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THE MEAN DERIVATIVES AND THE ABSOLUTE CONVERGENCE OF THE FOURIER SERIES OF A STOCHASTIC PROCESS

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ABSTRACT

A criterion for the almost sure absolute convergence of the Fourier series of a peroidic stochastic process which has the mean derivatives is given and the result is applied to the sample properties of the process. A theorem on the mean derivative of a sine series is also given.

1. Introduction

Let $X(t, \omega), t \in R^1$, be a complex valued stochastic process on a given probability space (Ω, F, P) . Suppose that $X(t, \omega)$ is measurable $L \times F$ on $R^1 \times \Omega$, L being the class of Lebesgue measurable sets on R^1 , and that for $r \geq 1$,

$$(1.1) \quad E|X(t, \omega)|^r < \infty,$$

for every t and $X(t, \omega)$ is 2π -periodic:

$$(1.2) \quad E|X(t+2\pi, \omega) - X(t, \omega)| = 0,$$

for $t \in R^1$.

Furthermore we throughout assume that $(T = [-\pi, \pi])$

$$(1.3) \quad \int_{-\pi}^{\pi} E|X(t, \omega)|^r dt < \infty.$$

In case we write $X(t, \omega) \in L_p^r(T \times \Omega)$.

Let

$$(1.4) \quad \sum_{n=-\infty}^{\infty} C_n(\omega) e^{int}$$

be the Fourier series of $X(t, \omega)$, where

$$(1.5) \quad C_n(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t, \omega) e^{-int} dt, \quad n=0, \pm 1, \pm 2, \dots$$

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In [1], the author has given some results on the almost sure convergence of the series

$$(1.6) \quad \sum_{n=-\infty}^{\infty} |n|^k \alpha_n |C_n(\omega)|$$

($\alpha_n > 0$) for some nonnegative integer k , and applied them to sample continuity or differentiability of $X(t, \omega)$.

Write

$$(1.7) \quad M_r^{*(p)}(\delta) = M_r^{*(p)}(\delta, X) = \sup_{|h| \leq \delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} E |A_h^{(p)} X(t, \omega)|^r dt \right)^{1/r},$$

where p is a positive integer and $A_h^{(p)} X(t, \omega)$ is the p -th difference of $X(t, \omega)$:

$$(1.8) \quad A_h^{(p)} X(t, \omega) = \sum_{k=0}^p (-1)^{p-k} \binom{p}{k} X(t + kh, \omega).$$

Let $\phi(t)$ be a nondecreasing function on $[0, 1]$ such that either $\phi(0) = 0$ and $\phi(t)/t$ is nonincreasing on $[0, 1]$, or $\phi(t)$ is identically 1 on $[0, 1]$.

One of the results obtained in [1] is the following.

THEOREM A. *Let $X(t, \omega)$ be of $L_P^r(T \times \Omega)$ for some $r > 1$. Let k be a given non-negative integer. If there exists a positive integer p such that*

$$(1.9) \quad \sum_{n=1}^{\infty} n^{k-1+1/r} \left[\phi\left(\frac{1}{n}\right) \right]^{-1} M_r^{*(p)}\left(\frac{1}{n}\right) < \infty,$$

then

$$(1.10) \quad \sum_{n=-\infty}^{\infty} |n|^k \left[\phi\left(\frac{1}{n}\right) \right]^{-1} |C_n(\omega)| < \infty,$$

almost surely.

In this paper we give a relationship between $M_r^{*(p)}(\delta)$ and the mean derivative of $X(t, \omega)$ and apply it to the absolute convergence of Fourier series and the sample properties of $X(t, \omega)$.

2. Mean derivatives.

Let $X(t, \omega) \in L_P^r(T \times \Omega)$ for some $r \geq 1$. If there is a stochastic process $X'_M(t, \omega)$ of $L_P^r(T \times \Omega)$ such that

$$(2.1) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} E \left| \frac{X(t+h, \omega) - X(t, \omega)}{h} - X'_M(t, \omega) \right|^r dt \rightarrow 0$$

as $h \rightarrow 0$, then $X'_M(t, \omega)$ is called the mean derivative in $L^r(T \times \Omega)$ of $X(t, \omega)$. The p -th mean derivative $X_M^{(p)}(t, \omega)$ is defined successively in an obvious way.

Write

$$(2.2) \quad \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} E|X(t, \omega)|^r dt \right)^{1/r} = \|X(t, \omega)\|_r$$

which is the norm of $X(t, \omega)$ in $L^r(T \times \Omega)$.

