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Author	金川, 秀也(Kanagawa, Shuya)
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ON THE RATE OF CONVERGENCE OF THE INVARIANCE PRINCIPLE FOR STATIONARY SEQUENCES

SHÔYA KANAGAWA

Dept. of Mathematics, Keio University, Yokohama 223, Japan

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ABSTRACT

In this paper, the author estimates the rate of convergence of the invariance principle for some strictly stationary sequences of random variables satisfying the ϕ-mixing condition.

1. Introduction and results

Let $\{X_i, i\geqq 1\}$ be a strictly stationary sequence of random variables on a probability space (Ω, F, P) and suppose $EX_1=0$ and $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. For positive integers a and b with $0<a<b$, let F_a^b denote the σ-field generateted by random variables $\{X_a, \cdots, X_b\}$. Suppose that the sequence $\{X_i\}$ satisfies the ϕ-mixing condition in the sense that

$$\sup_{a\geqq 1}\ \sup_{A\in F_1^a, B\in F_{a+n}^\infty} |P(A\cap B)-P(A)P(B)|/P(A)\equiv\phi(n)\rightarrow 0$$

as $n\rightarrow\infty$.

It is known that if

$$\phi(n)=0(n^{-\beta}) \text{ for some } \beta>(2+\delta)/(1+\delta)$$

then the limit

$$\sigma^2=\lim_{n\rightarrow\infty} n^{-1}E(\textstyle\sum_{i=1}^n X_i)^2 \tag{1.1}$$

exists. (See, e. g. [5].)

Let $S_n=\sum_{i=1}^n X_i$ and $S_0=0$. Suppose $\sigma^2>0$ and define a continuous polygonal line $\{X_n(t), 0\leqq t\leqq 1\}$ by

$$X_n(t)=(\sigma^2 n)^{-1/2}S_{[nt]}+(nt-[nt])\,(\sigma^2 n)^{-1/2}X_{[nt]+1}$$

where $[b]$ denotes an integer part of b.

Let C=C[0,1] be the space of continuous functions on [0,1] with the uniform metric $d(x,y)=\sup_{0\leq t\leq 1}|x(t)-y(t)|$ and C be the smallest σ-field containing all open sets in C.

Let P_n be the distribution of $\{X_n(t)\}$ and W be the Wiener measure on (C, C). The Prokhorov-Lévy metric $\rho(\cdot,\cdot)$ on the space of probability measures on (C, C) is defined by

$$\rho(R,Q)=\inf\{\varepsilon>0\,;R(B)\leqq\varepsilon+Q\{y\,;d(x,y)<\varepsilon,x\varepsilon B\},$$

$$Q(B)\leqq\varepsilon+R\{y\,;d(x,y)<\varepsilon,x\varepsilon B\}\text{ for all }B\varepsilon C\}$$

where R and Q are probability measures on (C, C).

In this paper we shall show the following results concerning the rate of convergence of $\rho(P_n,W)$ to zero.

Theorem 1. *Let $\{X_i\}$ be a strictly stationary sequence, which is ϕ-mixing with coefficient $\phi(n)$ satisfying*

$$\phi(n)=0(e^{-\gamma n}) \tag{1.2}$$

for some $\gamma>0$ as $n\to\infty$. Suppose that $EX_1=0$ and $\sigma=1$ in (1.1). *If $E|X_1|^{2+\delta}<\infty$ for some $0<\delta\leqq 2$, then as $n\to\infty$*

$$\rho(P_n,W)=0(n^{-\delta/2(3+2\delta)}\log n).$$

Theorem 2. *In* Theorem 1, *replace condition* (1.2) *by*

$$\phi(n)=0(n^{-\beta})$$

for some $\beta>2(2+\delta)/(1+\delta)$. Then we have

$$\rho(P_n,W)=0(n^{-\delta/\{2(3+2\delta)+3(2+\delta)/\beta\}}\log n).$$

It should be mentioned that Yoshihara [9] gave some results on the rate of convergence of $\rho(P_n,W)$ for an absolutely regular sequence under the moment condition $E|X_1|^{4+\varepsilon}<\infty$ for some $\varepsilon>0$. Our moment condition is weaker than his, although his mixing condition is weaker than ours.

2. Preliminaries

The two lemmas are due to Ibragimov [6]. (As to Lemma 2, see also Yokoyama [8].)

