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THE MEAN DERIVATIVES AND THE ABSOLUTE CONVERGENCE OF THE FOURIER SERIES OF A STOCHASTIC PROCESS

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ABSTRACT

A criterion for the almost sure absolute convergence of the Fourier series of a periodic stochastic process which has the mean derivatives is given and the result is applied to the sample properties of the process. A theorem on the mean derivative of a sine series is also given.

1. Introduction

Let $X(t, \omega), t \in R^1$, be a complex valued stochastic process on a given probability space (Ω, F, P). Suppose that $X(t, \omega)$ is measurable $L \times F$ on $R^1 \times \Omega, L$ being the class of Lebesgue measurable sets on R^1, and that for $r \geqq 1$,

(1.1) $$E|X(t, \omega)|^r < \infty,$$

for every t and $X(t, \omega)$ is 2π-periodic:

(1.2) $$E|X(t+2\pi, \omega) - X(t, \omega)| = 0,$$

for $t \in R^1$.

Furthermore we throughout assume that $(T = [-\pi, \pi])$

(1.3) $$\int_{-\pi}^{\pi} E|X(t, \omega)|^r dt < \infty.$$

In case we write $X(t, \omega) \in L_P^r(T \times \Omega)$.

Let

(1.4) $$\sum_{n=-\infty}^{\infty} C_n(\omega) e^{int}$$

be the Fourier series of $X(t, \omega)$, where

(1.5) $$C_n(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t, \omega) e^{-int} dt, \quad n = 0, \pm 1, \pm 2, \cdots$$

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In [1], the author has given some results on the almost sure convergence of the series

$$\sum_{n=-\infty}^{\infty} |n|^k \alpha_n |C_n(\omega)| \tag{1.6}$$

($\alpha_n > 0$) for some nonnegative integer k, and applied them to sample continuity or differentiability or $X(t, \omega)$.

Write

$$M_r^{*(p)}(\delta) = M_r^{*(p)}(\delta, X) = \sup_{|h| \leqq \delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} E|\Delta_h^{(p)} X(t, \omega)|^r dt \right)^{1/r}, \tag{1.7}$$

where p is a positive integer and $\Delta_h^{(p)} X(t, \omega)$ is the p-th difference of $X(t, \omega)$:

$$\Delta_h^{(p)} X(t, \omega) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} X(t+kh, \omega). \tag{1.8}$$

Let $\phi(t)$ be a nondecreasing function on $[0, 1]$ such that either $\phi(0)=0$ and $\phi(t)/t$ is noninereasing on $[0,1]$, or $\phi(t)$ is identically 1 on $[0, 1]$.

One of the resuls obtained in [1] is the following.

THEOREM A. *Let $X(t, \omega)$ be of $L_P^r(T \times \Omega)$ for some $r > 1$. Let k be a given nonnegative integer. If there exists a positive integer p such that*

$$\sum_{n=1}^{\infty} n^{k-1+1/r} \left[\phi\left(\frac{1}{n}\right) \right]^{-1} M_r^{*(p)}\left(\frac{1}{n}\right) < \infty, \tag{1.9}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^k \left[\phi\left(\frac{1}{n}\right) \right]^{-1} |C_n(\omega)| < \infty, \tag{1.10}$$

almost surely.

In this paper we give a relationship between $M_r^{*(p)}(\delta)$ and the mean derivative of $X(t, \omega)$ and apply it to the absolute convergence of Fourier series and the sample properties of X((t, ω).

2. Mean derivatives.

Let $X(t, \omega) \in L_P^r(T \times \Omega)$ for some $r \geqq 1$. If there is a stochastic process $X'_M(t, \omega)$ of $L_P^r(T \times \Omega)$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} E\left| \frac{X(t+h, \omega) - X(t, \omega)}{h} - X'_M(t, \omega) \right|^r dt \to 0 \tag{2.1}$$

as $h \to 0$, then $X'_M(t, \omega)$ is called the mean derivative in $L^r(T \times \Omega)$ of $X(t, \omega)$. The p-th mean derivative $X_M^{(p)}(t, \omega)$ is defined succesively in an obvious way.

Write

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} E|X(t,\omega)|^r dt\right)^{1/r} = ||X(t,\omega)||_r \tag{2.2}$$

which is the norm of $X(t,\omega)$ in $L^r(T\times\Omega)$.

We give two simple lemmas.

