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STEIN-TYPE ESTIMATORS FOR PARAMETERS IN TRUNCATED SPACES

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ABSTRACT

We give Stein-type estimators for p-normal means when their lower bounds are given, for scale parameters of p independent gamma distributions when their upper bounds are given, and for means of p independent Poisson distributions when their lower bounds are given.

Each of the Stein-type estimators is the shrunken of the estimator whose i-th component is an admissible estimator of the corresponding parameter suggested by Katz (1961), and has uniformly smaller sum of squared errors risk than the latter. This shows that a set of admissible estimators may not be admissible in simultaneous estimation in truncated parameter space.

1. Introduction.

Suppose that a p-variate random variable $\mathbf{X}$ is normal by distributed and has a mean vector $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_p)$ and the identity covariance matrix I. This assumption is denoted by $\mathbf{X}\sim N(\boldsymbol{\mu}, I)$. Discussing simultaneous estimation of the mean vector $\boldsymbol{\mu}$ under the sum of squared errors loss, James and Stein (1961) introduced an estimator, called Stein-type estimator, which has uniformly smaller risk than the usual estimator. Since then, the Stein-type estimator has been discussed by many authors. The author (1981) discussed Stein-type estimators when some constraints on $\boldsymbol{\mu}$ of linear inequalities are given in advance and gave Stein-type estimators for $\boldsymbol{\mu}$ for several cases by shrinking maximum likelihood estimators (MLE).

Katz (1961) gave a general formula of admissible estimator for mean of an exponential family of densities when a constraint is given on the mean. He applied the general formula to obtain an admissible estimator for the mean of the normal distribution and Poisson distribution. If X is a normal variable with mean $\mu\geqq 0$ and the variance 1, then

$$\hat{\mu}(X)=X+\phi_0(X), \text{ where } \phi_0(X)=e^{-(X^2/2)}\Big/\int_{-\infty}^{X} e^{-(t^2/2)}dt, \tag{1.1}$$

is an admissible estimator of μ. If X is Poisson variable with the mean λ and a constraint $\lambda \geqq a$ ($a>0$ is a fixed known constant) is given on λ, then

$$\hat{\lambda}(X)=X+\psi_a(X), \text{ where } \psi_a(X)=a^X e^{-a} \Big/ \int_a^{\infty} \lambda^{X-1} e^{-\lambda} d\lambda, \tag{1.2}$$

is an admissible estimator of λ.

In this paper we will construct Stein-type estimators based on the admissible estimators by Katz. In Section 2, we give some Stein-type estimators when $\mathbf{X} \sim N(\boldsymbol{\mu}, I)$ and constraints $\mu_i \geqq 0, i=1, \cdots, p$, are given. In Section 3, we apply the Katz's general formula to a gamma distribution with known shape parameter α and unknown scale parameter β, with constraint $0<\beta \leqq a$. Using this admissible estimator we give a Stein-type estimator for $(\beta_1, \cdots, \beta_p)$, when $X_1, \cdots, X_p$ are independent gamma random variables with shape parameters $\alpha_1, \cdots, \alpha_p$ and scale parameters $\beta_1, \cdots, \beta_p$ such that $0<\beta_i \leqq a, i=1, \cdots, p$. In Section 4 we give some Stein-type estimators for $(\lambda_1, \cdots, \lambda_p)$ when $X_1, \cdots, X_p$ are independent Poisson variables with means $\lambda_1, \cdots, \lambda_p$ such that $\lambda_i \geqq a, i=1, \cdots, p$.

2. Stein-type estimators for multivariate normal means when lower bounds for unknown means are given.

Suppose that $\mathbf{X} \sim N(\boldsymbol{\mu}, I)$, where $\mathbf{X}=(X_1, \cdots, X_p)$, $\boldsymbol{\mu}=(\mu_1, \cdots, \mu_p)$ and I is the identity matrix. From an observation of the random variable $\mathbf{X}$, it is required to estimate the mean vector $\boldsymbol{\mu}$ simultaneously. We try to improve an estimator of $\boldsymbol{\mu}$ with the i-th component $X_i+t(X_i), i=1, \cdots, p$, where $t(\cdot)$ is a non-negative real valued function. If

$$t(x_i)=\begin{cases} 0, & \text{if } \; x_i \geqq 0, \\ |x_i|, & \text{if } \; x_i<0, \end{cases}$$

$X_i+t(X_i)$ is the MLE of μ_i, and if $t(x_i)=\phi_0(x_i)$ in (1.1) $X_i+t(X_i)$ is an admissible estimator of μ_i.