We give two simple lemmas.

LEMMA 1. *If $X(t, \omega) \in L^r(T \times \Omega)$ has the mean derivative $X'_M(t, \omega)$ in $L^r(T \times \Omega)$, then, as $h \rightarrow 0$,*

$$(2.3) \quad \left\| \frac{1}{h} \int_t^{t+h} X(u, \omega) du - X(t, \omega) \right\|_r = O(|h|).$$

Proof. The left hand side of (2.3) is not greater than

$$\begin{aligned} & \left\| \frac{1}{h} \int_0^h [X(u+t, \omega) - X(t, \omega) - u X'_M(t, \omega)] du \right\|_r + \frac{|h|}{2} \left\| X'_M(t, \omega) \right\|_r \\ & \leq \int_0^{|h|} \left\| \frac{X(u+t, \omega) - X(t, \omega)}{u} - X'_M(t, \omega) \right\|_r du + \frac{|h|}{2} \left\| X'_M(t, \omega) \right\|_r = o(|h|) + O(|h|) = O(|h|). \end{aligned}$$

LEMMA 2. *If $X(t, \omega) \in L^r_P(T \times \Omega)$ has the mean derivative $X'_M(t, \omega)$ in $L^r(T \times \Omega)$, then*

$$(2.4) \quad X(t+h, \omega) - X(t, \omega) = \int_t^{t+h} X'_M(u, \omega) du$$

almost everywhere in $T \times \Omega$, for each h .

Namely there is a subset $G = G(h)$ of $T \times \Omega$ such that (2.4) holds for $(t, \omega) \in G$ and $\mu(G) = 2\pi$, μ being the $m \times P$ measure and m the Lebesgue measure. We note that we can easily see that (2.4) holds on G if t is replaced by $t+h$ by periodicity, when h is fixed. The dependence of G on h makes the difference between the almost sure absolute continuity and the existence of mean derivative in $L(T \times \Omega)$ of $X(t, \omega)$.

Proof. Let $h > 0$ without loss of generality and let $|k| < h$. Write

$$S_k = \|Y(t, \omega)\|_r,$$

where

$$Y(t, \omega) = \int_t^{t+h} \left[\frac{X(u+k, \omega) - X(u, \omega)}{k} - X'_M(u, \omega) \right] du.$$

Since

$$\begin{aligned} |Y(t, \omega)|^r & \leq \int_t^{t+h} \left| \frac{X(u+k, \omega) - X(u, \omega)}{k} - X'_M(u, \omega) \right|^r du \cdot h^{r-1} \\ & \leq \left\| \frac{X(u+k, \omega) - X(u, \omega)}{k} - X'_M(u, \omega) \right\|_r^r du \cdot h^{r-1}. \end{aligned}$$

Hence

$$(2.5) \quad S_k^r = \frac{1}{2\pi} \int_{-\pi}^{\pi} E|Y(t, \omega)|^r dt \leq 2\pi h^{r-1} \cdot \|Y(t, \omega)\|_r^r \rightarrow 0$$

as $k \rightarrow 0$ for a fixed h .

On the other hand,

$$\begin{aligned} S_k &= \left\| \frac{1}{k} \int_{t+k}^{t+k+h} X(u, \omega) du - \frac{1}{k} \int_t^{t+h} X(u, \omega) du - \int_t^{t+h} X'_M(u, \omega) du \right\|_r \\ &= \left\| \frac{1}{k} \int_{t+h}^{t+h+k} X(u, \omega) du - \frac{1}{k} \int_t^{t+k} X(u, \omega) du - \int_t^{t+h} X'_M(u, \omega) du \right\|_r. \end{aligned}$$

By Lemma 1, this converges to

$$\left\| X(t+h, \omega) - X(t, \omega) - \int_t^{t+h} X'_M(u, \omega) du \right\|_r$$

as $k \rightarrow 0$. Hence from (2.5) the last quantity should be zero. This gives us that (2.2) holds a.e. in $T \times \Omega$ for each h .

3. Mean derivatives and continuity modulus.

Let $X(t, \omega)$ be of $L_P^r(T \times \Omega)$, $r \geq 1$. Suppose $X_M^{(p)}(t, \omega)$, the mean derivative of order p of $X(t, \omega)$ in $L^r(T \times \Omega)$ exists, p being a positive integer.