LEMMA 1. *If* $X(t,\omega)\in L^r(T\times\Omega)$ *has the mean derivative* $X'_M(t,\omega)$ *in* $L^r(T\times\Omega)$, *then, as* $h\to 0$,

$$\left|\left|\frac{1}{h}\int_t^{t+h} X(u,\omega)du - X(t,\omega)\right|\right|_r = O(|h|). \tag{2.3}$$

Proof. The left hand side of (2.3) is not greater than

$$\left|\left|\frac{1}{h}\int_0^h [X(u+t,\omega)-X(t,\omega)-uX'_M(t,\omega)]du\right|\right|_r + \frac{|h|}{2}\left|\left|X'_M(t,\omega)\right|\right|_r$$

$$\leqq \int_0^{|h|}\left|\left|\frac{X(u+t,\omega)-X(t,\omega)}{u} - X'_M(t,\omega)\right|\right|_r du + \frac{|h|}{2}\left|\left|X'_M(t,\omega)\right|\right|_r = o(|h|)+O(|h|)=O(|h|).$$

LEMMA 2. *If* $X(t,\omega)\in L^r_P(T\times\Omega)$ *has the mean derivative* $X'_M(t,\omega)$ *in* $L^r(T\times\Omega)$, *then*

$$X(t+h,\omega)-X(t,\omega)=\int_t^{t+h} X'_M(u,\omega)du \tag{2.4}$$

almost everywhere in $T\times\Omega$, *for each* h.

Namely there is a subset $G=G(h)$ of $T\times\Omega$ such that (2.4) holds for $(t,\omega)\in G$ and $\mu(G)=2\pi$, μ being the $m\times P$ measure and m the Lebesgue measure. We note that we can easily see that (2.4) holds on G if t is replaced by $t+h$ by periodicity, when h is fixed. The dependence of G on h makes the difference between the almost sure absolute continuity and the existence of mean derivative in $L(T\times\Omega)$ of $X(t,\omega)$.

Proof. Let $h>0$ without loss of generality and let $|k|<h$. Write

$$S_k = ||Y(t,\omega)||_r,$$

where

$$Y(t,\omega)=\int_t^{t+h}\left[\frac{X(u+k,\omega)-X(u,\omega)}{k} - X'_M(u,\omega)\right]du.$$

Since

$$|Y(t,\omega)|^r \leqq \int_t^{t+h}\left|\frac{X(u+k,\omega)-X(u,\omega)}{k} - X'_M(u,\omega)\right|^r du\cdot h^{r-1}$$

$$\leqq \left|\left|\frac{X(u+k,\omega)-X(u,\omega)}{k} - X'_M(u,\omega)\right|\right|_r^r du\cdot h^{r-1}.$$

Hence

$$S_k^r = \frac{1}{2\pi}\int_{-\pi}^{\pi} E|Y(t,\omega)|^r dt \leqq 2\pi h^{r-1}\cdot\|Y(t,\omega)\|_r^r \to 0 \tag{2.5}$$

as $k\to 0$ for a fixed h.

On the other hand,

$$\begin{aligned} S_k &= \left\|\frac{1}{k}\int_{t+k}^{t+k+h} X(u,\omega)du - \frac{1}{k}\int_t^{t+h} X(u,\omega)du - \int_t^{t+h} X'_M(u,\omega)du\right\|_r \\ &= \left\|\frac{1}{k}\int_{t+h}^{t+h+k} X(u,\omega)du - \frac{1}{k}\int_t^{t+k} X(u,\omega)du - \int_t^{t+h} X'_M(u,\omega)du\right\|_r . \end{aligned}$$

By Lemma 1, this converges to

$$\left\|X(t+h,\omega) - X(t,\omega) - \int_t^{t+h} X'_M(u,\omega)du\right\|_r$$

as $k\to 0$. Hence from (2.5) the last quantity should be zero. This gives us that (2.2) holds a.e. in $T\times\Omega$. for each h.

3. Mean derivatives and continuity modulus.

Let $X(t,\omega)$ be of $L_P^r(T\times\Omega), r\geqq 1$. Suppose $X_M^{(p)}(t,\omega)$, the mean derivative of order p of $X(t,\omega)$ in $L^r(T\times\Omega)$ exists, p being a positive integer.

