Title	Evaluation of the solid angle of a cone defined by an arbitrary point and a circle
Sub Title	
Author	Tamagawa, Hajime
Publisher	慶應義塾大学理工学部
Publication year	1982
Jtitle	Keio Science and Technology Reports Vol.35, No.8 (1982. 8) ,p.139- 152
JaLC DOI	
Abstract	The problem of this article concerns the evaluation of the solid angle of an elliptical cone of any shape. It is not easy to get a mathematical formula for the solid angle that is convenient for computer calculation. It is necessary to use a third kind of complete elliptic integral, a method inconvenient for the computer. Then, usually, we have to evaluate by a numerical integration. But it is very difficult to get a value with high precision especially for compressed cones. The author has succeeded in getting a convenient series expansion formula by considering the difference between the circumscribed rectangular cone and the elliptical cone. He has succeeded also in writing a computer program to obtain the solid angle of any type of elliptical cone with any required precision better than 10^{-12} absolute error using the series expansion formula and numerical integration.
Notes	
Genre	Departmental Bulletin Paper
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=KO50001004-00350008-0139

KEIO SCIENCE AND TECHNOLOGY REPORTS
VOL. 35, NO. 8, pp. 139-152, 1981

EVALUATION OF THE SOLID ANGLE OF A CONE DEFINED BY AN ARBITRARY POINT AND A CIRCLE

HAJIME TAMAGAWA

Department of Physics, Faculty of Science and Technology, Keio University

(Received May 6, 1982)

SUMMARY

The problem of this article concerns the evaluation of the solid angle of an elliptical cone of any shape. It is not easy to get a mathematical formula for the solid angle that is convenient for computer calculation. It is necessary to use a third kind of complete elliptic integral, a method inconvenient for the computer. Then, usually, we have to evaluate by a numerical integration. But it is very difficult to get a value with high precision especially for compressed cones. The author has succeeded in getting a convenient series expansion formula by considering the difference between the circumscribed rectangular cone and the elliptical cone. He has succeeded also in writing a computer program to obtain the solid angle of any type of elliptical cone with any required precision better than 10^{-12} absolute error using the series expansion formula and numerical integration.

1. Introduction

The magnetic field which is induced by a single circuit coil current can be introduced by magnetic potential. The magnetic potential is determined by the solid angle which is defined by the point concerned and the circumference of the circuit. Calculating the magnetic potential of a circular current seems to be a simple problem, but since the integration for getting the potential includes a third kind of complete elliptical integral, the problem is not an easy one. Then, when we want to calculate by computer, we always have to use numerical integration. The potential may be often given by a series expansion using Legendre spherical polynomials, which is often given in text books,[1] but when the potentials concerned are at or near the radius of the circle from the center, the series cannot to be converged. Then, the method is not to be useful. The author has developed a new method to obtain the solid angle of an elliptical cone. He has gotten a convenient series expansion formula by considering the difference between the circumscribed rectangular cone and the elliptical cone. He has succeeded in making a computer program to get the solid angle of any type of elliptical cone, with any required precision better than 10^{-12} absolute error using the series expansion formula and numerical integration.

2. The Principle of evaluation

2-1. Determination of the parameters of the ellipse

One point and a circle define an elliptical cone. The problem comes in evaluating the solid angle of the vertex of the cone. The parameters which determine the vertical angle of the cone are the eccentricity ε of the base ellipse and the ratio α of the height of the cone to the major radius of the base ellipse. ε and α can be obtained by the following process. In Fig. 1, call the center of

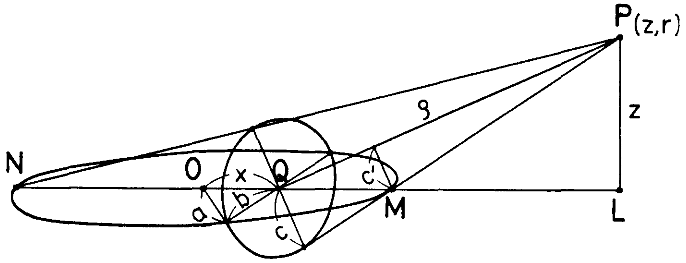

Fig. 1. Evaluation of parameters of elliptical cone.

the circle O, the vertex P, the foot of the perpendicular from the vertex onto the base L, the intersections of line OL and the circle M and N and the bisector of $\angle MPN$ and line MN Q. Define

a: radius of the circle,
z, r: cylinder co-ordinates of P,
ρ: length of PQ,
x: length of OQ.

Let b and c be the major and minor radii respectively, of the ellipse which is determined by the cross section of the plane containing Q and perpendicular to PQ and the elliptical cone which is defined by vertex P and the circle O. After some elementary calculations,

$$x=[(a^2+r^2+z^2)-\{(a^2+r^2+z^2)^2-4a^2r^2\}^{1/2}]/(2r),$$
$$\rho=\{z^2+(r-x)^2\}^{1/2},$$
$$b=\{a^2-x^2\}^{1/2},$$
$$PM=d=\{z^2+(r-a)^2\}^{1/2},$$
$$QM=e=a-x,$$

$s=(\rho+d+e)/2,$
$\triangle PMQ=S=\{s(s-\rho)(s-d)(s-e)\}^{1/2},$
$c'=2S/\rho,$
$c=c'\rho/(d^2-c^2)^{1/2},$

Then we can obtain the value ρ, b and c. The parameters are determined as

$$\varepsilon=c/b, \qquad \alpha=\rho/b.$$

Then the problem comes in evaluating the vertex solid angle of the rectangular elliptical cone with a major radius of unity, minor radius ε and height α.