Theorem 1. Suppose that $\mathbf{X} \sim N(\boldsymbol{\mu}, I)$ and that constraints $\mu_i \geqq 0, i=1, \cdots, p$, are given. Let two estimators $\boldsymbol{\delta}^0(\mathbf{X})=(\delta_1^0(\mathbf{X}), \cdots, \delta_p^0(\mathbf{X}))$ and $\boldsymbol{\delta}^1(\mathbf{X})=(\delta_1^1(\mathbf{X}), \cdots, \delta_p^1(\mathbf{X}))$ of $\boldsymbol{\mu}$ be defined by

$$\delta_i^0(\mathbf{X})=X_i+t(X_i), \quad i=1, \cdots, p,$$

and

$$\delta_i^1(\mathbf{X})=\begin{cases} X_i+t(X_i)-\dfrac{cX_i}{\sum_{j=1}^{p} X_j^2}, & \text{if } X_j \geqq 0, j=1, \cdots, p, \\ X_i+t(X_i), & \text{otherwise}, \end{cases} \quad , i=1, \cdots, p.$$

In simultaneous estimation for $\boldsymbol{\mu}$ under the sum of squared errors loss, $\boldsymbol{\delta}^1(\mathbf{X})$ has uniformly smaller risk than $\boldsymbol{\delta}^0(\mathbf{X})$ if $p \geqq 3$ and $0<c<2(p-2)$.

Proof: Let $S=\{\mathbf{x}=(x_1, \cdots, x_p) | x_i \geqq 0, i=1, \cdots, p\}$. The difference of risks of $\boldsymbol{\delta}^0(\mathbf{X})$ and $\boldsymbol{\delta}^1(\mathbf{X})$ is given by

$$\Delta R = R(\boldsymbol{\delta}^0, \boldsymbol{\mu}) - R(\boldsymbol{\delta}^1, \boldsymbol{\mu}) = \int_S \sum_{i=1}^{p} \left[\frac{2c(x_i-\mu_i)x_i}{\sum_{j=1}^{p} x_j^2} \right] f(\mathbf{x}, \boldsymbol{\mu}) d\mathbf{x}$$

$$+\int_S \sum_{i=1}^{p} \left[\frac{2ct(x_i)x_i}{\sum_{j=1}^{p} x_j^2} \right] f(\mathbf{x}, \boldsymbol{\mu}) d\mathbf{x} - \int_S \frac{c^2}{\sum_{j=1}^{p} x_j^2} f(\mathbf{x}, \boldsymbol{\mu}) d\mathbf{x}, \tag{2.1}$$

where $f(\mathbf{x}, \boldsymbol{\mu})$ is the probability density function of $N(\boldsymbol{\mu}, I)$. From the definitions of $t(x_i)$ and S the second term of (2.1) is non-negative. Integrating the first term of (2.1) by parts with respect to the i-th component of $\mathbf{X}$, we have

$$\int_S \sum_{i=1}^{p} \left[\frac{2c(x_i-\mu_i)x_i}{\sum_{j=1}^{p} x_j^2} \right] f(\mathbf{x}, \boldsymbol{\mu}) d\mathbf{x} = \int_S \left[\frac{2pc-4c}{\sum_{j=1}^{p} x_j^2} \right] f(\mathbf{x}, \boldsymbol{\mu}) d\mathbf{x}.$$

Therefore, provided that $p \geqq 3$ and $0<c<2(p-2)$,

$$\Delta R \geqq \int_S \left[\frac{2pc-(4c+c^2)}{\sum_{j=1}^{p} x_j^2} \right] f(\mathbf{x}, \boldsymbol{\mu}) d\mathbf{x} > 0.$$

Q. E. D.

Theorem 1 shows inadmissibility of the MLE of $\boldsymbol{\mu}$ and of the estimator whose i-th component is Katz's admissible estimator of μ_i, in simultaneous estimation of $\boldsymbol{\mu}$ under the constraints $\mu_i \geqq 0, i=1, \cdots, p$.