Now let $G_l = G_l(h)$ be the set on which (2.4) with $X_M^{(l-1)}(t,\omega)$ in place of $X(t,\omega)$ $l=1,2,\cdots,p$, holds.

$$\Delta_h^{(2)}X(t,\omega) = \Delta_h^{(1)}X(t+h,\omega) - \Delta_h^{(1)}X(t,\omega)$$

is, by Lemma 2, equal in G_1 to

$$\int_{t+h}^{t+2h} X'_M(t_1,\omega)dt_1 - \int_t^{t+h} X'_M(t_1,\omega)dt_1 = \int_t^{t+h}[X'_M(t_1+h,\omega) - X'_M(t_1,\omega)]dt$$

which is again by Lemma 2 equal in $G_1\cap G_2$ to

$$= \int_t^{t+h} dt_1 \int_{t_1}^{t_1+h} X''_M(t_2,\omega)dt_2 .$$

Repeating this procedure, we have

$$\Delta_h^{(p)}X(t,\omega) = \int_t^{t+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \cdots \int_{t_{p-1}}^{t_{p-1}+h} X_M^{(p)}(t_p,\omega)dt_p . \tag{3.1}$$

a.e. in $G_0 = \bigcap_{l=1}^{p} G_l$, $\mu(G_0) = 2\pi$.

We now prove

THEOREM 1. *If $X(t,\omega)$ belongs to $L_P^r(T\times\Omega)$ and has $X_M^{(p)}(t,\omega)$ in $L^r(T\times\Omega), r\geqq 1$, p being a positive integer, then*

$$M_r^{*(p)}(\delta) \leqq 2^{p/r} \|X_M^{(p)}(t, \omega)\|_r \cdot \delta^p. \tag{3.2}$$

Proof. From (3.1), we have, for any fixed $h>0$, in G_0,

$$\begin{aligned}
|\Delta_h^{(p)} X(t, \omega)|^r &= \left[\int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} X_M^{(p)}(t_p, \omega) | dt_p\right]^r \\
&\leqq \int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)|^r dt_p \cdot \left[\int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} dt_p\right]^{r-1} \\
&= h^{p(r-1)} \int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)|^r dt_p \\
&= h^{p(r-1)} \int_t^{t+h} Y(t_1, \omega) dt_1,
\end{aligned}$$

where

$$Y(t_1, \omega) = \int_{t_1}^{t_1+h} dt_2 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)^r dt_p.$$

Since for a 2π-periodic function $f(u) \in L^1(T)$,

$$\left|\int_{-\pi}^{\pi} \int_t^{t+h} f(u) du\right| \leqq |h| \int_{-\pi-h}^{\pi+h} |f(u)| du \leqq 2|h| \int_{-\pi}^{\pi} f(u)| du$$

for $|h|<\pi$, which is easily seen by the interchange of integration signs on the left hand side, we have, for small $|h|$,

$$\begin{aligned}
\|\Delta_h^{(p)} X(t, \omega)\|_r^r &= \frac{1}{2\pi} \int_{-\pi}^{\pi} E|\Delta X_h^{(p)}(t, \omega)|^r dt \\
&\leqq \frac{|h|^{p(r-1)}}{2\pi} \int_{-\pi}^{\pi} dt \int_t^{t+h} EY(t_1, \omega) dt_1 \\
&\leqq \frac{1}{\pi} |h|^{p(r-1)} |h| \int_{-\pi}^{\pi} EY(t_1, \omega) dt_1.
\end{aligned}$$

Repeating the same arguments $p-1$ more times, we have

$$\|\Delta_h^{(p)} X(t, \omega))\|_r^r \leqq 2^p |h|^{pr} \frac{1}{2\pi} \int_{-\pi}^{\pi} E|X_M^{(p)}(t_p, \omega)|^r dt_p = 2^p |h|^{pr} \|X_M^{(p)}(t, \omega)\|_r^r.$$

This is no more than (3.2).

4. Absolute convergence of the Fourier series of a periodic stochastic process.

Let $\phi(t)$ be the function in **1**. The combination of Theorem 1 and Theorem A immediately gives us the following theorem.

THEOREM 2. *If $X(t, \omega) \in L_P^r(T \times \Omega)$ and for some nonnegative integer k, $X_M^{(k+1)}(t, \omega)$ exists in $L^r(T \times \Omega)$, $1 < r \leqq 2$, and*

$$\sum_{n=1}^{\infty} n^{-2+1/r}\left[\phi\left(\frac{1}{n}\right)\right]^{-1}<\infty, \tag{4.1}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^{k}\left[\phi\left(\frac{1}{n}\right)\right]^{-1}|C_n(\omega)|<\infty, \tag{4.2}$$

almost surely.