2-2. Expression formula of the vertical solid angle of the rectangular elliptical cone

Several methods can be obtained to express the vertical solid angle. Here we consider two methods as follows. The first method is obtained by integration in the rectangular co-ordinate. The expression is

$$\Omega=4\int_0^1 dx\int_0^{\varepsilon(1-x^2)^{1/2}}\frac{\alpha dy}{(\alpha^2+x^2+y^2)^{3/2}}$$

$$=4\varepsilon\alpha\int_0^1\frac{(1-x^2)^{1/2}dx}{(\alpha^2+x^2)\{\alpha^2+\varepsilon^2+(1-\varepsilon^2)x^2\}^{1/2}}. \tag{I-1}$$

Here, putting $x=\sin\theta$, we obtain

$$=4\varepsilon\alpha\int_0^{\pi/2}\frac{\cos^2\theta\, d\theta}{(\alpha^2+\sin^2\theta)\{\alpha^2+\varepsilon^2+\varepsilon^2(1-\varepsilon^2)\sin^2\theta\}^{1/2}}. \tag{I-2}$$

This integral is the third kind of complete elliptic integral.

The second method is obtained by integration in the angular co-ordinate in the plane of the ellipse. We get

$$\Omega=4\int_0^{\pi/2}\left|1-\frac{\alpha\{1-(1-\varepsilon^2)\cos^2\theta\}^{1/2}}{[\varepsilon^2+\alpha^2\{1-(1-\varepsilon^2)\cos^2\theta\}]^{1/2}}\right|d\theta. \tag{II-1}$$

Obviously the expression is different from the first method. Putting $\tan\theta=\varepsilon\tan\theta'$ we get

$$\Omega=4\int_0^{\pi/2}\left[1-\frac{\alpha}{\{\alpha^2+\varepsilon^2+(1-\varepsilon^2)\cos^2\theta'\}^{1/2}}\right]\frac{\varepsilon d\theta'}{\varepsilon^2+(1-\varepsilon^2)\cos^2\theta'}. \tag{II-2}$$

Again, this becomes the third kind of complete integral.

In any case, we can not solve the problem using only the familiar functions. The difficulties of numerical integrations are not the same among the expressions, according to values of ε and α. Generally speaking, when $\alpha>1$, equations (I-1), (I-2) and (II-2) are more useful than the equation (II-1), but when $\alpha<1$ and $\varepsilon\sim1$, the equation (II-1) is the most useful. In the case of $\alpha<1$ and $\varepsilon\ll1$, all expressions are rather hard to evaluate. Since the change of integrand is great necessiting too many function values to get precise result, then we have to find some other approach.

2-3. Approximate formula for small ε and α

In the case of small values of ε and α, precision of numerical integration is not good as described in the former section. But in this case we can take the vertical solid angle of the rectangular cone which circumscribes the elliptical cone as a guess value. Under this consideration, the author introduced a series expansion formula which give the difference of the solid angle of the elliptic cone and the one of the rectangular cone (region of the hatched space of Fig. 2).

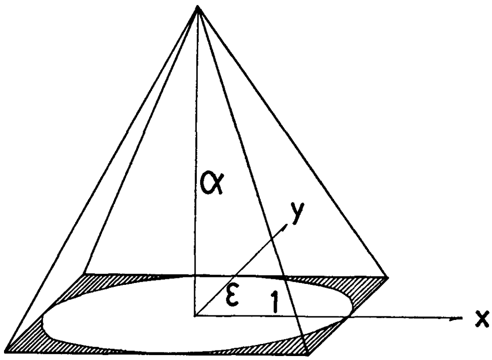

Fig. 2. Illustrate the elliptical cone and the circumscribed rectangular cone.

In Fig. 2, put the vertex angle of the rectangular cone as Ω_0. Then, Ω_0 is given as

$$\frac{\Omega_0}{4}=\tan^{-1}\frac{\varepsilon}{\alpha(1+\alpha^2+\varepsilon^2)^{1/2}},$$

Let Ω' be the difference of the vertex angles; then

$$\frac{\Omega'}{4}=\int_0^1 dx\int_{\varepsilon(1-x^2)^{1/2}}^{\varepsilon}\frac{dy}{(\alpha^2+x^2+y^2)^{3/2}}$$

$$=\int_0^1\left[\frac{\alpha\varepsilon}{(\alpha^2+x^2)(\alpha^2+\varepsilon^2+x^2)^{1/2}}-\frac{\alpha\varepsilon(1-x^2)^{1/2}}{(\alpha^2+x^2)\{\alpha^2+\varepsilon^2+(1-\varepsilon^2)x^2\}^{1/2}}\right]dx.$$