The component $\delta_i^1(\mathbf{X})$ is the shrinkage of $X_i+t(X_i)$ towards the origin by $cX_i/\sum_{j=1}^{p} X_j^2$, when $X_i \geqq 0, i=1, \cdots, p$. This quantity depends on X_i but not on $t(X_i)$.

This suggests another type of improvement of the admissible estimator $X_i+\phi_0(X_i), i=1, \cdots, p$.

Theorem 2. Suppose that $\mathbf{X} \sim N(\boldsymbol{\mu}, I)$ and that constraints $\mu_i \geqq 0, i=1, \cdots, p$, are given. Let two estimators $\boldsymbol{\delta}^N(X)=(\delta_1^N(\mathbf{X}), \cdots, \delta_p^N(\mathbf{X}))$ and $\boldsymbol{\delta}^2(\mathbf{X})=(\delta_1^2(\mathbf{X}), \cdots, \delta_p^2(\mathbf{X}))$ of $\boldsymbol{\mu}$ be defined by

$$\delta_i^N(\mathbf{X}) = X_i+\phi_0(X_i), \quad i=1, \cdots, p,$$

and

$$\delta_i^2(\mathbf{X}) = X_i+\phi_0(X_i) - \frac{c(X_i+\phi_0(X_i))}{\sum_{j=1}^{p} (X_j+\phi_0(X_j))^2}, \quad i=1, \cdots, p.$$

Then in simultaneous estimation of $\boldsymbol{\mu}$ under the sum of squared errors loss, $\boldsymbol{\delta}^2(\mathbf{X})$ has uniformly smaller risk than $\boldsymbol{\delta}^N(\mathbf{X})$ if $p \geqq 3$ and $0<c<2(p-2)$.

Proof: The difference of risks $\boldsymbol{\delta}^N(\mathbf{X})$ and $\boldsymbol{\delta}^2(\mathbf{X})$ is given by

$$\Delta R=R(\boldsymbol{\delta}^N,\boldsymbol{\mu})-R(\boldsymbol{\delta}^2,\boldsymbol{\mu})=\sum_{i=1}^p E\left[\frac{2c(X_i-\mu_i)(X_i+\phi_0(X_i))}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]$$
$$+\sum_{i=1}^p E\left[\frac{2c\phi_0(X_i)(X_i+\phi_0(X_i))}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]-E\left[\frac{c^2}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]. \tag{2.2}$$

Integrating the first term of (2.2) by parts with respect to the i-th component of $\mathbf{X}$, we have

$$E\left[\frac{2c(X_i-\mu_i)(X_i+\phi_0(X_i))}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]=E\left[\frac{2c+2c\phi_0'(X_i)}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]$$
$$-E\left[\frac{4c(X_i+\phi_0(X_i))^2}{\left[\sum_{j=1}^p (X_j+\phi_0(X_j))^2\right]^2}\right]-E\left[\frac{4c(X_i+\phi_0(X_i))^2\phi_0'(X_i)}{\left[\sum_{j=1}^p (X_j+\phi_0(X_j))^2\right]^2}\right],$$

Since $\phi_0'(x)=-\phi_0(x)[\phi_0(x)+x]\leqq 0$ the last expression is not less than

$$E\left[\frac{2c-2c\phi_0(X_i)(X_i+\phi_0(X_i))}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]-E\left[\frac{4c(X_i+\phi_0(X_i))^2}{\left[\sum_{j=1}^p (X_j+\phi_0(X_j))^2\right]^2}\right].$$

Therefore, if $p\geqq 3$ and $0<c<2(p-2)$, then

$$\Delta R\geqq E\left[\frac{2pc-(4c+c^2)}{\sum_{j=1}^p (X_j+\phi_0(X_j))^2}\right]>0.$$

Q. E. D.

3. Stein-type estimators for scale parameters of independent gamma distributions when upper bounds for unknown scale parameters are given.