Lemma 1. *Suppose that f is F_1^a-measurable and g is F_{a+n}^∞-measurable and that $E|f|^r<\infty$ and $E|g|^s<\infty$ for $r>1$ and $s>1$ with $r^{-1}+s^{-1}=1$. Then*

$$|E(fg)-E(f)E(g)|\leqq 2(\phi(n))^{1/r}\|f\|_r\|g\|_s.$$

Lemma 2. *Under the assumptions in* Theorem 1 *or* Theorem 2, *for each $\delta>0$,* *as* $n \to \infty$

$$E|S_n|^{2+\delta} = 0((ES_n^2)^{1+\delta/2}).$$

For some $0<\alpha<1$, let $M=[n/[n^\alpha]]+1$ and define $I_j=\{(j-1)[n^\alpha]+1, (j-1)[n^\alpha]+2, \cdots, j[n^\alpha]\}$ for $j=1, \cdots, M-1$ and $I_M=\{(M-1)[n^\alpha]+1, (M-1)[n^\alpha]+2, \cdots, n\}$. Let $y_j=\sum_{i \in I_j} n^{-1/2}X_i$ for $j=1, \cdots, M$. Moreover for positive θ, define $U_j=\{j[n^\alpha]-[\theta(\log n)]+1, j[n^\alpha]-[\theta(\log n)]+2, \cdots, j[n^\alpha]\}$ and $v_j=\sum_{i \in U_j} n^{-1/2}X_i$, $i=1, \cdots, M-1$ and $v_M=0$.

The basic idea of the proofs of Theorems 1 and 2 is using the following approximation theorem by Berkes and Philipp [1].

Lemma 3. *Let $\{X_i, i \geqq 1\}$ be a sequence of random variables and $\{L_i, i \geqq 1\}$ be a sequence of σ-fields such that X_i is L_i-measurable for each i and for some $n_k \geqq 0$*

$$|P(A \cap B)-P(A)P(B)| \leqq \eta_k P(A)$$

for all $A \in V_{i<k}L_i$ and $B \in L_k$. Then without changing its distribution we can redefine the sequence $\{X_i, i \geqq 1\}$ on a probability space together with a sequence $\{Y_i\}$ of independent random variables such that Y_i has the same distribution as X_i for each i and

$$P\{|X_i-Y_i| \geqq 6\eta_i\} \leqq 6\eta_i,\ i \geqq 1.$$

The following lemma is due to Borovkov [2].

Lemma 4. *Let $\{Y_i\}$ be a sequence of independent random variables with $EY_i=0$ and $E|Y_i|^{2+\delta}<\infty$ for some $\delta>0$ for all $i \geqq 1$. Then for each $t>0$,*

$$P\{\max_{1 \leqq k \leqq n}|\textstyle\sum_{i=1}^k Y_i|>t\} \leqq C_1 \exp\{-C_2 t^2/\textstyle\sum_{i=1}^n EY_i^2\}$$
$$+\textstyle\sum_{i=1}^n P(|Y_i|>t/4)+C_3(t^{-(2+\delta)}\textstyle\sum_{i=1}^n E|Y_i|^{2+\delta})^2,$$

where C_1, C_2 and C_3 are positive constants depending only on δ.

3. Proof of Theorem 1

Let

$$\alpha=\delta/(3+2\delta),$$
$$\varepsilon_n=Kn^{-\delta/2(3+2\delta)}\log n,\ n \geqq 1,$$
$$\lambda_n=n^{-\delta/(3+2\delta)}\log n,\ n \geqq 1,$$

where K is larger than $(\theta/C_2)^{1/2}$.

Using Lemmas 1 and 2, we can easily prove the following

Lemma 5. *For $j=1, \cdots, M-1$, as $n \to \infty$*

$$Ey_j^2=n^{\alpha-1}+0(n^{-1}),$$

$$E|y_j|^{2+\delta}=0(n^{(\alpha-1)(1+\delta/2)}),$$

$$Ev_j^2=\theta(\log n)/n+0(n^{-1})$$

and

$$E|v_j|^{2+\delta}=0(n^{-(1+\delta/2)}(\log n)^{1+\delta/2}).$$

Also

$$Ey_M^2=(n-(M-1)n^{\alpha})/n+0(n^{-1})$$

and

$$E|y_M|^{2+\delta}=0(n^{(\alpha-1)(1+\delta/2)}).$$

By the same argument as in the proof of Theorem 1 in [3] (p. 213), if we construct the Brownian motion $\{B(t), 0\leqq t\leqq 1\}$ on (Ω, F, P) such that

$$P\{\sup_{0\leqq t\leqq 1}|X_n(t)-B(t)|\geqq\varepsilon_n\}=0(\varepsilon_n) \tag{3.1}$$

as $n\to\infty$, the statement of the theorem is concluded. Hence first we have to define the Brownian motion.