The first term of the integrand is given by putting

$$B(x)=\alpha^2+\varepsilon^2+(1-\varepsilon^2/2)x^2,$$

$$y=\varepsilon^2/B(x),$$

then

$$\text{THE FIRST TERM}=\frac{\alpha\varepsilon}{B(x)^{3/2}}\{1-(1-x^2/2)y\}^{-1}(1+x^2/2\,y)^{-1/2}$$

$$=\frac{\alpha\varepsilon}{B(x)^{3/2}}\{1+(4-3x^2)y/4+(32-40x^2+15x^4)y^2/32+\cdots\}.$$

Next, the second term is given

$$\text{THE SECOND TERM}=\frac{\alpha\varepsilon(1-x^2)^{1/2}}{B(x)^{3/2}}\{1-(1-x^2/2)y\}^{-1}(1-x^2/2\,y)^{-1/2}$$

$$=\frac{\alpha\varepsilon(1-x^2)^{1/2}}{B(x)^{3/2}}\{1+(4-x^2)y/4+(32-24x^2+7x^4)y^2/32+\cdots\}.$$

After some arrangement,

$$\begin{aligned}\frac{\Omega'}{4}=&\varepsilon\alpha\int_0^1\frac{dx}{B(x)^{3/2}}-\int_0^1\frac{(1-x^2)^{1/2}dx}{B(x)^{3/2}}\\&+\frac{\varepsilon^3\alpha}{4}\left\{\int_0^1\frac{(4-3x^2)dx}{B(x)^{5/2}}-\int_0^1\frac{(4-x^2)(1-x^2)^{1/2}dx}{B(x)^{5/2}}\right\}\\&+\frac{\varepsilon^5\alpha}{32}\left\{\int_0^1\frac{(32-40x^2+15x^4)dx}{B(x)^{7/2}}\right.\\&\qquad\left.-\int_0^1\frac{(32-24x^2+7x^4)(1-x^2)^{1/2}dx}{B(x)^{7/2}}\right\}\\&+\cdots\cdots\cdots\cdots.\end{aligned}$$

In the above equation, the integrals which don't include the term $(1-x^2)^{1/2}$ can be executed with elementary integrations. The ones including the term $(1-x^2)^{1/2}$ can be executed with using the first and second kind of complete elliptical integrations. Therefore, changing variables as

$$u^2=1-x^2,$$

$$k^2=1/\{1+(\alpha^2+\varepsilon^2)/(1-\varepsilon^2/2)\},$$

The integrals can be transformed to standard formulas.

Define

$$\Phi_n(u)=(1-k^2u^2)^n\{(1-k^2u^2)(1-u^2)\}^{1/2},$$

then we get

$$\frac{\Omega'}{4}=A_1+A_2+A_3+\cdots\cdots,$$

and

$$A_1=\frac{\varepsilon\alpha}{(1+\varepsilon^2/2+\alpha^2)^{3/2}}\left\{\int_0^1\frac{u\,du}{\Phi_1(u)}-\int_0^1\frac{u^2du}{\Phi_1(u)}\right\},$$

$$A_2=\frac{\varepsilon^3\alpha}{4(1+\varepsilon^2/2+\alpha^2)^{5/2}}\left\{\int_0^1\frac{(1+3u^2)udu}{\Phi_2(u)}-\int_0^1\frac{(3+u^2)u^2du}{\Phi_2(u)}\right\},$$

$$A_3=\frac{\varepsilon^5\alpha}{32(1+\varepsilon^2/2+\alpha^2)^{7/2}}\left\{\int_0^1\frac{(7+10u^2+15u^4)udu}{\Phi_3(u)}-\int_0^1\frac{(15+10u^2+7u^4)u^2du}{\Phi_3(u)}\right\},$$

$$A_4=\frac{\varepsilon^7\alpha}{256(1+\varepsilon^2/2+\alpha^2)^{9/2}}\left\{\int_0^1\frac{(18+98u^2+70u^4+70u^6)udu}{\Phi_4(u)}\right.$$

$$\left.-\int_0^1\frac{(70+70u^2+98u^4+18u^6)u^2du}{\Phi_4(u)}\right\},$$

$$A_5=\frac{\varepsilon^9\alpha}{2048(1+\varepsilon^2/2+\alpha^2)^{11/2}}\left\{\int_0^1\frac{(75+452u^2+690u^4+548u^6+283u^8)udu}{\Phi_5(u)}\right.$$

$$\left.-\int_0^1\frac{(283+548u^2+690u^4+452u^6+75u^8)u^2du}{\Phi_5(u)}\right\},$$

$$A_6=\frac{\varepsilon^{11}\alpha}{16384(1+\varepsilon^2/2+\alpha^2)^{13/2}}$$

$$\times\left\{\int_0^1\frac{(302+2354u^2+3564u^4+6468u^6+2310u^8+1386u^{10})udu}{\Phi_6(u)}\right.$$

$$\left.-\int_0^1\frac{(1386+2310u^2+6468u^4+3564u^6+2354u^8+302u^{10})u^2du}{\Phi_6(u)}\right\}.$$