Now let $G_l = G_l(h)$ be the set on which (2.4) with $X_M^{(l-1)}(t, \omega)$ in place of $X(t, \omega)$ $l=1, 2, \dots, p$, holds.

$$\Delta_h^{(2)} X(t, \omega) = \Delta_h^{(1)} X(t+h, \omega) - \Delta_h^{(1)} X(t, \omega)$$

is, by Lemma 2, equal in G_1 to

$$\int_{t+h}^{t+2h} X'_M(t_1, \omega) dt_1 - \int_t^{t+h} X'_M(t_1, \omega) dt_1 = \int_t^{t+h} [X'_M(t_1+h, \omega) - X'_M(t_1, \omega)] dt$$

which is again by Lemma 2 equal in $G_1 \cap G_2$ to

$$= \int_t^{t+h} dt_1 \int_{t_1}^{t_1+h} X''_M(t_2, \omega) dt_2.$$

Repeating this procedure, we have

$$(3.1) \quad \Delta_h^{(p)} X(t, \omega) = \int_t^{t+h} dt_1 \int_{t_1}^{t_1+h} dt_2 \cdots \int_{t_{p-1}}^{t_{p-1}+h} X_M^{(p)}(t_p, \omega) dt_p.$$

a.e. in $G_0 = \bigcap_{l=1}^p G_l$, $\mu(G_0) = 2\pi$.

We now prove

THEOREM 1. *If $X(t, \omega)$ belongs to $L_P^r(T \times \Omega)$ and has $X_M^{(p)}(t, \omega)$ in $L^r(T \times \Omega)$, $r \geq 1$, p being a positive integer, then*

$$(3.2) \quad M_r^{*(p)}(\delta) \leq 2^{p/r} \|X_M^{(p)}(t, \omega)\|_r \cdot \delta^p.$$

Proof. From (3.1), we have, for any fixed $h > 0$, in G_0 ,

$$\begin{aligned} |A_h^{(p)} X(t, \omega)|^r &= \left[\int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)|^r dt_p \right]^r \\ &\leq \int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)|^r dt_p \cdot \left[\int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} dt_p \right]^{r-1} \\ &= h^{p(r-1)} \int_t^{t+h} dt_1 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)|^r dt_p \\ &= h^{p(r-1)} \int_t^{t+h} Y(t_1, \omega) dt_1, \end{aligned}$$

where

$$Y(t_1, \omega) = \int_{t_1}^{t_1+h} dt_2 \cdots \int_{t_{p-1}}^{t_{p-1}+h} |X_M^{(p)}(t_p, \omega)|^r dt_p.$$

Since for a 2π -periodic function $f(u) \in L^1(T)$,

$$\left| \int_{-\pi}^{\pi} \int_t^{t+h} f(u) du \right| \leq |h| \int_{-\pi-h}^{\pi+h} |f(u)| du \leq 2|h| \int_{-\pi}^{\pi} |f(u)| du$$

for $|h| < \pi$, which is easily seen by the interchange of integration signs on the left hand side, we have, for small $|h|$,

$$\begin{aligned} \|A_h^{(p)} X(t, \omega)\|_r^r &= \frac{1}{2\pi} \int_{-\pi}^{\pi} E |A_h^{(p)} X(t, \omega)|^r dt \\ &\leq \frac{|h|^{p(r-1)}}{2\pi} \int_{-\pi}^{\pi} dt \int_t^{t+h} E Y(t_1, \omega) dt_1 \\ &\leq \frac{1}{\pi} |h|^{p(r-1)} |h| \int_{-\pi}^{\pi} E Y(t_1, \omega) dt_1. \end{aligned}$$

Repeating the same arguments $p-1$ more times, we have

$$\|A_h^{(p)} X(t, \omega)\|_r \leq 2^p |h|^{pr} \frac{1}{2\pi} \int_{-\pi}^{\pi} E |X_M^{(p)}(t_p, \omega)|^r dt_p = 2^p |h|^{pr} \|X_M^{(p)}(t, \omega)\|_r.$$

This is no more than (3.2).

4. Absolute convergence of the Fourier series of a periodic stochastic process.

Let $\phi(t)$ be the function in 1. The combination of Theorem 1 and Theorem A immediately gives us the following theorem.

