KEIO SCIENCE AND TECHNOLOGY REPORTS
VOL. 35, NO. 1, pp. 1-36, 1982

ON SOLUTIONS OF $x''=e^{\alpha\lambda t}x^{1+\alpha}$

ICHIRO TSUKAMOTO, TOSHIRO MISHINA
and MAHITO ONO

**Department of Mathematics, Keio University,
Hiyoshi-cho, Kohoku-ku, Yokohama 223**

(Received August, 11, 1981)

§ 0. Introduction

Asymptotic behavior of solutions of a second order nonlinear differential equation

$$x''=t^{\beta}x^{1+\alpha}, \quad '=d/dt$$

was discussed by Saito in the papers [1], [2]. Main tool used in those papers is a certain change of variables which transforms this equation into a first order rational differential equation or equivalently, into a two-dimensional automous system.

In this paper, we shall investigate a second order nonlinear differential equation of the form

$$(1) \qquad x''=e^{\alpha\lambda t}x^{1+\alpha}, \quad \alpha>0, \lambda>0,$$

in a domain

$$G: \ -\infty<t<\infty, \quad 0\leq x<\infty$$

by using the same technique. Here $x^{1+\alpha}$ always represents its non-negative valued branch.

The solutions of (1) to be considered here are those which satisfy the initial condition

$$x(t_0)=a, \quad x'(t_0)=b, \quad -\infty<t_0<\infty, 0<a<\infty, -\infty<b<\infty.$$

where t_0 is supposed to be arbitrarily fixed. Such solutions of (1) will be denoted by $\phi(t, a, b)$ or simply $\phi(t, b)$ since a is fixed in the course of our discussions.

As can easily be verified, (1) has a particular solution

$$x=\lambda^{2/\alpha}e^{-\lambda t}.$$

This will be denoted by $\psi(t)$.

In § 1 we shall show the existence of a solution $\hat{\phi}(t)$ which is defined and bounded for $t_0\leq t<\infty$. Since $\hat{\phi}(t)$ is unique for fixed t_0 and a, we shall denote

$\hat{\phi}'(t_0)$ by $\hat{b}$. The solution $\phi(t, a, b)$ with $b>\hat{b}$ will be considered in §2. In the last section we consider the solution $\phi(t, a, b)$ with $b<\hat{b}$. Thus we can get the information about all kinds of solutions of (1).

§1. On bounded solutions as $t \to \infty$.

If $x(t)$ is a nontrivial solution of (1), then

$$x''(t)=e^{\alpha\lambda t}x(t)^{1+\alpha}>0\,,$$

i.e. the solution curve of $x(t)$ is convex. Hence (1) does not have any nontrivial solution which is defined and bounded for $-\infty<t<\infty$. So in this section we discuss the existence of the solution of (1) which remains bounded as $t\to\infty$. The main result of this section is as follows:

Theorem I. If a positive value a and a real value t_0 are given, then the equation (1):

$$x''=e^{\alpha\lambda t}x^{1+\alpha}\,,\quad \alpha>0, \lambda>0$$

has one and only one solution $\hat{\phi}(t)$ such that

(A) $\hat{\phi}(t)$ is defined for $\omega'<t<\infty$ $(-\infty\leq\omega'<t_0)$ where ω' depends on a,

(B) $\hat{\phi}(t_0)=a$,

(C) $\hat{\phi}(t)$ is bounded as $t\to\infty$.

Moreover the following statements are valid:

(D) If $0<a\leq\psi(t_0)$, then $\hat{\phi}(t)$ is defined for $-\infty<t<\infty$, i.e. $\omega'=-\infty$, and

$$\lim_{t\to\infty}\hat{\phi}(t)=0\,,\quad \lim_{t\to\infty}\hat{\phi}'(t)=0\,,$$

$$\lim_{t\to-\infty}\hat{\phi}(t)=\infty\,,\quad \lim_{t\to-\infty}\hat{\phi}'(t)=-\infty\,.$$

In the neighborhood of $t=\infty$, $\hat{\phi}(t)$ can be expressed as

$$\hat{\phi}(t)=\sum_{n=0}^{\infty}a_n\exp\{(n(1-\sqrt{1+\alpha})-1\}\lambda t\,,\quad a_0>0\,. \tag{2}$$

(E) If $a>\psi(t_0)$, then $\hat{\phi}(t)$ is defined for $\omega'<t<\infty$ $(-\infty<\omega'<t_0)$ and

$$\lim_{t\to\infty}\hat{\phi}(t)=0\,,\quad \lim_{t\to\infty}\hat{\phi}'(t)=0$$

$$\lim_{t\to\omega'}\hat{\phi}(t)=\infty\,,\quad \lim_{t\to\omega'}\hat{\phi}'(t)=-\infty\,.$$

In the neighborhood of $t=\infty$, $\hat{\phi}(t)$ is expressed as

$$\hat{\phi}(t)=\sum_{n=0}^{\infty}a_n\exp\{(n(1-\sqrt{1+\alpha})-1)\lambda t\}\,,\quad a_0>0\,. \tag{2$'$}$$

and in the neighborhood of $t=\omega'$, as

(3) $$\hat{\phi}(t)=\left(\frac{2}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha}e^{-\lambda\omega'}(t-\omega')$$
$$\times[1+\Sigma_{m+n>0}a_{mn}\{(t-\omega')^{2+(4/\alpha)}(C\log(t-\omega')+B)\}^n]$$

where B and C are constants and $C=0$ if $4/\alpha$ is not integer.

Proof will be carried out in the following order:

1) Uniqueness of the solution satisfying (A), (B) and (C).
2) Existence of the solution satisfying (D).
3) Existence of the solution satisfying (E).

Here we must notice that the existence of the solution of (1) satisfying (A), (B) and (C) can be shown directly from 2), 3).

Let us start to show 1).

Suppose that (1) admits a nontrivial solution $\hat{\phi}(t)$ which is defined and bounded for $t_0\leq t<\infty$. Since

$$\hat{\phi}''(t)=e^{\alpha\lambda t}\hat{\phi}(t)^{1+\alpha}\geq 0$$

in G by our assumption given at the outset, $\hat{\phi}'(t)$ is a nondecreasing function of t. Therefore we have

$$\hat{\phi}'(t)\geq\hat{\phi}'(\tau)\quad\text{for } t\geq\tau\,.$$

Then

$$\hat{\phi}(t)-\hat{\phi}(\tau)\geq\hat{\phi}'(\tau)(t-\tau)\quad\text{for } t\geq\tau\,.$$

Then if $\hat{\phi}'(\tau)>0$ for some $\tau>0$, this inequality contradicts the boundedness of $\hat{\phi}(t)$. Thus we get the following proposition.

Proposition 1. If $\hat{\phi}(t)$ is a nontrivial solution of (1) bounded as $t\to\infty$, then

$$\hat{\phi}'(t)\leq 0\,.$$

Lemma 1. Let $\phi(t)$ be a solution of (1) defined for $\omega'<t<\omega(-\infty\leq\omega'<t_0<\omega\leq\infty)$, then the following statements are valid:

(a) $\phi(t)$ has a limit as $t\to\omega$ or $t\to\omega'$, including the infinity.
(b) $\lim_{t\to\infty}\phi(t)=0$ implies $\lim_{t\to\infty}\phi'(t)=0$.
(c) $\lim_{t\to\omega}\phi(t)=\infty$ implies $\lim_{t\to\omega}\phi'(t)=\infty$.
(d) $\lim_{t\to\omega'}\phi(t)=\infty$ implies $\lim_{t\to\omega'}\phi'(t)=-\infty$.

Proof. (a) is obvious from the convexity of the solution curves of (1). If $\lim_{t\to\infty}\phi(t)=0$, then $\phi(t)$ is a bounded solution of (1) as $t\to\infty$, and hence from Proposition 1 we get

$$\phi'(t)\leq 0\,.$$

Therefore if $\lim_{t\to\infty}\phi'(t)\neq 0$, then there exists a number c such that

$$\phi'(t)<c<0\,.$$

Integrating both sides, we get

$$\phi(t) < \phi(t_0) + c(t - t_0)\,, \quad t > t_0$$

which implies

$$\lim_{t\to\infty} \phi(t) = -\infty$$

in contradiction with

$$\lim_{t\to\infty} \phi(t) = 0\,.$$

Hence (b) is valid.

If $-\infty < \omega' < \omega < \infty$, (c) and (d) are evident.

If $\omega = \infty$, there exists a number T_1, for an arbitrarily given positive number R_1, such that $t \geq T_1$ implies

$$\phi''(t) = e^{\alpha\lambda t}\phi(t)^{1+\alpha} > R_1$$

since $\lim_{t\to\omega} \phi(t) = \infty$. Hence

$$\phi'(t) - \phi'(T_1) > R_1(t - T_1)\,, \quad t \geq T_1$$

and so

$$\lim_{t\to\infty} \phi'(t) = \infty\,.$$

If $\omega' = -\infty$, there exists a number T_2 for an arbitrarily given positive number R_2 such that $t \leq T_2$ implies

$$\phi''(t) = e^{\alpha\lambda t}\phi(t)^{1+\alpha} > e^{\alpha\lambda t}R_2\,,$$

since $\lim_{t\to\omega'} \phi(t) = \infty$. Hence

$$\phi'(T_2) - \phi'(t) > \frac{R_2}{\alpha\lambda}(e^{\alpha\lambda T_2} - e^{\alpha\lambda t})\,, \quad t \leq T_2\,,$$

and so

$$\phi'(T_2) - \lim_{t\to-\infty} \phi'(t) > \frac{R_2}{\alpha\lambda} e^{\alpha\lambda T_2}\,.$$

Since R_2 is an arbitrary constant,

$$\lim_{t\to-\infty} \phi'(t) = -\infty\,. \qquad \text{Q.E.D.}$$

The next lemma will show the uniqueness of the solution $\hat{\phi}(t)$.

Lemma 2. Let $x = \hat{\phi}_1(t)$, $x = \hat{\phi}_2(t)$ be two nontrivial solutions such that

(A) $\hat{\phi}_i(t)$ is defined for (ω_i', ∞) $(-\infty \leq \omega_i' < t_0)$

(B) $\hat{\phi}_i(t)$ is bounded as $t \to \infty$,

(C) $\hat{\phi}_1(s) > \hat{\phi}_2(s)$ for some s such that $\max(\omega_1', \omega_2') < s < \infty$, $(i = 1, 2)$.

Then

$$\hat{\phi}_1(t) > \hat{\phi}_2(t)$$

for $\omega'<t<\omega$, $\omega'=\max(\omega_1', \omega_2')$.