Denote $\xi_j=y_j-v_j$, $j=1, \cdots, M$ and denote M_a^b the σ-field generated by the random variables $\{\xi_a, \cdots, \xi_b\}$ for $0<a<b$. For any $A\varepsilon M_1^a$ and $B\varepsilon M_{a+1}^M$ we have

$$|P(A\cap B)-P(A)P(B)|\leqq\phi([\theta\log n])P(A)=0(n^{-\theta\tau})P(A)$$

uniformly with respect to a with $1\leqq a\leqq M-1$. Hence, by applying Lemma 3, we see that there exist a sequence $\{\bar{\xi}_1, \cdots, \bar{\xi}_M\}$ and a sequence of independent random variables $\{\bar{Y}_1, \cdots, \bar{Y}_M\}$ on a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$ such that the joint distribution of $\{\bar{\xi}_1, \cdots, \bar{\xi}_M\}$ is same as $\{\xi_1, \cdots, \xi_M\}$ and $\bar{\xi}_i$ has the same distribution as Y_i with

$$P\{|\bar{\xi}_i-\bar{Y}_i|\geqq C_4 n^{-\theta\tau}\}\leqq C_5 n^{-\theta\tau} \tag{3.2}$$

for each $i=1, \cdots, M$, where C_4 and C_5 are absolute positive constants.

On the other hand, using the Skorokhod embedding theorem (Theorem 7 in [7]) we construct the Brownian motion $\{B^*(t), 0\leqq t\leqq 1\}$ and a sequence of independent and positive random variables $\{T_i^*, i=1, \cdots, M\}$ on another probability space (Ω^*, F^*, P^*) such that

$$\{(B^*(T_1^*), B^*(T_1^*+T_2^*)-B^*(T_1^*), \cdots, B^*(\textstyle\sum_{i=1}^M T_i^*)-B^*(\sum_{i=1}^{M-1} T_i^*)\}\overset{d}{=}(Y_1, Y_2, \cdots, Y_M),$$

and

$$ET_i^*=E\bar{Y}_i^2 \quad\text{and}\quad E|T_i^*|^{1+\delta/2}\leqq C_6 E|\bar{Y}_i|^{2+\delta} \tag{3.3}$$

for all $1\leqq i\leqq M$, where "$\overset{d}{=}$" means the equality of joint distributions and C_6 is a positive constant depending only on δ.

Using Lemma A1 in [1] we can redefine $(\{\xi_i\}, \{Y_i\}, \{T_i\}, \{B(t)\})$ on a common probability space (Ω, F, P) such that the distribution of $(\{\bar{\xi}_i\}, \{\bar{Y}_i\})$ and $(\{B^*(\sum_{k=1}^i T_k^*) - B^*(\sum_{k=1}^{i-1} T_k^*)\}, \{T_i^*\}, \{B^*(t)\})$ are same as those of $(\{\xi_i\}, \{Y_i\})$ and $(\{Y_i\}, \{T_i\}, \{B(t)\})$, respectively.

Define $\{a_0, a_1, \cdots, a_M\}$ by $a_i=i[n^{\alpha}]/n$ for $i=0, 1, \cdots, M-1$ and $a_M=1$. Let $\{\tilde{X}_n(t), 0\leqq t\leqq 1\}$ and $\{\tilde{B}_n(t), 0\leqq t\leqq 1\}$ be continuous polygonal lines defined by

$$\tilde{X}_n(t)=X_n(a_k)+(X_n(a_{k+1})-X_n(a_k))(t-a_k)/(a_{k+1}-a_k),$$

for $t\varepsilon[a_k, a_{k+1}]$, $k=0, 1, \cdots, M-1$, and

$$\tilde{B}_n(t)=B(a_k)+(B(a_{k+1})-B(a_k))(t-a_k)/(a_{k+1}-a_k),$$

for $t\varepsilon[a_k, a_{k+1}]$, $k=0, 1, M-1$, respectively.