THEOREM 2. *If $X(t, \omega) \in L^r(T \times \Omega)$ and for some nonnegative integer k , $X_M^{(k+1)}(t, \omega)$ exists in $L^r(T \times \Omega)$, $1 < r \leq 2$, and*

The mean derivatives

$$(4.1) \quad \sum_{n=1}^{\infty} n^{-2+1/r} \left[\phi\left(\frac{1}{n}\right) \right]^{-1} < \infty,$$

then

$$(4.2) \quad \sum_{n=-\infty}^{\infty} |n|^k \left[\phi\left(\frac{1}{n}\right) \right]^{-1} |C_n(\omega)| < \infty,$$

almost surely.

In particular, taking $\phi(t)=t^\alpha$, we have the following collorary. For nonrandom case, this is thought of as a version of a known result on the absolute convergence of an absolutely continuous function which has the derivative belonging to $L^r(T)$. ([1] Cor. 2. p. 161, [4] Theorem 3.8, p. 242)

COLLORARY 1. If $X(t, \omega) \in L_P^r(T \times \Omega)$ and for some nonnegative k , $X_M^{(k+1)}(t, \omega)$ exists in $L^r(T \times \Omega)$, and

$$(4.3) \quad 0 \leq \alpha < 1 - \frac{1}{r},$$

then

$$(4.4) \quad \sum_{n=-\infty}^{\infty} |n|^{k+\alpha} |C_n(\omega)| < \infty,$$

almost surely.

The condition (4.3) is of the best kind in some sense. Actually if $\alpha=1-1/r$, there is an $X(t, \omega) \in L_P^r(T \times \Omega)$ which has $X_M^{(k+1)}(t, \omega)$ in $L^r(T \times \Omega)$ and is such that the series in (4.4) diverges almost surely.

We show this for simplicity when $k=0$. Let

$$(4.5) \quad f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{n^{\alpha+1} \log n}, \quad 0 \leq \alpha < 1.$$

This is absolutely continuous and $f(x) = \int_0^x g(u) du$, where

$$(4.6) \quad g(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{n^{\alpha} \log n}$$

which is defined for all $-\pi \leq x \leq \pi$ except at $x=0$. We can show that $g(x) \in L^r(T)$ and is the mean derivative of $f(x)$ in $L^r(T)$ for $r>1$, if $\alpha \geq 1-1/r$. More precisely

$$(4.7) \quad \int_{-\pi}^{\pi} \left| \frac{f(x+h) - f(x)}{h} - g(x) \right|^r dx \leq C \frac{|h|^{(\alpha-1)r+1}}{|\log |h||^r},$$

as $h \rightarrow 0$, where C is a constant independent of h .

The proof of this fact will be shown in a more general form in 6.

Let $r>1$ and $\alpha=1-1/r$. Then from (4.7), $g(x)$ is the mean derivative of $f(x)$

in $L^r(T)$ and

$$\sum_{n=2}^{\infty} n^{\alpha} b_n = \infty$$

where $b_n = n^{-\alpha-1} (\log n)^{-1}$ is the sine coefficient of $f(x)$.

Theorem 2 and Corollary 1 seem to be new, even for nonrandom case. The above example shows that Corollary 1 with $k=0$ for nonrandom case is no more true if $\alpha=1-1/r$.

$$(4.9) \quad X(t, \omega) = X(\omega) f(t),$$

where $f(t)$ is the function (4.5) and $X(\omega)$ is any random variable of $L^r(\Omega)$, gives a counter example for Corollary 1 with $k=0, \alpha=1-1/r$.

5. Sample properties of a periodic stochastic process

The author has shown the following theorem.

THEOREM B. *Let $X(t, \omega) \in L^r_P, 1 < r \leq 2$. Let k be a given nonnegative integer. Suppose there exists a positive integer p such that*

$$(5.1) \quad \sum_{n=1}^{\infty} n^{k-1+1/r} \left[\phi\left(\frac{1}{n}\right) \right]^{-1} M_r^{*(p)}\left(\frac{1}{n}\right) < \infty.$$

If $X(t, \omega)$ is stochastically continuous, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ with the property that $X_0(t, \omega)$ has almost surely the k -th derivative belonging to the Lipschitz class Λ_ϕ .

Λ_ϕ is the class of functions f with continuity modulus ϕ .

This theorem is applied to the case where the mean derivative of $(k+1)$ -th order of $X(t, \omega)$ exists in $L^r(T \times \Omega)$ and the following theorem is immediately obtained, $p=k+1$ being taken, which corresponds to the critical case in some sense.