In particular, taking $\phi(t)=t^{\alpha}$, we have the following collorary. For nonrandom case, this is thought of as a version of a known result on the absolute convergence of an absolutely continuous function which has the derivative belonging to $L^r(T)$. ([1] Cor. 2. p. 161, [4] Theorem 3.8, p. 242)

COLLORARY 1. *If $X(t,\omega)\in L_P^r(T\times\Omega)$ and for some nonnegative k, $X_M^{(k+1)}(t,\omega)$ exists in $L^r(T\times\Omega)$, and*

$$0\leqq\alpha<1-\frac{1}{r}, \tag{4.3}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^{k+\alpha}|C_n(\omega)|<\infty, \tag{4.4}$$

almost surely.

The condition (4.3) is of the best kind in some sense. Actually if $\alpha=1-1/r$, there is an $X(t,\omega)\in L_P^r(T\times\Omega)$ which has $X_M^{(k+1)}(t,\omega)$ in $L^r(T\times\Omega)$ and is such that the series in (4.4) diverges almost surely.

We show this for simplicity when $k=0$. Let

$$f(x)=\sum_{n=2}^{\infty}\frac{\sin nx}{n^{\alpha+1}\log n},\quad 0\leqq\alpha<1. \tag{4.5}$$

This is abolutely continuous and $f(x)=\int_0^x g(u)du$, where

$$g(x)=\sum_{n=2}^{\infty}\frac{\cos nx}{n^{\alpha}\log n} \tag{4.6}$$

which is defined for all $-\pi\leqq x\leqq\pi$ except at $x=0$. We can show that $g(x)\in L^r(T)$ and is the mean derivative of $f(x)$ in $L^r(T)$ for $r>1$, if $\alpha\geqq 1-1/r$. More precisely

$$\int_{-\pi}^{\pi}\left|\frac{f(x+h)-f(x)}{h}-g(x)\right|^r dx\leqq C\frac{|h|^{(\alpha-1)r+1}}{|\log|h||^r}, \tag{4.7}$$

as $h\to 0$, where C is a constant independent of h.

The proof of this fact will be shown in a more general form in **6**.

Let $r>1$ and $\alpha=1-1/r$. Then from (4.7), $g(x)$ is the mean derivative of $f(x)$

in $L^r(T)$ and

$$\sum_{n=2}^{\infty} n^{\alpha} b_n = \infty$$

where $b_n = n^{-\alpha-1}(\log n)^{-1}$ is the sine coefficient of $f(x)$.

Theorem 2 and Collorary 1 seem to be new, even for nonrandom case. The above example shows that Collorary 1 with $k=0$ for nonrandom case is no more true if $\alpha=1-1/r$.

$$X(t,\omega)=X(\omega)f(t), \tag{4.9}$$

where $f(t)$ is the function (4.5) and $X(\omega)$ is any random variable of $L^r(\Omega)$, gives a counter example for Collorary 1 with $k=0, \alpha=1-1/r$.

5. Sample properties of a periodic stochastic process

The author has shown the following theorem.

THEORFM B. *Let $X(t,\omega)\in L_P^r$, $1<r\leqq 2$. Let k be a given nonnegative integer. Suppose there exists a positive integer p such that*

$$\sum_{n=1}^{\infty} n^{k-1+1/r}\left[\phi\left(\frac{1}{n}\right)\right]^{-1} M_r^{*(p)}\left(\frac{1}{n}\right)<\infty. \tag{5.1}$$

If $X(t,\omega)$ is stochastically continuous, then there is a modification $X_0(t,\omega)$ of $X(t,\omega)$ with the property that $X_0(t,\omega)$ has almost surely the k-th derivative belonging to the Lipschitz class Λ_ϕ.

Λ_ϕ is the class of functions f with continuity modulus ϕ.

This theorem is applied to the case where the mean derivative of $(k+1)$-th order of $X(t,\omega)$ exists in $L^r(T\times\Omega)$ and the following theorem is immediately obtained, $p=k+1$ being taken, which corresponds to the critical case in some sense.

THEOREM 3. *If the conditions in Theorem 2 are satisfied for $1<r\leqq 2$ and $X(t,\omega)$ is stochastically continuous, then there is a modification $X_0(t,\omega)$ of $X(t,\omega)$, of $X(t,\omega)$, which has almost surely the k-th derivative belonging to Λ_ϕ.*

6. A theorem on a sine series.

We shall prove a theorem on a sine series which implies (4.7) as a particular case.

Consider two series

$$f(x)=\sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx \tag{6.1}$$

and

(6.2) $$g(x)=\sum_{n=1}^{\infty} a_n \cos nx.$$

Let $r>1$ and suppose

(6.3) $$a_n>0,\ a_n\to 0,\ \Delta^2 a_n \geqq 0,$$

where $\Delta_n=a_n-a_{n+1}$, $\Delta^2 a_n=\Delta(\Delta a_n)$.