At first, suppose that X is a gamma variable with known shape parameter α and unknown scale parameter β. From an observation of the random variable X, we construct an admissible estimator for β by using the Katz's general formula when the constraint $0<\beta\leqq a$ is given on unknown parameter β.

The probability density function of X is given by

$$\beta^{-\alpha}x^{\alpha-1}e^{-x/\beta}/\Gamma(\alpha),\ x\in(0,\infty).$$

Applying the Katz's theorem an admissible estimator $\hat{\beta}$ of β is given by

$$\hat{\beta}(X)=X/\alpha-\xi_a(X),\ \text{where}\ \xi_a(X)=(1/\alpha)a^{-\alpha}e^{-X/a}\Big/\int_0^a \beta^{-(\alpha+2)}e^{-X/\beta}d\beta, \tag{3.1}$$

when the constraint $0<\beta\leqq a$ is given.

Now, we consider the simultaneous estimation problem. Assume that $X=(X_1,\cdots,X_p)$ is a p-variate random variable, where $X_i, i=1,\cdots,p$, are independent gamma random variables and X_i has known shape parameter $\alpha_i>1$ and unknown scale parameter β_i. This assumption is denoted by $X_i\sim ind$ Gamma(α_i,β_i),

$i=1, \cdots, p$. It is required to estimate $\beta_1, \cdots, \beta_p$, simultaneously, when constraints $0<\beta_i \leqq a, i=1, \cdots, p$, are given. Based on estimators $X_i/\alpha_i - t(X_i), i=1, \cdots, p$, we construct Stein-type estimator for $(\beta_1, \cdots, \beta_p)$, where $t(\cdot)$ is a non-negative real valued function. If we put $t(X_i)=\xi_a(X_i)$ then $X_i/\alpha_i - t(X_i)$ is an admissible estimator of β_i.

Theorem 3. Suppose that $X_i \sim ind$ Gamma$(\alpha_i, \beta_i), i=1, \cdots, p$, and that constraints $0<\beta_i \leqq a, i=1, \cdots, p$ are given. Let two estimators $\boldsymbol{\delta}^G(\mathbf{X})=(\delta_1^G(\mathbf{X}), \cdots, \delta_p^G(\mathbf{X}))$ and $\boldsymbol{\delta}^3=(\delta_1^3(\mathbf{X}), \cdots, \delta_p^3(\mathbf{X}))$ of $(\beta_1, \cdots, \beta_p)$ be defined by

$$\delta_i^G(\mathbf{X})=X_i/\alpha_i - t(X_i), i=1, \cdots, p,$$

and

$$\delta_i^3(\mathbf{X})=\frac{X_i}{\alpha_i}-t(X_i)+\frac{c\alpha_i(\alpha_i-1)X_i^{-1}}{\sum_{j=1}^{p}\alpha_j^2(\alpha_j+1)^2X_j^{-2}}+\frac{2c\alpha_i^3(\alpha_i+1)^2X_i^{-3}}{\left[\sum_{j=1}^{p}\alpha_j^2(\alpha_j+1)^2X_j^{-2}\right]^2}, i=1, \cdots, p.$$

Then in simultaneous estimation of $\boldsymbol{\beta}=(\beta_1, \cdots, \beta_p)$ under the sum of squared errors loss, $\boldsymbol{\delta}^3(\mathbf{X})$ has uniformly smaller risk than $\boldsymbol{\delta}^G(\mathbf{X})$ if $\alpha_i>1, p\geqq 3$ and $0<c<2(p-2)$.