Proof. Suppose that

$$\hat{\phi}_1(t)>\hat{\phi}_2(t)\,,\quad s<t<\tau<\infty\,,$$
$$\hat{\phi}_1(\tau)=\hat{\phi}_2(\tau)\,,$$

then we get

$$\hat{\phi}_1'(\tau)\leq\hat{\phi}_2'(\tau)\,.$$

If $\hat{\phi}_1'(\tau)=\hat{\phi}_2'(\tau)$, then $\hat{\phi}_1(t)\equiv\hat{\phi}_2(t)$ since they both satisfy the same initial condition at $t=\tau$. Hence

$$\hat{\phi}_1'(\tau)<\hat{\phi}_2'(\tau)\,.$$

Since $\hat{\phi}_i'(t)$ $(i=1,2)$ are continuous, $\phi_1'(t)\phi_2'(t)$ and therefore

$$\hat{\phi}_1(t)<\hat{\phi}_2(t)$$

if $t(>\tau)$ is sufficiently close to τ.

Let us suppose that

$$\hat{\phi}_1(t)<\hat{\phi}_2(t)\,,\quad \tau<t<\tau_1\,,$$
$$\hat{\phi}_1(\tau_1)=\hat{\phi}_2(\tau_1)\,.$$

Then

(4) $$\hat{\phi}_1'(\tau_1)>\hat{\phi}_2'(\tau_1)\,,$$

since the case $\hat{\phi}_1'(\tau_1)=\phi_2'(\tau_1)$ is excluded by the same reason as above. On the other hand, we get

$$\hat{\phi}_1''(t)=e^{\alpha\lambda t}\hat{\phi}_1(t)^{1+\alpha}<e^{\alpha\lambda t}\hat{\phi}_2(t)^{1+\alpha}=\hat{\phi}_2''(t)\,,\quad \tau<t<\tau_1\,,$$

since $\hat{\phi}_1(t)<\hat{\phi}_2(t)$, $\tau<t<\tau_1$. Integrating both sides of this inequality from τ to $t(<\tau_1)$,

(5) $$\hat{\phi}_2'(t)>\hat{\phi}_1'(t)+\hat{\phi}_2'(\tau)-\hat{\phi}_1'(\tau)\,.$$

Since $\hat{\phi}_2'(\tau)>\hat{\phi}_1'(\tau)$, we obtain

$$\hat{\phi}_2'(\tau_1)>\hat{\phi}_1'(\tau_1)$$

in contradiction with (4).

Hence

$$\hat{\phi}_1(t)<\hat{\phi}_2(t)$$

for $\tau<t<\infty$. Then we find (5) is valid for $\tau<t<\infty$. Integrating both sides of (5) we obtain

$$\hat{\phi}_2(t)-\hat{\phi}_2(\tau)>\hat{\phi}_1(t)-\hat{\phi}_1(\tau)+(\hat{\phi}_2'(\tau)-\hat{\phi}_1'(\tau))(t-\tau)\,,\quad \tau<t\,.$$

Since $\hat{\phi}_1(t)>0$, $\hat{\phi}_2'(\tau)-\hat{\phi}_1'(\tau)>0$, $\hat{\phi}_1(\tau)=\hat{\phi}_2(\tau)$, this implies

$$\lim_{t\to\infty}\hat{\phi}_2(t)=\infty .$$

This contradicts with the boundedness of $\hat{\phi}_2(t)$. Thus we get

$$\hat{\phi}_1(t)>\hat{\phi}_2(t)$$

for $s<t<\infty$.

If there exists τ such that

$$\hat{\phi}_1(\tau)<\hat{\phi}_2(\tau)\,, \quad \omega'<\tau<s\,,$$

then we are led to contradiction by the same argument as above and hence Lemma 2 is proved. Q.E.D.

If there exist two solutions $\hat{\phi}_1(t)$, $\hat{\phi}_2(t)$ satisfying the conditions (A), (B) and (C), then for some s we must have

$$\hat{\phi}_1(s)\neq\hat{\phi}_2(s)\,.$$

Hence we get from Lemma 2,

$$\hat{\phi}_1(t)\neq\hat{\phi}_2(t)$$

for all t, which contradicts with

$$\hat{\phi}_1(t_0)=\hat{\phi}_2(t_0)=a\,.$$

Thus we could prove the uniqueness of a solution satisfying (A), (B) and (C) of Theorem I.

Now we shall show that (1) can be transformed into a first order rational differential equation by a change of variables similar to that used in [1].

Lemma 3. Let $\phi(t)$ be a solution of (1) and we put

$$y=y(t)=\psi(t)^{-\alpha}\phi(t)^{\alpha}=\lambda^{-2}e^{\alpha\lambda t}\phi(t)^{\alpha} \tag{6}$$

$$z=y'(t) \tag{7}$$

Then, (1) is transformed into the first order rational differential equation

$$\frac{dz}{dy}=\frac{(\alpha-1)z^2+2\alpha\lambda yz-\alpha^2\lambda^2(y^2-y^3)}{\alpha yz} \tag{8}$$

Proof is immediate if we substitute (6) into (1) and notice that $z=y'(t)$ and $\psi'(t)=-\lambda\psi(t)$.

By investigating the equation (8) we prove the existence of a solution which is bounded as $t\to\infty$.

Proposition 2. The equation (8) has one and only one solution such that

(a) $z(y)>0$ for $0<y<1$,

(b) $\lim_{y\to 0}z(y)=\lim_{y\to 1}z(y)=0$.

Proof. Instead of (8), we consider an autonomous system

$$(9)\qquad \begin{cases} \dfrac{dy}{ds}=\alpha yz \\ \dfrac{dz}{ds}=(\alpha-1)z^2+2\alpha\lambda yz-\alpha^2\lambda^2(y^2-y^3) \end{cases}$$

whose critical points are

$$(y,z)=(0,0)\,,\quad (1,0)\,.$$

If $\alpha=1$, $(0,z)$ (z is arbitrary) is also a critical point of (9).

If we put $y=1+\eta$, $z=\zeta$, then (9) turns into

$$(10)\qquad \begin{cases} \dfrac{d\eta}{ds}=\alpha\zeta+\cdots \\ \dfrac{d\zeta}{ds}=\alpha^2\lambda^2\eta+2\alpha\lambda\zeta+\cdots \end{cases}$$

where the unwritten part represents the terms whose degrees are greater than 1. Since the eigenvalues of the matrix

$$\begin{bmatrix} 0 & \alpha \\ \alpha^2\lambda^2 & 2\alpha\lambda \end{bmatrix}$$

are

$$\mu_1=(1+\sqrt{1+\alpha})\alpha\lambda>0$$

and

$$\mu_2=(1-\sqrt{1+\alpha})\alpha\lambda<0\,,$$

$(\eta,\zeta)=(0,0)$, i.e. $(y,z)=(1,0)$ is a saddle point. A solution of (10) which tends to $(0,0)$ can be represented as

$$\eta=a_{i1}(Ce^{\mu_i s})+a_{i2}(Ce^{\mu_i s})^2+\cdots$$

$$\zeta=b_{i1}(Ce^{\mu_i s})+b_{i2}(Ce^{\mu_i s})^2+\cdots$$

($i=1,2$; C is an arbitrary constant ; a_{in} and b_{in} are constants)

where the power series in $Ce^{\mu_i s}$ in the right-hand members are convergent in the neighborhood of $s=-\infty$ ($i=1$) or $s=\infty$ ($i=2$). Then we find

$$\frac{b_{i1}}{a_{i1}}=\frac{\mu_i}{\alpha}$$

and so we get the power series expression of the orbit:

$$\zeta=\frac{b_{i1}}{a_{i1}}\eta+\cdots=\frac{\mu_i}{\alpha}\eta+\cdots\,,$$

or, returning to the original variables

(11) $$z=\frac{\mu_i}{\alpha}(y-1)+\cdots$$

which converges in the neighborhood of $y=1$. Since $\frac{\mu_1}{\alpha}>0$ and $\frac{\mu_2}{\alpha}<0$, only the curve

(12) $$z=\frac{\mu_2}{\alpha}(y-1)+\cdots$$

intersects the domain

$$0<y<1, \quad z\geq 0 .$$

Now we will show that this orbit tends to $(0,0)$ as $s\to-\infty$. We denote by C the arc of a parabola

$$z=f(y)=\alpha\lambda(y-y^2)$$

lying between $y=0$ and $y=1$ and by D the domain bounded by C and the segment $0\leq y\leq 1$ on the y-axis. On the open segment $0<y<1$, $z=1$, we have

$$\frac{dy}{ds}=0, \quad \frac{dz}{ds}=-\alpha^2\lambda^2(y^2-y^3)<0 .$$

On the arc C, we have

$$\frac{d}{ds}(z-f(y))=(\alpha+1)\alpha^2\lambda^2(y^3-y^4)>0$$

for $0<y<1$. From those we find that orbits passing through the boundary of D (except for $(0,0)$, $(1,0)$) go out of D as s increases. Therefore every orbit starting from inside of D can never leave D as s decreases to $-\infty$.

On the other hand,

$$\frac{\mu_2}{\alpha}=(1-\sqrt{1+\alpha})\lambda=\frac{-\alpha\lambda}{1+\sqrt{1+\alpha}}>-\alpha\lambda=\frac{d}{dy}f(y)\bigg|_{y=1} .$$

Hence the orbit belongs to D for $-\infty<s<\infty$. Then, since

$$\frac{dy}{ds}=\alpha yz>0$$

in D, Poincaré-Bendixson theory shows that the orbit tends to the critical point $(0,0)$ as $s\to-\infty$. Q.E.D.

We denote by $z_1(y)$ a solution whose existence was just shown above.

Proposition 3. Let $y(t)$ be a solution of the equation

$$\frac{dy}{dt}=z_1(y)$$

which satisfies an initial condition

$$y(t_0)=y_0=\phi(t_0)^{-\alpha}a^{\alpha}$$

where

$$a=\phi(t_0)<\psi(t_0)\,.$$

If we define $\phi_1(t)$ by

$$\phi_1(t)=\psi(t)y(t)^{1/\alpha}\,,$$

then $\phi_1(t)$ is the only solution of (1) such that

(a) $\phi_1(t)$ is defined for $-\infty<t<\infty$,

(b) $\lim_{t\to\infty}\phi_1(t)=0$, $\lim_{t\to\infty}\phi_1'(t)=0$,

(c) $\phi_1(t_0)=a$.

