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ON A THEOREM ON QUASIANALYTICITY OF A WEAKLY STATIONARY PROCESS

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1. Introduction

Let $X(t, \omega)$, $t \in R^1$, $\omega \in \Omega$, Ω being a given complete probability space, be a measurable weakly stationary process with covariance function

$$\rho(u)=\int_{-\infty}^{\infty} e^{iu\lambda}dF(\lambda), \tag{1.1}$$

where $F(\lambda)$ is the spectral distribution function of $X(t, \omega)$.

Let $C(m_n, (a, b))=C(m_n)$ be a class of complex valued infinitely many times differentiable functions $f(x)$ satisfying

$$\sup_{a<x<b} |f^{(n)}(x)| \leqq AK^n m_n, \qquad n=0, 1, 2, \cdots, \tag{1.2}$$

for some sequence $\{m_n, n=0, 1, 2, \cdots\}$ of positive numbers, ∞ being allowed, where A and K are constants independent of n. The class is called quasianalytic, if $f \in C(m_n)$ and $f^{(n)}(x_0)=0$, $n=0, 1, \cdots$ for some $x_0 \in (a, b)$ implies that $f(x)=0$ throughout (a, b). We also consider the class $C_2(l_n, (a, b))=C_2(l_n)$ of $f(x)$ which are infinitely many times differentiable and are such that

$$\sup_{a<x<b} |f^{(2n)}(x)| \leqq AK^n l_n, \qquad n=0, 1, 2, \cdots, \tag{1.3}$$

for some sequence $\{l_n, n=0, 1, \cdots\}$ of positive numbers. Writing $m_{2n}=l_n$, $m_{2n+1}=\infty$, $n=0, 1, \cdots$, $C_2(l_n)$ is identical with $C(m_n)$.

I. L. Ivanova [2] has given the theorem that if the covariance function $\rho(u)$ belongs to $C_2(l_n)$ which is supposed to be quasianalytic in R^1, then $X(t, \omega)$ almost surely belongs to the class $C(l_n^{1/2})$ which is also quasianalytic.

She concludes this from the fact that

$$\sum_{n=0}^{\infty} P(|X^{(n)}(t, \omega)| \geqq AK^n l_n^{1/2}) < \infty. \tag{1.4}$$

We here used slightly different notations from [2]. This derivation, however, involves a vague point, because the subset of Ω inside $P(\cdot)$ depends on t in general. The author has recently given a different proof [3] of the above theorem of Ivanova for the case of a periodic weakly stationary process. In this paper we shall give another proof for the general case, showing in place of (1.4) that

$$\sum_{n=0}^{\infty} P(\max_{|t| \leq B} |X_0^{(n)}(t, \omega)| \geqq AK_1^n l_{n+1}^{1/2}) < \infty, \tag{1.5}$$

for every fixed constant B for some modification $X_0(t,\omega)$ of $X(t,\omega)$ and for some K_1 (depending on B).

2. Lemmas

Lemma 1. *$C_2(l_n,(a,b))$ is quasianalytic if and only if $C(l_{n+1}^{1/2},(a,b))$ is quasianalytic.*

This was given in Lemma 7.2 [3].

Suppose without loss of generality that $EX(t,\omega)=0$, $t\in R^1$. Write

$$Y(t,\omega)=(1+t^2)^{-1}X(t,\omega)\,,\qquad t\in R^1\,. \tag{2.1}$$

Since

$$\begin{aligned}E\int_{-\infty}^{\infty}|Y(t,\omega)|dt&=\int_{-\infty}^{\infty}(1+t^2)^{-1}E|X(t,\omega)|dt\leqq\int_{-\infty}^{\infty}(1+t^2)^{-1}[E|X(t,\omega)|^2]^{1/2}dt\\&=\rho^{1/2}(0)\int_{-\infty}^{\infty}(1+t^2)^{-1}dt\,,\end{aligned}$$

$Y(t,\omega)\in L^1(R^1)$ almost surely. Namely there is a subset Ω' of Ω with $P(\Omega')=1$ such that $Y(t,\omega)\in L^1(R^1)$ for $\omega\in\Omega'$. For $\omega\in\Omega'$, we define the Fourier transform of $Y(t,\omega)$

$$\hat{Y}(t,\omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}Y(x,\omega)e^{-itx}dx\,, \tag{2.2}$$

which is measurable on $R^1\times\Omega'$.