Proof: Let $D=\sum_{j=1}^{p}\alpha_j^2(\alpha_j+1)^2X_j^{-2}$. Then the difference of risks of $\boldsymbol{\delta}^G(\mathbf{X})$ and $\boldsymbol{\delta}^3(\mathbf{X})$ is given by

$$\begin{aligned}\Delta R=R(\boldsymbol{\delta}^G, \boldsymbol{\beta})-R(\boldsymbol{\delta}^3, \boldsymbol{\beta})=&-\sum_{i=1}^{p}\left\{E\left[\frac{2c\alpha_i(\alpha_i-1)X_i^{-1}}{D}\left(\frac{X_i}{\alpha_i}-t(X_i)-\beta_i\right)\right]\right\}\\ &-\sum_{i=1}^{p}\left\{E\left[\frac{4c\alpha_i^3(\alpha_i+1)^2X_i^{-3}}{D^2}\left(\frac{X_i}{\alpha_i}-t(X_i)-\beta_i\right)\right]\right\}\\ &-\sum_{i=1}^{p}\left\{E\left[\frac{4c^2\alpha_i^4(\alpha_i-1)(\alpha_i+1)^2X_i^{-4}}{D^3}+\frac{c^2\alpha_i^2(\alpha_i-1)^2X_i^{-2}}{D^2}+\frac{4c^2\alpha_i^6(\alpha_i+1)^4X_i^{-6}}{D^4}\right]\right\}.\end{aligned}$$

From the assumptions,

$$\frac{2c\alpha_i(\alpha_i-1)X_i^{-1}}{D}t(X_i)\geqq 0, \frac{4c\alpha_i^3(\alpha_i+1)^2X_i^{-3}}{D^2}t(X_i)\geqq 0,$$

and by using the identity

$$E\left(\frac{X_i^{-1}\beta_i}{D}\right)=E\left(\frac{1}{(\alpha_i-1)D}-\frac{2\alpha_i^2(\alpha_i+1)^2X_i^{-3}\beta_i}{(\alpha_i-1)D^2}\right),$$

we get

$$\begin{aligned}\Delta R\geqq \sum_{i=1}^{p}E\bigg[&\frac{2c}{D}-\frac{4c\alpha_i^2(\alpha_i+1)^2X_i^{-2}}{D^2}-\frac{4c^2\alpha_i^4(\alpha_i+1)^2(\alpha_i-1)X_i^{-4}}{D^3}-\frac{c^2\alpha_i^2(\alpha_i-1)^2X_i^{-2}}{D^2}\\ &-\frac{4c^2\alpha_i^6(\alpha_i+1)^4X_i^{-6}}{D^4}\bigg].\end{aligned}$$

Also from the definition of D we have

$$\frac{\alpha_i^2(\alpha_i+1)^2X_i^{-2}}{D}\leqq 1,$$

$$\frac{\alpha_i^4(\alpha_i+1)^2X_i^{-4}}{D^3}\leqq\frac{\alpha_i^2X_i^{-2}}{D^2},$$

and

$$\frac{\alpha_i^6(\alpha_i+1)^4X_i^{-6}}{D^4}\leqq\frac{\alpha_i^2X_i^{-2}}{D^2}.$$

Therefore

$$\begin{aligned}\Delta R&\geqq E\left(\frac{2pc-4c}{D}\right)-E\left[\frac{c^2}{D^2}\sum_{i=1}^{p}(4\alpha_i^2(\alpha_i-1)+\alpha_i^2(\alpha_i-1)^2+4\alpha_i^2)X_i^{-2}\right]\\&=E\left(\frac{2pc-4c-c^2}{D}\right)>0,\end{aligned}$$

provided that $p\geqq 3$ and $0<c<2(p-2)$.

Q. E. D.

From Theorem 3, in simultaneous estimation of $\beta_1,\cdots,\beta_p$ when constraints $0<\beta_i\leqq a, i=1,\cdots,p$, are given, the estimator of $\beta_1,\cdots,\beta_p$ whose i-th component is an admissible estimator of β_i given by (3.1) is inadmissible.

Remark: The sufficient conditions $p\geqq 3$ and $0<c<2(p-2)$, in Theorem 3 are different from the sufficient conditions, $p\geqq 2$ and $0<c<4(p-1)$ for Stein-type estimator suggested by Berger (1981) when there is no constraint.