Proof. Since

$$\frac{dy}{dt}=z_1(y)$$

we get

$$\frac{dt}{dy}=\frac{1}{z_1(y)}>0$$

and hence t is an increasing fuction of y. Consequently it is sufficient to show

$$\lim_{y\to 0}t(y)=-\infty\,,\quad \lim_{y\to 1}t(y)=\infty$$

for the proof of (a).

From (12)

$$z_1(y)=\frac{\mu_2}{\alpha}(y-1)+\cdots$$

in the neighborhood of $y=1$. Integrating from y_0 to $y(t)$

(13) $$\int_{y_0}^{y(t)}\frac{dy}{z_1(y)}=\frac{\alpha}{\mu_2}\log(y-1)+\cdots=t-t_0$$

when $y(t)$ is sufficiently close to 1. Hence

$$\lim_{y\to 1}t=\infty\,.$$

As we know

$$z_1(y)<\alpha\lambda(y-y^2)\,,\quad 0<y<1$$

we get

$$\frac{1}{z_1(y)}>\frac{1}{\alpha\lambda(y-y^2)}>\frac{1}{\alpha\lambda y}$$

for $0<y<1$. Hence, if $t<t_0$ (i.e. $y(t)<y_0$)

$$t-t_0=\int_{y_0}^{y(t)}\frac{dy}{z_1(y)}<\frac{1}{\alpha\lambda}\int_{y_0}^{y(t)}\frac{dy}{y}=\frac{1}{\alpha\lambda}\log\frac{y(t)}{y_0}.$$

Therefore

$$\lim_{y\to 0}t=-\infty.$$

Since

$$0\leq\lim_{t\to\infty}\phi_1(t)=\lim_{t\to\infty}\lambda^{2/\alpha}e^{-\lambda t}y(t)^{1/\alpha}<\lim_{t\to\infty}\lambda^{2/\alpha}e^{-\lambda t}=0,$$

we have

$$\lim_{t\to\infty}\phi_1(t)=0. \tag{14}$$

Hence

$$\lim_{t\to\infty}\phi'(t)=0$$

by Lemma 1(b) and (b) was proved.

(c) is obvious from the definition of y_0. Q.E.D.

Now we construct an analytical expression of $\phi_1(t)$ valid in the neighborhood of $t=\infty$. From (13), we get

$$(y-1)\sum_{n=0}^{\infty}\hat{a}_n(y-1)^n=e^{(\mu_2/\alpha)t},\quad \hat{a}_0\neq 0,$$

in the neighborhood of $y=1$. Hence, solving it with respect to y, we obtain

$$y=\sum_{n=0}^{\infty}\tilde{a}_n e^{(\mu_2/\alpha)t}=\sum_{n=0}^{\infty}\tilde{a}_n\exp\{n(1-\sqrt{1+\alpha})\lambda t\}.$$

Therefore

$$\phi_1(t)=\sum_{n=0}^{\infty}a_n\exp\{n(1-\sqrt{1+\alpha})-1)\lambda t\},\quad a_0>0.$$

This is the desired analytical expression of $\phi_1(t)$ at $t=\infty$.

By putting

$$\hat{\phi}(t)=\phi_1(t)$$

we can prove that there exists a solution of (1) satisfying (A), (B), and (C) in Theorem I if $a<\phi(t_0)$. Moreover (D) is also verified.

If $a=\phi(t_0)$, it is sufficient to put $\hat{\phi}(t)=\phi(t)$.

Finally we must show that there exists a solution of (1) satisfying (A), (B), and (C) even if $a>\phi(t_0)$.

Proposition 4. The equation (8) has one and only one solution such that

(a) $z(y)<0$ for $1<y<\infty$,

(b) $\lim_{y\to 1}z(y)=0$, $\lim_{y\to\infty}z(y)=-\infty$.

Proof. We consider again (9), i.e.

$$\begin{cases} \dfrac{dy}{ds}=\alpha yz \\ \dfrac{dz}{ds}=(\alpha-1)z^2+2\alpha\lambda yz-\alpha^2\lambda^2(y^2-y^3)\,. \end{cases}$$

This system has the only solution which tends to a critical point (1, 0) and intersects a domain

$$y>1\,,\quad z<0\,.$$

This solution can be expressed by

$$y-1=a_{21}(Ce^{\mu_2 s})+a_{22}(Ce^{\mu_2 s})^2+\cdots$$

$$z=b_{21}(Ce^{\mu_2 s})+b_{22}(Ce^{\mu_2 s})+\cdots$$

where the power series in $Ce^{\mu_2 s}$ in the right-hand members converge in the neighborhood of $s=\infty$ as $\mu_2=(1-\sqrt{1+\alpha})\alpha\lambda<0$. The same solution is also represented as

$$z=\frac{\mu_2}{\alpha}(y-1)+\cdots$$

where the right-hand member is a power series in $y-1$ convergent in the neighborhood of $y=1$.

Now we will show that this orbit tends to $(\infty, -\infty)$ as $s\to-\infty$. If we put

$$z=f_1(y)=\alpha\lambda(y-y^2)\,,$$

then

$$\frac{d}{ds}(z-f_1(y))\bigg|_{z=f_1(y)}=(\alpha+1)\alpha^2\lambda^2(y^3-y^4)<0$$

for $y>1$. Also if we put

$$z=f_2(y)=-\frac{\alpha\lambda}{\alpha+2}(y-1)$$

then

$$\frac{d}{ds}(z-f_2(y))\bigg|_{z=f_2(y)}>\left(\frac{\alpha\lambda}{\alpha+2}\right)^2(y-1)\{(2\alpha+5)(y^2-y)+(\alpha-1)(y^2-1)\}>0$$

for $y>1$. On the other hand, we have

$$f_1'(1)=-\alpha\lambda<\frac{\mu_2}{\alpha}=(1-\sqrt{1+\alpha})\lambda<\frac{-\alpha\lambda}{\alpha+2}=f_2'(1)\,.$$

Consequently, this orbit is contained in a domain

$$y>1\,,\quad \alpha\lambda(y-y^2)<z<-\frac{\alpha\lambda}{\alpha+2}(y-1)$$

and hence tends to $(\infty, -\infty)$ as $s\to-\infty$, since

$$\frac{dy}{ds}=\alpha yz<0$$

in this domain. Q.E.D.

We denote this solution by $z_2(y)$.

Proposition 5. Let $y(t)$ be a solution of the equation

$$\frac{dy}{dt}=z_2(y)$$

which satisfies an initial condition

$$y(t_0)=y_0=\psi(t_0)^{-\alpha}a^{\alpha}\,,\quad a>\psi(t_0)\,.$$

If we define $\phi_2(t)$ by

$$\phi_2(t)=\psi(t)y(t)^{1/\alpha}$$

then $\phi_2(t)$ is the only solution of (1) such that

(a) $\phi_2(t)$ is defined for $\omega'<t<\infty$ $(-\infty\leq\omega'<t_0)$,

(b) $\lim_{t\to\infty}\phi_2(t)=0$, $\lim_{t\to\infty}\phi_2'(t)=0$, $\lim_{t\to\omega'}\phi_2(t)=\infty$, $\lim_{t\to\omega'}\phi_2'(t)=-\infty$,

(c) $\phi_2(t_0)=a$.

Proof. From

$$\frac{dy}{dt}=z_2(y)\,,$$

we get

$$\int_{y_0}^{y(t)}\frac{dy}{z_2(y)}=t-t_0\,,\quad y(t_0)=y_0\,.$$

Since

$$\frac{dt}{dy}=\frac{1}{z_2(y)}<0\,, \tag{15}$$

t is a decreasing function of y and hence it is sufficient to show

$$\lim_{y\to1}t(y)=\infty\,,$$

for the proof of (a). But this can be shown as in Proposition 3.

Since $y\to1$ as $t\to\infty$, we get

$$0<\lim_{t\to\infty}\phi_2(t)=\lim_{t\to\infty}\lambda^{2/\alpha}e^{-\lambda t}y(t)^{1/\alpha}=\lim_{t\to\infty}2\lambda^{2/\alpha}e^{-\lambda t}=0\,,$$

i.e.

$$\lim_{t\to\infty}\phi_2(t)=0$$

and by Lemma 1(b)

$$\lim_{t\to\infty}\phi_2'(t)=0\,.$$

On the other hand by Proposition 1 we know

$$\phi_2'(t)\le 0$$

and hence if $\omega'>-\infty$, then

$$\lim_{t\to\omega'}\phi_2(t)=\infty\,.$$

If $\omega'=-\infty$, then we get

$$\lim_{t\to\omega'}\phi_2(t)=\infty\,,$$

since (1) does not have a solution bounded for $-\infty<t<\infty$. Hence from Lemma 1(d) we obtain

$$\lim_{t\to\omega'}\phi'(t)=\infty\,. \qquad \text{Q.E.D.}$$

Just as above we get an analytecal expression (2)′ i.e.

$$\phi_2(t)=\sum_{n=0}^{\infty}a_n\exp\{n(1-\sqrt{1+\alpha})-1)\lambda t\}\,,\quad a_0>0\,.$$

which converges in the neighborhood of $t=\infty$.

Now we have to discuss whether $\omega'=-\infty$ or not, and construct an analytical expression of $\phi_2(t)$ valid in the neighborhood of $t=\omega'$. For this purpose we shall prove the following lemma, which will be also used in the following sections.

Lemma 4. Let $\phi(t)$ be a solution of (1) defined for (ω',ω) where $\omega'<\omega$. Suppose that

(a) $\lim_{t\to\tau}\phi(t)=\infty$,

(b) $\lim_{t\to\tau}y(t)=\lim_{t\to\tau}\phi(t)^{-\alpha}\phi(t)^{\alpha}=\infty$,

where $\tau=\omega'$ or ω, then the following statements are valid:

(c) $|\tau|<\infty$,

(d) $z(y)=O(y^{3/2})$ as $y\to\infty$,

(e) If $\tau=\omega'$, then we get

$$\phi(t)=\left(\frac{2}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha}e^{-\lambda\omega'}(t-\omega')^{-2/\alpha}\times[1+\Sigma_{m+n>0}a_{mn}(t-\omega')^m\{(t-\omega')^{2+(4/\alpha)}(C\log(t-\omega')+B)\}^n]\,,\tag{16}$$

and if $\tau=\omega$, then we get

$$\phi(t)=\left(\frac{2}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha}e^{-\lambda\omega}(\omega-t)^{-2/\alpha}\times[1+\Sigma_{m+n>0}a_{mn}(\omega-t)^m\{(\omega-t)^{2+(4/\alpha)}(C\log(\omega-t)+B)\}^n]\tag{17}$$

where B and C are constants and $C=0$ if $4/\alpha$ is not an integer. (16) and (17) converge in the neighborhood of $t=\tau$.