It is noted that (6.3) implies $\Delta a_n \geqq 0$. It is well known that (6.2) is convergent except at $x=0$ and $g(x)$ is of $L^1(T)$. $f(x)$ is then absolutely continuous and $f'(x)=g(x)$ almost everywhere. We shall give a condition which assures that $f(x)$ has the mean derivative $g(x)$ in $L^r(T)$.

We begin with

LEMMA 3. *Let $p>1$ and β be a real number >-1. For any $b_n \geqq 0, n=1,2,\cdots,$*

(6.4) $$\sum_{n=1}^{\infty}\left(n^{-\beta-1}\sum_{k=1}^{n} k^{\beta} b_k\right)^p \leqq C_p \sum_{n=1}^{\infty} b_n^p,$$

where C_p is a constant depending only on p.

This is a particular case of the following inequality with $a_n=\lambda_n^{-1/p} b_n$, $\lambda_n=n^{\beta p/(p-1)}$.

$$\sum_{n=1}^{\infty}\lambda_n\left(\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k a_k\right)^p \leqq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\lambda_n a_n^p,$$

for $p>1$, $a_n \geqq 0$, $\lambda_n>0$, where $\Lambda_n=\lambda_1+\lambda_2+\cdots+\lambda_n$. (See [2] p. 247, Theorem 332)

LEMMA 4. *Let $b_n \geqq 0, p>1$. Then*

(6.5) $$\sum_{n=1}^{\infty} n^{p-2}\left(\sum_{k=n}^{\infty}\frac{b_k}{k}\right)^p \leqq C_p \sum_{n=1}^{\infty} n^{p-2} b_n^p,$$

and if b_n is nonincreasing, then we moreover have

(6.6) $$\sum_{n=1}^{\infty} n^{p-2} b_n^p \leqq C'_p \sum_{n=1}^{\infty} n^{p-2}\left(\sum_{k=n}^{\infty}\frac{b_k}{k}\right)^p,$$

where C_p and C'_p are constants depending only on p.

Proof. For any $a_n \geqq 0$ with $\sum_{n=1}^{\infty} a_n^{p'}<\infty$, $1/p+1/p'=1$,

$$\sum_{k=1}^{\infty}\left(k^{1-2/p}\sum_{n=k}^{\infty}\frac{b_n}{n}\right)a_k=\sum_{n=1}^{\infty}\frac{b_n}{n}\sum_{k=1}^{n} k^{1-2/p} a_k$$

which is by the Hölder inequality not greater than

$$\left(\sum_{n=1}^{\infty} b_n^p n^{p-2}\right)^{1/p}\left[\sum_{n=1}^{\infty}\left(n^{-2(1-1/p)}\sum_{k=1}^{n} k^{1-2/p} a_k\right)^{p'}\right]^{1/p'}.$$

The second factor is, by Lemma 3, not greater then

$$\left(C_p \sum_{n=1}^{\infty} a_n^{p'}\right)^{1/p'}.$$

Therefore by the converse of the Hölder inequality we have (6.5). C_p may differ from each other.

(6.6) is easy to show. Actually

$$\sum_{k=n}^{\infty} \frac{b_k}{k} \geqq \sum_{k=n}^{2n} \frac{b_k}{k} \geqq b_{2n} \sum_{k=n}^{2n} \frac{1}{k} \geqq C b_{2n},$$

where C is an absolute constant. Hence

$$\sum_{n=1}^{\infty} n^{p-2}\left(\sum_{k=n}^{\infty} \frac{b_k}{k}\right)^p \geqq C \sum_{n=1}^{\infty} n^{p-2} b_{2n}^p \geqq C_p \sum_{n=1}^{\infty} (2n)^{p-2} b_{2n}^p,$$

Similarly

$$\sum_{n=1}^{\infty} n^{p-2}\left(\sum_{k=n}^{\infty} \frac{b_k}{k}\right)^p \geqq C_p \sum_{n=1}^{\infty} (2n+1)^{p-2} b_{2n+1}^p .$$

From both relations, we have (6.6).

We shall prove the following theorem.

THEOREM 4. *Consider the series* (6.1) *and* (6.2) *with the condition* (6.3). *Suppose* $r>1$ *and* na_n *is nondecreasing. If*

(6.7) $$\sum_{n=1}^{\infty} n^{r-2} a_n^r < \infty,$$

then $f(x)$ *has the mean derivative* $g(x)$ *in* $L^r(T)$.