Lemma 2. If

$$\int_{-\infty}^{\infty}\lambda^{2(n+1)}dF(\lambda)<\infty\,, \tag{2.3}$$

for some nonnegative integer n, then there is a set $\Omega_n\in\Omega'$ with $P(\Omega_n)=1$ such that

$$\int_{-\infty}^{\infty}|\hat{Y}(t,\omega)||t|^n dt<\infty\,, \tag{2.4}$$

for $\omega\in\Omega_n$ and moreover

$$E\left(\int_{-\infty}^{\infty}|\hat{Y}(t,\omega)||t|^n dt\right)^2\leqq C(2(n+1))!\,\rho(0)+C2^n\int_{-\infty}^{\infty}(|\lambda|+2)^{2(n+1)}dF(\lambda)\,, \tag{2.5}$$

where C's are absolute constants. Here and in what follows C's may be different on each occurrence.

Proof. It is sufficient to prove (2.5). The left hand side of (2.5) is, writing $E|X|^2=\|X\|^2$,

$$S=\left\|\sum_{k=-\infty}^{\infty}\int_k^{k+1}|\hat{Y}(t,\omega)||t|^n dt\right\|^2$$

$$\leqq\left\|\sum_{k=-\infty}^{\infty}(|k|+1)^n\int_k^{k+1}|\hat{Y}(t,\omega)|dt\right\|^2$$

$$\leqq\left\|\sum_{k=-\infty}^{\infty}(|k|+1)^n\left(\int_k^{k+1}|\hat{Y}(t,\omega)|^2dt\right)^{1/2}\right\|^2$$

$$\leqq E\sum_{k=-\infty}^{\infty}(|k|+1)^{-2}\cdot\sum_{k=-\infty}^{\infty}(|k|+1)^{2(n+1)}\int_k^{k+1}|\hat{Y}(t,\omega)|^2dt$$

$$=C\sum_{k=-\infty}^{\infty}(|k|+1)^{2(n+1)}\int_k^{k+1}\|\hat{Y}(t,\omega)\|^2dt\,. \tag{2.6}$$

Now

$$\|\hat{Y}(t,\omega)\|^2=(2\pi)^{-1}E\int_{-\infty}^{\infty}X(t,\omega)(1+x^2)^{-1}e^{-itx}dx\cdot\int_{-\infty}^{\infty}\overline{X(y,\omega)}(1+y^2)^{-1}e^{ity}dy$$

$$=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+x^2)^{-1}(1+y^2)^{-1}e^{-it(x-y)}\rho(x-y)dxdy$$

$$=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+x^2)^{-1}(1+y^2)^{-1}e^{-it(x-y)}dxdy\cdot\int_{-\infty}^{\infty}e^{i(x-y)\lambda}dF(\lambda)$$

$$=(2\pi)^{-1}\int_{-\infty}^{\infty}dF(\lambda)\left|\int_{-\infty}^{\infty}(1+x^2)^{-1}e^{-i(t-\lambda)x}dx\right|^2$$

$$=(\pi/2)\int_{-\infty}^{\infty}e^{-2|\lambda-t|}dF(\lambda)\,.$$

Hence

$$S\leqq C\sum_{k=-\infty}^{\infty}(|k|+1)^{2(n+1)}\int_{-\infty}^{\infty}dF(\lambda)\int_k^{k+1}e^{-2|\lambda-t|}dt$$

$$\leqq C\int_{-\infty}^{\infty}dF(\lambda)\sum_{k=-\infty}^{\infty}\int_k^{k+1}(|t|+2)^{2(n+1)}e^{-2|\lambda-t|}dt$$