Moreover, putting

$$M_n=\int_0^1\frac{udu}{\Phi_n(u)},\qquad K_n=\int_0^1\frac{u^2du}{\Phi_n(n)},$$

A_n can be expressed with using M_n and K_n, because

$$\int_0^1 u^2\frac{udu}{\Phi_n(u)}=k^{-2}(M_n-M_{n-1}),$$

$$\int_0^1 u^4\frac{udu}{\Phi_n(u)}=k^{-4}M_n-2M_{n-1}+M_{n-2}),$$

$$\int_0^1 u^6\frac{udu}{\Phi_n(u)}=k^{-6}(M_n-3M_{n-1}+3M_{n-2}-M_{n-3})$$

and so on. The coefficients of M in parenthesis are given by the binomial formula, and if we switch udu to du, each M_n will be switched to K_n.

Finally, M_n and K_n can be obtained using recurrence relations:

$$M_1=1/p,\ M_2=(1+2M_1)/(3p),\ M_3=(1+4M_2)/(5p),$$

$$\cdots\cdots\cdots\cdots,$$

$$M_{n+1}=(1+2nM_n)/(2n+1)p.$$

For K_n, putting

$$K(k)=\int_0^1 \frac{du}{\{(1-k^2u^2)(1-u^2)\}^{1/2}}$$

(the first kind of complete elliptic integral)

$$E(k)=\int_0^1 \{(1-k^2u^2)/(1-u^2)\}^{1/2}du$$

(the second kind of complete elliptic integral)

$K_n s$ are given as

$$K_0=K(k),$$
$$K_1=E(k)/p,$$
$$K_2=\{(4+4p)k_1-2K_0\}/(2\cdot 3p),$$
$$K_3=\{(16+19p)K_2-(14+2p)K_1+K_0\}/(4\cdot 5p),$$
$$K_4=\{(48+68p)K_3-(56+16p)K_2+12K_1\}/(8\cdot 7p),$$
$$K_5=\{(128+212p)K_4-(184+77p)K_3+(64+4p)K_2-3K_1\}/(16\cdot 9p),$$
$$K_6=\{(320+608p)K_5-(544+298p)K_4+(260+36p)K_3-30K_2\}/(32\cdot 11p).$$

Now, we have expressions for Ω', then the solid angle of the elliptical cone is given as

$$\Omega=\Omega_0-\Omega'$$

3. Decision of the applicable region for each method

3-1. Series expansion method

The applicable region of the new series expansion formula introduced in the former section was checked as follows. Making a computer program able to calculate according to the new series formula, the author computed the solid angle of an elliptical cone for wide ranges of ε and α. The values were compared with the values which are obtained by the Automatic Numerical Integration Subroutine "AQNN9D"[2] of Nagoya University Computing Center with the necessary precision using expression formulas (I-1) and/or (II-1). The results are shown in Fig. 3. In the figure, three cases are shown where absolute maximum errors are taken as 10^{-6}, 10^{-9} and 10^{-12}. The numbers in the figure are the number of terms needed to get the required precision at the values of ε and α, and symbol * shows that at the particular value of ε and α we can not obtain the precision by using less than five terms. The author has tried to use up to the ninth term, but he has found that with more than five terms the convergence of the series is not good in the region shown by the symbol *. Practically, we should consider the region where we can evaluate the solid angle by using the series expansion formula using no more than four or five terms.