More precisely

(6.8) $$\int_{-\pi}^{\pi} \left| \frac{f(x+h)-f(x)}{h} - g(x) \right|^r dx \leqq C_r \sum_{n \geqq |h|^{-1}} n^{r-2} a_n^r .$$

Before proving this theorem we give some notations and elementary facts we use. The proof of the fact that $g(x) \in L^r(T)$ under the conditions in the theorem is contained in the course of the proof of the theorem.

Denote by $D_n(x)$ and $\bar{D}_n(x)$ the Dirichlet and the conjugate Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin \frac{x}{2}},$$

$$\bar{D}_n(x) = \sum_{k=1}^{n} \sin kx = \frac{\sin \frac{nx}{2} \sin \frac{n+1}{2} x}{\sin \frac{x}{2}}$$

and

(6.9) $$J_n(h)=\Delta(a_n(1-\cos nh)/n),$$

(6.10) $$H_n(h)=\Delta(a_n(\sin nh-nh)/n).$$

Let $h>0$ without loss of generality. In what follows C's are constants which may be different on each occurrence. Note that a_n is nonincreasing.

Write

(6.11) $$b_n=a_n+n\Delta a_n, \quad n=1,2,\cdots$$

We see, for $nh<1$,

$$|J_n(h)|\leqq C\left[\Delta\left(\frac{a_n}{n}\right)n^2h^2+a_nh^2\right]\leqq C(a_n+n\Delta a_n)h^2=Cb_nh^2,$$

and for $nh\geqq 1$,

$$|J_n(h)|\leqq C\left[\Delta\left(\frac{a_n}{n}\right)+\frac{a_n}{n}h\right]\leqq C\frac{b_nh}{n}.$$

We also see that the same estimates for $H_n(h)$ hold. Namely

(6.12) $$|J_n(h)|\leqq Cb_nh^2, \quad \text{for } nh\leqq 1,$$
$$\leqq Cb_nh/n, \quad \text{for } nh>1,$$

(6.13) $$|H_n(h)|\leqq Cb_nh^2, \quad \text{for } nh\leqq 1,$$
$$\leqq Cb_nh/n, \quad \text{for } nh>1.$$

In this section we denote by $||\cdot||_r$ the norm in $L^r(T)$, $\left(\int_{-\pi}^{\pi}|\cdot|^r\right)^{1/r}$ (diffenent from those in **1**~**4**). We obviously have

(6.14) $$||D_n(x)||_r\leqq Cn^{1-1/r},\ ||\bar{D}_n(n)||_r\leqq Cn^{1-1/r}.$$

Finally we note that if (6.5) is true with a_n in place of b_n, then it is, with b_n in (6.11). Because

$$\sum_{k=n}^{\infty}\frac{b_k}{k}=\sum_{k=n}^{\infty}\left(\frac{a_k}{k}+\Delta a_k\right)=\sum_{k=n}^{\infty}\frac{a^k}{k}+a_n$$

and hence

$$\sum_{n=1}^{\infty}n^{p-2}\left(\sum_{k=n}^{\infty}\frac{b_k}{k}\right)^p\leqq C\sum_{n=1}^{\infty}n^{p-2}\left(\sum_{k=n}^{\infty}\frac{a_k}{k}\right)^p+C\sum_{n=1}^{\infty}n^{p-2}a_n^p$$

(6.15) $$\leqq C\sum_{n=1}^{\infty}n^{p-2}a_n^p\left(\leqq C\sum_{n=1}^{\infty}n^{p-2}b_n^p\right).$$

The similar thing is also true for (6.6), namely

$$\sum_{n=1}^{\infty}n^{p-2}b_n^p\leqq C\sum_{n=1}^{\infty}n^{p-2}\left(\sum_{k=n}^{\infty}\frac{a_k}{k}\right)^p+C\sum_{n=1}^{\infty}n^{p-2}a_n^p$$

(6.16) $$\leqq C\sum_{n=1}^{\infty} n^{p-2}\left(\sum_{n=k}^{\infty}\frac{a_k}{k}\right)^p,$$

by (6.6) with a_n in place of b_n.

We now proceed to prove the theorem.