THEOREM 3. *If the conditions in Theorem 2 are satisfied for $1 < r \leq 2$ and $X(t, \omega)$ is stochastically continuous, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$, of $X(t, \omega)$, which has almost surely the k -th derivative belonging to Λ_ϕ .*

6. A theorem on a sine series.

We shall prove a theorem on a sine series which implies (4.7) as a particular case.

Consider two series

$$(6.1) \quad f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx$$

and

The mean derivatives

$$(6.2) \quad g(x) = \sum_{n=1}^{\infty} a_n \cos nx.$$

Let $r > 1$ and suppose

$$(6.3) \quad a_n > 0, a_n \rightarrow 0, \Delta^2 a_n \geq 0,$$

where $\Delta_n = a_n - a_{n+1}$, $\Delta^2 a_n = \Delta(\Delta a_n)$.

It is noted that (6.3) implies $\Delta a_n \geq 0$. It is well known that (6.2) is convergent except at $x=0$ and $g(x)$ is of $L^1(T)$. $f(x)$ is then absolutely continuous and $f'(x)=g(x)$ almost everywhere. We shall give a condition which assures that $f(x)$ has the mean derivative $g(x)$ in $L^r(T)$.

We begin with

LEMMA 3. Let $p > 1$ and β be a real number > -1 . For any $b_n \geq 0, n=1, 2, \dots$,

$$(6.4) \quad \sum_{n=1}^{\infty} \left(n^{-\beta-1} \sum_{k=1}^n k^{\beta} b_k \right)^p \leq C_p \sum_{n=1}^{\infty} b_n^p,$$

where C_p is a constant depending only on p .

This is a particular case of the following inequality with $a_n = \lambda_n^{-1/p} b_n$, $\lambda_n = n^{\beta p / (p-1)}$.

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{\Lambda_n} \sum_{k=1}^n \lambda_k a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p,$$

for $p > 1, a_n \geq 0, \lambda_n > 0$, where $\Lambda_n = \lambda_1 + \lambda_2 + \dots + \lambda_n$. (See [2] p. 247, Theorem 332)

LEMMA 4. Let $b_n \geq 0, p > 1$. Then

$$(6.5) \quad \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k} \right)^p \leq C_p \sum_{n=1}^{\infty} n^{p-2} b_n^p,$$

and if b_n is nonincreasing, then we moreover have

$$(6.6) \quad \sum_{n=1}^{\infty} n^{p-2} b_n^p \leq C'_p \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k} \right)^p,$$

where C_p and C'_p are constants depending only on p .

Proof. For any $a_n \geq 0$ with $\sum_{n=1}^{\infty} a_n^{p'} < \infty, 1/p + 1/p' = 1$,

$$\sum_{k=1}^{\infty} \left(k^{1-2/p} \sum_{n=k}^{\infty} \frac{b_n}{n} \right) a_k = \sum_{n=1}^{\infty} \frac{b_n}{n} \sum_{k=1}^n k^{1-2/p} a_k$$

which is by the Hölder inequality not greater than

$$\left(\sum_{n=1}^{\infty} b_n^p n^{p-2} \right)^{1/p} \left[\sum_{n=1}^{\infty} \left(n^{-2(1-1/p)} \sum_{k=1}^n k^{1-2/p} a_k \right)^{p'} \right]^{1/p'}.$$

The second factor is, by Lemma 3, not greater than

$$\left(C_p \sum_{n=1}^{\infty} a_n^{p'}\right)^{1/p'}.$$

Therefore by the converse of the Hölder inequality we have (6.5). C_p may differ from each other.

(6.6) is easy to show. Actually

$$\sum_{k=n}^{\infty} \frac{b_k}{k} \geq \sum_{k=n}^{2n} \frac{b_k}{k} \geq b_{2n} \sum_{k=n}^{2n} \frac{1}{k} \geq C b_{2n},$$

where C is an absolute constant. Hence

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k} \right)^p \geq C \sum_{n=1}^{\infty} n^{p-2} b_{2n}^p \geq C_p \sum_{n=1}^{\infty} (2n)^{p-2} b_{2n}^p,$$

Similarly

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k} \right)^p \geq C_p \sum_{n=1}^{\infty} (2n+1)^{p-2} b_{2n+1}^p.$$

From both relations, we have (6.6).

We shall prove the following theorem.

THEOREM 4. *Consider the series (6.1) and (6.2) with the condition (6.3). Suppose $r > 1$ and na_n is nondecreasing. If*

$$(6.7) \quad \sum_{n=1}^{\infty} n^{r-2} a_n^r < \infty,$$

then $f(x)$ has the mean derivative $g(x)$ in $L^r(T)$.