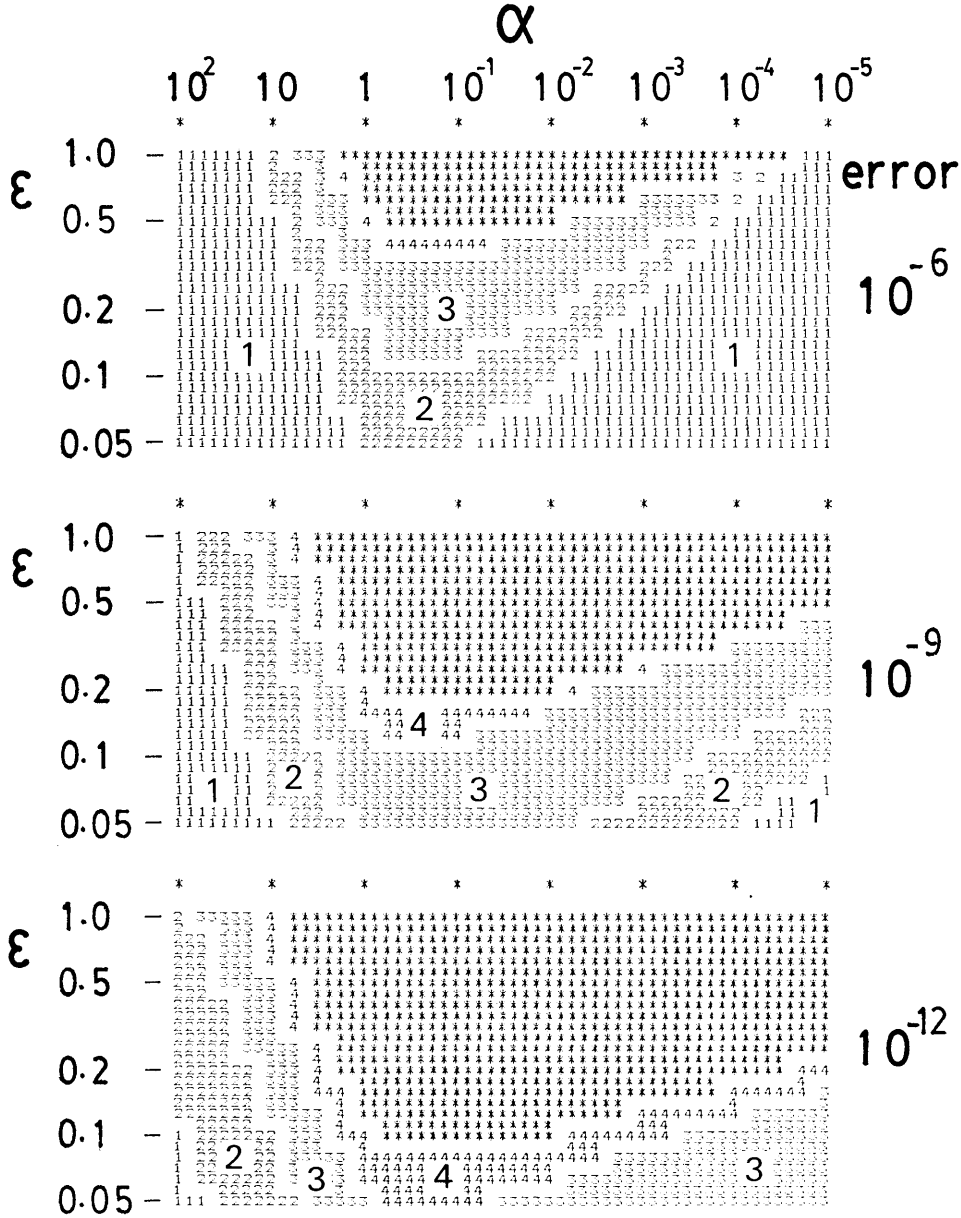

Fig. 3. Patterns of needed terms' number of series expansion formula for various precisions.

3-2. Numerical integrations

In the region where the series expansion formula cannot be applied, we can not help but use numerical integrations. But, there are some regions where the numerical integration methods are superior to that using the series expansion formula. Many methods are considered for numerical integration, but we have to select the most advantageous method taking into consideration precision and