Proof. From (6), (7), we get

$$z(y)=\alpha y(t)\left(\lambda+\frac{\phi'(t)}{\phi(t)}\right) \tag{18}$$

and hence

$$\lim_{y\to\infty} y^{-3/2}z(y)=\lim_{t\to\tau} y(t)^{-1/2}\frac{\phi'(t)}{\phi(t)}. \tag{19}$$

But from the assumption and Lemma 1, we get

$$y(t)\to\infty\,,\quad \phi(t)\to\infty \quad\text{and}\quad |\phi'(t)|\to\infty$$

as $t\to\tau$. Hence by l'Hospital's theorem

$$\begin{aligned}\lim_{t\to\tau} y(t)^{-1}\frac{\phi'(t)^2}{\phi(t)^2} &= \lambda^2 \lim_{t\to\tau}\frac{\phi'(t)^2}{e^{\alpha\lambda t}\phi(t)^{\alpha+2}}\\ &=\lambda^2 \lim_{t\to\tau}\frac{2\phi'(t)\phi''(t)}{\alpha\lambda e^{\alpha\lambda t}\phi(t)^{\alpha+2}+(\alpha+2)e^{\alpha\lambda t}\phi(t)^{\alpha+1}\phi'(t)}\\ &=\lambda^2 \lim_{t\to\tau}\frac{2\phi'(t)e^{\alpha\lambda t}\phi(t)^{1+\alpha}}{\alpha\lambda e^{\alpha\lambda t}\phi(t)^{\alpha+2}+(\alpha+2)e^{\alpha\lambda t}\phi(t)^{\alpha+1}\phi'(t)}\\ &=\lambda^2 \lim_{t\to\tau}\frac{2}{\alpha\lambda\dfrac{\phi(t)}{\phi'(t)}+\alpha+2}.\end{aligned}$$

But if we again use l'Hospital's theorem, then we get

$$\lim_{t\to\tau}\frac{\phi'(t)^2}{\phi(t)^2}=\lim_{t\to\tau}\lambda^{-2}y(t)=\infty\,.$$

Hence we get ultimately

$$\lim_{t\to\tau} y(t)^{-1}\frac{\phi'(t)^2}{\phi(t)^2}=\frac{2\lambda^2}{\alpha+2}$$

i.e.

$$\lim_{t\to\tau} y(t)^{-1/2}\frac{\phi'(t)}{\phi(t)}=\begin{cases}\lambda\sqrt{\dfrac{2}{\alpha+2}} & (\text{if } \tau=\omega)\\[2ex] -\lambda\sqrt{\dfrac{2}{\alpha+2}} & (\text{if } \tau=\omega').\end{cases} \tag{20}$$

From (19), (20), we obtain

$$\lim_{y\to\infty} y^{-3/2}z(y)=\begin{cases}\alpha\lambda\sqrt{\dfrac{2}{\alpha+2}} & (\text{if } \tau=\omega)\\[2ex] -\alpha\lambda\sqrt{\dfrac{2}{\alpha+2}} & (\text{if } \tau=\omega')\end{cases}$$

and we have proved (d).

Therefore if we put

$$y^{-1/2}=\eta\,,\quad z^{-1}=\eta^3(A+u)\,,\quad A=\pm\frac{1}{\alpha\lambda}\sqrt{\frac{\alpha+2}{2}}\,,$$

then we get from (8)

$$\eta\frac{du}{d\eta}=\frac{2(\alpha+\lambda)}{\alpha^2\lambda}\eta+\left(2+\frac{4}{\alpha}\right)u+\cdots \tag{21}$$

where the unwritten part is a polynomial of η and u beginning with the terms of the second degree. What we need is a solution of (21) which tends to zero as $\eta\to 0$. Since $\eta=0$ is a singularity of Briot-Bouquet type and $2+\frac{4}{\alpha}>0$, such a solution can be expressed as

$$u=\Sigma_{m+n>0}u_{mn}\eta^m[\eta^{2+(4/\alpha)}(C\log\eta+B)]^n$$

where B is an arbitrary constant and, if $2+\frac{4}{\alpha}$ is not an integer, $C=0$. Thus we get

$$z(y)^{-1}=y^{-3/2}F(y^{-1/2})$$

where

$$F(\eta)=A+\Sigma_{m+n>0}u_{mn}\eta^m[\eta^{2+(4/\alpha)}(C\log\eta+B)]^n\,.$$

Hence

$$\int_{y_0}^{y}\frac{dy}{z(y)}=-2\int_{\eta_0}^{\eta}F(\eta)d\eta\,,\quad \eta=y^{-1/2},\ \eta_0=y_0{}^{-1/2}\,.$$

From (7),

$$\frac{dy}{dt}=z(y)$$

and so

$$\int_{\eta_0}^{\eta}F(\eta)d\eta=-\frac{1}{2}(t-t_0)\,.$$

Since $y\to\infty$ as $t\to\tau$, we get

$$\lim_{\eta\to 0}\int_{\eta_0}^{\eta}F(\eta)d\eta=-\frac{1}{2}\lim_{t\to\tau}(t-t_0)=-\frac{1}{2}(\tau-t_0) \tag{22}$$

But

$$\left|\lim_{\eta\to 0}\int_{\eta_0}^{\eta}F(\eta)d\eta\right|<\infty$$

is valid since

$$F(\eta)=O(1) \quad (\eta\to 0)\,.$$

Therefore we get

$$|\tau|<\infty\,.$$

This proves (c).

From (22), we get

$$\int_0^\eta F(\eta)d\eta=A\eta[1+\Sigma_{m+n>0}\gamma_{mn}\eta^m\{\eta^{2+(4/\alpha)}(C\log\eta+B)\}^n]=\frac{1}{2}(\tau-t)\,.$$

If we put

$$L=-\frac{1}{2}\alpha\lambda\sqrt{\frac{2}{\alpha+2}}(\omega'-t)=\frac{1}{2}\alpha\lambda\sqrt{\frac{2}{\alpha+2}}(t-\omega')$$

for $\tau=\omega'$ and

$$L=\frac{1}{2}\alpha\lambda\sqrt{\frac{2}{\alpha+2}}(\omega-t)$$

for $\tau=\omega$, then

$$\eta[1+\Sigma_{m+n>0}\tilde{\gamma}_{mn}\eta^m\{\eta^{2+(4/\alpha)}(C\log\eta+B)\}^n]=L\,.$$

By Smith's lemma (See [2] Lemma 1, or [3]), we get

$$\eta=y^{-1/2}=L[1+\Sigma_{m+n>0}\gamma_{mn}L^m\{L^{2+(4/\alpha)}(C\log L+B)\}^n]\,.$$

Hence we obtain

$$y^{1/\alpha}=\left(\frac{2}{\alpha\lambda}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha}L^{-2/\alpha}[1+\Sigma_{m+n>0}\hat{a}_{mn}L^m\{L^{2+(4/\alpha)}(C\log L+B_1)\}^n]$$

where

$$B_1=B-C\log\frac{2}{\alpha\lambda}\sqrt{\frac{\alpha+2}{2}}\,.$$

Therefore if we notice that

$$\phi(t)=\psi(t)y(t)^{1/\alpha}$$

and that $\psi(t)$ has an analytical expression of the form

$$\psi(t)=\lambda^{2/\alpha}e^{-\lambda(t-\tau+\tau)}=\begin{cases}\lambda^{2/\alpha}e^{-\lambda\tau}\sum_{n=0}^{\infty}\dfrac{\lambda^n}{n!}(\tau-t)^n & (\text{if } \tau=\omega)\\[2ex] \lambda^{2/\alpha}e^{-\lambda\tau}\sum_{n=0}^{\infty}\dfrac{(-1)^n\lambda^n}{n!}(t-\tau)^n & (\text{if } \tau=\omega')\end{cases}$$

in the neighborhood of $t=\tau$. then we can obtain the desired analytical expression

(16), (17) of $\phi(t)$. Q.E.D.

Let us continue the discussion whether $\omega'=-\infty$ or not. From Proposition 5 we get

$$\lim_{t\to\omega'}\phi_2(t)=\infty\,,\quad \lim_{t\to\omega'}y(t)=\infty\,.$$

(Recall that $\phi_2(t)$ is constructed by $z_2(y)$, which is defined for $1\leq y<\infty$.) Therefore we can apply the last lemma, and hence we conclude

$$-\infty<\omega'<t_0\,.$$

Moreover we can obtain an analytical expression (16) i.e. (3).

Thus the proof of Theorem I was accomplished.

§2. The solution $\phi(t, a, b)$ with $b>\hat{\phi}'(t_0)$.

In §1 it is proved that there exists one and only one solution $\hat{\phi}(t)$ of (1) bounded as $t\to\infty$ such that

$$\hat{\phi}(t_0)=a>0\,.$$

Consequently, if we fix t_0 and a, the value of $b(=\hat{\phi}'(t_0))$ for such a solution is determined uniquely. We shall denote this value of b by $\hat{b}(t_0, a)$ or simply by $\hat{b}$. In this section we consider the case $b>\hat{b}(t_0, a)=\hat{b}$. The main result of this section is as follows:

Theorem II. Let $\phi(t, a, b)$ be a solution of (1)

$$x''=e^{\alpha\lambda t}x^{1+\alpha}\,,\quad \alpha>0,\ \lambda>0$$

with an initial condition

$$\phi(t_0, a, b)=a\,,\quad \phi'(t_0, a, b)=b\,,\quad b>\hat{b}\,,$$

and we assume $\phi(t, a, b)$ is defined for $\omega'<t<\omega$. Then the following statements are valid:

(A) ω is finite, and

$$\lim_{t\to\omega}\phi(t, a, b)=\infty\,,\quad \lim_{t\to\omega}\phi'(t, a, b)=\infty\,.$$

In the neighborhood of $t=\omega$

$$\phi(t, a, b)=\left(\frac{2}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha}e^{-\lambda\omega}(\omega-t)^{-2/\alpha}$$
$$\times[1+\Sigma_{m+n>0}a_{mn}(\omega-t)^m\{(\omega-t)^{2+(4/\alpha)}(C\log(\omega-t)+B)\}^n]$$

where B and C are constants and $C=0$ if $\dfrac{4}{\alpha}$ is not an integer.