More precisely

$$(6.8) \quad \int_{-\pi}^{\pi} \left| \frac{f(x+h) - f(x)}{h} - g(x) \right|^r dx \leq C_r \sum_{n \geq |h|^{-1}} n^{r-2} a_n^r.$$

Before proving this theorem we give some notations and elementary facts we use. The proof of the fact that $g(x) \in L^r(T)$ under the conditions in the theorem is contained in the course of the proof of the theorem.

Denote by $D_n(x)$ and $\bar{D}_n(x)$ the Dirichlet and the conjugate Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^n \cos kx = \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin \frac{x}{2}},$$

$$\bar{D}_n(x) = \sum_{k=1}^n \sin kx = \frac{\sin \frac{nx}{2} \sin \frac{n+1}{2}x}{\sin \frac{x}{2}}$$

and

The mean derivatives

$$(6.9) \quad J_n(h) = \Delta(a_n(1 - \cos nh)/n),$$

$$(6.10) \quad H_n(h) = \Delta(a_n(\sin nh - nh)/n).$$

Let $h > 0$ without loss of generality. In what follows C 's are constants which may be different on each occurrence. Note that a_n is nonincreasing.

Write

$$(6.11) \quad b_n = a_n + n\Delta a_n, \quad n = 1, 2, \dots$$

We see, for $nh < 1$,

$$|J_n(h)| \leq C \left[\Delta\left(\frac{a_n}{n}\right) n^2 h^2 + a_n h^2 \right] \leq C(a_n + n\Delta a_n) h^2 = C b_n h^2,$$

and for $nh \geq 1$,

$$|J_n(h)| \leq C \left[\Delta\left(\frac{a_n}{n}\right) + \frac{a_n}{n} h \right] \leq C \frac{b_n h}{n}.$$

We also see that the same estimates for $H_n(h)$ hold. Namely

$$(6.12) \quad |J_n(h)| \leq C b_n h^2, \quad \text{for } nh \leq 1,$$

$$\leq C b_n h/n, \quad \text{for } nh > 1,$$

$$(6.13) \quad |H_n(h)| \leq C b_n h^2, \quad \text{for } nh \leq 1,$$

$$\leq C b_n h/n, \quad \text{for } nh > 1.$$

In this section we denote by $\|\cdot\|_r$ the norm in $L^r(T)$, $\left(\int_{-\pi}^{\pi} |\cdot|^r\right)^{1/r}$ (different from those in 1~4). We obviously have

$$(6.14) \quad \|D_n(x)\|_r \leq C n^{1-1/r}, \quad \|\bar{D}_n(n)\|_r \leq C n^{1-1/r}.$$

Finally we note that if (6.5) is true with a_n in place of b_n , then it is, with b_n in (6.11). Because

$$\sum_{k=n}^{\infty} \frac{b_k}{k} = \sum_{k=n}^{\infty} \left(\frac{a_k}{k} + \Delta a_k \right) = \sum_{k=n}^{\infty} \frac{a_k}{k} + a_n$$

and hence

$$(6.15) \quad \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{b_k}{k} \right)^p \leq C \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{a_k}{k} \right)^p + C \sum_{n=1}^{\infty} n^{p-2} a_n^p$$

$$\leq C \sum_{n=1}^{\infty} n^{p-2} a_n^p \left(\leq C \sum_{n=1}^{\infty} n^{p-2} b_n^p \right).$$

The similar thing is also true for (6.6), namely

$$\sum_{n=1}^{\infty} n^{p-2} b_n^p \leq C \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{a_k}{k} \right)^p + C \sum_{n=1}^{\infty} n^{p-2} a_n^p$$

$$(6.16) \quad \leq C \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{k=n}^{\infty} \frac{a_k}{k} \right)^p,$$

by (6.6) with a_n in place of b_n .

We now proceed to prove the theorem.

Proof of Theorem 4.

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} - g(x) &= \sum_{n=1}^{\infty} a_n \sin nx \frac{\cos nh - 1}{nh} + \sum_{n=1}^{\infty} a_n \cos nx \frac{\sin nh - nh}{nh} \\ &= \sum_{n=1}^{\infty} A_n(x, h) + \sum_{n=1}^{\infty} B_n(x, h), \end{aligned}$$

say. We have

$$(6.17) \quad \left\| \frac{f(x+h)-f(x)}{h} - g(x) \right\|_r \leq S_1 + S_2,$$

where

$$S_1 = S_1(h) = \left\| \sum_{n=1}^{\infty} A_n(x, h) \right\|_r, \quad S_2 = S_2(h) = \left\| \sum_{n=1}^{\infty} B_n(x, h) \right\|_r.$$

We shall prove the theorem by direct computations of S_1 and S_2 .