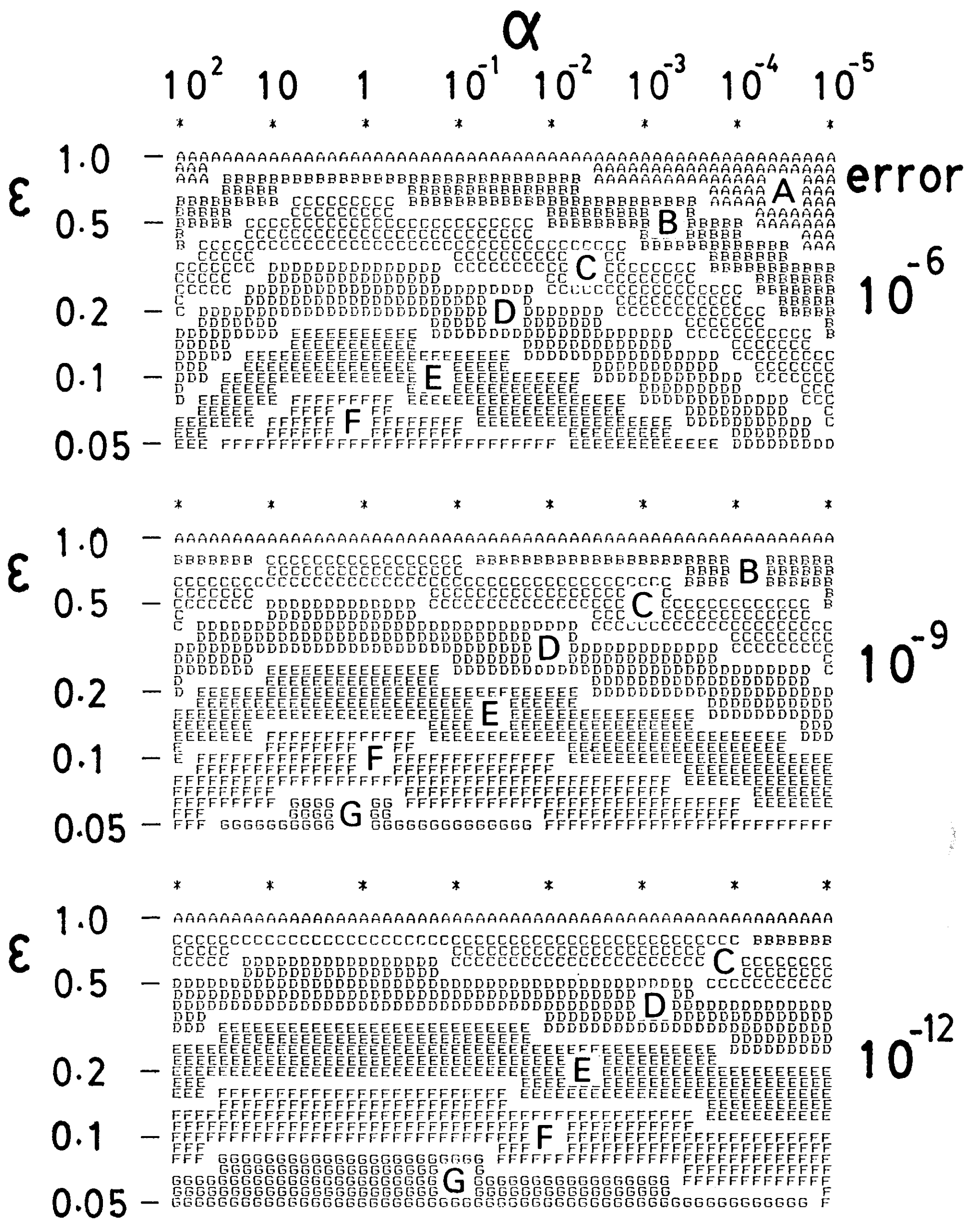

Fig. 4. Patterns of the needed sections' numbers of numerical integration with using eq. II-1, for various precisions. Character indicate number of sections for getting required precision, A : 2, B : 4, C : 8, D : 16, E : 32, F : 64, G : 128.

computing time. In the case of integration of a smooth periodic function over a period, or in the case where the differentiation coefficients of the upper and lower limits of a variable are zero, a simple trapezoidal formula integration method with equal divisions is the most advantageous. Then, he selects the simplest method. Among the four formulas of integral expression, excepting eq. (I-1), each inte-

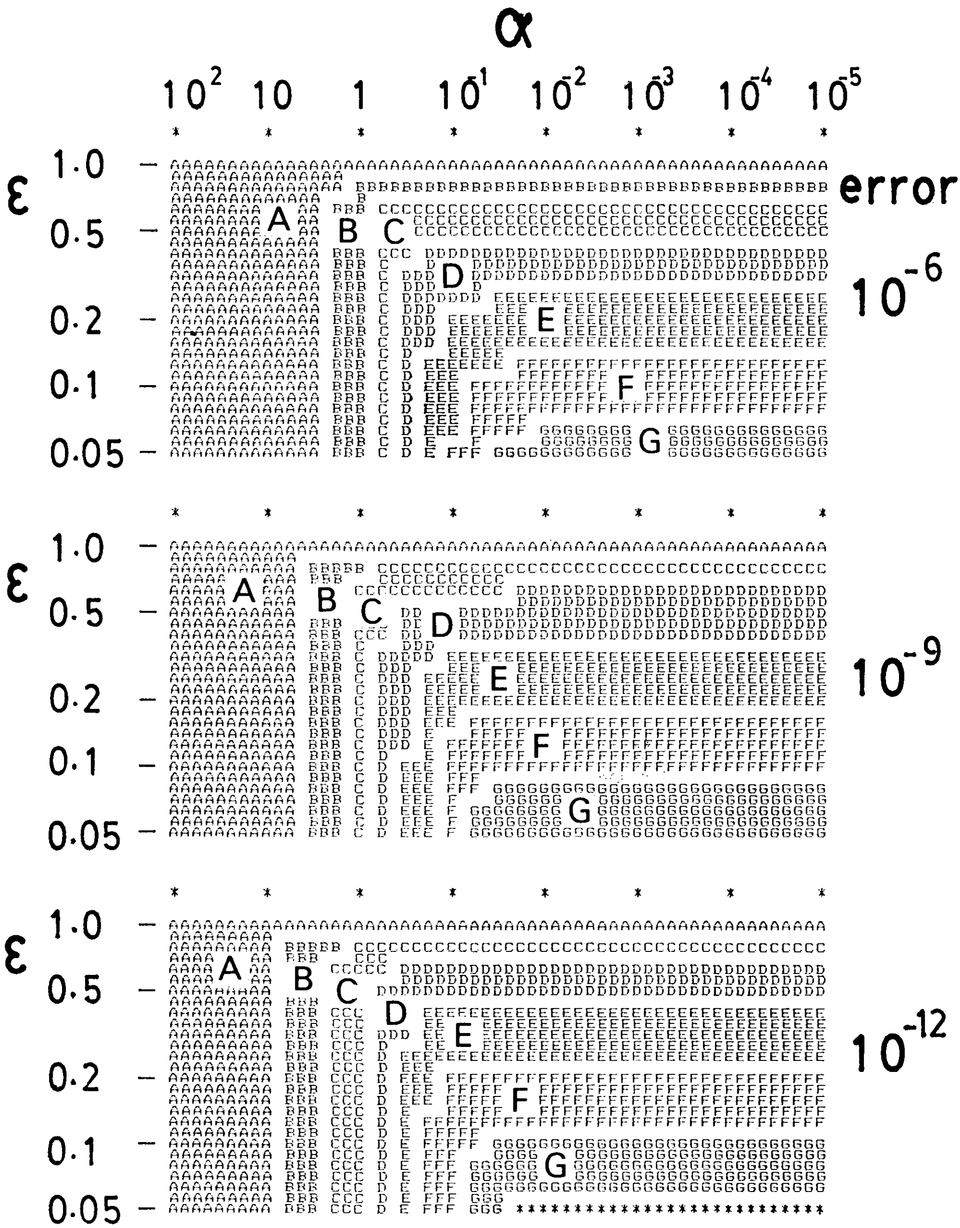

Fig. 5. The same pattern as Fig. 4 with using eq. II-2.