(B) If $0<a\leq\psi(t_0)$, then there exists a number $b_3(\hat{b}<b_3)$ satisfying the following conditions:

(a) If $\hat{b}<b<b_3$, then

$$\omega'=-\infty$$

and

$$\lim_{t\to-\infty}\phi(t,a,b)=\infty\,,\quad \lim_{t\to-\infty}\phi'(t,a,b)=-\infty\,.$$

(b) If $b=b_3$, then

$$\omega'=-\infty$$

and

$$\lim_{t\to-\infty}\phi(t,a,b)=c\quad(0<c<\infty)\,,\quad \lim_{t\to-\infty}\phi'(t,a,b)=0\,.$$

In the neighborhood of $t=-\infty$

$$\phi(t,a,b)=\sum_{n=1}^{\infty}a_n e^{\alpha\lambda(n-1)t}\,,\quad 0<a_1<\infty\,.$$

(c) If $b>b_3$, then

$$-\infty<\omega'<t_0$$

and

$$\lim_{t\to\omega'}\phi(t,a,b)=0\,,\quad \lim_{t\to\omega'}\phi'(t,a,b)=d\quad(0<d<\infty)\,.$$

In the neighborhood of $t=\omega'$

$$\phi(t,a,b)=A(t-\omega')(1+\Sigma_{m+n>0}a_{mn}(t-\omega')^m(t-\omega')^{\alpha n})$$

where A is a constant.

(C) If $a>\phi(t_0)$, then there exist numbers b_3 and b_4 $(\hat{b}<b_4<b_3)$ satisfying the following conditions:

(a) If $\hat{b}<b<b_4$, then

$$-\infty<\omega'<t_0$$

and

$$\lim_{t\to\omega'}\phi(t,a,b)=\infty\,,\quad \lim_{t\to\omega'}\phi'(t,a,b)=-\infty\,.$$

In the neighborhood of $t=\omega'$

$$\phi(t,a,b)=\left(\frac{2}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha}e^{-\lambda\omega'}(t-\omega')$$
$$\times[1+\Sigma_{m+n>0}a_{mn}(t-\omega')^m\{(t-\omega')^{2+(4/\alpha)}(C\log(t-\omega')+B)\}^n]$$

where B and C are constants and $C=0$ if $\dfrac{4}{\alpha}$ is not an integer.

(b) If $b=b_4$, then

$$\omega' = -\infty$$

and

$$\lim_{t\to-\infty}\phi(t, a, b)=\infty\,, \quad \lim_{t\to-\infty}\phi'(t, a, b)=-\infty\,.$$

In the neighborhood of $t=-\infty$

$$\phi(t, a, b)=\sum_{n=0}^{\infty} a_n \exp\{(n(1+\sqrt{1+\alpha})-1)\lambda t\}\,, \quad a_0>0\,.$$

(c) If $b_4<b<b_3$, then (a) of (B) is valid.
(d) If $b=b_3$, then (b) of (B) is valid.
(e) If $b>b_3$, then (c) of (B) is valid.

In the neighborhood of $t=\omega'$

$$\phi(t, a, b)=A(t-\omega')(1+\Sigma_{m+n>0}a_{mn}(t-\omega')^m(t-\omega')^{\alpha n})$$

where A is constant.

The proof of this theorem will be accomplished by proving the following propositions.

Let $\phi(t, a, b)$ be a solution of (1) satisfying an initial condition

$$\phi(t_0, a, b)=a\,, \quad \phi'(t_0, a, b)=b\,, \quad b>\hat{b}$$

and put

$$y(t)=\psi(t)^{-\alpha}\phi(t, a, b)^{\alpha}\,, \quad z(y, b)=y'(t)$$

as in (6), (7). Then $z=z(y, b)$ is a solution of (8), and from (18) we get

$$z(y, b)=\alpha y(t)\left(\lambda+\frac{\phi'(t, a, b)}{\phi(t, a, b)}\right).$$

If $t=t_0$, then

$$z(y_0, b)=\alpha y_0\left(\lambda+\frac{b}{a}\right)$$

where

$$y_0=y(t_0)\,.$$

Hence, we have

$$z(y, b)>z(y, \hat{b})=\begin{cases} z_1(y) & \text{if} \quad 0<a<\psi(t_0) \\ 0 & \text{if} \quad a=\psi(t_0) \\ z_2(y) & \text{if} \quad a>\psi(t_0) \end{cases} \tag{23}$$

in the neighborhood of $y=y_0$. But from the uniqueness of the solution of (8) we find (23) is valid as long as both solutions are defined and holomorphic. Therefore in the domain

$$\{(y, z);\quad z > z(y, \hat{b})\}$$

we consider (8). In the remaining part of this section $\phi(t, a, b)$ and $z(y, b)$ will be denoted by $\phi(t, b)$, $z(y)$ respectively. As in the previous section we assume $\phi(t, b)$ is defined for (ω', ω) $(-\infty \leq \omega' < \omega \leq \infty)$.

First we prove property (A) in Theorem II about the behavior of $\phi(t, a, b)$ as $t \to \omega$.

Proposition 6. If $b > \hat{b}$, then $\phi(t, b)$ is a solution of (1) such that

(a) $\phi(t, b)$ is defined for (ω', ω) and ω is finite,

(b) $\lim_{t\to\omega} \phi(t, b) = \infty$, $\lim_{t\to\omega} \phi'(t, b) = \infty$,

(c) $\phi(t, b)$ has an analytical expression of the form (17) in the neighborhood of $t = \omega$.

Proof. We have the following four possibilities

(i) $-\infty < \omega < \infty$ and $\lim_{t\to\omega} \phi(t, b) = \infty$,

(ii) $-\infty < \omega < \infty$ and $\lim_{t\to\omega} \phi(t, b) = 0$,

(iii) $\omega = \infty$ and $\lim_{t\to\omega} \phi(t, b) = \infty$,

(iv) $\omega = \infty$ and $0 < \lim_{t\to\omega} \phi(t, b) < \infty$.

In the case (ii) we get

$$\lim_{t\to\omega} y(t) = \lim_{t\to\omega} \phi(t)^{-\alpha} \phi(t)^{\alpha} = 0$$

and hence if t is sufficiently close to ω,

$$z(y) = y'(t) < 0$$

is valid, since $y(t) > 0$. But from (23) we have

$$z(y) > z(y, \hat{b})$$

in the neighborhood of $y = y_0$ and so the curve $z = z(y)$ have to intersect the curve $z = z(y, \hat{b})$ in order that $y(t)$ reaches 0 as $t \to \omega$. But this contradicts the uniqueness of the solution of the equation (8) and therefore the case (ii) can never take place.

In the case (iv) $\phi(t, b)$ is bounded as $t \to \infty$. This implies $b = \hat{b}$ since such a solution is unique. Hence the case (iv) must be excluded.

Finally we consider the case (i) and (iii). From Lemma 1, we get

$$\lim_{t\to\omega} \phi'(t, b) = \infty,$$

and we also get

$$\lim_{t\to\omega} y(t) = \lim_{t\to\omega} \phi(t)^{-\alpha} \phi(t, b)^{\alpha} = \infty.$$

Therefore we can apply Lemma 4 and we find that

$$-t_0 < \omega < \infty$$

and that $\phi(t, b)$ has an analytical expression of the form (17) in the neighborhood of $t = \omega$. Q.E.D.

Next we investigate the behavior of $\phi(t, b)$ as $t \to \omega'$.

First we shall prove (B) in Theorem II and for this purpose we assume

$$a(=\phi(t_0))<\psi(t_0)\,.$$

Proposition 7. If $a<\psi(t_0)$, then

$$\lim_{t\to\omega'} y(t)=\lim_{t\to\omega'} \psi(t)^{-\alpha}\phi(t,b)^{\alpha}=0$$

for each $b(>\hat{b})$.

Proof. If $-\infty<\omega'<\infty$, then

$$\lim_{t\to\omega'} \phi(t,b)=0 \quad \text{or} \quad \lim_{t\to\omega'} \phi(t,b)=\infty\,.$$

If $\lim_{t\to\omega'} \phi(t,b)=\infty$, then $\lim_{t\to\omega'} y(t)=\infty$. But from $a<\psi(t_0)$ we get $y(t)<1$ and from (23) we get

$$z(y)=y'(t)>z_1(y)>0\,.$$

Hence $y(t)$ is an increasing function and so we get

$$\lim_{t\to\omega'} y(t)\leq 1\,.$$

Therefore

$$\lim_{t\to\omega'} \phi(t)\neq\infty\,.$$

Hence

$$\lim_{t\to\omega'} \phi(t)=0$$

which implies

$$\lim_{t\to\omega'} y(t)=0\,.$$

If $\omega'=-\infty$, then

$$\lim_{t\to\omega'} \phi(t)=\infty \quad \text{or} \quad \lim_{t\to\omega'} \phi(t)<\infty\,.$$

If $\lim_{t\to-\infty} \phi(t)<\infty$, then we get

$$\lim_{t\to-\infty} y(t)=\lim_{t\to-\infty} \psi(t)^{-\alpha}\,\phi(t)^{\alpha}=0\,.$$

If $\lim_{t\to-\infty} \phi(t)=\infty$, then

$$\lim_{t\to-\infty} \phi'(t)=-\infty$$

by Lemma 1. Since we know that $y(t)$ is an increasing function and

$$0<y(t)<1\,,$$

we get

$$0\leq\lim_{t\to-\infty} y(t)=\lim_{t\to-\infty} \frac{\phi(t)}{\psi(t)}<1\,.$$

On the other hand, we have

$$\lim_{t\to-\infty}\frac{\phi(t)}{\psi(t)}=\lim_{t\to-\infty}\frac{\phi''(t)}{\psi''(t)}=\lim_{t\to-\infty}\frac{\phi(t)^{1+\alpha}}{\psi(t)^{1+\alpha}}$$

by l'Hospital's theorem. This shows that

$$c=\lim_{t\to-\infty}\frac{\phi(t)}{\psi(t)}=0 \quad \text{or} \quad 1$$

and

$$\lim_{t\to-\infty} y(t)=c^{\alpha}\,.$$

Since

$$\lim_{t\to-\infty} y(t)<1\,,$$

we must have

$$c=0\,,$$

and in this case we also get

$$\lim_{t\to-\infty} y(t)=0\,. \qquad \text{Q.E.D.}$$

Proposition 8. There exists one and only one solution $z=z_3(y)$ of (8) such that
(a) $\lim_{y\to0} z_3(y)=0$,
(b) $\lim_{y\to0}\dfrac{z_3(y)}{y}=\alpha\lambda+0$.