$$\leqq C2^{2n}\int_{-\infty}^{\infty}dF(\lambda)\int_0^{\infty}[u^{2(n+1)}+(|\lambda|+2)^{2(n+1)}]e^{-2u}du$$

in which

$$\int_0^{\infty}u^{2(n+1)}e^{-2u}du=2^{-2n-3}(2(n+1))!\,.$$

Thus we get

$$S\leqq C(2(n+1))!\,\rho(0)+C2^{2n}\int_{-\infty}^{\infty}(|\lambda|+2)^{2(n+1)}dF(\lambda)\,.$$

This is no more than (2.5).

Define for $\omega\in\Omega'$ and for $R>0$,

$$\sigma_R(x, \omega)=\int_{-\infty}^{\infty} Y(y, \omega)K_R(x-y)dy \,, \tag{2.7}$$

where $K_R(u)$ is the Fejer kernel $\pi^{-1}\sin^2(Ru/2)/(Ru^2/2)$.

Lemma 3. For every x,

$$\|\sigma_R(x, \omega)-Y(x, \omega)\| \to 0 \,, \qquad R\to\infty \,. \tag{2.8}$$

Proof. The quantity in (2.8) is

$$\begin{aligned}\|\sigma_R(x, \omega)-Y(x, \omega)\| &= \left\|\pi^{-1}\int_{-\infty}^{\infty}[Y(x+u, \omega)-Y(x, \omega)]K_R(u)du\right\| \\ &\leqq \pi^{-1}\int_{-\infty}^{\infty}\|Y(x+u, \omega)-Y(x, \omega)\|K_R(u)du\end{aligned}$$

which converges to zero as $R\to\infty$, because it is easy to see that $Y(x, \omega)$ is continuous in $L^2(\Omega)$, if one notes that any measurable weakly stationary process is continuous in $L^2(\Omega)$ [1].

3. The theorem and the proof

We consider the class $C_2(l_n, R^1)$ which is broader than the class of analytic functions and suppose substantially without loss of generality that

$$l_n \geqq (2n)!M^{-n} \,, \qquad n=0, 1, \cdots \tag{3.1}$$

for some constant $M>0$.

The theorem we are going to prove is stated in the following form.

Theorem. Suppose $C_2(l_n, R^1)$ *is a quasianalytic class and suppose* (3.1). *If the covariance function* $\rho(u)$ *of a measurable weakly stationary process* $X(t, \omega)$ *belongs to the class* $C_2(l_n, R^1)$, *then in every finite interval* $(-B, B)$ *there is a modification* $X_0(t, \omega)$ *of* $X(t, \omega)$ *with the property that* $X_0(t, \omega)$ *almost surely belongs to the quasianalytic class* $C(l_{n+1}^{1/2})$ *on* $(-B, B)$.

Note that $C(l_{n+1}^{1/2})$ is a quasianalytic class by Lemma 1.

Proof. Write $\tilde{\Omega}=\bigcap_{n=0}^{\infty}\Omega_n$. Obviously $P(\tilde{\Omega})=1$. Define $X_0(t, \omega)$ for $\omega\in\tilde{\Omega}$ by

$$X_0(t, \omega)=(1+t^2)\cdot(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{Y}(x, \omega)e^{itx}dx \tag{3.2}$$

which is measurable on $(-B, B)\times\tilde{\Omega}$. From the assumption that $\rho(u)\in C_2(l_n, R^1)$, $|\rho^{(2n)}(0)|<\infty$ for all n and actually, for some K,

$$\max_u|\rho^{(2n)}(u)| \leqq |\rho^{(2n)}(0)| = \int_{-\infty}^{\infty}\lambda^{2n}dF(\lambda) \leqq AK^n l_n \,. \tag{3.3}$$

From this and Lemma 2, $X_0(t,\omega)$ is well defined and continuous for each $\omega\in\tilde{\Omega}$. On the other hand by the inversion of Fourier transform we have