Proof of Theorem 4.

$$\frac{f(x+h)-f(x)}{h}-g(x)=\sum_{n=1}^{\infty} a_n \sin nx\,\frac{\cos nh-1}{nh}+\sum_{n=1}^{\infty} a_n \cos nx\,\frac{\sin nh-nh}{nh}$$
$$=\sum_{n=1}^{\infty} A_n(x,h)+\sum_{n=1}^{\infty} B_n(x,h),$$

say. We have

(6.17) $$\left\|\frac{f(x+h)-f(x)}{h}-g(x)\right\|_r\leqq S_1+S_2,$$

where

$$S_1=S_1(h)=\left\|\sum_{n=1}^{\infty} A_n(x,h)\right\|_r,\ S_2=S_2(h)=\left\|\sum_{n=1}^{\infty} B_n(x,h)\right\|_r.$$

We shall prove the theorem by direct computations of S_1 and S_2.

Let $h>0$ and write $N=[h^{-1}]$. we first deal with S_1.

$$S_1\leqq\left\|\sum_{n\leqq N} A_n(x,h)\right\|_r+\left\|\sum_{n>N} A_n(x,h)\right\|_r=I_1+I_2,$$

say. By summation by parts, we see that

$$I_1\leqq N\left\|\sum_{n=1}^{N-1}\bar{D}_n(x)J_n(h)\right\|_r+N\|\bar{D}_N(x)a_N(1-\cos Nh)/N\|_r$$
$$\leqq N\sum_{n=1}^{N-1}|J_n(h)|\|\bar{D}_n(x)\|_r+a_N\|\bar{D}_n(x)\|_r.$$

Using the first inequalities of (6.12) and (6.14), we have

$$I_1\leqq CN^{-1}\sum_{n=1}^{N-1} b_n n^{1-1/r}+a_N N^{1-1/r}$$
$$=CN^{-1}\sum_{n=1}^{N-1} a_n n^{1-1/r}+CN^{-1}\sum_{n=1}^{N-1}\Delta a_n n^{2-1/r}+a_N N^{1-1/r}$$
$$\leqq CN^{-1}\sum_{n=1}^{N-1} a_n n^{1-1/r}+a_N N^{1-1/r}$$

which is not greater than $Ca_N N^{1-1/r}$, since na is nondecreasing.

We thus have

(6.18) $$I_1\leqq Ca_N N^{1-1/r}\leqq C\left(\sum_{n=N}^{\infty} a_n^r n^{r-2}\right)^{1/r},$$

for

$$\sum_{n=N}^{\infty} a_n^r n^{r-2} \geqq a_n^r \sum_{n=N}^{2N} n^{r-2} \geqq C a_N^r N^{r-1}.$$

For I_2, we apply the summation by parts, and we have

$$\begin{aligned}
I_2 &\leqq N\left\|\sum_{n \geqq N} \bar{D}_n(x) J_n(h)\right\|_r + N\|\bar{D}_{N-1} a_N (1\text{-}\cos Nh)/N\|_r \\
&\leqq CN\left(\int_{|x|<N^{-1}} \left|\sum_{n \geqq N} \bar{D}_n(x) J_n(h)\right|^r dx\right)^{1/r} \\
&\quad + \left(\int_{|x|\geqq N^{-1}} \left|\sum_{n \geqq N} \bar{D}_n(x) J_n(h)\right|^r dx\right)^{1/r} \\
&\quad + N\|\bar{D}_{N-1}(x) a_N (1-\cos Nh)/N\|_r \\
&= I_{21} + I_{22} + I_{23},
\end{aligned}$$

say. We see as before

$$I_{23} \leqq a_N N^{1-1/r} \leqq C\left(\sum_{n=N}^{\infty} a_n^r n^{r-2}\right)^{1/r}, \tag{6.19}$$

and

$$\begin{aligned}
I_{21} &\leqq CN\left(\int_{|x|<N^{-1}} \left|\sum_{N<n\leqq|x|^{-1}}\right|^r dx\right)^{1/r} \\
&\quad + CN\left(\int_{|x|<N^{-1}} \left|\sum_{|x|^{-1}<n}\right|^r\right)^{1/r} \\
&= I_{211} + I_{212},
\end{aligned}$$

say. Since $|\bar{D}_n(x)| \leqq Cn^2|x|$, we have, using the second relation of (6.12) and noting $r>1$,

$$\begin{aligned}
I_{211} &\leqq CN\left[\int_{|x|<N^{-1}} \left(\sum_{N\leqq n\leqq|x|^{-1}} h|x| n b_n\right)^r dx\right]^{1/r} \\
&\leqq C\left[\int_{|x|<N^{-1}} |x|^r \left(\sum_{n\leqq|x|^{-1}} n b_n\right)^r dx\right]^{1/r}.
\end{aligned}$$