Proof. If we put

$$v=y^{-1}z-\alpha\lambda$$

then we get

$$\frac{dv}{dy}=\frac{\alpha^2\lambda^2 y-v^2}{\alpha y(v+\alpha\lambda)} \tag{24}$$

or a two-dimensional autonomous system

$$\begin{cases}\dfrac{dy}{ds}=\alpha^2\lambda y+\alpha yv\\[2ex]\dfrac{dv}{ds}=\alpha^2\lambda^2 y-v^2\end{cases} \tag{25}$$

The coefficient matrix

$$\begin{bmatrix}\alpha^2\lambda & 0\\ \alpha^2\lambda^2 & 0\end{bmatrix}$$

of the linear terms of right-hand members of (25) has eigenvalues 0, $\alpha^2\lambda$. Hence

(25) has a solution represented as

$$y=(Ce^{\alpha^2\lambda s})+a_2(Ce^{\alpha^2\lambda s})^2+\cdots$$
$$v=\lambda(Ce^{\alpha^2\lambda s})+b_2(Ce^{\alpha^2\lambda s})^2+\cdots$$

convergent in the neighborhood of $s=-\infty$. Hence we get

$$v=\lambda y+O(y^2)\,.$$

Since

$$v=y^{-1}z-\alpha\lambda$$

we get

(26) $$z=\alpha\lambda y+\lambda y^2+O(y^3)=\alpha\lambda y+\lambda y^2(1+O(1))\,.$$

Therefore, if we define $z_3(y)$ by (26), we obtain

$$z_3(y)>\alpha\lambda y$$

in the neighborhood of $y=0$, and $z_3(y)$ satisfies the conditions (a), (b).

Next we shall prove the uniqueness of such a solution. We assume that $z=z_0(y)$ is a solution of (8) which satisfies the conditions (a), (b). If we put

$$v=y^{-1}z_0-\alpha\lambda\,,$$

then $v=v(y)$ is a solution of (25) and v tends to 0 as $y\to 0$.

From (25) we get

$$\frac{dv}{dy}=\left(\alpha\lambda^2-\frac{v^2}{\alpha\lambda}\right)\Big/(v+\alpha\lambda)$$

and so

$$\frac{dv}{dy}<\frac{\alpha\lambda^2}{v+\alpha\lambda}\,.$$

Hence if $y>0$ is sufficiently close to 0, we get

$$\frac{dv}{dy}\leq\lambda$$

and so

$$v\leq\lambda y\,.$$

Since we get

$$\frac{v^2}{y}\to 0$$

as $y\to 0$ from this, we obtain

$$\lim_{y\to 0}\frac{dv}{dy}=\lambda$$

and

$$\lim_{y\to 0}\frac{v}{y}=\lambda\,.$$

If we put

$$w=y^{-1}v-\lambda\,,$$

then w tends to 0 as $y\to 0$ and we get

$$\frac{dw}{dy}=\frac{-(\alpha+1)y(w+\lambda)^2-\alpha^2\lambda w}{\alpha y(yw+\lambda y+\alpha\lambda)} \tag{27}$$

or

$$\begin{cases}\dfrac{dy}{ds}=\alpha^2\lambda y+\alpha\lambda y^2+\alpha y^2 w\\[2ex] \dfrac{dw}{ds}=-(\alpha+1)\lambda^2 y-\alpha^2\lambda w-2(\alpha+1)\lambda yw-(\alpha+1)yw^2\,.\end{cases} \tag{28}$$

The matrix

$$\begin{bmatrix}\alpha^2\lambda & 0\\ -(\alpha+1)\lambda^2 & -\alpha^2\lambda\end{bmatrix}$$

has eigenvalue $\pm\alpha^2\lambda$, and hence $(y,w)=(0,0)$ is a saddle point. Therefore there exists only two solutions which tend to $(0,0)$ and these are represented as

$$\begin{cases}y\equiv 0\\ w=Ce^{-\alpha^2\lambda s}\end{cases} \tag{29}$$

and

$$\begin{cases}y=a_1(Ce^{\alpha^2\lambda s})+a_2(Ce^{\alpha^2\lambda s})^2+\cdots\\ w=b_1(Ce^{\alpha^2\lambda s})+b_2(Ce^{\alpha^2\lambda s})^2+\cdots\end{cases} \tag{30}$$

But (29) is not a solution of (27). From (30) we get

$$w=-\frac{(\alpha+1)\lambda}{2\alpha^2}y+\sum_{n=2}^{\infty}c_n y^n\,,$$

where $\sum_{n=2}^{\infty}c_n y^n$ is a convergent power series and c_n are uniquely determined constants. Since

$$w=y^{-1}v-\lambda=y^{-1}(y^{-1}z_0-\alpha\lambda)-\lambda$$

is valid

(31) $$z_0=\alpha\lambda y+\lambda y^2-\frac{(\alpha+1)\lambda}{2\alpha^2}y^3+\sum_{n=2}^{\infty}c_{n-2}y^n\,.$$

$z_3(y)$ is also represented by the right-hand member of (31) and hence

$$z_0(y)=z_3(y)$$

which shows the uniqueness of $z_3(y)$. Q.E.D.

Proposition 9. Let $y(t)$ be a solution of

$$\frac{dy}{dt}=z_3(y)$$

with an initial condition

$$y(t_0)=\phi(t_0)^{-\alpha}a^{\alpha}$$

and put

$$\phi_3(t)=\phi(t)y(t)^{1/\alpha}\,.$$

Then the following statements are valid:

(a) $\phi_3(t)$ is defined for $-\infty<t<\omega$ $(t_0<\omega<\infty)$

(b) $\lim_{t\to-\infty}\phi_3(t)=c$ $(0<c<\infty)$, $\lim_{t\to-\infty}\phi_3'(t)=0$,

(c) $\phi_3(t_0)=a$.

Moreover in the neighborhood of $t=-\infty$,

$$\phi_3(t)=\sum_{n=1}^{\infty}a_ne^{\alpha\lambda(n-1)t}\,,\quad a_1>0\,.$$

Proof. (c) is evident. Since

$$\frac{dy}{dt}=z_3(y)=\alpha\lambda y+\lambda y^2+\cdots\,,$$

we get

$$y=\sum_{n=1}^{\infty}\tilde{a}_ne^{\alpha\lambda nt}\,,\quad \tilde{a}_1>0\,.$$

Therefore

(32) $$\phi_3(t)=\sum_{n=1}^{\infty}a_ne^{\alpha\lambda(n-1)t}\,,\quad a_1>0$$

where converges in the neighborhood of $t=-\infty$. Hence

$$\phi_3'(t)=\sum_{n=2}^{\infty}a_n\alpha\lambda(n-1)e^{\alpha\lambda(n-1)t}\to0$$

as $t\to-\infty$. Q.E.D.

The last proposition implies (b) of (B) in Theorem II.

The value $\varphi_3'(t_0)$ will be denoted by $b_3(>\hat{b})$ hereafter. For further discussion, we have to examine the critical point $(0,0)$ of (9) in more detail.

Lemma 5. If $z(y)$ is a solution of (8) which tends to 0 as $y\to 0$, then

$$\lim_{y\to 0}\frac{z(y)}{y}=\alpha\lambda \quad \text{or} \quad \pm\infty .$$

Proof. First we shall show the existence of the limit of $\frac{z(y)}{y}$ as $y\to 0$. For that purpose we consider the autonomous system (9).

If we put

$$f(y)=\gamma y ,$$

then

$$\frac{d}{ds}(z-f(y))\Big|_{z=f(y)}=\alpha^2\lambda^2y^2\left(y-\frac{(\gamma-\alpha\lambda)^2}{\alpha^2\lambda^2}\right).$$

Hence if $\gamma=\alpha\lambda$, then

$$\frac{d}{ds}(z-f(y))\Big|_{z=f(y)}>0$$

and if $\gamma\neq\alpha\lambda$, then

$$\frac{d}{ds}(z-f(y))\Big|_{z=f(y)}<0$$

when y is sufficiently close to 0. Therefore the slope $\frac{z(y)}{y}$ can not oscillate and the limit of $\frac{z(y)}{y}$ as $y\to\infty$ exists.

Now we put

$$d=\lim_{y\to 0}\frac{z(y)}{y}.$$

From (8) we get

$$\frac{dz}{dy}=\frac{\alpha-1}{\alpha}\cdot\frac{z}{y}+2\lambda-\alpha\lambda^2\cdot\frac{y}{z}+\alpha\lambda^2\cdot\frac{y}{z}y .$$

Hence if $-\infty<d<\infty$, then

$$\lim_{y\to 0}\frac{dz}{dy}=\frac{\alpha-1}{\alpha}d+2\lambda-\frac{\alpha\lambda^2}{d}$$

i.e. $\lim_{y\to 0}\frac{dz}{dy}$ exists. Therefore l'Hospital's theorem shows

$$\lim_{y\to 0}\frac{dz}{dy}=d ,$$

which implies

$$d=\frac{\alpha-1}{\alpha}d+2\lambda-\frac{\alpha\lambda^2}{d} \quad \text{i.e. } d=\alpha\lambda . \qquad \text{Q.E.D.}$$

Proposition 10 and Proposition 11 will show how the solution $\phi(t,b)$ behaves as $t\to\omega'$ when b satisfies either $\hat{b}<b<b_3$ or $b>b_3$.

Proposition 10. If $\hat{b}<b<b_3$, then $\phi(t,b)$ is a solution of (1) such that
(a) $\phi(t,b)$ is defined for $-\infty<t<\omega(t_0<\omega<\infty)$,
(b) $\lim_{t\to-\infty}\phi(t,b)=\infty$, $\lim_{t\to-\infty}\phi'(t,b)=-\infty$.

