1.

Proof. Write $\tilde{\Omega} = \bigcap_{n=0}^{\infty} \Omega_n$. Obviously $P(\tilde{\Omega}) = 1$. Define $X_0(t, \omega)$ for $\omega \in \tilde{\Omega}$ by

$$(3.2) \quad X_0(t, \omega) = (1+t^2) \cdot (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{Y}(x, \omega) e^{itx} dx$$

which is measurable on $(-B, B) \times \tilde{\Omega}$. From the assumption that $\rho(u) \in C_2(l_n, R^1)$, $|\rho^{(2n)}(0)| < \infty$ for all n and actually, for some K ,

$$(3.3) \quad \max_u |\rho^{(2n)}(u)| \leq |\rho^{(2n)}(0)| = \int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) \leq AK^n l_n.$$

From this and Lemma 2, $X_0(t, \omega)$ is well defined and continuous for each $\omega \in \tilde{\Omega}$. On the other hand by the inversion of Fourier transform we have

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{Y}(x, \omega) e^{itx} dx = Y(t, \omega) = X(t, \omega)(1+t^2)$$

for almost all t . Hence $X_0(t, \phi) = X(t, \omega)$ for almost all t for each $\omega \in \tilde{\Omega}$. Furthermore $X_0(t, \omega)$ is a modification of $X(t, \omega)$. In fact, because of Lemma 3, there is a sequence $\{R_k\}$ with $R_k \rightarrow \infty$ ($k \rightarrow \infty$) such that $\sigma_{R_k}(t, \omega) \rightarrow Y(t, \omega)$ ($k \rightarrow \infty$) on some $\Omega''(t)$ for each t and $P(\Omega''(t)) = 1$. Namely, as $k \rightarrow \infty$, for $\omega \in \Omega''(t)$

$$(3.4) \quad \sigma_{R_k}(t, \omega)(1+t^2) \rightarrow X(t, \omega)$$

for each t . Since

$$\sigma_{R_k}(t, \omega) = R_k^{-1} \int_0^{R_k} du \int_{-u}^u \hat{Y}(x, \omega) e^{itx} dx$$

and $Y(x, \omega) \in L^1(R)$ on $\Omega_0 \subset \Omega'$, (3.4) shows that

$$(1+t^2)(2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{Y}(x, \omega) e^{itx} dx = X(t, \omega)$$

on $\Omega''(t) \cap \tilde{\Omega} = \Omega'''(t)$ for each t . That is, from the definition (3.2) of $X_0(t, \omega)$, we have, for each t , $X_0(t, \omega) = X(t, \omega)$ on $\Omega'''(t)$.

Now we shall show the convergence of ΣJ_n for this modification, where

$$(3.5) \quad J_n = P\left(\max_{|t| \leq B} |X_0^{(n)}(t, \omega)| \geq AK_1^n n^{1/2}\right).$$

K_1 and B are any positive constants. It is to be noted that $\max_{|t| \leq B} |X_0^{(n)}(t, \omega)|$ is measurable on Ω , since $X_0^{(n)}(t, \omega)$ is continuous for each $\omega \in \tilde{\Omega}$.

From Lemma 2 and (3.2), $\int_{-\infty}^{\infty} \hat{Y}(x, \omega) e^{itx} dx$ is infinitely many times differentiable for $\omega \in \tilde{\Omega}$. Hence for $t \in (-B, B)$

$$\begin{aligned} X_0^{(n)}(t, \omega) &= \sum_{k=0}^n \binom{n}{k} \frac{d^k}{dt^k} (1+t^2) \frac{d^{n-k}}{dt^{n-k}} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{Y}(x, \omega) e^{itx} dx \\ &= (1+t^2)(2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{Y}(x, \omega) (ix)^n e^{itx} dx \\ &\quad + \sum_{k=1}^2 \binom{n}{k} 2t^{2-k} (2\pi)^{-1/2} \int_{-\infty}^{\infty} Y(x, \omega) (ix)^{n-k} dx \end{aligned}$$