Let $h > 0$ and write $N = [h^{-1}]$. we first deal with S_1 .

$$S_1 \leq \left\| \sum_{n \leq N} A_n(x, h) \right\|_r + \left\| \sum_{n > N} A_n(x, h) \right\|_r = I_1 + I_2,$$

say. By summation by parts, we see that

$$\begin{aligned} I_1 &\leq N \left\| \sum_{n=1}^{N-1} \bar{D}_n(x) f_n(h) \right\|_r + N \| \bar{D}_N(x) a_N (1 - \cos Nh) / N \|_r \\ &\leq N \sum_{n=1}^{N-1} |f_n(h)| \| \bar{D}_n(x) \|_r + a_N \| \bar{D}_N(x) \|_r. \end{aligned}$$

Using the first inequalities of (6.12) and (6.14), we have

$$\begin{aligned} I_1 &\leq CN^{-1} \sum_{n=1}^{N-1} b_n n^{1-1/r} + a_N N^{1-1/r} \\ &= CN^{-1} \sum_{n=1}^{N-1} a_n n^{1-1/r} + CN^{-1} \sum_{n=1}^{N-1} \Delta a_n n^{2-1/r} + a_N N^{1-1/r} \\ &\leq CN^{-1} \sum_{n=1}^{N-1} a_n n^{1-1/r} + a_N N^{1-1/r} \end{aligned}$$

which is not greater than $Ca_N N^{1-1/r}$, since na is nondecreasing.

We thus have

$$(6.18) \quad I_1 \leq Ca_N N^{1-1/r} \leq C \left(\sum_{n=N}^{\infty} a_n^r n^{r-2} \right)^{1/r},$$

for

The mean derivatives

$$\sum_{n=N}^{\infty} a_n^r n^{r-2} \geq a_n^r \sum_{n=N}^{2N} n^{r-2} \geq C a_N^r N^{r-1}.$$

For I_2 , we apply the summation by parts, and we have

$$\begin{aligned} I_2 &\leq N \left\| \sum_n \bar{D}_n(x) J_n(h) \right\|_r + N \| \bar{D}_{N-1} a_N (1 - \cos Nh) / N \|_r \\ &\leq CN \left(\int_{|x| < N^{-1}} \left| \sum_n \bar{D}_n(x) J_n(h) \right|^r dx \right)^{1/r} \\ &\quad + \left(\int_{|x| \geq N^{-1}} \left| \sum_n \bar{D}_n(x) J_n(h) \right|^r dx \right)^{1/r} \\ &\quad + N \| \bar{D}_{N-1}(x) a_N (1 - \cos Nh) / N \|_r \\ &= I_{21} + I_{22} + I_{23}, \end{aligned}$$

say. We see as before

$$(6.19) \quad I_{23} \leq a_N N^{1-1/r} \leq C \left(\sum_{n=N}^{\infty} a_n^r n^{r-2} \right)^{1/r},$$

and

$$\begin{aligned} I_{21} &\leq CN \left(\int_{|x| < N^{-1}} \left| \sum_{N \leq n \leq |x|^{-1}} \right|^r dx \right)^{1/r} \\ &\quad + CN \left(\int_{|x| < N^{-1}} \left| \sum_{|x|^{-1} < n} \right|^r dx \right)^{1/r} \\ &= I_{211} + I_{212}, \end{aligned}$$

say. Since $|\bar{D}_n(x)| \leq C n^2 |x|$, we have, using the second relation of (6.12) and noting $r > 1$,

$$\begin{aligned} I_{211} &\leq CN \left[\int_{|x| < N^{-1}} \left(\sum_{N \leq n \leq |x|^{-1}} h |x| n b_n \right)^r dx \right]^{1/r} \\ &\leq C \left[\int_{|x| < N^{-1}} |x|^r \left(\sum_{n \leq |x|^{-1}} n b_n \right)^r dx \right]^{1/r}. \end{aligned}$$