Proof. If for this $\phi(t,b)$ we define $y(t)$ and $z(y)$ by (6), (7), then $z(y)$ is a solution of (8) and, since $\hat{b}<b<b_3$, we get

$$z_1(y)<z(y)<z_3(y)$$

if $y<1$, which shows

$$\lim_{y\to0} z(y)=0 .$$

But from Lemma 5 and the uniqueness of $z_3(y)$ we get

$$\lim_{y\to0}\frac{z(y)}{y}=\alpha\lambda-0 .$$

Hence if y is sufficiently close to 0, then

$$z(y(t))<\alpha\lambda y(t) . \tag{33}$$

From (33) and (18) i.e.

$$z(y)=\alpha y(t)\left(\lambda+\frac{\phi'(t,b)}{\phi(t,b)}\right)$$

we find that there exists τ and $\varepsilon>0$ such that

$$\phi'(\tau,b)<-\varepsilon<0 .$$

Since $\phi''(t,b)\geq0$, we get

$$\phi'(t,b)<-\varepsilon$$

if $t<\tau$. Integrating both sides,

$$\phi(\tau,b)-\phi(t,b)<-\varepsilon(t-\tau)\,, \quad t<\tau$$

i.e.

$$\phi(t,b)>\phi(\tau,b)+\varepsilon(\tau-t)\,, \quad t<\tau$$

which shows the existence of an ω' such that

$$\lim_{t\to\omega'}\phi(t,b)=\infty \quad (-\infty\leq\omega'<t_0) .$$

But if $-\infty<\omega'<t_0$, then

$$\lim_{t\to\omega'}\phi(t, b)=\lim_{t\to\omega'}\psi(t)y(t)^{1/\alpha}=0 .$$

Hence

$$\omega'=-\infty .$$

The Lemma 1 shows

$$\lim_{t\to-\infty}\phi'(t, b)=-\infty . \qquad \text{Q.E.D.}$$

Proposition 11. If $b>b_3$, then $\phi(t, b)$ is a solution of (1) such that
(a) $\phi(t, b)$ is defined for $\omega'<t<\omega$ where $-\infty<\omega'<t_0<\omega<\infty$,
(b) $\lim_{t\to\omega'}\phi(t, b)=0$, $0<\lim_{t\to\omega'}\phi'(t, b)<\infty$.
Moreover in the neighborhood of $t=\omega'$,

$$\phi(t, b)=A(t-\omega')[1+\Sigma_{m+n>0}a_{mn}(t-\omega')^m(t-\omega')^{\alpha n}] .$$

Proof. For this $\phi(t, b)$ we define $y(t)$ and $z(y)$ by (6), (7). From Proposition 7, we get

$$\lim_{t\to\omega'}y(t)=0 .$$

Since $b>b_3$, we get

$$z(y)>z_3(y) .$$

Hence we have the following alternative;
(i) $\lim_{y\to0}z(y)=0$, $\lim_{y\to0}\dfrac{z(y)}{y}=\infty$,
(ii) $\lim_{y\to0}z(y)=c$, $0<c\leq\infty$.
(Notice Lemma 5.) In any of these two cases we get

$$\lim_{y\to0}\frac{z(y)}{y}=\infty$$

and hence from (18)

$$\lim_{t\to\omega'}\frac{\phi'(t, b)}{\phi(t, b)}=\infty . \tag{34}$$

Since

$$\phi(t, b)\geq0 ,$$

we get

$$\lim_{t\to\omega'}\phi'(t, b)\geq0 .$$

Hence if we observe that $\phi''(t)$ is positive, then

$$\lim_{t\to\omega'}\phi'(t, b)\neq\infty .$$

Therefore from (34) we have

$$\lim_{t\to\omega'}\phi(t,b)=0.$$

If we assume

$$\lim_{t\to\omega'}\phi'(t,b)=0,$$

we obtain

$$\lim_{t\to\omega'}\frac{\phi'(t,b)^2}{\phi(t,b)^2}=\lim_{t\to\omega'}\frac{2\phi'(t,b)\phi''(t,b)}{2\phi(t,b)\phi'(t,b)}=\lim_{t\to\omega'}e^{\alpha\lambda t}\phi(t,b)^{\alpha}=0$$

in contradiction with (34), and hence

$$0<\lim_{t\to\omega'}\phi(t,b)<\infty.$$

Consequently ω' must be finite.

We can construct an analytical expression of this $\phi(t,b)$ (where $b>b_3$) in the neighborhood of $t=\omega'$. In (8) we put

$$w=yz(y)^{-1}$$

and we get

$$y\frac{dw}{dy}=\frac{w}{\alpha}-2\lambda w^2+\alpha\lambda^2(1-y)w^3.$$

Since this equation is of Briot-Bouquet type, we obtain

$$w=\Sigma_{m+n>0}w_{mn}y^m(By^{1/\alpha})^n,\quad w_{01}=1$$

where B is an arbitrary constant and hence

$$z(y)^{-1}=By^{(1/\alpha)-1}(1+\Sigma_{m+n>0}w_{mn}y^m(By^{1/\alpha})^n)>0.$$

Therefore we obtain

$$\begin{aligned}t-\omega'&=\int_0^y By^{(1/\alpha)-1}(1+\Sigma_{m+n>0}w_{mn}y^m(By^{1/\alpha})^n)dy\\&=B\alpha y^{(1/\alpha)}(1+\Sigma_{m+n>0}\tilde{w}_{mn}y^m(By^{1/\alpha})^n).\end{aligned}$$

Now put

$$\eta=By^{1/\alpha},\quad \zeta=\frac{1}{\alpha}(t-\omega')$$

then

$$\eta[1+\Sigma_{m+n>0}\tilde{w}_{mn}(B^{-1}\eta)^{m\alpha}\eta^n]=\zeta.$$

By applying Smith's lemma we get

$$\eta=\zeta[1+\Sigma_{m+n>0}\hat{w}_{mn}\zeta^n(B^{-1}\zeta)^{m\alpha}]$$

i.e.

$$y^{1/\alpha}=\frac{\Gamma}{\alpha}(t-\omega')[1+\Sigma_{m+n>0}\hat{w}_{mn}(t-\omega')^{m}(t-w')^{\alpha n}]$$

where $\Gamma=B^{-1}$. Since

$$\psi(t)=\lambda^{-2/\alpha}e^{-\lambda t}=\lambda^{-2/\alpha}e^{-\lambda\omega}\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{n!}(t-\omega')^n$$

we obtain

(35)
$$\begin{aligned}\phi(t, b)&=\psi(t)y(t)^{1/\alpha}\\&=\frac{\Gamma}{\alpha}\lambda^{2/\alpha}e^{-\lambda\omega'}(t-\omega')[1+\Sigma_{m+n>0}a_{mn}(t-\omega')^{m}(t-\omega')^{\alpha n}]\end{aligned}$$

which is desired analytical expression. Q.E.D.

Thus we could prove (a) and (c) of (A) in the Theorem II.

Finally we must verify (B), where $a>\psi(t_0)$ is assumed.

Proposition 12. There exists one and only one solution $z=z_4(y)$ of (8) such that

(a) $z_4(y)$ is defined for $1<y<\infty$,

(b) $z_4(y)>0$,

(c) $\lim_{t\to 1+0}z_4(y)=+0$.

Proof. We consider the autonomous system (9) again. From (11) we find that a solution of (9) which tends to the saddle point $(1,0)$ can be represented as

$$z=\frac{\mu_i}{\alpha}(y-1)+\cdots, \quad (i=1,2).$$

If we put

$$z_4(y)=\frac{\mu_1}{\alpha}(y-1)+\cdots, \quad (y>1)$$

then $z_4(y)$ satisfies (c). Since $(1,0)$ is a saddle point, a solution satisfying (c) is unique.

On the line $y>1$, $z=0$, we have

$$\frac{dy}{ds}=0, \quad \frac{dz}{ds}=-\alpha^2\lambda^2(y^2-y^3)>0$$

from (9). Hence the orbit of $z=z_4(y)$ cannot pass through the line

$$y>1, \quad z=0$$

and so (b) is valid.

If we put $z=\frac{1}{\zeta}$, then (8) is transformed into

(36)
$$\frac{d\zeta}{dy}=-\frac{(\alpha-1)\zeta+2\alpha yz\zeta^2-\alpha^2\lambda^2(y^2-y^3)\zeta^3}{\alpha y}.$$

Suppose that there exists a number η such that

$$1<\eta<\infty, \quad \lim_{y\to\eta-0} z_4(y)=\infty,$$

then $\zeta=0$ if $y=\eta$. Since $(\zeta, 0)$ is a nonsingular point of (36) and $\zeta\equiv 0$ is a solution of (36), then the uniqueness of the solution of (8) shows

$$z_4(y)\equiv\infty$$

which is a contradiction. Therefore (a) is valid. Q.E.D.

Proposition 13. Let $y(t)$ be a solution of

$$\frac{dy}{dt}=z_4(y), \quad y(t_0)=\phi(t_0)^{-\alpha}a^{\alpha}, \quad (a>\phi(t_0))$$

and put

$$\phi_4(t)=\phi(t)y(t)^{1/\alpha}$$

then $\phi_4(t)$ is a solution of (1) such that

(a) $\phi_4(t)$ is defined for $-\infty<t<\omega(t_0<\omega<\infty)$,

(b) $\lim_{t\to-\infty}\phi_4(t)=\infty$, $\lim_{t\to-\infty}\phi_4'(t)=-\infty$,

(c) $\phi(t_0)=a$.

Moreover in the neighborhood of $t=-\infty$

$$\phi_4(t)=\sum_{n=0}^{\infty} a_n \exp\{n(1-\sqrt{1+\alpha})-1)\lambda t\}, \quad a_0>0.$$

Proof. Since

$$\frac{dy}{dt}=z_4(y)=(1+\sqrt{1+\alpha})\lambda(y-1)+\cdots$$

we get

$$\log(y-1)+\cdots+C=(1+\sqrt{1+\alpha})\lambda t \tag{37}$$

where C is a constant and the unwritten part represents a power series of $y-1$. Hence

$$t\to-\infty$$

as $y\to 1$. Since

$$\frac{dy}{dt}=z_4(y)>0,$$

we find

$$y\to 1$$

as $t\to-\infty$, which implies (a).

Since

$$y(t)>1\,,$$

we get

$$\lim_{t\to-\infty}\phi_4(t)=\lim_{t\to-\infty}\lambda^{2/\alpha}e^{-\lambda t}y(t)^{1/\alpha}=\infty$$

and from Lemma 1

$$\lim_{t\to-\infty}\phi'(t)=-\infty\,.$$

From (37) we get

$$y=1+\sum_{n=1}^{\infty}\tilde{a}_n\exp\{n(1+\sqrt{1+\alpha})\lambda t\}$$

and

$$\phi_4(t)=\sum_{n=0}^{\infty}a_n\exp\{n(1+\sqrt{1+\alpha})-1)\lambda t\}\,,\quad a_0=\lambda^{2/\alpha}>0$$

in the neighborhood of $t=-\infty$, which is a desired analytical expression of $\phi_4(t)$.
Q.E.D.