4. Stein-type estimators for the means of independent Poisson distributions when lower bounds for unknown means are given.

Suppose that $\mathbf{X}=(X_1,\cdots,X_p)$ is a p-variate random variable, where X_i is independent Poisson random variable with mean λ_i. This assumption is denoted by $X_i\sim ind\ P_0(\lambda_i), i=1,\cdots,p$. From an observation of the random variable $\mathbf{X}$ it is required to estimate $\boldsymbol{\lambda}=(\lambda_1,\cdots,\lambda_p)$ when constraints $\lambda_i\geqq a, i=1,\cdots,p$, are given. We consider an estimator of $\boldsymbol{\lambda}$ with i-th component $X_i+t(X_i), i=1,\cdots,p$. Then we try to improve it in simultaneous estimation, where $t(\cdot)$ is a non-negative real valued function. If we put $t(X_i)=\psi_a(X_i)$ in (1.2) then $X_i+t(X_i)$ is an admissible estimator of λ_i.

Theorem 4. Suppose that $X_i\sim ind\ P_0(\lambda_i), i=1,\cdots,p$, and that constraints $\lambda_i\geqq a$, $i=1,\cdots,p$, are given. Let two estimators $\boldsymbol{\delta}^P(\mathbf{X})=(\delta_1^P(\mathbf{X}),\cdots,\delta_p^P(\mathbf{X}))$ and $\boldsymbol{\delta}^4(\mathbf{X})=(\delta_1^4(\mathbf{X}),\cdots,\delta_p^4(\mathbf{X}))$ of $\boldsymbol{\lambda}$ be defined by

$$\delta_i^P(\mathbf{X})=X_i+t(X_i),\qquad i=1,\cdots,p,$$

and

$$\delta_i^4(\mathbf{X})=X_i+t(X_i)-\frac{\Phi(Z)X_i}{Z+p-1},\quad i=1,\cdots,p,$$

where $Z=\sum_{i=1}^p X_i$ and $\Phi:[0,\infty]\to[0,2(p-1)]$ is non-decreasing and not identically equal to zero. Then, in simultaneous estimation of $\boldsymbol{\lambda}$, under the standerized squared errors loss $\sum_{i=1}^{p} \lambda_i^{-1}(\hat{\lambda}_i-\lambda_i)^2$, where $\hat{\lambda}_i$ is an estimator of $\lambda_i, i=1, \cdots, p$, $\boldsymbol{\delta}^4(\mathbf{X})$ has uniformly smaller risk than $\boldsymbol{\delta}^P(\mathbf{X})$ if $p \geqq 2$.

Proof: The difference of risks of $\boldsymbol{\delta}^P(\mathbf{X})$ and $\boldsymbol{\delta}^4(\mathbf{X})$ is given by

$$\begin{aligned}\Delta R &= R(\boldsymbol{\delta}^P, \boldsymbol{\lambda}) - R(\boldsymbol{\delta}^4, \boldsymbol{\lambda}) \\ &= \sum_{i=1}^{p} \frac{1}{\lambda_i} E\left[2(X_i+t(X_i)-\lambda_i)\frac{\Phi(Z)X_i}{Z+p-1} - \frac{\Phi^2(Z)X_i^2}{(Z+p-1)^2}\right].\end{aligned}$$

From the assumptions, we have

$$t(X_i)\frac{\Phi(Z)X_i}{Z+p-1} \geqq 0, \quad i=1, \cdots, p,$$

therefore,

$$\Delta R \geqq \sum_{i=1}^{p} \frac{1}{\lambda_i} E\left[2(X_i-\lambda_i)\frac{\Phi(Z)X_i}{Z+p-1} - \frac{\Phi^2(Z)X_i^2}{(Z+p-1)^2}\right]. \tag{4.1}$$

if $p \geqq 2$ then from the result of Clevenson and Zidek (1975), the right hand of (4.1) is positive.