Lemma 6. *As* $n\to\infty$

$$P\{\sup_{0\le t\le 1}|\tilde{X}_n(t)-\tilde{B}_n(t)|\geqq\varepsilon_n\}=o(\varepsilon_n).$$

Proof. We have

$$\begin{aligned}
&P\{\sup_{0\le t\le 1}|\tilde{X}_n(t)-\tilde{B}_n(t)|\geqq\varepsilon_n\}\\
&=P\{\max_{1\le k\le M}|\textstyle\sum_{i=1}^k y_i-B(a_k)|\geqq\varepsilon_n\}\\
&\leqq P\{\max_{1\le k\le M}|\textstyle\sum_{i=1}^k (y_i-v_i)-B(a_k)|\geqq\varepsilon_n/2\}\\
&\qquad\qquad+P\{\max_{1\le k\le M}|\textstyle\sum_{i=1}^k v_i|\geqq\varepsilon_n/2\}\\
&\equiv A_1+A_2,
\end{aligned}$$

say. Applying Lemma 3 to A_2, we see that there exist independent and identically distributed random variables $\{V_i\}$ such that V_i has the same distribution as v_i and

$$P\{|v_i-V_i|\geqq C_7\, e^{-\gamma n^{\alpha/2}}\}\leqq C_8\, e^{-\gamma n^{\alpha/2}},$$

where C_7 and C_8 are absolute positive constants. Thus by Lemmas 4 and 5 we have

$$\begin{aligned}
A_2&\leqq P\{\max_{1\le k\le M-1}|\textstyle\sum_{i=1}^k (v_i-V_i)|\geqq\varepsilon_n/4\}\\
&\qquad+P\{\max_{1\le k\le M-1}|\textstyle\sum_{i=1}^k V_i|>\varepsilon_n/4\}\\
&\leqq(M-1)P\{|v_i-V_i|\geqq\varepsilon_n/4(M-1)\}+C_1\exp\{-C_2\,\varepsilon_n^2/\textstyle\sum_{i=1}^{M-1} EV_i^2\}\\
&\quad+\textstyle\sum_{i=1}^{M-1} P\{|V_i|\geqq\varepsilon_n/16\}+C_3(\textstyle\sum_{i=1}^{M-1} E|V_i|^{2+\delta}\varepsilon_n^{-(2+\delta)})^2\\
&=o(\varepsilon_n),
\end{aligned}$$

as $n\to\infty$.

We next estimate A_1. Denote $Z_k=\sum_{i=1}^k(T_i-ET_i)$. Recall $y_i-v_i=\xi_i$. Since $\sum_{i=1}^k Y_i=B(\sum_{i=1}^k T_i)$ by the construction of $\{Y_i\}$, we have

$$\begin{aligned}
A_1&\leqq P\{\max_{1\le k\le M}|\textstyle\sum_{i=1}^k(\xi_i-Y_i)|\geqq\varepsilon_n/4\}\\
&\qquad+P\{\max_{1\le k\le M}|\textstyle\sum_{i=1}^k Y_i-B(a_k)|\geqq\varepsilon_n/4\}\\
&\leqq P\{\textstyle\sum_{i=1}^M|\xi_i-Y_i|\geqq\varepsilon_n/4\}\\
&\qquad+P\{\max_{1\le k\le M}|B(Z_k+\textstyle\sum_{i=1}^k ET_i)-B(a_k)|\geqq\varepsilon_n/4\}\\
&\leqq\textstyle\sum_{i=1}^M P\{|\xi_i-Y_i|\geqq\varepsilon_n/4M\}\\
&\qquad+P\{\max_{1\le k\le M}|B(Z_k+\textstyle\sum_{i=1}^k ET_i)-B(a_k)|\geqq\varepsilon_n/4,\\
&\qquad\qquad\qquad\max_{1\le k\le M}|Z_k|\leqq\varepsilon_n\}\\
&\qquad+P\{\max_{1\le k\le M}|Z_k|>\varepsilon_n\}\\
&\equiv L_1+L_2+L_3,
\end{aligned}$$

say. By (3.2) we have for any sufficiently large θ

$$L_1=O(n^{1-\alpha-\theta\gamma})=o(\varepsilon_n).$$

Since $ET_i=EY_i^2$ and Y_i has the same distribution as $\xi_i=y_i-v_i$, we have $ET_i=E(y_i-v_i)^2$, thus

$$\max_{1\leq k\leq M}|\sum_{i=1}^{k} ET_i-a_k|$$
$$\leq \max_{1\leq k\leq M}|\sum_{i=1}^{k} Ey_i^2-a_k|+2\sum_{i=1}^{M}|E(y_iv_i)|+\sum_{i=1}^{M} Ev_i^2 .$$