The last proposition implies (b) of (C). $\phi_4'(t_0)$ will be denoted b_4 ($\hat{b}<b_4<b_3$) hereafter. The solution $z(y, b)$ of (8) with $b>b_4$ is sure to pour into the domain

$$0<y<1\,,\quad z>0$$

and hence the behavior of $\phi(t, b)$ has been already investigated in Proposition 9, 10 and 11. Therefore we shall discuss only the case when $\hat{b}<b<b_4$ is valid.

Proposition 14. If $\hat{b}<b<b_4$, then $\phi(t, b)$ is a solution of (1) such that
(a) $\phi(t, b)$ is defined for $\omega'<t<\omega$ ($-\infty<\omega'<t_0<\omega<\infty$),
(b) $\lim_{t\to\omega'}\phi(t, b)=\infty$, $\lim_{t\to\omega'}\phi'(t, b)=-\infty$.

Proof. If we define $y(t)$ and $z(y)$ by (6) and (7) for the $\phi(t, b)$ with $\hat{b}<b<b_4$, then $z(y)$ is a solution of (8) and from $\hat{b}<b<b_4$ we get

$$z_2(y)<z(y)<z_4(y)\,.$$

Hence $z(y)$ is defined only for $y>1$. Since y is defined by (6), we get

$$\phi(t, b)>\phi(t)\,.$$

But since $\phi(t)$ tends to ∞ as $t\to-\infty$, we get

$$\lim_{t\to\omega'}\phi(t, b)=\infty\,,\quad -\infty\leq\omega'<t_0\,.$$

From Lemma 1 we get also

$$\lim_{t\to\omega'}\phi'(t, b)=-\infty\,.$$

If $\omega'>-\infty$, then we get directly

$$\lim_{t\to\omega'}y(t)=\infty$$

from (6) and if $\omega'=-\infty$, then we get

$$\lim_{t\to\omega'} y(t)=\infty$$

by l'Hospital's theorem. Hence we can apply Lemma 4 and find $\omega'>-\infty$ and that $\phi(t, b)$ is represented as (16) in the neighborhood of $t=\omega'$. Q.E.D.

Thus the proof of Theorem II was completed.

§ 3. The solution $\phi(t, a, b)$ with $b<\hat{\phi}'(t_0)$.

Let $\phi(t, a, b)$ be a solution of (1) with an initial condition

$$\phi(t_0, a, b)=a\,, \quad \phi'(t_0, a, b)=b\,, \quad -\infty<t_0<\infty$$

where $b<\hat{b}$. As in the previous sections we assume $\phi(t, a, b)$ is defined for $\omega'<t<\omega$. Then the following theorem is valid.

Theorem III. (A) If $b<\hat{b}$, then ω is finite and

$$\lim_{t\to\omega}\phi(t, a, b)=0\,, \quad \lim_{t\to\omega}\phi'(t, a, b)=d \quad (-\infty<d<0)\,.$$

In the neighborhood of $t=\omega$,

$$\phi(t, a, b)=A(\omega-t)[1+\Sigma_{m+n>0}a_{mn}(\omega-t)^m(\omega-t)^{\alpha n}]$$

where A is constant.

(B) If $0<a\leq\hat{\phi}(t_0)$, then there exists a number b_5 satisfying the following conditions:

(a) If $b_5<b<\hat{b}$, then

$$\omega'=-\infty\,,$$

and

$$\lim_{t\to-\infty}\phi(t, a, b)=\infty\,, \quad \lim_{t\to-\infty}\phi'(t, a, b)=-\infty\,.$$

(b) If $b=b_5$, then

$$\omega'=-\infty$$

and

$$\lim_{t\to-\infty}\phi(t, a, b)=\infty\,, \quad \lim_{t\to-\infty}\phi'(t, a, b)=-\infty\,.$$

In the neighborhood of $t=-\infty$

$$\phi(t, a, b)=\sum_{n=0}^{\infty} a_n \exp\{(n(1+\sqrt{1+\alpha})-1)\lambda t\}\,, \quad a_0>0\,.$$

(c) If $b<b_5$, then

$$-\infty<\omega'<t_0$$

and

$$\lim_{t\to\omega'}\phi(t, a, b)=\infty, \quad \lim_{t\to\omega'}\phi'(t, a, b)=-\infty.$$

In the neighborhood of $t=\omega'$

$$\phi(t, a, b)=\left(\frac{2}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)^{2/\alpha} e^{-\lambda\omega'}(t-\omega')^{-2/\alpha}$$
$$\times[1+\Sigma_{m+n>0}a_{mn}(t-\omega')^m\{(t-\omega')^{2+(4/\alpha)}(C\log(t-\omega')+B)\}^n]$$

where B and C are constants and $C=0$ if $\dfrac{4}{\alpha}$ is not an integer.

(C) If $a>\phi(t_0)$, then, for an arbitrary $b(<\hat{b})$, (c) of (B) is valid.

Proof. For simplicity $\phi(t, a, b)$ will be denoted by $\phi(t, b)$. For $\phi(t, b)$ with $b<\hat{b}$ we define $y(t)$ and $z(y)$ by (6) and (7). Since $b<\hat{b}$, we get

$$z(y)<z(y, \hat{b})=\begin{cases} z_1(y) & (\text{if } 0<y<1) \\ 0 & (\text{if } y=1) \\ z_2(y) & (\text{if } y>1). \end{cases} \tag{38}$$

On the other hand, we have the following four possibilities:

(i) $-\infty<\omega<\infty$ and $\lim_{t\to\omega}\phi(t, b)=0$,

(ii) $-\infty<\omega<\infty$ and $\lim_{t\to\omega}\phi(t, b)=\infty$,

(iii) $\omega=\infty$ and $\lim_{t\to\omega}\phi(t, b)<\infty$,

(iv) $\omega=\infty$ and $\lim_{t\to\omega}\phi(t, b)=\infty$.

In the cases (ii) and (iv), we get

$$\lim_{t\to\omega} y(t)=\infty.$$

But from (38) we get

$$z(y)=y'(t)<0$$

for the sufficiently large y. Hence (ii) and (iv) cannot take place.

(iii) can be excluded by the uniqueness of a bounded solution of (1) as $t\to\infty$ which is showed in § 1.

Consequently (i) does take place.

In this case we get

$$\lim_{y\to 0}\frac{z(y)}{y}=\infty$$

from Lemma 5. Hence we obtain a desired analytical expression

$$\phi(t, b)=A(\omega-t)[1+\Sigma_{m+n>0}a_{mn}(\omega-t)^m(\omega-t)^{\alpha n}]$$

by following the process of constructing (35).

By the uniqueness of the solution of (1) we get also

$$\lim_{t\to\omega}\phi'(t, b)=d, \quad -\infty<d<0$$

and (A) was proved.

We shall show (B) where $0<a\leq\phi(t_0)$ is assumed. For this purpose we consider the autonomous system (9). Since $(1,0)$ is a saddle point of (9), we get a unique solution $z_5(y)$ of (8) such that

(a) $z_5(y)$ is defined for $0<y<1$,

(b) $z_5(y)<0$,

(c) $\lim_{y\to 1-0}z_5(y)=-0$.

In fact we obtain

$$z_5(y)=(1+\sqrt{1+\alpha})\lambda(y-1)+\cdots$$

from (11). If we define

$$\phi_5(t)=\phi(t)y(t)^{1/\alpha}, \quad b_5=\phi_5'(t_0)$$

where $y(t)$ is a solution of

$$\frac{dy}{dt}=z_5(y), \quad y(t_0)=\phi(t_0)^{-\alpha}a^{\alpha},$$

then the same argument as in Proposition 13 shows that $\phi_5(t)$ satisfies the conclusion of (b) of (B).

Here we consider the following four possibilities:

(i) $-\infty<\omega'<\infty$ and $\lim_{t\to\omega'}\phi(t,b)=0$,

(ii) $-\infty<\omega'<\infty$ and $\lim_{t\to\omega'}\phi(t,b)=\infty$,

(iii) $\omega'=-\infty$ and $\lim_{t\to\omega'}\phi(t,b)<\infty$,

(iv) $\omega'=-\infty$ and $\lim_{t\to\omega'}\phi(t,b)=\infty$.

In the case (iv), l'Hospital's theorem shows

$$\begin{aligned}\lim_{t\to\omega'}y(t)^{1/\alpha}&=\lim_{t\to\omega'}\frac{\phi(t,b)}{\phi(t)}=\lim_{t\to\omega'}\frac{\phi''(t,b)}{\phi''(t)}=\lim_{t\to\omega'}\frac{\phi(t,b)^{1+\alpha}}{\phi(t)^{1+\alpha}}\\&=\lim_{t\to\omega'}y(t)^{(1/\alpha)+1}.\end{aligned}$$

and hence we get

$$\lim_{t\to\omega'}y(t)=0,\quad 1 \text{ or } \infty.$$

On the other hand, in the cases (i) and (iii) we get

$$\lim_{t\to\omega'}y(t)=0$$

and in the case (ii) we get

$$\lim_{t\to\omega'}y(t)=\infty.$$

Therefore if $b_5<b<\hat{b}$ i.e. $z_5(y)<z(y)$, then we get

$$\lim_{t\to\omega'}y(t)=0,$$

and if $b<b_5$ i.e. $z(y)<z_5(y)$, then we get

$$\lim_{t\to\omega'}y(t)=\infty,$$

since the uniqueness of the solution of the equation (8) is valid.

Hence if $b_5 < b < \hat{b}$, then we get

$$\lim_{y\to 0} \frac{z(y)}{y} = \alpha\lambda - 0$$

from Lemma 5 and the uniqueness of $z_3(y)$, and the same argument as in Proposition 10 shows (a) of (B).

If $b < b_5$, then we can apply Lemma 4, which implies (c) of (B).

If $a > \psi(t_0)$, $b < \hat{b}$, then

$$\lim_{t\to\omega'} y(t) \neq 0$$

which implies that the cases (i) and (iii) cannot take place. Hence we can apply Lemma 4 and (C) was proved. Q.E.D.

Acknowledgment

Authors express their gratitude to Prof. T. Saito for his help and guidance in the preparation of this paper.

References

[1] T. Saito, On bounded solutions of $x'' = t^{\beta} x^{1+\alpha}$, Tokyo J. Math., vol. 1, no. 1 (1978), 57–75.

[2] T. Saito, Solutions of $x'' = t^{\alpha\lambda-2} x^{1+\alpha}$ with movable singularity, Tokyo J. Math., vol. 2, no. 2 (1979), 262–283.

[3] R. A. Smith, Singularities of solutions of certain plane autonomous systems, Proc. Roy. Soc. Edinburgh, Sect. A, vol. 72, 4 (1973/74), 307–315.