and thus

$$\begin{aligned} \max_{|t| \leq B} |X_0^{(n)}(t, \omega)| &\leq (1+B^2)(2\pi)^{-1/2} \int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^n dx \\ &\quad + 2 \sum_{k=1}^2 \binom{n}{k} B^{2-k} (2\pi)^{-1/2} \int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^{n-k} dx \\ &\leq C_B \int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^n dx \end{aligned}$$

$$+C_B \sum_{k=1}^2 \binom{n}{k} \int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^{n-k} dx = \Phi_n(B, \omega)$$

say, where C_B 's are constants depending only on B . We have

$$J_n \leq P(\Phi_n(B, \omega) \geq AK_1^n l_{n+1}^{1/2})$$

which is, by the Chebyshev inequality, not greater than

$$A^{-2} K_1^{-2n} l_{n+1}^{-1} E \Phi_n^2(B, \omega)$$

in which

$$\begin{aligned} E \Phi_n^2(B, \omega) &\leq C_B E \left[\int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^n dx \right]^2 + C_B n^2 E \left[\int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^{n-1} dx \right]^2 \\ &\quad + C_B n^4 \left[\int_{-\infty}^{\infty} |\hat{Y}(x, \omega)| |x|^{n-2} dx \right]^2. \end{aligned}$$

Hence, because of Lemma 2, J_n is not greater than

$$\begin{aligned} &C_B A^{-2} K_1^{-2n} l_{n+1}^{-1} \left\{ \rho(0) [(2(n+1))! + n^2(2n)! + n^4(2(n-1))!] \right. \\ &\quad + 2^n \int_{-\infty}^{\infty} (|\lambda| + 2)^{2(n+1)} dF(\lambda) + n^2 2^{n-1} \int_{-\infty}^{\infty} (|\lambda| + 2)^{2n} dF(\lambda) \\ &\quad \left. + n^4 2^{n-2} \int_{-\infty}^{\infty} (|\lambda| + 2)^{2(n-1)} dF(\lambda) \right\} \end{aligned}$$

which is, because of (3.3) and (3.1), not greater than

$$\begin{aligned} &C_B A^{-2} K_1^{-2n} l_{n+1}^{-1} \left[n^4 (2(n+1))! \rho(0) + C_B n^4 2^n \int_{-\infty}^{\infty} \lambda^{2(n+1)} dF(\lambda) \right] \\ &\leq C_B A^{-2} K_1^{-2n} l_{n+1}^{-1} n^4 (2(n+1))! \rho(0) + C_B A^{-2} K_1^{-2n} A K^{n+1} n^4 \\ &\leq C_B A^{-2} K_1^{-2n} n^4 M^{n+1} \rho(0) + C_B A^{-2} K (2K/K_1^2)^n n^4. \end{aligned}$$

Take K_1 so that

$$K_1 > \max (M, (2K)^{1/2}).$$

Then we see that $\sum_{n=1}^{\infty} J_n$ is convergent. Hence the Borel-Centelli lemma gives us that there exist a set $\Omega^0 \subset \tilde{\Omega}$ with $P(\Omega^0) = 1$ and an $n_0(\omega)$ for $\omega \in \Omega^0$ such that

$$\max_{|t| \leq B} |X_0^{(n)}(t, \omega)| \leq AK_1^n l_{n+1}^{1/2}$$

holds for $n \geq n_0(\omega)$. Writing

$$K_2 = K_2(\omega) = \max \{K_1, \max_{n \leq n_0(\omega)} [\max_{|t| \leq B} |X_0^{(n)}(t, \omega)| / (Al_{n+1}^{1/2})]^{1/n}\},$$

we have

$$\max_{|t| \leq B} |X_0^{(n)}(t, \omega)| \leq AK_2^n(\omega) l_{n+1}^{1/2}, \quad n = 0, 1, 2, \dots$$

for $\omega \in \Omega^0$. This means that $X_0(t, \omega) \in C(l_{n+1}^{1/2})$ in $(-B, B)$ for $\omega \in \Omega^0$. Since by assumption $C_2(l_n)$ is quasianalytic, $C(l_{n+1}^{1/2})$ is also quasianalytic in $(-B, B)$ by Lemma 1 and the proof of the theorem is complete.

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