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{Y}(x,\omega)e^{itx}dx=Y(t,\omega)=X(t,\omega)(1+t^2)$$

for almost all t. Hence $X_0(t,\psi)=X(t,\omega)$ for almost all t for each $\omega\in\tilde{\Omega}$. Furthermore $X_0(t,\omega)$ is a modification of $X(t,\omega)$. In fact, because of Lemma 3, there is a sequence $\{R_k\}$ with $R_k\to\infty$ $(k\to\infty)$ such that $\sigma_{R_k}(t,\omega)\to Y(t,\omega)$ $(k\to\infty)$ on some $\Omega''(t)$ for each t and $P(\Omega''(t))=1$. Namely, as $k\to\infty$, for $\omega\in\Omega''(t)$

$$\sigma_{R_k}(t,\omega)(1+t^2)\to X(t,\omega) \tag{3.4}$$

for each t. Since

$$\sigma_{R_k}(t,\omega)=R_k^{-1}\int_0^{R_k}du\int_{-u}^{u}\hat{Y}(x,\omega)e^{itx}dx$$

and $Y(x,\omega)\in L^1(R)$ on Ω_0 $(\subset\Omega')$, (3.4) shows that

$$(1+t^2)(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{Y}(x,\omega)e^{itx}dx=X(t,\omega)$$

on $\Omega''(t)\cap\tilde{\Omega}=\Omega'''(t)$ for each t. That is, from the definition (3.2) of $X_0(t,\omega)$, we have, for each t, $X_0(t,\omega)=X(t,\omega)$ on $\Omega'''(t)$.

Now we shall show the convergence of ΣJ_n for this modification, where

$$J_n=P(\max_{|t|\leqq B}|X_0^{(n)}(t,\omega)|\geqq AK_1^n l_{n+1}^{1/2})\,. \tag{3.5}$$

K_1 and B are any positive constants. It is to be noted that $\max_{|t|\leqq B}|X_0^{(n)}(t,\omega)|$ is measurable on Ω, since $X_0^{(n)}(t,\omega)$ is continuous for each $\omega\in\tilde{\Omega}$.

From Lemma 2 and (3.2), $\int_{-\infty}^{\infty}\hat{Y}(x,\omega)e^{itx}dx$ is infinitely many times differentiable for $\omega\in\tilde{\Omega}$. Hence for $t\in(-B,B)$

$$\begin{aligned}X_0^{(n)}(t,\omega)&=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dt^k}(1+t^2)\frac{d^{n-k}}{dt^{n-k}}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{Y}(x,\omega)e^{itx}dx\\&=(1+t^2)(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{Y}(x,\omega)(ix)^n e^{itx}dx\\&\quad+\sum_{k=1}^{2}\binom{n}{k}2t^{2-k}(2\pi)^{-1/2}\int_{-\infty}^{\infty}Y(x,\omega)(ix)^{n-k}dx\end{aligned}$$

and thus

$$\begin{aligned}\max_{|t|<B}|X_0^{(n)}(t,\omega)|&\leqq(1+B^2)(2\pi)^{-1/2}\int_{-\infty}^{\infty}|\hat{Y}(x,\omega)||x|^n dx\\&\quad+2\sum_{k=1}^{2}\binom{n}{k}B^{2-k}(2\pi)^{-1/2}\int_{-\infty}^{\infty}|\hat{Y}(x,\omega)||x|^{n-k}dx\\&\leqq C_B\int_{-\infty}^{\infty}|\hat{Y}(x,\omega)||x|^n dx\end{aligned}$$

$$+C_B\sum_{k=1}^{2}\binom{n}{k}\int_{-\infty}^{\infty}|\hat{Y}(x,\omega)||x|^{n-k}dx=\Phi_n(B,\omega)$$