grand is smooth periodic even function with period π. They thus satisfy the conditions. Then we can apply simple numerical integration for eq. (II-1) and eq. (II-2). In Fig. 4 and Fig. 5, the divieing numbers for getting enough precision are shown as the same pattern as Fig. 3. Fig. 4 is for the eq. (II-1) and Fig. 5 is for the eq. (II-2). In the figure, the numbers of calculations of the integrand required to obtain the desired precision, 3, 5, 9, 17, 33, 65 and 129 are represented by the letters A, B, C, D, E, F and G respectively. The symbol * show where more than 129 calculations are required.

4. An outline of the computer program

By considering the Fig. 3—Fig. 5, we can obtain a guideline for writing a computer program to calculate solid angles of elliptical cones of any shape.

For the method using the series expansion formula, calculation of M_n and K_n are necessary. M_n can be written using elementary functions. But since K_n are given by complete elliptical integrals, then we need the function library in computer. We can get all K_n values by using only two values, i. e. the first and second kind of complete elliptical integrals, and the four rules of arithmetic. Then the number of actual calculations of functions is three, i. e. the two complete elliptical integrals and the arc-tangent, but in some region of ε and α, the method loses virtue.

For the numerical integration method, using simple trapezoidal formula integration is very convenient as follows. First, divide integral region into two. Calculate the initial guess value with the two end points and the center point. Next, add the center of two regions and get a second value. Repeating this procedure, we can get as precise a value as we want. The calculating points of each step are as shown in Fig. 6. The values of the function for using numerical integration by the n'th step is 2^n+1. By this method all the values which have

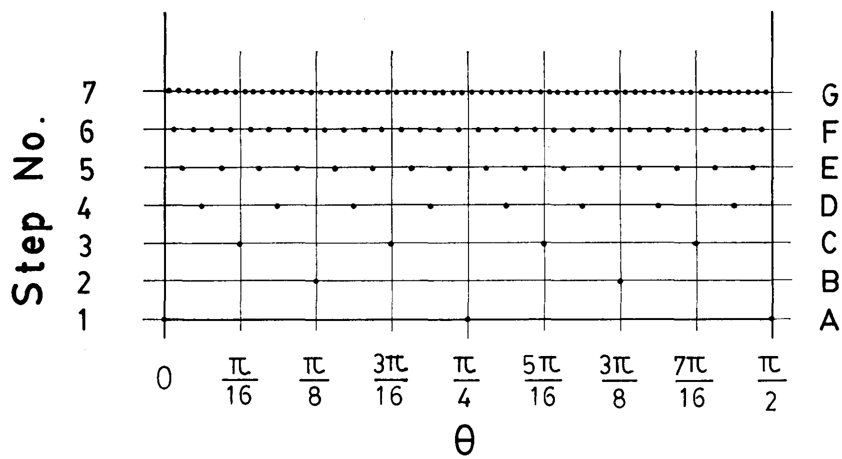

Fig. 6. Indicate calculating points for each step of numerical integration.

been calculated can be contributed to the result. This method needs values of cosine and square-root, but the values of cosine are needed only for definite discrete value of the arguments. Thus we can prepare all the values for the