Since

$$\sum_{n \leq |x|^{-1}} n b_n = \sum_{n \leq |x|^{-1}} n a_n + \sum_{n \leq |x|^{-1}} n^2 \Delta a_n,$$

we have

$$\begin{aligned} I_{211} &\leq C \left[\int_{|x| \leq N^{-1}} (|x|^{-1} a_{\lfloor |x|^{-1} \rfloor})^r dx \right]^{1/r} \\ &= C \left[\sum_{k=N}^{\infty} \int_{(k+1)^{-1}}^{k^{-1}} (|x|^{-1} a_{\lfloor |x|^{-1} \rfloor})^r dx \right]^{1/r} \\ &= C \left(\sum_{k=N}^{\infty} k^{r-2} a_k^r \right)^{1/r}. \end{aligned}$$

As to I_{212} , we have, using the second estimate of (6.12) and $|\bar{D}_n(x)| \leq |x|^{-1}$,

$$\begin{aligned} I_{212} &\leq CN \left[\int_{|x| \leq N^{-1}} \left(N^{-1} |x|^{-1} \sum_{|x|^{-1} \leq n} n^{-1} b_n \right)^r dx \right]^{1/r} \\ &\leq C \left[\int_{|x| < N^{-1}} |x|^{-r} \left(\sum_{n > |x|^{-1}} a_n n^{-1} + a_{\lfloor |x|^{-1} \rfloor} \right)^r dx \right]^{1/r} \\ &\leq \left[\sum_{k=N}^{\infty} \int_{(k+1)^{-1}}^{k^{-1}} |x|^{-r} \left(\sum_{n \geq k} a_n n^{-1} + a_k \right)^r dx \right]^{1/r} \\ &= C \left[\sum_{k=N}^{\infty} k^{r-2} \left(\sum_{n=k}^{\infty} a_n n^{-1} \right)^r \right]^{1/r} + C \left(\sum_{k=N}^{\infty} k^{r-2} a_k \right)^{1/r}. \end{aligned}$$

Because of (6.5), the last one is not greater than $C(\sum_{k=N}^{\infty} k^{r-2} a_k^r)^{1/r}$.

Hence we have obtained

$$(6.20) \quad I_{21} \leq C \left(\sum_{k=N}^{\infty} n^{r-2} a_n^r \right)^{1/r}.$$

For I_2 , we have, using the second of (6.12) and $|\bar{D}_n(x)| \leq |x|^{-1}$,

$$\begin{aligned} I_2 &\leq C \left[\int_{|x| > N^{-1}} \left(|x|^{-1} \sum_{n=N}^{\infty} b_n n^{-1} \right)^r \right]^{1/r} \\ &\leq CN^{1-1/r} \sum_{n=N}^{\infty} b_n n^{-1} \leq CN^{1-1/r} \left(\sum_{n=N}^{\infty} a_n n^{-1} + a_N \right) \\ &\leq CN^{1-1/r} \left(\sum_{n=N}^{\infty} a_n^r n^{r-2} \right)^{1/r} \left(\sum_{n=N}^{\infty} n^{-2+2/r} \right)^{1-1/r} \\ &\quad + CN^{1-1/r} a_N \\ (6.21) \quad &\leq C \left(\sum_{n=N}^{\infty} a_n^r n^{r-2} \right)^{1/r}. \end{aligned}$$

From (6.9), (6.20) and (6.21) we have

$$(9.22) \quad I_2 \leq C \left(\sum_{n=N}^{\infty} a_n^r n^{r-2} \right)^{1/r}.$$

Putting (6.18) and (6.22) together, we finally have obtained

$$(6.23) \quad S_1 \leq C \left(\sum_{n > h^{-1}} n^{r-2} a_n^r \right)^{1/r}.$$

Finally since $J_n(h)$ and $H_n(h)$ have the same estimates (6.12) and (6.13), and $\bar{D}_n(x)$ and $D_n(x)$ also have the similar estimates (6.14) and $|D_n(x)| \leq |x|^{-1}$, $|D_n(x)| \leq Cn$, we see that just the same manipulation gives us that

$$(6.24) \quad S_2 \leq C \left(\sum_{n > h^{-1}} n^{r-2} a_n^r \right)^{1/r}.$$

(6.23) and (6.24) now complete the proof of the theorem.

Now let $a_n = (n^\alpha \log(n+1))^{-1}$ and $1-1/r \leq \alpha < 1$ we easily see that all the conditions for a_n in Theorem 4 is satisfied for $r > 1$. This shows (4.7).

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