Q. E. D.

Although, in Theorem 4, we improved the estimator of $\boldsymbol{\lambda}, \boldsymbol{\delta}^P(\mathbf{X})$, when $p \geqq 2$, in next theorem we will improve $\boldsymbol{\delta}^P(\mathbf{X})$ when $p \geqq 3$.

Theorem 5. Suppose that $X_i \sim ind\, P_0(\lambda_i), i=1, \cdots, p$, and that constraints $\lambda_i \geqq a$, $i=1, \cdots p$, are given. Let $\boldsymbol{\delta}^5(\mathbf{X})=(\delta_1^5(\mathbf{X}), \cdots, \delta_p^5(\mathbf{X}))$ be an estimator of $\boldsymbol{\lambda}$ defined by

$$\delta_i^5(\mathbf{X}) = X_i + t(X_i) - \frac{c\left(p-\sum_{j=0}^{[a]+1} N_j - 2\right)^+}{\sum_{j=1}^{p} h^2(X_j)} h(X_i),$$

where $h(j)$ is a non-negative function defined by

$$h(j) = \begin{cases} \sum_{n=1}^{j-[a]-1} \frac{1}{[a]+1+n}, & j=[a]+2, \cdots, \max_{i=1}^{p} \{x_i\}, \\ 0, & j=0, \cdots, [a]+1, \end{cases}$$

$b^+ = \max(0, b)$, $[a]$ is the greatest integer less than or equal to a and $N_j = \#\{X_i | X_i = j\}$. If $p \geqq 3$ and $0 < c < 2$, then under the sum of squared errors loss $\boldsymbol{\delta}^5(\mathbf{X})$ has uniformly smaller risk than $\boldsymbol{\delta}^P(\mathbf{X})$.

Proof: The difference of risks $\boldsymbol{\delta}^P(\mathbf{X})$ and $\boldsymbol{\delta}^5(\mathbf{X})$ is given by

$$\Delta R = R(\boldsymbol{\delta}^P, \boldsymbol{\lambda}) - R(\boldsymbol{\delta}^5, \boldsymbol{\lambda}) = \sum_{i=1}^{p} E \frac{2c(X_i+t(X_i)-\lambda_i)\left(p-\sum_{j=0}^{[a]+1} N_j - 2\right)^+ h(X_i)}{\sum_{j=1}^{p} h^2(X_j)}$$

$$-\sum_{i=1}^{p} E\left[\frac{ch(X_i)\left(p-\sum_{j=0}^{[a]+1} N_j-2\right)^+}{\sum_{j=1}^{p} h^2(X_j)}\right]^2 .$$

From the assumptions,

$$\frac{2ct(X_i)\left(p-\sum_{j=0}^{[a]+1} N_j-2\right)^+ h(X_i)}{\sum_{j=1}^{p} h^2(X_j)} \geqq 0, \quad i=1, \cdots, p,$$

therefore we have

$$\Delta R \geqq \sum_{i=1}^{p} E\frac{2c(X_i-\lambda_i)\,h(X_i)\left(p-\sum_{j=0}^{[a]+1} N_j-2\right)^+}{\sum_{j=1}^{p} h^2(X_j)}$$

$$-\sum_{i=1}^{p} E\left[\frac{ch(X_i)\left(p-\sum_{j=0}^{[a]+1} N_j-2\right)^+}{\sum_{j=1}^{p} h^2(X_j)}\right]^2 . \tag{4.2}$$

If $p \geqq 3$ and $0<c<2$ then, from the result of Tsui (1978), the right hand of (4.2) is positive.

Q. E. D.

From Theorem 4 and Theorem 5, in simultaneous estimation of $\boldsymbol{\lambda}$ when constraints $\lambda_i \geqq a, i=1, \cdots, p$, are given, the estimator of $\boldsymbol{\lambda}$ whose i-th component is an admissible estimator suggested by Katz is inadmissible.

It should be noticed that in the proofs of all of the above theorems we have not used the constraints on parameters in showing the risk of Stein-type estimator to be smaller than that of the corresponding estimator. This means that Stein-type estimators dominate always the corresponding ones whether the parameters satisfy the constraints or not.

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