By Lemmas 1 and 5 we have

$$|E(y_1v_1)|\leq|E\{(\sum_{i=1}^{[n^\alpha]-[\theta(\log n)]}n^{-1/2}X_i)v_i\}|+Ev_1^2$$
$$\leq\sum_{i=1}^{[n^\alpha]-[\theta(\log n)]}2(\phi([n^\alpha]-[\theta(\log n)+1-i))^{(1+\delta)/(2+\delta)}$$
$$\times||n^{-1/2}X_i||_{(2+\delta)/(1+\delta)}||v_1||_{2+\delta}+Ev_1^2$$
$$=O(n^{-1}(\log n)),$$

thus

$$\sum_{i=1}^{M}|E(y_iv_i)|=O(\lambda_n).$$

And also we have from Lemma 5

$$\max_{1\leq k\leq M}|\sum_{i=1}^{k} Ey_i^2-a_k|=o(\lambda_n)$$

and

$$\sum_{i=1}^{M} Ev_i^2=o(\lambda_n).$$

Hence there exists a positive constant C_9 such that

$$\max_{1\leq k\leq M}|\sum_{i=1}^{k} ET_i-a_k|\leq C_9\lambda_n .$$

Then by the same argument as in the proof of Lemma 5 in [7], we have

$$L_2\leq\sum_{k=1}^{M} P\{\sup_{|t|\leq(1+C_9)\lambda_n}|B(t+a_k)-B(a_k)|\geq\varepsilon/4\}$$
$$\leq 2M P\{\sup_{0\leq t\leq(1+C_9)\lambda_n}|B(t)|\geq\varepsilon_n/4\}$$
$$\leq 8M P\{|B(1)|\geq\varepsilon_n/4((1+C_9)\lambda_n)^{1/2}\}$$
$$=o(\varepsilon_n).$$

On the other hand by the Kolmogorov inequlity and the Marcinkiewicz-Zygmund inequlity we have

$$L_3\leq\lambda_n^{-(1+\delta/2)}E|Z_M|^{1+\delta/2}\leq C_{10}\,\lambda_n^{-(1+\delta/2)}\sum_{i=1}^{M} E|T_i-ET_i|^{1+\delta/2},$$

where C_{10} is a positive constant depending only on δ. Since Y_i has the same distribution as ξ_i, using (3.3) and Lemma 5, we have

$$L_3\leq C_{11}\,\lambda_n^{-(1+\delta/2)}\sum_{i=1}^{M} E|\xi_i|^{2+\delta}=o(\varepsilon_n),$$

where C_{11} is an absolute constant. Thus we have $A_1=o(\varepsilon_n)$, which concludes the lemma.

The following lemma is due to Borovkov (Lemma 2 in [3]).

Lemma 7. *As* $n\to\infty$

$$P\{\sup_{0\leq t\leq 1}|\tilde{B}_n(t)-B(t)|\geqq\varepsilon_n\}=o(\varepsilon_n).$$

Finally we shall prove

Lemma 8. *As* $n\to\infty$

$$P\{\sup_{0\leq t\leq 1}|X_n(t)-\tilde{X}_n(t)|\geqq\varepsilon_n\}=o(\varepsilon_n).$$

Proof. Let q be a real number such that $\phi(q)<1/10$ and define

$$\chi(j,k)=\begin{cases}0 & \text{if } [k/q]q+j>k,\\ 1 & \text{if } [k/q]q+j\leqq k.\end{cases}$$