say, where C_B's are constants depending only on B. We have

$$J_n\leqq P(\Phi_n(B,\omega)\geqq AK_1^n l_{n+1}^{1/2})$$

which is, by the Chebyshev inequality, not greater than

$$A^{-2}K_1^{-2n}l_{n+1}^{-1}E\Phi_n^2(B,\omega)$$

in which

$$E\Phi_n^2(B,\omega)\leqq C_B E\left[\int_{-\infty}^{\infty}|\hat{Y}(x,\omega)||x|^n dx\right]^2+C_B n^2 E\left[\int_{-\infty}^{\infty}\hat{Y}(x,\omega)||x|^{n-1}dx\right]^2$$
$$+C_B n^4\left[\int_{-\infty}^{\infty}|\hat{Y}(x,\omega)||x|^{n-2}dx\right]^2.$$

Hence, because of Lemma 2, J_n is not greater than

$$C_B A^{-2}K_1^{-2n}l_{n+1}^{-1}\Big\{\rho(0)[(2(n+1))!+n^2(2n)!+n^4(2(n-1))!]$$
$$+2^n\int_{-\infty}^{\infty}(|\lambda|+2)^{2(n+1)}dF(\lambda)+n^2 2^{n-1}\int_{-\infty}^{\infty}(|\lambda|+2)^{2n}dF(\lambda)$$
$$+n^4 2^{n-2}\int_{-\infty}^{\infty}(|\lambda|+2)^{2(n-1)}dF(\lambda)\Big\}$$

which is, because of (3.3) and (3.1), not greater than

$$C_B A^{-2}K_1^{-2n}l_{n+1}^{-1}\left[n^4(2(n+1))!\,\rho(0)+C_B n^4 2^n\int_{-\infty}^{\infty}\lambda^{2(n+1)}dF(\lambda)\right]$$
$$\leqq C_B A^{-2}K_1^{-2n}l_{n+1}^{-1}n^4(2(n+1))!\,\rho(0)+C_B A^{-2}K_1^{-2n}AK^{n+1}n^4$$
$$\leqq C_B A^{-2}K_1^{-2n}n^4M^{n+1}\rho(0)+C_B A^{-2}K(2K/K_1^2)^n n^4\,.$$

Take K_1 so that

$$K_1>\max(M,(2K)^{1/2})\,.$$

Then we see that $\sum_{n=1}^{\infty}J_n$ is convergent. Hence the Borel-Centelli lemma gives us that there exist a set $\Omega^0\subset\tilde{\Omega}$ with $P(\Omega^0)=1$ and an $n_0(\omega)$ for $\omega\in\Omega^0$ such that

$$\max_{|t|\leqq B}|X_0^{(n)}(t,\omega)|\leqq AK_1^n l_{n+1}^{1/2}$$

holds for $n\geqq n_0(\omega)$. Writing

$$K_2=K_2(\omega)=\max\{K_1,\ \max_{n\leqq n_0(\omega)}[\max_{|t|\leqq B}|X_0^{(n)}(t,\omega)|/(Al_{n+1}^{1/2})]^{1/n}\}\,,$$

we have

$$\max_{|t|\leqq B}|X_0^{(n)}(t,\omega)|\leqq AK_2^n(\omega)l_{n+1}^{1/2}\,,\qquad n=0,1,2,\cdots$$

for $\omega \in \Omega^0$. This means that $X_0(t, \omega) \in C(l_{n+1}^{1/2})$ in $(-B, B)$ for $\omega \in \Omega^0$. Since by assumption $C_2(l_n)$ is quasianalytic, $C(l_{n+1}^{1/2})$ is also quasianalytic in $(-B, B)$ by Lemma 1 and the proof of the theorem is complete.

References

[1] M. M. Crum, On positive definite functions, Proc. London Math. Soc. (3) **6** (1956), 548-560.

[2] L. Ivanova, Quasianalytic random processes, Selected Transl. Math. Statist. Prob. **12** (1973), 99-108.

[3] T. Kawata, Absolute convergence of Fourier series of periodic stochastic processes and its applications, to appear in Tohoku Math. Jour.