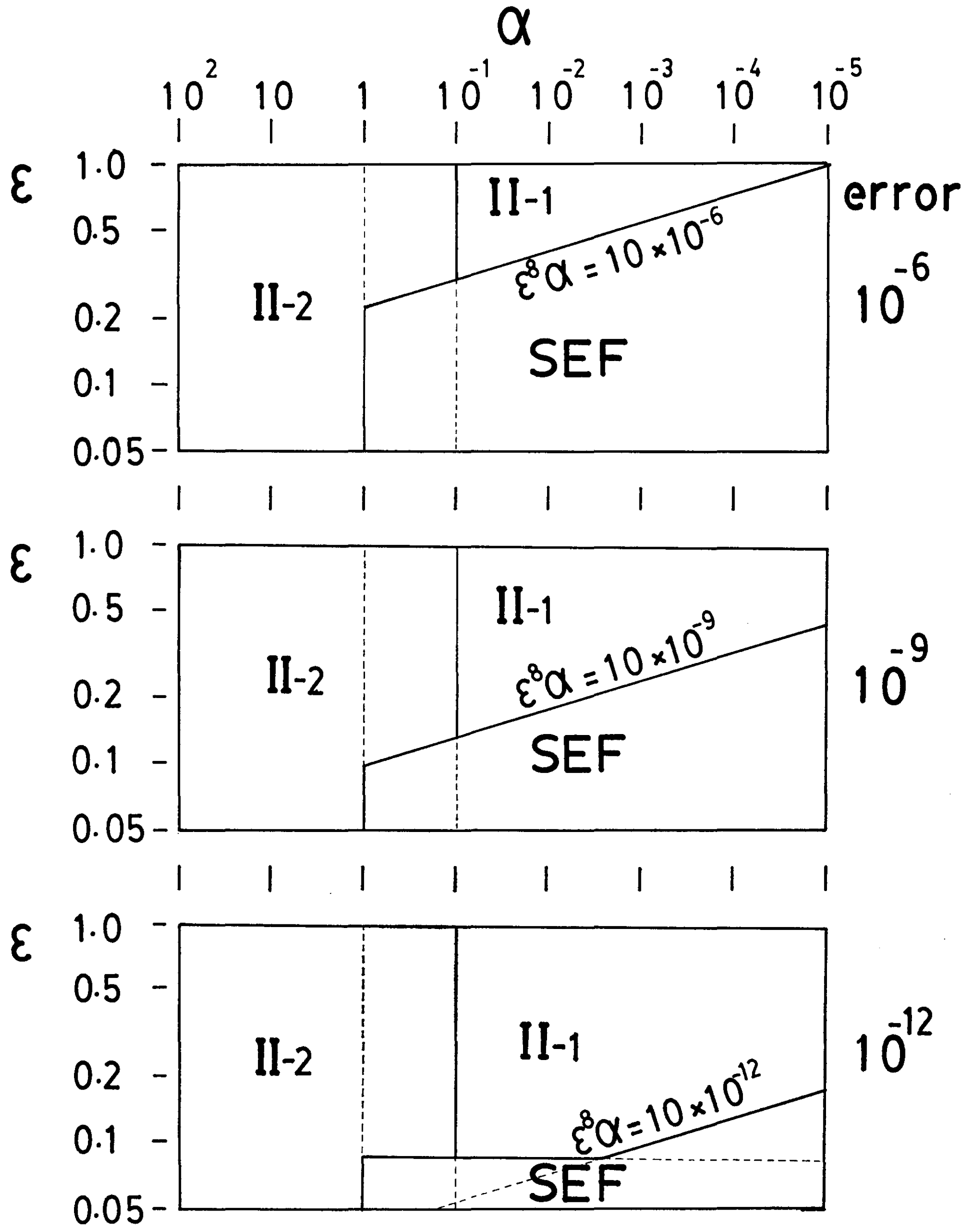

Fig. 7. Patterns of the calculating modes for various precisions.
(II-1) ; Numerical integration region using eq. (II-1).
(II-2) ; Numerical integration region using eq. (II-2).
(SEF); Series expansion formula region.

cosines beforehand as an array. With using the above method, we can save lots of computing time. As shown in Fig. 5, even when the desired precision is as small as 10^{-12}, the region marked by characters A, B and C (representing 9 or fewer calculations), covers the entire area where $\alpha>1$. Thus we can say that numerical integration of eq. (II-2) is more advantageous than the series expansion method.

Under above considerations, we can make computing program to get the value of the solid angle of an elliptical cone of arbitrary shape, using the following methods.

A) using series expansion formula.
B) using numerical integration of eq. (II-1).
C) using numerical integration of eq. (II-2).

Deciding regions may be arbitrary. But we should select as simple a way as we can. Here, the author would suggest the following:

1) In the case of $\alpha>1$; C.
2) In the case of $\varepsilon<0.09$; A.
3) In the case of $\alpha\varepsilon^8<10\times$required absolute maximum error; A.
4) In the case of $\alpha<0.1$; C.
5) Otherwise ; B.

Computing the values according to the above order, we can obtain the solid angle of elliptical cone of any shape with required precision up to less than 10^{-12} error. In Fig. 7 the pattern of calculating method is shown. It is obvious that, when the required precision is high, in the region near $\varepsilon\sim0.1$ and $\alpha\sim0.1$, with any method the calculating time is long.

5. Conclusion

As described above, the author has introduced a method to calculate the solid angle of an elliptical cone of any shape, a problem that looks simple but is not so easy. The author has introduced a series expansion formula, and has made a computer program using the new formula with numerical integrations. He can calculate the solid angle for any required precision up to less than 10^{-12} error.

This work was carried out initially at Nagoya University using FACOM 230/60 of Computing Center, and has been revised and improved at Keio University especially in the numerical integration formulas.

In conclusion, the author would like to express his sincere appreciation to Prof. I. Ninomiya and Prof. S. Kuwabara of Nagoya University for their beneficial discussion on this work. The author has been guided by Prof. Ninomiya on using the "AQNN9D" (Automatic Numerical Integration Subroutine Pragram), which has been developed by him. It was very helpful for carrying out this work. He also expresses gratitude to Associate Prof. K. Nishina and the members of his laboratory in Nagoya University for helpful discussion on this work.

REFERENCES

1) J. JEANS: *Electricity and Magnetism* (Cambridge, 1966) 5th ed., p. 431 (for example)
2) I. NINOMIYA et. al.: Nagoya University Computing Center News; 8 (1977) 209, in Japanese.