By the definition of $\{X_n(t)\}$ and $\{\tilde{X}_n(t)\}$ we have

$$\begin{aligned}
&P\{\sup_{0\leq t\leq 1}|X_n(t)-\tilde{X}_n(t)|\geqq\varepsilon_n\}\\
&\leqq\textstyle\sum_{i=1}^{M} P\{\sup_{a_{i-1}\leq t\leq a_i}|X_n(t)-\tilde{X}_n(t)|\geqq\varepsilon_n\}\\
&\leqq M\,P\{\max_{1\leq k\leq[n^\alpha]}|\textstyle\sum_{i=1}^{k} n^{-1/2}X_i-(k/[n^\alpha])\sum_{i=1}^{[n^\alpha]}n^{-1/2}X_i|\geqq\varepsilon_n\}\\
&\leqq M\,P\{\max_{1\leq k\leq[n^\alpha]}|\textstyle\sum_{i=1}^{k} n^{-1/2}X_i|\geqq\varepsilon_n/2\}\\
&\leqq M\,P\{\max_{1\leq k\leq[n^\alpha]}|\textstyle\sum_{j=1}^{q}\sum_{i=1}^{[k/q]+\chi(j,k)}n^{-1/2}X_{(i-1)q+j}|\geqq\varepsilon_n/2\}\\
&\leqq(Mq)P\{\max_{1\leq k\leq[[n^\alpha]/q]+1}|\textstyle\sum_{i=1}^{k} n^{-1/2}X_{(i-1)q+1}|\geqq\varepsilon_n/2q\}
\end{aligned}$$

Let $m=[[n^\alpha]/q]+1$. Using Theorem 1.2 in [4] and lemma 3 we have as $n\to\infty$

$$(3.4)\qquad \begin{aligned}&(Mq)P\{\max_{1\leq k\leq m}|\textstyle\sum_{i=1}^{k} n^{-1/2}X_{(i-1)q+1}|\geqq\varepsilon_n/2q\}\\
&\leqq\frac{(Mq)(\varepsilon_n/4q)^{-(2+\delta)}E|\sum_{i=1}^{m} n^{-1/2}X_{(i-1)q+1}|^{2+\delta}}{1-\phi(q)-(\varepsilon_n/4q)^{-(2+\delta)}\max_{1\leq k\leq m-1}E|\sum_{i=k}^{m} n^{-1/2}X_{(i-1)q+1}|^{2+\delta}}.\end{aligned}$$

Similarly as in the proof of Lemma 5, we have by Lemmas 1 and 2

$$E|\textstyle\sum_{i=1}^{m} n^{-1/2}X_{(i-1)q+1}|^{2+\delta}=O(n^{(\alpha-1)(1+\delta/2)}),$$

and also we have from the stationarity of $\{X_i\}$

$$\begin{aligned}&\max_{1\leq k\leq m-1}E|\textstyle\sum_{i=k}^{m} n^{-1/2}X_{(i-1)q+1}|^{2+\delta}\\
&=\max_{2\leq k\leq m}E|\textstyle\sum_{i=1}^{k} n^{-1/2}X_{(i-1)q+1}|^{2+\delta},\end{aligned}$$

which is $O(n^{(\alpha-1)/(1+\delta/2)})$ because

$$E|\textstyle\sum_{i=1}^{k} n^{-1/2}X_{(i-1)q+1}|^{2+\delta}=O(n^{(\alpha-1)/(1+\delta/2)})$$

uniformly with respect to k, $2 \leqq k \leqq m$. Thus the right hand side of (3.4) is the order of $o(\varepsilon_n)$ as $n \to \infty$. Hence the lemma is proved.

By Lemmas 6, 7 and 8 we obtain (3.1) and thus conclude the proof of Theorem 1.

4. Proof of Theorem 2

It is sufficient to modify some definitions in the proof of Theorem 1. We first define U_j by $U_j = \{j[n^\alpha] - [n^{(2-\alpha)/(1+2\beta)}]+1, j[n^\alpha] - [n^{(2-\alpha)/(1+2\beta)}]+2, \cdots, j[n^\alpha]\}$, $j=1, \cdots, M-1$ and put

$$\alpha = \{2\delta\beta + 3(2+\delta)\}/\{2(3+2\delta)\beta + 3(2+\delta)\},$$

$$\varepsilon_n = K n^{-\delta/\{2(3+2\delta)+3(2+\delta)/\beta\}}(\log n),$$

$$\lambda_n = n^{-2\delta/\{2(3+2\delta)+3(2+\delta)/\beta\}}(\log n).$$

We note that for $\beta > 2(2+\delta)/(1+\delta)$

$$Ey_1^2 = n^{-1+(2-\alpha)/(1+2\beta)} + O(n^{-1})$$

and

$$\max_{1 \leqq k \leqq M} \left| \sum_{i=1}^k ET_i - a_k \right| = O(n^{-\alpha+(2-\alpha)/(1+2\beta)}) = o(\lambda_n),$$

as $n \to \infty$. The rest of the proof is the same as that of Theorem 1 and so is omitted.

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