Since

$$\sum_{n\leqq|x|^{-1}} n b_n = \sum_{n\leqq|x|^{-1}} n a_n + \sum_{n\leqq|x|^{-1}} n^2 \varDelta a_n,$$

we have

$$\begin{aligned}
I_{211} &\leqq C\left[\int_{|x|\leqq N^{-1}} (|x|^{-1} a_{[|x|^{-1}]})^r dx\right]^{1/r} \\
&= C\left[\sum_{k=N}^{\infty} \int_{(k+1)^{-1}}^{k^{-1}} (|x|^{-1} a_{[|x|^{-1}]})^r dx\right]^{1/r} \\
&= C\left(\sum_{k=N}^{\infty} k^{r-2} a_k^r\right)^{1/r}.
\end{aligned}$$

As to I_{212}, we have, using the second estimate of (6.12) and $|\bar{D}_n(x)|\leqq|x|^{-1}$,

$$I_{212}\leqq CN\Big[\int_{|x|\leqq N^{-1}}\Big(N^{-1}|x|^{-1}\sum_{|x|^{-1}<n} n^{-1}b_n\Big)^r dx\Big]^{1/r}$$
$$\leqq C\Big[\int_{|x|<N^{-1}}|x|^{-r}\Big(\sum_{n>|x|^{-1}} a_n n^{-1}+a_{[|x|^{-1}]}\Big)^r dx\Big]^{1/r}$$
$$\leqq\Big[\sum_{k=N}^{\infty}\int_{(k+1)^{-1}}^{k^{-1}}|x|^{-r}\Big(\sum_{n\geqq k} a_n n^{-1}+a_k\Big)^r dx\Big]^{1/r}$$
$$=C\Big[\sum_{k=N}^{\infty}k^{r-2}\Big(\sum_{n=k}a_n n^{-1}\Big)^r\Big]^{1/r}+C\Big(\sum_{k=N}^{\infty}k^{r-2}a_k\Big)^{1/r}.$$

Because of (6.5), the last one is not greater than $C(\sum_{k=N}k^{r-2}a_k{}^r)^{1/r}$.

Hence we have obtained

(6.20)
$$I_{21}\leqq C\Big(\sum_{k=N}n^{r-2}a_n^r\Big)^{1/r}.$$

For I_2, we have, using the second of (6.12) and $|\bar{D}_n(x)|\leqq|x|^{-1}$,

$$I_2\leqq C\Big[\int_{|x|>N^{-1}}\Big(|x|^{-1}\sum_{n=N}^{\infty}b_n n^{-1}\Big)^r\Big]^{1/r}$$
$$\leqq CN^{1-1/r}\sum_{n=N}^{\infty}b_n n^{-1}\leqq CN^{1-1/r}\Big(\sum_{n=N}^{\infty}a_n n^{-1}+a_N\Big)$$
$$\leqq CN^{1-1/r}\Big(\sum_{n=N}^{\infty}a_n^r n^{r-2}\Big)^{1/r}\Big(\sum_{n=N}^{\infty}n^{-2+2/r}\Big)^{1-1/r}$$
$$+CN^{1-1/r}a_N$$

(6.21)
$$\leqq C\Big(\sum_{n=N}a_n^r n^{r-2}\Big)^{1/r}.$$

From (6.9), (6.20) and (6.21) we have

(9.22)
$$I_2\leqq C\Big(\sum_{n=N}^{\infty}a_n^r n^{r-2}\Big)^{1/r}.$$

Putting (6.18) and (6.22) together, we finally have obtained

(6.23)
$$S_1\leqq C\Big(\sum_{n>h^{-1}}n^{r-2}a_n^r\Big)^{1/r}.$$

Finally since $J_n(h)$ and $H_n(h)$ have the same estimates (6.12) and (6.13), and $\bar{D}_n(x)$ and $D_n(x)$ also have the similar estimates (6.14) and $|D_n(x)|\leqq|x|^{-1}$, $|D_n(x)|\leqq Cn$, we see that just the same manipulation gives us that

(6.24)
$$S_2\leqq C\Big(\sum_{n>h^{-1}}n^{r-2}a_n^r\Big)^{1/r}.$$

(6.23) and (6.24) now complete the proof of the theorem.

Now let $a_n=(n^\alpha\log(n+1))^{-1}$ and $1-1/r\leqq\alpha<1$ we easily see that all the conditions for a_n in Theorem 4 is satisfied for $r>1$. This shows (4.7).

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